# Relational Divisibility

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**Abstract**

We formalise key concepts and axioms of the divisibility relation on natural numbers using relation algebras. They use standard relational constructions for extrema, bounds, suprema, the univalent part and symmetric quotients, which we also formalise. We moreover prove that mono-atomic elements correspond to join-irreducible elements under the divisibility axioms.

## Contents



## 1 Relational Constructions

**theory** *Relational-Constructions*

**imports** *Stone-Relation-Algebras.Relation-Algebras*

**begin**

This theory defines relational constructions for extrema, bounds and suprema, the univalent part and symmetric quotients. All definitions and most properties are standard; for example, see [1, 3, 4, 5]. Some properties are new. We start with a few general properties of relations and orders.

**context** *bounded-distrib-allegory*
**begin**

**lemma** *transitive-mapping-idempotent*:
  *transitive* $x \Longrightarrow$ *mapping* $x \Longrightarrow$ *idempotent* $x$
  **by** (*smt* (*verit*, *ccfv-threshold*) *conv-dist-comp conv-involutive epm-3 inf.order-iff top-greatest total-conv-surjective transitive-conv-closed mult-assoc*)

**end**

**context** *pd-allegory*
**begin**

**lemma** *comp-univalent-complement*:
  **assumes** *univalent* $x$
    **shows** $x * -y = x * top \sqcap -(x * y)$
**proof** (*rule order.antisym*)
  **show** $x * -y \leq x * top \sqcap -(x * y)$
    **by** (*simp add*: *assms comp-isotone comp-univalent-below-complement*)
  **show** $x * top \sqcap -(x * y) \leq x * -y$
    **by** (*metis inf.sup-left-divisibility inf-top.left-neutral theorem24xxiii*)
**qed**

**lemma** *comp-injective-complement*:
  *injective* $x \Longrightarrow -y * x = top * x \sqcap -(y * x)$
  **by** (*smt* (*verit*, *ccfv-threshold*) *antisym-conv comp-injective-below-complement complement-conv-sub dedekind-2 inf.bounded-iff mult-left-isotone order-lesseq-imp top.extremum*)

**lemma** *strict-order-irreflexive*:
  *irreflexive* $(x \sqcap -1)$
  **by** *simp*

**lemma** *strict-order-transitive-1*:
  *antisymmetric* $x \Longrightarrow$ *transitive* $x \Longrightarrow x * (x \sqcap -1) \leq x \sqcap -1$
  **by** (*smt* (*verit*, *best*) *bot-unique inf.order-trans inf.semilattice-order-axioms mult.monoid-axioms p-shunting-swap schroeder-5-p semiring.add-decreasing2 semiring.mult-left-mono sup.bounded-iff symmetric-one-closed monoid.right-neutral semilattice-order.boundedI semilattice-order.cobounded1 semilattice-order.cobounded2*)

**lemma** *strict-order-transitive-2*:

*antisymmetric x* $\Longrightarrow$ *transitive x* $\Longrightarrow$ $(x \sqcap -1) * x \leq x \sqcap -1$
**by** (*smt* (*verit*, *ccfv-SIG*) *comp-commute-below-diversity dual-order.eq-iff inf.boundedE inf.order-iff inf.sup-monoid.add-assoc mult-left-isotone strict-order-transitive-1*)

**lemma** *strict-order-transitive*:
*antisymmetric x* $\Longrightarrow$ *transitive x* $\Longrightarrow$ $(x \sqcap -1) * (x \sqcap -1) \leq x \sqcap -1$
**using** *comp-isotone inf.cobounded1 inf.order-lesseq-imp strict-order-transitive-2*
**by** *blast*

**lemma** *strict-order-transitive-eq-1*:
*order x* $\Longrightarrow$ $(x \sqcap -1) * x = x \sqcap -1$
**by** (*metis comp-right-one dual-order.antisym mult-right-isotone strict-order-transitive-2*)

**lemma** *strict-order-transitive-eq-2*:
*order x* $\Longrightarrow$ $x * (x \sqcap -1) = x \sqcap -1$
**by** (*metis dual-order.antisym mult-1-left mult-left-isotone strict-order-transitive-1*)

**lemma** *strict-order-transitive-eq*:
*order x* $\Longrightarrow$ $(x \sqcap -1) * x = x * (x \sqcap -1)$
**by** (*simp add*: *strict-order-transitive-eq-1 strict-order-transitive-eq-2*)

**lemma** *strict-order-asymmetric*:
*antisymmetric x* $\Longrightarrow$ *asymmetric* $(x \sqcap -1)$
**by** (*metis antisymmetric-inf-closed antisymmetric-inf-diversity inf.order-iff inf.right-idem pseudo-complement*)

**end**

The following gives relational definitions for extrema, bounds, suprema, the univalent part and symmetric quotients.

**context** *relation-algebra-signature*
**begin**

**definition** *maximal* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*maximal r s* $\equiv s \sqcap -((r \sqcap -1) * s)$

**definition** *minimal* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*minimal r s* $\equiv s \sqcap -((r^T \sqcap -1) * s)$

**definition** *upperbound* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*upperbound r s* $\equiv -(-r^T * s)$

**definition** *lowerbound* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*lowerbound r s* $\equiv -(-r * s)$

**definition** *greatest* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**

*greatest* $r\ s \equiv s \sqcap -(-r^T * s)$

**definition** *least* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*least* $r\ s \equiv s \sqcap -(-r * s)$

**definition** *supremum* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*supremum* $r\ s \equiv$ *least* $r$ (*upperbound* $r\ s$)

**definition** *infimum* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*infimum* $r\ s \equiv$ *greatest* $r$ (*lowerbound* $r\ s$)

**definition** *univalent-part* :: $'a \Rightarrow 'a$ **where**
*univalent-part* $r \equiv r \sqcap -(r * -1)$

**definition** *symmetric-quotient* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
*symmetric-quotient* $r\ s \equiv -(r^T * -s) \sqcap -(-r^T * s)$

**abbreviation** *noyau* :: $'a \Rightarrow 'a$ **where**
*noyau* $r \equiv$ *symmetric-quotient* $r\ r$

**end**

**context** *relation-algebra*
**begin**

## 1.1 Extrema, bounds and suprema

**lemma** *maximal-comparable*:
$r \sqcap$ (*maximal* $r\ s$) $*$ (*maximal* $r\ s$)$^T \leq r^T$
**proof** $-$
**have** $r \sqcap -r^T \leq r \sqcap -1$
**by** (*metis inf-commute inf-le2 le-inf-iff one-inf-conv p-shunting-swap*)
**hence** *maximal* $r\ s \sqcap (r \sqcap -r^T) *$ *maximal* $r\ s \leq$ *maximal* $r\ s \sqcap (r \sqcap -1) * s$
**using** *comp-inf.mult-right-isotone comp-isotone dual-order.eq-iff maximal-def*
**by** *fastforce*
**also have** ... $\leq$ *bot*
**by** (*simp add*: *inf-commute maximal-def*)
**finally show** *?thesis*
**by** (*smt* (*verit*, *best*) *double-compl inf.sup-monoid.add-assoc inf-commute le-bot pseudo-complement schroeder-2*)
**qed**

**lemma** *maximal-comparable-same*:
**assumes** *antisymmetric* $r$
**shows** $r \sqcap$ (*maximal* $r\ s$) $*$ (*maximal* $r\ s$)$^T \leq 1$
**by** (*meson assms inf.sup-left-divisibility le-infI order-trans maximal-comparable*)

**lemma** *transitive-lowerbound*:
*transitive* $r \Longrightarrow r *$ *lowerbound* $r\ s \leq$ *lowerbound* $r\ s$

**by** (*metis comp-associative double-compl lowerbound-def mult-left-isotone schroeder-3-p*)

**lemma** *transitive-least*:
*transitive r* $\Longrightarrow$ *r* $*$ *least r top* $\leq$ *least r top*
**using** *least-def lowerbound-def transitive-lowerbound* **by** *auto*

**lemma** *transitive-minimal-not-least*:
**assumes** *transitive r*
**shows** $r^T$ $*$ *minimal r* $(-$*least r top*$)$ $\leq$ $-$*least r top*
**proof** $-$
**have** *least r top* $\leq$ $-$*minimal r* $(-$*least r top*$)$
**by** (*simp add*: *minimal-def*)
**hence** *r* $*$ *least r top* $\leq$ $-$*minimal r* $(-$*least r top*$)$
**using** *assms dual-order.trans transitive-least* **by** *blast*
**thus** *?thesis*
**using** *schroeder-3-p* **by** *auto*
**qed**

**lemma** *least-injective*:
**assumes** *antisymmetric r*
**shows** *injective* (*least r s*)
**proof** $-$
**have** (*least r s*) $*$ (*least r s*)$^T$ $\leq -(-r * s) * s^T \sqcap s * -(-r * s)^T$
**by** (*simp add*: *least-def comp-isotone conv-complement conv-dist-inf*)
**also have** ... $\leq r \sqcap r^T$
**by** (*metis comp-inf.comp-isotone conv-complement conv-dist-comp pp-increasing schroeder-3 schroeder-5*)
**also have** ... $\leq$ *1*
**by** (*simp add*: *assms*)
**finally show** *?thesis*
.
**qed**

**lemma** *least-conv-greatest*:
*least r* = *greatest* $(r^T)$
**using** *greatest-def least-def* **by** *fastforce*

**lemma** *greatest-injective*:
*antisymmetric r* $\Longrightarrow$ *injective* (*greatest r s*)
**by** (*metis antisymmetric-conv-closed least-injective least-conv-greatest conv-involutive*)

**lemma** *supremum-upperbound*:
**assumes** *antisymmetric r*
**and** $s \leq r$
**shows** *supremum r s* = *1* $\longleftrightarrow$ *upperbound r s* $\leq r^T$
**proof** (*rule iffI*)
**assume** *supremum r s* = *1*

**hence** *1* $\leq$ *lowerbound r* (*upperbound r s*)
**using** *least-def lowerbound-def supremum-def* **by** *auto*
**thus** *upperbound r s* $\leq r^T$
**by** (*metis comp-right-one compl-le-compl-iff compl-le-swap1 conv-complement schroeder-3-p lowerbound-def*)
**next**
**assume** *1*: *upperbound r s* $\leq r^T$
**hence** *2*: *1* $\leq$ *lowerbound r* (*upperbound r s*)
**by** (*simp add*: *compl-le-swap1 conv-complement schroeder-3-p lowerbound-def*)
**have** *3*: *1* $\leq$ *upperbound r s*
**by** (*simp add*: *assms*(*2*) *compl-le-swap1 conv-complement schroeder-3-p upperbound-def*)
**hence** *lowerbound r* (*upperbound r s*) $\leq$ *r*
**using** *brouwer.p-antitone-iff mult-right-isotone lowerbound-def* **by** *fastforce*
**hence** *supremum r s* $\leq$ *1*
**using** *1* **by** (*smt* (*verit*, *del-insts*) *assms*(*1*) *least-def inf.sup-mono inf-commute order.trans lowerbound-def supremum-def*)
**thus** *supremum r s* = *1*
**using** *2 3 least-def order.eq-iff lowerbound-def supremum-def* **by** *auto*
**qed**

## 1.2 Univalent part

**lemma** *univalent-part-idempotent*:
*univalent-part* (*univalent-part r*) = *univalent-part r*
**by** (*smt* (*verit*, *best*) *inf.absorb2 inf.coboundedl inf.order-iff inf-assoc mult-left-isotone p-antitone-inf univalent-part-def*)

**lemma** *univalent-part-univalent*:
*univalent* (*univalent-part r*)
**by** (*smt* (*verit*, *ccfv-SIG*) *inf.coboundedl inf.sup-monoid.add-commute mult-left-isotone order-lesseq-imp p-antitone-iff regular-one-closed schroeder-3-p univalent-part-def*)

**lemma** *univalent-part-times-converse*:
$r^T$ ∗ *univalent-part r* = (*univalent-part r*)$^T$ ∗ *univalent-part r*
**proof** −
**have** *1*: $(r \sqcap r * -1)^T$ ∗ *univalent-part r* $\leq$ *1*
**by** (*smt* (*verit*, *best*) *compl-le-swap1 inf.coboundedl inf.cobounded2 mult-left-isotone order-lesseq-imp regular-one-closed schroeder-3-p univalent-part-def*)
**hence** *2*: $(r \sqcap r * -1)^T$ ∗ *univalent-part r* $\leq -1$
**by** (*simp add*: *inf.coboundedI2 schroeder-3-p univalent-part-def*)
**have** $r^T$ ∗ *univalent-part r* = $(r \sqcap r * -1)^T$ ∗ *univalent-part r* ⊔ (*univalent-part r*)$^T$ ∗ *univalent-part r*
**by** (*metis conv-dist-sup maddux-3-11 mult-right-dist-sup univalent-part-def*)
**thus** *?thesis*
**using** *1 2* **by** (*metis inf.orderE inf-compl-bot-right maddux-3-13 pseudo-complement*)

**qed**

**lemma** *univalent-part-times-converse-1*:
$r^T * \mathit{univalent\text{-}part}\ r \le 1$
**by** (*simp add*: *univalent-part-times-converse univalent-part-univalent*)

**lemma** *minimal-univalent-part*:
**assumes** *reflexive r*
**and** *vector s*
**shows** $\mathit{minimal}\ r\ s = s \sqcap \mathit{univalent\text{-}part}\ ((r \sqcap s)^T) * \mathit{top}$
**proof** (*rule order.antisym*)
**have** $1 \sqcap r^T * (-1 \sqcap s) \le (r^T \sqcap -1 \sqcap s^T) * (-1 \sqcap s)$
**by** (*smt* (*z3*) *conv-complement conv-dist-inf dedekind-2 equivalence-one-closed inf.sup-monoid.add-assoc inf.sup-monoid.add-commute mult-1-left*)
**also have** ... $\le (r^T \sqcap -1) * s$
**using** *inf-le1 inf-le2 mult-isotone* **by** *blast*
**finally have** $1 \sqcap -((r^T \sqcap -1) * s) \le -(r^T * (-1 \sqcap s))$
**by** (*simp add*: *p-shunting-swap*)
**also have** *1*: ... $= -((r \sqcap s)^T * -1)$
**by** (*simp add*: *assms*(*2*) *conv-dist-inf covector-inf-comp-3 inf.sup-monoid.add-commute*)
**finally have** *2*: $1 \sqcap -((r^T \sqcap -1) * s) \le r^T \sqcap -((r \sqcap s)^T * -1)$
**by** (*simp add*: *assms*(*1*) *le-infI1 reflexive-conv-closed*)
**have** $\mathit{minimal}\ r\ s = (1 \sqcap -((r^T \sqcap -1) * s)) * s$
**by** (*metis assms*(*2*) *complement-vector inf-commute vector-export-comp-unit minimal-def mult-assoc*)
**also have** ... $\le (r^T \sqcap -((r \sqcap s)^T * -1)) * s$
**using** *2 mult-left-isotone* **by** *blast*
**also have** *3*: ... $= \mathit{univalent\text{-}part}\ ((r \sqcap s)^T) * \mathit{top}$
**by** (*smt* (*verit*, *ccfv-threshold*) *assms*(*2*) *comp-inf.vector-top-closed comp-inf-covector comp-inf-vector conv-dist-inf inf.sup-monoid.add-assoc inf.sup-monoid.add-commute surjective-one-closed vector-conv-covector univalent-part-def*)
**finally show** $\mathit{minimal}\ r\ s \le s \sqcap \mathit{univalent\text{-}part}\ ((r \sqcap s)^T) * \mathit{top}$
**by** (*simp add*: *minimal-def*)
**have** $s \sqcap (r^T \sqcap -1) * s \sqcap 1 \le (r^T \sqcap -1) * s \sqcap 1$
**using** *comp-inf.comp-isotone inf.cobounded2* **by** *blast*
**also have** ... $\le (r^T \sqcap -1) * (s \sqcap (r^T \sqcap -1)^T)$
**by** (*metis comp-right-one dedekind-1*)
**also have** ... $\le r^T * (s \sqcap -1)$
**using** *comp-inf.mult-right-isotone conv-complement conv-dist-inf mult-isotone* **by** *auto*
**finally have** *4*: $s \sqcap (r^T \sqcap -1) * s \sqcap 1 \le r^T * (s \sqcap -1)$
**.**
**have** $s \sqcap (r^T \sqcap -1) * s \sqcap -1 \le r^T * (s \sqcap -1)$
**by** (*metis assms*(*1*) *comp-inf.comp-left-subdist-inf inf.coboundedI1 inf.order-trans mult-1-left mult-left-isotone order.refl reflexive-conv-closed*)
**hence** *5*: $s \sqcap (r^T \sqcap -1) * s \le r^T * (s \sqcap -1)$
**using** *4 comp-inf.case-split-right heyting.implies-itself-top* **by** *blast*

**have** $s \sqcap (r^T \sqcap -1) * s \sqcap (r^T \sqcap -(r^T * (s \sqcap -1))) * s = (s \sqcap (r^T \sqcap -1) * s \sqcap r^T \sqcap -(r^T * (s \sqcap -1))) * s$
**using** *assms*(*2*) *inf-assoc vector-inf-comp mult-assoc* **by** *simp*
**also have** ... $=$ *bot*
**using** *5 le-infI1 semiring.mult-not-zero shunting-1* **by** *blast*
**finally have** $s \sqcap$ *univalent-part* $((r \sqcap s)^T) *$ *top* $\leq -((r^T \sqcap -1) * s)$
**using** *1 3* **by** (*simp add*: *inf.sup-monoid.add-commute p-shunting-swap pseudo-complement*)
**thus** $s \sqcap$ *univalent-part* $((r \sqcap s)^T) *$ *top* $\leq$ *minimal r s*
**by** (*simp add*: *minimal-def*)
**qed**

## 1.3 Symmetric quotients

**lemma** *univalent-part-syq*:
*univalent-part r* $=$ *symmetric-quotient* $(r^T)$ *1*
**by** (*simp add*: *inf-commute symmetric-quotient-def univalent-part-def*)

**lemma** *minimal-syq*:
**assumes** *reflexive r*
**and** *vector s*
**shows** *minimal r s* $= s \sqcap$ *symmetric-quotient* $(r \sqcap s)$ *1* $*$ *top*
**by** (*simp add*: *assms minimal-univalent-part univalent-part-syq*)

**lemma** *syq-complement*:
*symmetric-quotient* $(-r)$ $(-s)$ $=$ *symmetric-quotient r s*
**by** (*simp add*: *conv-complement inf.sup-monoid.add-commute symmetric-quotient-def*)

**lemma** *syq-converse*:
$($*symmetric-quotient r s*$)^T$ $=$ *symmetric-quotient s r*
**by** (*simp add*: *conv-complement conv-dist-comp conv-dist-inf inf.sup-monoid.add-commute symmetric-quotient-def*)

**lemma** *syq-comp-transitive*:
*symmetric-quotient r s* $*$ *symmetric-quotient s p* $\leq$ *symmetric-quotient r p*
**proof** $-$
**have** $r * -(r^T * -s) * -(s^T * -p) \leq s * -(s^T * -p)$
**by** (*metis complement-conv-sub conv-complement mult-left-isotone schroeder-5*)
**also have** ... $\leq p$
**by** (*simp add*: *schroeder-3*)
**finally have** *1*: $-(r^T * -s) * -(s^T * -p) \leq -(r^T * -p)$
**by** (*simp add*: *p-antitone-iff schroeder-3-p mult-assoc*)
**have** $-(-r^T * s) * -(-s^T * p) * p^T \leq -(-r^T * s) * s^T$
**by** (*metis complement-conv-sub double-compl mult-right-isotone mult-assoc*)
**also have** ... $\leq r^T$
**using** *brouwer.pp-increasing complement-conv-sub inf.order-trans* **by** *blast*
**finally have** *2*: $-(-r^T * s) * -(-s^T * p) \leq -(-r^T * p)$
**by** (*metis compl-le-swap1 double-compl schroeder-4*)

**have** *symmetric-quotient r s* $*$ *symmetric-quotient s p* $\leq -(r^T * -s) * -(s^T * -p) \sqcap -(-r^T * s) * -(-s^T * p)$
**by** (*simp add*: *mult-isotone symmetric-quotient-def*)
**also have** ... $\leq -(r^T * -p) \sqcap -(-r^T * p)$
**using** *1 2 inf-mono* **by** *blast*
**finally show** *?thesis*
**by** (*simp add*: *symmetric-quotient-def*)
**qed**

**lemma** *syq-comp-syq-top*:
*symmetric-quotient r s* $*$ *symmetric-quotient s p* = *symmetric-quotient r p* $\sqcap$ *symmetric-quotient r s* $*$ *top*
**proof** (*rule order.antisym*)
**show** *symmetric-quotient r s* $*$ *symmetric-quotient s p* $\leq$ *symmetric-quotient r p* $\sqcap$ *symmetric-quotient r s* $*$ *top*
**by** (*simp add*: *mult-right-isotone syq-comp-transitive*)
**have** *symmetric-quotient r p* $\sqcap$ *symmetric-quotient r s* $*$ *top* $\leq$ *symmetric-quotient r s* $*$ *symmetric-quotient s r* $*$ *symmetric-quotient r p*
**by** (*metis comp-right-one dedekind-1 inf-top-left inf-vector-comp mult-assoc syq-converse*)
**also have** ... $\leq$ *symmetric-quotient r s* $*$ *symmetric-quotient s p*
**by** (*simp add*: *mult-right-isotone mult-assoc syq-comp-transitive*)
**finally show** *symmetric-quotient r p* $\sqcap$ *symmetric-quotient r s* $*$ *top* $\leq$ *symmetric-quotient r s* $*$ *symmetric-quotient s p*
**.**
**qed**

**lemma** *syq-comp-top-syq*:
*symmetric-quotient r s* $*$ *symmetric-quotient s p* = *symmetric-quotient r p* $\sqcap$ *top* $*$ *symmetric-quotient s p*
**by** (*metis conv-dist-comp conv-dist-inf symmetric-top-closed syq-comp-syq-top syq-converse*)

**lemma** *comp-syq-below*:
*r* $*$ *symmetric-quotient r s* $\leq$ *s*
**by** (*simp add*: *schroeder-3 symmetric-quotient-def*)

**lemma** *comp-syq-top*:
*r* $*$ *symmetric-quotient r s* = *s* $\sqcap$ *top* $*$ *symmetric-quotient r s*
**proof** (*rule order.antisym*)
**show** *r* $*$ *symmetric-quotient r s* $\leq$ *s* $\sqcap$ *top* $*$ *symmetric-quotient r s*
**by** (*simp add*: *comp-syq-below mult-left-isotone*)
**have** *s* $\sqcap$ *top* $*$ *symmetric-quotient r s* $\leq$ *s* $*$ *symmetric-quotient s r* $*$ *symmetric-quotient r s*
**by** (*metis dedekind-2 inf-commute inf-top.right-neutral syq-converse*)
**also have** ... $\leq$ *r* $*$ *symmetric-quotient r s*
**by** (*simp add*: *comp-syq-below mult-left-isotone*)
**finally show** *s* $\sqcap$ *top* $*$ *symmetric-quotient r s* $\leq$ *r* $*$ *symmetric-quotient r s*
**.**

**qed**

**lemma** *syq-comp-isotone*:
*symmetric-quotient* $r\ s \le$ *symmetric-quotient* $(q * r)\ (q * s)$
**proof** −
**have** $q^T * -(q * s) \le -s$
**by** (*simp add*: *conv-complement-sub-leq*)
**hence** $(q * r)^T * -(q * s) \le r^T * -s$
**by** (*simp add*: *comp-associative conv-dist-comp mult-right-isotone*)
**hence** *1*: $-(r^T * -s) \le -((q * r)^T * -(q * s))$
**by** *simp*
**have** $-(q * r)^T * q \le -r^T$
**using** *schroeder-6* **by** *auto*
**hence** $-(q * r)^T * q * s \le -r^T * s$
**using** *mult-left-isotone* **by** *auto*
**hence** $-(-r^T * s) \le -(-(q * r)^T * q * s)$
**by** *simp*
**thus** *?thesis*
**using** *1* **by** (*metis comp-inf.comp-isotone mult-assoc symmetric-quotient-def*)
**qed**

**lemma** *syq-comp-isotone-eq*:
**assumes** *univalent* $q$
**and** *surjective* $q$
**shows** *symmetric-quotient* $r\ s =$ *symmetric-quotient* $(q * r)\ (q * s)$
**proof** −
**have** *symmetric-quotient* $(q * r)\ (q * s) \le$ *symmetric-quotient* $(q^T * q * r)\ (q^T * q * s)$
**by** (*simp add*: *mult-assoc syq-comp-isotone*)
**also have** ... = *symmetric-quotient* $r\ s$
**using** *assms antisym-conv mult-left-one surjective-var* **by** *auto*
**finally show** *?thesis*
**by** (*simp add*: *dual-order.antisym syq-comp-isotone*)
**qed**

**lemma** *univalent-comp-syq*:
**assumes** *univalent* $p$
**shows** $p *$ *symmetric-quotient* $r\ s = p *$ *top* $\sqcap$ *symmetric-quotient* $(r * p^T)\ s$
**proof** −
**have** $p *$ *symmetric-quotient* $r\ s = p *$ *top* $\sqcap -(p * r^T * -s) \sqcap -(p * -r^T * s)$
**by** (*metis assms comp-associative comp-univalent-complement inf.sup-monoid.add-assoc mult-left-dist-sup p-dist-sup symmetric-quotient-def*)
**also have** ... $= p *$ *top* $\sqcap -(p * r^T * -s) \sqcap -(p *$ *top* $\sqcap -(p * r^T) * s)$
**using** *assms comp-univalent-complement vector-export-comp* **by** *auto*
**also have** ... $= p *$ *top* $\sqcap -(p * r^T * -s) \sqcap -(-(p * r^T) * s)$
**by** (*simp add*: *comp-inf.coreflexive-comp-inf-complement*)
**also have** ... $= p *$ *top* $\sqcap -((r * p^T)^T * -s) \sqcap -(-(r * p^T)^T * s)$
**by** (*simp add*: *conv-dist-comp*)
**also have** ... $= p *$ *top* $\sqcap$ *symmetric-quotient* $(r * p^T)\ s$

**by** (*simp add*: *inf.sup-monoid.add-assoc symmetric-quotient-def*)
**finally show** *?thesis*
**.**
**qed**

**lemma** *coreflexive-comp-syq*:
*coreflexive p* $\Longrightarrow$ *p* $*$ *symmetric-quotient r s* = *p* $*$ *symmetric-quotient* (*r* $*$ *p*) *s*
**by** (*metis coreflexive-comp-top-inf coreflexive-injective coreflexive-symmetric univalent-comp-syq*)

**lemma** *injective-comp-syq*:
*injective p* $\Longrightarrow$ *symmetric-quotient r s* $*$ *p* = *top* $*$ *p* $\sqcap$ *symmetric-quotient r* (*s* $*$ *p*)
**by** (*metis univalent-comp-syq*[*of* $p^T$ *s r*] *conv-dist-comp conv-dist-inf conv-involutive symmetric-top-closed syq-converse*)

**lemma** *syq-comp-coreflexive*:
*coreflexive p* $\Longrightarrow$ *symmetric-quotient r s* $*$ *p* = *symmetric-quotient r* (*s* $*$ *p*) $*$ *p*
**by** (*simp add*: *injective-comp-syq coreflexive-idempotent coreflexive-symmetric mult-assoc*)

**lemma** *coreflexive-comp-syq-comp-coreflexive*:
*coreflexive p* $\Longrightarrow$ *coreflexive q* $\Longrightarrow$ *p* $*$ *symmetric-quotient r s* $*$ *q* = *p* $*$ *symmetric-quotient* (*r* $*$ *p*) (*s* $*$ *q*) $*$ *q*
**by** (*metis coreflexive-comp-syq comp-associative syq-comp-coreflexive*)

**lemma** *surjective-syq*:
*surjective* (*symmetric-quotient r s*) $\Longrightarrow$ *r* $*$ *symmetric-quotient r s* = *s*
**by** (*metis comp-syq-top inf-top.right-neutral*)

**lemma** *comp-syq-surjective*:
**assumes** *total* ($-$(*top* $*$ *r*))
**shows** *surjective* (*symmetric-quotient r s*) $\longleftrightarrow$ *r* $*$ *symmetric-quotient r s* = *s*
**proof** (*rule iffI*, *fact surjective-syq*)
**assume** *r* $*$ *symmetric-quotient r s* = *s*
**hence** *1*: *top* $*$ *s* $\leq$ *top* $*$ *symmetric-quotient r s*
**by** (*metis comp-syq-top comp-inf-covector inf.absorb-iff1*)
**have** $-$(*top* $*$ *s*) = $-$(*top* $*$ *r*) $*$ $-$(*top* $*$ *s*)
**by** (*metis assms comp-associative complement-covector vector-top-closed*)
**also have** ... = *top* $*$ ($-$($r^T$ $*$ *top*) $\sqcap$ $-$(*top* $*$ *s*))
**by** (*metis assms conv-complement conv-dist-comp covector-comp-inf covector-complement-closed inf-top.left-neutral symmetric-top-closed vector-top-closed mult-assoc*)
**also have** ... $\leq$ *top* $*$ ($-$($r^T$ $*$ $-$*s*) $\sqcap$ $-$($-r^T$ $*$ *s*))
**by** (*meson comp-inf.mult-isotone comp-isotone order-refl p-antitone top-greatest*)
**finally have** $-$(*top* $*$ *s*) $\leq$ *top* $*$ *symmetric-quotient r s*
**by** (*simp add*: *symmetric-quotient-def*)
**thus** *surjective* (*symmetric-quotient r s*)

**using** *1* **by** (*metis compl-inter-eq inf.order-iff top-greatest*)
**qed**

**lemma** *noyau-reflexive*:
*reflexive* (*noyau r*)
**by** (*simp add*: *compl-le-swap1 conv-complement schroeder-3 symmetric-quotient-def*)

**lemma** *noyau-equivalence*:
*equivalence* (*noyau r*)
**by** (*smt* (*z3*) *comp-associative comp-right-one conv-complement conv-dist-comp conv-dist-inf conv-involutive inf.antisym-conv inf.boundedI inf.cobounded1 inf.sup-monoid.add-commute mult-right-isotone schroeder-5-p symmetric-quotient-def noyau-reflexive*)

**lemma** *noyau-reflexive-comp*:
$r * noyau\ r = r$
**proof** (*rule order.antisym*)
**show** $r * noyau\ r \le r$
**by** (*simp add*: *schroeder-3 symmetric-quotient-def*)
**show** $r \le r * noyau\ r$
**using** *mult-right-isotone noyau-reflexive* **by** *fastforce*
**qed**

**lemma** *syq-comp-reflexive*:
$noyau\ r * symmetric\text{-}quotient\ r\ s = symmetric\text{-}quotient\ r\ s$
**by** (*simp add*: *inf-absorb1 top-left-mult-increasing syq-comp-top-syq*)

**lemma** *reflexive-antisymmetric-noyau*:
**assumes** *reflexive r*
**and** *antisymmetric r*
**shows** $noyau\ r = 1$
**proof** −
**have** *1*: $-(r^T * -r) \le r$
**using** *assms*(*1*) *brouwer.p-antitone-iff mult-left-isotone reflexive-conv-closed* **by** *fastforce*
**have** $-(-r^T * r) \le r^T$
**by** (*metis assms*(*1*) *compl-le-swap2 mult-1-right mult-right-isotone*)
**thus** *?thesis*
**using** *1* **by** (*smt* (*verit*, *ccfv-threshold*) *assms*(*2*) *inf.sup-mono inf.sup-monoid.add-assoc inf.sup-monoid.add-commute inf-absorb1 symmetric-quotient-def noyau-equivalence*)
**qed**

**end**

**end**

## 2 Divisibility

**theory** *Relational-Divisibility*

**imports** *Relational-Constructions*

**begin**

This theory gives relational axioms and definitions for divisibility. We start with the definitions, which are based on standard relational constructions. Then follow the axioms, which are relational formulations of axioms expressed in predicate logic in [2].

**context** *bounded-distrib-allegory-signature*
**begin**

**definition** *antichain* :: $'a \Rightarrow 'a \Rightarrow bool$ **where**
$antichain\ r\ s \equiv vector\ s \wedge r \sqcap s \sqcap s^T \leq 1$

**end**

**class** *divisibility-op* =
**fixes** *divisibility* :: $'a$ ($D$)

**class** *divisibility-def* = *relation-algebra* + *divisibility-op*
**begin**

*Dbot* is the least element of the divisibility order, which represents the number 1.

**definition** *Dbot* :: $'a$ **where**
$Dbot \equiv least\ D\ top$

*Datoms* are the atoms of the divisibility order, which represent the prime numbers.

**definition** *Datoms* :: $'a$ **where**
$Datoms \equiv minimal\ D\ (-Dbot)$

*Datoms* are the mono-atomic elements of the divisibility order, which represent the prime powers.

**definition** *Dmono* :: $'a$ **where**
$Dmono \equiv univalent\text{-}part\ ((D \sqcap Datoms)^T) * top$

*Dfactor* relates $p$ to $x$ if and only if $p$ is maximal prime power factor of $x$.

**definition** *Dfactor* :: $'a$ **where**
$Dfactor \equiv maximal\ D\ (D \sqcap Dmono)$

*Dsupport* relates $x$ to $y$ if and only if $y$ is the product of all primes below $x$.

**definition** *Dsupport* :: *'a* **where**
$Dsupport \equiv symmetric\text{-}quotient\ (Datoms \sqcap D)\ Dfactor$

*Dsucc* relates $x$ to $y$ if and only if $y$ is the product of prime power $x$ with its base prime.

**definition** *Dsucc* :: *'a* **where**
$Dsucc \equiv greatest\ D\ (D \sqcap -1)$

*Dinc* relates $x$ to $y$ if and only if $y$ is the product of $x$ with all its base primes.

**definition** *Dinc* :: *'a* **where**
$Dinc \equiv symmetric\text{-}quotient\ Dfactor\ (Dsucc * Dfactor)$

*Datomsbot* includes the number 1 with the prime numbers.

**definition** *Datomsbot* :: *'a* **where**
$Datomsbot \equiv Datoms \sqcup Dbot$

*Dmonobot* includes the number 1 with the prime powers.

**definition** *Dmonobot* :: *'a* **where**
$Dmonobot \equiv Dmono \sqcup Dbot$

*Dfactorbot* is like *Dfactor* except it also relates 1 to 1.

**definition** *Dfactorbot* :: *'a* **where**
$Dfactorbot \equiv maximal\ D\ (D \sqcap Dmonobot)$

We consider the following axioms for $D$. They correspond to axioms A1–A3, A6–A9, A11–A13 and A15–A16 of [2].

**abbreviation** *D1-reflexive* :: *'a* ⇒ *bool* **where** *D1-reflexive* - $\equiv reflexive\ D$
**abbreviation** *D2-antisymmetric* :: *'a* ⇒ *bool* **where** *D2-antisymmetric* - $\equiv antisymmetric\ D$
**abbreviation** *D3-transitive* :: *'a* ⇒ *bool* **where** *D3-transitive* - $\equiv transitive\ D$
**abbreviation** *D6-least-surjective* :: *'a* ⇒ *bool* **where** *D6-least-surjective* - $\equiv surjective\ Dbot$
**abbreviation** *D7-pre-f-decomposable* :: *'a* ⇒ *bool* **where** *D7-pre-f-decomposable* - $\equiv supremum\ D\ (D \sqcap Dmono) = 1$
**abbreviation** *D8-fibered* :: *'a* ⇒ *bool* **where** *D8-fibered* - $\equiv Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T \le D \sqcup D^T$
**abbreviation** *D9-f-decomposable* :: *'a* ⇒ *bool* **where** *D9-f-decomposable* - $\equiv Datoms \sqcap D \le D * Dfactor$
**abbreviation** *D11-atomic* :: *'a* ⇒ *bool* **where** *D11-atomic* - $\equiv D^T * Datoms = -Dbot$
**abbreviation** *D12-infinite-base* :: *'a* ⇒ *bool* **where** *D12-infinite-base* - $\equiv -D^T * Datoms = top$
**abbreviation** *D13-supportable* :: *'a* ⇒ *bool* **where** *D13-supportable* - $\equiv total\ Dsupport$
**abbreviation** *D15a-discrete-fibers-succ* :: *'a* ⇒ *bool* **where** *D15a-discrete-fibers-succ* - $\equiv Dmono \le Dsucc * top$

**abbreviation** *D15b-discrete-fibers-pred* :: *'a* ⇒ *bool* **where**
*D15b-discrete-fibers-pred* - ≡ *Dmono* ≤ *Dsucc*$^T$ ∗ *top*
**abbreviation** *D16-incrementable* :: *'a* ⇒ *bool* **where** *D16-incrementable*
- ≡ *total Dinc*

## 2.1 Partial order

**lemma** *div-antisymmetric-equal*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**shows** *D* ⊓ *D*$^T$ = *1*
**by** (*simp add*: *assms dual-order.antisym reflexive-conv-closed*)

**lemma** *div-idempotent*:
**assumes** *D1-reflexive* -
**and** *D3-transitive* -
**shows** *idempotent D*
**using** *assms preorder-idempotent* **by** *auto*

**lemma** *div-total*:
**assumes** *D1-reflexive* -
**shows** *D* ∗ *top* = *top*
**by** (*simp add*: *assms reflexive-conv-closed reflexive-mult-closed total-var*)

**lemma** *div-surjective*:
**assumes** *D1-reflexive* -
**shows** *top* ∗ *D* = *top*
**by** (*simp add*: *assms reflexive-conv-closed reflexive-mult-closed surjective-var*)

**lemma** *div-below-div-converse*:
**assumes** *D2-antisymmetric* -
**and** *x* ≤ *D*
**shows** *D* ⊓ *x*$^T$ ≤ *x*
**by** (*smt assms conv-dist-inf conv-involutive coreflexive-symmetric inf.cobounded2 inf.orderE inf-left-commute*)

## 2.2 Bounds

The least element can be introduced equivalently by

- defining *Dbot* = *least D top* and axiomatising either *surjective Dbot* or *Dbot* ≠ *bot*, or
- axiomatising *point Dbot* and *Dbot* ≤ *D*.

**lemma** *div-least-div*:
*Dbot* ≤ *D*
**by** (*simp add*: *Dbot-def compl-le-swap2 least-def top-right-mult-increasing*)

**lemma** *div-least-vector*:

*vector Dbot*
**by** (*simp add*: *Dbot-def complement-vector least-def mult-assoc*)

**lemma** *div-least-injective*:
**assumes** *D2-antisymmetric -*
**shows** *injective Dbot*
**by** (*metis assms div-least-div div-least-vector antisymmetric-inf-closed inf.absorb2 vector-covector*)

**lemma** *div-least-point*:
**assumes** *D2-antisymmetric -*
**and** *D6-least-surjective -*
**shows** *point Dbot*
**using** *assms div-least-injective div-least-vector* **by** *blast*

**lemma** *div-point-least*:
**assumes** *D2-antisymmetric -*
**and** *point x*
**and** $x \leq D$
**shows** $x =$ *least D top*
**proof** (*rule order.antisym*)
**show** $x \leq$ *least D top*
**by** (*smt* (*verit*, *ccfv-SIG*) *assms*(*2*,*3*) *comp-associative double-compl inf-top.left-neutral least-def schroeder-4 vector-covector*)
**have** *1*: $D \sqcap x^T \leq x$
**by** (*smt* (*verit*, *best*) *assms*(*1*,*3*) *conv-dist-inf inf.absorb1 inf.sup-same-context inf-assoc inf-le2 one-inf-conv*)
**have** $-x = (-x \sqcap x^T) * top$
**using** *assms*(*2*) *complement-vector surjective-conv-total vector-inf-comp* **by** *auto*
**also have** ... $\leq -D * top$
**using** *1* **by** (*simp add*: *inf.sup-monoid.add-commute mult-left-isotone p-shunting-swap*)
**finally show** *least D top* $\leq x$
**by** (*simp add*: *compl-le-swap2 least-def*)
**qed**

**lemma** *div-least-surjective-iff*:
**assumes** *D2-antisymmetric -*
**shows** *D6-least-surjective -* $\longleftrightarrow (\exists x \,.\, point\ x \wedge x \leq D)$
**using** *Dbot-def assms div-least-div div-point-least div-least-point* **by** *auto*

**lemma** *div-least-converse*:
**assumes** *D2-antisymmetric -*
**shows** $D \sqcap Dbot^T \leq Dbot$
**using** *assms div-below-div-converse div-least-div* **by** *blast*

**lemma** *bot-div-bot*:
**assumes** *D1-reflexive -*

**and** *D3-transitive* -
**shows** $D * Dbot = Dbot$
**by** (*metis assms div-idempotent Dbot-def antisym-conv mult-1-left mult-left-isotone transitive-least*)

**lemma** *all-div-bot*:
**assumes** *D2-antisymmetric* -
**and** *D6-least-surjective* -
**shows** $D^T * Dbot = top$
**using** *assms div-least-div div-least-point inf.order-eq-iff schroeder-4-p schroeder-6 shunt-bijective* **by** *fastforce*

**lemma** *div-strict-bot*:
**assumes** *D2-antisymmetric* -
**shows** $(D \sqcap -1) * Dbot = bot$
**proof** –
**have** $(D^T \sqcap -1) * top \leq -D * top$
**using** *assms inf-commute mult-left-isotone p-shunting-swap* **by** *force*
**thus** *?thesis*
**by** (*smt* (*verit*, *ccfv-threshold*) *Dbot-def conv-complement conv-dist-comp conv-dist-inf conv-involutive equivalence-one-closed inf-p inf-top.left-neutral le-bot least-def regular-in-p-image-iff schroeder-6*)
**qed**

## 2.3 Atoms

The atoms can be introduced equivalently by

* defining $Datoms = minimal\ D\ (-Dbot)$ and axiomatising either $D^T * Datoms = -Dbot$ or $-Dbot \leq D^T * Datoms$ or $-D \leq D^T * Datoms$, or
* axiomatising $antichain\ D\ Datoms$ and $D^T * Datoms = -Dbot$.

**lemma** *div-atoms-vector*:
*vector Datoms*
**by** (*simp add*: *Datoms-def div-least-vector comp-associative minimal-def vector-complement-closed vector-inf-closed*)

**lemma** *div-atoms-bot-vector*:
*vector Datomsbot*
**by** (*simp add*: *div-atoms-vector Datomsbot-def div-least-vector mult-right-dist-sup*)

**lemma** *div-least-not-atom*:
$Dbot \leq -Datoms$
**by** (*simp add*: *Datoms-def minimal-def*)

**lemma** *div-atoms-antichain*:

*antichain D Datoms*
**proof** (*unfold antichain-def*, *rule conjI*, *fact div-atoms-vector*)
**have** $(D \sqcap -1) * Datoms \leq (D \sqcap -1) * -((D^T \sqcap -1) * -Dbot)$
**by** (*simp add*: *Datoms-def minimal-def mult-right-isotone*)
**also have** ... $\leq Dbot$
**by** (*metis complement-conv-sub conv-complement conv-dist-inf equivalence-one-closed schroeder-5*)
**also have** ... $\leq -Datoms$
**by** (*simp add*: *Datoms-def minimal-def*)
**finally have** $Datoms * Datoms^T \leq -D \sqcup 1$
**by** (*simp add*: *schroeder-4-p*)
**thus** $D \sqcap Datoms \sqcap Datoms^T \leq 1$
**by** (*simp add*: *div-atoms-vector heyting.implies-galois-var inf-assoc vector-covector*)
**qed**

**lemma** *div-atomic-bot*:
**assumes** *D2-antisymmetric* -
**and** *D6-least-surjective* -
**shows** $D^T * Datomsbot = top$
**using** *assms all-div-bot Datomsbot-def semiring.distrib-left* **by** *auto*

**lemma** *div-via-atom*:
**assumes** *D3-transitive* -
**and** *D11-atomic* -
**shows** $-Dbot \sqcap D \leq D^T * (D \sqcap Datoms)$
**proof** −
**have** $D^T * Datoms \sqcap D \leq D^T * (D \sqcap Datoms)$
**by** (*smt* (*verit*, *ccfv-SIG*) *assms*(*1*) *conv-involutive dedekind-1 inf.sup-monoid.add-commute inf.boundedI inf.order-lesseq-imp inf-le1 mult-right-isotone*)
**thus** *?thesis*
**by** (*simp add*: *assms*(*2*))
**qed**

**lemma** *div-via-atom-bot*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D6-least-surjective* -
**shows** $D \leq D^T * (D \sqcap Datomsbot)$
**proof** −
**have** $D^T * Datomsbot \sqcap D \leq D^T * (D \sqcap Datomsbot)$
**by** (*metis assms*(*1*,*3*) *div-idempotent conv-involutive dedekind-1 inf.sup-monoid.add-commute*)
**thus** *?thesis*
**by** (*simp add*: *assms*(*2*,*4*) *div-atomic-bot*)
**qed**

**lemma** *div-converse-via-atom*:
  **assumes** *D3-transitive* -
    **and** *D11-atomic* -
  **shows** $-Dbot^T \sqcap D^T \leq D^T * (D \sqcap Datoms)$
**proof** –
  **have** *symmetric* $(D^T * (D \sqcap Datoms))$
    **by** (*simp add*: *div-atoms-vector conv-dist-comp conv-dist-inf covector-inf-comp-3 inf.sup-monoid.add-commute*)
  **thus** *?thesis*
    **by** (*metis assms div-via-atom conv-complement conv-dist-inf conv-isotone*)
**qed**

**lemma** *div-converse-via-atom-bot*:
  **assumes** *D1-reflexive* -
    **and** *D2-antisymmetric* -
    **and** *D3-transitive* -
    **and** *D6-least-surjective* -
  **shows** $D^T \leq D^T * (D \sqcap Datomsbot)$
  **by** (*metis assms div-atoms-bot-vector div-idempotent div-via-atom-bot comp-inf-vector conv-dist-comp conv-dist-inf conv-involutive schroeder-4 schroeder-6 symmetric-top-closed*)

**lemma** *div-comparable-via-atom*:
  **assumes** *D3-transitive* -
    **and** *D11-atomic* -
  **shows** $-Dbot \sqcap -Dbot^T \sqcap (D \sqcup D^T) \leq D^T * (D \sqcap Datoms)$
**proof** –
  **have** $-Dbot \sqcap -Dbot^T \sqcap (D \sqcup D^T) = (-Dbot \sqcap -Dbot^T \sqcap D) \sqcup (-Dbot \sqcap -Dbot^T \sqcap D^T)$
    **by** (*simp add*: *comp-inf.semiring.distrib-left*)
  **also have** ... $\leq (-Dbot \sqcap D) \sqcup (-Dbot^T \sqcap D^T)$
    **by** (*metis comp-inf.coreflexive-comp-inf-comp comp-inf.semiring.add-mono inf.cobounded1 inf.cobounded2 top.extremum*)
  **also have** ... $\leq D^T * (D \sqcap Datoms)$
    **by** (*simp add*: *assms div-converse-via-atom div-via-atom*)
  **finally show** *?thesis*
    .
**qed**

**lemma** *div-comparable-via-atom-bot*:
  **assumes** *D1-reflexive* -
    **and** *D2-antisymmetric* -
    **and** *D3-transitive* -
    **and** *D6-least-surjective* -
  **shows** $D \sqcup D^T \leq D^T * (D \sqcap Datomsbot)$
  **by** (*simp add*: *assms div-converse-via-atom-bot div-via-atom-bot*)

**lemma** *div-atomic-iff-1*:
  **assumes** *D3-transitive* -

**shows** *D11-atomic* - $\longleftrightarrow -Dbot \leq D^T * Datoms$
**using** *Datoms-def Dbot-def assms transitive-minimal-not-least* **by** *force*

**lemma** *div-atomic-iff-2*:
**assumes** *D3-transitive* -
**shows** *D11-atomic* - $\longleftrightarrow -D \leq D^T * Datoms$
**by** (*metis Dbot-def assms div-atomic-iff-1 div-atoms-vector div-least-div brouwer.p-antitone comp-associative double-compl inf-top.left-neutral least-def mult-left-isotone*)

**lemma** *div-atoms-antichain-minimal*:
**assumes** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *antichain D x*
**and** $D^T * x = -Dbot$
**shows** $x = minimal\ D\ (-Dbot)$
**proof** (*rule order.antisym*)
**have** *1*: $x \leq -Dbot$
**by** (*smt* (*verit*, *del-insts*) *assms*(*4*) *Dbot-def ex231d div-least-vector compl-le-compl-iff conv-complement-sub-leq conv-involutive double-compl inf-top.left-neutral least-def order-lesseq-imp schroeder-4-p top-right-mult-increasing*)
**have** $-Dbot \leq ((D^T \sqcap -1) \sqcup 1) * x$
**by** (*metis assms*(*4*) *heyting.implies-galois-increasing mult-left-isotone regular-one-closed sup-commute*)
**also have** ... $\leq x \sqcup (D^T \sqcap -1) * x$
**by** (*simp add*: *mult-right-dist-sup*)
**also have** ... $\leq x \sqcup (D^T \sqcap -1) * -Dbot$
**using** *1 mult-isotone sup-right-isotone* **by** *blast*
**finally have** $-Dbot \sqcap -((D^T \sqcap -1) * -Dbot) \leq x$
**using** *half-shunting sup-neg-inf* **by** *blast*
**thus** $minimal\ D\ (-Dbot) \leq x$
**by** (*simp add*: *minimal-def*)
**have** *2*: $D \sqcap -1 \sqcap x \leq -x^T$
**using** *assms*(*3*) *antichain-def inf.sup-monoid.add-commute inf-left-commute shunting-1* **by** *auto*
**have** $D * (D \sqcap -1) \leq D \sqcap -1$
**by** (*smt* (*verit*, *ccfv-threshold*) *assms*(*1*,*2*) *antisymmetric-inf-diversity conv-complement conv-involutive conv-order le-inf-iff mult-left-one mult-right-isotone order-lesseq-imp schroeder-4-p*)
**hence** $(D^T \sqcap -1) * D^T \leq D^T \sqcap -1$
**by** (*metis* (*mono-tags*, *opaque-lifting*) *conv-complement conv-dist-comp conv-dist-inf conv-order equivalence-one-closed*)
**hence** $(D^T \sqcap -1) * -Dbot \leq (D^T \sqcap -1) * x$
**by** (*metis assms*(*4*) *comp-associative mult-left-isotone*)
**also have** ... $= (D \sqcap -1 \sqcap x)^T * top$
**using** *assms*(*3*) *antichain-def conv-complement conv-dist-inf covector-inf-comp-3* **by** *auto*
**also have** ... $\leq -x * top$

**using** *2* **by** (*metis conv-complement conv-involutive conv-order mult-left-isotone*)
**also have** ... $= -x$
**using** *assms*(*3*) *antichain-def complement-vector* **by** *auto*
**finally show** $x \le$ *minimal D* ($-Dbot$)
**using** *1* **by** (*simp add*: *minimal-def p-antitone-iff*)
**qed**

**lemma** *div-atomic-iff-3*:
**assumes** *D2-antisymmetric* -
**and** *D3-transitive* -
**shows** *D11-atomic* - $\longleftrightarrow$ ($\exists x$ . *antichain D x* $\wedge D^T * x = -Dbot$)
**using** *Datoms-def assms div-atoms-antichain-minimal div-atoms-antichain* **by** *fastforce*

The literal translation of axiom A12 is $-Dbot \le -D^T * Datoms$. However, this allows a model without atoms, where $Dbot = top$ and $Datoms = Dmono = Dfactor = bot$. Nitpick finds a counterexample to *surjective Datoms*. With A2 and A12 the latter is equivalent to $-D^T * Datoms = top$, which we use as a replacement for axiom A12.

**lemma** *div-atom-surjective*:
**assumes** *D12-infinite-base* -
**shows** *surjective Datoms*
**by** (*metis assms invertible-surjective top-greatest*)

**lemma** *div-infinite-base-upperbound*:
**assumes** *D12-infinite-base* -
**shows** *upperbound D Datoms* = *bot*
**by** (*simp add*: *assms upperbound-def*)

**lemma** *div-atom-surjective-iff-infinite-base*:
**assumes** *D2-antisymmetric* -
**and** $-Dbot \le -D^T * Datoms$
**shows** *surjective Datoms* $\longleftrightarrow$ *D12-infinite-base* -
**proof** (*rule iffI*)
**assume** *1*: *surjective Datoms*
**have** *2*: $Dbot \sqcap -Dbot^T \le -D^T$
**by** (*metis assms*(*1*) *div-least-converse conv-dist-inf conv-involutive conv-order double-compl inf.sup-monoid.add-commute p-shunting-swap*)
**have** $top = top * Datoms$
**using** *1* **by** *simp*
**also have** ... $= top * (-Dbot \sqcap Datoms)$
**by** (*simp add*: *Datoms-def minimal-def*)
**also have** ... $= -Dbot^T * Datoms$
**by** (*metis complement-vector conv-complement covector-inf-comp-3 div-least-vector inf-top.left-neutral*)
**finally have** $Dbot = Dbot \sqcap -Dbot^T * Datoms$
**by** *simp*
**also have** ... $= (Dbot \sqcap -Dbot^T) * Datoms$

**by** (*simp add*: *div-least-vector vector-inf-comp*)
**also have** ... $\leq -D^T * Datoms$
**using** *2 mult-left-isotone* **by** *auto*
**finally have** $Dbot \leq -D^T * Datoms$
.
**thus** $-D^T * Datoms = top$
**by** (*metis assms*(*2*) *sup-absorb2 sup-shunt*)
**next**
**assume** $-D^T * Datoms = top$
**thus** *surjective Datoms*
**using** *div-atom-surjective* **by** *auto*
**qed**

## 2.4 Fibers

**lemma** *div-mono-vector*:
*vector Dmono*
**by** (*simp add*: *Dmono-def comp-associative*)

**lemma** *div-mono-bot-vector*:
*vector Dmonobot*
**by** (*simp add*: *div-mono-vector Dmonobot-def div-least-vector vector-sup-closed*)

**lemma** *div-atom-mono-atom*:
$Datoms \sqcap D * (D^T \sqcap Dmono) \sqcap Datoms^T \leq 1$
**proof** −
**let** $?u = univalent\text{-}part\ ((D \sqcap Datoms)^T)$
**have** *1*: $D^T \sqcap ?u * top \leq ?u * (?u^T * D^T)$
**by** (*metis dedekind-1 inf.absorb-iff1 inf-commute top-greatest*)
**have** *2*: $(Datoms \sqcap D) * ?u \leq 1$
**by** (*metis conv-involutive inf.sup-monoid.add-commute univalent-part-times-converse-1*)
**have** $Datoms \sqcap D * (D^T \sqcap Dmono) \sqcap Datoms^T = (Datoms \sqcap D) * (D^T \sqcap ?u * top) \sqcap Datoms^T$
**by** (*metis div-atoms-vector Dmono-def vector-export-comp*)
**also have** ... $\leq (Datoms \sqcap D) * ?u * ?u^T * D^T \sqcap Datoms^T$
**using** *1* **by** (*simp add*: *comp-associative inf.sup-monoid.add-commute le-infI2 mult-right-isotone*)
**also have** ... $\leq ?u^T * D^T \sqcap Datoms^T$
**using** *2* **by** (*metis comp-inf.mult-left-isotone mult-left-isotone comp-associative comp-left-one*)
**also have** ... $= ?u^T * (D \sqcap Datoms)^T$
**using** *div-atoms-vector conv-dist-inf covector-comp-inf vector-conv-covector* **by** *force*
**also have** ... $= ?u^T * ?u$
**by** (*metis conv-dist-comp conv-involutive univalent-part-times-converse*)
**also have** ... $\leq 1$
**by** (*simp add*: *univalent-part-univalent*)
**finally show** *?thesis*

.
**qed**

**lemma** *div-atoms-mono*:
**assumes** *D1-reflexive* -
**shows** *Datoms* $\leq$ *Dmono*
**proof** $-$
**have** $D^T \sqcap$ *Datoms* $\sqcap$ *Datoms*$^T \leq$ *1*
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-antichain antichain-def conv-dist-inf conv-involutive coreflexive-symmetric inf.left-commute inf.sup-monoid.add-commute*)
**hence** *1* $\sqcap$ ($D^T \sqcap$ *Datoms* $\sqcap$ *Datoms*$^T$) $*$ $-$*1* $\leq$ *bot*
**by** (*metis bot-least coreflexive-comp-top-inf inf-compl-bot-right*)
**hence** *1* $\sqcap$ *Datoms* $\sqcap$ ($D^T \sqcap$ *Datoms*$^T$) $*$ $-$*1* $\leq$ *bot*
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-vector inf.sup-monoid.add-commute inf-assoc vector-inf-comp*)
**hence** *1* $\sqcap$ *Datoms* $\leq$ $-$(($D^T \sqcap$ *Datoms*$^T$) $*$ $-$*1*)
**using** *le-bot pseudo-complement* **by** *blast*
**hence** *1* $\sqcap$ *Datoms* $\sqcap$ *Datoms*$^T \leq D^T \sqcap$ $-$(($D^T \sqcap$ *Datoms*$^T$) $*$ $-$*1*) $\sqcap$ *Datoms*$^T$
**using** *comp-inf.mult-left-isotone assms reflexive-conv-closed* **by** *fastforce*
**hence** (*1* $\sqcap$ *Datoms* $\sqcap$ *Datoms*$^T$) $*$ *top* $\leq$ ($D^T \sqcap$ $-$(($D^T \sqcap$ *Datoms*$^T$) $*$ $-$*1*) $\sqcap$ *Datoms*$^T$) $*$ *top*
**using** *mult-left-isotone* **by** *blast*
**hence** *Datoms* $\leq$ ($D^T \sqcap$ $-$(($D^T \sqcap$ *Datoms*$^T$) $*$ $-$*1*) $\sqcap$ *Datoms*$^T$) $*$ *top*
**by** (*smt* (*verit*) *div-atoms-vector inf.absorb2 inf.cobounded2 inf.left-commute inf-top.right-neutral one-inf-conv vector-export-comp-unit*)
**also have** ... = (($D \sqcap$ *Datoms*)$^T \sqcap$ $-$(($D \sqcap$ *Datoms*)$^T$ $*$ $-$*1*)) $*$ *top*
**by** (*smt* (*verit*) *conv-dist-inf inf.sup-monoid.add-assoc inf.sup-monoid.add-commute univalent-part-def*)
**finally show** *?thesis*
**by** (*simp add*: *Dmono-def univalent-part-def*)
**qed**

**lemma** *div-mono-downclosed*:
**assumes** *D3-transitive* -
**and** *D11-atomic* -
**shows** $-$*Dbot* $\sqcap$ $D$ $*$ *Dmono* $\leq$ *Dmono*
**proof** $-$
**let** *?u* = *univalent-part* (($D \sqcap$ *Datoms*)$^T$)
**have** $-$*Dbot* $\sqcap$ $D$ $*$ *?u* = ($-$*Dbot* $\sqcap$ $D$) $*$ *?u*
**by** (*simp add*: *Dbot-def least-def vector-export-comp*)
**also have** ... $\leq D^T$ $*$ ($D \sqcap$ *Datoms*) $*$ *?u*
**by** (*simp add*: *assms div-via-atom mult-left-isotone*)
**also have** ... $\leq D^T$
**by** (*metis comp-associative comp-right-one conv-involutive mult-right-isotone univalent-part-times-converse-1*)
**finally have** *1*: $-$*Dbot* $\sqcap$ $D$ $*$ *?u* $\leq D^T$
.
**have** (*Datoms* $\sqcap$ $D$) $*$ $D$ $\leq$ *Datoms* $\sqcap$ $D$

**using** *assms*(*1*) *div-atoms-vector inf-mono vector-inf-comp* **by** *auto*
**hence** $D^T * (D \sqcap Datoms)^T * -1 \leq (D \sqcap Datoms)^T * -1$
**by** (*metis conv-dist-comp conv-isotone inf-commute mult-left-isotone*)
**hence** *2*: $D * -((D \sqcap Datoms)^T * -1) \leq -((D \sqcap Datoms)^T * -1)$
**by** (*simp add*: *comp-associative schroeder-3-p*)
**have** $D * ?u \leq D * (D^T \sqcap Datoms^T) \sqcap D * -((D \sqcap Datoms)^T * -1)$
**using** *comp-right-subdist-inf conv-dist-inf univalent-part-def* **by** *auto*
**also have** ... $\leq Datoms^T \sqcap D * -((D \sqcap Datoms)^T * -1)$
**using** *div-atoms-vector comp-inf.mult-left-isotone covector-comp-inf vector-conv-covector* **by** *force*
**finally have** $-Dbot \sqcap D * ?u \leq D^T \sqcap D * -((D \sqcap Datoms)^T * -1) \sqcap Datoms^T$
**using** *1* **by** (*simp add*: *inf.coboundedI2*)
**also have** ... $\leq D^T \sqcap -((D \sqcap Datoms)^T * -1) \sqcap Datoms^T$
**using** *2 comp-inf.comp-isotone* **by** *blast*
**also have** ... $= ?u$
**by** (*smt* (*verit*, *ccfv-threshold*) *conv-dist-inf inf.sup-monoid.add-assoc inf.sup-monoid.add-commute univalent-part-def*)
**finally have** $-Dbot \sqcap D * ?u * top \leq ?u * top$
**by** (*metis div-least-vector mult-left-isotone vector-complement-closed vector-inf-comp*)
**thus** $-Dbot \sqcap D * Dmono \leq Dmono$
**by** (*simp add*: *Dmono-def comp-associative*)
**qed**

**lemma** *div-mono-bot-downclosed*:
**assumes** *D1-reflexive* -
**and** *D3-transitive* -
**and** *D11-atomic* -
**shows** $D * Dmonobot \leq Dmonobot$
**proof** −
**have** $D * Dmonobot = D * Dmono \sqcup D * Dbot$
**using** *Dmonobot-def comp-left-dist-sup* **by** *auto*
**also have** ... $= (-Dbot \sqcap D * Dmono) \sqcup Dbot$
**by** (*simp add*: *assms*(*1*,*2*) *bot-div-bot sup-commute*)
**also have** ... $\leq Dmonobot$
**using** *assms div-mono-downclosed Dmonobot-def sup-left-isotone* **by** *auto*
**finally show** *?thesis*
**.**
**qed**

**lemma** *div-least-not-mono*:
**assumes** *D2-antisymmetric* -
**shows** $Dbot \leq -Dmono$
**proof** −
**let** $?u = univalent\text{-}part\ ((D \sqcap Datoms)^T)$
**have** *1*: $Dbot \sqcap D^T \leq Dbot^T$
**by** (*metis assms div-least-converse conv-dist-inf conv-involutive conv-order inf.sup-monoid.add-commute*)

**have** *Dbot* $\sqcap$ *?u* $\leq$ *Dbot* $\sqcap$ $D^T$ $\sqcap$ $Datoms^T$
**using** *conv-dist-inf inf.sup-left-divisibility inf-assoc univalent-part-def* **by** *auto*
**also have** ... $\leq$ $Dbot^T$ $\sqcap$ $Datoms^T$
**using** *1 inf.sup-left-isotone* **by** *blast*
**also have** ... $\leq$ *bot*
**by** (*metis div-least-not-atom bot-least conv-dist-inf coreflexive-symmetric pseudo-complement*)
**finally show** *?thesis*
**by** (*metis Dmono-def compl-le-swap1 div-least-vector inf-top.right-neutral mult-left-isotone p-top pseudo-complement vector-complement-closed*)
**qed**

**lemma** *div-fibered-transitive-1*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D11-atomic* -
**shows** $Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T = Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**proof** (*rule order.antisym*)
**show** $Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T \leq Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**using** *assms*(*1*) *div-atoms-mono comp-inf.mult-right-isotone inf.sup-left-isotone inf.sup-mono mult-isotone sup.cobounded1 sup-ge2* **by** *auto*
**have** $Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T = (Dmono \sqcap (D \sqcup D^T) \sqcap Dmono^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**by** (*metis div-mono-vector covector-inf-comp-2 vector-export-comp*)
**also have** ... $\leq (-Dbot \sqcap (D \sqcup D^T) \sqcap Dmono^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**using** *assms*(*2*) *div-least-not-mono comp-inf.mult-left-isotone compl-le-swap1 mult-left-isotone* **by** *auto*
**also have** ... $\leq (-Dbot \sqcap (D \sqcup D^T) \sqcap -Dbot^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**by** (*smt* (*verit*) *assms*(*2*) *div-least-not-mono compl-le-compl-iff conv-complement conv-order double-compl inf.sup-monoid.add-commute inf.sup-right-isotone mult-left-isotone*)
**also have** ... $\leq D^T * (D \sqcap Datoms) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T$
**by** (*smt* (*verit, del-insts*) *assms*(*3*,*4*) *div-comparable-via-atom inf-commute inf-left-commute inf-sup-distrib1 mult-right-dist-sup sup.order-iff*)
**also have** ... $= D^T * (D \sqcap Datoms \sqcap Dmono^T) * (Dmono \sqcap (D \sqcup D^T) \sqcap Dmono^T)$
**by** (*smt* (*verit, ccfv-SIG*) *div-mono-vector covector-comp-inf covector-inf-comp-2 vector-conv-covector*)
**also have** ... $\leq D^T * (D \sqcap Datoms \sqcap Dmono^T) * (-Dbot \sqcap (D \sqcup D^T) \sqcap Dmono^T)$
**using** *assms*(*2*) *div-least-not-mono comp-inf.mult-left-isotone compl-le-swap1 mult-right-isotone* **by** *auto*
**also have** ... $\leq D^T * (D \sqcap Datoms \sqcap Dmono^T) * (-Dbot \sqcap (D \sqcup D^T) \sqcap -Dbot^T)$

**by** (*metis assms*(*2*) *div-least-not-mono comp-inf.mult-right-isotone compl-le-swap1 conv-complement conv-order mult-right-isotone*)
**also have** ... $= D^T * (D \sqcap Datoms \sqcap Dmono^T) * (-Dbot \sqcap (D \sqcup D^T) \sqcap -Dbot^T)^T$
**using** *conv-complement conv-dist-inf conv-dist-sup conv-involutive inf.sup-monoid.add-commute inf-assoc sup-commute* **by** *auto*
**also have** ... $\leq D^T * (D \sqcap Datoms \sqcap Dmono^T) * (D^T * (D \sqcap Datoms))^T$
**using** *assms*(*3*,*4*) *div-comparable-via-atom conv-order inf.sup-monoid.add-commute inf-assoc mult-right-isotone* **by** *auto*
**also have** ... $= D^T * (D \sqcap Datoms \sqcap Dmono^T) * (D \sqcap Datoms)^T * D$
**by** (*simp add*: *comp-associative conv-dist-comp*)
**also have** ... $= D^T * (Datoms \sqcap D * (Dmono \sqcap D^T) \sqcap Datoms^T) * D$
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-vector div-mono-vector comp-associative conv-dist-inf covector-inf-comp-3 inf.sup-monoid.add-commute*)
**also have** ... $= D^T * (Datoms \sqcap D * (Dmono \sqcap D^T) \sqcap Datoms^T) * (Datoms \sqcap D)$
**using** *div-atoms-vector covector-comp-inf covector-inf-comp-3 vector-conv-covector* **by** *auto*
**also have** ... $\leq D^T * (Datoms \sqcap D)$
**by** (*metis div-atom-mono-atom comp-right-one inf.sup-monoid.add-commute mult-left-isotone mult-right-isotone*)
**finally show** $Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T \leq Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T$
**by** (*simp add*: *inf.coboundedI2 inf.sup-monoid.add-commute*)
**qed**

**lemma** *div-fibered-iff*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D11-atomic* -
**shows** *D8-fibered* - $\longleftrightarrow Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T \leq D \sqcup D^T$
**using** *assms div-fibered-transitive-1* **by** *auto*

**lemma** *div-fibered-transitive*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D8-fibered* -
**and** *D11-atomic* -
**shows** $Dmono \sqcap (D \sqcup D^T) * (Dmono \sqcap (D \sqcup D^T)) \sqcap Dmono^T \leq D \sqcup D^T$
**using** *assms div-fibered-transitive-1* **by** *auto*

## 2.5 Fiber decomposition

**lemma** *div-factor-div-mono*:
$Dfactor \leq D \sqcap Dmono$
**by** (*metis Dfactor-def inf.cobounded1 maximal-def*)

**lemma** *div-factor-div*:
*Dfactor* $\leq$ *D*
**using** *div-factor-div-mono* **by** *auto*

**lemma** *div-factor-mono*:
*Dfactor* $\leq$ *Dmono*
**using** *div-factor-div-mono* **by** *auto*

**lemma** *div-factor-one-mono*:
*Dfactor* $\sqcap$ *1* $\leq$ *Dmono*
**using** *div-factor-mono inf.coboundedI1* **by** *blast*

**lemma** *div-pre-f-decomposable-1*:
**assumes** *D2-antisymmetric -*
**and** *D7-pre-f-decomposable -*
**shows** *upperbound D* (*D* $\sqcap$ *Dmono*) $\leq$ $D^T$
**using** *assms supremum-upperbound* **by** *force*

**lemma** *div-pre-f-decomposable-iff*:
**assumes** *D2-antisymmetric -*
**shows** *D7-pre-f-decomposable -* $\longleftrightarrow$ *upperbound D* (*D* $\sqcap$ *Dmono*) $\leq$ $D^T$
**using** *assms supremum-upperbound* **by** *force*

**lemma** *div-pre-f-decomposable-char*:
**assumes** *D2-antisymmetric -*
**and** *D7-pre-f-decomposable -*
**shows** *upperbound D* (*D* $\sqcap$ *Dmono*) $\sqcap$ (*upperbound D* (*D* $\sqcap$ *Dmono*))$^T$ = *1*
**proof** (*rule order.antisym*)
**have** *1* $\leq$ *upperbound D* (*D* $\sqcap$ *Dmono*)
**by** (*simp add*: *compl-le-swap1 conv-complement schroeder-3-p upperbound-def*)
**thus** *1* $\leq$ *upperbound D* (*D* $\sqcap$ *Dmono*) $\sqcap$ (*upperbound D* (*D* $\sqcap$ *Dmono*))$^T$
**using** *le-inf-iff reflexive-conv-closed* **by** *blast*
**have** *upperbound D* (*D* $\sqcap$ *Dmono*) $\sqcap$ (*upperbound D* (*D* $\sqcap$ *Dmono*))$^T$ $\leq$ $D^T \sqcap D$
**by** (*metis assms comp-inf.comp-isotone conv-involutive conv-order div-pre-f-decomposable-1*)
**thus** *upperbound D* (*D* $\sqcap$ *Dmono*) $\sqcap$ (*upperbound D* (*D* $\sqcap$ *Dmono*))$^T$ $\leq$ *1*
**by** (*metis assms*(*1*) *inf.absorb2 inf.boundedE inf-commute*)
**qed**

**lemma** *div-factor-bot*:
**assumes** *D1-reflexive -*
**and** *D2-antisymmetric -*
**and** *D3-transitive -*
**and** *D11-atomic -*
**shows** *Dfactorbot* = *Dfactor* $\sqcup$ (*Dbot* $\sqcap$ $Dbot^T$)
**proof** −
**have** *Dbot* $\sqcap$ $Datoms^T$ $\leq$ $-1$
**by** (*metis comp-inf.semiring.mult-not-zero div-least-not-atom*

*inf.sup-monoid.add-commute inf-left-commute one-inf-conv pseudo-complement*)
**hence** $Dbot \sqcap Datoms^T * D = (Dbot \sqcap Datoms^T \sqcap -1) * D$
**by** (*simp add*: *div-least-vector inf.absorb1 vector-inf-comp*)
**also have** ... $= (Dbot \sqcap -1) * (D \sqcap Datoms)$
**by** (*smt* (*verit*, *del-insts*) *div-atoms-vector comp-inf-vector conv-dist-comp inf.sup-monoid.add-assoc inf.sup-monoid.add-commute symmetric-top-closed*)
**also have** ... $\leq (D \sqcap -1) * (D \sqcap Dmono)$
**using** *assms*(*1*) *div-atoms-mono div-least-div comp-isotone inf.sup-mono order-refl* **by** *blast*
**finally have** *1*: $Dbot \sqcap Datoms^T * D \leq (D \sqcap -1) * (D \sqcap Dmono)$
·
**hence** *1*: $Dbot \sqcap -((D \sqcap -1) * (D \sqcap Dmono)) \leq Dbot^T$
**by** (*metis assms*(*4*) *conv-complement conv-dist-comp conv-involutive double-compl p-shunting-swap*)
**have** $Dbot \sqcap Dbot^T \sqcap (D \sqcap -1) * (D \sqcap Dmono) \leq Dbot^T \sqcap (D \sqcap -1) * D$
**using** *comp-inf.mult-right-isotone comp-right-subdist-inf inf.sup-monoid.add-assoc* **by** *force*
**also have** ... $=$ *bot*
**by** (*smt* (*verit*, *best*) *assms bot-div-bot div-atoms-vector div-strict-bot comp-associative comp-inf.vector-bot-closed complement-vector conv-dist-comp schroeder-2 symmetric-top-closed*)
**finally have** $Dbot \sqcap Dbot^T \leq -((D \sqcap -1) * (D \sqcap Dmono))$
**using** *le-bot pseudo-complement* **by** *blast*
**hence** *2*: $Dbot \sqcap -((D \sqcap -1) * (D \sqcap Dmono)) = Dbot \sqcap Dbot^T$
**using** *1* **by** (*smt* (*verit*, *del-insts*) *inf.absorb1 inf.sup-monoid.add-assoc inf-commute*)
**have** $Dfactorbot = D \sqcap (Dmono \sqcup Dbot) \sqcap -((D \sqcap -1) * (D \sqcap Dmono)) \sqcap -((D \sqcap -1) * (D \sqcap Dbot))$
**by** (*simp add*: *Dfactorbot-def Dmonobot-def comp-inf.vector-inf-comp inf-sup-distrib1 maximal-def mult-left-dist-sup*)
**also have** ... $= D \sqcap (Dmono \sqcup Dbot) \sqcap -((D \sqcap -1) * (D \sqcap Dmono))$
**by** (*simp add*: *assms*(*2*) *div-strict-bot div-least-div inf.absorb2*)
**also have** ... $= Dfactor \sqcup (Dbot \sqcap -((D \sqcap -1) * (D \sqcap Dmono)))$
**using** *div-least-div Dfactor-def comp-inf.mult-right-dist-sup inf.absorb2 inf-sup-distrib1 maximal-def* **by** *auto*
**finally show** *?thesis*
**using** *2* **by** *auto*
**qed**

**lemma** *div-factor-surjective*:
**assumes** *D1-reflexive* -
**and** *D3-transitive* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *surjective* $(Dbot^T \sqcup Dfactor)$
**proof** −
**have** $D \sqcap Datoms \leq top * Dfactor$
**by** (*metis assms*(*3*) *inf.sup-monoid.add-commute mult-left-isotone order-lesseq-imp top-greatest*)

**hence** $D^T * (D \sqcap Datoms) \leq top * Dfactor$
**by** (*metis covector-mult-closed mult-isotone top-greatest vector-top-closed*)
**hence** $-Dbot \sqcap D \leq top * Dfactor$
**using** *assms*(*2*,*4*) *div-via-atom* **by** *auto*
**hence** $top * (-Dbot \sqcap D) \leq top * Dfactor$
**by** (*metis comp-associative mult-right-isotone vector-top-closed*)
**hence** $-Dbot^T * D \leq top * Dfactor$
**by** (*simp add*: *Dbot-def comp-inf-vector conv-complement conv-dist-comp inf.sup-monoid.add-commute least-def*)
**hence** $-Dbot^T \leq top * Dfactor$
**by** (*metis assms*(*1*) *bot-least case-split-right inf.sup-monoid.add-commute maddux-3-21 semiring.mult-not-zero shunting-1 sup.cobounded2*)
**hence** $top \leq Dbot^T \sqcup top * Dfactor$
**by** (*simp add*: *sup-neg-inf*)
**thus** *?thesis*
**using** *div-least-vector mult-left-dist-sup top-le vector-conv-covector* **by** *auto*
**qed**

**lemma** *div-factor-bot-surjective*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *surjective Dfactorbot*
**proof** −
**have** $top * (Dbot \sqcap Dbot^T) = top * Dbot^T$
**by** (*smt* (*verit*) *conv-dist-comp covector-inf-comp-3 div-least-vector ex231d mult-right-isotone order.eq-iff top-greatest vector-conv-covector vector-covector vector-top-closed*)
**thus** *?thesis*
**using** *assms div-factor-bot div-factor-surjective mult-left-dist-sup sup-monoid.add-commute* **by** *force*
**qed**

**lemma** *div-factor-surjective-2*:
**assumes** *D1-reflexive* -
**and** *D3-transitive* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $-D \leq Dfactor^T * top$
**proof** −
**have** $-D^T \leq -Dbot^T$
**using** *div-least-div conv-order* **by** *auto*
**also have** ... $\leq top * Dfactor$
**by** (*metis assms div-factor-surjective conv-dist-comp equivalence-top-closed mult-left-dist-sup sup-shunt div-least-vector*)
**finally show** *?thesis*
**by** (*metis conv-complement conv-dist-comp conv-involutive conv-order*

*equivalence-top-closed*)
**qed**

**lemma** *div-conv-factor-div-factor*:
  **assumes** *D1-reflexive* -
    **shows** $Dmono \sqcap D^T * Dfactor \sqcap D \le D * Dfactor$
**proof** −
  **have** $-(1 \sqcup -D^T)^T * (Dmono \sqcap D) \le (D \sqcap -1) * (D \sqcap Dmono)$
    **by** (*simp add*: *conv-complement conv-dist-sup inf.sup-monoid.add-commute*)
  **hence** $(Dmono \sqcap D) * -((D \sqcap -1) * (D \sqcap Dmono))^T \le 1 \sqcup -D^T$
    **using** *schroeder-5 schroeder-6* **by** *blast*
  **hence** *1*: $Dmono \sqcap D^T \sqcap D * -((D \sqcap -1) * (D \sqcap Dmono))^T \le 1$
    **by** (*simp add*: *Dmono-def heyting.implies-galois-var*
*inf.sup-monoid.add-commute inf-assoc inf-vector-comp sup-commute*)
  **have** $Dfactor^T \le -((D \sqcap -1) * (D \sqcap Dmono))^T$
    **by** (*metis Dfactor-def conv-complement conv-order inf.sup-right-divisibility*
*maximal-def*)
  **hence** *2*: $Dmono \sqcap D^T \sqcap D * Dfactor^T \le 1$
    **using** *1* **by** (*meson inf.sup-right-isotone mult-right-isotone order-trans*)
  **hence** $(Dmono \sqcap D^T \sqcap D * Dfactor^T) * Dfactor \sqcap D \le D * Dfactor$
    **using** *assms*(*1*) *dual-order.trans inf.coboundedI1 mult-left-isotone* **by** *blast*
  **thus** *?thesis*
    **using** *2* **by** (*smt* (*verit*, *del-insts*) *div-factor-div Dmono-def*
*coreflexive-comp-top-inf dedekind-2 dual-order.trans inf.absorb1 inf-assoc*
*vector-export-comp*)
**qed**

**lemma** *div-f-decomposable-mono*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
      **and** *D3-transitive* -
      **and** *D8-fibered* -
      **and** *D9-f-decomposable* -
      **and** *D11-atomic* -
    **shows** $Dmono \sqcap D \le D * Dfactor$
**proof** −
  **have** $Dmono \sqcap D = Dmono \sqcap -Dbot \sqcap D$
    **by** (*metis assms*(*2*) *div-least-not-mono compl-le-swap1 inf.order-iff*)
  **also have** ... $= Dmono \sqcap D^T * (D \sqcap Datoms) \sqcap D$
    **by** (*smt* (*verit*, *ccfv-SIG*) *assms*(*2*,*3*,*6*) *div-least-not-mono div-via-atom*
*compl-le-swap1 inf.le-iff-sup inf-assoc inf-left-commute*)
  **also have** ... $= Dmono \sqcap D^T * (D \sqcap Datoms \sqcap D * Dfactor) \sqcap D$
    **using** *assms*(*5*) *inf.le-iff-sup inf.sup-monoid.add-commute* **by** *auto*
  **also have** ... $= Dmono \sqcap D^T * (D \sqcap Datoms \sqcap D * (Dmono \sqcap Dfactor)) \sqcap D$
    **using** *div-factor-mono inf.le-iff-sup* **by** *fastforce*
  **also have** ... $= Dmono \sqcap D^T * (D \sqcap (Datoms \sqcap D \sqcap Dmono^T) * Dfactor) \sqcap D$
    **using** *div-atoms-vector div-mono-vector covector-inf-comp-3*
*inf.sup-monoid.add-assoc vector-inf-comp* **by** *auto*
  **also have** ... $\le Dmono \sqcap D^T * (Datoms \sqcap D \sqcap Dmono^T) * Dfactor \sqcap D$

**by** (*simp add*: *comp-associative inf.coboundedI2 inf.sup-monoid.add-commute mult-right-isotone*)
**also have** ... $= Dmono \sqcap (Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T) * Dfactor \sqcap D$
**by** (*metis div-mono-vector covector-comp-inf inf.left-idem vector-conv-covector vector-export-comp*)
**also have** ... $\leq Dmono \sqcap (D \sqcup D^T) * Dfactor \sqcap D$
**by** (*metis assms*(*4*) *inf.sup-monoid.add-commute inf.sup-right-isotone mult-left-isotone*)
**also have** ... $= (Dmono \sqcap D * Dfactor \sqcap D) \sqcup (Dmono \sqcap D^T * Dfactor \sqcap D)$
**by** (*simp add*: *inf-sup-distrib1 inf-sup-distrib2 mult-right-dist-sup*)
**also have** ... $\leq D * Dfactor$
**by** (*simp add*: *assms*(*1*) *div-conv-factor-div-factor inf.coboundedI1*)
**finally show** *?thesis*
**.**
**qed**

**lemma** *div-pre-f-decomposable-2*:
**assumes** *D2-antisymmetric* -
**and** *D7-pre-f-decomposable* -
**shows** $-D \leq (D \sqcap Dmono)^T * -D$
**by** (*metis assms brouwer.p-antitone-iff conv-complement conv-dist-comp conv-involutive conv-order div-pre-f-decomposable-1 upperbound-def*)

**lemma** *div-f-decomposable-char-1*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $Dfactor^T * -D = -D$
**proof** (*rule order.antisym*)
**have** $Dfactor * D \leq D$
**using** *assms*(*3*) *div-factor-div dual-order.trans mult-left-isotone* **by** *blast*
**thus** $Dfactor^T * -D \leq -D$
**by** (*simp add*: *schroeder-3-p*)
**have** $-D \leq (D \sqcap Dmono)^T * -D$
**by** (*simp add*: *assms*(*2*,*4*) *div-pre-f-decomposable-2*)
**also have** ... $\leq Dfactor^T * D^T * -D$
**by** (*metis assms*(*1*−*3*,*5*−*7*) *div-f-decomposable-mono conv-dist-comp conv-order inf-commute mult-left-isotone*)
**also have** ... $\leq Dfactor^T * -D$
**using** *assms*(*3*) *comp-associative mult-right-isotone schroeder-3* **by** *auto*
**finally show** $-D \leq Dfactor^T * -D$
**.**
**qed**

**lemma** *div-f-decomposable-char-2*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
      **and** *D3-transitive* -
      **and** *D7-pre-f-decomposable* -
      **and** *D8-fibered* -
      **and** *D9-f-decomposable* -
      **and** *D11-atomic* -
    **shows** *noyau Dfactor* = *1*
**proof** (*rule order.antisym*)
  **show** *reflexive* (*noyau Dfactor*)
    **by** (*simp add*: *noyau-reflexive*)
  **have** $-(Dfactor^T * -Dfactor) \leq -(Dfactor^T * -D)$
    **by** (*simp add*: *div-factor-div mult-right-isotone*)
  **also have** ... = *D*
    **by** (*simp add*: *assms div-f-decomposable-char-1*)
  **finally have** *1*: $-(Dfactor^T * -Dfactor) \leq D$
    .
  **hence** $-(-Dfactor^T * Dfactor) \leq D^T$
    **by** (*metis conv-complement conv-dist-comp conv-involutive conv-order*)
  **thus** *noyau Dfactor* ≤ *1*
    **using** *1 assms*(*1*,*2*) *div-antisymmetric-equal comp-inf.comp-isotone symmetric-quotient-def* **by** *force*
**qed**

**lemma** *div-mono-one-div-factor*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
    **shows** $Dmono \sqcap 1 \leq Dfactor$
**proof** −
  **have** $Dmono \sqcap 1 \sqcap (D \sqcap -1) * (D \sqcap Dmono) \leq 1 \sqcap (D \sqcap -1) * D$
    **by** (*meson comp-inf.mult-right-isotone comp-right-subdist-inf inf.bounded-iff*)
  **also have** ... ≤ *bot*
    **by** (*metis assms*(*2*) *compl-le-swap1 dual-order.eq-iff inf-shunt mult-1-left p-shunting-swap schroeder-4-p double-compl*)
  **finally have** *1*: $Dmono \sqcap 1 \leq -((D \sqcap -1) * (D \sqcap Dmono))$
    **using** *le-bot pseudo-complement* **by** *auto*
  **have** $Dmono \sqcap 1 \leq D \sqcap Dmono$
    **by** (*simp add*: *assms*(*1*) *le-infI2*)
  **thus** *?thesis*
    **using** *1* **by** (*simp add*: *Dfactor-def maximal-def*)
**qed**

**lemma** *div-mono-one-div-factor-one*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
    **shows** $Dmono \sqcap 1 = Dfactor \sqcap 1$
  **using** *assms div-mono-one-div-factor div-factor-mono inf.sup-same-context le-infI1* **by** *blast*

**lemma** *div-factor-div-mono-div-factor*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $Dfactor * D = Dmono \sqcap D * Dfactor$
**proof** (*rule order.antisym*)
**have** $Dfactor * D \le D * Dfactor$
**by** (*smt* (*verit*, *best*) *assms div-f-decomposable-mono div-factor-div-mono div-idempotent div-mono-vector comp-isotone inf-commute order-trans vector-export-comp*)
**thus** $Dfactor * D \le Dmono \sqcap D * Dfactor$
**by** (*metis div-factor-mono div-mono-vector inf.boundedI mult-isotone top-greatest*)
**have** $Dmono \sqcap D * Dfactor \le Dmono \sqcap D * D$
**using** *div-factor-div comp-inf.mult-isotone mult-isotone* **by** *blast*
**also have** ... $\le Dmono \sqcap D$
**by** (*simp add*: *assms*(*3*) *inf.coboundedI1 inf-commute*)
**also have** ... $= (Dmono \sqcap 1) * D$
**by** (*simp add*: *div-mono-vector vector-inf-one-comp*)
**also have** ... $\le Dfactor * D$
**by** (*simp add*: *assms*(*1*,*2*) *div-mono-one-div-factor mult-left-isotone*)
**finally show** $Dmono \sqcap D * Dfactor \le Dfactor * D$
**.**
**qed**

**lemma** *div-mono-strict-div-factor*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**shows** $Dmono \sqcap (D \sqcap -1) * Dfactor \le Dfactor * (D \sqcap -1)$
**proof** −
**have** $Dmono \sqcap (D \sqcap -1) * Dfactor \le Dmono \sqcap (D \sqcap -1) * D$
**using** *div-factor-div comp-inf.mult-isotone mult-isotone* **by** *blast*
**also have** ... $\le Dmono \sqcap D \sqcap -1$
**using** *assms*(*2*,*3*) *comp-inf.semiring.mult-left-mono strict-order-transitive-2* **by** *auto*
**also have** ... $= (Dmono \sqcap 1) * (D \sqcap -1)$
**by** (*simp add*: *div-mono-vector inf.sup-monoid.add-assoc vector-inf-one-comp*)
**also have** ... $\le Dfactor * (D \sqcap -1)$
**by** (*simp add*: *assms*(*1*,*2*) *div-mono-one-div-factor mult-left-isotone*)
**finally show** *?thesis*
**.**
**qed**

**lemma** *div-factor-div-strict*:

**assumes** *D1-reflexive* -
  **and** *D2-antisymmetric* -
  **and** *D3-transitive* -
  **and** *D8-fibered* -
  **and** *D9-f-decomposable* -
  **and** *D11-atomic* -
 **shows** $Dfactor * D \sqcap -1 = Dfactor * (D \sqcap -1)$
**proof** (*rule order.antisym*)
 **have** $Dfactor * D \sqcap -1 \leq Dmono \sqcap D \sqcap -1$
  **by** (*metis assms div-factor-div div-factor-div-mono-div-factor div-idempotent inf.bounded-iff inf.cobounded1 inf.sup-left-isotone mult-left-isotone*)
 **also have** ... $= (Dmono \sqcap 1) * (D \sqcap -1)$
  **by** (*simp add*: *div-mono-vector inf.sup-monoid.add-assoc vector-inf-one-comp*)
 **also have** ... $\leq Dfactor * (D \sqcap -1)$
  **using** *assms*(*1*,*2*) *div-mono-one-div-factor mult-left-isotone* **by** *auto*
 **finally show** $Dfactor * D \sqcap -1 \leq Dfactor * (D \sqcap -1)$
  **.**
 **have** $Dfactor * (D \sqcap -1) \leq D * (D \sqcap -1)$
  **by** (*simp add*: *div-factor-div mult-left-isotone*)
 **also have** ... $\leq -1$
  **by** (*simp add*: *assms*(*1*,*2*,*3*) *strict-order-transitive-eq-2*)
 **finally show** $Dfactor * (D \sqcap -1) \leq Dfactor * D \sqcap -1$
  **by** (*simp add*: *mult-right-isotone*)
**qed**

**lemma** *div-factor-strict*:
 **assumes** *D1-reflexive* -
  **and** *D2-antisymmetric* -
  **and** *D3-transitive* -
  **and** *D8-fibered* -
  **and** *D9-f-decomposable* -
  **and** *D11-atomic* -
 **shows** $Dfactor \sqcap -1 \leq Dfactor * (D \sqcap -1)$
 **by** (*metis assms div-factor-div-strict comp-right-one inf.sup-monoid.add-commute inf.sup-right-isotone mult-right-isotone*)

**lemma** *div-factor-div-mono-div*:
 **assumes** *D1-reflexive* -
  **and** *D2-antisymmetric* -
  **and** *D3-transitive* -
 **shows** $Dfactor * D = Dmono \sqcap D$
**proof** (*rule order.antisym*)
 **show** $Dfactor * D \leq Dmono \sqcap D$
  **by** (*smt* (*verit*, *ccfv-SIG*) *assms*(*3*) *div-factor-div div-factor-mono div-mono-vector comp-isotone inf.boundedI inf.order-trans order.refl top-greatest*)
 **show** $Dmono \sqcap D \leq Dfactor * D$
  **by** (*metis assms*(*1*,*2*) *div-mono-one-div-factor div-mono-vector mult-left-isotone vector-export-comp-unit*)
**qed**

**lemma** *div-factor-div-div-factor*:
  **assumes** *D1-reflexive* -
    **and** *D2-antisymmetric* -
    **and** *D3-transitive* -
    **and** *D8-fibered* -
    **and** *D9-f-decomposable* -
    **and** *D11-atomic* -
  **shows** $Dfactor * D \leq D * Dfactor$
  **by** (*simp add*: *assms div-factor-div-mono-div-factor*)

**lemma** *div-f-decomposable-eq*:
  **assumes** *D3-transitive* -
    **and** *D9-f-decomposable* -
  **shows** $Datoms \sqcap D = Datoms \sqcap D * Dfactor$
  **by** (*smt* (*verit*, *ccfv-threshold*) *assms div-factor-div inf.absorb2 inf.sup-monoid.add-assoc inf-commute mult-isotone mult-right-isotone*)

**lemma** *div-f-decomposable-iff-1*:
  **assumes** *D3-transitive* -
  **shows** *D9-f-decomposable* - $\longleftrightarrow Datoms \sqcap D = Datoms \sqcap D * Dfactor$
  **using** *assms div-f-decomposable-eq* **by** *fastforce*

**lemma** *div-f-decomposable-iff-2*:
  **assumes** *D3-transitive* -
  **shows** $Dmono \sqcap D \leq D * Dfactor \longleftrightarrow Dmono \sqcap D = Dmono \sqcap D * Dfactor$
  **by** (*smt* (*verit*, *ccfv-SIG*) *assms div-factor-div div-mono-vector inf.absorb1 inf.cobounded2 inf.le-iff-sup inf.sup-monoid.add-assoc mult-isotone vector-inf-comp*)

**lemma** *div-factor-not-bot-conv*:
  **assumes** *D2-antisymmetric* -
  **shows** $Dfactor \leq -Dbot^T$
  **by** (*smt* (*verit*, *best*) *assms div-least-converse div-least-not-mono div-factor-div-mono inf.absorb2 inf.coboundedI1 p-shunting-swap*)

**lemma** *div-total-top-factor*:
  **assumes** *D2-antisymmetric* -
    **and** *D6-least-surjective* -
  **shows** *total* $(-(top * Dfactor))$
**proof** −
  **have** $top = -(top * -Dbot^T) * top$
    **using** *assms div-least-point surjective-conv-total vector-conv-compl* **by** *auto*
  **also have** ... $\leq -(top * Dfactor) * top$
    **by** (*simp add*: *assms*(*1*) *div-factor-not-bot-conv mult-isotone*)
  **finally show** *?thesis*
    **by** (*simp add*: *dual-order.antisym*)
**qed**

**lemma** *dif-f-decomposable-iff-3*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D8-fibered* -
**and** *D11-atomic* -
**shows** *D9-f-decomposable* - $\longleftrightarrow$ *Dmono* $\sqcap$ *D* $\leq$ *D* $*$ *Dfactor*
**using** *assms div-atoms-mono div-f-decomposable-iff-1 div-f-decomposable-iff-2 div-f-decomposable-mono inf.sup-relative-same-increasing* **by** *blast*

## 2.6 Support

**lemma** *div-support-div*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *Dsupport* $\leq D^T$
**proof** −
**have** $-D^T = -D^T$ $*$ *Dfactor*
**by** (*metis assms div-f-decomposable-char-1 conv-complement conv-dist-comp conv-involutive*)
**also have** ... $\leq -($*Datoms* $\sqcap D)^T$ $*$ *Dfactor*
**by** (*simp add*: *conv-isotone mult-left-isotone*)
**finally have** $-(-($*Datoms* $\sqcap D)^T$ $*$ *Dfactor*$) \leq D^T$
**using** *compl-le-swap2* **by** *blast*
**thus** *?thesis*
**using** *Dsupport-def symmetric-quotient-def inf.coboundedI2* **by** *auto*
**qed**

**lemma** *div-support-univalent*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *univalent Dsupport*
**by** (*metis assms div-f-decomposable-char-2 Dsupport-def syq-comp-transitive syq-converse*)

**lemma** *div-support-mapping*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -

**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D13-supportable* -
**shows** *mapping Dsupport*
**by** (*simp add*: *assms div-support-univalent*)

**lemma** *div-support-2*:
**assumes** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $Dsupport = -((Datoms \sqcap D)^T * -Dfactor) \sqcap -(-D^T * (Datoms \sqcap D))$
**proof** (*rule order.antisym*)
**have** $-D^T * (Datoms \sqcap D) \leq -D^T * D * Dfactor$
**by** (*simp add*: *assms*(*3*) *comp-associative mult-right-isotone*)
**also have** ... $\leq -(Datoms \sqcap D)^T * Dfactor$
**by** (*meson assms*(*2*) *comp-isotone inf-le2 order.refl order-trans schroeder-6*)
**finally show** $Dsupport \leq -((Datoms \sqcap D)^T * -Dfactor) \sqcap -(-D^T * (Datoms \sqcap D))$
**using** *Dsupport-def symmetric-quotient-def comp-inf.mult-right-isotone* **by** *auto*
**have** $D \sqcap Dfactor * Dfactor^T \leq D^T$
**using** *Dfactor-def maximal-comparable* **by** *auto*
**hence** $Datoms \sqcap D \sqcap Dfactor * Dfactor^T \leq Datoms \sqcap D \sqcap D^T$
**by** (*simp add*: *inf.coboundedI2 inf.sup-monoid.add-assoc*)
**also have** ... $\leq Datoms^T$
**by** (*smt* (*verit*, *ccfv-threshold*) *assms*(*1*) *comp-inf.covector-comp-inf comp-inf.mult-left-isotone inf-commute inf-top.left-neutral le-inf-iff one-inf-conv*)
**finally have** $Dfactor * Dfactor^T \leq Datoms^T \sqcup -D \sqcup -Datoms$
**by** (*simp add*: *heyting.implies-galois-var sup.left-commute sup-monoid.add-commute*)
**hence** $Dfactor^T * -(Datoms^T \sqcup -D \sqcup -Datoms)^T \leq -Dfactor^T$
**using** *schroeder-5* **by** *auto*
**hence** *1*: $Dfactor^T * (-Datoms \sqcap D^T \sqcap Datoms^T) \leq -Dfactor^T$
**by** (*simp add*: *conv-complement conv-dist-sup*)
**have** $Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap D)) = Dfactor^T * (-Datoms \sqcap D^T \sqcap Datoms^T) * (Datoms \sqcap D)$
**by** (*metis div-atoms-vector covector-inf-comp-2 vector-complement-closed vector-inf-comp mult-assoc*)
**also have** ... $\leq -Dfactor^T * (Datoms \sqcap D)$
**using** *1 mult-left-isotone* **by** *blast*
**finally have** *2*: $Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap D)) \leq -Dfactor^T * (Datoms \sqcap D)$
.
**have** $Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap -D)) \leq D^T * D^T * (Datoms \sqcap -D)$

**by** (*simp add*: *div-factor-div comp-associative comp-isotone conv-isotone*)
**also have** ... $\leq D^T * (Datoms \sqcap -D)$
**by** (*simp add*: *assms*(*2*) *mult-left-isotone transitive-conv-closed*)
**finally have** *3*: $Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap -D)) \leq D^T * (Datoms \sqcap -D)$
.
**have** $Dfactor^T * -(Datoms \sqcap D) = Dfactor^T * (-D \sqcap Datoms) \sqcup Dfactor^T * -Datoms$
**by** (*smt* (*verit*, *del-insts*) *double-compl inf-import-p mult-left-dist-sup p-dist-sup sup-monoid.add-commute*)
**also have** ... $\leq D^T * (-D \sqcap Datoms) \sqcup (Dfactor \sqcap Dmono)^T * -Datoms$
**using** *div-factor-div div-factor-mono comp-inf.semiring.add-right-mono conv-order mult-left-isotone sup-right-isotone* **by** *auto*
**also have** ... $= D^T * (-D \sqcap Datoms) \sqcup Dfactor^T * (-Datoms \sqcap Dmono)$
**by** (*simp add*: *Dmono-def comp-inf-vector conv-dist-comp conv-dist-inf*)
**also have** ... $\leq D^T * (-D \sqcap Datoms) \sqcup Dfactor^T * (-Datoms \sqcap -Dbot)$
**using** *assms*(*1*) *div-least-not-mono mult-right-isotone p-antitone-iff sup-right-isotone* **by** *force*
**also have** ... $\leq D^T * (-D \sqcap Datoms) \sqcup Dfactor^T * (-Datoms \sqcap D^T * Datoms)$
**by** (*simp add*: *assms*(*4*))
**also have** ... $= D^T * (-D \sqcap Datoms) \sqcup Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap D)) \sqcup Dfactor^T * (-Datoms \sqcap D^T * (Datoms \sqcap -D))$
**by** (*metis inf-sup-distrib1 inf-top-right mult-left-dist-sup sup-commute sup-compl-top sup-left-commute*)
**also have** ... $\leq D^T * (-D \sqcap Datoms) \sqcup -Dfactor^T * (Datoms \sqcap D)$
**using** *2 3* **by** (*smt* (*verit*, *best*) *comp-inf.semiring.add-right-mono inf.sup-monoid.add-commute sup-absorb2 sup-commute sup-monoid.add-assoc*)
**also have** ... $= -Dfactor^T * (Datoms \sqcap D) \sqcup (Datoms \sqcap D)^T * -D$
**by** (*simp add*: *div-atoms-vector conv-dist-inf covector-inf-comp-3 inf-commute sup-commute*)
**finally have** $Dfactor^T * -(Datoms \sqcap D) \leq -Dfactor^T * (Datoms \sqcap D) \sqcup (Datoms \sqcap D)^T * -D$
.
**hence** $-(Datoms \sqcap D)^T * Dfactor \leq (Datoms \sqcap D)^T * -Dfactor \sqcup -D^T * (Datoms \sqcap D)$
**by** (*smt* (*verit*, *best*) *conv-complement conv-dist-comp conv-dist-sup conv-involutive conv-order*)
**hence** $-((Datoms \sqcap D)^T * -Dfactor) \sqcap -(-D^T * (Datoms \sqcap D)) \leq -(-(Datoms \sqcap D)^T * Dfactor)$
**using** *brouwer.p-antitone* **by** *fastforce*
**thus** $-((Datoms \sqcap D)^T * -Dfactor) \sqcap -(-D^T * (Datoms \sqcap D)) \leq Dsupport$
**using** *Dsupport-def symmetric-quotient-def* **by** *simp*
**qed**

**lemma** *noyau-div-support*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -

**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D13-supportable* -
**shows** *noyau* ($Datoms \sqcap D$) = $Dsupport * Dsupport^T$
**using** *assms div-support-mapping Dsupport-def syq-comp-syq-top syq-converse*
**by** *auto*

**lemma** *div-support-transitive*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D13-supportable* -
**shows** *idempotent Dsupport*
**proof** −
**let** $?r = Datoms \sqcap D$
**let** $?s = Datoms \sqcap Dfactor$
**have** $?r * -(?r^T * -?s) * -(?r^T * -?s) \leq ?s * -(?r^T * -?s)$
**by** (*metis complement-conv-sub conv-complement mult-left-isotone schroeder-5*)
**also have** ... $\leq ?r * -(?r^T * -?s)$
**by** (*simp add*: *div-factor-div le-infI2 mult-left-isotone*)
**also have** ... $\leq ?s$
**using** *pp-increasing schroeder-3* **by** *blast*
**finally have** $-(?r^T * -?s) * -(?r^T * -?s) \leq -(?r^T * -?s)$
**by** (*simp add*: *p-antitone-iff schroeder-3-p mult-assoc*)
**hence** *1*: $-(?r^T * -Dfactor) * -(?r^T * -Dfactor) \leq -(?r^T * -Dfactor)$
**by** (*simp add*: *div-atoms-vector conv-dist-inf covector-inf-comp-3 inf.sup-monoid.add-commute*)
**have** $-(-?r^T * ?r) * -(-?r^T * ?r) * ?r^T \leq -(-?r^T * ?r) * ?r^T$
**by** (*metis complement-conv-sub double-compl mult-right-isotone mult-assoc*)
**also have** ... $\leq ?r^T$
**using** *brouwer.pp-increasing complement-conv-sub inf.order-trans* **by** *blast*
**finally have** $-(-?r^T * ?r) * -(-?r^T * ?r) \leq -(-?r^T * ?r)$
**by** (*simp add*: *p-antitone-iff schroeder-4-p*)
**hence** *2*: $-(-D^T * ?r) * -(-D^T * ?r) \leq -(-D^T * ?r)$
**by** (*smt* (*verit, del-insts*) *div-atoms-vector conv-dist-inf covector-inf-comp-3 inf.sup-monoid.add-commute inf-import-p*)
**have** $Dsupport * Dsupport \leq -(?r^T * -Dfactor) * -(?r^T * -Dfactor) \sqcap -(-D^T * ?r) * -(-D^T * ?r)$
**by** (*simp add*: *assms*(*2*,*3*,*6*,*7*) *div-support-2 mult-isotone*)
**also have** ... $\leq Dsupport$
**using** *1 2 assms*(*2*,*3*,*6*,*7*) *div-support-2 inf.sup-mono* **by** *auto*
**finally have** *transitive Dsupport*
.
**thus** *?thesis*

**using** *assms div-support-mapping transitive-mapping-idempotent* **by** *blast*
**qed**

**lemma** *div-support-below-noyau*:
**assumes** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $Dsupport \le noyau\ (Datoms \sqcap D)$
**proof** $-$
**have** $-((Datoms \sqcap D)^T * -Dfactor) \le -((Datoms \sqcap D)^T * -D)$
**by** (*simp add*: *div-factor-div mult-right-isotone*)
**also have** ... $= -((Datoms \sqcap D)^T * -(Datoms \sqcap D))$
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-vector comp-inf-vector conv-dist-comp conv-dist-inf inf-commute inf-import-p symmetric-top-closed*)
**finally have** *1*: $-((Datoms \sqcap D)^T * -Dfactor) \le -((Datoms \sqcap D)^T * -(Datoms \sqcap D))$
.
**have** $-(-D^T * (Datoms \sqcap D)) = -(-(Datoms \sqcap D)^T * (Datoms \sqcap D))$
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-vector comp-inf-vector conv-dist-comp conv-dist-inf inf-commute inf-import-p symmetric-top-closed*)
**thus** $Dsupport \le noyau\ (Datoms \sqcap D)$
**using** *1 assms div-support-2 symmetric-quotient-def inf.sup-left-isotone* **by** *auto*
**qed**

**lemma** *div-support-least-noyau*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D13-supportable* -
**shows** $Dsupport = (least\ D\ (noyau\ (Datoms \sqcap D)))^T$
**proof** (*rule order.antisym*)
**let** $?n = noyau\ (Datoms \sqcap D)$
**have** $?n \le Dsupport * D$
**by** (*metis assms div-support-div conv-involutive conv-order mult-right-isotone noyau-div-support*)
**hence** $Dsupport^T * ?n \le D$
**using** *assms div-support-mapping shunt-mapping* **by** *blast*
**hence** $Dsupport \le -(?n * -D^T)$
**by** (*simp add*: *compl-le-swap1 conv-complement schroeder-6-p*)
**hence** $Dsupport \le -(-D * ?n)^T$
**by** (*simp add*: *conv-complement conv-dist-comp syq-converse*)
**thus** $Dsupport \le (least\ D\ ?n)^T$
**using** *assms*(*2*,*3*,*6*,*7*) *div-support-below-noyau least-def syq-converse*

*conv-complement conv-dist-inf* **by** *auto*
**have** *Dsupport* $\sqcap -1 \leq$ (*Dsupport* $\sqcap -1$) $*$ (*Dsupport* $\sqcap -1)^T *$ (*Dsupport* $\sqcap$ $-1$)
**using** *ex231c* **by** *auto*
**also have** ... $\leq (D^T \sqcap -1) *$ *Dsupport*$^T *$ *Dsupport*
**using** *assms*(*1*−*7*) *div-support-div comp-inf.mult-left-isotone comp-isotone conv-order* **by** *auto*
**also have** ... $\leq -D *$ *Dsupport*
**by** (*metis assms div-antisymmetric-equal div-support-mapping div-support-transitive inf-commute mult-isotone order-refl p-shunting-swap pp-increasing shunt-mapping mult-assoc*)
**finally have** *1*: *least D Dsupport* $\leq$ *1*
**by** (*metis double-compl least-def p-shunting-swap*)
**have** *least D ?n* = *Dsupport* $*$ *Dsupport*$^T \sqcap -(-D *$ *Dsupport* $*$ *Dsupport*$^T$)
**by** (*simp add*: *assms comp-associative least-def noyau-div-support*)
**also have** ... = (*Dsupport* $\sqcap -(-D *$ *Dsupport*)) $*$ *Dsupport*$^T$
**using** *assms div-support-univalent comp-bijective-complement injective-comp-right-dist-inf total-conv-surjective* **by** *auto*
**also have** ... $\leq$ *Dsupport*$^T$
**using** *1 least-def mult-left-isotone* **by** *fastforce*
**finally show** (*least D ?n*)$^T \leq$ *Dsupport*
**using** *conv-order* **by** *fastforce*
**qed**

**lemma** *div-factor-support*:
**assumes** *D13-supportable* -
**shows** *Datoms* $\sqcap$ *D* = *Dfactor* $*$ *Dsupport*$^T$
**by** (*metis assms Dsupport-def comp-syq-top inf.sup-monoid.add-commute inf-top.left-neutral surjective-conv-total syq-converse*)

**lemma** *div-supportable-iff*:
**assumes** *D2-antisymmetric* -
**and** *D6-least-surjective* -
**shows** *D13-supportable* - $\longleftrightarrow$ *Datoms* $\sqcap$ *D* = *Dfactor* $*$ *Dsupport*$^T$
**by** (*metis assms Dsupport-def div-total-top-factor comp-syq-surjective conv-dist-comp symmetric-top-closed syq-converse*)

## 2.7 Increments

**lemma** *least-div-atoms-succ*:
*Dbot* $\sqcap$ *Datoms*$^T \leq$ *Dsucc*
**proof** −
**have** *1*: *Dbot* $\sqcap$ *Datoms*$^T \leq D$
**using** *div-least-div inf.coboundedI1* **by** *blast*
**have** *2*: *Dbot* $\sqcap$ *Datoms*$^T \leq -1$
**by** (*metis div-least-not-atom comp-inf.semiring.mult-not-zero inf.sup-monoid.add-assoc inf.sup-monoid.add-commute one-inf-conv pseudo-complement*)
**have** $(D \sqcap -1)^T * -$*Dbot* $\leq -$*Datoms*

**by** (*simp add*: *Datoms-def minimal-def conv-complement conv-dist-inf*)
**hence** $(D \sqcap -1) * Datoms \leq Dbot$
**by** (*simp add*: *schroeder-3-p*)
**hence** $(D \sqcap -1) * Datoms * Dbot^T \leq Dbot$
**by** (*metis div-least-vector mult-isotone top-greatest*)
**also have** ... $\leq D$
**by** (*simp add*: *div-least-div*)
**finally have** $Dbot * Datoms^T \leq -(-D^T * (D \sqcap -1))$
**by** (*metis comp-associative compl-le-swap1 conv-dist-comp conv-involutive schroeder-6*)
**hence** $Dbot \sqcap Datoms^T \leq -(-D^T * (D \sqcap -1))$
**by** (*simp add*: *div-atoms-vector div-least-vector vector-covector*)
**thus** *?thesis*
**using** *1 2 Dsucc-def greatest-def* **by** *auto*
**qed**

**lemma** *least-div-succ*:
**assumes** *D12-infinite-base* -
**shows** $Dbot \leq Dsucc * top$
**proof** −
**have** $Dbot \leq (Dbot \sqcap Datoms^T) * top$
**using** *assms div-atom-surjective div-least-vector surjective-conv-total vector-inf-comp* **by** *auto*
**also have** ... $\leq Dsucc * top$
**using** *least-div-atoms-succ mult-left-isotone* **by** *blast*
**finally show** *?thesis*
**.**
**qed**

**lemma** *noyau-div*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**shows** $noyau\ D = 1$
**by** (*simp add*: *assms reflexive-antisymmetric-noyau*)

**lemma** *div-discrete-fibers-pred-geq*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** $Dsucc^T * top \leq Dmono$
**proof** −
**have** $Dfactor \sqcap -Dmono^T \leq -1$
**by** (*metis brouwer.p-antitone conv-complement div-factor-mono inf.coboundedI2 inf-commute one-inf-conv p-shunting-swap*)
**hence** $Dfactor \sqcap -Dmono^T \leq D \sqcap -1$

**by** (*simp add*: *div-factor-div le-infI1*)
**hence** $Dsucc \leq D \sqcap -1 \sqcap -(-D^T * (Dfactor \sqcap -Dmono^T))$
**by** (*metis Dsucc-def greatest-def inf.sup-right-isotone mult-right-isotone p-antitone*)
**also have** ... $= D \sqcap -1 \sqcap -(-D^T * Dfactor \sqcap -Dmono^T)$
**by** (*simp add*: *covector-comp-inf div-mono-vector vector-conv-compl*)
**also have** ... $= (D \sqcap -1 \sqcap -(-D^T * Dfactor)) \sqcup (D \sqcap -1 \sqcap Dmono^T)$
**by** (*simp add*: *comp-inf.semiring.distrib-left*)
**also have** ... $= (D \sqcap -1 \sqcap D^T) \sqcup (D \sqcap -1 \sqcap Dmono^T)$
**by** (*metis assms div-f-decomposable-char-1 conv-complement conv-dist-comp conv-involutive double-compl*)
**also have** ... $= D \sqcap -1 \sqcap Dmono^T$
**using** *assms*(*1*,*2*) *div-antisymmetric-equal inf-commute* **by** *fastforce*
**also have** ... $\leq Dmono^T$
**by** *simp*
**finally show** *?thesis*
**by** (*metis conv-involutive conv-order div-mono-vector mult-left-isotone*)
**qed**

**lemma** *div-discrete-fibers-pred-eq*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D15b-discrete-fibers-pred* -
**shows** $Dmono = Dsucc^T * top$
**by** (*simp add*: *assms div-discrete-fibers-pred-geq dual-order.eq-iff*)

**lemma** *div-discrete-fibers-pred-iff*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *D15b-discrete-fibers-pred* - $\longleftrightarrow Dmono = Dsucc^T * top$
**using** *assms div-discrete-fibers-pred-geq* **by** *force*

**lemma** *div-succ-univalent*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -

**and** *D11-atomic* -
**and** *D15b-discrete-fibers-pred* -
**shows** $Dsucc^T * (-Dbot \sqcap Dsucc) \leq 1$
**proof** −
**have** *1*: $Dsucc \leq D$
**by** (*simp add*: *Dsucc-def greatest-def inf-assoc*)
**have** *2*: $D^T * Datoms \sqcap D \leq D^T * (Datoms \sqcap D)$
**using** *assms*(*3*,*7*) *div-via-atom comp-inf.coreflexive-commutative* **by** *auto*
**have** *3*: $(D \sqcap -1) * Dsucc^T \leq D$
**by** (*metis Dsucc-def conv-involutive double-compl greatest-def p-dist-sup schroeder-6 sup.cobounded2*)
**have** $Dsucc^T * Dsucc \sqcap D \sqcap -1 \leq (Dsucc^T \sqcap (D \sqcap -1) * Dsucc^T) * Dsucc$
**by** (*simp add*: *comp-inf.vector-inf-comp dedekind-2*)
**also have** ... $\leq (Dsucc^T \sqcap D) * Dsucc$
**using** *3 inf.sup-right-isotone mult-left-isotone* **by** *blast*
**also have** ... $\leq ((D \sqcap -1)^T \sqcap D) * Dsucc$
**using** *Dsucc-def conv-dist-inf greatest-def inf.cobounded1 inf.sup-left-isotone mult-left-isotone* **by** *auto*
**also have** ... $=$ *bot*
**by** (*metis assms*(*2*) *antisymmetric-inf-diversity conv-inf-bot-iff equivalence-one-closed inf-compl-bot-right mult-1-right mult-left-zero schroeder-2*)
**finally have** *4*: $Dsucc^T * Dsucc \sqcap D \leq 1$
**by** (*simp add*: *shunting-1*)
**hence** *5*: $Dsucc^T * Dsucc \sqcap D^T \leq 1$
**by** (*metis conv-dist-comp conv-dist-inf conv-involutive coreflexive-symmetric*)
**have** $Dsucc^T * (-Dbot \sqcap Dsucc) \leq top * Dsucc$
**by** (*simp add*: *mult-isotone*)
**also have** ... $= Dmono^T$
**using** *assms div-discrete-fibers-pred-eq conv-dist-comp* **by** *fastforce*
**finally have** $Dsucc^T * (-Dbot \sqcap Dsucc) = Dsucc^T * (-Dbot \sqcap Dsucc) \sqcap Dmono^T$
**using** *inf.order-iff* **by** *auto*
**also have** ... $= Dmono \sqcap Dsucc^T * (-Dbot \sqcap Dsucc) \sqcap Dmono^T$
**by** (*metis assms div-discrete-fibers-pred-eq div-mono-vector domain-comp vector-export-comp-unit vector-inf-comp*)
**also have** ... $\leq Dmono \sqcap D^T * (-Dbot \sqcap D) \sqcap Dmono^T$
**using** *1 conv-order inf.sup-left-isotone inf.sup-right-isotone mult-isotone* **by** *auto*
**also have** ... $= Dmono \sqcap D^T * (D^T * Datoms \sqcap D) \sqcap Dmono^T$
**using** *assms*(*7*) **by** *auto*
**also have** ... $\leq Dmono \sqcap D^T * D^T * (Datoms \sqcap D) \sqcap Dmono^T$
**using** *2* **by** (*metis comp-inf.mult-left-isotone inf-commute mult-right-isotone mult-assoc*)
**also have** ... $= Dmono \sqcap D^T * (Datoms \sqcap D) \sqcap Dmono^T$
**by** (*metis assms*(*1*,*3*) *div-idempotent conv-dist-comp*)
**also have** ... $\leq D \sqcup D^T$
**using** *assms*(*5*) **by** *force*
**finally have** $Dsucc^T * (-Dbot \sqcap Dsucc) = Dsucc^T * (-Dbot \sqcap Dsucc) \sqcap (D \sqcup D^T)$

**using** *inf.absorb1* **by** *auto*
**also have** ... $\leq$ $Dsucc^T * Dsucc \sqcap (D \sqcup D^T)$
**using** *comp-inf.mult-left-isotone comp-isotone* **by** *force*
**also have** ... $= (Dsucc^T * Dsucc \sqcap D) \sqcup (Dsucc^T * Dsucc \sqcap D^T)$
**using** *inf-sup-distrib1* **by** *blast*
**also have** ... $\leq$ *1*
**using** *4 5 le-sup-iff* **by** *blast*
**finally show** *?thesis*
**.**
**qed**

**lemma** *div-succ-injective*:
**assumes** *D2-antisymmetric* -
**shows** *injective Dsucc*
**by** (*simp add*: *assms Dsucc-def greatest-injective*)

**lemma** *div-succ-below-div-irreflexive*:
$Dsucc \leq D \sqcap -1$
**by** (*metis Dsucc-def greatest-def inf-le1*)

**lemma** *div-succ-below-div*:
$Dsucc \leq D$
**using** *div-succ-below-div-irreflexive* **by** *auto*

**lemma** *div-succ-mono-bot*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D12-infinite-base* -
**and** *D15a-discrete-fibers-succ* -
**shows** $Dsucc * top = Dmonobot$
**proof** –
**have** $Dsucc * top \leq Dsucc * Dsucc^T * Dsucc * top$
**by** (*simp add*: *comp-isotone ex231c*)
**also have** ... $\leq Dsucc * Dsucc^T * top$
**by** (*simp add*: *mult-right-isotone mult-assoc*)
**also have** ... $\leq Dsucc * Dmono$
**by** (*simp add*: *assms div-discrete-fibers-pred-geq mult-right-isotone mult-assoc*)
**also have** ... $\leq D * Dmono$
**using** *div-succ-below-div mult-left-isotone* **by** *auto*
**also have** ... $\leq Dmonobot$
**using** *assms*(*3*,*7*) *div-mono-downclosed Dmonobot-def heyting.implies-galois-var sup-commute* **by** *auto*
**finally have** $Dsucc * top \leq Dmonobot$
**.**

**thus** *?thesis*
**by** (*simp add*: *assms*(*8*,*9*) *least-div-succ Dmonobot-def order*.*antisym*)
**qed**

**lemma** *div-discrete-fibers-succ-iff*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D12-infinite-base* -
**shows** *D15a-discrete-fibers-succ* - $\longleftrightarrow$ $Dsucc * top = Dmonobot$
**using** *Dmonobot-def assms div-succ-mono-bot* **by** *force*

**lemma** *div-succ-bot-atoms*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D6-least-surjective* -
**shows** $Dsucc^T * Dbot = Datoms$
**proof** (*rule order*.*antisym*)
**have** $Dsucc^T * Dbot \leq (D \sqcap -1)^T * top$
**using** *div-succ-below-div-irreflexive conv-order mult-isotone* **by** *auto*
**also have** ... $\leq -Dbot$
**by** (*simp add*: *assms*(*2*) *div-strict-bot schroeder-3-p*)
**finally have** *1*: $Dsucc^T * Dbot \leq -Dbot$
.
**have** $-Dbot * Dbot^T \leq -D$
**by** (*metis assms*(*1*,*3*) *bot-div-bot complement-conv-sub*)
**hence** $(D \sqcap -1)^T * -Dbot * Dbot^T \leq (D \sqcap -1)^T * -D$
**by** (*simp add*: *comp-isotone mult-assoc*)
**hence** *2*: $-((D \sqcap -1)^T * -D) * Dbot \leq -((D \sqcap -1)^T * -Dbot)$
**by** (*simp add*: *schroeder-4-p*)
**have** $Dsucc \leq -(-D^T * (D \sqcap -1))$
**by** (*simp add*: *Dsucc-def greatest-def*)
**hence** $Dsucc^T \leq -((D \sqcap -1)^T * -D)$
**by** (*simp add*: *Dsucc-def conv-complement conv-dist-comp conv-dist-inf greatest-def*)
**hence** $Dsucc^T * Dbot \leq -((D \sqcap -1)^T * -D) * Dbot$
**using** *mult-left-isotone* **by** *blast*
**also have** ... $\leq -((D \sqcap -1)^T * -Dbot)$
**using** *2* **by** *blast*
**finally show** $Dsucc^T * Dbot \leq Datoms$
**using** *1* **by** (*simp add*: *Datoms-def conv-complement conv-dist-inf minimal-def*)
**have** $Datoms * Dbot^T \leq Dsucc^T$
**by** (*metis div-atoms-vector least-div-atoms-succ double-compl schroeder-3-p*

*schroeder-5 vector-covector div-least-vector*)
  **thus** *Datoms* $\leq$ *Dsucc*$^T$ $*$ *Dbot*
    **using** *assms*(*2*,*4*) *div-least-point shunt-bijective* **by** *blast*
**qed**

**lemma** *div-succ-inverse-poly*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
      **and** *D3-transitive* -
      **and** *D6-least-surjective* -
      **and** *D7-pre-f-decomposable* -
      **and** *D8-fibered* -
      **and** *D9-f-decomposable* -
      **and** *D11-atomic* -
      **and** *D15b-discrete-fibers-pred* -
    **shows** *Dsucc*$^T$ $*$ *Dsucc* $*$ (*Dmono* $\sqcap$ $-$*Datoms* $\sqcap$ *1*) = *Dmono* $\sqcap$ $-$*Datoms* $\sqcap$ *1*
**proof** (*rule order.antisym*)
  **let** *?q* = *Dmono* $\sqcap$ $-$*Datoms* $\sqcap$ *1*
  **have** *?q* = *?q* $\sqcap$ *Dsucc*$^T$ $*$ *top*
    **using** *assms*(*1*$-$*3*,*5*$-$*9*) *div-discrete-fibers-pred-eq inf-commute*
*inf-left-commute* **by** *auto*
  **also have** ... = *?q* $\sqcap$ (*Dsucc*$^T$ $\sqcap$ *?q* $*$ *top*) $*$ (*top* $\sqcap$ *Dsucc* $*$ *?q*)
    **by** (*simp add*: *dedekind-eq inf.sup-monoid.add-commute*)
  **also have** ... $\leq$ *Dsucc*$^T$ $*$ *Dsucc* $*$ *?q*
    **using** *comp-associative inf.coboundedI2 inf-vector-comp* **by** *auto*
  **finally show** *?q* $\leq$ *Dsucc*$^T$ $*$ *Dsucc* $*$ *?q*
    .
  **have** *Dsucc* $*$ *?q* $\leq$ *Dsucc* $*$ $-$*Datoms*
    **by** (*simp add*: *inf.coboundedI1 mult-right-isotone*)
  **also have** ... $\leq$ $-$*Dbot*
    **by** (*metis assms*(*1*$-$*4*) *div-succ-bot-atoms conv-complement-sub-leq*
*conv-involutive*)
  **finally have** *Dsucc*$^T$ $*$ *Dsucc* $*$ *?q* = *Dsucc*$^T$ $*$ ($-$*Dbot* $\sqcap$ *Dsucc* $*$ *?q*)
    **by** (*simp add*: *inf.le-iff-sup mult-assoc*)
  **also have** ... = *Dsucc*$^T$ $*$ ($-$*Dbot* $\sqcap$ *Dsucc*) $*$ *?q*
    **by** (*simp add*: *Dbot-def comp-associative least-def vector-export-comp*)
  **also have** ... $\leq$ *?q*
    **by** (*metis assms*(*1*$-$*3*,*5*$-$*9*) *div-succ-univalent coreflexive-comp-inf*
*inf.sup-right-divisibility*)
  **finally show** *Dsucc*$^T$ $*$ *Dsucc* $*$ *?q* $\leq$ *?q*
    .
**qed**

**lemma** *div-inc-injective*:
  **assumes** *D1-reflexive* -
      **and** *D2-antisymmetric* -
      **and** *D3-transitive* -
      **and** *D7-pre-f-decomposable* -
      **and** *D8-fibered* -

**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**shows** *injective Dinc*
**using** *assms div-f-decomposable-char-2 Dinc-def syq-comp-top-syq syq-converse*
**by** *force*

**lemma** *div-factor-not-bot*:
**assumes** *D2-antisymmetric* -
**shows** $Dfactor \leq -Dbot$
**using** *assms div-factor-mono div-least-not-mono compl-le-swap1 inf.order-trans*
**by** *blast*

**lemma** *div-factor-conv-inc*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D6-least-surjective* -
**shows** $Dfactor * Dinc^T \leq Dmono \sqcap -Datoms$
**proof** −
**have** *1*: $Dfactor * Dinc^T \leq Dmono$
**by** (*metis div-factor-mono div-mono-vector mult-isotone top-greatest*)
**have** $Dfactor * Dinc^T = Dfactor * symmetric\text{-}quotient\ (Dsucc * Dfactor)$
*Dfactor*
**by** (*simp add*: *Dinc-def syq-converse*)
**also have** ... $\leq Dfactor * -((Dsucc * Dfactor)^T * -Dfactor)$
**using** *mult-right-isotone symmetric-quotient-def* **by** *force*
**also have** ... $\leq Dfactor * -((Dsucc * Dfactor)^T * Dbot)$
**using** *assms*(*2*) *div-factor-not-bot mult-right-isotone p-antitone-iff* **by** *auto*
**also have** ... $= Dfactor * -(Dfactor^T * Datoms)$
**by** (*simp add*: *assms div-succ-bot-atoms conv-dist-comp mult-assoc*)
**also have** ... $\leq -Datoms$
**by** (*simp add*: *schroeder-3*)
**finally show** *?thesis*
**using** *1* **by** *auto*
**qed**

**lemma** *div-inc-univalent*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D6-least-surjective* -
**and** *D7-pre-f-decomposable* -
**and** *D8-fibered* -
**and** *D9-f-decomposable* -
**and** *D11-atomic* -
**and** *D15b-discrete-fibers-pred* -
**shows** *univalent Dinc*
**proof** −
**let** $?sf = Dsucc * Dfactor$

**let** *?p* = *symmetric-quotient ?sf Dfactor* ∗ *top* ⊓ *1*
**let** *?q* = *Dmono* ⊓ −*Datoms* ⊓ *1*
**have** *Dfactor* ∗ *?p* ≤ *Dfactor* ∗ $Dinc^T$ ∗ *top*
**by** (*simp add*: *Dinc-def mult-right-isotone syq-converse mult-assoc*)
**also have** ... ≤ *Dmono* ⊓ −*Datoms*
**by** (*metis assms*(*1*−*4*) *div-atoms-vector div-factor-conv-inc Dmono-def mult-left-isotone vector-complement-closed vector-export-comp*)
**finally have** *Dfactor* ∗ *?p* ≤ *?q* ∗ *Dfactor* ∗ *?p*
**by** (*smt* (*verit*, *ccfv-threshold*) *div-atoms-vector div-mono-vector complement-vector inf.le-sup-iff mult-left-one order-refl vector-inf-comp*)
**hence** *Dfactor* ∗ *?p* = *?q* ∗ *Dfactor* ∗ *?p*
**by** (*simp add*: *inf.absorb2 test-comp-test-inf*)
**hence** *1*: $Dsucc^T$ ∗ *?sf* ∗ *?p* = *Dfactor* ∗ *?p*
**by** (*metis assms div-succ-inverse-poly mult-assoc*)
**have** $Dinc^T$ ∗ *Dinc* = *symmetric-quotient ?sf Dfactor* ∗ *symmetric-quotient Dfactor ?sf*
**by** (*simp add*: *Dinc-def syq-converse*)
**also have** ... = *symmetric-quotient ?sf Dfactor* ∗ *top* ⊓ *symmetric-quotient ?sf ?sf* ⊓ *top* ∗ *symmetric-quotient Dfactor ?sf*
**by** (*smt* (*verit*) *comp-isotone inf.absorb-iff2 inf.sup-monoid.add-assoc order.refl syq-comp-top-syq top.extremum*)
**also have** ... = *?p* ∗ *symmetric-quotient ?sf ?sf* ⊓ *top* ∗ *symmetric-quotient Dfactor ?sf*
**using** *vector-export-comp-unit* **by** *auto*
**also have** ... = *?p* ∗ *symmetric-quotient ?sf ?sf* ∗ *?p*
**by** (*simp add*: *comp-inf-vector inf-commute syq-converse*)
**also have** ... = *?p* ∗ *symmetric-quotient* (*?sf* ∗ *?p*) (*?sf* ∗ *?p*) ∗ *?p*
**using** *coreflexive-comp-syq-comp-coreflexive inf-le2* **by** *blast*
**also have** ... ≤ *?p* ∗ *symmetric-quotient* ($Dsucc^T$ ∗ *?sf* ∗ *?p*) ($Dsucc^T$ ∗ *?sf* ∗ *?p*) ∗ *?p*
**using** *comp-isotone order.refl syq-comp-isotone mult-assoc* **by** *auto*
**also have** ... = *?p* ∗ *symmetric-quotient* (*Dfactor* ∗ *?p*) (*Dfactor* ∗ *?p*) ∗ *?p*
**using** *1* **by** *auto*
**also have** ... = *?p* ∗ *symmetric-quotient Dfactor Dfactor* ∗ *?p*
**by** (*metis coreflexive-comp-syq-comp-coreflexive inf.cobounded2*)
**also have** ... ≤ *symmetric-quotient Dfactor Dfactor*
**by** (*simp add*: *assms*(*1*−*3*,*5*−*8*) *div-f-decomposable-char-2 vector-export-comp-unit*)
**also have** ... = *1*
**using** *assms*(*1*−*3*,*5*−*8*) *div-f-decomposable-char-2* **by** *blast*
**finally show** *?thesis*
.
**qed**

**lemma** *div-inc-mapping*:
**assumes** *D1-reflexive* -
**and** *D2-antisymmetric* -
**and** *D3-transitive* -
**and** *D6-least-surjective* -

```
    and D7-pre-f-decomposable -
    and D8-fibered -
    and D9-f-decomposable -
    and D11-atomic -
    and D15b-discrete-fibers-pred -
    and D16-incrementable -
  shows mapping Dinc
 using assms div-inc-univalent by blast

lemma div-inc-mapping:
 assumes D1-reflexive -
    and D2-antisymmetric -
    and D3-transitive -
    and D6-least-surjective -
    and D7-pre-f-decomposable -
    and D8-fibered -
    and D9-f-decomposable -
    and D11-atomic -
    and D13-supportable -
    and D15a-discrete-fibers-succ -
    and D15b-discrete-fibers-pred -
    and D16-incrementable -
  shows surjective Datoms
 nitpick[expect=genuine,card=2]
 oops

end

end
```

# 3 Mono-Atomic Elements

**theory** *Mono-Atomic*

**imports** *Stone-Relation-Algebras.Relation-Algebras*

**begin**

This theory defines mono-atomic elements in a bounded semilattice and shows that they correspond to join-irreducible elements under the divisibility axioms A1–A17 of [2]. In the model of natural numbers both types of elements correspond to prime powers.

## 3.1 Mono-atomic

**context** *order-bot*
**begin**

Divisibility axioms A1 (reflexivity), A2 (antisymmetry), A3 (transitivity) and A6 (least element) are the axioms of class *order-bot*, so not mentioned explicitly.

An *atom* in a partial order is an element that is strictly above only the least element *bot*.

**definition** *atom a* $\equiv a \neq bot \wedge (\forall x \,.\, x \leq a \longrightarrow x = bot \vee x = a)$
**abbreviation** *atom-below a x* $\equiv atom\ a \wedge a \leq x$

A mono-atomic element has exactly one atom below it.

**definition** *mono-atomic x* $\equiv (\exists! a \,.\, atom\text{-}below\ a\ x)$
**definition** *mono-atomic-with x a* $\equiv atom\text{-}below\ a\ x \wedge (\forall b \,.\, atom\text{-}below\ b\ x \longrightarrow b = a)$
**abbreviation** *mono-atomic-below x y* $\equiv mono\text{-}atomic\ x \wedge x \leq y$
**abbreviation** *mono-atomic-above x y* $\equiv mono\text{-}atomic\ x \wedge y \leq x$
**definition** *mono-atomic-above-or-bot x y* $\equiv x = bot \vee mono\text{-}atomic\text{-}above\ x\ y$

Divisibility axiom A11 (atomicity) states that every element except *bot* is above some atom.

**abbreviation** *A11-atomic* :: *'a* ⇒ *bool* **where** *A11-atomic* - $\equiv (\forall x \,.\, x \neq bot \longrightarrow (\exists a \,.\, atom\text{-}below\ a\ x))$

**lemma** *mono-atomic-above*:
*mono-atomic x* $\longleftrightarrow (\exists a \,.\, mono\text{-}atomic\text{-}with\ x\ a)$
**by** (*metis mono-atomic-with-def mono-atomic-def*)

Among others, the following divisibility axioms are considered in [2]. In the model of natural numbers,

* A7 (pre-f-decomposability) expresses that every number $x$ is the least upper bound of the prime powers below $x$;
* A8 (fibered) expresses that the prime powers can be partitioned into chains;
* A9 (f-decomposability) expresses that for every number $x$ above an atom $a$ there is a maximal prime power of $a$ below $x$;
* A14 (truncability) express that the prime powers contained in a number $y$ can be restricted to those whose atoms are not below a number $x$.

Their definitions are based on join-irreducible elements and given in class *bounded-semilattice-sup-bot* below. Here we introduce corresponding axioms B7, B8, B9 and B14 based on mono-atomic elements.

**abbreviation** *B7-pre-f-decomposable* :: *'a* ⇒ *bool* **where** *B7-pre-f-decomposable* - $\equiv (\forall x\ y \,.\, (\forall z \,.\, mono\text{-}atomic\text{-}below\ z\ x \longrightarrow z \leq y) \longrightarrow x \leq y)$

**abbreviation** *B8-fibered* :: *'a* ⇒ *bool* **where** *B8-fibered* - ≡ (∀ *x y z* . *mono-atomic x* ∧ *mono-atomic y* ∧ *mono-atomic z* ∧ ((*x* ≤ *z* ∧ *y* ≤ *z*) ∨ (*z* ≤ *x* ∧ *z* ≤ *y*)) ⟶ *x* ≤ *y* ∨ *y* ≤ *x*)
**abbreviation** *B9-f-decomposable* :: *'a* ⇒ *bool* **where** *B9-f-decomposable* - ≡ (∀ *x a* . *atom a* ⟶ (∃ *z* . *mono-atomic-above-or-bot z a* ∧ *z* ≤ *x* ∧ (∀ *w* . *mono-atomic-above-or-bot w a* ∧ *w* ≤ *x* ⟶ *w* ≤ *z*)))

Function *mval* returns the value whose existence is asserted by axiom B9.

**definition** *mval a x* ≡ *SOME z* . *mono-atomic-above-or-bot z a* ∧ *z* ≤ *x* ∧ (∀ *w* . *mono-atomic-above-or-bot w a* ∧ *w* ≤ *x* ⟶ *w* ≤ *z*)

**lemma** *mval-char*:
  **assumes** *B9-f-decomposable* -
    **and** *atom a*
   **shows** *mono-atomic-above-or-bot* (*mval a x*) *a* ∧ *mval a x* ≤ *x* ∧ (∀ *w* . *mono-atomic-above-or-bot w a* ∧ *w* ≤ *x* ⟶ *w* ≤ *mval a x*)
**proof** −
  **obtain** *z* **where** *mono-atomic-above-or-bot z a* ∧ *z* ≤ *x* ∧ (∀ *w* . *mono-atomic-above-or-bot w a* ∧ *w* ≤ *x* ⟶ *w* ≤ *z*)
   **using** *assms* **by** *blast*
  **thus** *?thesis*
   **using** *mval-def someI* **by** *simp*
**qed**

**lemma** *mval-unique*:
  **assumes** *B9-f-decomposable* -
    **and** *atom a*
    **and** *mono-atomic-above-or-bot z a* ∧ *z* ≤ *x* ∧ (∀ *w* . *mono-atomic-above-or-bot w a* ∧ *w* ≤ *x* ⟶ *w* ≤ *z*)
   **shows** *z* = *mval a x*
  **by** (*simp add*: *assms dual-order.antisym mval-char*)

**lemma** *atom-below-mval*:
  **assumes** *B9-f-decomposable* -
    **and** *atom a*
    **and** *a* ≤ *x*
   **shows** *a* ≤ *mval a x*
**proof** −
  **have** *mono-atomic-above-or-bot a a*
   **using** *assms*(*2*) *atom-def mono-atomic-above-or-bot-def mono-atomic-def* **by** *auto*
  **thus** *?thesis*
   **by** (*simp add*: *assms mval-char*)
**qed**

**abbreviation** *B14-truncability* :: *'a* ⇒ *bool* **where** *B14-truncability* - ≡ (∀ *x y* . ∃ *z* . ∀ *a* . *atom a* ⟶ (*if a* ≤ *x then mval a z* = *bot else mval a z* = *mval a y*))

Function *mtrunc* returns the value whose existence is asserted by axiom B14.

**definition** *mtrunc x y* ≡ *SOME z* . ∀ *a* . *atom a* ⟶ (*if a* ≤ *x then mval a z* = *bot else mval a z* = *mval a y*)

**lemma** *mtrunc-char*:
**assumes** *B14-truncability* -
**shows** ∀ *a* . *atom a* ⟶ (*if a* ≤ *x then mval a* (*mtrunc x y*) = *bot else mval a* (*mtrunc x y*) = *mval a y*)
**proof** –
**obtain** *z* **where** ∀ *a* . *atom a* ⟶ (*if a* ≤ *x then mval a z* = *bot else mval a z* = *mval a y*)
**using** *assms* **by** *blast*
**thus** *?thesis*
**by** (*smt mtrunc-def someI*)
**qed**

**lemma** *mtrunc-char-1*:
**assumes** *B14-truncability* -
**and** *atom a*
**and** *a* ≤ *x*
**shows** *mval a* (*mtrunc x y*) = *bot*
**by** (*simp add*: *assms mtrunc-char*)

**lemma** *mtrunc-char-2*:
**assumes** *B14-truncability* -
**and** *atom a*
**and** ¬ *a* ≤ *x*
**shows** *mval a* (*mtrunc x y*) = *mval a y*
**by** (*simp add*: *assms mtrunc-char*)

**lemma** *mtrunc-unique*:
**assumes** *B14-truncability* -
**and** ∀ *a* . *atom a* ⟶ (*if a* ≤ *x then mval a z* = *bot else mval a z* = *mval a y*)
**and** *atom a*
**shows** *mval a z* = *mval a* (*mtrunc x y*)
**by** (*smt* (*z3*) *assms mtrunc-char*)

**lemma** *lesseq-iff-mval-below*:
**assumes** *B7-pre-f-decomposable* -
**and** *B9-f-decomposable* -
**and** *atom a*
**shows** *x* ≤ *y* ⟷ (∀ *a* . *atom a* ⟶ *mval a x* ≤ *y*)
**proof** (*rule iffI*)
**assume** *1*: *x* ≤ *y*
**show** ∀ *a* . *atom a* ⟶ *mval a x* ≤ *y*
**proof** (*rule allI*, *rule impI*)
**fix** *a*
**assume** *atom a*

**thus** *mval a x* $\leq$ *y*
  **using** *1 assms*(*2*) *dual-order.trans mval-char* **by** *blast*
**qed**
**next**
**assume** *2*: $\forall$ *a* . *atom a* $\longrightarrow$ *mval a x* $\leq$ *y*
**have** $\forall$ *z* . *mono-atomic-below z x* $\longrightarrow$ *z* $\leq$ *y*
**proof** (*rule allI*, *rule impI*)
  **fix** *z*
  **assume** *3*: *mono-atomic-below z x*
  **from** *this* **obtain** *a* **where** *4*: *atom-below a z*
    **using** *mono-atomic-def* **by** *blast*
  **hence** *z* $\leq$ *mval a x*
    **using** *3 assms*(*2*) *mono-atomic-above-or-bot-def mval-char* **by** *auto*
  **thus** *z* $\leq$ *y*
    **using** *2 4* **by** *auto*
**qed**
**thus** *x* $\leq$ *y*
  **using** *assms*(*1*) **by** *blast*
**qed**

**end**

## 3.2 Join-irreducible

**context** *bounded-semilattice-sup-bot*
**begin**

Divisibility axioms A1 (reflexivity), A2 (antisymmetry), A3 (transitivity), A5 (least upper bound) and A6 (least element) are the axioms of class *bounded-semilattice-sup-bot*, so not mentioned explicitly.

A join-irreducible element cannot be expressed as the join of two incomparable elements.

**definition** *join-irreducible x* $\equiv$ *x* $\neq$ *bot* $\wedge$ ($\forall$ *y z* . *x* = *y* $\sqcup$ *z* $\longrightarrow$ *x* = *y* $\vee$ *x* = *z*)
**abbreviation** *join-irreducible-below x y* $\equiv$ *join-irreducible x* $\wedge$ *x* $\leq$ *y*
**abbreviation** *join-irreducible-above x y* $\equiv$ *join-irreducible x* $\wedge$ *y* $\leq$ *x*
**definition** *join-irreducible-above-or-bot x y* $\equiv$ *x* = *bot* $\vee$ *join-irreducible-above x y*

Divisibility axioms A7, A8 and A9 based on join-irreducible elements are introduced here; axiom A14 is not needed for this development.

**abbreviation** *A7-pre-f-decomposable* :: *'a* $\Rightarrow$ *bool* **where** *A7-pre-f-decomposable -* $\equiv$ ($\forall$ *x y* . ($\forall$ *z* . *join-irreducible-below z x* $\longrightarrow$ *z* $\leq$ *y*) $\longrightarrow$ *x* $\leq$ *y*)
**abbreviation** *A8-fibered* :: *'a* $\Rightarrow$ *bool* **where** *A8-fibered -* $\equiv$ ($\forall$ *x y z* . *join-irreducible x* $\wedge$ *join-irreducible y* $\wedge$ *join-irreducible z* $\wedge$ ((*x* $\leq$ *z* $\wedge$ *y* $\leq$ *z*) $\vee$ (*z* $\leq$ *x* $\wedge$ *z* $\leq$ *y*)) $\longrightarrow$ *x* $\leq$ *y* $\vee$ *y* $\leq$ *x*)
**abbreviation** *A9-f-decomposable* :: *'a* $\Rightarrow$ *bool* **where** *A9-f-decomposable -* $\equiv$ ($\forall$ *x a* . *atom a* $\longrightarrow$ ($\exists$ *z* . *join-irreducible-above-or-bot z a* $\wedge$ *z* $\leq$ *x* $\wedge$ ($\forall$ *w* . *join-irreducible-above-or-bot w a* $\wedge$ *w* $\leq$ *x* $\longrightarrow$ *w* $\leq$ *z*)))

**lemma** *atom-join-irreducible*:
  **assumes** *atom a*
    **shows** *join-irreducible a*
  **by** (*metis assms join-irreducible-def atom-def sup.cobounded1 sup-bot-left*)

**lemma** *mono-atomic-with-downclosed*:
  **assumes** *A11-atomic -*
      **and** *mono-atomic-with x a*
      **and** $y \neq bot$
      **and** $y \leq x$
    **shows** *mono-atomic-with y a*
  **using** *assms mono-atomic-with-def*[*of y a*] *mono-atomic-with-def*[*of x a*] *order-lesseq-imp*[*of y*] **by** *blast*

## 3.3 Equivalence

The following result shows that under divisibility axioms A1–A3, A5–A9 and A11, join-irreducible elements coincide with mono-atomic elements.

**lemma** *join-irreducible-iff-mono-atomic*:
  **assumes** *A7-pre-f-decomposable -*
      **and** *A8-fibered -*
      **and** *A9-f-decomposable -*
      **and** *A11-atomic -*
    **shows** *join-irreducible x* $\longleftrightarrow$ *mono-atomic x*
**proof** (*rule iffI*)
  **assume** *1*: *join-irreducible x*
  **from** *this* **obtain** *a* **where** *2*: *atom-below a x*
    **using** *assms*(*4*) *join-irreducible-def* **by** *blast*
  **have** $\forall b$ . *atom-below b x* $\longrightarrow b = a$
  **proof** (*rule allI*, *rule impI*)
    **fix** *b*
    **assume** *3*: *atom-below b x*
    **hence** *join-irreducible a* $\wedge$ *join-irreducible b*
      **using** *2 atom-join-irreducible* **by** *auto*
    **hence** $a \leq b \vee b \leq a$
      **using** *1 2 3 assms*(*2*) **by** *blast*
    **thus** $b = a$
      **using** *2 3 atom-def* **by** *auto*
  **qed**
  **thus** *mono-atomic x*
    **using** *2 mono-atomic-def* **by** *auto*
**next**
  **assume** *mono-atomic x*
  **from** *this* **obtain** *a* **where** *4*: *mono-atomic-with x a*
    **using** *mono-atomic-above* **by** *blast*
  **hence** *5*: $x \neq bot$
    **using** *atom-def le-bot mono-atomic-with-def* **by** *blast*
  **have** $\forall y\ z\ .\ x = y \sqcup z \longrightarrow x = y \vee x = z$

**proof** (*intro allI*, *rule impI*)
**fix** *y z*
**assume** *6*: $x = y \sqcup z$
**show** $x = y \vee x = z$
**proof** (*cases* $y = bot \vee z = bot$)
**case** *True*
**thus** $x = y \vee x = z$
**using** *6* **by** *auto*
**next**
**case** *False*
**hence** *7*: *mono-atomic-with y a* ∧ *mono-atomic-with z a*
**using** *4 6 assms*(*4*) *sup.cobounded1 sup.cobounded2*
*mono-atomic-with-downclosed* **by** *blast*
**from** *this* **obtain** *u* **where** *8*: *join-irreducible-above-or-bot u a* ∧ $u \le y$ ∧
(∀ *w* . *join-irreducible-above-or-bot w a* ∧ $w \le y \longrightarrow w \le u$)
**using** *assms*(*3*) *mono-atomic-with-def* **by** *blast*
**from** *7* **obtain** *v* **where** *9*: *join-irreducible-above-or-bot v a* ∧ $v \le z$ ∧ (∀ *w*
. *join-irreducible-above-or-bot w a* ∧ $w \le z \longrightarrow w \le v$)
**using** *assms*(*3*) *mono-atomic-with-def* **by** *blast*
**have** *join-irreducible a*
**using** *4 atom-join-irreducible mono-atomic-with-def* **by** *blast*
**hence** *10*: $u \le v \vee v \le u$
**using** *8 9 assms*(*2*) *join-irreducible-above-or-bot-def* **by** *auto*
**have** *11*: $u \le v \Longrightarrow y \le z$
**proof** −
**assume** *12*: $u \le v$
**have** ∀ *w* . *join-irreducible-below w y* ⟶ $w \le z$
**proof** (*rule allI*, *rule impI*)
**fix** *w*
**assume** *13*: *join-irreducible-below w y*
**hence** *mono-atomic-with w a*
**using** *7* **by** (*metis assms*(*4*) *join-irreducible-def*
*mono-atomic-with-downclosed*)
**hence** $w \le u$
**using** *8 13* **by** (*simp add*: *join-irreducible-above-or-bot-def*
*mono-atomic-with-def*)
**thus** $w \le z$
**using** *9 12* **by** *force*
**qed**
**thus** $y \le z$
**using** *assms*(*1*) **by** *blast*
**qed**
**have** $v \le u \Longrightarrow z \le y$
**proof** −
**assume** *14*: $v \le u$
**have** ∀ *w* . *join-irreducible-below w z* ⟶ $w \le y$
**proof** (*rule allI*, *rule impI*)
**fix** *w*
**assume** *15*: *join-irreducible-below w z*

```
          hence mono-atomic-with w a
            using 7 by (metis assms(4) join-irreducible-def
mono-atomic-with-downclosed)
          hence w ≤ v
            using 9 15 by (simp add: join-irreducible-above-or-bot-def
mono-atomic-with-def)
          thus w ≤ y
            using 8 14 by force
        qed
        thus z ≤ y
          using assms(1) by blast
      qed
      thus ?thesis
        using 6 10 11 sup.order-iff sup-monoid.add-commute by force
    qed
  qed
  thus join-irreducible x
    using 5 join-irreducible-def by blast
qed
```

The following result shows that under divisibility axioms A1–A3, A5–A6, B7–B9, A11 and B14, join-irreducible elements coincide with mono-atomic elements.

```
lemma mono-atomic-iff-join-irreducible:
  assumes B7-pre-f-decomposable -
      and B8-fibered -
      and B9-f-decomposable -
      and A11-atomic -
      and B14-truncability -
    shows mono-atomic x ⟷ join-irreducible x
proof (rule iffI)
  assume 1: mono-atomic x
  from this obtain a where mono-atomic-below a x
    by blast
  hence 2: x ≠ bot
    using atom-def bot-unique mono-atomic-def by force
  have ∀ y z . x = y ⊔ z ⟶ x = y ∨ x = z
  proof (intro allI, rule impI)
    fix y z
    assume 3: x = y ⊔ z
    show x = y ∨ x = z
    proof (cases y = bot ∨ z = bot)
      case True
      thus ?thesis
        using 3 by fastforce
    next
      case False
      hence mono-atomic y ∧ mono-atomic z
        using 1 3 by (metis assms(4) mono-atomic-above sup.cobounded1
```

*sup-right-divisibility mono-atomic-with-downclosed*)
      **hence** $y \leq z \vee z \leq y$
        **using** *1 3 assms*(*2*) **by** *force*
      **thus** *?thesis*
        **using** *3 sup.order-iff sup-monoid.add-commute* **by** *force*
    **qed**
  **qed**
  **thus** *join-irreducible x*
    **using** *2 join-irreducible-def* **by** *blast*
**next**
  **assume** *join-irreducible x*
  **from** *this* **obtain** *a* **where** *4*: *atom a* ∧ *join-irreducible-above x a*
    **using** *assms*(*4*) *join-irreducible-def* **by** *blast*
  **let** *?y* = *mval a x*
  **let** *?z* = *mtrunc ?y x*
  **have** *5*: *mval a ?z* = *bot*
    **using** *4* **by** (*smt* (*z3*) *assms*(*3*,*5*) *mono-atomic-above-or-bot-def mtrunc-char*
*mval-char*)
  **have** *6*: *mono-atomic-above-or-bot ?y a*
    **using** *4 assms*(*3*) *mval-char* **by** *simp*
  **hence** ∀ *b* . *atom b* ∧ *b* ≠ *a* ⟶ ¬ *b* ≤ *?y*
    **using** *4* **by** (*metis atom-def bot-unique mono-atomic-above-or-bot-def*
*mono-atomic-def*)
  **hence** *7*: ∀ *b* . *atom b* ∧ *b* ≠ *a* ⟶ *mval b ?z* = *mval b x*
    **by** (*simp add*: *assms*(*5*) *mtrunc-char*)
  **have** *8*: *?y* ≤ *x*
    **using** *4 assms*(*3*) *mval-char* **by** *blast*
  **have** ∀ *b* . *atom b* ⟶ *mval b ?z* ≤ *x*
  **proof** (*rule allI*, *rule impI*)
    **fix** *b*
    **assume** *9*: *atom b*
    **show** *mval b ?z* ≤ *x*
    **proof** (*cases b* = *a*)
      **case** *True*
      **thus** *?thesis*
        **using** *5* **by** *auto*
    **next**
      **case** *False*
      **thus** *?thesis*
        **using** *7 9* **by** (*simp add*: *assms*(*3*) *mval-char*)
    **qed**
  **qed**
  **hence** *10*: *?z* ≤ *x*
    **using** *4 assms*(*1*,*3*) *lesseq-iff-mval-below* **by** *blast*
  **have** ∀ *w* . *?y* ≤ *w* ∧ *?z* ≤ *w* ⟶ *x* ≤ *w*
  **proof** (*rule allI*, *rule impI*)
    **fix** *w*
    **assume** *11*: *?y* ≤ *w* ∧ *?z* ≤ *w*
    **have** ∀ *c* . *atom c* ⟶ *mval c x* ≤ *w*

```
    proof (rule allI, rule impI)
      fix c
      assume 12: atom c
      show mval c x ≤ w
      proof (cases c = a)
        case True
        thus ?thesis
          using 11 by blast
      next
        case False
        thus ?thesis
          using 7 11 12 by (smt (z3) assms(3) dual-order.trans mval-char)
      qed
    qed
    thus x ≤ w
      using 4 assms(1,3) lesseq-iff-mval-below by blast
  qed
  hence 13: x = ?y ⊔ ?z
    using 8 10 order.ordering-axioms ordering.antisym by force
  have x ≠ ?z
  proof (rule notI)
    assume x = ?z
    hence ?y = bot
      using 5 by force
    hence a = bot
      using 4 assms(3) atom-below-mval bot-unique by fastforce
    thus False
      using 4 atom-def by blast
  qed
  hence x = ?y
    using 4 13 join-irreducible-def by force
  thus mono-atomic x
    using 4 6 join-irreducible-def mono-atomic-above-or-bot-def by auto
qed

end

end
```

# References


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