# Regular Algebras

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**Abstract**

Regular algebras axiomatise the equational theory of regular expressions as induced by regular language identity. We use Isabelle/HOL for a detailed systematic study of regular algebras given by Boffa, Conway, Kozen and Salomaa. We investigate the relationships between these classes, formalise a soundness proof for the smallest class (Salomaa's) and obtain completeness of the largest one (Boffa's) relative to a deep result by Krob. In addition we provide a large collection of regular identities in the general setting of Boffa's axiom.

## Contents

# 1 Introductory Remarks

These Isabelle theories complement the article on *On the Fine-Structure of Regular Algebra* [5]. For an introduction to the topic, conceptual explanations and references we refer to this article. Our regular algebra hierarchy is orthogonal to the Kleene algebra hierarchy in the Archive of Formal Proofs [1]; we have not aimed at an integration for pragmatic reasons.

# 2 Dioids, Powers and Finite Sums

**theory** *Dioid-Power-Sum*
 **imports** *Kleene-Algebra.Dioid Kleene-Algebra.Finite-Suprema*

**begin**

We add a few facts about powers and finite sums—in fact, finite suprema—to an existing theory field for dioids.

**context** *dioid-one-zero*

**begin**

**lemma** *add-iso-r*: $y \leq z \Longrightarrow x + y \leq x + z$
 ⟨*proof*⟩

**notation** *power* (-$^-$ [*101*,*50*] *100*)

**lemma** *power-subdist*: $x^n \leq (x + y)^n$
 ⟨*proof*⟩

**lemma** *power-inductl-var*: $x \cdot y \leq y \Longrightarrow x^n \cdot y \leq y$
 ⟨*proof*⟩

**lemma** *power-inductr-var*: $y \cdot x \leq y \Longrightarrow y \cdot x^n \leq y$
 ⟨*proof*⟩

**definition** *powsum* :: $'a \Rightarrow nat \Rightarrow nat \Rightarrow 'a$ (-$_-^-$ [*101*,*50*,*50*] *100*) **where**
 *powsum x m n = sum* $((\widehat{\ }) \ x)$ $\{m..n + m\}$

**lemmas** *powsum-simps = powsum-def atLeastAtMostSuc-conv numerals*

**lemma** *powsum1* [*simp*]: $x_n{}^0 = x^n$
 ⟨*proof*⟩

**lemma** *powsum2*: $x_n{}^{Suc\ m} = x_n{}^m + x^{n+Suc\ m}$
⟨*proof*⟩

**lemma** *powsum-00* [*simp*]: $x_0{}^0 = 1$
 ⟨*proof*⟩

**lemma** *powsum-01* [*simp*]: $x_0{}^1 = 1 + x$
⟨*proof*⟩

**lemma** *powsum-10* [*simp*]: $x_1{}^0 = x$
⟨*proof*⟩

**lemma** *powsum-split*: $x_m{}^{i+Suc\ n} = x_m{}^i + x_{m+Suc\ i}{}^n$
⟨*proof*⟩

**lemma** *powsum-split-var1*: $x_0{}^{n+1} = 1 + x_1{}^n$
⟨*proof*⟩

**lemma** *powsum-split-var2* [*simp*]: $x^m + x_0{}^m = x_0{}^m$
⟨*proof*⟩

**lemma** *powsum-split-var3*: $x_0{}^{m+Suc\ n} = x_0{}^m + x_{0+Suc\ m}{}^n$
⟨*proof*⟩

**lemma** *powsum-split-var4* [*simp*]: $x_0{}^{m+n} + x_m{}^n = x_0{}^{m+n}$
⟨*proof*⟩

**lemma** *powsum-split-var6*: $x_0{}^{(Suc\ k)+Suc\ n} = x_0{}^{Suc\ k} + x_{0+Suc\ (Suc\ k)}{}^n$
⟨*proof*⟩

**lemma** *powsum-ext*: $x \leq x_0{}^{Suc\ n}$
⟨*proof*⟩

**lemma** *powsum-one*: $1 \leq x_0{}^{Suc\ n}$
⟨*proof*⟩

**lemma** *powsum-shift1*: $x \cdot x_m{}^n = x_{m+1}{}^n$
⟨*proof*⟩

**lemma** *powsum-shift*: $x^k \cdot x_m{}^n = x_{k+m}{}^n$
⟨*proof*⟩

**lemma** *powsum-prod-suc*: $x_0{}^m \cdot x_0{}^{Suc\ n} = x_0{}^{Suc\ (m+n)}$
⟨*proof*⟩

**lemma** *powsum-prod*: $x_0{}^m \cdot x_0{}^n = x_0{}^{m+n}$
⟨*proof*⟩

**end**

**end**

# 3 Regular Algebras

**theory** *Regular-Algebras*
 **imports** *Dioid-Power-Sum Kleene-Algebra.Finite-Suprema Kleene-Algebra.Kleene-Algebra*
**begin**

## 3.1 Conway's Classical Axioms

Conway's classical axiomatisation of Regular Algebra from [4].

**class** *star-dioid* = *dioid-one-zero* + *star-op* + *plus-ord*

**class** *conway-dioid* = *star-dioid* +
  **assumes** *C11*: $(x + y)^\star = (x^\star \cdot y)^\star \cdot x^\star$
  **and** *C12*: $(x \cdot y)^\star = 1 + x \cdot (y \cdot x)^\star \cdot y$

**class** *strong-conway-dioid* = *conway-dioid* +
  **assumes** *C13*: $(x^\star)^\star = x^\star$

**class** *C-algebra* = *strong-conway-dioid* +
  **assumes** *C14*: $x^\star = (x^{n+1})^\star \cdot x_0{}^n$

We tried to dualise using sublocales, but this causes an infinite loop on dual.dual.dual....

**lemma** (**in** *conway-dioid*) *C11-var*: $(x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

**lemma** (**in** *conway-dioid*) *dual-conway-dioid*:
  *class.conway-dioid* (+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

**lemma** (**in** *strong-conway-dioid*) *dual-strong-conway-dioid*: *class.strong-conway-dioid* ((+) ) ((⊙) ) *1* *0* (≤) (<) *star*
⟨*proof*⟩

Nitpick finds counterexamples to the following claims.

**lemma** (**in** *conway-dioid*) $1^\star = 1$
  **nitpick** [*expect=genuine*] — 3-element counterexample
⟨*proof*⟩

**lemma** (**in** *conway-dioid*) $(x^\star)^\star = x^\star$
  **nitpick** [*expect=genuine*] — 3-element counterexample
⟨*proof*⟩

**context** *C-algebra*
**begin**

**lemma** *C-unfoldl* [*simp*]: $1 + x \cdot x^\star = x^\star$
  ⟨*proof*⟩

**lemma** *C-slide*: $(x \cdot y)^\star \cdot x = x \cdot (y \cdot x)^\star$
$\langle proof \rangle$

**lemma** *powsum-ub*: $i \leq n \Longrightarrow x^i \leq x_0{}^n$
$\langle proof \rangle$

**lemma** *C14-aux*: $m \leq n \Longrightarrow\ x^m \cdot (x^n)^\star = (x^n)^\star \cdot x^m$
$\langle proof \rangle$

**end**

**context** *dioid-one-zero*
**begin**

**lemma** *opp-power-def*:
  $power.power\ 1\ (\odot)\ x\ n = x^n$
$\langle proof \rangle$

**lemma** *opp-powsum-def*:
  $dioid\text{-}one\text{-}zero.powsum\ (+)\ (\odot)\ 1\ 0\ x\ m\ n = x_m{}^n$
$\langle proof \rangle$

**end**

**lemma** *C14-dual*:
  **fixes** $x::'a::C\text{-}algebra$
  **shows** $x^\star = x_0{}^n \cdot (x^{n+1})^\star$
$\langle proof \rangle$

**lemma** *C-algebra*: $class.C\text{-}algebra\ (+)\ (\odot)\ (1::'a::C\text{-}algebra)\ 0\ (\leq)\ (<)\ star$
$\langle proof \rangle$

## 3.2 Boffa's Axioms

Boffa's two axiomatisations of Regular Algebra from [2, 3].

**class** *B1-algebra* = *conway-dioid* +
  **assumes** *R*: $x \cdot x = x \Longrightarrow x^\star = 1 + x$

**class** *B2-algebra* = *star-dioid* +
  **assumes** *B21*: $1 + x \leq x^\star$
  **and** *B22* [*simp*]: $x^\star \cdot x^\star = x^\star$
  **and** *B23*: $[\![\ 1 + x \leq y;\ y \cdot y = y\ ]\!] \Longrightarrow x^\star \leq y$

**lemma** (**in** *B1-algebra*) *B1-algebra*:
  $class.B1\text{-}algebra\ (+)\ (\odot)\ 1\ 0\ (\leq)\ (<)\ star$
$\langle proof \rangle$

**lemma** (**in** *B2-algebra*) *B2-algebra*:

*class.B2-algebra* $(+)$ $(\odot)$ *1* *0* $(\leq)$ $(<)$ *star*
⟨*proof*⟩

**instance** *B1-algebra* $\subseteq$ *B2-algebra*
⟨*proof*⟩

**context** *B2-algebra*
**begin**

**lemma** *star-ref*: $1 \leq x^\star$
⟨*proof*⟩

**lemma** *star-plus-one* [*simp*]: $1 + x^\star = x^\star$
⟨*proof*⟩

**lemma** *star-trans*: $x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

**lemma** *star-trans-eq* [*simp*]: $x^\star \cdot x^\star = x^\star$
⟨*proof*⟩

**lemma** *star-invol* [*simp*]: $(x^\star)^\star = x^\star$
⟨*proof*⟩

**lemma** *star-1l*: $x \cdot x^\star \leq x^\star$
⟨*proof*⟩

**lemma** *star-one* [*simp*]: $1^\star = 1$
⟨*proof*⟩

**lemma** *star-subdist*: $x^\star \leq (x + y)^\star$
⟨*proof*⟩

**lemma** *star-iso*: $x \leq y \Longrightarrow x^\star \leq y^\star$
⟨*proof*⟩

**lemma** *star2*: $(1 + x)^\star = x^\star$
⟨*proof*⟩

**lemma** *star-unfoldl*: $1 + x \cdot x^\star \leq x^\star$
⟨*proof*⟩

**lemma** *star-unfoldr*: $1 + x^\star \cdot x \leq x^\star$
⟨*proof*⟩

**lemma** *star-ext*: $x \leq x^\star$
⟨*proof*⟩

**lemma** *star-1r*: $x^\star \cdot x \leq x^\star$

⟨*proof*⟩

**lemma** *star-unfoldl-eq* [*simp*]: $1 + x \cdot x^\star = x^\star$
⟨*proof*⟩

**lemma** *star-unfoldr-eq* [*simp*]: $1 + x^\star \cdot x = x^\star$
⟨*proof*⟩

**lemma** *star-prod-unfold-le*: $(x \cdot y)^\star \leq 1 + x \cdot (y \cdot x)^\star \cdot y$
⟨*proof*⟩

**lemma** *star-prod-unfold* [*simp*]: $1 + x \cdot (y \cdot x)^\star \cdot y = (x \cdot y)^\star$
⟨*proof*⟩

**lemma** *star-slide1*: $(x \cdot y)^\star \cdot x \leq x \cdot (y \cdot x)^\star$
⟨*proof*⟩

**lemma** *star-slide-var1*: $x^\star \cdot x \leq x \cdot x^\star$
⟨*proof*⟩

**lemma** *star-slide*: $(x \cdot y)^\star \cdot x = x \cdot (y \cdot x)^\star$
⟨*proof*⟩

**lemma** *star-rtc1*: $1 + x + x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

**lemma** *star-rtc1-eq*: $1 + x + x^\star \cdot x^\star = x^\star$
⟨*proof*⟩

**lemma** *star-subdist-var-1*: $x \leq (x + y)^\star$
⟨*proof*⟩

**lemma** *star-subdist-var-2*: $x \cdot y \leq (x + y)^\star$
⟨*proof*⟩

**lemma** *star-subdist-var-3*: $x^\star \cdot y^\star \leq (x + y)^\star$
⟨*proof*⟩

**lemma** *R-lemma*:
**assumes** $x \cdot x = x$
**shows** $x^\star = 1 + x$
⟨*proof*⟩

**lemma** *star-denest-var-0*: $(x + y)^\star = (x^\star \cdot y)^\star \cdot x^\star$
⟨*proof*⟩

**lemma** *star-denest*: $(x + y)^\star = (x^\star \cdot y^\star)^\star$
⟨*proof*⟩

**lemma** *star-sum-var*: $(x + y)^\star = (x^\star + y^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var*: $(x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-2*: $(x^\star \cdot y^\star)^\star = x^\star \cdot (y \cdot x^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-3*: $(x^\star \cdot y^\star)^\star = x^\star \cdot (y^\star \cdot x^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-4*: $(x^\star \cdot y^\star)^\star = (y^\star \cdot x^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-5*: $x^\star \cdot (y \cdot x^\star)^\star = y^\star \cdot (x \cdot y^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-6*: $(x + y)^\star = x^\star \cdot y^\star \cdot (x + y)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-7*: $(x + y)^\star = (x + y)^\star \cdot x^\star \cdot y^\star$
⟨*proof*⟩

**lemma** *star-denest-var-8*: $(x^\star \cdot y^\star)^\star = x^\star \cdot y^\star \cdot (x^\star \cdot y^\star)^\star$
⟨*proof*⟩

**lemma** *star-denest-var-9*: $(x^\star \cdot y^\star)^\star = (x\ ^\star \cdot y^\star)^\star \cdot x^\star \cdot y^\star$
⟨*proof*⟩

**lemma** *star-slide-var*: $x^\star \cdot x = x \cdot x^\star$
⟨*proof*⟩

**lemma** *star-sum-unfold*: $(x + y)^\star = x^\star + x^\star \cdot y \cdot (x + y)^\star$
⟨*proof*⟩

**lemma** *troeger*: $x^\star \cdot (y \cdot ((x + y)^\star \cdot z) + z) = (x + y)^\star \cdot z$
⟨*proof*⟩

**lemma** *meyer-1*: $x^\star = (1 + x) \cdot (x \cdot x)^\star$
⟨*proof*⟩

**lemma** *star-zero* [*simp*]: $0^\star = 1$
⟨*proof*⟩

**lemma** *star-subsum* [*simp*]: $x^\star + x^\star \cdot x = x^\star$
⟨*proof*⟩

**lemma** *prod-star-closure*: $x \leq z^\star \Longrightarrow y \leq z^\star \Longrightarrow x \cdot y \leq z^\star$

⟨*proof*⟩

**end**

**sublocale** *B2-algebra* $\subseteq$ *B1-algebra*
⟨*proof*⟩

**context** *B2-algebra*
**begin**

**lemma** *power-le-star*: $x^n \leq x^\star$
⟨*proof*⟩

**lemma** *star-power-slide*:
**assumes** $k \leq n$
**shows** $x^k \cdot (x^n)^\star = (x^n)^\star \cdot x^k$
⟨*proof*⟩

**lemma** *powsum-le-star*: ${x_m}^n \leq x^\star$
⟨*proof*⟩

**lemma** *star-sum-power-slide*:
**assumes** $m \leq n$
**shows** ${x_0}^m \cdot (x^n)^\star = (x^n)^\star \cdot {x_0}^m$
⟨*proof*⟩

**lemma** *aarden-aux*:
**assumes** $y \leq y \cdot x + z$
**shows** $y \leq y \cdot x^{(Suc\ n)} + z \cdot x^\star$
⟨*proof*⟩

**lemma** *conway-powerstar1*: $(x^{n+1})^\star \cdot {x_0}^n \cdot (x^{n+1})^\star \cdot {x_0}^n = (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

**lemma** *conway-powerstar2*: $1 + x \leq (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

**theorem** *powerstar*: $x^\star = (x^{n+1})^\star \cdot {x_0}^n$
⟨*proof*⟩

**end**

**sublocale** *B2-algebra* $\subseteq$ *strong-conway-dioid*
⟨*proof*⟩

**sublocale** *B2-algebra* $\subseteq$ *C-algebra*
⟨*proof*⟩

The following fact could neither be verified nor falsified in Isabelle. It does not hold for other reasons.

**lemma** (**in** *C-algebra*) $x{\cdot}x = x \longrightarrow x^\star = 1{+}x$
⟨*proof*⟩

### 3.3 Boffa Monoid Identities

**typedef** ($'a$ , $'b$) *boffa-mon* = {$f$ :: $'a$::{*finite*,*monoid-mult*} ⇒ $'b$::*B1-algebra*. *True*}
⟨*proof*⟩

**notation**
*Rep-boffa-mon* (-$_{-}$)

**lemma** *finite* (*range* (*Rep-boffa-mon* $M$))
⟨*proof*⟩

**abbreviation** *boffa-pair* :: ($'a$, $'b$) *boffa-mon* ⇒ $'a$::{*finite*,*monoid-mult*} ⇒ $'a$ ⇒ $'b$::*B1-algebra* **where**
*boffa-pair* $x\ i\ j \equiv \sum\ \{\ x_k \mid k.\ i{\cdot}k = j\}$

**notation**
*boffa-pair* (-$_{-,-}$)

**abbreviation** *conway-assms* **where**
*conway-assms* $x \equiv (\forall\ i\ j.\ (x_i \cdot x_j \le x_{i \cdot j}) \wedge (x_{i,i})^\star = x_{i,i})$

**lemma** *pair-one*: $x_{1,1} = x_1$
⟨*proof*⟩

**definition** *conway-assm1* **where** *conway-assm1* $x = (\forall\ i\ j.\ x_i \cdot x_j \le x_{i \cdot j})$
**definition** *conway-assm2* **where** *conway-assm2* $x = (\forall\, i.\ x_{i,i}{}^\star = x_{i,i})$

**lemma** *pair-star*:
**assumes** *conway-assm2* $x$
**shows** $x_1{}^\star = x_1$
⟨*proof*⟩

**lemma** *conway-monoid-one*:
**assumes** *conway-assm2* $x$
**shows** $x_1 = 1 + x_1$
⟨*proof*⟩

**lemma** *conway-monoid-split*:
**assumes** *conway-assm2* $x$
**shows** $\sum\ \{x_i \mid i\ .\ i \in \mathit{UNIV}\} = 1 + \sum\ \{x_i \mid i\ .\ i \in \mathit{UNIV}\}$
⟨*proof*⟩

**lemma** *boffa-mon-aux1*: $\{x_{i \cdot j} \mid i\ j.\ i \in \mathit{UNIV} \wedge j \in \mathit{UNIV}\} = \{x_i \mid i.\ i \in \mathit{UNIV}\}$
⟨*proof*⟩

**lemma** *sum-intro'* [*intro*]:
$[\![$*finite* ($A$ :: *'a::join-semilattice-zero set*); *finite* $B$; $\forall\, a{\in}A.\ \exists\, b{\in}B.\ a \le b$ $]\!] \Longrightarrow \sum A \le \sum B$
$\langle proof \rangle$

**lemma** *boffa-aux2*:
*conway-assm1* $x \Longrightarrow$
$\sum\{x_i{\cdot}x_j \mid i\ j.\ i \in UNIV \wedge j \in UNIV\} \le \sum\{x_{i \cdot j} \mid i\ j.\ i \in UNIV \wedge j \in UNIV\}$
$\langle proof \rangle$

**lemma** *boffa-aux3*:
**assumes** *conway-assm1 x*
**shows** $(\sum\ \{x_i \mid i.\ i{\in}UNIV\}) + (\sum\ \{x_i \cdot x_j \mid i\ j\ .\ i{\in}UNIV \wedge j{\in}UNIV\}) = (\sum\ \{x_i \mid i.\ i{\in}UNIV\})$
$\langle proof \rangle$

**lemma** *conway-monoid-identity*:
**assumes** *conway-assm1 x conway-assm2 x*
**shows** $(\sum\ \{x_i | i.\ i{\in}UNIV\})^\star = (\sum\ \{x_i |\ i.\ i{\in}UNIV\})$
$\langle proof \rangle$

### 3.4 Conway's Conjectures

**class** *C0-algebra* = *strong-conway-dioid* +
**assumes** *C0*: $x \cdot y = y \cdot z \Longrightarrow x^\star \cdot y = y \cdot z^\star$

**class** *C1l-algebra* = *strong-conway-dioid* +
**assumes** *C1l*: $x \cdot y \le y \cdot z \Longrightarrow x^\star \cdot y \le y \cdot z^\star$

**class** *C1r-algebra* = *strong-conway-dioid* +
**assumes** *C1r*: $y \cdot x \le z \cdot y \Longrightarrow y \cdot x^\star \le z^\star \cdot y$

**class** *C2l-algebra* = *conway-dioid* +
**assumes** *C2l*: $x = y \cdot x \Longrightarrow x = y^\star \cdot x$

**class** *C2r-algebra* = *conway-dioid* +
**assumes** *C2r*: $x = x \cdot y \Longrightarrow x = x \cdot y^\star$

**class** *C3l-algebra* = *conway-dioid* +
**assumes** *C3l*: $x \cdot y \le y \Longrightarrow x^\star \cdot y \le y$

**class** *C3r-algebra* = *conway-dioid* +
**assumes** *C3r*: $y \cdot x \le y \Longrightarrow y \cdot x^\star \le y$

**sublocale** *C1r-algebra* $\subseteq$ *dual*: *C1l-algebra*
$(+)$ $(\odot)$ *1 0* $(\le)$ $(<)$ *star*
$\langle proof \rangle$

**sublocale** *C2r-algebra* ⊆ *dual*: *C2l-algebra*
(+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

**sublocale** *C3r-algebra* ⊆ *dual*: *C3l-algebra*
(+) (⊙) *1* *0* (≤) (<) *star*
⟨*proof*⟩

**lemma** (**in** *C3l-algebra*) *k2-var*: $z + x \cdot y \leq y \Longrightarrow x^{\star} \cdot z \leq y$
⟨*proof*⟩

**instance** *C2l-algebra* ⊆ *B1-algebra*
⟨*proof*⟩

**instance** *C2r-algebra* ⊆ *B1-algebra*
⟨*proof*⟩

The following claims are refuted by Nitpick

**lemma** (**in** *conway-dioid*)
**assumes** $x \cdot y = y \cdot z \Longrightarrow x^{\star} \cdot y = y \cdot z^{\star}$
**shows** $1^{\star} = 1$

⟨*proof*⟩

**lemma** (**in** *conway-dioid*)
**assumes** $x \cdot y \leq y \cdot z \Longrightarrow x^{\star} \cdot y \leq y \cdot z^{\star}$
**shows** $1^{\star} = 1$

⟨*proof*⟩

The following fact could not be refuted by Nitpick or Quickcheck; but an infinite counterexample exists.

**lemma** (**in** *B1-algebra*) $x = x \cdot y \longrightarrow x = x \cdot y^{\star}$
⟨*proof*⟩

**instance** *C3l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

**sublocale** *C2l-algebra* ⊆ *C3l-algebra*
⟨*proof*⟩

**sublocale** *C1l-algebra* ⊆ *C3l-algebra*
⟨*proof*⟩

**sublocale** *C3l-algebra* ⊆ *C1l-algebra*
⟨*proof*⟩

**sublocale** *C1l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

**sublocale** *C3r-algebra* $\subseteq$ *C2r-algebra*
⟨*proof*⟩

**sublocale** *C2r-algebra* $\subseteq$ *C3r-algebra*
⟨*proof*⟩

**sublocale** *C1r-algebra* $\subseteq$ *C3r-algebra*
⟨*proof*⟩

**sublocale** *C3r-algebra* $\subseteq$ *C1r-algebra*
⟨*proof*⟩

**class** *C1-algebra* = *C1l-algebra* + *C1r-algebra*

**class** *C2-algebra* = *C2l-algebra* + *C2r-algebra*

**class** *C3-algebra* = *C3l-algebra* + *C3r-algebra*

**sublocale** *C0-algebra* $\subseteq$ *C2-algebra*
⟨*proof*⟩

**sublocale** *C2-algebra* $\subseteq$ *C0-algebra*
⟨*proof*⟩

**sublocale** *C2-algebra* $\subseteq$ *C1-algebra* ⟨*proof*⟩

**sublocale** *C1-algebra* $\subseteq$ *C2-algebra* ⟨*proof*⟩

**sublocale** *C2-algebra* $\subseteq$ *C3-algebra* ⟨*proof*⟩

**sublocale** *C3-algebra* $\subseteq$ *C2-algebra* ⟨*proof*⟩

## 3.5 Kozen's Kleene Algebras

Kozen's Kleene Algebras [7, 6].

**class** *Kl-base* = *star-dioid* +
**assumes** *Kl*: $1 + x \cdot x^\star \leq x^\star$

**class** *Kr-base* = *star-dioid* +
**assumes** *Kr*: $1 + x^\star \cdot x \leq x^\star$

**class** *K1l-algebra* = *Kl-base* +
**assumes** *star-inductl*: $x \cdot y \leq y \Longrightarrow x^\star \cdot y \leq y$

**class** *K1r-algebra* = *Kr-base* +
**assumes** *star-inductr*: $y \cdot x \leq y \Longrightarrow y \cdot x^\star \leq y$

**class** *K2l-algebra* = *Kl-base* +

**assumes** *star-inductl-var*: $z + x \cdot y \leq y \Longrightarrow x^\star \cdot z \leq y$

**class** *K2r-algebra* = *Kr-base* +
**assumes** *star-inductr-var*: $z + y \cdot x \leq y \Longrightarrow z \cdot x^\star \leq y$

**class** *K1-algebra* = *K1l-algebra* + *K1r-algebra*

**class** *K2-algebra* = *K2l-algebra* + *K2r-algebra*

**sublocale** *K1r-algebra* ⊆ *dual*: *K1l-algebra*
(+) (⊙) *1 0* (≤) (<) *star*
⟨*proof*⟩

**sublocale** *K1l-algebra* ⊆ *B2-algebra*
⟨*proof*⟩

**sublocale** *K1r-algebra* ⊆ *B2-algebra*
⟨*proof*⟩

**sublocale** *K1l-algebra* ⊆ *C2l-algebra*
⟨*proof*⟩

**sublocale** *C2l-algebra* ⊆ *K1l-algebra*
⟨*proof*⟩

**sublocale** *K1r-algebra* ⊆ *C2r-algebra*
⟨*proof*⟩

**sublocale** *C2r-algebra* ⊆ *K1r-algebra*
⟨*proof*⟩

**sublocale** *K1-algebra* ⊆ *C0-algebra*
⟨*proof*⟩

**sublocale** *C0-algebra* ⊆ *K1l-algebra* ⟨*proof*⟩

**sublocale** *K2r-algebra* ⊆ *dual*: *K2l-algebra*
(+) (⊙) *1 0* (≤) (<) *star*
⟨*proof*⟩

**sublocale** *K1l-algebra* ⊆ *K2l-algebra*
⟨*proof*⟩

**sublocale** *K2l-algebra* ⊆ *K1l-algebra*
⟨*proof*⟩

**sublocale** *K1r-algebra* ⊆ *K2r-algebra*
⟨*proof*⟩

**sublocale** *K2r-algebra* $\subseteq$ *K1r-algebra*
⟨*proof*⟩

**sublocale** *kleene-algebra* $\subseteq$ *K1-algebra*
⟨*proof*⟩

**sublocale** *K1-algebra* $\subseteq$ *K2-algebra* ⟨*proof*⟩

**sublocale** *K2-algebra* $\subseteq$ *koz*: *kleene-algebra*
⟨*proof*⟩

### 3.6 Salomaa's Axioms

Salomaa's axiomatisations of Regular Algebra [9].

**class** *salomaa-base* = *star-dioid* +
**fixes** $ewp :: 'a \Rightarrow bool$
**assumes** *S11*: $(1 + x)^\star = x^\star$
**and** *EWP* : $ewp\ x \longleftrightarrow (\exists\, y.\ x = 1 + y \wedge \neg\ ewp\ y)$

**class** *Sr-algebra* = *salomaa-base* +
**assumes** *S12r*: $1 + x^\star \cdot x = x^\star$
**and** *Ar* : $[\![\ \neg\ ewp\ y;\ x = x \cdot y + z\ ]\!] \Longrightarrow x = z \cdot y^\star$

The following claim is ruled out by Nitpick. The unfold law cannot be weakened as in Kleene algebra.

**lemma** (**in** *salomaa-base*)
**assumes** *S12r'*: $1 + x^\star \cdot x \leq x^\star$
**and** *Ar'* : $[\![\ \neg\ ewp\ y;\ x = x \cdot y + z\ ]\!] \Longrightarrow x = z \cdot y^\star$
**shows** $x^\star \leq 1 + x^\star \cdot x$

⟨*proof*⟩

**class** *Sl-algebra* = *salomaa-base* +
**assumes** *S12l*: $1 + x \cdot x^\star = x^\star$
**and** *Al* : $[\![\ \neg\ ewp\ y;\ x = y{\cdot}x{+}z\ ]\!] \Longrightarrow x = y^\star{\cdot}z$

**class** *S-algebra* = *Sl-algebra* + *Sr-algebra*

**sublocale** *Sl-algebra* $\subseteq$ *dual*: *Sr-algebra*
$(+)\ (\odot)\ 1\ 0\ (\leq)\ (<)$ *star ewp*
⟨*proof*⟩

**context** *Sr-algebra*
**begin**

**lemma** *kozen-induct-r*:
**assumes** $y \cdot x + z \leq y$
**shows** $z \cdot x^\star \leq y$

⟨*proof*⟩

**end**

**context** *Sl-algebra*
**begin**

**lemma** *kozen-induct-l*:
  **assumes** $x \cdot y + z \leq y$
  **shows** $x^\star \cdot z \leq y$
  ⟨*proof*⟩

**end**

**sublocale** *Sr-algebra* ⊆ *K2r-algebra*
  ⟨*proof*⟩

**sublocale** *Sr-algebra* ⊆ *K1r-algebra* ⟨*proof*⟩

**sublocale** *Sl-algebra* ⊆ *K2l-algebra*
  ⟨*proof*⟩

**sublocale** *Sl-algebra* ⊆ *K1l-algebra* ⟨*proof*⟩

**sublocale** *S-algebra* ⊆ *K1-algebra* ⟨*proof*⟩

**sublocale** *S-algebra* ⊆ *K2-algebra* ⟨*proof*⟩

The following claim could be refuted.

**lemma** (**in** *K2-algebra*) $(\neg\ 1 \leq\ x) \longrightarrow\ x = x \cdot y + z \longrightarrow x = z \cdot y^\star$
⟨*proof*⟩

**class** *salomaa-conj-r* = *salomaa-base* +
  **assumes** *salomaa-small-unfold*: $1 + x^\star \cdot x = x^\star$
  **assumes** *salomaa-small-r*: $[\![ \neg\ ewp\ y\ ;\ x = x \cdot y + 1\ ]\!] \Longrightarrow x = y^\star$

**sublocale** *Sr-algebra* ⊆ *salomaa-conj-r*
  ⟨*proof*⟩

**lemma** (**in** *salomaa-conj-r*) $(\neg\ ewp\ y) \wedge (x = x \cdot y + z) \longrightarrow x = z \cdot y^\star$

⟨*proof*⟩

**end**

# 4 Models of Regular Algebra

**theory** *Regular-Algebra-Models*
  **imports** *Regular-Algebras Kleene-Algebra.Kleene-Algebra-Models*

**begin**

## 4.1 Language Model of Salomaa Algebra

**abbreviation** *w-length* :: *'a list* $\Rightarrow$ *nat* ( |-|)
  **where** $|x| \equiv$ *length x*

**definition** *l-ewp* :: *'a lan* $\Rightarrow$ *bool* **where**
  *l-ewp X* $\longleftrightarrow$ $\{[]\} \subseteq X$

**interpretation** *lan-kozen*: *K2-algebra* $(+)$ $(\cdot)$ *1* :: *'a lan 0* $(\subseteq)$ $(\subset)$ *star* ⟨*proof*⟩

**interpretation** *lan-boffa*: *B1-algebra* $(+)$ $(\cdot)$ *1* :: *'a lan 0* $(\subseteq)$ $(\subset)$ *star* ⟨*proof*⟩

**lemma** *length-lang-pow-lb*:
  **assumes** $\forall x{\in}X.\ |x| \geq k$ $x \in X\hat{}n$
  **shows** $|x| \geq k{*}n$
⟨*proof*⟩

**lemma** *l-prod-elim*: $w{\in}X \cdot Y \longleftrightarrow (\exists\, u\ v.\ w = u@v \wedge u{\in}X \wedge v{\in}Y)$
  ⟨*proof*⟩

**lemma** *power-minor-var*:
  **assumes** $\forall\, w{\in}X.\ k{\leq}|w|$
  **shows** $\forall\, w{\in}X^{Suc\ n}.\ n{*}k{\leq}|w|$
  ⟨*proof*⟩

**lemma** *power-lb*: $(\forall\, w{\in}X.\ k{\leq}|w|\ ) \longrightarrow (\forall\, w.\ w{\in}X^{Suc\ n} \longrightarrow n{*}k{\leq}|w|\ )$
  ⟨*proof*⟩

**lemma** *prod-lb*:
  $[\![\ (\forall\, w{\in}X.\ m \leq \mathit{length}\ w);\ (\forall\, w{\in}Y.\ n \leq \mathit{length}\ w)\ ]\!] \Longrightarrow (\forall\, w{\in}(X{\cdot}Y).\ (m{+}n) \leq \mathit{length}\ w)$
  ⟨*proof*⟩

**lemma** *suicide-aux-l*:
  $[\![\ (\forall\, w{\in}Y.\ 0{\leq}\mathit{length}\ w);\ (\forall\, w{\in}X^{Suc\ n}.\ n \leq \mathit{length}\ w)\ ]\!] \Longrightarrow (\forall\, w{\in}X^{Suc\ n} \cdot Y.\ n \leq \mathit{length}\ w)$
  ⟨*proof*⟩

**lemma** *suicide-aux-r*:
  $[\![\ (\forall\, w{\in}Y.\ 0{\leq}\mathit{length}\ w);\ (\forall\, w{\in}X^{Suc\ n}.\ n \leq \mathit{length}\ w)\ ]\!] \Longrightarrow (\forall\, w{\in}Y \cdot X^{Suc\ n}.\ n \leq \mathit{length}\ w)$
  ⟨*proof*⟩

**lemma** *word-suicide-l*:
  **assumes** $\neg$ *l-ewp X* $Y \neq \{\}$
  **shows** $(\forall\, w{\in}Y.\ \exists\, n.\ w{\notin}X^{Suc\ n} \cdot Y)$
⟨*proof*⟩

**lemma** *word-suicide-r*:
  **assumes** $\neg$ *l-ewp* $X$ $Y \neq \{\}$
  **shows** $(\forall\, w \in Y.\ \exists\, n.\ w \notin Y \cdot X^{Suc\ n})$
⟨*proof*⟩

**lemma** *word-suicide-lang-l*: $[\![\ \neg\ l\text{-}ewp\ X;\ Y \neq \{\}\ ]\!] \Longrightarrow \exists\ n.\ \neg\ (Y \leq X^{Suc\ n} \cdot Y)$
  ⟨*proof*⟩

**lemma** *word-suicide-lang-r*: $[\![\ \neg\ l\text{-}ewp\ X;\ Y \neq \{\}\ ]\!] \Longrightarrow \exists\ n.\ \neg\ (Y \leq Y \cdot X^{Suc\ n})$
  ⟨*proof*⟩

These duality results cannot be relocated easily

**context** *K1-algebra*
**begin**

**lemma** *power-dual-transfer* [*simp*]:
  $power.power\ (1{::}'a)\ (\odot)\ x\ n = x^n$
  ⟨*proof*⟩

**lemma** *aarden-aux-l*:
  $y \leq x \cdot y + z \Longrightarrow y \leq\ x^{Suc\ n} \cdot y + x^\star \cdot z$
  ⟨*proof*⟩

**end**

**lemma** *arden-l*:
  **assumes** $\neg$ *l-ewp* $y$ $x = y{\cdot}x + z$
  **shows** $x = y^\star \cdot z$
⟨*proof*⟩

**lemma** *arden-r*:
  **assumes** $\neg$ *l-ewp* $y$ $x = x \cdot y + z$
  **shows** $x = z \cdot y^\star$
⟨*proof*⟩

The following two facts provide counterexamples to Arden's rule if the empty word property is not considered.

**lemma** *arden-l-counter*: $\exists\ (x{::}'a\ lan)\ (y{::}'a\ lan)\ (z{::}'a\ lan).\ x = y \cdot x + z \wedge x \neq y^\star \cdot z$
⟨*proof*⟩

**lemma** *arden-r-counter*: $\exists\ (x{::}'a\ lan)\ (y{::}'a\ lan)\ (z{::}'a\ lan).\ x = x \cdot y + z \wedge x \neq z \cdot y^\star$
⟨*proof*⟩

**interpretation** *lan-salomaa-l*: *Sl-algebra* $(+)\ (\cdot)\ 1\ ::\ 'a\ lan\ 0\ (\subseteq)\ (\subset)$ *star l-ewp*
⟨*proof*⟩

**interpretation** *lan-salomaa-r*: *Sr-algebra* $(+)$ $(\cdot)$ *1* :: *'a lan* *0* $(\subseteq)$ $(\subset)$ *star* *l-ewp*
⟨*proof*⟩

## 4.2 Regular Language Model of Salomaa Algebra

**notation**
*Atom* (⟨-⟩) **and**
*Plus* (**infixl** $+_r$ *65*) **and**
*Times* (**infixl** $\cdot_r$ *70*) **and**
*Star* ($-^{\star}{}_r$ [*101*] *100*) **and**
*Zero* ($0_r$) **and**
*One* ($1_r$)

**fun** *rexp-ewp* :: *'a rexp* ⇒ *bool* **where**
*rexp-ewp* $0_r$ = *False* |
*rexp-ewp* $1_r$ = *True* |
*rexp-ewp* ⟨*a*⟩ = *False* |
*rexp-ewp* ($s +_r t$) = (*rexp-ewp s* ∨ *rexp-ewp t*) |
*rexp-ewp* ($s \cdot_r t$) = (*rexp-ewp s* ∧ *rexp-ewp t*) |
*rexp-ewp* ($s^{\star}{}_r$) = *True*

**abbreviation** *ro*(*s*) ≡ (*if* (*rexp-ewp s*) *then* $1_r$ *else* $0_r$)

**lift-definition** *r-ewp* :: *'a reg-lan* ⇒ *bool* **is** *l-ewp* ⟨*proof*⟩

**lift-definition** *r-lang* :: *'a rexp* ⇒ *'a reg-lan* **is** *lang*
⟨*proof*⟩

**abbreviation** *r-sim* :: *'a rexp* ⇒ *'a rexp* ⇒ *bool* (**infix** ∼ *50*) **where**
*p* ∼ *q* ≡ *r-lang p* = *r-lang q*

**declare** *Rep-reg-lan* [*simp*]
**declare** *Rep-reg-lan-inverse* [*simp*]
**declare** *Abs-reg-lan-inverse* [*simp*]

**lemma** *rexp-ewp-l-ewp*: *l-ewp* (*lang x*) = *rexp-ewp x*
⟨*proof*⟩

**theorem** *regexp-ewp*:
**defines** *P-def*: $P(t) \equiv \exists\ t'.\ t \sim ro(t) +_r t' \wedge ro(t') = 0_r$
**shows** *P t*
⟨*proof*⟩

**instantiation** *reg-lan* :: (*type*) *Sr-algebra*
**begin**

**lift-definition** *ewp-reg-lan* :: *'a reg-lan* ⇒ *bool* **is** *l-ewp* ⟨*proof*⟩

**instance** ⟨*proof*⟩

**end**

**instantiation** *reg-lan* :: (*type*) *Sl-algebra*
**begin**

**instance** ⟨*proof*⟩
**end**

**instance** *reg-lan* :: (*type*) *S-algebra* ⟨*proof*⟩

**theorem** *arden-regexp-l*:
  **assumes** $ro(y) = 0_r\ x \sim y \cdot_r x +_r z$
  **shows** $x \sim y^{\star}{}_r \cdot_r z$
  ⟨*proof*⟩

**theorem** *arden-regexp-r*:
  **assumes** $ro(y) = 0_r\ x \sim x \cdot_r y +_r z$
  **shows** $x \sim z \cdot_r y^{\star}{}_r$
  ⟨*proof*⟩

**end**

# 5 Pratt's Counterexamples

**theory** *Pratts-Counterexamples*
  **imports** *Regular-Algebras*
**begin**

We create two regular algebra models due to Pratt [8] which are used to distinguish K1 algebras from K1l and K1r algebras.

**datatype** *pratt1* =
  *P1Bot* ($\bot_1$) |
  *P1Nat nat* ($[\text{-}]_1$) |
  *P1Infty* ($\infty_1$) |
  *P1Top* ($\top_1$)

**datatype** *pratt2* =
  *P2Bot* ($\bot_2$) |
  *P2Nat nat* ($[\text{-}]_2$) |
  *P2Infty* ($\infty_2$) |
  *P2Top* ($\top_2$)

**fun** *pratt1-max* **where**
  $\mathit{pratt1\text{-}max}\ [x]_1\ [y]_1 = [\mathit{max}\ x\ y]_1$ |
  $\mathit{pratt1\text{-}max}\ x\ \bot_1 = x$ |
  $\mathit{pratt1\text{-}max}\ \bot_1\ y = y$ |
  $\mathit{pratt1\text{-}max}\ \infty_1\ [y]_1 = \infty_1$ |
  $\mathit{pratt1\text{-}max}\ [y]_1\ \infty_1 = \infty_1$ |
  $\mathit{pratt1\text{-}max}\ \infty_1\ \infty_1 = \infty_1$ |

$pratt1\text{-}max\ \top_1\ x = \top_1\ |$
$pratt1\text{-}max\ x\ \top_1 = \top_1$

**fun** *pratt1-plus* :: *pratt1* $\Rightarrow$ *pratt1* $\Rightarrow$ *pratt1* (**infixl** $+_1$ *65*) **where**
$[x]_1 +_1 [y]_1 = [x + y]_1\ |$
$\bot_1 +_1 x = \bot_1\ |$
$x +_1 \bot_1 = \bot_1\ |$
$\infty_1 +_1 [0]_1 = \infty_1\ |$
$[0]_1 +_1 \infty_1 = \infty_1\ |$
$\infty_1 +_1 \infty_1 = \top_1\ |$
$\top_1 +_1 x = \top_1\ |$
$x +_1 \top_1 = \top_1\ |$
$[x]_1 +_1 \infty_1 = \infty_1\ |$
$\infty_1 +_1 x = \top_1$

**lemma** *plusl-top-infty*: $\top_1 +_1 \infty_1 = \top_1$
⟨*proof*⟩

**lemma** *plusl-infty-top*: $\infty_1 +_1 \top_1 = \top_1$
⟨*proof*⟩

**lemma** *plusl-bot-infty*: $\bot_1 +_1 \infty_1 = \bot_1$
⟨*proof*⟩

**lemma** *plusl-infty-bot*: $\infty_1 +_1 \bot_1 = \bot_1$
⟨*proof*⟩

**lemma** *plusl-zero-infty*: $[0]_1 +_1 \infty_1 = \infty_1$
⟨*proof*⟩

**lemma** *plusl-infty-zero*: $\infty_1 +_1 [0]_1 = \infty_1$
⟨*proof*⟩

**lemma** *plusl-infty-num* [*simp*]: $x > 0 \Longrightarrow \infty_1 +_1 [x]_1 = \top_1$
⟨*proof*⟩

**lemma** *plusl-num-infty* [*simp*]: $x > 0 \Longrightarrow [x]_1 +_1 \infty_1 = \infty_1$
⟨*proof*⟩

**fun** *pratt2-max* **where**
$pratt2\text{-}max\ [x]_2\ [y]_2 = [max\ x\ y]_2\ |$
$pratt2\text{-}max\ x\ \bot_2 = x\ |$
$pratt2\text{-}max\ \bot_2\ y = y\ |$
$pratt2\text{-}max\ \infty_2\ [y]_2 = \infty_2\ |$
$pratt2\text{-}max\ [y]_2\ \infty_2 = \infty_2\ |$
$pratt2\text{-}max\ \infty_2\ \infty_2 = \infty_2\ |$
$pratt2\text{-}max\ \top_2\ x = \top_2\ |$
$pratt2\text{-}max\ x\ \top_2 = \top_2$

**fun** *pratt2-plus* :: *pratt2* ⇒ *pratt2* ⇒ *pratt2* (**infixl** $+_2$ *65*) **where**
$[x]_2 +_2 [y]_2 = [x + y]_2$ |
$\bot_2 +_2 x = \bot_2$ |
$x +_2 \bot_2 = \bot_2$ |
$\infty_2 +_2 [0]_2 = \infty_2$ |
$[0]_2 +_2 \infty_2 = \infty_2$ |
$\infty_2 +_2 \infty_2 = \top_2$ |
$\top_2 +_2 x = \top_2$ |
$x +_2 \top_2 = \top_2$ |
$x +_2 \infty_2 = \top_2$ |
$\infty_2 +_2 [x]_2 = \infty_2$

**instantiation** *pratt1* :: *selective-semiring*
**begin**

**definition** *zero-pratt1-def*:
$0 \equiv \bot_1$

**definition** *one-pratt1-def*:
$1 \equiv [0]_1$

**definition** *plus-pratt1-def*:
$x + y \equiv$ *pratt1-max* $x\ y$

**definition** *times-pratt1-def*:
$x * y \equiv x +_1 y$

**definition** *less-eq-pratt1-def*:
$(x::pratt1) \leq y \equiv x + y = y$

**definition** *less-pratt1-def*:
$(x::pratt1) < y \equiv x \leq y \wedge x \neq y$

**instance**
⟨*proof*⟩
**end**

**instance** *pratt1* :: *dioid-one-zero* ⟨*proof*⟩

**instantiation** *pratt2* :: *selective-semiring*
**begin**

**definition** *zero-pratt2-def*:
$0 \equiv \bot_2$

**definition** *one-pratt2-def*:
$1 \equiv [0]_2$

**definition** *times-pratt2-def*:

$x * y \equiv x +_2 y$

**definition** *plus-pratt2* :: *pratt2* $\Rightarrow$ *pratt2* $\Rightarrow$ *pratt2* **where**
*plus-pratt2* $x\ y \equiv$ *pratt2-max* $x\ y$

**definition** *less-eq-pratt2-def*:
$(x::pratt2) \leq y \equiv x + y = y$

**definition** *less-pratt2-def*:
$(x::pratt2) < y \equiv x \leq y \wedge x \neq y$

**instance**
⟨*proof*⟩
**end**

**lemma** *top-greatest*: $x \leq \top_1$
⟨*proof*⟩

**instantiation** *pratt1* :: *star-op*
**begin**

**definition** *star-pratt1* **where**
$x^\star \equiv if\ (x = \bot_1 \vee x = [0]_1)\ then\ [0]_1\ else\ \top_1$
**instance** ⟨*proof*⟩
**end**

**instantiation** *pratt2* :: *star-op*
**begin**
**definition** *star-pratt2* **where**
$x^\star \equiv if\ (x = \bot_2 \vee x = [0]_2)\ then\ [0]_2\ else\ \top_2$
**instance** ⟨*proof*⟩
**end**

**instance** *pratt1* :: *K1r-algebra*
⟨*proof*⟩

**instance** *pratt2* :: *K1l-algebra*
⟨*proof*⟩

**lemma** *one-star-top*: $[1]_1{}^\star = \top_1$
⟨*proof*⟩

**lemma** *pratt1-kozen-1l-counterexample*:
$\exists\ x\ y :: pratt1.\ \neg\ (x \cdot y \leq y \wedge x^\star \cdot y \leq y)$
⟨*proof*⟩

**lemma** *pratt2-kozen-1r-counterexample*:
$\exists\ x\ y :: pratt2.\ \neg\ (y \cdot x \leq y \wedge y \cdot x^\star \leq y)$
⟨*proof*⟩

**end**

# 6 Variants of Regular Algebra

**theory** *Regular-Algebra-Variants*
  **imports** *Regular-Algebras Pratts-Counterexamples*
**begin**

Replacing Kozen's induction axioms by Boffa's leads to incompleteness.

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
  **and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
  **and** $\bigwedge x.\ x \cdot x = x \Longrightarrow x^\star = 1 + x$
  **shows** $\bigwedge x\ y.\ (x + y)^\star = x^\star \cdot (y \cdot x^\star)^\star$

⟨*proof*⟩

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
  **and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
  **and** $\bigwedge x.\ x \cdot x = x \Longrightarrow x^\star = 1 + x$
  **shows** $\bigwedge x\ y.\ x \leq y \longrightarrow x^\star \leq y^\star$

⟨*proof*⟩

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
  **and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
  **and** $\bigwedge x\ y.\ 1 + x \leq y \wedge y \cdot y \leq y \longrightarrow x^\star \leq y$
  **shows** $\bigwedge x.\ 1 + x \leq x^\star$
  ⟨*proof*⟩

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
  **and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
  **and** $\bigwedge x\ y.\ 1 + x \leq y \wedge y \cdot y \leq y \longrightarrow x^\star \leq y$
  **shows** $\bigwedge x.\ x^\star = (1 + x)^\star$
⟨*proof*⟩

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
  **and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
  **and** $\bigwedge x\ y.\ 1 + x + y \cdot y \leq y \Longrightarrow x^\star \leq y$
  **shows** $\bigwedge x.\ x^\star \cdot x^\star \leq x^\star$
⟨*proof*⟩

**lemma** (**in** *star-dioid*)
  **assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$

**and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
**and** $\bigwedge x\ y\ z.\ x \cdot y = y \cdot z \Longrightarrow x^\star \cdot y = y \cdot z^\star$
**shows** $1^\star = 1$

⟨*proof*⟩

**lemma** (**in** *star-dioid*)
**assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
**and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
**and** $\bigwedge x\ y\ z.\ x \cdot y \leq y \cdot z \Longrightarrow x^\star \cdot y \leq y \cdot z^\star$
**and** $\bigwedge x\ y\ z.\ y \cdot x \leq z \cdot y \Longrightarrow y \cdot x^\star \leq z^\star \cdot y$
**shows** $1^\star = 1$

⟨*proof*⟩

**lemma** (**in** *star-dioid*)
**assumes** $\bigwedge x.\ 1 + x \cdot x^\star = x^\star$
**and** $\bigwedge x.\ 1 + x^\star \cdot x = x^\star$
**and** $\bigwedge x\ y.\ x = y \cdot x \Longrightarrow x = y^\star \cdot x$
**and** $\bigwedge x\ y.\ x = x \cdot y \Longrightarrow x = x \cdot y^\star$
**shows** $\bigwedge x\ y.\ y \cdot x \leq y \Longrightarrow y \cdot x^\star \leq y$
⟨*proof*⟩

**class** *C3l-var* = *conway-dioid* +
**assumes** *C3l-var*: $z + x \cdot y \leq y \Longrightarrow x^\star \cdot z \leq y$

**class** *C3r-var* = *conway-dioid* +
**assumes** *C3r-var*: $z + y \cdot x \leq y \Longrightarrow z \cdot x^\star \leq y$

**class** *C3-var* = *C3l-var* + *C3r-var*

**sublocale** *C3l-var* ⊆ *C3l-algebra*
⟨*proof*⟩

**sublocale** *C3l-algebra* ⊆ *C3l-var*
⟨*proof*⟩

**sublocale** *C3-var* ⊆ *C3-algebra*
⟨*proof*⟩

**sublocale** *C3-algebra* ⊆ *C3-var*
⟨*proof*⟩

**class** *Brtc-algebra* = *star-dioid* +
**assumes** *rtc1*: $1 + x^\star \cdot x^\star + x \leq x^\star$
**and** *rtc2*: $1 + x + y \cdot y \leq y \Longrightarrow x^\star \leq y$

**sublocale** *B2-algebra* ⊆ *Brtc-algebra*
⟨*proof*⟩

**sublocale** *Brtc-algebra* $\subseteq$ *B2-algebra*
$\langle proof \rangle$

**class** *wB1-algebra* = *conway-dioid* +
**assumes** *wR*: $x \cdot x \leq x \Longrightarrow x^\star = 1 + x$

**sublocale** *wB1-algebra* $\subseteq$ *B1-algebra*
$\langle proof \rangle$

**lemma** (**in** *B1-algebra*) *one-plus-star*: $x^\star = (1 + x)^\star$
$\langle proof \rangle$

**sublocale** *B1-algebra* $\subseteq$ *wB1-algebra*
$\langle proof \rangle$

**end**

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