Regular Algebras

Simon Foster and Georg Struth

March 17, 2025

Abstract

Regular algebras axiomatise the equational theory of regular expressions as induced by regular language identity. We use Isabelle/HOL for a detailed systematic study of regular algebras given by Boffa, Conway, Kozen and Salomaa. We investigate the relationships between these classes, formalise a soundness proof for the smallest class (Salomaa's) and obtain completeness of the largest one (Boffa's) relative to a deep result by Krob. In addition we provide a large collection of regular identities in the general setting of Boffa's axiom.

Contents

1 Introductory Remarks

These Isabelle theories complement the article on *On the Fine-Structure of Regular Algebra* [5]. For an introduction to the topic, conceptual explanations and references we refer to this article. Our regular algebra hierarchy is orthogonal to the Kleene algebra hierarchy in the Archive of Formal Proofs [1]; we have not aimed at an integration for pragmatic reasons.

2 Dioids, Powers and Finite Sums

theory *Dioid-Power-Sum*
imports *Kleene-Algebra.Dioid Kleene-Algebra.Finite-Suprema*

begin

We add a few facts about powers and finite sums—in fact, finite suprema—to an existing theory field for dioids.

context *dioid-one-zero*

begin

lemma *add-iso-r*: $y \leq z \Longrightarrow x + y \leq x + z$
⟨*proof*⟩

notation *power* (‹-^-› [*101*,*50*] *100*)

lemma *power-subdist*: $x^n \leq (x + y)^n$
⟨*proof*⟩

lemma *power-inductl-var*: $x \cdot y \leq y \Longrightarrow x^n \cdot y \leq y$
⟨*proof*⟩

lemma *power-inductr-var*: $y \cdot x \leq y \Longrightarrow y \cdot x^n \leq y$
⟨*proof*⟩

definition *powsum* :: *'a* ⇒ *nat* ⇒ *nat* ⇒ *'a* (‹-_-^-› [*101*,*50*,*50*] *100*) **where**
powsum x m n = sum ((^) x) {m..n + m}

lemmas *powsum-simps = powsum-def atLeastAtMostSuc-conv numerals*

lemma *powsum1* [*simp*]: ${x_n}^0 = x^n$
⟨*proof*⟩

lemma *powsum2*: ${x_n}^{Suc\ m} = {x_n}^m + x^{n+Suc\ m}$
⟨*proof*⟩

lemma *powsum-00* [*simp*]: ${x_0}^0 = 1$
⟨*proof*⟩

lemma *powsum-01* [*simp*]: $x_0^1 = 1 + x$
⟨*proof*⟩

lemma *powsum-10* [*simp*]: $x_1^0 = x$
⟨*proof*⟩

lemma *powsum-split*: $x_m^{i+Suc\ n} = x_m^i + x_{m+Suc\ i}^n$
⟨*proof*⟩

lemma *powsum-split-var1*: $x_0^{n+1} = 1 + x_1^n$
⟨*proof*⟩

lemma *powsum-split-var2* [*simp*]: $x^m + x_0^m = x_0^m$
⟨*proof*⟩

lemma *powsum-split-var3*: $x_0^{m+Suc\ n} = x_0^m + x_{0+Suc\ m}^n$
⟨*proof*⟩

lemma *powsum-split-var4* [*simp*]: $x_0^{m+n} + x_m^n = x_0^{m+n}$
⟨*proof*⟩

lemma *powsum-split-var6*: $x_0^{(Suc\ k)+Suc\ n} = x_0^{Suc\ k} + x_{0+Suc\ (Suc\ k)}^n$
⟨*proof*⟩

lemma *powsum-ext*: $x \leq x_0^{Suc\ n}$
⟨*proof*⟩

lemma *powsum-one*: $1 \leq x_0^{Suc\ n}$
⟨*proof*⟩

lemma *powsum-shift1*: $x \cdot x_m^n = x_{m+1}^n$
⟨*proof*⟩

lemma *powsum-shift*: $x^k \cdot x_m^n = x_{k+m}^n$
⟨*proof*⟩

lemma *powsum-prod-suc*: $x_0^m \cdot x_0^{Suc\ n} = x_0^{Suc\ (m+n)}$
⟨*proof*⟩

lemma *powsum-prod*: $x_0^m \cdot x_0^n = x_0^{m+n}$
⟨*proof*⟩

end

end

3 Regular Algebras

theory *Regular-Algebras*
 imports *Dioid-Power-Sum Kleene-Algebra.Finite-Suprema Kleene-Algebra.Kleene-Algebra*
begin

3.1 Conway's Classical Axioms

Conway's classical axiomatisation of Regular Algebra from [4].

class *star-dioid* = *dioid-one-zero* + *star-op* + *plus-ord*

class *conway-dioid* = *star-dioid* +
  assumes *C11*: $(x + y)^\star = (x^\star \cdot y)^\star \cdot x^\star$
  and *C12*: $(x \cdot y)^\star = 1 + x \cdot (y \cdot x)^\star \cdot y$

class *strong-conway-dioid* = *conway-dioid* +
  assumes *C13*: $(x^\star)^\star = x^\star$

class *C-algebra* = *strong-conway-dioid* +
  assumes *C14*: $x^\star = (x^{n+1})^\star \cdot {x_0}^n$

We tried to dualise using sublocales, but this causes an infinite loop on dual.dual.dual....

lemma (**in** *conway-dioid*) *C11-var*: $(x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

lemma (**in** *conway-dioid*) *dual-conway-dioid*:
  class.conway-dioid (+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

lemma (**in** *strong-conway-dioid*) *dual-strong-conway-dioid*: *class.strong-conway-dioid* ((+)) ((⊙)) *1* *0* (≤) (<) *star*
⟨*proof*⟩

Nitpick finds counterexamples to the following claims.

lemma (**in** *conway-dioid*) $1^\star = 1$
  nitpick [*expect=genuine*] — 3-element counterexample
⟨*proof*⟩

lemma (**in** *conway-dioid*) $(x^\star)^\star = x^\star$
  nitpick [*expect=genuine*] — 3-element counterexample
⟨*proof*⟩

context *C-algebra*
begin

lemma *C-unfoldl* [*simp*]: $1 + x \cdot x^\star = x^\star$
  ⟨*proof*⟩

lemma *C-slide*: $(x \cdot y)^\star \cdot x = x \cdot (y \cdot x)^\star$
⟨*proof*⟩

lemma *powsum-ub*: $i \leq n \Longrightarrow x^i \leq x_0{}^n$
⟨*proof*⟩

lemma *C14-aux*: $m \leq n \Longrightarrow\ x^m \cdot (x^n)^\star = (x^n)^\star \cdot x^m$
⟨*proof*⟩

end

context *dioid-one-zero*
begin

lemma *opp-power-def*:
  $power.power\ 1\ (\odot)\ x\ n = x^n$
⟨*proof*⟩

lemma *opp-powsum-def*:
  $dioid\text{-}one\text{-}zero.powsum\ (+)\ (\odot)\ 1\ 0\ x\ m\ n = x_m{}^n$
⟨*proof*⟩

end

lemma *C14-dual*:
  fixes $x::'a::C\text{-}algebra$
  shows $x^\star = x_0{}^n \cdot (x^{n+1})^\star$
⟨*proof*⟩

lemma *C-algebra*: $class.C\text{-}algebra\ (+)\ (\odot)\ (1::'a::C\text{-}algebra)\ 0\ (\leq)\ (<)\ star$
⟨*proof*⟩

3.2 Boffa's Axioms

Boffa's two axiomatisations of Regular Algebra from [2, 3].

class *B1-algebra* = *conway-dioid* +
  assumes *R*: $x \cdot x = x \Longrightarrow x^\star = 1 + x$

class *B2-algebra* = *star-dioid* +
  assumes *B21*: $1 + x \leq x^\star$
  and *B22* [*simp*]: $x^\star \cdot x^\star = x^\star$
  and *B23*: $[\![\ 1 + x \leq y;\ y \cdot y = y\ ]\!] \Longrightarrow x^\star \leq y$

lemma (**in** *B1-algebra*) *B1-algebra*:
  $class.B1\text{-}algebra\ (+)\ (\odot)\ 1\ 0\ (\leq)\ (<)\ star$
⟨*proof*⟩

lemma (**in** *B2-algebra*) *B2-algebra*:

class.B2-algebra (+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

instance *B1-algebra* ⊆ *B2-algebra*
⟨*proof*⟩

context *B2-algebra*
begin

lemma *star-ref*: $1 \leq x^\star$
⟨*proof*⟩

lemma *star-plus-one* [*simp*]: $1 + x^\star = x^\star$
⟨*proof*⟩

lemma *star-trans*: $x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

lemma *star-trans-eq* [*simp*]: $x^\star \cdot x^\star = x^\star$
⟨*proof*⟩

lemma *star-invol* [*simp*]: $(x^\star)^\star = x^\star$
⟨*proof*⟩

lemma *star-1l*: $x \cdot x^\star \leq x^\star$
⟨*proof*⟩

lemma *star-one* [*simp*]: $1^\star = 1$
⟨*proof*⟩

lemma *star-subdist*: $x^\star \leq (x + y)^\star$
⟨*proof*⟩

lemma *star-iso*: $x \leq y \Longrightarrow x^\star \leq y^\star$
⟨*proof*⟩

lemma *star2*: $(1 + x)^\star = x^\star$
⟨*proof*⟩

lemma *star-unfoldl*: $1 + x \cdot x^\star \leq x^\star$
⟨*proof*⟩

lemma *star-unfoldr*: $1 + x^\star \cdot x \leq x^\star$
⟨*proof*⟩

lemma *star-ext*: $x \leq x^\star$
⟨*proof*⟩

lemma *star-1r*: $x^\star \cdot x \leq x^\star$

⟨*proof*⟩

lemma *star-unfoldl-eq* [*simp*]: $1 + x \cdot x^\star = x^\star$
⟨*proof*⟩

lemma *star-unfoldr-eq* [*simp*]: $1 + x^\star \cdot x = x^\star$
⟨*proof*⟩

lemma *star-prod-unfold-le*: $(x \cdot y)^\star \leq 1 + x \cdot (y \cdot x)^\star \cdot y$
⟨*proof*⟩

lemma *star-prod-unfold* [*simp*]: $1 + x \cdot (y \cdot x)^\star \cdot y = (x \cdot y)^\star$
⟨*proof*⟩

lemma *star-slide1*: $(x \cdot y)^\star \cdot x \leq x \cdot (y \cdot x)^\star$
⟨*proof*⟩

lemma *star-slide-var1*: $x^\star \cdot x \leq x \cdot x^\star$
⟨*proof*⟩

lemma *star-slide*: $(x \cdot y)^\star \cdot x = x \cdot (y \cdot x)^\star$
⟨*proof*⟩

lemma *star-rtc1*: $1 + x + x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

lemma *star-rtc1-eq*: $1 + x + x^\star \cdot x^\star = x^\star$
⟨*proof*⟩

lemma *star-subdist-var-1*: $x \leq (x + y)^\star$
⟨*proof*⟩

lemma *star-subdist-var-2*: $x \cdot y \leq (x + y)^\star$
⟨*proof*⟩

lemma *star-subdist-var-3*: $x^\star \cdot y^\star \leq (x + y)^\star$
⟨*proof*⟩

lemma *R-lemma*:
assumes $x \cdot x = x$
shows $x^\star = 1 + x$
⟨*proof*⟩

lemma *star-denest-var-0*: $(x + y)^\star = (x^\star \cdot y)^\star \cdot x^\star$
⟨*proof*⟩

lemma *star-denest*: $(x + y)^\star = (x^\star \cdot y^\star)^\star$
⟨*proof*⟩

lemma *star-sum-var*: $(x + y)^\star = (x^\star + y^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var*: $(x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-2*: $(x^\star \cdot y^\star)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-3*: $(x^\star \cdot y^\star)^\star = x^\star \cdot (y^\star \cdot x^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-4*: $(x^\star \cdot y^\star)^\star = (y^\star \cdot x^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-5*: $x^\star \cdot (y \cdot x^\star)^\star = y^\star \cdot (x \cdot y^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-6*: $(x + y)^\star = x^\star \cdot y^\star \cdot (x + y)^\star$
⟨*proof*⟩

lemma *star-denest-var-7*: $(x + y)^\star = (x + y)^\star \cdot x^\star \cdot y^\star$
⟨*proof*⟩

lemma *star-denest-var-8*: $(x^\star \cdot y^\star)^\star = x^\star \cdot y^\star \cdot (x^\star \cdot y^\star)^\star$
⟨*proof*⟩

lemma *star-denest-var-9*: $(x^\star \cdot y^\star)^\star = (x^\star \cdot y^\star)^\star \cdot x^\star \cdot y^\star$
⟨*proof*⟩

lemma *star-slide-var*: $x^\star \cdot x = x \cdot x^\star$
⟨*proof*⟩

lemma *star-sum-unfold*: $(x + y)^\star = x^\star + x^\star \cdot y \cdot (x + y)^\star$
⟨*proof*⟩

lemma *troeger*: $x^\star \cdot (y \cdot ((x + y)^\star \cdot z) + z) = (x + y)^\star \cdot z$
⟨*proof*⟩

lemma *meyer-1*: $x^\star = (1 + x) \cdot (x \cdot x)^\star$
⟨*proof*⟩

lemma *star-zero* [*simp*]: $0^\star = 1$
⟨*proof*⟩

lemma *star-subsum* [*simp*]: $x^\star + x^\star \cdot x = x^\star$
⟨*proof*⟩

lemma *prod-star-closure*: $x \leq z^\star \Longrightarrow y \leq z^\star \Longrightarrow x \cdot y \leq z^\star$

⟨*proof*⟩

end

sublocale *B2-algebra* $\subseteq$ *B1-algebra*
⟨*proof*⟩

context *B2-algebra*
begin

lemma *power-le-star*: $x^n \leq x^\star$
⟨*proof*⟩

lemma *star-power-slide*:
assumes $k \leq n$
shows $x^k \cdot (x^n)^\star = (x^n)^\star \cdot x^k$
⟨*proof*⟩

lemma *powsum-le-star*: ${x_m}^n \leq x^\star$
⟨*proof*⟩

lemma *star-sum-power-slide*:
assumes $m \leq n$
shows ${x_0}^m \cdot (x^n)^\star = (x^n)^\star \cdot {x_0}^m$
⟨*proof*⟩

lemma *aarden-aux*:
assumes $y \leq y \cdot x + z$
shows $y \leq y \cdot x^{(Suc\ n)} + z \cdot x^\star$
⟨*proof*⟩

lemma *conway-powerstar1*: $(x^{n+1})^\star \cdot {x_0}^n \cdot (x^{n+1})^\star \cdot {x_0}^n = (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

lemma *conway-powerstar2*: $1 + x \leq (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

theorem *powerstar*: $x^\star = (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

end

sublocale *B2-algebra* $\subseteq$ *strong-conway-dioid*
⟨*proof*⟩

sublocale *B2-algebra* $\subseteq$ *C-algebra*
⟨*proof*⟩

The following fact could neither be verified nor falsified in Isabelle. It does not hold for other reasons.

lemma (**in** *C-algebra*) $x{\cdot}x = x \longrightarrow x^\star = 1{+}x$
⟨*proof*⟩

3.3 Boffa Monoid Identities

typedef ($'a$, $'b$) *boffa-mon* = {f :: $'a$::{*finite*,*monoid-mult*} ⇒ $'b$::*B1-algebra*. *True*}
⟨*proof*⟩

notation
Rep-boffa-mon (‹-›)

lemma *finite* (*range* (*Rep-boffa-mon* M))
⟨*proof*⟩

abbreviation *boffa-pair* :: ($'a$, $'b$) *boffa-mon* ⇒ $'a$::{*finite*,*monoid-mult*} ⇒ $'a$ ⇒ $'b$::*B1-algebra* **where**
boffa-pair $x\ i\ j \equiv \sum\ \{\ x_k \mid k.\ i{\cdot}k = j\}$

notation
boffa-pair (‹-,-›)

abbreviation *conway-assms* **where**
conway-assms $x \equiv (\forall\ i\ j.\ (x_i \cdot x_j \le x_{i \cdot j}) \wedge (x_{i,i})^\star = x_{i,i})$

lemma *pair-one*: $x_{1,1} = x_1$
⟨*proof*⟩

definition *conway-assm1* **where** *conway-assm1* $x = (\forall\ i\ j.\ x_i \cdot x_j \le x_{i \cdot j})$
definition *conway-assm2* **where** *conway-assm2* $x = (\forall\, i.\ x_{i,i}{}^\star = x_{i,i})$

lemma *pair-star*:
assumes *conway-assm2* x
shows $x_1{}^\star = x_1$
⟨*proof*⟩

lemma *conway-monoid-one*:
assumes *conway-assm2* x
shows $x_1 = 1 + x_1$
⟨*proof*⟩

lemma *conway-monoid-split*:
assumes *conway-assm2* x
shows $\sum\ \{x_i \mid i\ .\ i \in UNIV\} = 1 + \sum\ \{x_i \mid i\ .\ i \in UNIV\}$
⟨*proof*⟩

lemma *boffa-mon-aux1*: $\{x_{i \cdot j} \mid i\ j.\ i \in UNIV \wedge j \in UNIV\} = \{x_i \mid i.\ i \in UNIV\}$
⟨*proof*⟩

lemma *sum-intro′* [*intro*]:
  $[\![$*finite* (*A* :: *′a::join-semilattice-zero set*); *finite B*; $\forall a \in A.\ \exists b \in B.\ a \le b$ $]\!] \Longrightarrow$ $\sum A \le \sum B$
  ⟨*proof*⟩

lemma *boffa-aux2*:
  conway-assm1 x $\Longrightarrow$
  $\sum \{x_i \cdot x_j \mid i\, j.\ i \in UNIV \wedge j \in UNIV\} \le \sum \{x_{i \cdot j} \mid i\, j.\ i \in UNIV \wedge j \in UNIV\}$
  ⟨*proof*⟩

lemma *boffa-aux3*:
  assumes *conway-assm1 x*
  shows $(\sum \{x_i \mid i.\ i \in UNIV\}) + (\sum \{x_i \cdot x_j \mid i\, j\, .\ i \in UNIV \wedge j \in UNIV\}) = (\sum \{x_i \mid i.\ i \in UNIV\})$
⟨*proof*⟩

lemma *conway-monoid-identity*:
  assumes *conway-assm1 x conway-assm2 x*
  shows $(\sum \{x_i | i.\ i \in UNIV\})^\star = (\sum \{x_i | \ i.\ i \in UNIV\})$
⟨*proof*⟩

3.4 Conway's Conjectures

class *C0-algebra* = *strong-conway-dioid* +
  assumes *C0*: $x \cdot y = y \cdot z \Longrightarrow x^\star \cdot y = y \cdot z^\star$

class *C1l-algebra* = *strong-conway-dioid* +
  assumes *C1l*: $x \cdot y \le y \cdot z \Longrightarrow x^\star \cdot y \le y \cdot z^\star$

class *C1r-algebra* = *strong-conway-dioid* +
  assumes *C1r*: $y \cdot x \le z \cdot y \Longrightarrow y \cdot x^\star \le z^\star \cdot y$

class *C2l-algebra* = *conway-dioid* +
  assumes *C2l*: $x = y \cdot x \Longrightarrow x = y^\star \cdot x$

class *C2r-algebra* = *conway-dioid* +
  assumes *C2r*: $x = x \cdot y \Longrightarrow x = x \cdot y^\star$

class *C3l-algebra* = *conway-dioid* +
  assumes *C3l*: $x \cdot y \le y \Longrightarrow x^\star \cdot y \le y$

class *C3r-algebra* = *conway-dioid* +
  assumes *C3r*: $y \cdot x \le y \Longrightarrow y \cdot x^\star \le y$

sublocale *C1r-algebra* ⊆ *dual*: *C1l-algebra*
  (+) (⊙) *1 0* (≤) (<) *star*
⟨*proof*⟩

sublocale *C2r-algebra* ⊆ *dual*: *C2l-algebra*
(+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

sublocale *C3r-algebra* ⊆ *dual*: *C3l-algebra*
(+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

lemma (**in** *C3l-algebra*) *k2-var*: $z + x \cdot y \leq y \Longrightarrow x^\star \cdot z \leq y$
⟨*proof*⟩

instance *C2l-algebra* ⊆ *B1-algebra*
⟨*proof*⟩

instance *C2r-algebra* ⊆ *B1-algebra*
⟨*proof*⟩

The following claims are refuted by Nitpick

lemma (**in** *conway-dioid*)
assumes $x \cdot y = y \cdot z \Longrightarrow x^\star \cdot y = y \cdot z^\star$
shows $1^\star = 1$

⟨*proof*⟩

lemma (**in** *conway-dioid*)
assumes $x \cdot y \leq y \cdot z \Longrightarrow x^\star \cdot y \leq y \cdot z^\star$
shows $1^\star = 1$

⟨*proof*⟩

The following fact could not be refuted by Nitpick or Quickcheck; but an infinite counterexample exists.

lemma (**in** *B1-algebra*) $x = x \cdot y \longrightarrow x = x \cdot y^\star$
⟨*proof*⟩

instance *C3l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

sublocale *C2l-algebra* ⊆ *C3l-algebra*
⟨*proof*⟩

sublocale *C1l-algebra* ⊆ *C3l-algebra*
⟨*proof*⟩

sublocale *C3l-algebra* ⊆ *C1l-algebra*
⟨*proof*⟩

sublocale *C1l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

sublocale *C3r-algebra* ⊆ *C2r-algebra*
⟨*proof*⟩

sublocale *C2r-algebra* ⊆ *C3r-algebra*
⟨*proof*⟩

sublocale *C1r-algebra* ⊆ *C3r-algebra*
⟨*proof*⟩

sublocale *C3r-algebra* ⊆ *C1r-algebra*
⟨*proof*⟩

class *C1-algebra* = *C1l-algebra* + *C1r-algebra*

class *C2-algebra* = *C2l-algebra* + *C2r-algebra*

class *C3-algebra* = *C3l-algebra* + *C3r-algebra*

sublocale *C0-algebra* ⊆ *C2-algebra*
⟨*proof*⟩

sublocale *C2-algebra* ⊆ *C0-algebra*
⟨*proof*⟩

sublocale *C2-algebra* ⊆ *C1-algebra* ⟨*proof*⟩

sublocale *C1-algebra* ⊆ *C2-algebra* ⟨*proof*⟩

sublocale *C2-algebra* ⊆ *C3-algebra* ⟨*proof*⟩

sublocale *C3-algebra* ⊆ *C2-algebra* ⟨*proof*⟩

3.5 Kozen's Kleene Algebras

Kozen's Kleene Algebras [7, 6].

class *Kl-base* = *star-dioid* +
assumes *Kl*: $1 + x \cdot x^\star \leq x^\star$

class *Kr-base* = *star-dioid* +
assumes *Kr*: $1 + x^\star \cdot x \leq x^\star$

class *K1l-algebra* = *Kl-base* +
assumes *star-inductl*: $x \cdot y \leq y \Longrightarrow x^\star \cdot y \leq y$

class *K1r-algebra* = *Kr-base* +
assumes *star-inductr*: $y \cdot x \leq y \Longrightarrow y \cdot x^\star \leq y$

class *K2l-algebra* = *Kl-base* +

assumes *star-inductl-var*: $z + x \cdot y \leq y \Longrightarrow x^\star \cdot z \leq y$

class *K2r-algebra* = *Kr-base* +
assumes *star-inductr-var*: $z + y \cdot x \leq y \Longrightarrow z \cdot x^\star \leq y$

class *K1-algebra* = *K1l-algebra* + *K1r-algebra*

class *K2-algebra* = *K2l-algebra* + *K2r-algebra*

sublocale *K1r-algebra* ⊆ *dual*: *K1l-algebra*
(+) (⊙) *1 0* (≤) (<) *star*
⟨*proof*⟩

sublocale *K1l-algebra* ⊆ *B2-algebra*
⟨*proof*⟩

sublocale *K1r-algebra* ⊆ *B2-algebra*
⟨*proof*⟩

sublocale *K1l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

sublocale *C2l-algebra* ⊆ *K1l-algebra*
⟨*proof*⟩

sublocale *K1r-algebra* ⊆ *C2r-algebra*
⟨*proof*⟩

sublocale *C2r-algebra* ⊆ *K1r-algebra*
⟨*proof*⟩

sublocale *K1-algebra* ⊆ *C0-algebra*
⟨*proof*⟩

sublocale *C0-algebra* ⊆ *K1l-algebra* ⟨*proof*⟩

sublocale *K2r-algebra* ⊆ *dual*: *K2l-algebra*
(+) (⊙) *1 0* (≤) (<) *star*
⟨*proof*⟩

sublocale *K1l-algebra* ⊆ *K2l-algebra*
⟨*proof*⟩

sublocale *K2l-algebra* ⊆ *K1l-algebra*
⟨*proof*⟩

sublocale *K1r-algebra* ⊆ *K2r-algebra*
⟨*proof*⟩

sublocale *K2r-algebra* ⊆ *K1r-algebra*
⟨*proof*⟩

sublocale *kleene-algebra* ⊆ *K1-algebra*
⟨*proof*⟩

sublocale *K1-algebra* ⊆ *K2-algebra* ⟨*proof*⟩

sublocale *K2-algebra* ⊆ *koz*: *kleene-algebra*
⟨*proof*⟩

3.6 Salomaa's Axioms

Salomaa's axiomatisations of Regular Algebra [9].

class *salomaa-base* = *star-dioid* +
fixes *ewp* :: *'a* ⇒ *bool*
assumes *S11*: $(1 + x)^\star = x^\star$
and *EWP* : $ewp\ x \longleftrightarrow (\exists y.\ x = 1 + y \wedge \neg\ ewp\ y)$

class *Sr-algebra* = *salomaa-base* +
assumes *S12r*: $1 + x^\star \cdot x = x^\star$
and *Ar* : $[\![\neg\ ewp\ y;\ x = x \cdot y + z]\!] \Longrightarrow x = z \cdot y^\star$

The following claim is ruled out by Nitpick. The unfold law cannot be weakened as in Kleene algebra.

lemma (**in** *salomaa-base*)
assumes *S12r'*: $1 + x^\star \cdot x \leq x^\star$
and *Ar'* : $[\![\neg\ ewp\ y;\ x = x \cdot y + z]\!] \Longrightarrow x = z \cdot y^\star$
shows $x^\star \leq 1 + x^\star \cdot x$

⟨*proof*⟩

class *Sl-algebra* = *salomaa-base* +
assumes *S12l*: $1 + x \cdot x^\star = x^\star$
and *Al* : $[\![\neg\ ewp\ y;\ x = y{\cdot}x{+}z]\!] \Longrightarrow x = y^\star{\cdot}z$

class *S-algebra* = *Sl-algebra* + *Sr-algebra*

sublocale *Sl-algebra* ⊆ *dual*: *Sr-algebra*
(+) (⊙) *1 0* (≤) (<) *star ewp*
⟨*proof*⟩

context *Sr-algebra*
begin

lemma *kozen-induct-r*:
assumes $y \cdot x + z \leq y$
shows $z \cdot x^\star \leq y$

⟨*proof*⟩

end

context *Sl-algebra*
begin

lemma *kozen-induct-l*:
  assumes $x \cdot y + z \leq y$
  shows $x^\star \cdot z \leq y$
  ⟨*proof*⟩

end

sublocale *Sr-algebra* ⊆ *K2r-algebra*
  ⟨*proof*⟩

sublocale *Sr-algebra* ⊆ *K1r-algebra* ⟨*proof*⟩

sublocale *Sl-algebra* ⊆ *K2l-algebra*
  ⟨*proof*⟩

sublocale *Sl-algebra* ⊆ *K1l-algebra* ⟨*proof*⟩

sublocale *S-algebra* ⊆ *K1-algebra* ⟨*proof*⟩

sublocale *S-algebra* ⊆ *K2-algebra* ⟨*proof*⟩

The following claim could be refuted.

lemma (**in** *K2-algebra*) $(\neg\ 1 \leq\ x) \longrightarrow\ x = x \cdot y + z \longrightarrow x = z \cdot y^\star$
⟨*proof*⟩

class *salomaa-conj-r* = *salomaa-base* +
  assumes *salomaa-small-unfold*: $1 + x^\star \cdot x = x^\star$
  assumes *salomaa-small-r*: $[\![\ \neg\ ewp\ y\ ;\ x = x \cdot y + 1\ ]\!] \Longrightarrow x = y^\star$

sublocale *Sr-algebra* ⊆ *salomaa-conj-r*
  ⟨*proof*⟩

lemma (**in** *salomaa-conj-r*) $(\neg\ ewp\ y) \wedge (x = x \cdot y + z) \longrightarrow x = z \cdot y^\star$

⟨*proof*⟩

end

4 Models of Regular Algebra

theory *Regular-Algebra-Models*
  imports *Regular-Algebras Kleene-Algebra.Kleene-Algebra-Models*

begin

4.1 Language Model of Salomaa Algebra

abbreviation *w-length* :: *'a list* ⇒ *nat* (‹|-|›)
  where $|x| \equiv$ *length x*

definition *l-ewp* :: *'a lan* ⇒ *bool* **where**
  l-ewp X $\longleftrightarrow \{[]\} \subseteq X$

interpretation *lan-kozen*: *K2-algebra* (+) (·) *1* :: *'a lan 0* (⊆) (⊂) *star* ⟨*proof*⟩

interpretation *lan-boffa*: *B1-algebra* (+) (·) *1* :: *'a lan 0* (⊆) (⊂) *star* ⟨*proof*⟩

lemma *length-lang-pow-lb*:
  assumes $\forall x \in X.\ |x| \geq k$ $x \in X\hat{\ }n$
  shows $|x| \geq k * n$
⟨*proof*⟩

lemma *l-prod-elim*: $w \in X \cdot Y \longleftrightarrow (\exists u\ v.\ w = u@v \wedge u \in X \wedge v \in Y)$
  ⟨*proof*⟩

lemma *power-minor-var*:
  assumes $\forall w \in X.\ k \leq |w|$
  shows $\forall w \in X^{Suc\ n}.\ n*k \leq |w|$
  ⟨*proof*⟩

lemma *power-lb*: $(\forall w \in X.\ k \leq |w|\ ) \longrightarrow (\forall w.\ w \in X^{Suc\ n} \longrightarrow n*k \leq |w|\ )$
  ⟨*proof*⟩

lemma *prod-lb*:
  $[\![\ (\forall w \in X.\ m \leq length\ w); (\forall w \in Y.\ n \leq length\ w)\ ]\!] \Longrightarrow (\forall w \in (X \cdot Y).\ (m+n) \leq length\ w)$
  ⟨*proof*⟩

lemma *suicide-aux-l*:
  $[\![\ (\forall w \in Y.\ 0 \leq length\ w); (\forall w \in X^{Suc\ n}.\ n \leq length\ w)\ ]\!] \Longrightarrow (\forall w \in X^{Suc\ n} \cdot Y.\ n \leq length\ w)$
  ⟨*proof*⟩

lemma *suicide-aux-r*:
  $[\![\ (\forall w \in Y.\ 0 \leq length\ w); (\forall w \in X^{Suc\ n}.\ n \leq length\ w)\ ]\!] \Longrightarrow (\forall w \in Y \cdot X^{Suc\ n}.\ n \leq length\ w)$
  ⟨*proof*⟩

lemma *word-suicide-l*:
  assumes $\neg$ *l-ewp X* $Y \neq \{\}$
  shows $(\forall w \in Y.\ \exists n.\ w \notin X^{Suc\ n} \cdot Y)$
⟨*proof*⟩

lemma *word-suicide-r*:
assumes $\neg$ *l-ewp* X $Y \neq \{\}$
shows $(\forall\, w{\in}Y.\ \exists\, n.\ w{\notin}\, Y \cdot X^{Suc\ n})$
⟨*proof*⟩

lemma *word-suicide-lang-l*: $[\![\ \neg$ *l-ewp* $X;\ Y \neq \{\}\ ]\!] \Longrightarrow \exists\ n. \neg\,(Y \leq X^{Suc\ n} \cdot Y)$
⟨*proof*⟩

lemma *word-suicide-lang-r*: $[\![\ \neg$ *l-ewp* $X;\ Y \neq \{\}\ ]\!] \Longrightarrow \exists\ n. \neg\,(Y \leq Y \cdot X^{Suc\ n})$
⟨*proof*⟩

These duality results cannot be relocated easily

context *K1-algebra*
begin

lemma *power-dual-transfer* [*simp*]:
power.power $(1::'a)$ $(\odot)$ $x\ n = x^n$
⟨*proof*⟩

lemma *aarden-aux-l*:
$y \leq x \cdot y + z \Longrightarrow y \leq\ x^{Suc\ n} \cdot y + x^\star \cdot z$
⟨*proof*⟩

end

lemma *arden-l*:
assumes $\neg$ *l-ewp* y $x = y{\cdot}x + z$
shows $x = y^\star \cdot z$
⟨*proof*⟩

lemma *arden-r*:
assumes $\neg$ *l-ewp* y $x = x \cdot y + z$
shows $x = z \cdot y^\star$
⟨*proof*⟩

The following two facts provide counterexamples to Arden's rule if the empty word property is not considered.

lemma *arden-l-counter*: $\exists\ (x::'a\ lan)\ (y::'a\ lan)\ (z::'a\ lan).\ x = y \cdot x + z \wedge x \neq y^\star \cdot z$
⟨*proof*⟩

lemma *arden-r-counter*: $\exists\ (x::'a\ lan)\ (y::'a\ lan)\ (z::'a\ lan).\ x = x \cdot y + z \wedge x \neq z \cdot y^\star$
⟨*proof*⟩

interpretation *lan-salomaa-l*: *Sl-algebra* $(+)\ (\cdot)\ 1 :: 'a\ lan\ 0\ (\subseteq)\ (\subset)$ *star l-ewp*
⟨*proof*⟩

interpretation *lan-salomaa-r*: *Sr-algebra* (+) (·) *1* :: *'a lan* *0* (⊆) (⊂) *star* *l-ewp*
⟨*proof*⟩

4.2 Regular Language Model of Salomaa Algebra

notation
Atom (‹⟨-⟩›) **and**
Plus (**infixl** ‹$+_r$› *65*) **and**
Times (**infixl** ‹$\cdot_r$› *70*) **and**
Star (‹$-^{\star}{}_r$› [*101*] *100*) **and**
Zero (‹0_r›) **and**
One (‹1_r›)

fun *rexp-ewp* :: *'a rexp* ⇒ *bool* **where**
rexp-ewp 0_r = *False* |
rexp-ewp 1_r = *True* |
rexp-ewp ⟨*a*⟩ = *False* |
rexp-ewp ($s +_r t$) = (*rexp-ewp* *s* ∨ *rexp-ewp* *t*) |
rexp-ewp ($s \cdot_r t$) = (*rexp-ewp* *s* ∧ *rexp-ewp* *t*) |
rexp-ewp ($s^{\star}{}_r$) = *True*

abbreviation *ro*(*s*) ≡ (*if* (*rexp-ewp* *s*) *then* 1_r *else* 0_r)

lift-definition *r-ewp* :: *'a reg-lan* ⇒ *bool* **is** *l-ewp* ⟨*proof*⟩

lift-definition *r-lang* :: *'a rexp* ⇒ *'a reg-lan* **is** *lang*
⟨*proof*⟩

abbreviation *r-sim* :: *'a rexp* ⇒ *'a rexp* ⇒ *bool* (**infix** ‹∼› *50*) **where**
p ∼ *q* ≡ *r-lang* *p* = *r-lang* *q*

declare *Rep-reg-lan* [*simp*]
declare *Rep-reg-lan-inverse* [*simp*]
declare *Abs-reg-lan-inverse* [*simp*]

lemma *rexp-ewp-l-ewp*: *l-ewp* (*lang* *x*) = *rexp-ewp* *x*
⟨*proof*⟩

theorem *regexp-ewp*:
defines *P-def*: $P(t) \equiv \exists\ t'.\ t \sim ro(t) +_r t' \wedge ro(t') = 0_r$
shows *P* *t*
⟨*proof*⟩

instantiation *reg-lan* :: (*type*) *Sr-algebra*
begin

lift-definition *ewp-reg-lan* :: *'a reg-lan* ⇒ *bool* **is** *l-ewp* ⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *reg-lan* :: (*type*) *Sl-algebra*
begin

instance ⟨*proof*⟩
end

instance *reg-lan* :: (*type*) *S-algebra* ⟨*proof*⟩

theorem *arden-regexp-l*:
  assumes $ro(y) = 0_r\ x \sim y \cdot_r x +_r z$
  shows $x \sim y^{\star}{}_r \cdot_r z$
  ⟨*proof*⟩

theorem *arden-regexp-r*:
  assumes $ro(y) = 0_r\ x \sim x \cdot_r y +_r z$
  shows $x \sim z \cdot_r y^{\star}{}_r$
  ⟨*proof*⟩

end

5 Pratt's Counterexamples

theory *Pratts-Counterexamples*
  imports *Regular-Algebras*
begin

We create two regular algebra models due to Pratt [8] which are used to distinguish K1 algebras from K1l and K1r algebras.

datatype *pratt1* =
  P1Bot (‹$\bot_1$›) |
  P1Nat nat (‹$[\text{-}]_1$›) |
  P1Infty (‹∞_1›) |
  P1Top (‹$\top_1$›)

datatype *pratt2* =
  P2Bot (‹$\bot_2$›) |
  P2Nat nat (‹$[\text{-}]_2$›) |
  P2Infty (‹∞_2›) |
  P2Top (‹$\top_2$›)

fun *pratt1-max* **where**
  pratt1-max $[x]_1\ [y]_1 = [max\ x\ y]_1$ |
  pratt1-max $x\ \bot_1 = x$ |
  pratt1-max $\bot_1\ y = y$ |
  pratt1-max $\infty_1\ [y]_1 = \infty_1$ |
  pratt1-max $[y]_1\ \infty_1 = \infty_1$ |
  pratt1-max $\infty_1\ \infty_1 = \infty_1$ |

$pratt1\text{-}max\ \top_1\ x = \top_1\ |$
$pratt1\text{-}max\ x\ \top_1 = \top_1$

fun *pratt1-plus* :: *pratt1* ⇒ *pratt1* ⇒ *pratt1* (**infixl** ‹$+_1$› *65*) **where**
$[x]_1 +_1 [y]_1 = [x + y]_1\ |$
$\bot_1 +_1 x = \bot_1\ |$
$x +_1 \bot_1 = \bot_1\ |$
$\infty_1 +_1 [0]_1 = \infty_1\ |$
$[0]_1 +_1 \infty_1 = \infty_1\ |$
$\infty_1 +_1 \infty_1 = \top_1\ |$
$\top_1 +_1 x = \top_1\ |$
$x +_1 \top_1 = \top_1\ |$
$[x]_1 +_1 \infty_1 = \infty_1\ |$
$\infty_1 +_1 x = \top_1$

lemma *plusl-top-infty*: $\top_1 +_1 \infty_1 = \top_1$
⟨*proof*⟩

lemma *plusl-infty-top*: $\infty_1 +_1 \top_1 = \top_1$
⟨*proof*⟩

lemma *plusl-bot-infty*: $\bot_1 +_1 \infty_1 = \bot_1$
⟨*proof*⟩

lemma *plusl-infty-bot*: $\infty_1 +_1 \bot_1 = \bot_1$
⟨*proof*⟩

lemma *plusl-zero-infty*: $[0]_1 +_1 \infty_1 = \infty_1$
⟨*proof*⟩

lemma *plusl-infty-zero*: $\infty_1 +_1 [0]_1 = \infty_1$
⟨*proof*⟩

lemma *plusl-infty-num* [*simp*]: $x > 0 \Longrightarrow \infty_1 +_1 [x]_1 = \top_1$
⟨*proof*⟩

lemma *plusl-num-infty* [*simp*]: $x > 0 \Longrightarrow [x]_1 +_1 \infty_1 = \infty_1$
⟨*proof*⟩

fun *pratt2-max* **where**
$pratt2\text{-}max\ [x]_2\ [y]_2 = [max\ x\ y]_2\ |$
$pratt2\text{-}max\ x\ \bot_2 = x\ |$
$pratt2\text{-}max\ \bot_2\ y = y\ |$
$pratt2\text{-}max\ \infty_2\ [y]_2 = \infty_2\ |$
$pratt2\text{-}max\ [y]_2\ \infty_2 = \infty_2\ |$
$pratt2\text{-}max\ \infty_2\ \infty_2 = \infty_2\ |$
$pratt2\text{-}max\ \top_2\ x = \top_2\ |$
$pratt2\text{-}max\ x\ \top_2 = \top_2$

fun *pratt2-plus* :: *pratt2* ⇒ *pratt2* ⇒ *pratt2* (**infixl** ‹$+_2$› *65*) **where**
$[x]_2 +_2 [y]_2 = [x + y]_2$ |
$\bot_2 +_2 x = \bot_2$ |
$x +_2 \bot_2 = \bot_2$ |
$\infty_2 +_2 [0]_2 = \infty_2$ |
$[0]_2 +_2 \infty_2 = \infty_2$ |
$\infty_2 +_2 \infty_2 = \top_2$ |
$\top_2 +_2 x = \top_2$ |
$x +_2 \top_2 = \top_2$ |
$x +_2 \infty_2 = \top_2$ |
$\infty_2 +_2 [x]_2 = \infty_2$

instantiation *pratt1* :: *selective-semiring*
begin

definition *zero-pratt1-def*:
$0 \equiv \bot_1$

definition *one-pratt1-def*:
$1 \equiv [0]_1$

definition *plus-pratt1-def*:
$x + y \equiv$ *pratt1-max* x y

definition *times-pratt1-def*:
$x * y \equiv x +_1 y$

definition *less-eq-pratt1-def*:
$(x::pratt1) \leq y \equiv x + y = y$

definition *less-pratt1-def*:
$(x::pratt1) < y \equiv x \leq y \wedge x \neq y$

instance
⟨*proof*⟩
end

instance *pratt1* :: *dioid-one-zero* ⟨*proof*⟩

instantiation *pratt2* :: *selective-semiring*
begin

definition *zero-pratt2-def*:
$0 \equiv \bot_2$

definition *one-pratt2-def*:
$1 \equiv [0]_2$

definition *times-pratt2-def*:

$x * y \equiv x +_2 y$

definition *plus-pratt2* :: *pratt2* ⇒ *pratt2* ⇒ *pratt2* **where**
plus-pratt2 x y ≡ *pratt2-max x y*

definition *less-eq-pratt2-def*:
$(x::pratt2) \leq y \equiv x + y = y$

definition *less-pratt2-def*:
$(x::pratt2) < y \equiv x \leq y \wedge x \neq y$

instance
⟨*proof*⟩
end

lemma *top-greatest*: $x \leq \top_1$
⟨*proof*⟩

instantiation *pratt1* :: *star-op*
begin

definition *star-pratt1* **where**
$x^\star \equiv$ *if* $(x = \bot_1 \vee x = [0]_1)$ *then* $[0]_1$ *else* $\top_1$
instance ⟨*proof*⟩
end

instantiation *pratt2* :: *star-op*
begin
definition *star-pratt2* **where**
$x^\star \equiv$ *if* $(x = \bot_2 \vee x = [0]_2)$ *then* $[0]_2$ *else* $\top_2$
instance ⟨*proof*⟩
end

instance *pratt1* :: *K1r-algebra*
⟨*proof*⟩

instance *pratt2* :: *K1l-algebra*
⟨*proof*⟩

lemma *one-star-top*: $[1]_1{}^\star = \top_1$
⟨*proof*⟩

lemma *pratt1-kozen-1l-counterexample*:
$\exists\ x\ y :: pratt1.\ \neg\ (x \cdot y \leq y \wedge x^\star \cdot y \leq y)$
⟨*proof*⟩

lemma *pratt2-kozen-1r-counterexample*:
$\exists\ x\ y :: pratt2.\ \neg\ (y \cdot x \leq y \wedge y \cdot x^\star \leq y)$
⟨*proof*⟩

end

6 Variants of Regular Algebra

theory *Regular-Algebra-Variants*
imports *Regular-Algebras Pratts-Counterexamples*
begin

Replacing Kozen's induction axioms by Boffa's leads to incompleteness.

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x.\ x \cdot x = x \Longrightarrow x^\star = 1 + x$
shows $\bigwedge x\ y.\ (x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$

⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x.\ x \cdot x = x \Longrightarrow x^\star = 1 + x$
shows $\bigwedge x\ y.\ x \leq y \longrightarrow x^\star \leq y^\star$

⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y.\ 1 + x \leq y \wedge y \cdot y \leq y \longrightarrow x^\star \leq y$
shows $\bigwedge x.\ 1 + x \leq x^\star$
⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y.\ 1 + x \leq y \wedge y \cdot y \leq y \longrightarrow x^\star \leq y$
shows $\bigwedge x.\ x^\star = (1 + x)^\star$
⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y.\ 1 + x + y \cdot y \leq y \Longrightarrow x^\star \leq y$
shows $\bigwedge x.\ x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$

and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y\ z.\ x \cdot y = y \cdot z \Longrightarrow x^\star \cdot y = y \cdot z^\star$
shows $1^\star = 1$

⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y\ z.\ x \cdot y \leq y \cdot z \Longrightarrow x^\star \cdot y \leq y \cdot z^\star$
and $\bigwedge x\ y\ z.\ y \cdot x \leq z \cdot y \Longrightarrow y \cdot x^\star \leq z^\star \cdot y$
shows $1^\star = 1$

⟨*proof*⟩

lemma (**in** *star-dioid*)
assumes $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
and $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
and $\bigwedge x\ y.\ x = y \cdot x \Longrightarrow x = y^\star \cdot x$
and $\bigwedge x\ y.\ x = x \cdot y \Longrightarrow x = x \cdot y^\star$
shows $\bigwedge x\ y.\ y \cdot x \leq y \Longrightarrow y \cdot x^\star \leq y$
⟨*proof*⟩

class *C3l-var* = *conway-dioid* +
assumes *C3l-var*: $z + x \cdot y \leq y \Longrightarrow x^\star \cdot z \leq y$

class *C3r-var* = *conway-dioid* +
assumes *C3r-var*: $z + y \cdot x \leq y \Longrightarrow z \cdot x^\star \leq y$

class *C3-var* = *C3l-var* + *C3r-var*

sublocale *C3l-var* ⊆ *C3l-algebra*
⟨*proof*⟩

sublocale *C3l-algebra* ⊆ *C3l-var*
⟨*proof*⟩

sublocale *C3-var* ⊆ *C3-algebra*
⟨*proof*⟩

sublocale *C3-algebra* ⊆ *C3-var*
⟨*proof*⟩

class *Brtc-algebra* = *star-dioid* +
assumes *rtc1*: $1 + x^\star \cdot x^\star + x \leq x^\star$
and *rtc2*: $1 + x + y \cdot y \leq y \Longrightarrow x^\star \leq y$

sublocale *B2-algebra* ⊆ *Brtc-algebra*
⟨*proof*⟩

sublocale *Brtc-algebra* $\subseteq$ *B2-algebra*
$\langle proof \rangle$

class *wB1-algebra* = *conway-dioid* +
assumes *wR*: $x \cdot x \leq x \Longrightarrow x^\star = 1 + x$

sublocale *wB1-algebra* $\subseteq$ *B1-algebra*
$\langle proof \rangle$

lemma (**in** *B1-algebra*) *one-plus-star*: $x^\star = (1 + x)^\star$
$\langle proof \rangle$

sublocale *B1-algebra* $\subseteq$ *wB1-algebra*
$\langle proof \rangle$

end

References

[1] A. Armstrong, G. Struth, and T. Weber. Kleene algebra. *Archive of Formal Proofs*, 2013. http://isa-afp.org/entries/Kleene_Algebra.shtml, Formal proof development.

[2] M. Boffa. Une remarque sur les systèmes complets d'identités rationnelles. *Informatique théorique et applications*, 24(4):419–423, 1990.

[3] M. Boffa. Une condition impliquant toutes les identités rationnelles. *Informatique théorique et applications*, 29(6):515–518, 1995.

[4] J. H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, 1971.

[5] S. Foster and G. Struth. On the fine-structure of regular algebra. *J. Automated Reasoning*, 54(2):165–197, 2015.

[6] D. Kozen. On Kleene algebras and closed semirings. In B. Rovan, editor, *MFCS'90*, volume 452 of *LNCS*, pages 26–47. Springer, 1990.

[7] D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366–390, 1994.

[8] V. Pratt. Action logic and pure induction. Technical Report STAN-CS-90-1343, Department of Computer Science, Stanford University, 1990.

[9] A. Salomaa. Two complete axiom systems for the algebra of regular events. *J. ACM*, 13(1):158–169, 1966.