The Imperative Refinement Framework

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Abstract

We present the Imperative Refinement Framework (IRF), a tool that supports a stepwise refinement based approach to imperative programs. This entry is based on the material we presented in [ITP-2015, CPP-2016].

It uses the Monadic Refinement Framework as a frontend for the specification of the abstract programs, and Imperative/HOL as a backend to generate executable imperative programs.

The IRF comes with tool support to synthesize imperative programs from more abstract, functional ones, using efficient imperative implementations for the abstract data structures.

This entry also includes the Imperative Isabelle Collection Framework (IICF), which provides a library of re-usable imperative collection data structures.

Moreover, this entry contains a quickstart guide and a reference manual, which provide an introduction to using the IRF for Isabelle/HOL experts. It also provids a collection of (partly commented) practical examples, some highlights being Dijkstra's Algorithm, Nested-DFS, and a generic worklist algorithm with subsumption.

Finally, this entry contains benchmark scripts that compare the runtime of some examples against reference implementations of the algorithms in Java and C++.

[ITP-2015] Peter Lammich: Refinement to Imperative/HOL. ITP 2015: 253–269

[CPP-2016] Peter Lammich: Refinement based verification of imperative data structures. CPP 2016: 27–36

Contents

Chapter 1

The Sepref Tool

This chapter contains the Sepref tool and related tools.

1.1 Operation Identification Phase

theory *Sepref-Id-Op*
imports
  Main
  Automatic-Refinement.Refine-Lib
  Automatic-Refinement.Autoref-Tagging
  Lib/Named-Theorems-Rev
begin

The operation identification phase is adapted from the Autoref tool. The basic idea is to have a type system, which works on so called interface types (also called conceptual types). Each conceptual type denotes an abstract data type, e.g., set, map, priority queue.

Each abstract operation, which must be a constant applied to its arguments, is assigned a conceptual type. Additionally, there is a set of *p*attern rewrite rules, which are applied to subterms before type inference takes place, and which may be backtracked over. This way, encodings of abstract operations in Isabelle/HOL, like λ-. *None* for the empty map, or *fun-upd m k* (*Some v*) for map update, can be rewritten to abstract operations, and get properly typed.

1.1.1 Proper Protection of Term

The following constants are meant to encode abstraction and application as proper HOL-constants, and thus avoid strange effects with HOL's higher-order unification heuristics and automatic beta and eta-contraction.

The first step of operation identification is to protect the term by replacing all function applications and abstractions be the constants defined below.

definition [*simp*]: $PROTECT2\ x\ (y{::}prop) \equiv x$
consts *DUMMY* :: *prop*

abbreviation *PROTECT2-syn* (‹′(#-#′)›) **where** *PROTECT2-syn t* $\equiv$ *PROTECT2 t DUMMY*

abbreviation (*input*)*ABS2* :: $('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**binder** ‹λ_2› *10*)
where *ABS2 f* $\equiv$ ($\lambda x.\ PROTECT2\ (f\ x)\ DUMMY$)

lemma *beta*: $(\lambda_2 x.\ f\ x)\$x \equiv f\ x$ **by** *simp*

Another version of ($). Treated like ($) by our tool. Required to avoid infinite pattern rewriting in some cases, e.g., map-lookup.

definition *APP′* (**infixl** ‹$″› *900*) **where** [*simp*, *autoref-tag-defs*]: $f\$'a \equiv f\ a$

Sometimes, whole terms should be protected from being processed by our tool. For example, our tool should not look into numerals. For this reason, the *PR-CONST* tag indicates terms that our tool shall handle as atomic constants, an never look into them.

The special form *UNPROTECT* can be used inside pattern rewrite rules. It has the effect to revert the protection from its argument, and then wrap it into a *PR-CONST*.

definition [*simp*, *autoref-tag-defs*]: *PR-CONST x* $\equiv$ *x* — Tag to protect constant
definition [*simp*, *autoref-tag-defs*]: *UNPROTECT x* $\equiv$ *x* — Gets converted to *PR-CONST*, after unprotecting its content

1.1.2 Operation Identification

Indicator predicate for conceptual typing of a constant

definition *intf-type* :: $'a \Rightarrow 'b\ itself \Rightarrow bool$ (**infix** ‹$::_i$› *10*) **where**
[*simp*]: $c{::}_i I \equiv True$

lemma *itypeI*: $c{::}_i I$ **by** *simp*
lemma *itypeI′*: *intf-type c TYPE*(′*T*) **by** (*rule itypeI*)

lemma *itype-self*: $(c{::}'a)\ {::}_i\ TYPE('a)$ **by** *simp*

definition *CTYPE-ANNOT* :: $'b \Rightarrow 'a\ itself \Rightarrow 'b$ (**infix** ‹$:::_i$› *10*) **where**
[*simp*]: $c{:::}_i I \equiv c$

Wrapper predicate for an conceptual type inference

definition *ID* :: $'a \Rightarrow 'a \Rightarrow 'c\ itself \Rightarrow bool$
where [*simp*]: $ID\ t\ t'\ T \equiv t{=}t'$

Conceptual Typing Rules

lemma *ID-unfold-vars*: $ID\ x\ y\ T \Longrightarrow x{\equiv}y$ **by** *simp*

lemma *ID-PR-CONST-trigger*: *ID* (*PR-CONST* *x*) *y* *T* $\Longrightarrow$ *ID* (*PR-CONST* *x*) *y* *T* **.**

lemma *pat-rule*:
  $[\![$ $p \equiv p'$; *ID* p' t' *T* $]\!]$ $\Longrightarrow$ *ID* *p* t' *T* **by** *simp*

lemma *app-rule*:
  $[\![$ *ID* *f* f' *TYPE*($'a \Rightarrow 'b$); *ID* *x* x' *TYPE*($'a$)$]\!]$ $\Longrightarrow$ *ID* ($f\$x$) ($f'\$x'$) *TYPE*($'b$)
  by *simp*

lemma *app'-rule*:
  $[\![$ *ID* *f* f' *TYPE*($'a \Rightarrow 'b$); *ID* *x* x' *TYPE*($'a$)$]\!]$ $\Longrightarrow$ *ID* ($f\$'x$) ($f'\$x'$) *TYPE*($'b$)
  by *simp*

lemma *abs-rule*:
  $[\![$ $\bigwedge x\ x'$. *ID* *x* x' *TYPE*($'a$) $\Longrightarrow$ *ID* (*t* *x*) (t' *x* x') *TYPE*($'b$) $]\!]$ $\Longrightarrow$
    ID ($\lambda_2 x.\ t\ x$) ($\lambda_2 x'.\ t'\ x'\ x'$) *TYPE*($'a \Rightarrow 'b$)
  by *simp*

lemma *id-rule*: $c{::}_i I$ $\Longrightarrow$ *ID* *c* *c* *I* **by** *simp*

lemma *annot-rule*: *ID* *t* t' *I* $\Longrightarrow$ *ID* ($t{:::}_i I$) t' *I*
  by *simp*

lemma *fallback-rule*:
  ID ($c{::}'a$) *c* *TYPE*($'c$)
  by *simp*

lemma *unprotect-rl1*: *ID* (*PR-CONST* *x*) *t* *T* $\Longrightarrow$ *ID* (*UNPROTECT* *x*) *t* *T*
  by *simp*

1.1.3 ML-Level code

ML ‹

```
infix 0 THEN-ELSE-COMB'

signature ID-OP-TACTICAL = sig
  val SOLVE-FWD: tactic' −> tactic'
  val DF-SOLVE-FWD: bool −> tactic' −> tactic'
end

structure Id-Op-Tactical :ID-OP-TACTICAL = struct

  fun SOLVE-FWD tac i st = SOLVED' (
    tac
    THEN-ALL-NEW-FWD (SOLVE-FWD tac)) i st


  (∗ Search for solution with DFS−strategy. If dbg−flag is given,
```

```
    return sequence of stuck states if no solution is found.
  *)
  fun DF-SOLVE-FWD dbg tac = let
    val stuck-list-ref = Unsynchronized.ref []

    fun stuck-tac - st = if dbg then (
      stuck-list-ref := st :: !stuck-list-ref;
      Seq.empty
    ) else Seq.empty

    fun rec-tac i st = (
        (tac THEN-ALL-NEW-FWD (SOLVED' rec-tac))
        ORELSE' stuck-tac
      ) i st

    fun fail-tac - - = if dbg then
      Seq.of-list (rev (!stuck-list-ref))
    else Seq.empty
  in
    rec-tac ORELSE' fail-tac
  end

end
›
```

named-theorems-rev *id-rules Operation identification rules*
named-theorems-rev *pat-rules Operation pattern rules*
named-theorems-rev *def-pat-rules Definite operation pattern rules (not back-tracked over)*

ML ‹

```
  structure Id-Op = struct

    fun id-a-conv cnv ct = case Thm.term-of ct of
      @{mpat ID - - -} => Conv.fun-conv (Conv.fun-conv (Conv.arg-conv cnv)) ct
    | - => raise CTERM(id-a-conv,[ct])

    fun
      protect env (@{mpat ?t:::_i ?I}) = let
        val t = protect env t
      in
        @{mk-term env: ?t:::_i ?I}
      end
    | protect - (t as @{mpat PR-CONST -}) = t
    | protect env (t1$t2) = let
```

```
      val t1 = protect env t1
      val t2 = protect env t2
    in
      @{mk-term env: ?t1.0 $ ?t2.0}
    end
  | protect env (Abs (x,T,t)) = let
      val t = protect (T::env) t
    in
      @{mk-term env: λv-x::?'v-T. PROTECT2 ?t DUMMY}
    end
  | protect - t = t

  fun protect-conv ctxt = Refine-Util.f-tac-conv ctxt
    (protect [])
    (fn goal-ctxt =>
         simp-tac (put-simpset HOL-basic-ss goal-ctxt addsimps @{thms PRO-
TECT2-def APP-def}) 1)

  fun unprotect-conv ctxt
    = Simplifier.rewrite (put-simpset HOL-basic-ss ctxt
      addsimps @{thms PROTECT2-def APP-def})

  fun do-unprotect-tac ctxt =
    resolve-tac ctxt @{thms unprotect-rl1} THEN'
   CONVERSION (Refine-Util.HOL-concl-conv (fn ctxt => id-a-conv (unprotect-conv
ctxt)) ctxt)

  val cfg-id-debug =
    Attrib.setup-config-bool @{binding id-debug} (K false)

  val cfg-id-trace-fallback =
    Attrib.setup-config-bool @{binding id-trace-fallback} (K false)

  fun dest-id-rl thm = case Thm.concl-of thm of
    @{mpat (typs) Trueprop (?c::i TYPE(?'v-T))} => (c,T)
  | - => raise THM(dest-id-rl,~1,[thm])


 val add-id-rule = snd oo Thm.proof-attributes [Named-Theorems-Rev.add @{named-theorems-rev
id-rules}]

  datatype id-tac-mode = Init | Step | Normal | Solve

  fun id-tac ss ctxt = let
    open Id-Op-Tactical
    val certT = Thm.ctyp-of ctxt
    val cert = Thm.cterm-of ctxt

    val thy = Proof-Context.theory-of ctxt
```

```
val id-rules = Named-Theorems-Rev.get ctxt @{named-theorems-rev id-rules}
val pat-rules = Named-Theorems-Rev.get ctxt @{named-theorems-rev pat-rules}
val def-pat-rules = Named-Theorems-Rev.get ctxt @{named-theorems-rev
def-pat-rules}

val rl-net = Tactic.build-net (
  (pat-rules |> map (fn thm => thm RS @{thm pat-rule}))
  @ @{thms annot-rule app-rule app'-rule abs-rule}
  @ (id-rules |> map (fn thm => thm RS @{thm id-rule}))
)

val def-rl-net = Tactic.build-net (
  (def-pat-rules |> map (fn thm => thm RS @{thm pat-rule}))
)

val id-pr-const-rename-tac =
    resolve-tac ctxt @{thms ID-PR-CONST-trigger} THEN'
    Subgoal.FOCUS (fn { context=ctxt, prems, ... } =>
      let
        fun is-ID @{mpat Trueprop (ID - - -)} = true | is-ID - = false
        val prems = filter (Thm.prop-of #> is-ID) prems
        val eqs = map (fn thm => thm RS @{thm ID-unfold-vars}) prems
        val conv = Conv.rewrs-conv eqs
        val conv = fn ctxt => (Conv.top-sweep-conv (K conv) ctxt)
        val conv = fn ctxt => Conv.fun2-conv (Conv.arg-conv (conv ctxt))
        val conv = Refine-Util.HOL-concl-conv conv ctxt
      in CONVERSION conv 1 end
    ) ctxt THEN'
    resolve-tac ctxt @{thms id-rule} THEN'
    resolve-tac ctxt id-rules

val ityping = id-rules
  |> map dest-id-rl
  |> filter (is-Const o #1)
  |> map (apfst (#1 o dest-Const))
  |> Symtab.make-list

val has-type = Symtab.defined ityping

fun mk-fallback name cT =
  case try (Sign.the-const-constraint thy) name of
    SOME T => try (Thm.instantiate'
                [SOME (certT cT), SOME (certT T)] [SOME (cert (Const
(name,cT)))])
              @{thm fallback-rule}
  | NONE => NONE

fun trace-fallback thm =
```

```
        Config.get ctxt cfg-id-trace-fallback
        andalso let
          open Pretty
            val p = block [str ID-OP: Applying fallback rule: , Thm.pretty-thm ctxt
thm]
        in
          string-of p |> tracing;
          false
        end

      val fallback-tac = CONVERSION Thm.eta-conversion THEN′ IF-EXGOAL
(fn i => fn st =>
        case Logic.concl-of-goal (Thm.prop-of st) i of
           @{mpat Trueprop (ID (mpaq-STRUCT (mpaq-Const ?name ?cT)) - -)}
=> (
          if not (has-type name) then
            case mk-fallback name cT of
              SOME thm => (trace-fallback thm; resolve-tac ctxt [thm] i st)
            | NONE => Seq.empty
          else Seq.empty
        )
      | - => Seq.empty)

    val init-tac = CONVERSION (
      Refine-Util.HOL-concl-conv (fn ctxt => (id-a-conv (protect-conv ctxt)))
        ctxt
    )

    val step-tac = (FIRST′ [
      assume-tac ctxt,
      eresolve-tac ctxt @{thms id-rule},
      resolve-from-net-tac ctxt def-rl-net,
      resolve-from-net-tac ctxt rl-net,
      id-pr-const-rename-tac,
      do-unprotect-tac ctxt,
      fallback-tac])

    val solve-tac = DF-SOLVE-FWD (Config.get ctxt cfg-id-debug) step-tac

  in
    case ss of
      Init => init-tac
    | Step => step-tac
    | Normal => init-tac THEN′ solve-tac
    | Solve => solve-tac

  end

end
```

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1.1.4 Default Setup

Numerals

lemma *pat-numeral*[*def-pat-rules*]: *numeral*\$*x* ≡ *UNPROTECT* (*numeral*\$*x*) **by** *simp*

lemma *id-nat-const*[*id-rules*]: (*PR-CONST* (*a*::*nat*)) $::_i$ *TYPE*(*nat*) **by** *simp*
lemma *id-int-const*[*id-rules*]: (*PR-CONST* (*a*::*int*)) $::_i$ *TYPE*(*int*) **by** *simp*

end

1.2 Basic Definitions

theory *Sepref-Basic*
imports
  HOL−Eisbach.Eisbach
  Separation-Logic-Imperative-HOL.Sep-Main
  Refine-Monadic.Refine-Monadic
  Lib/Sepref-Misc
  Lib/Structured-Apply
  Sepref-Id-Op
begin
no-notation *i-ANNOT* (**infixr** ‹$:::_i$› *10*)
no-notation *CONST-INTF* (**infixr** ‹$::_i$› *10*)

In this theory, we define the basic concept of refinement from a nondeterministic program specified in the Isabelle Refinement Framework to an imperative deterministic one specified in Imperative/HOL.

1.2.1 Values on Heap

We tag every refinement assertion with the tag *hn-ctxt*, to avoid higher-order unification problems when the refinement assertion is schematic.

definition *hn-ctxt* :: (′*a*⇒′*c*⇒*assn*) ⇒ ′*a* ⇒ ′*c* ⇒ *assn*
  — Tag for refinement assertion
  where
  hn-ctxt P a c ≡ *P a c*

definition *pure* :: (′*b* × ′*a*) *set* ⇒ ′*a* ⇒ ′*b* ⇒ *assn*
  — Pure binding, not involving the heap
  where *pure R* ≡ (λ*a c*. ↑((*c*,*a*)∈*R*))

lemma *pure-app-eq*: *pure R a c* = $\uparrow$((*c*,*a*)$\in$*R*) **by** (*auto simp*: *pure-def*)

lemma *pure-eq-conv*[*simp*]: *pure R* = *pure R'* $\longleftrightarrow$ *R*=*R'*
  unfolding *pure-def*
  apply (*rule iffI*)
  apply *safe*
  apply (*meson pure-assn-eq-conv*)
  apply (*meson pure-assn-eq-conv*)
  done

lemma *pure-rel-eq-false-iff*: *pure R x y* = *false* $\longleftrightarrow$ (*y*,*x*)$\notin$*R*
  by (*auto simp*: *pure-def*)

definition *is-pure P* $\equiv$ $\exists$ *P'*. $\forall$ *x x'*. *P x x'*=$\uparrow$(*P' x x'*)
lemma *is-pureI*[*intro?*]:
  assumes $\bigwedge$*x x'*. *P x x'* = $\uparrow$(*P' x x'*)
  shows *is-pure P*
  using *assms* **unfolding** *is-pure-def* **by** *blast*

lemma *is-pureE*:
  assumes *is-pure P*
  obtains *P'* **where** $\bigwedge$*x x'*. *P x x'* = $\uparrow$(*P' x x'*)
  using *assms* **unfolding** *is-pure-def* **by** *blast*

lemma *pure-pure*[*simp*]: *is-pure* (*pure P*)
  unfolding *pure-def* **by** *rule blast*
lemma *pure-hn-ctxt*[*intro*!]: *is-pure P* $\Longrightarrow$ *is-pure* (*hn-ctxt P*)
  unfolding *hn-ctxt-def*[*abs-def*] **.**

definition *the-pure P* $\equiv$ *THE P'*. $\forall$ *x x'*. *P x x'*=$\uparrow$((*x'*,*x*)$\in$*P'*)

lemma *the-pure-pure*[*simp*]: *the-pure* (*pure R*) = *R*
  unfolding *pure-def the-pure-def*
  by (*rule theI2*[**where** *a*=*R*]) *auto*

lemma *is-pure-alt-def*: *is-pure R* $\longleftrightarrow$ ($\exists$ *Ri*. $\forall$ *x y*. *R x y* = $\uparrow$((*y*,*x*)$\in$*Ri*))
  unfolding *is-pure-def*
  apply *auto*
  apply (*rename-tac P'*)
  apply (*rule-tac x*={(*x*,*y*). *P' y x*} **in** *exI*)
  apply *auto*
  done

lemma *pure-the-pure*[*simp*]: *is-pure R* $\Longrightarrow$ *pure* (*the-pure R*) = *R*
  unfolding *is-pure-alt-def pure-def the-pure-def*
  apply (*intro ext*)
  apply *clarsimp*

apply (*rename-tac a c Ri*)
apply (*rule-tac a=Ri* **in** *theI2*)
apply *auto*
done

lemma *is-pure-conv*: *is-pure* $R \longleftrightarrow (\exists R'. R = pure\ R')$
unfolding *pure-def is-pure-alt-def* **by** *force*

lemma *is-pure-the-pure-id-eq*[*simp*]: *is-pure* $R \Longrightarrow$ *the-pure* $R = Id \longleftrightarrow R{=}pure$ Id
by (*auto simp*: *is-pure-conv*)

lemma *is-pure-iff-pure-assn*: *is-pure* $P = (\forall x\ x'.$ *is-pure-assn* $(P\ x\ x'))$
unfolding *is-pure-def is-pure-assn-def* **by** *metis*

abbreviation *hn-val* $R \equiv$ *hn-ctxt* (*pure* R)

lemma *hn-val-unfold*: *hn-val* $R\ a\ b = \uparrow((b,a){\in}R)$
by (*simp add*: *hn-ctxt-def pure-def*)

definition *invalid-assn* $R\ x\ y \equiv \uparrow(\exists h.\ h{\models}R\ x\ y) * true$

abbreviation *hn-invalid* $R \equiv$ *hn-ctxt* (*invalid-assn* R)

lemma *invalidate-clone*: $R\ x\ y \Longrightarrow_A$ *invalid-assn* $R\ x\ y * R\ x\ y$
apply (*rule entailsI*)
unfolding *invalid-assn-def*
apply (*auto simp*: *models-in-range mod-star-trueI*)
done

lemma *invalidate-clone'*: $R\ x\ y \Longrightarrow_A$ *invalid-assn* $R\ x\ y * R\ x\ y * true$
apply (*rule entailsI*)
unfolding *invalid-assn-def*
apply (*auto simp*: *models-in-range mod-star-trueI*)
done

lemma *invalidate*: $R\ x\ y \Longrightarrow_A$ *invalid-assn* $R\ x\ y$
apply (*rule entailsI*)
unfolding *invalid-assn-def*
apply (*auto simp*: *models-in-range mod-star-trueI*)
done

lemma *invalid-pure-recover*: *invalid-assn* (*pure* R) $x\ y =$ *pure* $R\ x\ y * true$
apply (*rule ent-iffI*)
subgoal
apply (*rule entailsI*)

unfolding *invalid-assn-def*
by (*auto simp*: *pure-def*)
subgoal
unfolding *invalid-assn-def*
by (*auto simp*: *pure-def*)
done

lemma *hn-invalidI*: $h \models hn\text{-}ctxt\ P\ x\ y \Longrightarrow hn\text{-}invalid\ P\ x\ y = true$
apply (*cases h*)
apply (*rule ent-iffI*)
apply (*auto simp*: *invalid-assn-def hn-ctxt-def*)
done

lemma *invalid-assn-cong*[*cong*]:
assumes $x \equiv x'$
assumes $y \equiv y'$
assumes $R\ x'\ y' \equiv R'\ x'\ y'$
shows $invalid\text{-}assn\ R\ x\ y = invalid\text{-}assn\ R'\ x'\ y'$
using *assms* **unfolding** *invalid-assn-def*
by *simp*

1.2.2 Constraints in Refinement Relations

lemma *mod-pure-conv*[*simp*]: $(h,as) \models pure\ R\ a\ b \longleftrightarrow (as=\{\} \wedge (b,a) \in R)$
by (*auto simp*: *pure-def*)

definition $rdomp :: ('a \Rightarrow 'c \Rightarrow assn) \Rightarrow 'a \Rightarrow bool$ **where**
$rdomp\ R\ a \equiv \exists h\ c.\ h \models R\ a\ c$

abbreviation $rdom\ R \equiv Collect\ (rdomp\ R)$

lemma *rdomp-ctxt*[*simp*]: $rdomp\ (hn\text{-}ctxt\ R) = rdomp\ R$
by (*simp add*: *hn-ctxt-def*[*abs-def*])

lemma *rdomp-pure*[*simp*]: $rdomp\ (pure\ R)\ a \longleftrightarrow a \in Range\ R$
unfolding *rdomp-def pure-def* **by** *auto*

lemma *rdom-pure*[*simp*]: $rdom\ (pure\ R) = Range\ R$
unfolding *rdomp-def*[*abs-def*] *pure-def* **by** *auto*

lemma *Range-of-constraint-conv*[*simp*]: $Range\ (A \cap UNIV \times C) = Range\ A \cap C$
by *auto*

1.2.3 Heap-Nres Refinement Calculus

Predicate that expresses refinement. Given a heap Γ, program c produces a heap Γ' and a concrete result that is related with predicate R to some abstract result from m

definition $hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ m \equiv nofail\ m \longrightarrow$

$<\Gamma>\ c\ <\lambda r.\ \Gamma' * (\exists_A x.\ R\ x\ r * \uparrow(RETURN\ x \le m))\ >_t$

simproc-setup *assn-simproc-hnr* (*hn-refine* Γ *c* Γ')
= ‹*K Seplogic-Auto.assn-simproc-fun*›

lemma *hn-refineI*[*intro?*]:
assumes *nofail m*
$\Longrightarrow <\Gamma>\ c\ <\lambda r.\ \Gamma' * (\exists_A x.\ R\ x\ r * \uparrow(RETURN\ x \le m))\ >_t$
shows *hn-refine* $\Gamma\ c\ \Gamma'\ R\ m$
using *assms* **unfolding** *hn-refine-def* **by** *blast*

lemma *hn-refineD*:
assumes *hn-refine* $\Gamma\ c\ \Gamma'\ R\ m$
assumes *nofail m*
shows $<\Gamma>\ c\ <\lambda r.\ \Gamma' * (\exists_A x.\ R\ x\ r * \uparrow(RETURN\ x \le m))\ >_t$
using *assms* **unfolding** *hn-refine-def* **by** *blast*

lemma *hn-refine-preI*:
assumes $\bigwedge h.\ h \models \Gamma \Longrightarrow$ *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
shows *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
using *assms* **unfolding** *hn-refine-def*
by (*auto intro*: *hoare-triple-preI*)

lemma *hn-refine-nofailI*:
assumes *nofail a* $\Longrightarrow$ *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
shows *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
using *assms* **by** (*auto simp*: *hn-refine-def*)

lemma *hn-refine-false*[*simp*]: *hn-refine false* $c\ \Gamma'\ R\ m$
by *rule auto*

lemma *hn-refine-fail*[*simp*]: *hn-refine* $\Gamma\ c\ \Gamma'\ R$ *FAIL*
by *rule auto*

lemma *hn-refine-frame*:
assumes *hn-refine* $P'\ c\ Q'\ R\ m$
assumes $P \Longrightarrow_t F * P'$
shows *hn-refine* $P\ c\ (F * Q')\ R\ m$
using *assms*
unfolding *hn-refine-def entailst-def*
apply *clarsimp*
apply (*erule cons-pre-rule*)
apply (*rule cons-post-rule*)
apply (*erule fi-rule*, *frame-inference*)
apply (*simp only*: *star-aci*)
apply *simp*
**done*

lemma *hn-refine-cons*:
  assumes I: $P \Longrightarrow_t P'$
  assumes R: *hn-refine* P' c Q R m
  assumes I': $Q \Longrightarrow_t Q'$
  assumes R': $\bigwedge x\ y.\ R\ x\ y \Longrightarrow_t R'\ x\ y$
  shows *hn-refine* P c Q' R' m
  using R **unfolding** *hn-refine-def*
  apply *clarify*
  apply (*rule cons-pre-rulet*[*OF I*])
  apply (*rule cons-post-rulet*)
  apply *assumption*
  apply (*sep-auto simp*: *entailst-def*)
  apply (*rule enttD*)
  apply (*intro entt-star-mono* I' R')
  done

lemma *hn-refine-cons-pre*:
  assumes I: $P \Longrightarrow_t P'$
  assumes R: *hn-refine* P' c Q R m
  shows *hn-refine* P c Q R m
  by (*rule hn-refine-cons*[*OF I R*]) *sep-auto+*

lemma *hn-refine-cons-post*:
  assumes R: *hn-refine* P c Q R m
  assumes I: $Q \Longrightarrow_t Q'$
  shows *hn-refine* P c Q' R m
  using *assms*
  by (*rule hn-refine-cons*[*OF entt-refl - - entt-refl*])

lemma *hn-refine-cons-res*:
  $\llbracket$ *hn-refine* Γ f Γ' R g; $\bigwedge a\ c.\ R\ a\ c \Longrightarrow_t R'\ a\ c$ $\rrbracket \Longrightarrow$ *hn-refine* Γ f Γ' R' g
  by (*erule hn-refine-cons*[*OF entt-refl*]) *sep-auto+*

lemma *hn-refine-ref*:
  assumes LE: $m \le m'$
  assumes R: *hn-refine* P c Q R m
  shows *hn-refine* P c Q R m'
  apply *rule*
  apply (*rule cons-post-rule*)
  apply (*rule hn-refineD*[*OF R*])
  using LE **apply** (*simp add*: *pw-le-iff*)
  apply (*sep-auto intro*: *order-trans*[*OF - LE*])
  done

lemma *hn-refine-cons-complete*:
  assumes I: $P \Longrightarrow_t P'$
  assumes R: *hn-refine* P' c Q R m
  assumes I': $Q \Longrightarrow_t Q'$

assumes R': $\bigwedge x\ y.\ R\ x\ y \Longrightarrow_t R'\ x\ y$
assumes LE: $m \le m'$
shows *hn-refine* $P\ c\ Q'\ R'\ m'$
apply (*rule hn-refine-ref*[*OF LE*])
apply (*rule hn-refine-cons*[*OF I R I' R'*])
done

lemma *hn-refine-augment-res*:
assumes A: *hn-refine* $\Gamma\ f\ \Gamma'\ R\ g$
assumes B: $g \le_n$ *SPEC* Φ
shows *hn-refine* $\Gamma\ f\ \Gamma'\ (\lambda a\ c.\ R\ a\ c * \uparrow(\Phi\ a))\ g$
apply (*rule hn-refineI*)
apply (*rule cons-post-rule*)
apply (*erule A*[*THEN hn-refineD*])
using B
apply (*sep-auto simp*: *pw-le-iff pw-leof-iff*)
done

1.2.4 Product Types

Some notion for product types is already defined here, as it is used for currying and uncurrying, which is fundamental for the sepref tool

definition *prod-assn* :: $('a1 \Rightarrow 'c1 \Rightarrow assn) \Rightarrow ('a2 \Rightarrow 'c2 \Rightarrow assn)$
$\Rightarrow 'a1*'a2 \Rightarrow 'c1*'c2 \Rightarrow assn$ **where**
prod-assn $P1\ P2\ a\ c \equiv$ *case* (a,c) *of* $((a1,a2),(c1,c2)) \Rightarrow$
$P1\ a1\ c1 * P2\ a2\ c2$

notation *prod-assn* (**infixr** ‹$\times_a$› *70*)

lemma *prod-assn-pure-conv*[*simp*]: *prod-assn* (*pure R1*) (*pure R2*) = *pure* ($R1 \times_r R2$)
by (*auto simp*: *pure-def prod-assn-def intro*!: *ext*)

lemma *prod-assn-pair-conv*[*simp*]:
prod-assn $A\ B\ (a1,b1)\ (a2,b2) = A\ a1\ a2 * B\ b1\ b2$
unfolding *prod-assn-def* **by** *auto*

lemma *prod-assn-true*[*simp*]: *prod-assn* (λ- -. *true*) (λ- -. *true*) = (λ- -. *true*)
by (*auto intro*!: *ext simp*: *hn-ctxt-def prod-assn-def*)

1.2.5 Convenience Lemmas

lemma *hn-refine-guessI*:
assumes *hn-refine* $P\ f\ P'\ R\ f'$
assumes f=*f-conc*
shows *hn-refine* P *f-conc* $P'\ R\ f'$
— To prove a refinement, first synthesize one, and then prove equality
using *assms* **by** *simp*

lemma *imp-correctI*:
assumes R: *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
assumes C: $a \leq SPEC\ \Phi$
shows $<\Gamma>\ c\ <\lambda r'.\ \exists_A r.\ \Gamma' * R\ r\ r' * \uparrow(\Phi\ r)>_t$
apply (*rule cons-post-rule*)
apply (*rule hn-refineD*[*OF R*])
apply (*rule le-RES-nofailI*[*OF C*])
apply (*sep-auto dest*: *order-trans*[*OF - C*])
done

lemma *hnr-pre-ex-conv*:
shows *hn-refine* $(\exists_A x.\ \Gamma\ x)\ c\ \Gamma'\ R\ a \longleftrightarrow (\forall x.\ hn\text{-}refine\ (\Gamma\ x)\ c\ \Gamma'\ R\ a)$
unfolding *hn-refine-def*
apply *safe*
apply (*erule cons-pre-rule*[*rotated*])
apply (*rule ent-ex-postI*)
apply (*rule ent-refl*)
apply *sep-auto*
done

lemma *hnr-pre-pure-conv*:
shows *hn-refine* $(\Gamma * \uparrow P)\ c\ \Gamma'\ R\ a \longleftrightarrow (P \longrightarrow hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ a)$
unfolding *hn-refine-def*
by *auto*

lemma *hn-refine-split-post*:
assumes *hn-refine* $\Gamma\ c\ \Gamma'\ R\ a$
shows *hn-refine* $\Gamma\ c\ (\Gamma' \vee_A \Gamma'')\ R\ a$
apply (*rule hn-refine-cons-post*[*OF assms*])
by (*rule entt-disjI1-direct*)

lemma *hn-refine-post-other*:
assumes *hn-refine* $\Gamma\ c\ \Gamma''\ R\ a$
shows *hn-refine* $\Gamma\ c\ (\Gamma' \vee_A \Gamma'')\ R\ a$
apply (*rule hn-refine-cons-post*[*OF assms*])
by (*rule entt-disjI2-direct*)

Return

lemma *hnr-RETURN-pass*:
hn-refine (*hn-ctxt R x p*) (*return p*) (*hn-invalid R x p*) *R* (*RETURN x*)
— Pass on a value from the heap as return value
apply *rule*
apply (*sep-auto simp*: *hn-ctxt-def eintros*: *invalidate-clone'*)
done

lemma *hnr-RETURN-pure*:
assumes $(c,a) \in R$

shows *hn-refine emp* (*return c*) *emp* (*pure R*) (*RETURN a*)
— Return pure value
unfolding *hn-refine-def* **using** *assms*
by (*sep-auto simp*: *pure-def*)

Assertion

lemma *hnr-FAIL*[*simp*, *intro*!]: *hn-refine* Γ *c* Γ' *R FAIL*
unfolding *hn-refine-def*
by *simp*

lemma *hnr-ASSERT*:
assumes $\Phi \Longrightarrow$ *hn-refine* Γ *c* Γ' *R* c'
shows *hn-refine* Γ *c* Γ' *R* (*do* { *ASSERT* Φ; c'})
using *assms*
apply (*cases* Φ)
by *auto*

Bind

lemma *bind-det-aux*: $[\![$ *RETURN* $x \leq m$; *RETURN* $y \leq f\ x$ $]\!] \Longrightarrow$ *RETURN* $y \leq m \gg\!= f$
apply (*rule order-trans*[*rotated*])
apply (*rule Refine-Basic.bind-mono*)
apply *assumption*
apply (*rule order-refl*)
apply *simp*
done

lemma *hnr-bind*:
assumes *D1*: *hn-refine* Γ m' $\Gamma 1$ *Rh m*
assumes *D2*:
$\bigwedge x\ x'$. *RETURN* $x \leq m \Longrightarrow$ *hn-refine* ($\Gamma 1$ $*$ *hn-ctxt Rh* $x\ x'$) ($f'\ x'$) ($\Gamma 2\ x\ x'$) *R* ($f\ x$)
assumes *IMP*: $\bigwedge x\ x'$. $\Gamma 2\ x\ x' \Longrightarrow_t \Gamma'$ $*$ *hn-ctxt Rx* $x\ x'$
shows *hn-refine* Γ ($m' \gg\!= f'$) Γ' *R* ($m \gg\!= f$)
using *assms*
unfolding *hn-refine-def*
apply (*clarsimp simp add*: *pw-bind-nofail*)
apply (*rule Hoare-Triple.bind-rule*)
apply *assumption*
apply (*clarsimp intro*!: *normalize-rules simp*: *hn-ctxt-def*)
proof −
fix $x'\ x$
assume *1*: *RETURN* $x \leq m$
and *nofail m* $\forall x$. *inres m x* $\longrightarrow$ *nofail* ($f\ x$)
hence *nofail* ($f\ x$) **by** (*auto simp*: *pw-le-iff*)
moreover assume $\bigwedge x\ x'$. *RETURN* $x \leq m \Longrightarrow$
nofail ($f\ x$) $\longrightarrow$ <$\Gamma 1$ $*$ *Rh* $x\ x'$> $f'\ x'$
<$\lambda r'$. $\exists_A r$. $\Gamma 2\ x\ x'$ $*$ *R r r'* $*$ *true* $*$ $\uparrow$ (*RETURN* $r \leq f\ x$)>

ultimately have $\bigwedge x'.\ <\Gamma 1 * Rh\ x\ x'>\ f'\ x'$
$<\lambda r'.\ \exists_A r.\ \Gamma 2\ x\ x' * R\ r\ r' * true * \uparrow (RETURN\ r \le f\ x)>$
using *1* **by** *simp*
also have $\bigwedge r'.\ \exists_A r.\ \Gamma 2\ x\ x' * R\ r\ r' * true * \uparrow (RETURN\ r \le f\ x) \Longrightarrow_A$
$\exists_A r.\ \Gamma' * R\ r\ r' * true * \uparrow (RETURN\ r \le f\ x)$
apply (*sep-auto*)
apply (*rule ent-frame-fwd*[*OF IMP*[*THEN enttD*]])
apply *frame-inference*
apply (*solve-entails*)
done
finally (*cons-post-rule*) **have**
R: $<\Gamma 1 * Rh\ x\ x'>\ f'\ x'$
$<\lambda r'.\ \exists_A r.\ \Gamma' * R\ r\ r' * true * \uparrow(RETURN\ r \le f\ x)>$
.
show $<\Gamma 1 * Rh\ x\ x' * true>\ f'\ x'$
$<\lambda r'.\ \exists_A r.\ \Gamma' * R\ r\ r' * true * \uparrow (RETURN\ r \le m \gg= f)>$
by (*sep-auto heap*: *R intro*: *bind-det-aux*[*OF 1*])
qed

Recursion

definition *hn-rel P m* $\equiv \lambda r.\ \exists_A x.\ P\ x\ r * \uparrow(RETURN\ x \le m)$

lemma *hn-refine-alt*: *hn-refine Fpre c Fpost P m* $\equiv$ *nofail m* $\longrightarrow$
$<Fpre>\ c\ <\lambda r.\ hn\text{-}rel\ P\ m\ r * Fpost>_t$
apply (*rule eq-reflection*)
unfolding *hn-refine-def hn-rel-def*
apply (*simp add*: *hn-ctxt-def*)
apply (*simp only*: *star-aci*)
done

lemma *wit-swap-forall*:
assumes *W*: $<P>\ c\ <\lambda\text{-}.\ true>$
assumes *T*: $(\forall x.\ A\ x \longrightarrow <P>\ c\ <Q\ x>)$
shows $<P>\ c\ <\lambda r.\ \neg_A (\exists_A x.\ \uparrow(A\ x) * \neg_A\ Q\ x\ r)>$
unfolding *hoare-triple-def Let-def*
apply (*intro conjI impI allI*)
subgoal by (*elim conjE*) (*rule hoare-tripleD*[*OF W*], *assumption+*) []

subgoal
apply (*clarsimp*, *intro conjI allI*)
apply1 (*rule models-in-range*)
applyS (*rule hoare-tripleD*[*OF W*]; *assumption*; *fail*)
apply1 (*simp only*: *disj-not2*, *intro impI*)
apply1 (*drule spec*[*OF T*, *THEN mp*])
apply1 (*drule* (*2*) *hoare-tripleD*(*2*))
by *assumption*

subgoal by (*elim conjE*) (*rule hoare-tripleD*[*OF W*], *assumption+*)

subgoal by (*elim conjE*) (*rule hoare-tripleD*[*OF W*], *assumption+*)
done

lemma *hn-admissible*:
assumes *PREC*: *precise Ry*
assumes *E*: $\forall f{\in}A.\ nofail\ (f\,x) \longrightarrow {<}P{>}\ c\ {<}\lambda r.\ hn\text{-}rel\ Ry\ (f\,x)\ r * F{>}$
assumes *NF*: $nofail\ (INF\ f{\in}A.\ f\,x)$
shows ${<}P{>}\ c\ {<}\lambda r.\ hn\text{-}rel\ Ry\ (INF\ f{\in}A.\ f\,x)\ r * F{>}$
proof −
from *NF* **obtain** *f* **where** $f{\in}A$ **and** $nofail\ (f\,x)$
by (*simp only*: *refine-pw-simps*) *blast*

with *E* **have** ${<}P{>}\ c\ {<}\lambda r.\ hn\text{-}rel\ Ry\ (f\,x)\ r * F{>}$ **by** *blast*
hence *W*: ${<}P{>}\ c\ {<}\lambda\text{-}.\ true{>}$ **by** (*rule cons-post-rule*, *simp*)

from *E* **have**
E': $\forall f.\ f{\in}A \wedge nofail\ (f\,x) \longrightarrow {<}P{>}\ c\ {<}\lambda r.\ hn\text{-}rel\ Ry\ (f\,x)\ r * F{>}$
by *blast*
from *wit-swap-forall*[*OF W E'*] **have**
E'': ${<}P{>}\ c$
${<}\lambda r.\ \neg_A\ (\exists_A xa.\ \uparrow (xa \in A \wedge nofail\ (xa\ x)) *$
$\neg_A\ (hn\text{-}rel\ Ry\ (xa\ x)\ r * F)){>}$ **.**

thus *?thesis*
apply (*rule cons-post-rule*)
unfolding *entails-def hn-rel-def*
apply *clarsimp*
proof −
fix *h as p*
assume *A*: $\forall f.\ f{\in}A \longrightarrow (\exists a.$
$((h,\ as) \models Ry\ a\ p * F \wedge RETURN\ a \le f\,x)) \vee \neg\ nofail\ (f\,x)$
with ‹$f{\in}A$› **and** ‹$nofail\ (f\,x)$› **obtain** *a* **where**
1: $(h,\ as) \models Ry\ a\ p * F$ **and** $RETURN\ a \le f\,x$
by *blast*
have
$\forall f{\in}A.\ nofail\ (f\,x) \longrightarrow (h,\ as) \models Ry\ a\ p * F \wedge RETURN\ a \le f\,x$
proof *clarsimp*
fix f'
assume $f'{\in}A$ **and** $nofail\ (f'\ x)$
with *A* **obtain** a' **where**
2: $(h,\ as) \models Ry\ a'\ p * F$ **and** $RETURN\ a' \le f'\ x$
by *blast*

moreover note *preciseD'*[*OF PREC 1 2*]
ultimately show $(h,\ as) \models Ry\ a\ p * F \wedge RETURN\ a \le f'\ x$ **by** *simp*
qed
hence $RETURN\ a \le (INF\ f{\in}A.\ f\,x)$
by (*metis* (*mono-tags*) *le-INF-iff le-nofailI*)

with *1* **show** ∃ *a*. (*h*, *as*) ⊨ *Ry a p* ∗ *F* ∧ *RETURN a* ≤ (*INF f*∈*A*. *f x*)
by *blast*
qed
qed

lemma *hn-admissible′*:
assumes *PREC*: *precise Ry*
assumes *E*: ∀ *f*∈*A*. *nofail* (*f x*) ⟶ <*P*> *c* <λ*r*. *hn-rel Ry* (*f x*) *r* ∗ *F*>$_t$
assumes *NF*: *nofail* (*INF f*∈*A*. *f x*)
shows <*P*> *c* <λ*r*. *hn-rel Ry* (*INF f*∈*A*. *f x*) *r* ∗ *F*>$_t$
apply (*rule hn-admissible*[*OF PREC*, **where** *F*=*F*∗*true*, *simplified*])
apply *simp*
by *fact*+

lemma *hnr-RECT-old*:
assumes *S*: ⋀*cf af ax px*. ⟦
⋀*ax px*. *hn-refine* (*hn-ctxt Rx ax px* ∗ *F*) (*cf px*) (*F′ ax px*) *Ry* (*af ax*)⟧
⟹ *hn-refine* (*hn-ctxt Rx ax px* ∗ *F*) (*cB cf px*) (*F′ ax px*) *Ry* (*aB af ax*)
assumes *M*: (⋀*x*. *mono-Heap* (λ*f*. *cB f x*))
assumes *PREC*: *precise Ry*
shows *hn-refine*
(*hn-ctxt Rx ax px* ∗ *F*) (*heap.fixp-fun cB px*) (*F′ ax px*) *Ry* (*RECT aB ax*)
unfolding *RECT-gfp-def*
proof (*simp*, *intro conjI impI*)
assume *trimono aB*
hence *mono aB* **by** (*simp add*: *trimonoD*)
have ∀ *ax px*.
hn-refine (*hn-ctxt Rx ax px* ∗ *F*) (*heap.fixp-fun cB px*) (*F′ ax px*) *Ry*
(*gfp aB ax*)
apply (*rule gfp-cadm-induct*[*OF - -* ‹*mono aB*›])

apply *rule*
apply (*auto simp*: *hn-refine-alt intro*: *hn-admissible′*[*OF PREC*]) []

apply (*auto simp*: *hn-refine-alt*) []

apply *clarsimp*
apply (*subst heap.mono-body-fixp*[*of cB*, *OF M*])
apply (*rule S*)
apply *blast*
done
thus *hn-refine* (*hn-ctxt Rx ax px* ∗ *F*)
(*ccpo.fixp* (*fun-lub Heap-lub*) (*fun-ord Heap-ord*) *cB px*) (*F′ ax px*) *Ry*
(*gfp aB ax*) **by** *simp*
qed

lemma *hnr-RECT*:
assumes *S*: ⋀*cf af ax px*. ⟦
⋀*ax px*. *hn-refine* (*hn-ctxt Rx ax px* ∗ *F*) (*cf px*) (*F′ ax px*) *Ry* (*af ax*)⟧

$\Longrightarrow$ *hn-refine* (*hn-ctxt Rx ax px* $*$ *F*) (*cB cf px*) (*F′ ax px*) *Ry* (*aB af ax*)
assumes *M*: ($\bigwedge$*x. mono-Heap* (λ*f. cB f x*))
shows *hn-refine*
(*hn-ctxt Rx ax px* $*$ *F*) (*heap.fixp-fun cB px*) (*F′ ax px*) *Ry* (*RECT aB ax*)
unfolding *RECT-def*
proof (*simp*, *intro conjI impI*)
assume *trimono aB*
hence *flatf-mono-ge aB* **by** (*simp add*: *trimonoD*)
have ∀ *ax px*.
hn-refine (*hn-ctxt Rx ax px* $*$ *F*) (*heap.fixp-fun cB px*) (*F′ ax px*) *Ry*
(*flatf-gfp aB ax*)

apply (*rule flatf-ord.fixp-induct*[*OF* - ‹*flatf-mono-ge aB*›])

apply (*rule flatf-admissible-pointwise*)
apply *simp*

apply (*auto simp*: *hn-refine-alt*) []

apply *clarsimp*
apply (*subst heap.mono-body-fixp*[*of cB*, *OF M*])
apply (*rule S*)
apply *blast*
done
thus *hn-refine* (*hn-ctxt Rx ax px* $*$ *F*)
(*ccpo.fixp* (*fun-lub Heap-lub*) (*fun-ord Heap-ord*) *cB px*) (*F′ ax px*) *Ry*
(*flatf-gfp aB ax*) **by** *simp*
qed

lemma *hnr-If*:
assumes *P*: $\Gamma \Longrightarrow_t \Gamma 1 * $ *hn-val bool-rel a a′*
assumes *RT*: *a* $\Longrightarrow$ *hn-refine* (Γ*1* $*$ *hn-val bool-rel a a′*) *b′* Γ*2b R b*
assumes *RE*: $\neg$*a* $\Longrightarrow$ *hn-refine* (Γ*1* $*$ *hn-val bool-rel a a′*) *c′* Γ*2c R c*
assumes *IMP*: $\Gamma 2b \vee_A \Gamma 2c \Longrightarrow_t \Gamma'$
shows *hn-refine* Γ (*if a′ then b′ else c′*) Γ' *R* (*if a then b else c*)
apply (*rule hn-refine-cons*[*OF P*])
apply1 (*rule hn-refine-preI*)
applyF (*cases a*; *simp add*: *hn-ctxt-def pure-def*)
focus
apply1 (*rule hn-refine-split-post*)
applyF (*rule hn-refine-cons-pre*[*OF* - *RT*])
applyS (*simp add*: *hn-ctxt-def pure-def*)
applyS *simp*
solved
solved
apply1 (*rule hn-refine-post-other*)
applyF (*rule hn-refine-cons-pre*[*OF* - *RE*])
applyS (*simp add*: *hn-ctxt-def pure-def*)
applyS *simp*

solved
solved
applyS (*rule IMP*)
applyS (*rule entt-refl*)
done

1.2.6 ML-Level Utilities

ML ‹
signature SEPREF-BASIC = sig
(∗ *Destroy lambda term, return function to reconstruct. Bound var is replaced by free.* ∗)
val dest-lambda-rc: *Proof.context* −> *term* −> ((*term* ∗ (*term* −> *term*)) ∗ *Proof.context*)
(∗ *Apply function under lambda. Bound var is replaced by free.* ∗)
val apply-under-lambda: (*Proof.context* −> *term* −> *term*) −> *Proof.context* −> *term* −> *term*

(∗ *'a nres type* ∗)
val is-nresT: *typ* −> *bool*
val mk-nresT: *typ* −> *typ*
val dest-nresT: *typ* −> *typ*

(∗ *Make certified* == ∗)
val mk-cequals: *cterm* ∗ *cterm* −> *cterm*
(∗ *Make* $\Longrightarrow_A$ ∗)
val mk-entails: *term* ∗ *term* −> *term*

(∗ *Operations on pre−terms* ∗)
val constrain-type-pre: *typ* −> *term* −> *term* (∗ *t*::*T* ∗)

val mk-pair-in-pre: *term* −> *term* −> *term* −> *term* (∗ (*c*,*a*) ∈ *R* ∗)

val mk-compN-pre: *int* −> *term* −> *term* −> *term* (∗ *f o*...*o g*∗)

val mk-curry0-pre: *term* −> *term* (∗ *curry0 f* ∗)
val mk-curry-pre: *term* −> *term* (∗ *curry f* ∗)
val mk-curryN-pre: *int* −> *term* −> *term* (∗ *curry* (...(*curry f*)...) ∗)

val mk-uncurry0-pre: *term* −> *term* (∗ *uncurry0 f* ∗)
val mk-uncurry-pre: *term* −> *term* (∗ *uncurry f* ∗)
val mk-uncurryN-pre: *int* −> *term* −> *term* (∗ *uncurry* (...(*uncurry f*)...) ∗)

(∗ *Conversion for hn-refine* − *term*∗)
val hn-refine-conv: *conv* −> *conv* −> *conv* −> *conv* −> *conv* −> *conv*

```
(* Conversion on abstract value (last argument) of hn-refine - term *)
val hn-refine-conv-a: conv -> conv

(* Conversion on abstract value of hn-refine term in conclusion of theorem *)
val hn-refine-concl-conv-a: (Proof.context -> conv) -> Proof.context -> conv

(* Destruct hn-refine term *)
val dest-hn-refine: term -> term * term * term * term * term
(* Make hn-refine term *)
val mk-hn-refine: term * term * term * term * term -> term
(* Check if given term is Trueprop (hn-refine ...). Use with CONCL-COND'. *)
val is-hn-refine-concl: term -> bool

(* Destruct abs-fun, returns RETURN-flag, (f, args) *)
val dest-hnr-absfun: term -> bool * (term * term list)
(* Make abs-fun. *)
val mk-hnr-absfun: bool * (term * term list) -> term
(* Make abs-fun. Guess RETURN-flag from type. *)
val mk-hnr-absfun': (term * term list) -> term

(* Prove permutation of *. To be used with f-tac-conv. *)
val star-permute-tac: Proof.context -> tactic

(* Make separation conjunction *)
val mk-star: term * term -> term
(* Make separation conjunction from list. [] yields emp. *)
val list-star: term list -> term
(* Decompose separation conjunction. emp yields []. *)
val strip-star: term -> term list

(* Check if true-assertion *)
val is-true: term -> bool

(* Check if term is hn-ctxt-assertion *)
val is-hn-ctxt: term -> bool
(* Decompose hn-ctxt-assertion *)
val dest-hn-ctxt: term -> term * term * term
(* Decompose hn-ctxt-assertion, NONE if term has wrong format *)
val dest-hn-ctxt-opt: term -> (term * term * term) option


type phases-ctrl = {
  trace: bool,            (* Trace phases *)
  int-res: bool,          (* Stop with intermediate result *)
  start: string option,   (* Start with this phase. NONE: First phase *)
  stop: string option     (* Stop after this phase. NONE: Last phase *)
}
```

```
(* No tracing or intermediate result, all phases *)
val dflt-phases-ctrl: phases-ctrl
(* Tracing, intermediate result, all phases *)
val dbg-phases-ctrl: phases-ctrl
val flag-phases-ctrl: bool -> phases-ctrl

(* Name, tactic, expected number of created goals (may be negative for solved
goals) *)
type phase = string * (Proof.context -> tactic') * int

(* Perform sequence of tactics (tac,n), each expected to create n new goals,
   or solve goals if n is negative.
   Debug-flag: Stop with intermediate state after tactic
   fails or produces less/more goals as expected. *)
val PHASES': phase list -> phases-ctrl -> Proof.context -> tactic'

end

structure Sepref-Basic: SEPREF-BASIC = struct

  fun is-nresT (Type (@{type-name nres},[-])) = true | is-nresT - = false
  fun mk-nresT T = Type(@{type-name nres},[T])
  fun dest-nresT (Type (@{type-name nres},[T])) = T | dest-nresT T = raise
TYPE(dest-nresT,[T],[])


  fun dest-lambda-rc ctxt (Abs (x,T,t)) = let
      val (u,ctxt) = yield-singleton Variable.variant-fixes x ctxt
      val u = Free (u,T)
      val t = subst-bound (u,t)
      val reconstruct = Term.lambda-name (x,u)
    in
      ((t,reconstruct),ctxt)
    end
  | dest-lambda-rc - t = raise TERM(dest-lambda-rc,[t])

  fun apply-under-lambda f ctxt t = let
    val ((t,rc),ctxt) = dest-lambda-rc ctxt t
    val t = f ctxt t
  in
    rc t
  end


  (* Functions on pre-terms *)
  fun mk-pair-in-pre x y r = Const (@{const-name Set.member}, dummyT) $
    (Const (@{const-name Product-Type.Pair}, dummyT) $ x $ y) $ r
```

```
fun mk-uncurry-pre t = Const(@{const-name uncurry}, dummyT)$t
fun mk-uncurry0-pre t = Const(@{const-name uncurry0}, dummyT)$t
fun mk-uncurryN-pre 0 = mk-uncurry0-pre
  | mk-uncurryN-pre 1 = I
  | mk-uncurryN-pre n = mk-uncurry-pre o mk-uncurryN-pre (n−1)

fun mk-curry-pre t = Const(@{const-name curry}, dummyT)$t
fun mk-curry0-pre t = Const(@{const-name curry0}, dummyT)$t
fun mk-curryN-pre 0 = mk-curry0-pre
  | mk-curryN-pre 1 = I
  | mk-curryN-pre n = mk-curry-pre o mk-curryN-pre (n−1)


fun mk-compN-pre 0 f g = f $ g
  | mk-compN-pre n f g = let
      val g = fold (fn i => fn t => t$Bound i) (n−2 downto 0) g
      val t = Const(@{const-name Fun.comp},dummyT) $ f $ g

       val t = fold (fn i => fn t => Abs (x^string-of-int i,dummyT,t)) (n−1
downto 1) t
    in
      t
    end

  fun constrain-type-pre T t = Const(@{syntax-const -type-constraint-},T−−>T)
$ t



local open Conv in
  fun hn-refine-conv c1 c2 c3 c4 c5 ct = case Thm.term-of ct of
    @{mpat hn-refine - - - - -} => let
      val cc = combination-conv
    in
      cc (cc (cc (cc (cc all-conv c1) c2) c3) c4) c5 ct
    end
  | - => raise CTERM (hn-refine-conv,[ct])

  val hn-refine-conv-a = hn-refine-conv all-conv all-conv all-conv all-conv

  fun hn-refine-concl-conv-a conv ctxt = Refine-Util.HOL-concl-conv
    (fn ctxt => hn-refine-conv-a (conv ctxt)) ctxt

end

(∗ FIXME: Strange dependency! ∗)
val mk-cequals = uncurry SMT-Util.mk-cequals
```

```
val mk-entails = HOLogic.mk-binrel @{const-name entails}

val mk-star = HOLogic.mk-binop @{const-name Groups.times-class.times}

fun list-star [] = @{term emp::assn}
  | list-star [a] = a
  | list-star (a::l) = mk-star (list-star l,a)

fun strip-star @{mpat ?a∗?b} = strip-star a @ strip-star b
  | strip-star @{mpat emp} = []
  | strip-star t = [t]

fun is-true @{mpat true} = true | is-true - = false

fun is-hn-ctxt @{mpat hn-ctxt - - -} = true | is-hn-ctxt - = false
fun dest-hn-ctxt @{mpat hn-ctxt ?R ?a ?p} = (R,a,p)
  | dest-hn-ctxt t = raise TERM(dest-hn-ctxt,[t])

fun dest-hn-ctxt-opt @{mpat hn-ctxt ?R ?a ?p} = SOME (R,a,p)
  | dest-hn-ctxt-opt - = NONE

fun strip-abs-args (t as @{mpat PR-CONST -}) = (t,[])
    | strip-abs-args @{mpat ?f$?a} = (case strip-abs-args f of (f,args) =>
(f,args@[a]))
  | strip-abs-args t = (t,[])

fun dest-hnr-absfun @{mpat RETURN$?a} = (true, strip-abs-args a)
  | dest-hnr-absfun f = (false, strip-abs-args f)

 fun mk-hnr-absfun (true,fa) = Autoref-Tagging.list-APP fa |> (fn a => @{mk-term
RETURN$?a})
  | mk-hnr-absfun (false,fa) = Autoref-Tagging.list-APP fa

fun mk-hnr-absfun' fa = let
  val t = Autoref-Tagging.list-APP fa
  val T = fastype-of t
in
  case T of
    Type (@{type-name nres},-) => t
  | - => @{mk-term RETURN$?t}

end

fun dest-hn-refine @{mpat hn-refine ?P ?c ?Q ?R ?a} = (P,c,Q,R,a)
  | dest-hn-refine t = raise TERM(dest-hn-refine,[t])

fun mk-hn-refine (P,c,Q,R,a) = @{mk-term hn-refine ?P ?c ?Q ?R ?a}

val is-hn-refine-concl = can (HOLogic.dest-Trueprop #> dest-hn-refine)
```

fun star-permute-tac ctxt = ALLGOALS (simp-tac (put-simpset HOL-basic-ss ctxt addsimps @{thms star-aci}))

```
type phases-ctrl = {
  trace: bool,
  int-res: bool,
  start: string option,
  stop: string option
}

val dflt-phases-ctrl = {trace=false,int-res=false,start=NONE,stop=NONE}
val dbg-phases-ctrl = {trace=true,int-res=true,start=NONE,stop=NONE}
fun flag-phases-ctrl dbg = if dbg then dbg-phases-ctrl else dflt-phases-ctrl

type phase = string * (Proof.context -> tactic') * int

local
  fun ph-range phases start stop = let
    fun find-phase name = let
      val i = find-index (fn (n,-,-) => n=name) phases
      val - = if i<0 then error (No such phase: ^name) else ()
    in
      i
    end

    val i = case start of NONE => 0 | SOME n => find-phase n
    val j = case stop of NONE => length phases − 1 | SOME n => find-phase
n

    val phases = take (j+1) phases |> drop i

    val - = case phases of [] => error No phases selected, range is empty | - =>
()
  in
    phases
  end
in

  fun PHASES' phases ctrl ctxt = let
    val phases = ph-range phases (#start ctrl) (#stop ctrl)
    val phases = map (fn (n,tac,d) => (n,tac ctxt,d)) phases

    fun r [] - st = Seq.single st
      | r ((name,tac,d)::tacs) i st = let
          val n = Thm.nprems-of st
          val bailout-tac = if #int-res ctrl then all-tac else no-tac
        fun trace-tac msg st = (if #trace ctrl then tracing msg else (); Seq.single
```

```
st)
            val trace-start-tac = trace-tac (Phase ^ name)
          in
            K trace-start-tac THEN' IF-EXGOAL (tac)
            THEN-ELSE' (
              fn i => fn st =>
                  (* Bail out if a phase does not solve/create exactly the expected
subgoals *)
                if Thm.nprems-of st = n+d then
                  ((trace-tac   Done THEN r tacs i) st)
                else
                  (trace-tac *** Wrong number of produced goals THEN bailout-tac)
st
            ,
              K (trace-tac *** Phase tactic failed THEN bailout-tac))
          end i st

      in
        r phases
      end


    end

(*    (* Perform sequence of tactics (tac,n), each expected to create n new goals,
       or solve goals if n is negative.
       Debug−flag: Stop with intermediate state after tactic
       fails or produces less/more goals as expected. *)
    val PHASES': phase list −> phases-ctrl −> Proof.context −> tactic'
*)



(*


    fun xPHASES' dbg tacs ctxt = let
      val tacs = map (fn (tac,d) => (tac ctxt,d)) tacs

      fun r [] - st = Seq.single st
        | r ((tac,d)::tacs) i st = let
            val n = Thm.nprems-of st
            val bailout-tac = if dbg then all-tac else no-tac
          in
            IF-EXGOAL (tac)
            THEN-ELSE' (
              fn i => fn st =>
              (* Bail out if a phase does not solve/create exactly the expected subgoals
*)
                if Thm.nprems-of st = n+d then
```

```
          (r tacs i st)
        else
          bailout-tac st
      ,
        K bailout-tac)
    end i st

  in
    r tacs
  end
*)
 end


 signature SEPREF-DEBUGGING = sig
  (************************)
  (* Debugging *)
  (* Centralized debugging mode flag *)
  val cfg-debug-all: bool Config.T

  val is-debug: bool Config.T -> Proof.context -> bool
  val is-debug': Proof.context -> bool

  (* Conversion, trace errors if custom or central debugging flag is activated *)
  val DBG-CONVERSION: bool Config.T -> Proof.context -> conv -> tactic'

  (* Conversion, trace errors if central debugging flag is activated *)
  val DBG-CONVERSION': Proof.context -> conv -> tactic'

  (* Tracing message and current subgoal *)
  val tracing-tac': string -> Proof.context -> tactic'
  (* Warning message and current subgoal *)
  val warning-tac': string -> Proof.context -> tactic'
  (* Error message and current subgoal *)
  val error-tac': string -> Proof.context -> tactic'

  (* Trace debug message *)
  val dbg-trace-msg: bool Config.T -> Proof.context -> string -> unit
  val dbg-trace-msg': Proof.context -> string -> unit

  val dbg-msg-tac: bool Config.T -> (Proof.context -> int -> thm -> string)
-> Proof.context -> tactic'
  val dbg-msg-tac': (Proof.context -> int -> thm -> string) -> Proof.context
-> tactic'

  val msg-text: string -> Proof.context -> int -> thm -> string
  val msg-subgoal: string -> Proof.context -> int -> thm -> string
   val msg-from-subgoal: string -> (term -> Proof.context -> string) ->
Proof.context -> int -> thm -> string
```

```
  val msg-allgoals: string -> Proof.context -> int -> thm -> string

end

structure Sepref-Debugging: SEPREF-DEBUGGING = struct

  val cfg-debug-all =
    Attrib.setup-config-bool @{binding sepref-debug-all} (K false)

  fun is-debug cfg ctxt = Config.get ctxt cfg orelse Config.get ctxt cfg-debug-all
  fun is-debug' ctxt = Config.get ctxt cfg-debug-all

  fun dbg-trace cfg ctxt obj =
    if is-debug cfg ctxt then
      tracing (@{make-string} obj)
    else ()

  fun dbg-trace' ctxt obj =
    if is-debug' ctxt then
      tracing (@{make-string} obj)
    else ()

  fun dbg-trace-msg cfg ctxt msg =
    if is-debug cfg ctxt then
      tracing msg
    else ()
  fun dbg-trace-msg' ctxt msg =
    if is-debug' ctxt then
      tracing msg
    else ()

  fun DBG-CONVERSION cfg ctxt cv i st =
    Seq.single (Conv.gconv-rule cv i st)
    handle e as THM - => (dbg-trace cfg ctxt e; Seq.empty)
         | e as CTERM - => (dbg-trace cfg ctxt e; Seq.empty)
         | e as TERM - => (dbg-trace cfg ctxt e; Seq.empty)
         | e as TYPE - => (dbg-trace cfg ctxt e; Seq.empty);

  fun DBG-CONVERSION' ctxt cv i st =
    Seq.single (Conv.gconv-rule cv i st)
    handle e as THM - => (dbg-trace' ctxt e; Seq.empty)
         | e as CTERM - => (dbg-trace' ctxt e; Seq.empty)
         | e as TERM - => (dbg-trace' ctxt e; Seq.empty)
         | e as TYPE - => (dbg-trace' ctxt e; Seq.empty);


  local
    fun gen-subgoal-msg-tac do-msg msg ctxt = IF-EXGOAL (fn i => fn st =>
let
```

```
      val t = nth (Thm.prems-of st) (i−1)
      val - = Pretty.block [Pretty.str msg, Pretty.fbrk, Syntax.pretty-term ctxt t]
        |> Pretty.string-of |> do-msg

    in
      Seq.single st
    end)
  in
    val tracing-tac' = gen-subgoal-msg-tac tracing
    val warning-tac' = gen-subgoal-msg-tac warning
    val error-tac' = gen-subgoal-msg-tac error
  end


  fun dbg-msg-tac cfg msg ctxt =
    if is-debug cfg ctxt then (fn i => fn st => (tracing (msg ctxt i st); Seq.single
st))
    else K all-tac
  fun dbg-msg-tac' msg ctxt =
     if is-debug' ctxt then (fn i => fn st => (tracing (msg ctxt i st); Seq.single
st))
    else K all-tac

  fun msg-text msg - - - = msg

  fun msg-from-subgoal msg sgmsg ctxt i st =
    case try (nth (Thm.prems-of st)) (i−1) of
      NONE => msg ^\n ^Subgoal out of range
    | SOME t => msg ^\n ^sgmsg t ctxt

  fun msg-subgoal msg = msg-from-subgoal msg (fn t => fn ctxt =>
    Syntax.pretty-term ctxt t |> Pretty.string-of
  )

  fun msg-allgoals msg ctxt - st =
    msg ^\n ^ Goal-Display.string-of-goal ctxt st

end
›
```

ML ‹

```
(* Tactics for produced subgoals *)
infix 1 THEN-NEXT THEN-ALL-NEW-LIST THEN-ALL-NEW-LIST'
signature STACTICAL = sig
  (* Apply first tactic on this subgoal, and then second tactic on next subgoal *)
  val THEN-NEXT: tactic' * tactic' −> tactic'
  (* Apply tactics to the current and following subgoals *)
  val APPLY-LIST: tactic' list −> tactic'
```

```
  (* Apply list of tactics on subgoals emerging from tactic.
    Requires exactly one tactic per emerging subgoal.*)
  val THEN-ALL-NEW-LIST: tactic' * tactic' list -> tactic'
 (* Apply list of tactics to subgoals emerging from tactic, use fallback for additional subgoals. *)
  val THEN-ALL-NEW-LIST': tactic' * (tactic' list * tactic') -> tactic'

end

structure STactical : STACTICAL = struct
  infix 1 THEN-WITH-GOALDIFF
  fun (tac1 THEN-WITH-GOALDIFF tac2) st = let
    val n1 = Thm.nprems-of st
  in
    st |> (tac1 THEN (fn st => tac2 (Thm.nprems-of st - n1) st ))
  end

  fun (tac1 THEN-NEXT tac2) i =
    tac1 i THEN-WITH-GOALDIFF (fn d => (
      if d < ~1 then
        (error THEN-NEXT: Tactic solved more than one goal; no-tac)
      else
        tac2 (i+1+d)
    ))

  fun APPLY-LIST [] = K all-tac
    | APPLY-LIST (tac::tacs) = tac THEN-NEXT APPLY-LIST tacs

  fun (tac1 THEN-ALL-NEW-LIST tacs) i =
    tac1 i
    THEN-WITH-GOALDIFF (fn d =>
      if d+1 <> length tacs then (
          error THEN-ALL-NEW-LIST: Tactic produced wrong number of goals; no-tac
      ) else APPLY-LIST tacs i
    )

  fun (tac1 THEN-ALL-NEW-LIST' (tacs,rtac)) i =
    tac1 i
    THEN-WITH-GOALDIFF (fn d => let
       val - = if d+1 < length tacs then error THEN-ALL-NEW-LIST': Tactic produced too few goals else ();
      val tacs' = tacs @ replicate (d + 1 - length tacs) rtac
    in
      APPLY-LIST tacs' i
    end)


end
```

open STactical
›

end

1.3 Monadify

theory *Sepref-Monadify*
imports *Sepref-Basic Sepref-Id-Op*
begin

In this phase, a monadic program is converted to complete monadic form, that is, computation of compound expressions are made visible as top-level operations in the monad.

The monadify process is separated into 2 steps.

1. In a first step, eta-expansion is used to add missing operands to operations and combinators. This way, operators and combinators always occur with the same arity, which simplifies further processing.

2. In a second step, computation of compound operands is flattened, introducing new bindings for the intermediate values.

definition *SP* — Tag to protect content from further application of arity and combinator equations
where [*simp*]: $SP\ x \equiv x$
lemma *SP-cong*[*cong*]: $SP\ x \equiv SP\ x$ **by** *simp*
lemma *PR-CONST-cong*[*cong*]: $PR\text{-}CONST\ x \equiv PR\text{-}CONST\ x$ **by** *simp*

definition *RCALL* — Tag that marks recursive call
where [*simp*]: $RCALL\ D \equiv D$
definition *EVAL* — Tag that marks evaluation of plain expression for monadify phase
where [*simp*]: $EVAL\ x \equiv RETURN\ x$

Internally, the package first applies rewriting rules from *sepref-monadify-arity*, which use eta-expansion to ensure that every combinator has enough actual parameters. Moreover, this phase will mark recursive calls by the tag *RCALL*.

Next, rewriting rules from *sepref-monadify-comb* are used to add *EVAL*-tags to plain expressions that should be evaluated in the monad. The *EVAL* tags are flattened using a default simproc that generates left-to-right argument order.

lemma *monadify-simps*:
$Refine\text{-}Basic.bind\$(RETURN\$x)\$(\lambda_2 x.\ f\ x) = f\ x$

EVAL\$*x* ≡ *RETURN*\$*x*
by *simp-all*

definition [*simp*]: *PASS* ≡ *RETURN*
— Pass on value, invalidating old one

lemma *remove-pass-simps*:
Refine-Basic.bind\$(*PASS*\$*x*)\$(λ_2*x*. *f x*) ≡ *f x*
Refine-Basic.bind\$*m*\$(λ_2*x*. *PASS*\$*x*) ≡ *m*
by *simp-all*

definition *COPY* :: *'a* ⇒ *'a*
— Marks required copying of parameter
where [*simp*]: *COPY x* ≡ *x*
lemma *RET-COPY-PASS-eq*: *RETURN*\$(*COPY*\$*p*) = *PASS*\$*p* **by** *simp*

named-theorems-rev *sepref-monadify-arity Sepref.Monadify*: *Arity alignment equations*
named-theorems-rev *sepref-monadify-comb Sepref.Monadify*: *Combinator equations*

ML ‹
```
structure Sepref-Monadify = struct
  local
    fun cr-var (i,T) = (v^string-of-int i, Free (--v^string-of-int i,T))

    fun lambda2-name n t = let
      val t = @{mk-term PROTECT2 ?t DUMMY}
    in
      Term.lambda-name n t
    end



    fun
      bind-args exp0 [] = exp0
    | bind-args exp0 ((x,m)::xms) = let
        val lr = bind-args exp0 xms
          |> incr-boundvars 1
          |> lambda2-name x
      in @{mk-term Refine-Basic.bind$?m$?lr} end

    fun monadify t = let
      val (f,args) = Autoref-Tagging.strip-app t
      val - = not (is-Abs f) orelse
        raise TERM (monadify: higher−order,[t])

      val argTs = map fastype-of args
```

```
(*val args = map monadify args*)
val args = map (fn a => @{mk-term EVAL$?a}) args

(*val fT = fastype-of f
val argTs = binder-types fT*)

val argVs = tag-list 0 argTs
  |> map cr-var

val res0 = let
  val x = Autoref-Tagging.list-APP (f,map #2 argVs)
in
  @{mk-term SP (RETURN$?x)}
end

val res = bind-args res0 (argVs ~~ args)
in
  res
end

fun monadify-conv-aux ctxt ct = case Thm.term-of ct of
  @{mpat EVAL$-} => let
    fun tac goal-ctxt =
      simp-tac (put-simpset HOL-basic-ss goal-ctxt addsimps @{thms monad-
ify-simps SP-def}) 1
  in (*Refine-Util.monitor-conv monadify*) (
    Refine-Util.f-tac-conv ctxt (dest-comb #> #2 #> monadify) tac) ct
  end
| t => raise TERM (monadify-conv,[t])

(*fun extract-comb-conv ctxt = Conv.rewrs-conv
(Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-monadify-evalcomb})
*)
in
  (*
  val monadify-conv = Conv.top-conv
    (fn ctxt =>
      Conv.try-conv (
        extract-comb-conv ctxt else-conv monadify-conv-aux ctxt
      )
    )
  *)

  val monadify-simproc =
  simproc-setup ‹passive monadify (EVAL$a) = ‹K (try o monadify-conv-aux)››;

end

local
```

```
  open Sepref-Basic
  fun mark-params t = let
    val (P,c,Q,R,a) = dest-hn-refine t
    val pps = strip-star P |> map-filter (dest-hn-ctxt-opt #> map-option #2)

    fun tr env (t as @{mpat RETURN$?x}) =
          if is-Bound x orelse member (aconv) pps x then
            @{mk-term env: PASS$?x}
          else t
      | tr env (t1$t2) = tr env t1 $ tr env t2
      | tr env (Abs (x,T,t)) = Abs (x,T,tr (T::env) t)
      | tr - t = t

    val a = tr [] a
  in
    mk-hn-refine (P,c,Q,R,a)
  end

in
fun mark-params-conv ctxt = Refine-Util.f-tac-conv ctxt
  (mark-params)
(fn goal-ctxt => simp-tac (put-simpset HOL-basic-ss goal-ctxt addsimps @{thms
PASS-def}) 1)

end

local

  open Sepref-Basic

  fun dp ctxt (@{mpat Refine-Basic.bind$(PASS$?p)$(?t' AS_p (λ-. PROTECT2
- DUMMY))}) =
      let
        val (t',ps) = let
            val ((t',rc),ctxt) = dest-lambda-rc ctxt t'
           val f = case t' of @{mpat PROTECT2 ?f -} => f | - => raise Match

            val (f,ps) = dp ctxt f
            val t' = @{mk-term PROTECT2 ?f DUMMY}
            val t' = rc t'
          in
            (t',ps)
          end

        val dup = member (aconv) ps p
        val t = if dup then
          @{mk-term Refine-Basic.bind$(RETURN$(COPY$?p))$?t'}
        else
          @{mk-term Refine-Basic.bind$(PASS$?p)$?t'}
```

```
    in
      (t,p::ps)
    end
  | dp ctxt (t1$t2) = (#1 (dp ctxt t1) $ #1 (dp ctxt t2),[])
  | dp ctxt (t as (Abs -)) = (apply-under-lambda (#1 oo dp) ctxt t,[])
  | dp - t = (t,[])

fun dp-conv ctxt = Refine-Util.f-tac-conv ctxt
  (#1 o dp ctxt)
  (fn goal-ctxt =>
  ALLGOALS (simp-tac (put-simpset HOL-basic-ss goal-ctxt addsimps @{thms
RET-COPY-PASS-eq})))


in
fun dup-tac ctxt = CONVERSION (Sepref-Basic.hn-refine-concl-conv-a dp-conv
ctxt)
end



fun arity-tac ctxt = let
  val arity1-ss = put-simpset HOL-basic-ss ctxt
  addsimps ((Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-monadify-arity}))
    |> Simplifier.add-cong @{thm SP-cong}
    |> Simplifier.add-cong @{thm PR-CONST-cong}

  val arity2-ss = put-simpset HOL-basic-ss ctxt
    addsimps @{thms beta SP-def}
in
  simp-tac arity1-ss THEN' simp-tac arity2-ss
end

fun comb-tac ctxt = let
  val comb1-ss = put-simpset HOL-basic-ss ctxt
  addsimps (Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-monadify-comb})
  (*addsimps (Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-monadify-evalcomb})*)
    addsimprocs [monadify-simproc]
    |> Simplifier.add-cong @{thm SP-cong}
    |> Simplifier.add-cong @{thm PR-CONST-cong}

  val comb2-ss = put-simpset HOL-basic-ss ctxt
    addsimps @{thms SP-def}
in
  simp-tac comb1-ss THEN' simp-tac comb2-ss
end

(*fun ops-tac ctxt = CONVERSION (
  Sepref-Basic.hn-refine-concl-conv-a monadify-conv ctxt)*)
```

```
  fun mark-params-tac ctxt = CONVERSION (
    Refine-Util.HOL-concl-conv mark-params-conv ctxt)

  fun contains-eval @{mpat Trueprop (hn-refine - - - - ?a)} =
    Term.exists-subterm (fn @{mpat EVAL} => true | - => false) a
  | contains-eval t = raise TERM(contains-eval,[t]);

  fun remove-pass-tac ctxt =
   simp-tac (put-simpset HOL-basic-ss ctxt addsimps @{thms remove-pass-simps})

  fun monadify-tac dbg ctxt = let
    open Sepref-Basic
  in
    PHASES' [
      (arity, arity-tac, 0),
      (comb, comb-tac, 0),
      (*(ops, ops-tac, 0),*)
      (check-EVAL, K (CONCL-COND' (not o contains-eval)), 0),
      (mark-params, mark-params-tac, 0),
      (dup, dup-tac, 0),
      (remove-pass, remove-pass-tac, 0)
    ] (flag-phases-ctrl dbg) ctxt
  end

end
›
```

lemma *dflt-arity*[*sepref-monadify-arity*]:
$RETURN \equiv \lambda_2 x.\ SP\ RETURN\$x$
$RECT \equiv \lambda_2 B\ x.\ SP\ RECT\$(\lambda_2 D\ x.\ B\$(\lambda_2 x.\ RCALL\$D\$x)\$x)\$x$
$case\text{-}list \equiv \lambda_2 fn\ fc\ l.\ SP\ case\text{-}list\$fn\$(\lambda_2 x\ xs.\ fc\$x\$xs)\$l$
$case\text{-}prod \equiv \lambda_2 fp\ p.\ SP\ case\text{-}prod\$(\lambda_2 a\ b.\ fp\$a\$b)\$p$
$case\text{-}option \equiv \lambda_2 fn\ fs\ ov.\ SP\ case\text{-}option\$fn\$(\lambda_2 x.\ fs\$x)\$ov$
$If \equiv \lambda_2 b\ t\ e.\ SP\ If\$b\$t\$e$
$Let \equiv \lambda_2 x\ f.\ SP\ Let\$x\$(\lambda_2 x.\ f\$x)$
by (*simp-all only*: *SP-def APP-def PROTECT2-def RCALL-def*)

lemma *dflt-comb*[*sepref-monadify-comb*]:
$\bigwedge B\ x.\ RECT\$B\$x \equiv Refine\text{-}Basic.bind\$(EVAL\$x)\$(\lambda_2 x.\ SP\ (RECT\$B\$x))$
$\bigwedge D\ x.\ RCALL\$D\$x \equiv Refine\text{-}Basic.bind\$(EVAL\$x)\$(\lambda_2 x.\ SP\ (RCALL\$D\$x))$
$\bigwedge fn\ fc\ l.\ case\text{-}list\$fn\$fc\$l \equiv Refine\text{-}Basic.bind\$(EVAL\$l)\$(\lambda_2 l.\ (SP\ case\text{-}list\$fn\$fc\$l))$
$\bigwedge fp\ p.\ case\text{-}prod\$fp\$p \equiv Refine\text{-}Basic.bind\$(EVAL\$p)\$(\lambda_2 p.\ (SP\ case\text{-}prod\$fp\$p))$
$\bigwedge fn\ fs\ ov.\ case\text{-}option\$fn\$fs\$ov$
$\quad \equiv Refine\text{-}Basic.bind\$(EVAL\$ov)\$(\lambda_2 ov.\ (SP\ case\text{-}option\$fn\$fs\$ov))$
$\bigwedge b\ t\ e.\ If\$b\$t\$e \equiv Refine\text{-}Basic.bind\$(EVAL\$b)\$(\lambda_2 b.\ (SP\ If\$b\$t\$e))$
$\bigwedge x.\ RETURN\$x \equiv Refine\text{-}Basic.bind\$(EVAL\$x)\$(\lambda_2 x.\ SP\ (RETURN\$x))$
$\bigwedge x\ f.\ Let\$x\$f \equiv Refine\text{-}Basic.bind\$(EVAL\$x)\$(\lambda_2 x.\ (SP\ Let\$x\$f))$
by (*simp-all*)

lemma *dflt-plain-comb*[*sepref-monadify-comb*]:
EVAL\$(*If*\$*b*\$*t*\$*e*) ≡ *Refine-Basic.bind*\$(*EVAL*\$*b*)\$(λ_2*b*. *If*\$*b*\$(*EVAL*\$*t*)\$(*EVAL*\$*e*))
EVAL\$(*case-list*\$*fn*\$(λ_2*x xs*. *fc x xs*)\$*l*) ≡
Refine-Basic.bind\$(*EVAL*\$*l*)\$(λ_2*l*. *case-list*\$(*EVAL*\$*fn*)\$(λ_2*x xs*. *EVAL*\$(*fc x xs*))\$*l*)
EVAL\$(*case-prod*\$(λ_2*a b*. *fp a b*)\$*p*) ≡
Refine-Basic.bind\$(*EVAL*\$*p*)\$(λ_2*p*. *case-prod*\$(λ_2*a b*. *EVAL*\$(*fp a b*))\$*p*)
EVAL\$(*case-option*\$*fn*\$(λ_2*x*. *fs x*)\$*ov*) ≡
Refine-Basic.bind\$(*EVAL*\$*ov*)\$(λ_2*ov*. *case-option*\$(*EVAL*\$*fn*)\$(λ_2*x*. *EVAL*\$(*fs x*))\$*ov*)
EVAL \$ (*Let* \$ *v* \$ (λ_2*x*. *f x*)) ≡ (⤜) \$ (*EVAL* \$ *v*) \$ (λ_2*x*. *EVAL* \$ (*f x*))
apply (*rule eq-reflection*, *simp split*: *list.split prod.split option.split*)+
done

lemma *evalcomb-PR-CONST*[*sepref-monadify-comb*]:
EVAL\$(*PR-CONST x*) ≡ *SP* (*RETURN*\$(*PR-CONST x*))
by *simp*

end
theory *Sepref-Constraints*
imports *Main Automatic-Refinement.Refine-Lib Sepref-Basic*
begin

definition *CONSTRAINT-SLOT* (*x*::*prop*) ≡ *x*

lemma *insert-slot-rl1*:
assumes *PROP P* ⟹ *PROP* (*CONSTRAINT-SLOT* (*Trueprop True*)) ⟹ *PROP Q*
shows *PROP* (*CONSTRAINT-SLOT* (*PROP P*)) ⟹ *PROP Q*
using *assms* **unfolding** *CONSTRAINT-SLOT-def* **by** *simp*

lemma *insert-slot-rl2*:
assumes *PROP P* ⟹ *PROP* (*CONSTRAINT-SLOT S*) ⟹ *PROP Q*
shows *PROP* (*CONSTRAINT-SLOT* (*PROP S* &&& *PROP P*)) ⟹ *PROP Q*
using *assms* **unfolding** *CONSTRAINT-SLOT-def conjunction-def* **.**

lemma *remove-slot*: *PROP* (*CONSTRAINT-SLOT* (*Trueprop True*))
unfolding *CONSTRAINT-SLOT-def* **by** (*rule TrueI*)

definition *CONSTRAINT* **where** [*simp*]: *CONSTRAINT P x* ≡ *P x*

lemma *CONSTRAINT-D*:
assumes *CONSTRAINT* (*P*::′*a* => *bool*) *x*
shows *P x*
using *assms* **unfolding** *CONSTRAINT-def* **by** *simp*

lemma *CONSTRAINT-I*:
assumes *P x*
shows *CONSTRAINT* (*P*::′*a* => *bool*) *x*
using *assms* **unfolding** *CONSTRAINT-def* **by** *simp*

Special predicate to indicate unsolvable constraint. The constraint solver refuses to put those into slot. Thus, adding safe rules introducing this can be used to indicate unsolvable constraints early.

definition *CN-FALSE* :: (′*a*⇒*bool*) ⇒ ′*a* ⇒ *bool* **where** [*simp*]: *CN-FALSE P x* ≡ *False*
lemma *CN-FALSEI*: *CN-FALSE P x* ⟹ *P x* **by** *simp*

named-theorems *constraint-simps* ‹*Simplification of constraints*›

named-theorems *constraint-abbrevs* ‹*Constraint Solver*: *Abbreviations*›
lemmas *split-constraint-rls*
= *atomize-conj*[*symmetric*] *imp-conjunction all-conjunction conjunction-imp*

ML ‹
```
signature SEPREF-CONSTRAINTS = sig
  (******** Constraint Slot *)
  (* Tactic with slot subgoal *)
  val WITH-SLOT: tactic' -> tactic
  (* Process all goals in slot *)
  val ON-SLOT: tactic -> tactic
  (* Create slot as last subgoal. Fail if slot already present. *)
  val create-slot-tac: tactic
  (* Create slot if there isn't one already *)
  val ensure-slot-tac: tactic
  (* Remove empty slot *)
  val remove-slot-tac: tactic
  (* Move slot to first subgoal *)
  val prefer-slot-tac: tactic
  (* Destruct slot *)
  val dest-slot-tac: tactic'
  (* Check if goal state has slot *)
  val has-slot: thm -> bool
  (* Defer subgoal to slot *)
  val to-slot-tac: tactic'
  (* Print slot constraints *)
  val print-slot-tac: Proof.context -> tactic

  (* Focus on goals in slot *)
  val focus: tactic
  (* Unfocus goals in slot *)
  val unfocus: tactic
  (* Unfocus goals, and insert them as first subgoals *)
```

```
val unfocus-ins:tactic

(* Focus on some goals in slot *)
val cond-focus: (term -> bool) -> tactic
(* Move some goals to slot *)
val some-to-slot-tac: (term -> bool) -> tactic


(******** Constraints *)
(* Check if subgoal is a constraint. To be used with COND' *)
val is-constraint-goal: term -> bool
(* Identity on constraint subgoal, no-tac otherwise *)
val is-constraint-tac: tactic'
(* Defer constraint to slot *)
val slot-constraint-tac: int -> tactic

(******** Constraint solving *)

val add-constraint-rule: thm -> Context.generic -> Context.generic
val del-constraint-rule: thm -> Context.generic -> Context.generic
val get-constraint-rules: Proof.context -> thm list

val add-safe-constraint-rule: thm -> Context.generic -> Context.generic
val del-safe-constraint-rule: thm -> Context.generic -> Context.generic
val get-safe-constraint-rules: Proof.context -> thm list

(* Solve constraint subgoal *)
val solve-constraint-tac: Proof.context -> tactic'
(* Solve constraint subgoal if solvable, fail if definitely unsolvable,
  apply simplification and unique rules otherwise. *)
val safe-constraint-tac: Proof.context -> tactic'

(* CONSTRAINT tag on goal is optional *)
val solve-constraint'-tac: Proof.context -> tactic'
(* CONSTRAINT tag on goal is optional *)
val safe-constraint'-tac: Proof.context -> tactic'

(* Solve, or apply safe-rules and defer to constraint slot *)
val constraint-tac: Proof.context -> tactic'

(* Apply safe rules to all constraint goals in slot *)
val process-constraint-slot: Proof.context -> tactic

(* Solve all constraint goals in slot, insert unsolved ones as first subgoals *)
val solve-constraint-slot: Proof.context -> tactic


val setup: theory -> theory
```

```
end

structure Sepref-Constraints: SEPREF-CONSTRAINTS = struct
  fun is-slot-goal @{mpat CONSTRAINT-SLOT -} = true | is-slot-goal - = false

  fun slot-goal-num st = let
    val i = find-index is-slot-goal (Thm.prems-of st) + 1
  in
    i
  end

  fun has-slot st = slot-goal-num st > 0

  fun WITH-SLOT tac st = let
    val si = slot-goal-num st
  in
    if si>0 then tac si st else (warning Constraints: No slot; Seq.empty)
  end

  val to-slot-tac = IF-EXGOAL (fn i => WITH-SLOT (fn si =>
    if i<si then
      prefer-tac si THEN prefer-tac (i+1)
      THEN (
           PRIMITIVE (fn st => Drule.comp-no-flatten (st, 0) 1 @{thm in-
sert-slot-rl1})
        ORELSE PRIMITIVE (fn st => Drule.comp-no-flatten (st, 0) 1 @{thm
insert-slot-rl2})
      )
      THEN defer-tac 1
    else no-tac))

  val create-slot-tac =
    COND (has-slot) no-tac
     (PRIMITIVE (Thm.implies-intr @{cterm CONSTRAINT-SLOT (Trueprop
True)})
      THEN defer-tac 1)

  val ensure-slot-tac = TRY create-slot-tac


  val prefer-slot-tac = WITH-SLOT prefer-tac

  val dest-slot-tac = SELECT-GOAL (
    ALLGOALS (
      CONVERSION (Conv.rewr-conv @{thm CONSTRAINT-SLOT-def})
      THEN' Goal.conjunction-tac
      THEN' TRY o resolve0-tac @{thms TrueI})
    THEN distinct-subgoals-tac
```

```
)

val remove-slot-tac = WITH-SLOT (resolve0-tac @{thms remove-slot})

val focus = WITH-SLOT (fn i =>
  PRIMITIVE (Goal.restrict i 1)
  THEN ALLGOALS dest-slot-tac
  THEN create-slot-tac)

val unfocus-ins =
  PRIMITIVE (Goal.unrestrict 1)
  THEN WITH-SLOT defer-tac

fun some-to-slot-tac cond = (ALLGOALS (COND' (fn t => is-slot-goal t orelse
not (cond t)) ORELSE' to-slot-tac))

val unfocus =
  some-to-slot-tac (K true)
  THEN unfocus-ins

fun cond-focus cond =
  focus
  THEN some-to-slot-tac (not o cond)


fun ON-SLOT tac = focus THEN tac THEN unfocus

fun print-slot-tac ctxt = ON-SLOT (print-tac ctxt SLOT:)

local
  (*fun prepare-constraint-conv ctxt = let
    open Conv
    fun CONSTRAINT-conv ct = case Thm.term-of ct of
      @{mpat Trueprop (- -)} =>
        HOLogic.Trueprop-conv
          (rewr-conv @{thm CONSTRAINT-def[symmetric]}) ct
      | - => raise CTERM (CONSTRAINT-conv, [ct])

    fun rec-conv ctxt ct = (
      CONSTRAINT-conv
      else-conv
      implies-conv (rec-conv ctxt) (rec-conv ctxt)
      else-conv
      forall-conv (rec-conv o #2) ctxt
    ) ct
  in
    rec-conv ctxt
  end*)
```

```
fun unfold-abbrevs ctxt =
  Local-Defs.unfold0 ctxt (
    @{thms split-constraint-rls CONSTRAINT-def}
    @ Named-Theorems.get ctxt @{named-theorems constraint-abbrevs}
    @ Named-Theorems.get ctxt @{named-theorems constraint-simps})
  #> Conjunction.elim-conjunctions

fun check-constraint-rl thm = let
  fun ck (t as @{mpat Trueprop (?C -)}) =
        if is-Var (Term.head-of C) then
          raise TERM (Schematic head in constraint rule,[t,Thm.prop-of thm])
        else ()
    | ck @{mpat ⋀-. PROP ?t} = ck t
    | ck @{mpat PROP ?s ⟹ PROP ?t} = (ck s; ck t)
    | ck t = raise TERM (Invalid part of constraint rule,[t,Thm.prop-of thm])

in
  ck (Thm.prop-of thm); thm
end

fun check-unsafe-constraint-rl thm = let
  val - = Thm.nprems-of thm = 0
     andalso raise TERM(Unconditional constraint rule must be safe (register
this as safe rule),[Thm.prop-of thm])
in
  thm
end

in
  structure constraint-rules = Named-Sorted-Thms (
    val name = @{binding constraint-rules}
    val description = Constraint rules
    val sort = K I
    fun transform context = let
      open Conv
      val ctxt = Context.proof-of context
    in
    unfold-abbrevs ctxt #> map (check-constraint-rl o check-unsafe-constraint-rl)
    end
  )

  structure safe-constraint-rules = Named-Sorted-Thms (
    val name = @{binding safe-constraint-rules}
    val description = Safe Constraint rules
    val sort = K I
    fun transform context = let
      open Conv
      val ctxt = Context.proof-of context
    in
```

```
      unfold-abbrevs ctxt #> map check-constraint-rl
    end
  )

end

val add-constraint-rule = constraint-rules.add-thm
val del-constraint-rule = constraint-rules.del-thm
val get-constraint-rules = constraint-rules.get

val add-safe-constraint-rule = safe-constraint-rules.add-thm
val del-safe-constraint-rule = safe-constraint-rules.del-thm
val get-safe-constraint-rules = safe-constraint-rules.get

fun is-constraint-goal t = case Logic.strip-assums-concl t of
  @{mpat Trueprop (CONSTRAINT - -)} => true
| - => false

val is-constraint-tac = COND' is-constraint-goal

fun is-slottable-constraint-goal t = case Logic.strip-assums-concl t of
  @{mpat Trueprop (CONSTRAINT (CN-FALSE -) -)} => false
| @{mpat Trueprop (CONSTRAINT - -)} => true
| - => false

val slot-constraint-tac = COND' is-slottable-constraint-goal THEN' to-slot-tac

datatype 'a seq-cases = SC-NONE | SC-SINGLE of 'a Seq.seq | SC-MULTIPLE
of 'a Seq.seq

fun seq-cases seq =
  case Seq.pull seq of
    NONE => SC-NONE
  | SOME (st1,seq) => case Seq.pull seq of
      NONE => SC-SINGLE (Seq.single st1)
    | SOME (st2,seq) => SC-MULTIPLE (Seq.cons st1 (Seq.cons st2 seq))

fun SEQ-CASES tac (single-tac, multiple-tac) st = let
  val res = tac st
in
  case seq-cases res of
    SC-NONE => Seq.empty
  | SC-SINGLE res => Seq.maps single-tac res
  | SC-MULTIPLE res => Seq.maps multiple-tac res
end

fun SAFE tac = SEQ-CASES tac (all-tac, no-tac)
fun SAFE' tac = SAFE o tac
```

```
local
  fun simp-constraints-tac ctxt = let
    val ctxt = put-simpset HOL-basic-ss ctxt
      addsimps (Named-Theorems.get ctxt @{named-theorems constraint-simps})
  in
    simp-tac ctxt
  end

  fun unfold-abbrevs-tac ctxt =  let
    val ctxt = put-simpset HOL-basic-ss ctxt
    addsimps (Named-Theorems.get ctxt @{named-theorems constraint-abbrevs})
    val ethms = @{thms conjE}
    val ithms = @{thms conjI}
  in
    full-simp-tac ctxt
    THEN-ALL-NEW TRY o REPEAT-ALL-NEW (ematch-tac ctxt ethms)
    THEN-ALL-NEW TRY o REPEAT-ALL-NEW (match-tac ctxt ithms)
  end

  fun WITH-RULE-NETS tac ctxt = let
    val scn-net = safe-constraint-rules.get ctxt |> Tactic.build-net
    val cn-net = constraint-rules.get ctxt |> Tactic.build-net
  in
    tac (scn-net,cn-net) ctxt
  end

  fun wrap-tac step-tac ctxt = REPEAT-ALL-NEW (
    simp-constraints-tac ctxt
    THEN-ALL-NEW unfold-abbrevs-tac ctxt
    THEN-ALL-NEW step-tac ctxt
  )

  fun solve-step-tac (scn-net,cn-net) ctxt = REPEAT-ALL-NEW (
    DETERM o resolve-from-net-tac ctxt scn-net
    ORELSE' resolve-from-net-tac ctxt cn-net
  )

  fun safe-step-tac (scn-net,cn-net) ctxt = REPEAT-ALL-NEW (
    DETERM o resolve-from-net-tac ctxt scn-net
    ORELSE' SAFE' (resolve-from-net-tac ctxt cn-net)
  )

  fun solve-tac cn-nets ctxt = SOLVED' (wrap-tac (solve-step-tac cn-nets) ctxt)
  fun safe-tac cn-nets ctxt =
    simp-constraints-tac ctxt
    THEN-ALL-NEW unfold-abbrevs-tac ctxt
        THEN-ALL-NEW (solve-tac cn-nets ctxt ORELSE' TRY o wrap-tac
(safe-step-tac cn-nets) ctxt)
```

```
  in
    val solve-constraint-tac = TRADE (fn ctxt =>
      is-constraint-tac
      THEN' resolve-tac ctxt @{thms CONSTRAINT-I}
      THEN' WITH-RULE-NETS solve-tac ctxt)

    val safe-constraint-tac = TRADE (fn ctxt =>
      is-constraint-tac
      THEN' resolve-tac ctxt @{thms CONSTRAINT-I}
      THEN' WITH-RULE-NETS safe-tac ctxt
    THEN-ALL-NEW fo-resolve-tac @{thms CONSTRAINT-D} ctxt) (* TODO/FIXME:
fo-resolve-tac has non−canonical parameter order *)

    val solve-constraint'-tac = TRADE (fn ctxt =>
      TRY o resolve-tac ctxt @{thms CONSTRAINT-I}
      THEN' WITH-RULE-NETS solve-tac ctxt)

    val safe-constraint'-tac = TRADE (fn ctxt =>
      TRY o resolve-tac ctxt @{thms CONSTRAINT-I}
      THEN' WITH-RULE-NETS safe-tac ctxt)


  end

  fun constraint-tac ctxt =
    safe-constraint-tac ctxt THEN-ALL-NEW slot-constraint-tac

 fun process-constraint-slot ctxt = ON-SLOT (ALLGOALS (TRY o safe-constraint-tac
ctxt))

  fun solve-constraint-slot ctxt =
    cond-focus is-constraint-goal
      THEN ALLGOALS (
        COND' is-slot-goal
        ORELSE' (
          solve-constraint-tac ctxt
          ORELSE' TRY o safe-constraint-tac ctxt
        )
      )
    THEN unfocus-ins


  val setup = I
    #> constraint-rules.setup
    #> safe-constraint-rules.setup

 end
›
```

setup *Sepref-Constraints.setup*

method-setup *print-slot* = ‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD* (*Sepref-Constraints.print-slot-tac ctxt*))›

method-setup *solve-constraint* = ‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Sepref-Constraints.solve-constraint′-tac ctxt*))›
method-setup *safe-constraint* = ‹*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′* (*Sepref-Constraints.safe-constraint′-tac ctxt*))›

end

1.4 Frame Inference

theory *Sepref-Frame*
imports *Sepref-Basic Sepref-Constraints*
begin

In this theory, we provide a specific frame inference tactic for Sepref.

The first tactic, *frame-tac*, is a standard frame inference tactic, based on the assumption that only *hn-ctxt*-assertions need to be matched.

The second tactic, *merge-tac*, resolves entailments of the form $F1 \vee_A F2 \Longrightarrow_t ?F$ that occur during translation of if and case statements. It synthesizes a new frame ?F, where refinements of variables with equal refinements in *F1* and *F2* are preserved, and the others are set to *hn-invalid*.

definition *mismatch-assn* :: $('a \Rightarrow 'c \Rightarrow assn) \Rightarrow ('a \Rightarrow 'c \Rightarrow assn) \Rightarrow 'a \Rightarrow 'c \Rightarrow assn$
  where *mismatch-assn R1 R2 x y* $\equiv$ *R1 x y* $\vee_A$ *R2 x y*

abbreviation *hn-mismatch R1 R2* $\equiv$ *hn-ctxt* (*mismatch-assn R1 R2*)

lemma *recover-pure-aux*: *CONSTRAINT is-pure R* $\Longrightarrow$ *hn-invalid R x y* $\Longrightarrow_t$ *hn-ctxt R x y*
  by (*auto simp*: *is-pure-conv invalid-pure-recover hn-ctxt-def*)

lemma *frame-thms*:
  $P \Longrightarrow_t P$
  $P \Longrightarrow_t P' \Longrightarrow F \Longrightarrow_t F' \Longrightarrow F*P \Longrightarrow_t F'*P'$
  hn-ctxt R x y $\Longrightarrow_t$ *hn-invalid R x y*
  hn-ctxt R x y $\Longrightarrow_t$ *hn-ctxt* (λ- -. *true*) *x y*
  CONSTRAINT is-pure R $\Longrightarrow$ *hn-invalid R x y* $\Longrightarrow_t$ *hn-ctxt R x y*
  apply −
  applyS *simp*
  applyS (*rule entt-star-mono*; *assumption*)

subgoal
  apply (*simp add*: *hn-ctxt-def*)
  apply (*rule enttI*)
  apply (*rule ent-trans*[*OF invalidate*[*of R*]])
  by *solve-entails*
applyS (*sep-auto simp*: *hn-ctxt-def*)
applyS (*erule recover-pure-aux*)
done

named-theorems-rev *sepref-frame-match-rules* ‹*Sepref*: *Additional frame rules*›

Rules to discharge unmatched stuff

lemma *frame-rem1*: $P \Longrightarrow_t P$ **by** *simp*

lemma *frame-rem2*: $F \Longrightarrow_t F' \Longrightarrow F * \textit{hn-ctxt}\ A\ x\ y \Longrightarrow_t F' * \textit{hn-ctxt}\ A\ x\ y$
apply (*rule entt-star-mono*) **by** *auto*

lemma *frame-rem3*: $F \Longrightarrow_t F' \Longrightarrow F * \textit{hn-ctxt}\ A\ x\ y \Longrightarrow_t F'$
using *frame-thms*(*2*) **by** *fastforce*

lemma *frame-rem4*: $P \Longrightarrow_t \textit{emp}$ **by** *simp*

lemmas *frame-rem-thms* = *frame-rem1 frame-rem2 frame-rem3 frame-rem4*

named-theorems-rev *sepref-frame-rem-rules*
‹*Sepref*: *Additional rules to resolve remainder of frame−pairing*›

lemma *ent-disj-star-mono*:
$[\![\ A \vee_A C \Longrightarrow_A E;\ B \vee_A D \Longrightarrow_A F\ ]\!] \Longrightarrow A{*}B \vee_A C{*}D \Longrightarrow_A E{*}F$
by (*metis ent-disjI1 ent-disjI2 ent-disjE ent-star-mono*)

lemma *entt-disj-star-mono*:
$[\![\ A \vee_A C \Longrightarrow_t E;\ B \vee_A D \Longrightarrow_t F\ ]\!] \Longrightarrow A{*}B \vee_A C{*}D \Longrightarrow_t E{*}F$
proof −
  assume *a1*: $A \vee_A C \Longrightarrow_t E$
  assume $B \vee_A D \Longrightarrow_t F$
  then have $A * B \vee_A C * D \Longrightarrow_A \textit{true} * E * (\textit{true} * F)$
    using *a1* **by** (*simp add*: *ent-disj-star-mono enttD*)
  then show *?thesis*
    by (*metis* (*no-types*) *assn-times-comm enttI merge-true-star-ctx star-aci*(*3*))
qed

lemma *hn-merge1*:

$F \vee_A F \Longrightarrow_t F$
$[\![\ \textit{hn-ctxt}\ R1\ x\ x' \vee_A \textit{hn-ctxt}\ R2\ x\ x' \Longrightarrow_t \textit{hn-ctxt}\ R\ x\ x';\ Fl \vee_A Fr \Longrightarrow_t F\ ]\!]$
$\Longrightarrow Fl * \textit{hn-ctxt}\ R1\ x\ x' \vee_A Fr * \textit{hn-ctxt}\ R2\ x\ x' \Longrightarrow_t F * \textit{hn-ctxt}\ R\ x\ x'$

apply *simp*
by (*rule entt-disj-star-mono*; *simp*)

lemma *hn-merge2*:
$hn\text{-}invalid\ R\ x\ x' \vee_A hn\text{-}ctxt\ R\ x\ x' \Longrightarrow_t hn\text{-}invalid\ R\ x\ x'$
$hn\text{-}ctxt\ R\ x\ x' \vee_A hn\text{-}invalid\ R\ x\ x' \Longrightarrow_t hn\text{-}invalid\ R\ x\ x'$
by (*sep-auto eintros*: *invalidate ent-disjE intro*!: *ent-imp-entt simp*: *hn-ctxt-def*)+

lemma *invalid-assn-mono*: $hn\text{-}ctxt\ A\ x\ y \Longrightarrow_t hn\text{-}ctxt\ B\ x\ y$
$\Longrightarrow hn\text{-}invalid\ A\ x\ y \Longrightarrow_t hn\text{-}invalid\ B\ x\ y$
by (*clarsimp simp*: *invalid-assn-def entailst-def entails-def hn-ctxt-def*)
(*force simp*: *mod-star-conv*)

lemma *hn-merge3*:
$[\![NO\text{-}MATCH\ (hn\text{-}invalid\ XX)\ R2;\ hn\text{-}ctxt\ R1\ x\ x' \vee_A hn\text{-}ctxt\ R2\ x\ x' \Longrightarrow_t hn\text{-}ctxt\ Rm\ x\ x']\!] \Longrightarrow hn\text{-}invalid\ R1\ x\ x' \vee_A hn\text{-}ctxt\ R2\ x\ x' \Longrightarrow_t hn\text{-}invalid\ Rm\ x\ x'$
$[\![NO\text{-}MATCH\ (hn\text{-}invalid\ XX)\ R1;\ hn\text{-}ctxt\ R1\ x\ x' \vee_A hn\text{-}ctxt\ R2\ x\ x' \Longrightarrow_t hn\text{-}ctxt\ Rm\ x\ x']\!] \Longrightarrow hn\text{-}ctxt\ R1\ x\ x' \vee_A hn\text{-}invalid\ R2\ x\ x' \Longrightarrow_t hn\text{-}invalid\ Rm\ x\ x'$
apply (*meson entt-disjD1 entt-disjD2 entt-disjE entt-trans frame-thms*(*3*) *invalid-assn-mono*)
apply (*meson entt-disjD1 entt-disjD2 entt-disjE entt-trans frame-thms*(*3*) *invalid-assn-mono*)
done

lemmas *merge-thms* = *hn-merge1 hn-merge2*

named-theorems *sepref-frame-merge-rules* ‹*Sepref*: *Additional merge rules*›

lemma *hn-merge-mismatch*: $hn\text{-}ctxt\ R1\ x\ x' \vee_A hn\text{-}ctxt\ R2\ x\ x' \Longrightarrow_t hn\text{-}mismatch\ R1\ R2\ x\ x'$
by (*sep-auto simp*: *hn-ctxt-def mismatch-assn-def*)

lemma *is-merge*: $P1 \vee_A P2 \Longrightarrow_t P \Longrightarrow P1 \vee_A P2 \Longrightarrow_t P$ **.**

lemma *merge-mono*: $[\![A \Longrightarrow_t A';\ B \Longrightarrow_t B';\ A' \vee_A B' \Longrightarrow_t C]\!] \Longrightarrow A \vee_A B \Longrightarrow_t C$
by (*meson entt-disjE entt-disjI1-direct entt-disjI2-direct entt-trans*)

Apply forward rule on left or right side of merge

lemma *gen-merge-cons1*: $[\![A \Longrightarrow_t A';\ A' \vee_A B \Longrightarrow_t C]\!] \Longrightarrow A \vee_A B \Longrightarrow_t C$
by (*meson merge-mono entt-refl*)

lemma *gen-merge-cons2*: $[\![B \Longrightarrow_t B';\ A \vee_A B' \Longrightarrow_t C]\!] \Longrightarrow A \vee_A B \Longrightarrow_t C$
by (*meson merge-mono entt-refl*)

lemmas *gen-merge-cons* = *gen-merge-cons1 gen-merge-cons2*

These rules are applied to recover pure values that have been destroyed by rule application

definition $\mathit{RECOVER\text{-}PURE}\ P\ Q \equiv P \Longrightarrow_t Q$

lemma *recover-pure*:
$\mathit{RECOVER\text{-}PURE}\ \mathit{emp}\ \mathit{emp}$
$[\![\mathit{RECOVER\text{-}PURE}\ \mathit{P2}\ \mathit{Q2};\ \mathit{RECOVER\text{-}PURE}\ \mathit{P1}\ \mathit{Q1}]\!] \Longrightarrow \mathit{RECOVER\text{-}PURE}\ (\mathit{P1}{*}\mathit{P2})\ (\mathit{Q1}{*}\mathit{Q2})$
$\mathit{CONSTRAINT}\ \mathit{is\text{-}pure}\ R \Longrightarrow \mathit{RECOVER\text{-}PURE}\ (\mathit{hn\text{-}invalid}\ R\ x\ y)\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)$
$\mathit{RECOVER\text{-}PURE}\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)$
unfolding *RECOVER-PURE-def*
subgoal by *sep-auto*
subgoal by (*drule* (*1*) *entt-star-mono*)
subgoal by (*rule recover-pure-aux*)
subgoal by *sep-auto*
done

lemma *recover-pure-triv*:
$\mathit{RECOVER\text{-}PURE}\ P\ P$
unfolding *RECOVER-PURE-def* **by** *sep-auto*

Weakening the postcondition by converting *invalid-assn* to λ- -. *true*

definition $\mathit{WEAKEN\text{-}HNR\text{-}POST}\ \Gamma\ \Gamma'\ \Gamma'' \equiv (\exists\, h.\ h{\models}\Gamma) \longrightarrow (\Gamma'' \Longrightarrow_t \Gamma')$

lemma *weaken-hnr-postI*:
assumes $\mathit{WEAKEN\text{-}HNR\text{-}POST}\ \Gamma\ \Gamma''\ \Gamma'$
assumes $\mathit{hn\text{-}refine}\ \Gamma\ c\ \Gamma'\ R\ a$
shows $\mathit{hn\text{-}refine}\ \Gamma\ c\ \Gamma''\ R\ a$
apply (*rule hn-refine-preI*)
apply (*rule hn-refine-cons-post*)
apply (*rule assms*)
using *assms*(*1*) **unfolding** *WEAKEN-HNR-POST-def* **by** *blast*

lemma *weaken-hnr-post-triv*: $\mathit{WEAKEN\text{-}HNR\text{-}POST}\ \Gamma\ P\ P$
unfolding *WEAKEN-HNR-POST-def*
by *sep-auto*

lemma *weaken-hnr-post*:
$[\![\mathit{WEAKEN\text{-}HNR\text{-}POST}\ \Gamma\ P\ P';\ \mathit{WEAKEN\text{-}HNR\text{-}POST}\ \Gamma'\ Q\ Q']\!] \Longrightarrow \mathit{WEAKEN\text{-}HNR\text{-}POST}\ (\Gamma{*}\Gamma')\ (P{*}Q)\ (P'{*}Q')$
$\mathit{WEAKEN\text{-}HNR\text{-}POST}\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)$
$\mathit{WEAKEN\text{-}HNR\text{-}POST}\ (\mathit{hn\text{-}ctxt}\ R\ x\ y)\ (\mathit{hn\text{-}invalid}\ R\ x\ y)\ (\mathit{hn\text{-}ctxt}\ (\lambda\text{- -}.\ \mathit{true})\ x\ y)$
proof (*goal-cases*)
case *1* **thus** *?case*
unfolding *WEAKEN-HNR-POST-def*
apply *clarsimp*
apply (*rule entt-star-mono*)
by (*auto simp*: *mod-star-conv*)

next
  case *2* **thus** *?case* **by** (*rule weaken-hnr-post-triv*)
next
  case *3* **thus** *?case*
    unfolding *WEAKEN-HNR-POST-def*
    by (*sep-auto simp*: *invalid-assn-def hn-ctxt-def*)
qed

lemma *reorder-enttI*:
  assumes $A{*}true = C{*}true$
  assumes $B{*}true = D{*}true$
  shows $(A \Longrightarrow_t B) \equiv (C \Longrightarrow_t D)$
  apply (*intro eq-reflection*)
  unfolding *entt-def-true*
  by (*simp add*: *assms*)

lemma *merge-sat1*: $(A \vee_A A' \Longrightarrow_t Am) \Longrightarrow (A \vee_A Am \Longrightarrow_t Am)$
  using *entt-disjD1 entt-disjE* **by** *blast*
lemma *merge-sat2*: $(A \vee_A A' \Longrightarrow_t Am) \Longrightarrow (Am \vee_A A' \Longrightarrow_t Am)$
  using *entt-disjD2 entt-disjE* **by** *blast*

ML ‹
signature SEPREF-FRAME = sig

  (∗ *Check if subgoal is a frame obligation* ∗)
  (∗*val is-frame* : *term* −> *bool* ∗)
  (∗ *Check if subgoal is a merge obligation* ∗)
  val is-merge: *term* −> *bool*
  (∗ *Perform frame inference* ∗)
  val frame-tac: (*Proof.context* −> *tactic′*) −> *Proof.context* −> *tactic′*
  (∗ *Perform merging* ∗)
  val merge-tac: (*Proof.context* −> *tactic′*) −> *Proof.context* −> *tactic′*

  val frame-step-tac: (*Proof.context* −> *tactic′*) −> *bool* −> *Proof.context* −> *tactic′*

  (∗ *Reorder frame* ∗)
  val prepare-frame-tac : *Proof.context* −> *tactic′*
  (∗ *Solve a RECOVER-PURE goal, inserting constraints as necessary* ∗)
  val recover-pure-tac: *Proof.context* −> *tactic′*

```
  (* Split precondition of hnr−goal into frame and arguments *)
  val align-goal-tac: Proof.context −> tactic′
  (* Normalize goal′s precondition *)
  val norm-goal-pre-tac: Proof.context −> tactic′
  (* Rearrange precondition of hnr−term according to parameter order, normalize
all relations *)
  val align-rl-conv: Proof.context −> conv

  (* Convert hn-invalid to λ- -. true in postcondition of hnr−goal. Makes proving
the goal easier.*)
  val weaken-post-tac: Proof.context −> tactic′

  val add-normrel-eq : thm −> Context.generic −> Context.generic
  val del-normrel-eq : thm −> Context.generic −> Context.generic
  val get-normrel-eqs : Proof.context −> thm list

  val cfg-debug: bool Config.T

  val setup: theory −> theory
end


structure Sepref-Frame : SEPREF-FRAME = struct

  val cfg-debug =
    Attrib.setup-config-bool @{binding sepref-debug-frame} (K false)

  val DCONVERSION = Sepref-Debugging.DBG-CONVERSION cfg-debug
  val dbg-msg-tac = Sepref-Debugging.dbg-msg-tac cfg-debug


  structure normrel-eqs = Named-Thms (
    val name = @{binding sepref-frame-normrel-eqs}
    val description = Equations to normalize relations for frame matching
  )

  val add-normrel-eq = normrel-eqs.add-thm
  val del-normrel-eq = normrel-eqs.del-thm
  val get-normrel-eqs = normrel-eqs.get

  val mk-entailst = HOLogic.mk-binrel @{const-name entailst}


  local
    open Sepref-Basic Refine-Util Conv

    fun assn-ord p = case apply2 dest-hn-ctxt-opt p of
        (NONE,NONE) => EQUAL
```

```
    | (SOME -, NONE) => LESS
    | (NONE, SOME -) => GREATER
    | (SOME (-,a,-), SOME (-,a',-)) => Term-Ord.fast-term-ord (a,a')

in
  fun reorder-ctxt-conv ctxt ct = let
    val cert = Thm.cterm-of ctxt

    val new-ct = Thm.term-of ct
      |> strip-star
      |> sort assn-ord
      |> list-star
      |> cert

    val thm = Goal.prove-internal ctxt [] (mk-cequals (ct,new-ct))
      (fn - => simp-tac
        (put-simpset HOL-basic-ss ctxt addsimps @{thms star-aci}) 1)

  in
    thm
  end

  fun prepare-fi-conv ctxt ct = case Thm.term-of ct of
    (t as @{mpat ?P ⟹t ?Q}) => let

      (* Build table from abs−vars to ctxt *)
     val (Qm, Qum) = strip-star Q |> filter-out is-true |> List.partition is-hn-ctxt

      val Qtab = (
        Qm |> map (fn x => (#2 (dest-hn-ctxt x),(NONE,x)))
        |> Termtab.make
      ) handle
          e as (Termtab.DUP -) => (
            tracing (Dup heap:  ^ @{make-string} ct); raise e)

      (* Go over entries in P and try to find a partner *)
      val (Qtab,Pum) = fold (fn a => fn (Qtab,Pum) =>
        case dest-hn-ctxt-opt a of
          NONE => (Qtab,a::Pum)
        | SOME (-,p,-) => ( case Termtab.lookup Qtab p of
            SOME (NONE,tg) => (Termtab.update (p,(SOME a,tg)) Qtab, Pum)
          | - => (Qtab,a::Pum)
          )
      ) (strip-star P) (Qtab,[])

      val Pum = filter-out is-true Pum

      (* Read out information from Qtab *)
      val (pairs,Qum2) = Termtab.dest Qtab |> map #2
```

```
          |> List.partition (is-some o #1)
          |> apfst (map (apfst the))
          |> apsnd (map #2)

      (* Build reordered terms: P' = fst pairs * Pum, Q' = snd pairs * (Qum2*Qum)
*)
        val P' = mk-star (list-star (map fst pairs), list-star Pum)
        val Q' = mk-star (list-star (map snd pairs), list-star (Qum2@Qum))

        val new-t = mk-entailst (P', Q')
        val goal-t = Logic.mk-equals (t,new-t)

        val goal-ctxt = Variable.declare-term goal-t ctxt

          val msg-tac = dbg-msg-tac (Sepref-Debugging.msg-allgoals Solving frame
permutation) goal-ctxt 1
        val tac =
          msg-tac
          THEN ALLGOALS (resolve-tac goal-ctxt @{thms reorder-enttI})
          THEN star-permute-tac goal-ctxt

        val goal-ct = Thm.cterm-of ctxt goal-t

        val thm = Goal.prove-internal ctxt [] goal-ct (fn - => tac)

      in
        thm
      end
    | - => no-conv ct

  end

  fun is-merge @{mpat Trueprop (- ∨A - ⟹t -)} = true | is-merge - = false
  fun is-gen-frame @{mpat Trueprop (- ⟹t -)} = true | is-gen-frame - = false


  fun prepare-frame-tac ctxt = let
    open Refine-Util Conv
    val frame-ss = put-simpset HOL-basic-ss ctxt addsimps
      @{thms mult-1-right[where 'a=assn] mult-1-left[where 'a=assn]}
  in
    CONVERSION Thm.eta-conversion THEN'
    (*CONCL-COND' is-frame THEN'*)
    simp-tac frame-ss THEN'
    CONVERSION (HOL-concl-conv prepare-fi-conv ctxt)
  end


  local
```

```
  fun wrap-side-tac side-tac dbg tac = tac THEN-ALL-NEW-FWD (
    CONCL-COND' is-gen-frame
    ORELSE' (if dbg then TRY-SOLVED' else SOLVED') side-tac
  )
in
  fun frame-step-tac side-tac dbg ctxt = let
    open Refine-Util Conv

    (* Constraint solving is built−in *)
    val side-tac = Sepref-Constraints.constraint-tac ctxt ORELSE' side-tac ctxt

    val frame-thms = @{thms frame-thms} @
    Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-frame-match-rules}

    val merge-thms = @{thms merge-thms} @
      Named-Theorems.get ctxt @{named-theorems sepref-frame-merge-rules}
    val ss = put-simpset HOL-basic-ss ctxt addsimps normrel-eqs.get ctxt
  fun frame-thm-tac dbg = wrap-side-tac side-tac dbg (resolve-tac ctxt frame-thms)
  fun merge-thm-tac dbg = wrap-side-tac side-tac dbg (resolve-tac ctxt merge-thms)

    fun thm-tac dbg = CONCL-COND' is-merge THEN-ELSE' (merge-thm-tac
dbg, frame-thm-tac dbg)
  in
    full-simp-tac ss THEN' thm-tac dbg
  end
end

fun frame-loop-tac side-tac ctxt = let

in
  TRY o (
    REPEAT-ALL-NEW (DETERM o frame-step-tac side-tac false ctxt)
  )
end


fun frame-tac side-tac ctxt = let
  open Refine-Util Conv
  val frame-rem-thms = @{thms frame-rem-thms}
  @ Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-frame-rem-rules}
 val solve-remainder-tac = TRY o REPEAT-ALL-NEW (DETERM o resolve-tac
ctxt frame-rem-thms)
in
  (prepare-frame-tac ctxt
    THEN' resolve-tac ctxt @{thms ent-star-mono entt-star-mono})
  THEN-ALL-NEW-LIST [
    frame-loop-tac side-tac ctxt,
    solve-remainder-tac
  ]
```

```
end

fun merge-tac side-tac ctxt = let
  open Refine-Util Conv
  fun merge-conv ctxt = arg1-conv (binop-conv (reorder-ctxt-conv ctxt))
in
  CONVERSION Thm.eta-conversion THEN'
  CONCL-COND' is-merge THEN'
  simp-tac (put-simpset HOL-basic-ss ctxt addsimps @{thms star-aci}) THEN'
  CONVERSION (HOL-concl-conv merge-conv ctxt) THEN'
  frame-loop-tac side-tac ctxt
end

val setup = normrel-eqs.setup

local
  open Sepref-Basic
  fun is-invalid @{mpat hn-invalid - - - :: assn} = true | is-invalid - = false
   fun contains-invalid @{mpat Trueprop (RECOVER-PURE ?Q -)} = exists
is-invalid (strip-star Q)
    | contains-invalid - = false

in
  fun recover-pure-tac ctxt =
    CONCL-COND' contains-invalid THEN-ELSE' (
      REPEAT-ALL-NEW (DETERM o (resolve-tac ctxt @{thms recover-pure}
ORELSE' Sepref-Constraints.constraint-tac ctxt)),
      resolve-tac ctxt @{thms recover-pure-triv}
    )
end

local
  open Sepref-Basic Refine-Util
  datatype cte = Other of term | Hn of term * term * term
  fun dest-ctxt-elem @{mpat hn-ctxt ?R ?a ?c} = Hn (R,a,c)
    | dest-ctxt-elem t = Other t

  fun mk-ctxt-elem (Other t) = t
    | mk-ctxt-elem (Hn (R,a,c)) = @{mk-term hn-ctxt ?R ?a ?c}

  fun match x (Hn (-,y,-)) = x aconv y
    | match - - = false

  fun dest-with-frame (*ctxt*) - t = let
    val (P,c,Q,R,a) = dest-hn-refine t

    val (-,(-,args)) = dest-hnr-absfun a
    val pre-ctes = strip-star P |> map dest-ctxt-elem
```

```
  val (pre-args,frame) =
    (case split-matching match args pre-ctes of
      NONE => raise TERM(align-conv: Could not match all arguments,[P,a])
     | SOME x => x)

in
  ((frame,pre-args),c,Q,R,a)
end

fun align-goal-conv-aux ctxt t = let
  val ((frame,pre-args),c,Q,R,a) = dest-with-frame ctxt t
  val P' = apply2 (list-star o map mk-ctxt-elem) (frame,pre-args) |> mk-star
  val t' = mk-hn-refine (P',c,Q,R,a)
in t' end

fun align-rl-conv-aux ctxt t = let
  val ((frame,pre-args),c,Q,R,a) = dest-with-frame ctxt t

  val - = frame = [] orelse raise TERM (align-rl-conv: Extra preconditions in
rule,[t,list-star (map mk-ctxt-elem frame)])

  val P' = list-star (map mk-ctxt-elem pre-args)
  val t' = mk-hn-refine (P',c,Q,R,a)
in t' end


fun normrel-conv ctxt = let
  val ss = put-simpset HOL-basic-ss ctxt addsimps normrel-eqs.get ctxt
in
  Simplifier.rewrite ss
end

in
fun align-goal-conv ctxt = f-tac-conv ctxt (align-goal-conv-aux ctxt) star-permute-tac

fun norm-goal-pre-conv ctxt = let
  open Conv

  fun conv ctxt = let
    val nr-conv = normrel-conv ctxt
  in
    hn-refine-conv nr-conv all-conv all-conv all-conv all-conv
  end
in
  HOL-concl-conv conv ctxt
end

fun norm-goal-pre-tac ctxt = CONVERSION (norm-goal-pre-conv ctxt)
```

fun align-rl-conv ctxt = let
open Conv
fun conv ctxt = let
val nr-conv = normrel-conv ctxt
in
hn-refine-conv nr-conv all-conv nr-conv nr-conv all-conv
end
in
HOL-concl-conv (*fn ctxt => f-tac-conv ctxt* (*align-rl-conv-aux ctxt*) *star-permute-tac*)
ctxt
then-conv HOL-concl-conv conv ctxt
end

fun align-goal-tac ctxt =
CONCL-COND′ is-hn-refine-concl
THEN′ DCONVERSION ctxt (*HOL-concl-conv align-goal-conv ctxt*)
end

fun weaken-post-tac ctxt = TRADE (*fn ctxt =>*
resolve-tac ctxt @{*thms weaken-hnr-postI*}
THEN′ SOLVED′ (*REPEAT-ALL-NEW* (*DETERM o resolve-tac ctxt* @{*thms*
weaken-hnr-post weaken-hnr-post-triv}))
) *ctxt*

end
›

setup *Sepref-Frame.setup*

method-setup *weaken-hnr-post* = ‹*Scan.succeed* (*fn ctxt => SIMPLE-METHOD′*
(*Sepref-Frame.weaken-post-tac ctxt*))›
‹*Convert hn-invalid to hn-ctxt* (λ- -. *true*) *in postcondition of hn-refine goal*›

method *extract-hnr-invalids* = (
rule hn-refine-preI,
((*drule mod-starD hn-invalidI* | *elim conjE exE*)+)*?*
) — Extract *hn-invalid* - - - = *true* preconditions from *hn-refine* goal.

lemmas [*sepref-frame-normrel-eqs*] = *the-pure-pure pure-the-pure*

end

1.5 Refinement Rule Management

theory *Sepref-Rules*

imports *Sepref-Basic Sepref-Constraints*
begin

This theory contains tools for managing the refinement rules used by Sepref

The theories are based on uncurried functions, i.e., every function has type $'a \Rightarrow 'b$, where $'a$ is the tuple of parameters, or unit if there are none.

1.5.1 Assertion Interface Binding

Binding of interface types to refinement assertions

definition $\mathit{intf\text{-}of\text{-}assn} :: ('a \Rightarrow \text{-} \Rightarrow \mathit{assn}) \Rightarrow 'b\ \mathit{itself} \Rightarrow \mathit{bool}$ **where**
$[\mathit{simp}]\colon \mathit{intf\text{-}of\text{-}assn}\ a\ b = \mathit{True}$

lemma $\mathit{intf\text{-}of\text{-}assnI}\colon \mathit{intf\text{-}of\text{-}assn}\ R\ \mathit{TYPE}('a)$ **by** *simp*

named-theorems-rev *intf-of-assn* ‹*Links between refinement assertions and interface types*›

lemma $\mathit{intf\text{-}of\text{-}assn\text{-}fallback}\colon \mathit{intf\text{-}of\text{-}assn}\ (R :: 'a \Rightarrow \text{-} \Rightarrow \mathit{assn})\ \mathit{TYPE}('a)$ **by** *simp*

1.5.2 Function Refinement with Precondition

definition $\mathit{fref} :: ('c \Rightarrow \mathit{bool}) \Rightarrow ('a \times 'c)\ \mathit{set} \Rightarrow ('b \times 'd)\ \mathit{set}$
$\Rightarrow (('a \Rightarrow 'b) \times ('c \Rightarrow 'd))\ \mathit{set}$
$(‹[\text{-}]_f\ \text{-} \rightarrow \text{-}›\ [0,60,60]\ 60)$
where $[P]_f\ R \rightarrow S \equiv \{(f,g).\ \forall x\ y.\ P\ y \wedge (x,y) \in R \longrightarrow (f\ x,\ g\ y) \in S\}$

abbreviation $\mathit{freft}\ (‹\text{-} \rightarrow_f \text{-}›\ [60,60]\ 60)$ **where** $R \rightarrow_f S \equiv ([\lambda\text{-}.\ \mathit{True}]_f\ R \rightarrow S)$

lemma $\mathit{rel2p\text{-}fref}[\mathit{rel2p}]\colon \mathit{rel2p}\ (\mathit{fref}\ P\ R\ S)$
$= (\lambda f\ g.\ (\forall x\ y.\ P\ y \longrightarrow \mathit{rel2p}\ R\ x\ y \longrightarrow \mathit{rel2p}\ S\ (f\ x)\ (g\ y)))$
by $(\mathit{auto\ simp}\colon \mathit{fref\text{-}def}\ \mathit{rel2p\text{-}def}[\mathit{abs\text{-}def}])$

lemma *fref-cons*:
assumes $(f,g) \in [P]_f\ R \rightarrow S$
assumes $\bigwedge c\ a.\ (c,a) \in R' \Longrightarrow Q\ a \Longrightarrow P\ a$
assumes $R' \subseteq R$
assumes $S \subseteq S'$
shows $(f,g) \in [Q]_f\ R' \rightarrow S'$
using *assms*
unfolding *fref-def*
by *fastforce*

lemmas $\mathit{fref\text{-}cons}' = \mathit{fref\text{-}cons}[\mathit{OF}\ \text{-}\ \text{-}\ \mathit{order\text{-}refl}\ \mathit{order\text{-}refl}]$

lemma *frefI*[*intro?*]:

assumes $\bigwedge x\ y.\ [\![P\ y;\ (x,y)\in R]\!] \Longrightarrow (f\ x,\ g\ y)\in S$
shows $(f,g)\in fref\ P\ R\ S$
using *assms*
unfolding *fref-def*
by *auto*

lemma *fref-ncI*: $(f,g)\in R\to S \Longrightarrow (f,g)\in R\to_f S$
apply (*rule frefI*)
apply *parametricity*
done

lemma *frefD*:
assumes $(f,g)\in fref\ P\ R\ S$
shows $[\![P\ y;\ (x,y)\in R]\!] \Longrightarrow (f\ x,\ g\ y)\in S$
using *assms*
unfolding *fref-def*
by *auto*

lemma *fref-ncD*: $(f,g)\in R\to_f S \Longrightarrow (f,g)\in R\to S$
apply (*rule fun-relI*)
apply (*drule frefD*)
apply *simp*
apply *assumption+*
done

lemma *fref-compI*:
$fref\ P\ R1\ R2\ O\ fref\ Q\ S1\ S2 \subseteq$
$fref\ (\lambda x.\ Q\ x \wedge (\forall y.\ (y,x)\in S1 \longrightarrow P\ y))\ (R1\ O\ S1)\ (R2\ O\ S2)$
unfolding *fref-def*
apply (*auto*)
apply *blast*
done

lemma *fref-compI'*:
$[\![\ (f,g)\in fref\ P\ R1\ R2;\ (g,h)\in fref\ Q\ S1\ S2\ ]\!]$
$\Longrightarrow (f,h) \in fref\ (\lambda x.\ Q\ x \wedge (\forall y.\ (y,x)\in S1 \longrightarrow P\ y))\ (R1\ O\ S1)\ (R2\ O\ S2)$
using *fref-compI*[*of P R1 R2 Q S1 S2*]
by *auto*

lemma *fref-unit-conv*:
$(\lambda\text{-}.\ c,\ \lambda\text{-}.\ a) \in fref\ P\ unit\text{-}rel\ S \longleftrightarrow (P\ () \longrightarrow (c,a)\in S)$
by (*auto simp*: *fref-def*)

lemma *fref-uncurry-conv*:
$(uncurry\ c,\ uncurry\ a) \in fref\ P\ (R1\times_r R2)\ S$
$\longleftrightarrow (\forall x1\ y1\ x2\ y2.\ P\ (y1,y2) \longrightarrow (x1,y1)\in R1 \longrightarrow (x2,y2)\in R2 \longrightarrow (c\ x1\ x2,\ a\ y1\ y2) \in S)$
by (*auto simp*: *fref-def*)

lemma *fref-mono*: $\llbracket \bigwedge x.\ P'\ x \Longrightarrow P\ x;\ R' \subseteq R;\ S \subseteq S' \rrbracket$
$\Longrightarrow$ *fref P R S* $\subseteq$ *fref P' R' S'*
unfolding *fref-def*
by *auto blast*

lemma *fref-composeI*:
assumes *FR1*: $(f,g) \in$ *fref P R1 R2*
assumes *FR2*: $(g,h) \in$ *fref Q S1 S2*
assumes *C1*: $\bigwedge x.\ P'\ x \Longrightarrow Q\ x$
assumes *C2*: $\bigwedge x\ y.\ \llbracket P'\ x;\ (y,x) \in S1 \rrbracket \Longrightarrow P\ y$
assumes *R1*: $R' \subseteq R1\ O\ S1$
assumes *R2*: $R2\ O\ S2 \subseteq S'$
assumes *FH*: $f'=f\ h'=h$
shows $(f',h') \in$ *fref P' R' S'*
unfolding *FH*
apply (*rule subsetD*[*OF fref-mono fref-compI'*[*OF FR1 FR2*]])
using *C1 C2* **apply** *blast*
using *R1* **apply** *blast*
using *R2* **apply** *blast*
done

lemma *fref-triv*: $A \subseteq Id \Longrightarrow (f,f) \in [P]_f\ A \rightarrow Id$
by (*auto simp*: *fref-def*)

1.5.3 Heap-Function Refinement

The following relates a heap-function with a pure function. It contains a precondition, a refinement assertion for the arguments before and after execution, and a refinement relation for the result.

definition *hfref*
::
$('a \Rightarrow bool)$
$\Rightarrow (('a \Rightarrow 'ai \Rightarrow assn) \times ('a \Rightarrow 'ai \Rightarrow assn))$
$\Rightarrow ('b \Rightarrow 'bi \Rightarrow assn)$
$\Rightarrow (('ai \Rightarrow 'bi\ Heap) \times ('a \Rightarrow 'b\ nres))\ set$
(‹[-]$_a$ - → -› [*0*,*60*,*60*] *60*)
where
$[P]_a\ RS \rightarrow T \equiv \{\ (f,g)\ .\ \forall\ c\ a.\ \ P\ a \longrightarrow$ *hn-refine* (*fst RS a c*) (*f c*) (*snd RS a c*) *T* (*g a*)}

abbreviation *hfreft* (‹- $\rightarrow_a$ -› [*60*,*60*] *60*) **where** $RS \rightarrow_a T \equiv ([\lambda\text{-}.\ True]_a\ RS \rightarrow T)$

lemma *hfrefI*[*intro?*]:
assumes $\bigwedge c\ a.\ P\ a \Longrightarrow$ *hn-refine* (*fst RS a c*) (*f c*) (*snd RS a c*) *T* (*g a*)
shows $(f,g) \in$ *hfref P RS T*
using *assms* **unfolding** *hfref-def* **by** *blast*

lemma *hfrefD*:
assumes $(f,g) \in hfref\ P\ RS\ T$
shows $\bigwedge c\ a.\ P\ a \Longrightarrow hn\text{-}refine\ (fst\ RS\ a\ c)\ (f\ c)\ (snd\ RS\ a\ c)\ T\ (g\ a)$
using *assms* **unfolding** *hfref-def* **by** *blast*

lemma *hfref-to-ASSERT-conv*:
$NO\text{-}MATCH\ (\lambda\text{-}.\ True)\ P \Longrightarrow (a,b) \in [P]_a\ R \to S \longleftrightarrow (a,\lambda x.\ ASSERT\ (P\ x)$
$\gg b\ x) \in R \to_a S$
unfolding *hfref-def*
apply (*clarsimp*; *safe*; *clarsimp?*)
apply (*rule hn-refine-nofailI*)
apply (*simp add*: *refine-pw-simps*)
subgoal for *xc xa*
apply (*drule spec*[*of - xc*])
apply (*drule spec*[*of - xa*])
by *simp*
done

A pair of argument refinement assertions can be created by the input assertion and the information whether the parameter is kept or destroyed by the function.

primrec *hf-pres*
$:: ('a \Rightarrow 'b \Rightarrow assn) \Rightarrow bool \Rightarrow ('a \Rightarrow 'b \Rightarrow assn) \times ('a \Rightarrow 'b \Rightarrow assn)$
where
$hf\text{-}pres\ R\ True = (R,R) \mid hf\text{-}pres\ R\ False = (R,invalid\text{-}assn\ R)$

abbreviation *hfkeep*
$:: ('a \Rightarrow 'b \Rightarrow assn) \Rightarrow ('a \Rightarrow 'b \Rightarrow assn) \times ('a \Rightarrow 'b \Rightarrow assn)$
$(\langle(\text{-}^k)\rangle\ [1000]\ 999)$
where $R^k \equiv hf\text{-}pres\ R\ True$
abbreviation *hfdrop*
$:: ('a \Rightarrow 'b \Rightarrow assn) \Rightarrow ('a \Rightarrow 'b \Rightarrow assn) \times ('a \Rightarrow 'b \Rightarrow assn)$
$(\langle(\text{-}^d)\rangle\ [1000]\ 999)$
where $R^d \equiv hf\text{-}pres\ R\ False$

abbreviation $hn\text{-}kede\ R\ kd \equiv hn\text{-}ctxt\ (snd\ (hf\text{-}pres\ R\ kd))$
abbreviation $hn\text{-}keep\ R \equiv hn\text{-}kede\ R\ True$
abbreviation $hn\text{-}dest\ R \equiv hn\text{-}kede\ R\ False$

lemma *keep-drop-sels*[*simp*]:
$fst\ (R^k) = R$
$snd\ (R^k) = R$
$fst\ (R^d) = R$
$snd\ (R^d) = invalid\text{-}assn\ R$
by *auto*

lemma *hf-pres-fst*[*simp*]: $fst\ (hf\text{-}pres\ R\ k) = R$ **by** (*cases k*) *auto*

The following operator combines multiple argument assertion-pairs to argument assertion-pairs for the product. It is required to state argument assertion-pairs for uncurried functions.

definition *hfprod* ::
  $(('a \Rightarrow 'b \Rightarrow assn)\times('a \Rightarrow 'b \Rightarrow assn))$
  $\Rightarrow (('c \Rightarrow 'd \Rightarrow assn)\times('c \Rightarrow 'd \Rightarrow assn))$
  $\Rightarrow ((('a\times'c) \Rightarrow ('b \times 'd) \Rightarrow assn) \times (('a\times'c) \Rightarrow ('b \times 'd) \Rightarrow assn))$
  (**infixl** ‹$*_a$› *65*)
  where $RR *_a SS \equiv (prod\text{-}assn\ (fst\ RR)\ (fst\ SS),\ prod\text{-}assn\ (snd\ RR)\ (snd\ SS))$

lemma *hfprod-fst-snd*[*simp*]:
  $fst\ (A *_a B) = prod\text{-}assn\ (fst\ A)\ (fst\ B)$
  $snd\ (A *_a B) = prod\text{-}assn\ (snd\ A)\ (snd\ B)$
  unfolding *hfprod-def* **by** *auto*

Conversion from fref to hfref

lemma *fref-to-pure-hfref'*:
  assumes $(f,g) \in [P]_f\ R\rightarrow\langle S\rangle nres\text{-}rel$
  assumes $\bigwedge x.\ x\in Domain\ R \cap R^{-1}\ ``Collect\ P \Longrightarrow f\ x = RETURN\ (f'\ x)$
  shows $(return\ o\ f',\ g) \in [P]_a\ (pure\ R)^k\rightarrow pure\ S$
  apply (*rule hfrefI*) **apply** (*rule hn-refineI*)
  using *assms*
  apply ((*sep-auto simp*: *fref-def pure-def pw-le-iff pw-nres-rel-iff*
    refine-pw-simps eintros del: *exI*))
  apply *force*
  done

Conversion from hfref to hnr

This section contains the lemmas. The ML code is further down.

lemma *hf2hnr*:
  assumes $(f,g) \in [P]_a\ R \rightarrow S$
  shows $\forall x\ xi.\ P\ x \longrightarrow hn\text{-}refine\ (emp * hn\text{-}ctxt\ (fst\ R)\ x\ xi)\ (f\$xi)\ (emp * hn\text{-}ctxt\ (snd\ R)\ x\ xi)\ S\ (g\$x)$
  using *assms*
  unfolding *hfref-def*
  by (*auto simp*: *hn-ctxt-def*)

definition [*simp*]: $to\text{-}hnr\text{-}prod \equiv prod\text{-}assn$

lemma *to-hnr-prod-fst-snd*:
  $fst\ (A *_a B) = to\text{-}hnr\text{-}prod\ (fst\ A)\ (fst\ B)$
  $snd\ (A *_a B) = to\text{-}hnr\text{-}prod\ (snd\ A)\ (snd\ B)$
  unfolding *hfprod-def* **by** *auto*

lemma *hnr-uncurry-unfold*:
$(\forall x\ xi.\ P\ x \longrightarrow$
$\quad hn\text{-}refine$
$\quad\quad (\Gamma * hn\text{-}ctxt\ (to\text{-}hnr\text{-}prod\ A\ B)\ x\ xi)$
$\quad\quad (fi\ xi)$
$\quad\quad (\Gamma' * hn\text{-}ctxt\ (to\text{-}hnr\text{-}prod\ A'\ B')\ x\ xi)$
$\quad\quad R$
$\quad\quad (f\ x))$
$\longleftrightarrow (\forall b\ bi\ a\ ai.\ P\ (a,b) \longrightarrow$
$\quad hn\text{-}refine$
$\quad\quad (\Gamma * hn\text{-}ctxt\ B\ b\ bi * hn\text{-}ctxt\ A\ a\ ai)$
$\quad\quad (fi\ (ai,bi))$
$\quad\quad (\Gamma' * hn\text{-}ctxt\ B'\ b\ bi * hn\text{-}ctxt\ A'\ a\ ai)$
$\quad\quad R$
$\quad\quad (f\ (a,b))$
$)$
by (*auto simp*: *hn-ctxt-def prod-assn-def star-aci*)

lemma *hnr-intro-dummy*:
$\forall x\ xi.\ P\ x \longrightarrow hn\text{-}refine\ (\Gamma\ x\ xi)\ (c\ xi)\ (\Gamma'\ x\ xi)\ R\ (a\ x) \Longrightarrow \forall x\ xi.\ P\ x \longrightarrow hn\text{-}refine\ (emp{*}\Gamma\ x\ xi)\ (c\ xi)\ (emp{*}\Gamma'\ x\ xi)\ R\ (a\ x)$
by *simp*

lemma *hn-ctxt-ctxt-fix-conv*: *hn-ctxt* (*hn-ctxt R*) = *hn-ctxt R*
by (*simp add*: *hn-ctxt-def*[*abs-def*])

lemma *uncurry-APP*: *uncurry* $f\$(a,b) = f\$a\$b$ **by** *auto*

lemma *norm-RETURN-o*:
$\bigwedge f.\ (RETURN\ o\ f)\$x = (RETURN\$(f\$x))$
$\bigwedge f.\ (RETURN\ oo\ f)\$x\$y = (RETURN\$(f\$x\$y))$
$\bigwedge f.\ (RETURN\ ooo\ f)\$x\$y\$z = (RETURN\$(f\$x\$y\$z))$
$\bigwedge f.\ (\lambda x.\ RETURN\ ooo\ f\ x)\$x\$y\$z\$a = (RETURN\$(f\$x\$y\$z\$a))$
$\bigwedge f.\ (\lambda x\ y.\ RETURN\ ooo\ f\ x\ y)\$x\$y\$z\$a\$b = (RETURN\$(f\$x\$y\$z\$a\$b))$
by *auto*

lemma *norm-return-o*:
$\bigwedge f.\ (return\ o\ f)\$x = (return\$(f\$x))$
$\bigwedge f.\ (return\ oo\ f)\$x\$y = (return\$(f\$x\$y))$
$\bigwedge f.\ (return\ ooo\ f)\$x\$y\$z = (return\$(f\$x\$y\$z))$
$\bigwedge f.\ (\lambda x.\ return\ ooo\ f\ x)\$x\$y\$z\$a = (return\$(f\$x\$y\$z\$a))$
$\bigwedge f.\ (\lambda x\ y.\ return\ ooo\ f\ x\ y)\$x\$y\$z\$a\$b = (return\$(f\$x\$y\$z\$a\$b))$
by *auto*

lemma *hn-val-unit-conv-emp*[*simp*]: *hn-val unit-rel x y* = *emp*

by (*auto simp*: *hn-ctxt-def pure-def*)

Conversion from hnr to hfref

This section contains the lemmas. The ML code is further down.

abbreviation *id-assn* ≡ *pure Id*
abbreviation (*input*) *unit-assn* ≡ *id-assn* :: *unit* ⇒ -

lemma *pure-unit-rel-eq-empty*: *unit-assn x y* = *emp*
by (*auto simp*: *pure-def*)

lemma *uc-hfprod-sel*:
fst (*A* $*_a$ *B*) *a c* = (*case* (*a*,*c*) *of* ((*a1*,*a2*),(*c1*,*c2*)) ⇒ *fst A a1 c1* ∗ *fst B a2 c2*)
snd (*A* $*_a$ *B*) *a c* = (*case* (*a*,*c*) *of* ((*a1*,*a2*),(*c1*,*c2*)) ⇒ *snd A a1 c1* ∗ *snd B a2 c2*)
unfolding *hfprod-def prod-assn-def*[*abs-def*] **by** *auto*

Conversion from relation to fref

This section contains the lemmas. The ML code is further down.

definition *CURRY R* ≡ { (*f*,*g*). (*uncurry f*, *uncurry g*) ∈ *R* }

lemma *fref-param1*: *R*→*S* = *fref* (λ-. *True*) *R S*
by (*auto simp*: *fref-def fun-relD*)

lemma *fref-nest*: *fref P1 R1* (*fref P2 R2 S*)
≡ *CURRY* (*fref* (λ(*a*,*b*). *P1 a* ∧ *P2 b*) ($R1 \times_r R2$) *S*)
apply (*rule eq-reflection*)
by (*auto simp*: *fref-def CURRY-def*)

lemma *in-CURRY-conv*: (*f*,*g*) ∈ *CURRY R* ⟷ (*uncurry f*, *uncurry g*) ∈ *R*
unfolding *CURRY-def* **by** *auto*

lemma *uncurry0-APP*[*simp*]: *uncurry0 c* $ *x* = *c* **by** *auto*

lemma *fref-param0I*: (*c*,*a*)∈*R* ⟹ (*uncurry0 c*, *uncurry0 a*) ∈ *fref* (λ-. *True*) *unit-rel R*
by (*auto simp*: *fref-def*)

Composition

definition *hr-comp* :: ('*b* ⇒ '*c* ⇒ *assn*) ⇒ ('*b* × '*a*) *set* ⇒ '*a* ⇒ '*c* ⇒ *assn*
— Compose refinement assertion with refinement relation
where *hr-comp R1 R2 a c* ≡ $\exists_A$*b*. *R1 b c* ∗ ↑((*b*,*a*)∈*R2*)

definition *hrp-comp*
:: ('*d* ⇒ '*b* ⇒ *assn*) × ('*d* ⇒ '*c* ⇒ *assn*)

$\Rightarrow ('d \times 'a)\ set \Rightarrow ('a \Rightarrow 'b \Rightarrow assn) \times ('a \Rightarrow 'c \Rightarrow assn)$
— Compose argument assertion-pair with refinement relation
where *hrp-comp RR′ S* ≡ (*hr-comp* (*fst RR′*) *S*, *hr-comp* (*snd RR′*) *S*)

lemma *hr-compI*: $(b,a) \in R2 \Longrightarrow R1\ b\ c \Longrightarrow_A hr\text{-}comp\ R1\ R2\ a\ c$
unfolding *hr-comp-def*
by *sep-auto*

lemma *hr-comp-Id1*[*simp*]: *hr-comp* (*pure Id*) *R* = *pure R*
unfolding *hr-comp-def*[*abs-def*] *pure-def*
apply (*intro ext ent-iffI*)
by *sep-auto+*

lemma *hr-comp-Id2*[*simp*]: *hr-comp R Id* = *R*
unfolding *hr-comp-def*[*abs-def*]
apply (*intro ext ent-iffI*)
by *sep-auto+*

lemma *hr-comp-emp*[*simp*]: *hr-comp* ($\lambda a\ c.\ emp$) *R a c* = $\uparrow(\exists\, b.\ (b,a) \in R)$
unfolding *hr-comp-def*[*abs-def*]
apply (*intro ext ent-iffI*)
apply *sep-auto+*
done

lemma *hr-comp-prod-conv*[*simp*]:
hr-comp (*prod-assn Ra Rb*) ($Ra' \times_r Rb'$)
= *prod-assn* (*hr-comp Ra Ra′*) (*hr-comp Rb Rb′*)
unfolding *hr-comp-def*[*abs-def*] *prod-assn-def*[*abs-def*]
apply (*intro ext ent-iffI*)
apply *solve-entails* **apply** *clarsimp* **apply** *sep-auto*
apply *clarsimp* **apply** (*intro ent-ex-preI*)
apply (*rule ent-ex-postI*) **apply** (*sep-auto split*: *prod.splits*)
done

lemma *hr-comp-pure*: *hr-comp* (*pure R*) *S* = *pure* (*R O S*)
apply (*intro ext*)
apply (*rule ent-iffI*)
unfolding *hr-comp-def*[*abs-def*]
apply (*sep-auto simp*: *pure-def*)+
done

lemma *hr-comp-is-pure*[*safe-constraint-rules*]: *is-pure A* $\Longrightarrow$ *is-pure* (*hr-comp A B*)
by (*auto simp*: *hr-comp-pure is-pure-conv*)

lemma *hr-comp-the-pure*: *is-pure A* $\Longrightarrow$ *the-pure* (*hr-comp A B*) = *the-pure A O B*

unfolding *is-pure-conv*
by (*clarsimp simp*: *hr-comp-pure*)

lemma *rdomp-hrcomp-conv*: *rdomp* (*hr-comp A R*) *x* $\longleftrightarrow$ ($\exists$ *y*. *rdomp A y* $\wedge$ (*y*,*x*)$\in$*R*)
by (*auto simp*: *rdomp-def hr-comp-def*)

lemma *hn-rel-compI*:
$[\![$*nofail a*; (*b*,*a*)$\in\langle$*R2*$\rangle$*nres-rel*$]\!]$ $\Longrightarrow$ *hn-rel R1 b c* $\Longrightarrow_A$ *hn-rel* (*hr-comp R1 R2*) *a c*
unfolding *hr-comp-def hn-rel-def nres-rel-def*
apply (*clarsimp intro*!: *ent-ex-preI*)
apply (*drule* (*1*) *order-trans*)
apply (*simp add*: *ret-le-down-conv*)
by *sep-auto*

lemma *hr-comp-precise*[*constraint-rules*]:
assumes [*safe-constraint-rules*]: *precise R*
assumes *SV*: *single-valued S*
shows *precise* (*hr-comp R S*)
apply (*rule preciseI*)
unfolding *hr-comp-def*
apply *clarsimp*
by (*metis SV assms*(*1*) *preciseD single-valuedD*)

lemma *hr-comp-assoc*: *hr-comp* (*hr-comp R S*) *T* = *hr-comp R* (*S O T*)
apply (*intro ext*)
unfolding *hr-comp-def*
apply (*rule ent-iffI*; *clarsimp*)
apply *sep-auto*
apply (*rule ent-ex-preI*; *clarsimp*)
apply *sep-auto*
done

lemma *hnr-comp*:
assumes *R*: $\bigwedge$*b1 c1*. *P b1* $\Longrightarrow$ *hn-refine* (*R1 b1 c1* $*$ Γ) (*c c1*) (*R1p b1 c1* $*$ Γ') *R* (*b b1*)
assumes *S*: $\bigwedge$*a1 b1*. $[\![$*Q a1*; (*b1*,*a1*)$\in$*R1'*$]\!]$ $\Longrightarrow$ (*b b1*,*a a1*)$\in\langle$*R'*$\rangle$*nres-rel*
assumes *PQ*: $\bigwedge$*a1 b1*. $[\![$*Q a1*; (*b1*,*a1*)$\in$*R1'*$]\!]$ $\Longrightarrow$ *P b1*
assumes *Q*: *Q a1*
shows *hn-refine*
(*hr-comp R1 R1' a1 c1* $*$ Γ)
(*c c1*)
(*hr-comp R1p R1' a1 c1* $*$ Γ')
(*hr-comp R R'*)
(*a a1*)
unfolding *hn-refine-alt*
proof *clarsimp*

assume *NF*: *nofail* (*a a1*)
show
<*hr-comp R1 R1′ a1 c1* ∗ Γ>
c c1
<λ*r*. *hn-rel* (*hr-comp R R′*) (*a a1*) *r* ∗ (*hr-comp R1p R1′ a1 c1* ∗ Γ′)>$_t$
apply (*subst hr-comp-def*)
apply (*clarsimp intro*!: *norm-pre-ex-rule*)
proof −
fix *b1*
assume *R1*: (*b1*, *a1*) ∈ *R1′*

from *S R1 Q* **have** *R′*: (*b b1*, *a a1*) ∈ ⟨*R′*⟩*nres-rel* **by** *blast*
with *NF* **have** *NFB*: *nofail* (*b b1*)
by (*simp add*: *nres-rel-def pw-le-iff refine-pw-simps*)

from *PQ R1 Q* **have** *P*: *P b1* **by** *blast*
with *NFB R* **have** <*R1 b1 c1* ∗ Γ> *c c1* <λ*r*. *hn-rel R* (*b b1*) *r* ∗ (*R1p b1 c1* ∗ Γ′)>$_t$
unfolding *hn-refine-alt* **by** *auto*
thus <*R1 b1 c1* ∗ Γ>
c c1
<λ*r*. *hn-rel* (*hr-comp R R′*) (*a a1*) *r* ∗ (*hr-comp R1p R1′ a1 c1* ∗ Γ′)>$_t$
apply (*rule cons-post-rule*)
apply (*solve-entails*)
by (*intro ent-star-mono hn-rel-compI*[*OF NF R′*] *hr-compI*[*OF R1*] *ent-refl*)
qed
qed

lemma *hnr-comp1-aux*:
assumes *R*: ⋀*b1 c1*. *P b1* ⟹ *hn-refine* (*hn-ctxt R1 b1 c1*) (*c c1*) (*hn-ctxt R1p b1 c1*) *R* (*b*$*b1*)
assumes *S*: ⋀*a1 b1*. ⟦*Q a1*; (*b1*,*a1*)∈*R1′*⟧ ⟹ (*b*$*b1*,*a*$*a1*)∈⟨*R′*⟩*nres-rel*
assumes *PQ*: ⋀*a1 b1*. ⟦*Q a1*; (*b1*,*a1*)∈*R1′*⟧ ⟹ *P b1*
assumes *Q*: *Q a1*
shows *hn-refine*
(*hr-comp R1 R1′ a1 c1*)
(*c c1*)
(*hr-comp R1p R1′ a1 c1*)
(*hr-comp R R′*)
(*a a1*)
using *assms hnr-comp*[**where** Γ=*emp* **and** Γ′=*emp* **and** *a*=*a* **and** *b*=*b* **and** *c*=*c* **and** *P*=*P* **and** *Q*=*Q*]
unfolding *hn-ctxt-def*
by *auto*

lemma *hfcomp*:
assumes *A*: (*f*,*g*) ∈ $[P]_a$ *RR′* → *S*
assumes *B*: (*g*,*h*) ∈ $[Q]_f$ *T* → ⟨*U*⟩*nres-rel*
shows (*f*,*h*) ∈ [λ*a*. *Q a* ∧ (∀ *a′*. (*a′*,*a*)∈*T* ⟶ *P a′*)]$_a$

hrp-comp RR' $T \to$ *hr-comp* S U
using *assms*
unfolding *fref-def hfref-def hrp-comp-def*
apply *clarsimp*
apply (*rule hnr-comp1-aux*[*of*
P *fst* RR' f *snd* RR' S g $\lambda a.\ Q\ a \wedge (\forall a'.\ (a',a) {\in} T \longrightarrow P\ a')\ T\ h\ U$])
apply (*auto simp*: *hn-ctxt-def*)
done

lemma *hfref-weaken-pre-nofail*:
assumes $(f,g) \in [P]_a\ R \to S$
shows $(f,g) \in [\lambda x.\ nofail\ (g\ x) \longrightarrow P\ x]_a\ R \to S$
using *assms*
unfolding *hfref-def hn-refine-def*
by *auto*

lemma *hfref-cons*:
assumes $(f,g) \in [P]_a\ R \to S$
assumes $\bigwedge x.\ P'\ x \Longrightarrow P\ x$
assumes $\bigwedge x\ y.\ fst\ R'\ x\ y \Longrightarrow_t fst\ R\ x\ y$
assumes $\bigwedge x\ y.\ snd\ R\ x\ y \Longrightarrow_t snd\ R'\ x\ y$
assumes $\bigwedge x\ y.\ S\ x\ y \Longrightarrow_t S'\ x\ y$
shows $(f,g) \in [P']_a\ R' \to S'$
unfolding *hfref-def*
apply *clarsimp*
apply (*rule hn-refine-cons*)
apply (*rule assms*(*3*))
defer
apply (*rule entt-trans*[*OF assms*(*4*)]; *sep-auto*)
apply (*rule assms*(*5*))
apply (*frule assms*(*2*))
using *assms*(*1*)
unfolding *hfref-def*
apply *auto*
done

Composition Automation

This section contains the lemmas. The ML code is further down.

lemma *prod-hrp-comp*:
hrp-comp $(A *_a B)\ (C \times_r D) =$ *hrp-comp* $A\ C *_a$ *hrp-comp* $B\ D$
unfolding *hrp-comp-def hfprod-def* **by** *simp*

lemma *hrp-comp-keep*: *hrp-comp* $(A^k)\ B = (hr\text{-}comp\ A\ B)^k$
by (*auto simp*: *hrp-comp-def*)

lemma *hr-comp-invalid*: *hr-comp* (*invalid-assn R1*) *R2* = *invalid-assn* (*hr-comp R1 R2*)
apply (*intro ent-iffI entailsI ext*)

unfolding *invalid-assn-def hr-comp-def*
by *auto*

lemma *hrp-comp-dest*: *hrp-comp* (A^d) $B = (hr\text{-}comp\ A\ B)^d$
by (*auto simp*: *hrp-comp-def hr-comp-invalid*)

definition *hrp-imp RR RR'* $\equiv$
$\forall\, a\ b.\ (fst\ RR'\ a\ b \Longrightarrow_t fst\ RR\ a\ b) \wedge (snd\ RR\ a\ b \Longrightarrow_t snd\ RR'\ a\ b)$

lemma *hfref-imp*: *hrp-imp RR RR'* $\Longrightarrow [P]_a\ RR \rightarrow S \subseteq [P]_a\ RR' \rightarrow S$
apply *clarsimp*
apply (*erule hfref-cons*)
apply (*simp-all add*: *hrp-imp-def*)
done

lemma *hrp-imp-refl*: *hrp-imp RR RR*
unfolding *hrp-imp-def* **by** *auto*

lemma *hrp-imp-reflI*: $RR = RR' \Longrightarrow$ *hrp-imp RR RR'*
unfolding *hrp-imp-def* **by** *auto*

lemma *hrp-comp-cong*: *hrp-imp A A'* $\Longrightarrow B{=}B' \Longrightarrow$ *hrp-imp* (*hrp-comp A B*) (*hrp-comp A' B'*)
by (*sep-auto simp*: *hrp-imp-def hrp-comp-def hr-comp-def entailst-def*)

lemma *hrp-prod-cong*: *hrp-imp A A'* $\Longrightarrow$ *hrp-imp B B'* $\Longrightarrow$ *hrp-imp* $(A{*_a}B)$ $(A'{*_a}B')$
by (*sep-auto simp*: *hrp-imp-def prod-assn-def intro*: *entt-star-mono*)

lemma *hrp-imp-trans*: *hrp-imp A B* $\Longrightarrow$ *hrp-imp B C* $\Longrightarrow$ *hrp-imp A C*
unfolding *hrp-imp-def*
by (*fastforce intro*: *entt-trans*)

lemma *fcomp-norm-dflt-init*: $x{\in}[P]_a\ R \rightarrow T \Longrightarrow$ *hrp-imp R S* $\Longrightarrow x{\in}[P]_a\ S \rightarrow T$
apply (*erule rev-subsetD*)
by (*rule hfref-imp*)

definition *comp-PRE R P Q S* $\equiv \lambda x.\ S\ x \longrightarrow (P\ x \wedge (\forall\, y.\ (y,x){\in}R \longrightarrow Q\ x\ y))$

lemma *comp-PRE-cong*[*cong*]:
assumes $R{\equiv}R'$
assumes $\bigwedge x.\ P\ x \equiv P'\ x$
assumes $\bigwedge x.\ S\ x \equiv S'\ x$
assumes $\bigwedge x\ y.\ [\![P\ x;\ (y,x){\in}R;\ y{\in}Domain\ R;\ S'\ x\ ]\!] \Longrightarrow Q\ x\ y \equiv Q'\ x\ y$
shows *comp-PRE R P Q S* $\equiv$ *comp-PRE R' P' Q' S'*

using *assms*
by (*fastforce simp*: *comp-PRE-def intro*!: *eq-reflection ext*)

lemma *fref-compI-PRE*:
$\llbracket$ $(f,g) \in$ *fref P R1 R2*; $(g,h) \in$ *fref Q S1 S2* $\rrbracket$
$\Longrightarrow$ $(f,h) \in$ *fref* (*comp-PRE S1 Q* (λ-. *P*) (λ-. *True*)) (*R1 O S1*) (*R2 O S2*)
using *fref-compI*[*of P R1 R2 Q S1 S2*]
unfolding *comp-PRE-def*
by *auto*

lemma *PRE-D1*: $(Q\ x \wedge P\ x) \longrightarrow$ *comp-PRE S1 Q* (λx -. *P x*) *S x*
by (*auto simp*: *comp-PRE-def*)

lemma *PRE-D2*: $(Q\ x \wedge (\forall\, y.\ (y,x) \in S1 \longrightarrow S\ x \longrightarrow P\ x\ y)) \longrightarrow$ *comp-PRE S1 Q P S x*
by (*auto simp*: *comp-PRE-def*)

lemma *fref-weaken-pre*:
assumes $\bigwedge x.\ P\ x \longrightarrow P'\ x$
assumes $(f,h) \in$ *fref P′ R S*
shows $(f,h) \in$ *fref P R S*
apply (*rule rev-subsetD*[*OF assms*(*2*) *fref-mono*])
using *assms*(*1*) **by** *auto*

lemma *fref-PRE-D1*:
assumes $(f,h) \in$ *fref* (*comp-PRE S1 Q* (λx -. *P x*) *X*) *R S*
shows $(f,h) \in$ *fref* ($\lambda x.\ Q\ x \wedge P\ x$) *R S*
by (*rule fref-weaken-pre*[*OF PRE-D1 assms*])

lemma *fref-PRE-D2*:
assumes $(f,h) \in$ *fref* (*comp-PRE S1 Q P X*) *R S*
shows $(f,h) \in$ *fref* ($\lambda x.\ Q\ x \wedge (\forall\, y.\ (y,x) \in S1 \longrightarrow X\ x \longrightarrow P\ x\ y)$) *R S*
by (*rule fref-weaken-pre*[*OF PRE-D2 assms*])

lemmas *fref-PRE-D* = *fref-PRE-D1 fref-PRE-D2*

lemma *hfref-weaken-pre*:
assumes $\bigwedge x.\ P\ x \longrightarrow P'\ x$
assumes $(f,h) \in$ *hfref P′ R S*
shows $(f,h) \in$ *hfref P R S*
using *assms*
by (*auto simp*: *hfref-def*)

lemma *hfref-weaken-pre′*:
assumes $\bigwedge x.\ \llbracket P\ x;\ rdomp\ (fst\ R)\ x \rrbracket \Longrightarrow P'\ x$
assumes $(f,h) \in$ *hfref P′ R S*
shows $(f,h) \in$ *hfref P R S*
apply (*rule hfrefI*)
apply (*rule hn-refine-preI*)

using *assms*
by (*auto simp*: *hfref-def rdomp-def*)

lemma *hfref-weaken-pre-nofail′*:
assumes $(f,g) \in [P]_a\ R \to S$
assumes $\bigwedge x.\ [\![\mathit{nofail}\ (g\ x);\ Q\ x]\!] \Longrightarrow P\ x$
shows $(f,g) \in [Q]_a\ R \to S$
apply (*rule hfref-weaken-pre*[*OF - assms*(*1*)[*THEN hfref-weaken-pre-nofail*]])
using *assms*(*2*)
by *blast*

lemma *hfref-compI-PRE-aux*:
assumes *A*: $(f,g) \in [P]_a\ RR' \to S$
assumes *B*: $(g,h) \in [Q]_f\ T \to \langle U\rangle \mathit{nres\text{-}rel}$
shows $(f,h) \in [\mathit{comp\text{-}PRE}\ T\ Q\ (\lambda\text{-}.\ P)\ (\lambda\text{-}.\ \mathit{True})]_a$
hrp-comp RR' $T \to$ *hr-comp* S U
apply (*rule hfref-weaken-pre*[*OF - hfcomp*[*OF A B*]])
by (*auto simp*: *comp-PRE-def*)

lemma *hfref-compI-PRE*:
assumes *A*: $(f,g) \in [P]_a\ RR' \to S$
assumes *B*: $(g,h) \in [Q]_f\ T \to \langle U\rangle \mathit{nres\text{-}rel}$
shows $(f,h) \in [\mathit{comp\text{-}PRE}\ T\ Q\ (\lambda x\ y.\ P\ y)\ (\lambda x.\ \mathit{nofail}\ (h\ x))]_a$
hrp-comp RR' $T \to$ *hr-comp* S U
using *hfref-compI-PRE-aux*[*OF A B*, *THEN hfref-weaken-pre-nofail*]
apply (*rule hfref-weaken-pre*[*rotated*])
apply (*auto simp*: *comp-PRE-def*)
done

lemma *hfref-PRE-D1*:
assumes $(f,h) \in \mathit{hfref}\ (\mathit{comp\text{-}PRE}\ S1\ Q\ (\lambda x\ \text{-}.\ P\ x)\ X)\ R\ S$
shows $(f,h) \in \mathit{hfref}\ (\lambda x.\ Q\ x \wedge P\ x)\ R\ S$
by (*rule hfref-weaken-pre*[*OF PRE-D1 assms*])

lemma *hfref-PRE-D2*:
assumes $(f,h) \in \mathit{hfref}\ (\mathit{comp\text{-}PRE}\ S1\ Q\ P\ X)\ R\ S$
shows $(f,h) \in \mathit{hfref}\ (\lambda x.\ Q\ x \wedge (\forall y.\ (y,x)\in S1 \longrightarrow X\ x \longrightarrow P\ x\ y))\ R\ S$
by (*rule hfref-weaken-pre*[*OF PRE-D2 assms*])

lemma *hfref-PRE-D3*:
assumes $(f,h) \in \mathit{hfref}\ (\mathit{comp\text{-}PRE}\ S1\ Q\ P\ X)\ R\ S$
shows $(f,h) \in \mathit{hfref}\ (\mathit{comp\text{-}PRE}\ S1\ Q\ P\ X)\ R\ S$
using *assms* **.**

lemmas *hfref-PRE-D* = *hfref-PRE-D1 hfref-PRE-D3*

1.5.4 Automation

Purity configuration for constraint solver

lemmas [*safe-constraint-rules*] = *pure-pure*

Configuration for hfref to hnr conversion

named-theorems *to-hnr-post* ‹*to-hnr converter*: *Postprocessing unfold rules*›

lemma *uncurry0-add-app-tag*: *uncurry0* (*RETURN c*) = *uncurry0* (*RETURN*$*c*)
by *simp*

lemmas [*to-hnr-post*] = *norm-RETURN-o norm-return-o*
uncurry0-add-app-tag uncurry0-apply uncurry0-APP hn-val-unit-conv-emp
mult-1[*of x*::*assn* **for** *x*] *mult-1-right*[*of x*::*assn* **for** *x*]

named-theorems *to-hfref-post* ‹*to-hfref converter*: *Postprocessing unfold rules*›
lemma *prod-casesK*[*to-hfref-post*]: *case-prod* (λ- -. *k*) = (λ-. *k*) **by** *auto*
lemma *uncurry0-hfref-post*[*to-hfref-post*]: *hfref* (*uncurry0 True*) *R S* = *hfref* (λ-. *True*) *R S*
apply (*fo-rule arg-cong fun-cong*)+ **by** *auto*

Configuration for relation normalization after composition

named-theorems *fcomp-norm-unfold* ‹*fcomp*−*normalizer*: *Unfold theorems*›
named-theorems *fcomp-norm-simps* ‹*fcomp*−*normalizer*: *Simplification theorems*›
named-theorems *fcomp-norm-init fcomp*−*normalizer*: *Initialization rules*
named-theorems *fcomp-norm-trans fcomp*−*normalizer*: *Transitivity rules*
named-theorems *fcomp-norm-cong fcomp*−*normalizer*: *Congruence rules*
named-theorems *fcomp-norm-norm fcomp*−*normalizer*: *Normalization rules*
named-theorems *fcomp-norm-refl fcomp*−*normalizer*: *Reflexivity rules*

Default Setup

lemmas [*fcomp-norm-unfold*] = *prod-rel-comp nres-rel-comp Id-O-R R-O-Id*
lemmas [*fcomp-norm-unfold*] = *hr-comp-Id1 hr-comp-Id2*
lemmas [*fcomp-norm-unfold*] = *hr-comp-prod-conv*
lemmas [*fcomp-norm-unfold*] = *prod-hrp-comp hrp-comp-keep hrp-comp-dest hr-comp-pure*

lemma [*fcomp-norm-simps*]: *CONSTRAINT is-pure P* ⟹ *pure* (*the-pure P*) = *P* **by** *simp*
lemmas [*fcomp-norm-simps*] = *True-implies-equals*

lemmas [*fcomp-norm-init*] = *fcomp-norm-dflt-init*
lemmas [*fcomp-norm-trans*] = *hrp-imp-trans*
lemmas [*fcomp-norm-cong*] = *hrp-comp-cong hrp-prod-cong*

lemmas [*fcomp-norm-refl*] = *refl hrp-imp-refl*

lemma *ensure-fref-nresI*: $(f,g) \in [P]_f\ R \rightarrow S \Longrightarrow (RETURN\ o\ f,\ RETURN\ o\ g) \in [P]_f\ R \rightarrow \langle S \rangle nres\text{-}rel$
by (*auto intro*: *nres-relI simp*: *fref-def*)

lemma *ensure-fref-nres-unfold*:
$\bigwedge f.\ RETURN\ o\ (uncurry0\ f) = uncurry0\ (RETURN\ f)$
$\bigwedge f.\ RETURN\ o\ (uncurry\ f) = uncurry\ (RETURN\ oo\ f)$
$\bigwedge f.\ (RETURN\ ooo\ uncurry)\ f = uncurry\ (RETURN\ ooo\ f)$
by *auto*

Composed precondition normalizer

named-theorems *fcomp-prenorm-simps* ‹*fcomp precondition−normalizer*: *Simplification theorems*›

Support for preconditions of the form $\text{-}\in Domain\ R$, where R is the relation of the next more abstract level.

declare *DomainI*[*fcomp-prenorm-simps*]

lemma *auto-weaken-pre-init-hf*:
assumes $\bigwedge x.\ PROTECT\ P\ x \longrightarrow P'\ x$
assumes $(f,h) \in hfref\ P'\ R\ S$
shows $(f,h) \in hfref\ P\ R\ S$
using *assms*
by (*auto simp*: *hfref-def*)

lemma *auto-weaken-pre-init-f*:
assumes $\bigwedge x.\ PROTECT\ P\ x \longrightarrow P'\ x$
assumes $(f,h) \in fref\ P'\ R\ S$
shows $(f,h) \in fref\ P\ R\ S$
using *assms*
by (*auto simp*: *fref-def*)

lemmas *auto-weaken-pre-init* = *auto-weaken-pre-init-hf auto-weaken-pre-init-f*

lemma *auto-weaken-pre-uncurry-step*:
assumes $PROTECT\ f\ a \equiv f'$
shows $PROTECT\ (\lambda(x,y).\ f\ x\ y)\ (a,b) \equiv f'\ b$
using *assms*
by (*auto simp*: *curry-def dest*!: *meta-eq-to-obj-eq intro*!: *eq-reflection*)

lemma *auto-weaken-pre-uncurry-finish*:
$PROTECT\ f\ x \equiv f\ x$ **by** (*auto*)

lemma *auto-weaken-pre-uncurry-start*:
assumes $P \equiv P'$
assumes $P' \longrightarrow Q$
shows $P \longrightarrow Q$
using *assms* **by** (*auto*)

lemma *auto-weaken-pre-comp-PRE-I*:
  assumes $S\ x \Longrightarrow P\ x$
  assumes $\bigwedge y.\ [\![(y,x)\in R;\ P\ x;\ S\ x]\!] \Longrightarrow Q\ x\ y$
  shows *comp-PRE R P Q S x*
  using *assms* **by** (*auto simp*: *comp-PRE-def*)

lemma *auto-weaken-pre-to-imp-nf*:
  $(A \longrightarrow B \longrightarrow C) = (A \wedge B \longrightarrow C)$
  $((A \wedge B) \wedge C) = (A \wedge B \wedge C)$
  by *auto*

lemma *auto-weaken-pre-add-dummy-imp*:
  $P \Longrightarrow True \longrightarrow P$ **by** *simp*

Synthesis for hfref statements

definition *hfsynth-ID-R* :: $('a \Rightarrow \text{-} \Rightarrow assn) \Rightarrow 'a \Rightarrow bool$ **where**
  [*simp*]: *hfsynth-ID-R* - - $\equiv$ *True*

lemma *hfsynth-ID-R-D*:
  fixes I :: *'a itself*
  assumes *hfsynth-ID-R R a*
  assumes *intf-of-assn R I*
  shows $a ::_i I$
  by *simp*

lemma *hfsynth-hnr-from-hfI*:
  assumes $\forall x\ xi.\ P\ x \wedge$ *hfsynth-ID-R* (*fst R*) $x \longrightarrow$ *hn-refine* (*emp* $*$ *hn-ctxt* (*fst R*) *x xi*) (*f*\$*xi*) (*emp* $*$ *hn-ctxt* (*snd R*) *x xi*) *S* (*g*\$*x*)
  shows $(f,g) \in [P]_a\ R \rightarrow S$
  using *assms*
  unfolding *hfref-def*
  by (*auto simp*: *hn-ctxt-def*)

lemma *hfsynth-ID-R-uncurry-unfold*:
  hfsynth-ID-R (*to-hnr-prod R S*) $(a,b) \equiv$ *hfsynth-ID-R R a* $\wedge$ *hfsynth-ID-R S b*
  hfsynth-ID-R (*fst* (*hf-pres R k*)) $\equiv$ *hfsynth-ID-R R*
  by (*auto intro*!: *eq-reflection*)

ML ‹

```
signature SEPREF-RULES = sig
  (* Analysis of relations, both fref and fun-rel *)
  (* R1→...→Rn→- / [-]_f ((R1×_r R2)...×_r Rn)  ↦  [R1,...,Rn] *)
  val binder-rels: term −> term list
  (* -→...→-→S / [-]_f - → S  ↦  S *)
  val body-rel: term −> term
  (* Map →/fref to (precond,args,res). NONE if no/trivial precond. *)
  val analyze-rel: term −> term option * term list * term
```

```
(* Make trivial (λ-. True) precond *)
val mk-triv-precond: term list -> term
(* Make [P]_f ((R1 ×_r R2)...×_r Rn) → S. Insert trivial precond if NONE. *)
val mk-rel: term option * term list * term -> term
(* Map relation to (args,res) *)
val strip-rel: term -> term list * term

(* Make hfprod (op *_a) *)
val mk-hfprod : term * term -> term
val mk-hfprods : term list -> term

(* Determine interface type of refinement assertion, using default fallback
  if necessary. Use named-thms intf-of-assn for configuration. *)
val intf-of-assn : Proof.context -> term -> typ

(*
  Convert a parametricity theorem in higher-order form to
  uncurried fref-form. For functions without arguments,
  a unit-argument is added.

  TODO/FIXME: Currently this only works for higher-order theorems,
    i.e., theorems of the form (f,g)∈R1→...→Rn.

    First-order theorems are silently treated as refinement theorems
    for functions with zero arguments, i.e., a unit-argument is added.
*)
val to-fref : Proof.context -> thm -> thm

(* Convert a parametricity or fref theorem to first order form *)
val to-foparam : Proof.context -> thm -> thm

(* Convert schematic hfref goal to hnr-goal *)
val prepare-hfref-synth-tac : Proof.context -> tactic'

(* Convert theorem in hfref-form to hnr-form *)
val to-hnr : Proof.context -> thm -> thm

(* Convert theorem in hnr-form to hfref-form *)
val to-hfref: Proof.context -> thm -> thm

(* Convert theorem to given form, if not yet in this form *)
val ensure-fref : Proof.context -> thm -> thm
val ensure-fref-nres : Proof.context -> thm -> thm
val ensure-hfref : Proof.context -> thm -> thm
val ensure-hnr : Proof.context -> thm -> thm


type hnr-analysis = {
  thm: thm,                    (* Original theorem, may be normalized *)
```

```
    precond: term,                (* Precondition, abstracted over abs−arguments
*)
    prems : term list,            (* Premises not depending on arguments *)
    ahead: term * bool,           (* Abstract function, has leading RETURN *)
    chead: term * bool,           (* Concrete function, has leading return *)
    argrels: (term * bool) list,  (* Argument relations, preserved (keep−flag) *)
    result-rel: term              (* Result relation *)
  }

  val analyze-hnr: Proof.context −> thm −> hnr-analysis
  val pretty-hnr-analysis: Proof.context −> hnr-analysis −> Pretty.T
  val mk-hfref-thm: Proof.context −> hnr-analysis −> thm



  (* Simplify precondition of fref/hfref−theorem *)
  val simplify-precond: Proof.context −> thm −> thm

  (* Normalize hfref−theorem after composition *)
  val norm-fcomp-rule: Proof.context −> thm −> thm

  (* Replace pure ?A by ?A' and is-pure constraint, then normalize *)
  val add-pure-constraints-rule: Proof.context −> thm −> thm

  (* Compose fref/hfref and fref theorem, to produce hfref theorem.
    The input theorems may also be in ho−param or hnr form, and
    are converted accordingly.
  *)
  val gen-compose : Proof.context −> thm −> thm −> thm

  (* FCOMP−attribute *)
  val fcomp-attrib: attribute context-parser
end

structure Sepref-Rules: SEPREF-RULES = struct

  local open Refine-Util Relators in
    fun binder-rels @{mpat ?F → ?G} = F::binder-rels G
      | binder-rels @{mpat fref - ?F -} = strip-prodrel-left F
      | binder-rels - = []

    local
      fun br-aux @{mpat - → ?G} = br-aux G
        | br-aux R = R
    in
      fun body-rel @{mpat fref - - ?G} = G
        | body-rel R = br-aux R
    end
```

```
fun strip-rel R = (binder-rels R, body-rel R)

 fun analyze-rel @{mpat fref (λ-. True) ?R ?S} = (NONE,strip-prodrel-left
R,S)
   | analyze-rel @{mpat fref ?P ?R ?S} = (SOME P,strip-prodrel-left R,S)
   | analyze-rel R = let
       val (args,res) = strip-rel R
     in
       (NONE,args,res)
     end

 fun mk-triv-precond Rs = absdummy (map rel-absT Rs |> list-prodT-left)
@{term True}

fun mk-rel (P,Rs,S) = let
  val R = list-prodrel-left Rs

  val P = case P of
      SOME P => P
    | NONE => mk-triv-precond Rs

in
  @{mk-term fref ?P ?R ?S}
end
end


fun mk-hfprod (a, b) = @{mk-term ?a*_a ?b}

local
  fun mk-hfprods-rev [] = @{mk-term unit-assn^k}
    | mk-hfprods-rev [Rk] = Rk
    | mk-hfprods-rev (Rkn::Rks) = mk-hfprod (mk-hfprods-rev Rks, Rkn)
in
  val mk-hfprods = mk-hfprods-rev o rev
end


fun intf-of-assn ctxt t = let
  val orig-ctxt = ctxt
  val (t,ctxt) = yield-singleton (Variable.import-terms false) t ctxt

 val v = TVar ((T,0),Proof-Context.default-sort ctxt (T,0)) |> Logic.mk-type
  val goal = @{mk-term Trueprop (intf-of-assn ?t ?v)}

  val i-of-assn-rls =
    Named-Theorems-Rev.get ctxt @{named-theorems-rev intf-of-assn}
    @ @{thms intf-of-assn-fallback}
```

```
    fun tac ctxt = REPEAT-ALL-NEW (resolve-tac ctxt i-of-assn-rls)

    val thm = Goal.prove ctxt [] [] goal (fn {context,...} => ALLGOALS (tac
context))
    val intf = case Thm.concl-of thm of
        @{mpat Trueprop (intf-of-assn - (?v AS_p TYPE (-)))} => v
      | - => raise THM(Intf-of-assn: Proved a different theorem?,~1,[thm])

    val intf = singleton (Variable.export-terms ctxt orig-ctxt) intf
      |> Logic.dest-type

  in
    intf
  end

  datatype rthm-type =
    RT-HOPARAM    (* (-,-) ∈ - → . . . → - *)
  | RT-FREF       (* (-,-) ∈ [-]_f - → - *)
  | RT-HNR        (* hn-refine - - - - - *)
  | RT-HFREF      (* (-,-) ∈ [-]_a - → - *)
  | RT-OTHER

  fun rthm-type thm =
    case Thm.concl-of thm |> HOLogic.dest-Trueprop of
      @{mpat (-,-) ∈ fref - - -} => RT-FREF
    | @{mpat (-,-) ∈ hfref - - -} => RT-HFREF
    | @{mpat hn-refine - - - - -} => RT-HNR
    | @{mpat (-,-) ∈ -} => RT-HOPARAM (* TODO: Distinction between
ho−param and fo−param *)
    | - => RT-OTHER


  fun to-fref ctxt thm = let
    open Conv
  in
    case Thm.concl-of thm |> HOLogic.dest-Trueprop of
      @{mpat (-,-)∈-→-} =>
        Local-Defs.unfold0 ctxt @{thms fref-param1} thm
        |> fconv-rule (repeat-conv (Refine-Util.ftop-conv (K (rewr-conv @{thm
fref-nest})) ctxt))
        |> Local-Defs.unfold0 ctxt @{thms in-CURRY-conv}
    | @{mpat (-,-)∈-} => thm RS @{thm fref-param0I}
    | - => raise THM (to-fref: Expected theorem of form (-,-)∈-,~1,[thm])
  end

  fun to-foparam ctxt thm = let
    val unf-thms = @{thms
   split-tupled-all prod-rel-simp uncurry-apply cnv-conj-to-meta Product-Type.split}
  in
```

```
case Thm.concl-of thm of
  @{mpat Trueprop ((-,-) ∈ fref - - -)} =>
    (@{thm frefD} OF [thm])
    |> Thm.forall-intr-vars
    |> Local-Defs.unfold0 ctxt unf-thms
    |> Variable.gen-all ctxt
| @{mpat Trueprop ((-,-) ∈ -)} =>
    Parametricity.fo-rule thm
| - => raise THM(Expected parametricity or fref theorem,~1,[thm])
end

fun to-hnr ctxt thm =
  (thm RS @{thm hf2hnr})
    |> Local-Defs.unfold0 ctxt @{thms to-hnr-prod-fst-snd keep-drop-sels} (* Resolve fst and snd over *_a and R^k, R^d *)
    |> Local-Defs.unfold0 ctxt @{thms hnr-uncurry-unfold} (* Resolve products for uncurried parameters *)
    |> Local-Defs.unfold0 ctxt @{thms uncurry-apply uncurry-APP assn-one-left split} (* Remove the uncurry modifiers, the emp−dummy, and unfold product cases *)
    |> Local-Defs.unfold0 ctxt @{thms hn-ctxt-ctxt-fix-conv} (* Remove duplicate hn-ctxt tagging *)
    |> Local-Defs.unfold0 ctxt @{thms all-to-meta imp-to-meta HOL.True-implies-equals HOL.implies-True-equals Pure.triv-forall-equality cnv-conj-to-meta} (* Convert to meta−level, remove vacuous condition *)
    |> Local-Defs.unfold0 ctxt (Named-Theorems.get ctxt @{named-theorems to-hnr-post}) (* Post−Processing *)
    |> Goal.norm-result ctxt
    |> Conv.fconv-rule Thm.eta-conversion

(* Convert schematic hfref−goal to hn-refine goal *)
fun prepare-hfref-synth-tac ctxt = let
  val i-of-assn-rls =
    Named-Theorems-Rev.get ctxt @{named-theorems-rev intf-of-assn}
    @ @{thms intf-of-assn-fallback}

  val to-hnr-post-rls =
    Named-Theorems.get ctxt @{named-theorems to-hnr-post}

  val i-of-assn-tac = (
    REPEAT' (
      DETERM o dresolve-tac ctxt @{thms hfsynth-ID-R-D}
      THEN' DETERM o SOLVED' (REPEAT-ALL-NEW (resolve-tac ctxt i-of-assn-rls))
    )
  )
in
  (* Note: To re−use the to-hnr infrastructure, we first work with
    $−tags on the abstract function, which are finally removed.
```

```
    *)
    resolve-tac ctxt @{thms hfsynth-hnr-from-hfI} THEN-ELSE' (
      SELECT-GOAL (
        unfold-tac ctxt @{thms to-hnr-prod-fst-snd keep-drop-sels hf-pres-fst} (*
Distribute fst,snd over product and hf-pres *)
      THEN unfold-tac ctxt @{thms hnr-uncurry-unfold hfsynth-ID-R-uncurry-unfold}
(* Curry parameters *)
        THEN unfold-tac ctxt @{thms uncurry-apply uncurry-APP assn-one-left
split} (* Curry parameters (II) and remove emp assertion *)
       (*THEN unfold-tac ctxt @{thms hn-ctxt-ctxt-fix-conv} (* Remove duplicate
hn-ctxt (Should not be necessary) *)*)
      THEN unfold-tac ctxt @{thms all-to-meta imp-to-meta HOL.True-implies-equals
HOL.implies-True-equals Pure.triv-forall-equality cnv-conj-to-meta} (* Convert pre-
condition to meta−level *)
        THEN ALLGOALS i-of-assn-tac (* Generate -::_i- premises*)
        THEN unfold-tac ctxt to-hnr-post-rls (* Postprocessing *)
        THEN unfold-tac ctxt @{thms APP-def} (* Get rid of $ − tags *)
      )
    ,
      K all-tac
    )
  end


  (**********************************)
  (* Analyze hnr *)
  structure Termtab2 = Table(
    type key = term * term
    val ord = prod-ord Term-Ord.fast-term-ord Term-Ord.fast-term-ord);

  type hnr-analysis = {
    thm: thm,
    precond: term,
    prems : term list,
    ahead: term * bool,
    chead: term * bool,
    argrels: (term * bool) list,
    result-rel: term
  }


  fun analyze-hnr (ctxt:Proof.context) thm = let

    (* Debug information: Stores string*term pairs, which are pretty−printed on
error *)
    val dbg = Unsynchronized.ref []
    fun add-dbg msg ts = (
      dbg := (msg,ts) :: !dbg;
      ()
```

```
  )
  fun pretty-dbg (msg,ts) = Pretty.block [
    Pretty.str msg,
    Pretty.str :,
    Pretty.brk 1,
    Pretty.list [ ] (map (Syntax.pretty-term ctxt) ts)
  ]
  fun pretty-dbgs l = map pretty-dbg l |> Pretty.fbreaks |> Pretty.block

 fun trace-dbg msg = Pretty.block [Pretty.str msg, Pretty.fbrk, pretty-dbgs (rev
(!dbg))] |> Pretty.string-of |> tracing

  fun fail msg = (trace-dbg msg; raise THM(msg,~1,[thm]))
  fun assert cond msg = cond orelse fail msg;


  (* Heads may have a leading return/RETURN.
    The following code strips off the leading return, unless it has the form
    return x for an argument x
  *)
  fun check-strip-leading args t f = (* Handle the case RETURN x, where x is
an argument *)
    if Termtab.defined args f then (t,false) else (f,true)

 fun strip-leading-RETURN args (t as @{mpat RETURN$(?f)}) = check-strip-leading
args t f
  | strip-leading-RETURN args (t as @{mpat RETURN ?f}) = check-strip-leading
args t f
    | strip-leading-RETURN - t = (t,false)

 fun strip-leading-return args (t as @{mpat return$(?f)}) = check-strip-leading
args t f
     | strip-leading-return args (t as @{mpat return ?f}) = check-strip-leading
args t f
     | strip-leading-return - t = (t,false)


  (* The following code strips the arguments of the concrete or abstract
 function. It knows how to handle APP−tags ($), and stops at PR-CONST−tags.

    Moreover, it only strips actual arguments that occur in the
    precondition−section of the hn-refine−statement. This ensures
    that non−arguments, like maxsize, are treated correctly.
  *)
  fun strip-fun - (t as @{mpat PR-CONST -}) = (t,[])
    | strip-fun s (t as @{mpat ?f$?x}) = check-arg s t f x
    | strip-fun s (t as @{mpat ?f ?x}) = check-arg s t f x
    | strip-fun - f = (f,[])
  and check-arg s t f x =
```

```
    if Termtab.defined s x then
      strip-fun s f |> apsnd (curry op :: x)
    else (t,[])

(* Arguments in the pre/postcondition are wrapped into hn-ctxt tags.
  This function strips them off. *)
fun dest-hn-ctxt @{mpat hn-ctxt ?R ?a ?c} = ((a,c),R)
  | dest-hn-ctxt - = fail Invalid hn-ctxt parameter in pre or postcondition


fun dest-hn-refine @{mpat (hn-refine ?G ?c ?G' ?R ?a)} = (G,c,G',R,a)
  | dest-hn-refine - = fail Conclusion is not a hn-refine statement

(*
  Strip separation conjunctions. Special case for emp, which is ignored.
*)
fun is-emp @{mpat emp} = true | is-emp - = false

val strip-star' = Sepref-Basic.strip-star #> filter (not o is-emp)

(* Compare Termtab2s for equality of keys *)
fun pairs-eq pairs1 pairs2 =
          Termtab2.forall (Termtab2.defined pairs1 o fst) pairs2
  andalso Termtab2.forall (Termtab2.defined pairs2 o fst) pairs1


fun atomize-prem @{mpat Trueprop ?p} = p
  | atomize-prem - = fail Non−atomic premises

(* Make HOL conjunction list *)
fun mk-conjs [] = Const ‹True›
  | mk-conjs [p] = p
  | mk-conjs (p::ps) = Const ‹conj for p ‹mk-conjs ps››


(*********************)
(* Start actual analysis *)

val - = add-dbg thm [Thm.prop-of thm]
val prems = Thm.prems-of thm
val concl = Thm.concl-of thm |> HOLogic.dest-Trueprop
val (G,c,G',R,a) = dest-hn-refine concl

val pre-pairs = G
  |> strip-star'
  |> tap (add-dbg precondition)
  |> map dest-hn-ctxt
  |> Termtab2.make
```

```
val post-pairs = G'
  |> strip-star'
  |> tap (add-dbg postcondition)
  |> map dest-hn-ctxt
  |> Termtab2.make

val - = assert (pairs-eq pre-pairs post-pairs)
  Parameters in precondition do not match postcondition

val aa-set = pre-pairs |> Termtab2.keys |> map fst |> Termtab.make-set
val ca-set = pre-pairs |> Termtab2.keys |> map snd |> Termtab.make-set

val (a,leading-RETURN) = strip-leading-RETURN aa-set a
val (c,leading-return) = strip-leading-return ca-set c

val - = add-dbg stripped abstract term [a]
val - = add-dbg stripped concrete term [c]

val (ahead,aargs) = strip-fun aa-set a;
val (chead,cargs) = strip-fun ca-set c;

val - = add-dbg abstract head [ahead]
val - = add-dbg abstract args aargs
val - = add-dbg concrete head [chead]
val - = add-dbg concrete args cargs


val - = assert (length cargs = length aargs) Different number of abstract and concrete arguments;

val - = assert (not (has-duplicates op aconv aargs)) Duplicate abstract arguments
val - = assert (not (has-duplicates op aconv cargs)) Duplicate concrete arguments

val argpairs = aargs ~~ cargs
val ap-set = Termtab2.make-set argpairs
val - = assert (pairs-eq pre-pairs ap-set) Arguments from pre/postcondition do not match operation's arguments

val pre-rels = map (the o (Termtab2.lookup pre-pairs)) argpairs
val post-rels = map (the o (Termtab2.lookup post-pairs)) argpairs

val - = add-dbg pre−rels pre-rels
val - = add-dbg post−rels post-rels

fun adjust-hf-pres @{mpat snd (?R^k)} = R
  | adjust-hf-pres t = t
```

```
    val post-rels = map adjust-hf-pres post-rels

    fun is-invalid R @{mpat invalid-assn ?R'} = R aconv R'
      | is-invalid - @{mpat snd (-^d)} = true
      | is-invalid - - = false

    fun is-keep (R,R') =
      if R aconv R' then true
      else if is-invalid R R' then false
      else fail Mismatch between pre and post relation for argument

    val keep = map is-keep (pre-rels ~~ post-rels)

    val argrels = pre-rels ~~ keep

    val aa-set = Termtab.make-set aargs
    val ca-set = Termtab.make-set cargs

    fun is-precond t =
        (exists-subterm (Termtab.defined ca-set) t andalso fail Premise contains
concrete argument)
      orelse exists-subterm (Termtab.defined aa-set) t

    val (preconds, prems) = split is-precond prems

    val precond =
      map atomize-prem preconds
      |> mk-conjs
      |> fold lambda aargs

    val - = add-dbg precond [precond]
    val - = add-dbg prems prems

  in
    {
      thm = thm,
      precond = precond,
      prems = prems,
      ahead = (ahead,leading-RETURN),
      chead = (chead,leading-return),
      argrels = argrels,
      result-rel = R
    }
  end

  fun pretty-hnr-analysis
    ctxt
    ({thm,precond,ahead,chead,argrels,result-rel,...})
    : Pretty.T =
```

```
let
  val _ = thm (* Suppress unused warning for thm *)

  fun pretty_argrel (R,k) = Pretty.block [
    Syntax.pretty_term ctxt R,
    if k then Pretty.str ^k else Pretty.str ^d
  ]

  val pretty_chead = case chead of
    (t,false) => Syntax.pretty_term ctxt t
  | (t,true) => Pretty.block [Pretty.str return , Syntax.pretty_term ctxt t]

  val pretty_ahead = case ahead of
    (t,false) => Syntax.pretty_term ctxt t
  | (t,true) => Pretty.block [Pretty.str RETURN , Syntax.pretty_term ctxt t]

in
  Pretty.fbreaks [
    (*Display.pretty_thm ctxt thm,*)
    Pretty.block [
      Pretty.enclose [ ] [pretty_chead, pretty_ahead],
      Pretty.enclose [ ] [Syntax.pretty_term ctxt precond],
      Pretty.brk 1,
   Pretty.block (Pretty.separate → (map pretty_argrel argrels @ [Syntax.pretty_term
ctxt result_rel]))
    ]
  ] |> Pretty.block

end


fun mk_hfref_thm
  ctxt
  ({thm,precond,prems,ahead,chead,argrels,result_rel}) =
let

  fun mk_keep (R,true) = @{mk_term ?R^k}
    | mk_keep (R,false) = @{mk_term ?R^d}

  (* TODO: Move, this is of general use! *)
  fun mk_uncurry f = @{mk_term uncurry ?f}

  (* Uncurry function for the given number of arguments.
    For zero arguments, add a unit-parameter.
  *)
  fun rpt_uncurry n t =
    if n=0 then @{mk_term uncurry0 ?t}
    else if n=1 then t
    else funpow (n-1) mk_uncurry t
```

```
(* Rewrite uncurried lambda's to λ(-,-). - form. Use top−down rewriting
  to correctly handle nesting to the left.

  TODO: Combine with abstraction and  uncurry−procedure,
    and mark the deviation about uncurry as redundant
    intermediate step to be eliminated.
*)
fun rew-uncurry-lambda t = let
     val rr = map (Logic.dest-equals o Thm.prop-of) @{thms uncurry-def
uncurry0-def}
  val thy = Proof-Context.theory-of ctxt
in
  Pattern.rewrite-term-topdown thy rr [] t
end

(* Shortcuts for simplification tactics *)
fun gsimp-only ctxt sec = let
  val ss = put-simpset HOL-basic-ss ctxt |> sec
in asm-full-simp-tac ss end

fun simp-only ctxt thms = gsimp-only ctxt (fn ctxt => ctxt addsimps thms)


(*******************************)
(* Build theorem statement *)
(* ⟦prems⟧ ⟹ (chead,ahead) ∈ [precond] rels → R *)

(* Uncurry precondition *)
val num-args = length argrels
val precond = precond
  |> rpt-uncurry num-args
  |> rew-uncurry-lambda (* Convert to nicer λ((...,-),-) − form*)

(* Re−attach leading RETURN/return *)
fun mk-RETURN (t,r) = if r then
    let
      val T = funpow num-args range-type (fastype-of (fst ahead))
          val tRETURN = Const (@{const-name RETURN}, T −−>
Type(@{type-name nres},[T]))
    in
      Refine-Util.mk-compN num-args tRETURN t
    end
  else t

fun mk-return (t,r) = if r then
    let
      val T = funpow num-args range-type (fastype-of (fst chead))
  val tRETURN = Const (@{const-name return}, T −−> Type(@{type-name
```

```
Heap},[T]))
        in
          Refine-Util.mk-compN num-args tRETURN t
        end
      else t

    (* Hrmpf!: Gone for good from 2015→2016. Inserting ctxt−based substitute
here. *)
    fun certify-inst ctxt (instT, inst) =
     (TVars.map (K (Thm.ctyp-of ctxt)) instT,
      Vars.map (K (Thm.cterm-of ctxt)) inst);

    (*
    fun mk-RETURN (t,r) = if r then @{mk-term RETURN o ?t} else t
    fun mk-return (t,r) = if r then @{mk-term return o ?t} else t
    *)

    (* Uncurry abstract and concrete function, append leading return *)
    val ahead = ahead |> mk-RETURN |> rpt-uncurry num-args
    val chead = chead |> mk-return |> rpt-uncurry num-args

    (* Add keep−flags and summarize argument relations to product *)
     val argrel = map mk-keep argrels |> rev (* TODO: Why this rev? *) |>
mk-hfprods

    (* Produce final result statement *)
    val result = @{mk-term Trueprop ((?chead,?ahead) ∈ [?precond]_a ?argrel →
?result-rel)}
    val result = Logic.list-implies (prems,result)

    (******************************)
    (* Prove theorem *)

    (* Create context and import result statement and original theorem *)
    val orig-ctxt = ctxt
    (*val thy = Proof-Context.theory-of ctxt*)
    val (insts, ctxt) = Variable.import-inst true [result] ctxt
    val insts' = certify-inst ctxt insts
    val result = Term-Subst.instantiate insts result
    val thm = Thm.instantiate insts' thm

     (* Unfold APP tags. This is required as some APP−tags have also been
unfolded by analysis *)
    val thm = Local-Defs.unfold0 ctxt @{thms APP-def} thm

    (* Tactic to prove the theorem.
      A first step uses hfrefI to get a hnr−goal.
      This is then normalized in several consecutive steps, which
       get rid of uncurrying. Finally, the original theorem is used for resolution,
```

```
    where the pre− and postcondition, and result relation are connected with
   a consequence rule, to handle unfolded hn-ctxt−tags, re−ordered relations,
    and introduced unit−parameters (TODO:
    Mark artificially introduced unit−parameter specially, it may get confused

      with intentional unit−parameter, e.g., functional empty-set ()!)

  *)
 fun tac ctxt =
       resolve-tac ctxt @{thms hfrefI}
  THEN' gsimp-only ctxt (fn c => c
    addsimps @{thms uncurry-def hn-ctxt-def uncurry0-def
                 keep-drop-sels uc-hfprod-sel o-apply
                 APP-def}
    |> Splitter.add-split @{thm prod.split}
  )

  THEN' TRY o (
    REPEAT-ALL-NEW (match-tac ctxt @{thms allI impI})
    THEN' simp-only ctxt @{thms Product-Type.split prod.inject})

  THEN' TRY o REPEAT-ALL-NEW (ematch-tac ctxt @{thms conjE})
  THEN' TRY o hyp-subst-tac ctxt
  THEN' simp-only ctxt @{thms triv-forall-equality}
  THEN' (
    resolve-tac ctxt @{thms hn-refine-cons[rotated]}
    THEN' (resolve-tac ctxt [thm] THEN-ALL-NEW assume-tac ctxt))
  THEN-ALL-NEW simp-only ctxt
    @{thms hn-ctxt-def entt-refl pure-unit-rel-eq-empty
      mult-ac mult-1 mult-1-right keep-drop-sels}

 (* Prove theorem *)
 val result = Thm.cterm-of ctxt result
   val rthm = Goal.prove-internal ctxt [] result (fn - => ALLGOALS (tac
ctxt))

 (* Export statement to original context *)
 val rthm = singleton (Variable.export ctxt orig-ctxt) rthm

 (* Post−processing *)
val rthm = Local-Defs.unfold0 ctxt (Named-Theorems.get ctxt @{named-theorems
to-hfref-post}) rthm

in
  rthm
end

fun to-hfref ctxt = analyze-hnr ctxt #> mk-hfref-thm ctxt
```

```
(*********************************)
(* Composition *)

local
  fun norm-set-of ctxt = {
  trans-rules = Named-Theorems.get ctxt @{named-theorems fcomp-norm-trans},
  cong-rules = Named-Theorems.get ctxt @{named-theorems fcomp-norm-cong},
  norm-rules = Named-Theorems.get ctxt @{named-theorems fcomp-norm-norm},
   refl-rules = Named-Theorems.get ctxt @{named-theorems fcomp-norm-refl}
  }

fun init-rules-of ctxt = Named-Theorems.get ctxt @{named-theorems fcomp-norm-init}
   fun unfold-rules-of ctxt = Named-Theorems.get ctxt @{named-theorems
fcomp-norm-unfold}
   fun simp-rules-of ctxt = Named-Theorems.get ctxt @{named-theorems
fcomp-norm-simps}

in
  fun norm-fcomp-rule ctxt = let
    open PO-Normalizer Refine-Util
    val norm1 = gen-norm-rule (init-rules-of ctxt) (norm-set-of ctxt) ctxt
    val norm2 = Local-Defs.unfold0 ctxt (unfold-rules-of ctxt)
    val norm3 = Conv.fconv-rule (
      Simplifier.asm-full-rewrite
        (put-simpset HOL-basic-ss ctxt addsimps simp-rules-of ctxt))

      val norm = changed-rule (try-rule norm1 o try-rule norm2 o try-rule
norm3)
  in
    repeat-rule norm
  end
end

fun add-pure-constraints-rule ctxt thm = let
  val orig-ctxt = ctxt

  val t = Thm.prop-of thm

  fun
   cnv (@{mpat (typs) pure (mpaq-STRUCT (mpaq-Var ?x -) :: (?'v-c×?'v-a)
set)}) =
    let
      val T = a --> c --> @{typ assn}
      val t = Var (x,T)
      val t = @{mk-term (the-pure ?t)}
    in
```

```
        [(x,T,t)]
      end
    | cnv (t$u) = union op= (cnv t) (cnv u)
    | cnv (Abs (-,-,t)) = cnv t
    | cnv - = []

    val pvars = cnv t

    val - = (pvars |> map #1 |> has-duplicates op=)
      andalso raise TERM (Duplicate indexname with different type,[t]) (* This
should not happen *)

    val substs = map (fn (x,-,t) => (x,t)) pvars

    val t' = subst-Vars substs t

    fun mk-asm (x,T,-) = let
      val t = Var (x,T)
      val t = @{mk-term Trueprop (CONSTRAINT is-pure ?t)}
    in
      t
    end

    val assms = map mk-asm pvars

    fun add-prems prems t = let
      val prems' = Logic.strip-imp-prems t
      val concl = Logic.strip-imp-concl t
    in
      Logic.list-implies (prems@prems', concl)
    end

    val t' = add-prems assms t'

    val (t',ctxt) = yield-singleton (Variable.import-terms true) t' ctxt

    val thm' = Goal.prove-internal ctxt [] (Thm.cterm-of ctxt t') (fn - =>
      ALLGOALS (resolve-tac ctxt [thm] THEN-ALL-NEW assume-tac ctxt))

    val thm' = norm-fcomp-rule ctxt thm'

    val thm' = singleton (Variable.export ctxt orig-ctxt) thm'
  in
    thm'
  end


  val cfg-simp-precond =
    Attrib.setup-config-bool @{binding fcomp-simp-precond} (K true)
```

```
local
  fun mk-simp-thm ctxt t = let
    val st = t
      |> HOLogic.mk-Trueprop
      |> Thm.cterm-of ctxt
      |> Goal.init

    val ctxt = Context-Position.set-visible false ctxt
    val ctxt = ctxt addsimps (
        refine-pw-simps.get ctxt
      @ Named-Theorems.get ctxt @{named-theorems fcomp-prenorm-simps}
      @ @{thms split-tupled-all cnv-conj-to-meta}
      )

    val trace-incomplete-transfer-tac =
    COND (Thm.prems-of #> exists (strip-all-body #> Logic.strip-imp-concl
#> Term.is-open))
        (print-tac ctxt Failed transfer from intermediate level:) all-tac

    val tac =
      ALLGOALS (resolve-tac ctxt @{thms auto-weaken-pre-comp-PRE-I} )
      THEN ALLGOALS (Simplifier.asm-full-simp-tac ctxt)
      THEN trace-incomplete-transfer-tac
      THEN ALLGOALS (TRY o filter-prems-tac ctxt (K false))
      THEN Local-Defs.unfold0-tac ctxt [Drule.triv-forall-equality]

    val st' = tac st |> Seq.take 1 |> Seq.list-of
  val thm = case st' of [st'] => Goal.conclude st' | - => raise THM(Simp-Precond:
Simp−Tactic failed,~1,[st])

    (* Check generated premises for leftover intermediate stuff *)
    val - = exists (Logic.is-all) (Thm.prems-of thm)
        andalso raise THM(Simp-Precond: Transfer from intermediate level
failed,~1,[thm])

    val thm =
      thm
    (*|> map (Simplifier.asm-full-simplify ctxt)*)
    |> Conv.fconv-rule (Object-Logic.atomize ctxt)
    |> Local-Defs.unfold0 ctxt @{thms auto-weaken-pre-to-imp-nf}

    val thm = case Thm.concl-of thm of
      @{mpat Trueprop (- ⟶ -)} => thm
  | @{mpat Trueprop -} => thm RS @{thm auto-weaken-pre-add-dummy-imp}

    | - => raise THM(Simp-Precond: Generated odd theorem, expected form
'P⟶Q',~1,[thm])
```

```
    in
      thm
    end
  in
    fun simplify-precond ctxt thm = let
      val orig-ctxt = ctxt
      val thm = Refine-Util.OF-fst @{thms auto-weaken-pre-init} [asm-rl,thm]
      val thm =
        Local-Defs.unfold0 ctxt @{thms split-tupled-all} thm
        OF @{thms auto-weaken-pre-uncurry-start}

      fun rec-uncurry thm =
       case try (fn () => thm OF @{thms auto-weaken-pre-uncurry-step}) () of
         NONE => thm OF @{thms auto-weaken-pre-uncurry-finish}
       | SOME thm => rec-uncurry thm

      val thm = rec-uncurry thm
        |> Conv.fconv-rule Thm.eta-conversion

      val t = case Thm.prems-of thm of
            t::- => t | - => raise THM(Simp−Precond: Expected at least one
premise,~1,[thm])

      val (t,ctxt) = yield-singleton (Variable.import-terms false) t ctxt
      val ((-,t),ctxt) = Variable.focus NONE t ctxt
      val t = case t of
        @{mpat Trueprop (- ⟶ ?t)} => t | - => raise TERM(Simp-Precond:
Expected implication,[t])

      val simpthm = mk-simp-thm ctxt t
        |> singleton (Variable.export ctxt orig-ctxt)

      val thm = thm OF [simpthm]
      val thm = Local-Defs.unfold0 ctxt @{thms prod-casesK} thm
    in
      thm
    end

    fun simplify-precond-if-cfg ctxt =
      if Config.get ctxt cfg-simp-precond then
        simplify-precond ctxt
      else I

  end

  (* fref O fref *)
  fun compose-ff ctxt A B =
      (@{thm fref-compI-PRE} OF [A,B])
```

```
      |> norm-fcomp-rule ctxt
      |> simplify-precond-if-cfg ctxt
      |> Conv.fconv-rule Thm.eta-conversion

    (* hfref O fref *)
    fun compose-hf ctxt A B =
        (@{thm hfref-compI-PRE} OF [A,B])
      |> norm-fcomp-rule ctxt
      |> simplify-precond-if-cfg ctxt
      |> Conv.fconv-rule Thm.eta-conversion
      |> add-pure-constraints-rule ctxt
      |> Conv.fconv-rule Thm.eta-conversion

    fun ensure-fref ctxt thm = case rthm-type thm of
      RT-HOPARAM => to-fref ctxt thm
    | RT-FREF => thm
    | - => raise THM(Expected parametricity or fref theorem,~1,[thm])

    fun ensure-fref-nres ctxt thm = let
      val thm = ensure-fref ctxt thm
    in
      case Thm.concl-of thm of
        @{mpat (typs) Trueprop (-∈fref - - (-::(- nres×-)set))} => thm
      | @{mpat Trueprop ((-,-)∈fref - - -)} =>
          (thm RS @{thm ensure-fref-nresI}) |> Local-Defs.unfold0 ctxt @{thms
ensure-fref-nres-unfold}
      | - => raise THM(Expected fref−theorem,~1,[thm])
    end

    fun ensure-hfref ctxt thm = case rthm-type thm of
      RT-HNR => to-hfref ctxt thm
    | RT-HFREF => thm
    | - => raise THM(Expected hnr or hfref theorem,~1,[thm])

    fun ensure-hnr ctxt thm = case rthm-type thm of
      RT-HNR => thm
    | RT-HFREF => to-hnr ctxt thm
    | - => raise THM(Expected hnr or hfref theorem,~1,[thm])

    fun gen-compose ctxt A B = let
      val rtA = rthm-type A
    in
      if rtA = RT-HOPARAM orelse rtA = RT-FREF then
        compose-ff ctxt (ensure-fref ctxt A) (ensure-fref ctxt B)
      else
        compose-hf ctxt (ensure-hfref ctxt A) ((ensure-fref-nres ctxt B))

    end
```

```
    val parse-fcomp-flags = Refine-Util.parse-paren-lists
      (Refine-Util.parse-bool-config prenorm cfg-simp-precond)

  val fcomp-attrib = parse-fcomp-flags |-- Attrib.thm >> (fn B => Thm.rule-attribute
[] (fn context => fn A =>
    let
      val ctxt = Context.proof-of context
    in
      gen-compose ctxt A B
    end))

  end
›

attribute-setup to-fref = ‹
  Scan.succeed (Thm.rule-attribute [] (Sepref-Rules.to-fref o Context.proof-of))
› Convert parametricity theorem to uncurried fref−form

attribute-setup to-foparam = ‹
  Scan.succeed (Thm.rule-attribute [] (Sepref-Rules.to-foparam o Context.proof-of))
› ‹Convert param or fref rule to first order rule›

attribute-setup param-fo = ‹
  Scan.succeed (Thm.rule-attribute [] (Sepref-Rules.to-foparam o Context.proof-of))
› ‹Convert param or fref rule to first order rule›

attribute-setup to-hnr = ‹
  Scan.succeed (Thm.rule-attribute [] (Sepref-Rules.to-hnr o Context.proof-of))
› Convert hfref−rule to hnr−rule

attribute-setup to-hfref = ‹Scan.succeed (
    Thm.rule-attribute [] (Context.proof-of #> Sepref-Rules.to-hfref)
  )› ‹Convert hnr to hfref theorem›


attribute-setup ensure-fref-nres = ‹Scan.succeed (
    Thm.rule-attribute [] (Context.proof-of #> Sepref-Rules.ensure-fref-nres)
  )›

attribute-setup sepref-dbg-norm-fcomp-rule = ‹Scan.succeed (
    Thm.rule-attribute [] (Context.proof-of #> Sepref-Rules.norm-fcomp-rule)
  )›

attribute-setup sepref-simplify-precond = ‹Scan.succeed (
    Thm.rule-attribute [] (Context.proof-of #> Sepref-Rules.simplify-precond)
  )› ‹Simplify precondition of fref/hfref−theorem›

attribute-setup FCOMP = Sepref-Rules.fcomp-attrib Composition of refine-
ment rules
```

end

1.6 Setup for Combinators

theory *Sepref-Combinator-Setup*
imports *Sepref-Rules Sepref-Monadify*
keywords *sepref-register* :: *thy-decl*
and *sepref-decl-intf* :: *thy-decl*
begin

1.6.1 Interface Types

This tool allows the declaration of interface types. An interface type is a new type, and a rewriting rule to an existing (logic) type, which is used to encode objects of the interface type in the logic.

```
context begin
  private definition T :: string ⇒ unit list ⇒ unit where T - - ≡ ()
  private lemma unit-eq: (a::unit) ≡ b by simp
  named-theorems --itype-rewrite

  ML ‹
    signature SEPREF-INTF-TYPES = sig
      (* Declare new interface type *)
     val decl-intf-type-cmd: ((string list * binding) * mixfix) * string −> local-theory
−> local-theory
      (* Register interface type rewrite rule *)
      val register-itype-rewrite: typ −> typ −> Proof.context −> local-theory

      (* Convert interface type to logical type*)
      val norm-intf-type: Proof.context −> typ −> typ

      (* Check whether interface type matches operation's type *)
      val check-intf-type: Proof.context −> typ −> typ −> bool
     (* Invoke msg with (normalized) non−matching types in case of no−match *)
      val check-intf-type-msg: (typ * typ −> unit) −> Proof.context −> typ −>
typ −> unit
      (* Trigger error message if no match *)
      val check-intf-type-err: Proof.context −> typ −> typ −> unit

    end

    structure Sepref-Intf-Types: SEPREF-INTF-TYPES = struct
      fun t2t (Type(name,args)) =
        @{term T}
          $HOLogic.mk-string name
          $HOLogic.mk-list @{typ unit} (map t2t args)
      | t2t (TFree (name,-)) = Var ((F^name,0),HOLogic.unitT)
```

```
| t2t (TVar ((name,i),-)) = Var ((V^name,i),HOLogic.unitT)

fun tt2 (t as (Var ((name,i),-))) =
  if match-string F∗ name then TFree (unprefix F name, dummyS)
  else if match-string V∗ name then TVar ((unprefix V name,i), dummyS)
  else raise TERM(tt2: Invalid var,[t])
  | tt2 @{mpat T ?name ?args} = Type (HOLogic.dest-string name, HO-
Logic.dest-list args |> map tt2)
| tt2 t = raise TERM(tt2: Invalid,[t])

fun mk-t2t-rew ctxt T1 T2 = let
  fun chk-vars T = exists-subtype is-TVar T andalso raise TYPE(Type must
not contain schematics,[T],[])
  val - = chk-vars T1
  val - = chk-vars T2

  val free1 = Term.add-tfreesT T1 []
  val free2 = Term.add-tfreesT T2 []

   val - = subset (=) (free2,free1) orelse raise TYPE(Free variables on RHS
must also occur on LHS,[T1,T2],[])

in
  Thm.instantiate' [] [
     t2t T1 |> Thm.cterm-of ctxt |> SOME,
     t2t T2 |> Thm.cterm-of ctxt |> SOME
   ]
   @{thm unit-eq}
end

fun register-itype-rewrite T1 T2 lthy =
  lthy
|> Local-Theory.note ((Binding.empty,@{attributes [--itype-rewrite]}),[mk-t2t-rew
lthy T1 T2])
  |> #2

val decl-intf-type-parser =
  Parse.type-args -- Parse.binding -- Parse.opt-mixfix --| @{keyword is}
-- Parse.typ

fun decl-intf-type-cmd (((args,a),mx),T2-raw) lthy = let
   val (T1,lthy) = Typedecl.typedecl {final = true} (a, map (rpair dummyS)
args, mx) lthy
  val T2 = Syntax.read-typ lthy T2-raw
in
  register-itype-rewrite T1 T2 lthy
end

fun norm-intf-typet ctxt T = let
```

```
  val rew-rls = Named-Theorems.get ctxt @{named-theorems --itype-rewrite}
in
     t2t T
  |> Thm.cterm-of ctxt
  |> Drule.mk-term
  |> Local-Defs.unfold0 ctxt rew-rls
  |> Drule.dest-term
  |> Thm.term-of
end

fun norm-intf-type ctxt T = norm-intf-typet ctxt T |> tt2

fun check-intf-type ctxt iT cT = let
  val it = norm-intf-typet ctxt iT
  val ct = t2t cT
  val thy = Proof-Context.theory-of ctxt
in
  Pattern.matches thy (it,ct)
end

fun check-intf-type-msg msg ctxt iT cT = let
  val it = norm-intf-typet ctxt iT
  val ct = t2t cT
  val thy = Proof-Context.theory-of ctxt
in
  if Pattern.matches thy (it,ct) then ()
  else msg (tt2 it, tt2 ct)
end

fun check-intf-type-err ctxt iT cT = let
  fun msg (iT',cT') = Pretty.block [
    Pretty.str Interface type and logical type do not match,
    Pretty.fbrk,
    Pretty.str Interface: ,Syntax.pretty-typ ctxt iT, Pretty.brk 1,
    Pretty.str   is   , Syntax.pretty-typ ctxt iT', Pretty.fbrk,
    Pretty.str Logical: ,Syntax.pretty-typ ctxt cT, Pretty.brk 1,
    Pretty.str   is   , Syntax.pretty-typ ctxt cT', Pretty.fbrk
  ] |> Pretty.string-of |> error

in
  check-intf-type-msg msg ctxt iT cT
end

val - =
  Outer-Syntax.local-theory
    @{command-keyword sepref-decl-intf}
    Declare interface type
    ( decl-intf-type-parser >> decl-intf-type-cmd);
end
```

›

end

1.6.2 Rewriting Inferred Interface Types

definition *map-type-eq* :: *'a itself* ⇒ *'b itself* ⇒ *bool*
  (**infixr** ‹$\rightarrow_{nt}$› *60*)
  where [*simp*]: *map-type-eq* - - ≡ *True*
lemma *map-type-eqI*: *map-type-eq L R* **by** *auto*

named-theorems-rev *map-type-eqs*

1.6.3 ML-Code

context begin

private lemma *start-eval*: *x* ≡ *SP x* **by** *auto*
private lemma *add-eval*: *f x* ≡ (⋙=)\$(*EVAL*\$*x*)\$(λ_2*x*. *f x*) **by** *auto*

private lemma *init-mk-arity*: *f* ≡ *id* (*SP f*) **by** *simp*
private lemma *add-mk-arity*: *id f* ≡ (λ_2*x*. *id* (*f*\$*x*)) **by** *auto*
private lemma *finish-mk-arity*: *id f* ≡ *f* **by** *simp*

ML ‹

```
structure Sepref-Combinator-Setup = struct

  (* Check whether this term is a valid abstract operation *)
  fun is-valid-abs-op - (Const -) = true
    | is-valid-abs-op ctxt (Free (name,-)) = Variable.is-fixed ctxt name
    | is-valid-abs-op - @{mpat PR-CONST -} = true
    | is-valid-abs-op - - = false

  fun mk-itype ctxt t tyt = let
    val cert = Thm.cterm-of ctxt
    val t = cert t
    val tyt = cert tyt
  in
    Drule.infer-instantiate' ctxt [SOME t, SOME tyt] @{thm itypeI}
  end

  (* Generate mcomb−theorem, required for monadify transformation.
    t$x1$...$xn = x1←EVAL x1; ...; xn←EVAL xn; SP (t$x1$...$xn)
  *)
  fun mk-mcomb ctxt t n = let
    val T = fastype-of t
    val (argsT,-) = strip-type T
    val - = length argsT >= n orelse raise TERM(Too few arguments,[t])
    val effT = take n argsT
```

```
    val orig-ctxt = ctxt
    val names = map (fn i => x^string-of-int i) (1 upto n)
    val (names,ctxt) = Variable.variant-fixes names ctxt
    val vars = map Free (names ~~ effT)

    val lhs = Autoref-Tagging.list-APP (t,vars)
      |> Thm.cterm-of ctxt

    fun add-EVAL x thm =
      case Thm.prop-of thm of
        @{mpat - ≡ ?rhs} => let
          val f = lambda x rhs |> Thm.cterm-of ctxt
          val x = Thm.cterm-of ctxt x
          val eval-thm = Drule.infer-instantiate' ctxt
            [SOME f, SOME x] @{thm add-eval}
          val thm = @{thm transitive} OF [thm,eval-thm]
        in thm end
      | - => raise THM (mk-mcomb internal: Expected lhs==rhs,~1,[thm])

    val thm = Drule.infer-instantiate' ctxt [SOME lhs] @{thm start-eval}
    val thm = fold add-EVAL (rev vars) thm
    val thm = singleton (Proof-Context.export ctxt orig-ctxt) thm
  in
    thm
  end;

  (*
    Generate arity−theorem: t = λx1...xn. SP t$x1$...$xn
  *)
  fun mk-arity ctxt t n = let
    val t = Thm.cterm-of ctxt t
    val thm = Drule.infer-instantiate' ctxt [SOME t] @{thm init-mk-arity}
    val add-mk-arity = Conv.fconv-rule (
      Refine-Util.ftop-conv (K (Conv.rewr-conv @{thm add-mk-arity})) ctxt)
    val thm = funpow n add-mk-arity thm
    val thm = Conv.fconv-rule (
        Refine-Util.ftop-conv (K (Conv.rewr-conv @{thm finish-mk-arity})) ctxt)
thm
  in
    thm
  end;

  datatype opkind = PURE | COMB


  fun analyze-decl c tyt = let
    fun add-tcons-of (Type (name,args)) l = fold add-tcons-of args (name::l)
      | add-tcons-of - l = l
```

```
  fun all-tcons-of P T = forall P (add-tcons-of T [])

  val T = Logic.dest-type tyt
  val (argsT,resT) = strip-type T

  val - = forall (all-tcons-of (fn tn => tn <> @{type-name nres})) argsT
    orelse raise TYPE (
      Arguments contain nres−type
    ^(currently not supported by this attribute),
    argsT,[c,tyt])

  val kind = case resT of
    Type (@{type-name nres},-) => COMB
  | T => let
      val - = all-tcons-of (fn tn => tn <> @{type-name nres}) T
        orelse raise TYPE (
          Result contains inner nres−type,
        argsT,[c,tyt])
    in
      PURE
    end

in (kind,(argsT,resT)) end


fun analyze-itype-thm thm =
  case Thm.prop-of thm of
    @{mpat (typs) Trueprop (intf-type ?c (-::?'v-T itself))} => let
      val tyt = Logic.mk-type T
      val (kind,(argsT,resT)) = analyze-decl c tyt
    in (c,kind,(argsT,resT)) end
  | - => raise THM(Invalid itype−theorem,~1,[thm])


(*fun register-combinator itype-thm context = let
  val ctxt = Context.proof-of context
  val (t,kind,(argsT,-)) = analyze-itype-thm itype-thm
  val n = length argsT
in
  case kind of
    PURE => context
   |> Named-Theorems-Rev.add-thm @{named-theorems-rev id-rules} itype-thm
  | COMB => let
      val arity-thm = mk-arity ctxt t n
      (*val skel-thm = mk-skel ctxt t n*)
      val mcomb-thm = mk-mcomb ctxt t n
    in
      context
   |> Named-Theorems-Rev.add-thm @{named-theorems-rev id-rules} itype-thm
```

```
        |> Named-Theorems-Rev.add-thm @{named-theorems-rev sepref-monadify-arity}
arity-thm
        |> Named-Theorems-Rev.add-thm @{named-theorems-rev sepref-monadify-comb}
mcomb-thm
          (*|> Named-Theorems-Rev.add-thm @{named-theorems-rev sepref-la-skel}
skel-thm*)
        end
    end
    *)

    fun generate-basename ctxt t = let
      fun fail () = raise TERM (Basename generation heuristics failed. Specify a
basename.,[t])
      fun gb (Const (n,-)) =
       (* TODO: There should be a cleaner way than handling this on string level!*)
        n |> space-explode . |> List.last
        | gb (@{mpat PR-CONST ?t}) = gb t
        | gb (t as (-$-)) = let
            val h = head-of t
            val - = is-Const h orelse is-Free h orelse fail ()
          in
            gb h
          end
        | gb (Free (n,-)) =
            if Variable.is-fixed ctxt n then n
            else fail ()
        | gb - = fail ()
    in
      gb t
    end

    fun map-type-raw ctxt rls T = let
      val thy = Proof-Context.theory-of ctxt

      fun rewr-this (lhs,rhs) T = let
        val env = Sign.typ-match thy (lhs,T) Vartab.empty
      in
        Envir.norm-type env rhs
      end

      fun map-Targs f (Type (name,args)) = Type (name,map f args)
        | map-Targs - T = T

      fun
        rewr-thiss (r::rls) T =
          (SOME (rewr-this r T) handle Type.TYPE-MATCH => rewr-thiss rls T)
      | rewr-thiss [] - = NONE

      fun
```

```
    map-type-aux T =
      let
        val T = map-Targs map-type-aux T
      in
        case rewr-thiss rls T of
          SOME T => map-type-aux T
        | NONE => T
      end
in
  map-type-aux T
end

fun get-nt-rule thm = case Thm.prop-of thm of
  @{mpat (typs) Trueprop (map-type-eq (-::?'v-L itself) (-::?'v-R itself))} =>
  let
    val Lvars = Term.add-tvar-namesT L []
    val Rvars = Term.add-tvar-namesT R []

    val - = subset (=) (Rvars, Lvars) orelse (
      let
        val frees = subtract (=) Lvars Rvars
          |> map (Term.string-of-vname)
          |> Pretty.str-list [ ]
          |> Pretty.string-of
      in
        raise THM (Free variables on RHS: ^frees,~1,[thm])
      end)

  in
    (L,R)
  end

| - => raise THM(No map-type-eq theorem,~1,[thm])

fun map-type ctxt T = let
  val rls =
      Named-Theorems-Rev.get ctxt @{named-theorems-rev map-type-eqs}
   |> map get-nt-rule
in map-type-raw ctxt rls T end

fun read-term-type ts tys lthy = case tys of
  SOME ty => let
    val ty = Syntax.read-typ lthy ty
    val ctxt = Variable.declare-typ ty lthy
    val t = Syntax.read-term ctxt ts
    val ctxt = Variable.declare-term t ctxt
  in
    ((t,ty),ctxt)
  end
```

```
| NONE => let
    val t = Syntax.read-term lthy ts
    val ctxt = Variable.declare-term t lthy

    val tyt = fastype-of t |> map-type ctxt |> Logic.mk-type

    val tyt = tyt |> singleton (Variable.export-terms ctxt lthy)
    val (tyt,ctxt) = yield-singleton (Variable.import-terms true) tyt ctxt
    val ty = Logic.dest-type tyt
  in
    ((t,ty),ctxt)
  end

fun check-type-intf ctxt Tc Ti = let
  fun type2term (TFree (name,-)) = Var ((F^name,0),HOLogic.unitT)
    | type2term (TVar ((name,i),-)) = Var ((V^name,i),HOLogic.unitT)
    | type2term (Type (@{type-name fun},[T1,T2])) =
        Free (F,HOLogic.unitT --> HOLogic.unitT --> HOLogic.unitT)
          $type2term T1$type2term T2
    | type2term (Type (name,argsT)) = let
        val args = map type2term argsT
        val n = length args
        val T = replicate n HOLogic.unitT ---> HOLogic.unitT
        val v = Var ((T^name,0),T)
      in list-comb (v, args) end

  val c = type2term Tc
  val i = type2term Ti
  val thy = Proof-Context.theory-of ctxt
in
  Pattern.matches thy (i,c)
end

(* Import all terms into context, with disjoint free variables *)
fun import-terms-disj ts ctxt = let
  fun exp ctxt t = let
    val new-ctxt = Variable.declare-term t ctxt
    val t = singleton (Variable.export-terms new-ctxt ctxt) t
  in t end

  val ts = map (exp ctxt) ts

  fun cons-fst f a (l,b) = let val (a,b) = f a b in (a::l,b) end

 val (ts,ctxt) = fold-rev (cons-fst (yield-singleton (Variable.import-terms true)))
ts ([],ctxt)
in
  (ts,ctxt)
end
```

```
type reg-thms = {
  itype-thm: thm,
  arity-thm: thm option,
  mcomb-thm: thm option
}

fun cr-reg-thms t ty ctxt = let
  val orig-ctxt = ctxt
  val tyt = Logic.mk-type ty
  val ([t,tyt],ctxt) = import-terms-disj [t,tyt] ctxt

  val (kind,(argsT,-)) = analyze-decl t tyt
  val n = length argsT

  val - = Sepref-Intf-Types.check-intf-type-err ctxt ty (fastype-of t)

  val - = is-valid-abs-op ctxt t
     orelse raise TERM(Malformed abstract operation. Use PR-CONST for complex terms.,[t])

  val itype-thm = mk-itype ctxt t tyt
    |> singleton (Variable.export ctxt orig-ctxt)
in
  case kind of
    PURE => {itype-thm = itype-thm, arity-thm = NONE, mcomb-thm = NONE}
  | COMB => let
      val arity-thm = mk-arity ctxt t n
        |> singleton (Variable.export ctxt orig-ctxt)
      val mcomb-thm = mk-mcomb ctxt t n
        |> singleton (Variable.export ctxt orig-ctxt)
    in
      {itype-thm = itype-thm, arity-thm = SOME arity-thm, mcomb-thm = SOME mcomb-thm}
    end
end

fun gen-pr-const-pat ctxt t =
  if is-valid-abs-op ctxt t then (NONE,t)
  else
    let
      val ct = Thm.cterm-of ctxt t
      val thm = Drule.infer-instantiate' ctxt [SOME ct] @{thm UNPROTECT-def[symmetric]}
        |> Conv.fconv-rule (Conv.arg1-conv (Id-Op.protect-conv ctxt))
    in
      (SOME thm,@{mk-term PR-CONST ?t})
    end
```

```
fun sepref-register-single basename t ty lthy = let
  fun mk-qualified basename q = Binding.qualify true basename (Binding.name
q);
  fun
    do-note - - NONE = I
  | do-note q attrs (SOME thm) =
      Local-Theory.note ((mk-qualified basename q,attrs),[thm]) #> snd

  val (pat-thm,t) = gen-pr-const-pat lthy t

  val {itype-thm, arity-thm, mcomb-thm} = cr-reg-thms t ty lthy

  val lthy = lthy
    |> do-note pat @{attributes [def-pat-rules]} pat-thm
    |> do-note itype @{attributes [id-rules]} (SOME itype-thm)
    |> do-note arity @{attributes [sepref-monadify-arity]} arity-thm
    |> do-note mcomb @{attributes [sepref-monadify-comb]} mcomb-thm

in
  (((arity-thm,mcomb-thm),itype-thm),lthy)
end

fun sepref-register-single-cmd ((basename,ts),tys) lthy = let
  val t = Syntax.read-term lthy ts
  val ty = map-option (Syntax.read-typ lthy) tys

  val ty = case ty of SOME ty => ty | NONE => fastype-of t |> map-type lthy

  val basename = case basename of
    NONE => generate-basename lthy t
    | SOME n => n

  val ((-,itype-thm),lthy) = sepref-register-single basename t ty lthy
   val - = Document-Output.pretty-thm lthy itype-thm |> Pretty.string-of |>
writeln

in
  lthy
end

val sepref-register-cmd = fold sepref-register-single-cmd

val sepref-register-parser = Scan.repeat1 (
    Scan.option (Parse.name --| @{keyword :})
    -- Parse.term
    -- Scan.option (@{keyword ::} |-- Parse.typ)
  )
```

```
val - =
  Outer-Syntax.local-theory
    @{command-keyword sepref-register}
    Register operation for sepref
    ( sepref-register-parser
      >> sepref-register-cmd);

val sepref-register-adhoc-parser = Scan.repeat1 (
  Args.term -- Scan.option (Scan.lift (Args.$$$ ::) |-- Args.typ)
)

fun sepref-register-adhoc-single (t,ty) context = let
  val ctxt = Context.proof-of context

    (* TODO: Map-type probably not clean, as it draws info from (current)
context,
    which may have changed if registered elsewhere ...
  *)
  val ty = case ty of SOME ty => ty | NONE => fastype-of t |> map-type ctxt

  val (pat-thm,t) = gen-pr-const-pat ctxt t

  val {itype-thm, arity-thm, mcomb-thm} = cr-reg-thms t ty ctxt

  fun app - NONE = I
    | app attr (SOME thm) = Thm.apply-attribute attr thm #> snd

in
  context
    |> app (Named-Theorems-Rev.add @{named-theorems-rev def-pat-rules})
pat-thm
    |> app (Named-Theorems-Rev.add @{named-theorems-rev id-rules}) (SOME
itype-thm)
  |> app (Named-Theorems-Rev.add @{named-theorems-rev sepref-monadify-arity})
arity-thm
  |> app (Named-Theorems-Rev.add @{named-theorems-rev sepref-monadify-comb})
mcomb-thm
end

val sepref-register-adhoc = fold sepref-register-adhoc-single

fun sepref-register-adhoc-attr ttys = Thm.declaration-attribute (K (sepref-register-adhoc
ttys))

val sepref-register-adhoc-attr-decl = sepref-register-adhoc-parser >> sepref-register-adhoc-attr

end
›
```

end

attribute-setup *sepref-register-adhoc* = *Sepref-Combinator-Setup.sepref-register-adhoc-attr-decl*
⟨*Register operations in ad−hoc manner. Improper if this gets exported*!⟩

1.6.4 Obsolete Manual Setup Rules

lemma
mk-mcomb1: $\bigwedge c.\ c\$x1 \equiv (\gg\!=)\$(EVAL\$x1)\$(\lambda_2 x1.\ SP\ (c\$x1))$
and *mk-mcomb2*: $\bigwedge c.\ c\$x1\$x2 \equiv (\gg\!=)\$(EVAL\$x1)\$(\lambda_2 x1.\ (\gg\!=)\$(EVAL\$x2)\$(\lambda_2 x2.\ SP\ (c\$x1\$x2)))$
and *mk-mcomb3*: $\bigwedge c.\ c\$x1\$x2\$x3 \equiv (\gg\!=)\$(EVAL\$x1)\$(\lambda_2 x1.\ (\gg\!=)\$(EVAL\$x2)\$(\lambda_2 x2.\ (\gg\!=)\$(EVAL\$x3)\$(\lambda_2 x3.\ SP\ (c\$x1\$x2\$x3))))$
by *auto*

end

1.7 Translation

theory *Sepref-Translate*
imports
Sepref-Monadify
Sepref-Constraints
Sepref-Frame
Lib/Pf-Mono-Prover
Sepref-Rules
Sepref-Combinator-Setup
Lib/User-Smashing
begin

This theory defines the translation phase.

The main functionality of the translation phase is to apply refinement rules. Thereby, the linearity information is exploited to create copies of parameters that are still required, but would be destroyed by a synthesized operation. These *frame-based* rules are in the named theorem collection *sepref-fr-rules*, and the collection *sepref-copy-rules* contains rules to handle copying of parameters.

Apart from the frame-based rules described above, there is also a set of rules for combinators, in the collection *sepref-comb-rules*, where no automatic copying of parameters is applied.

Moreover, this theory contains

- A setup for the basic monad combinators and recursion.
- A tool to import parametricity theorems.

- Some setup to identify pure refinement relations, i.e., those not involving the heap.
- A preprocessor that identifies parameters in refinement goals, and flags them with a special tag, that allows their correct handling.

Tag to keep track of abstract bindings. Required to recover information for side-condition solving.

definition *bind-ref-tag* $x\ m \equiv RETURN\ x \leq m$

Tag to keep track of preconditions in assertions

definition *vassn-tag* $\Gamma \equiv \exists h.\ h \models \Gamma$

lemma *vassn-tagI*: $h \models \Gamma \Longrightarrow$ *vassn-tag* Γ
  unfolding *vassn-tag-def* **..**

lemma *vassn-dest*[*dest*!]:
  vassn-tag $(\Gamma_1 * \Gamma_2) \Longrightarrow$ *vassn-tag* $\Gamma_1 \wedge$ *vassn-tag* Γ_2
  vassn-tag (*hn-ctxt* $R\ a\ b$) $\Longrightarrow a \in rdom\ R$
  unfolding *vassn-tag-def rdomp-def*[*abs-def*]
  by (*auto simp*: *mod-star-conv hn-ctxt-def*)

lemma *entails-preI*:
  assumes *vassn-tag* $A \Longrightarrow A \Longrightarrow_A B$
  shows $A \Longrightarrow_A B$
  using *assms*
  by (*auto simp*: *entails-def vassn-tag-def*)

lemma *invalid-assn-const*:
  invalid-assn $(\lambda\text{- -}.\ P)\ x\ y = \uparrow($*vassn-tag* $P) * true$
  by (*simp-all add*: *invalid-assn-def vassn-tag-def*)

lemma *vassn-tag-simps*[*simp*]:
  vassn-tag emp
  vassn-tag true
  by (*sep-auto simp*: *vassn-tag-def mod-emp*)+

definition *GEN-ALGO* $f\ \Phi \equiv \Phi\ f$
— Tag to synthesize f with property Φ.

lemma *is-GEN-ALGO*: *GEN-ALGO* $f\ \Phi \Longrightarrow$ *GEN-ALGO* $f\ \Phi$ **.**

Tag for side-condition solver to discharge by assumption

definition *RPREM* :: *bool* $\Rightarrow$ *bool* **where** [*simp*]: *RPREM* $P = P$
lemma *RPREMI*: $P \Longrightarrow$ *RPREM* P **by** *simp*

lemma *trans-frame-rule*:
  assumes *RECOVER-PURE* $\Gamma\ \Gamma'$

assumes *vassn-tag* $\Gamma' \Longrightarrow$ *hn-refine* Γ' *c* Γ'' *R a*
shows *hn-refine* $(F{*}\Gamma)$ *c* $(F{*}\Gamma'')$ *R a*
apply (*rule hn-refine-frame*[*OF - entt-refl*])
applyF (*rule hn-refine-cons-pre*)
focus using *assms*(*1*) **unfolding** *RECOVER-PURE-def* **apply** *assumption*
solved

apply1 (*rule hn-refine-preI*)
apply1 (*rule assms*)
applyS (*auto simp add*: *vassn-tag-def*)
solved
done

lemma *recover-pure-cons*: — Used for debugging
assumes *RECOVER-PURE* Γ Γ'
assumes *hn-refine* Γ' *c* Γ'' *R a*
shows *hn-refine* (Γ) *c* (Γ'') *R a*
using *trans-frame-rule*[**where** *F*=*emp*, *OF assms*] **by** *simp*

— Tag to align structure of refinement assertions for consequence rule
definition *CPR-TAG* :: *assn* ⇒ *assn* ⇒ *bool* **where** [*simp*]: *CPR-TAG y x* ≡ *True*
lemma *CPR-TAG-starI*:
assumes *CPR-TAG P1 Q1*
assumes *CPR-TAG P2 Q2*
shows *CPR-TAG* (*P1*∗*P2*) (*Q1*∗*Q2*)
by *simp*
lemma *CPR-tag-ctxtI*: *CPR-TAG* (*hn-ctxt R x xi*) (*hn-ctxt R′ x xi*) **by** *simp*
lemma *CPR-tag-fallbackI*: *CPR-TAG P Q* **by** *simp*

lemmas *CPR-TAG-rules* = *CPR-TAG-starI CPR-tag-ctxtI CPR-tag-fallbackI*

lemma *cons-pre-rule*: — Consequence rule to be applied if no direct operation rule matches
assumes *CPR-TAG P P′*
assumes $P \Longrightarrow_t P'$
assumes *hn-refine P′ c Q R m*
shows *hn-refine P c Q R m*
using *assms*(*2*−) **by** (*rule hn-refine-cons-pre*)

named-theorems-rev *sepref-gen-algo-rules* ‹*Sepref*: *Generic algorithm rules*›

ML ‹

structure Sepref-Translate = *struct*

val cfg-debug =
Attrib.setup-config-bool @{*binding sepref-debug-translate*} (*K false*)

```
val dbg-msg-tac = Sepref-Debugging.dbg-msg-tac cfg-debug

fun gen-msg-analyze t ctxt = let
  val t = Logic.strip-assums-concl t
in
  case t of
    @{mpat Trueprop ?t} => (case t of
          @{mpat - ∨A - ⟹t -} => t-merge
        | @{mpat - ⟹t -} => t-frame
        | @{mpat INDEP -} => t-indep
        | @{mpat CONSTRAINT - -} => t-constraint
        | @{mpat mono-Heap -} => t-mono
        | @{mpat PREFER-tag -} => t-prefer
        | @{mpat DEFER-tag -} => t-defer
        | @{mpat RPREM -} => t-rprem
       | @{mpat hn-refine - - - - ?a} => Pretty.block [Pretty.str t-hnr: ,Pretty.brk
1, Syntax.pretty-term ctxt a] |> Pretty.string-of
        | - => Unknown goal type
      )
  | - => Non−Trueprop goal
end

fun msg-analyze msg = Sepref-Debugging.msg-from-subgoal msg gen-msg-analyze

fun check-side-conds thm = let
  open Sepref-Basic
  (* Check that term is no binary operator on assertions *)
  fun is-atomic (Const (-,@{typ assn⇒assn⇒assn})$-$-) = false
    | is-atomic - = true

   val is-atomic-star-list = (Expected atoms separated by star,forall is-atomic o
strip-star)

  val is-trueprop = (Expected Trueprop conclusion,can HOLogic.dest-Trueprop)

  fun assert t′ (msg,p) t = if p t then () else raise TERM(msg,[t′,t])

  fun chk-prem t = let
    val assert = assert t

    fun chk @{mpat ?l ∨A ?r ⟹t ?m} = (
          assert is-atomic-star-list l;
          assert is-atomic-star-list r;
          assert is-atomic-star-list m
        )
      | chk (t as @{mpat - ⟹A -}) = raise TERM(Invalid frame side condition
(old−style ent),[t])
      | chk @{mpat ?l ⟹t ?r} = (
```

```
        assert is-atomic-star-list l;
        assert is-atomic-star-list r
      )
    | chk - = ()

    val t = Logic.strip-assums-concl t
  in
    assert is-trueprop t;
    chk (HOLogic.dest-Trueprop t)
  end

in
  map chk-prem (Thm.prems-of thm)
end

structure sepref-comb-rules = Named-Sorted-Thms (
  val name = @{binding sepref-comb-rules}
  val description = Sepref: Combinator rules
  val sort = K I
  fun transform - thm = let
    val - = check-side-conds thm
  in
    [thm]
  end
)

structure sepref-fr-rules = Named-Sorted-Thms (
  val name = @{binding sepref-fr-rules}
  val description = Sepref: Frame−based rules
  val sort = K I
  fun transform context thm = let
    val ctxt = Context.proof-of context
    val thm = Sepref-Rules.ensure-hnr ctxt thm
      |> Conv.fconv-rule (Sepref-Frame.align-rl-conv ctxt)

    val - = check-side-conds thm
    val - = case try (Sepref-Rules.analyze-hnr ctxt) thm of
        NONE =>
          (Pretty.block [
            Pretty.str hnr−analysis failed,
            Pretty.str :,
            Pretty.brk 1,
            Thm.pretty-thm ctxt thm])
          |> Pretty.string-of |> error
      | SOME ana => let
          val - = Sepref-Combinator-Setup.is-valid-abs-op ctxt (fst (#ahead ana))
            orelse Pretty.block [
              Pretty.str Invalid abstract head:,
              Pretty.brk 1,
```

```
              Pretty.enclose ( ) [Syntax.pretty-term ctxt (fst (#ahead ana))],
              Pretty.brk 1,
              Pretty.str in thm,
              Pretty.brk 1,
              Thm.pretty-thm ctxt thm
            ]
          |> Pretty.string-of |> error
        in () end
    in
      [thm]
    end
  )

  (***** Side Condition Solving *)
  local
    open Sepref-Basic
  in

    fun side-unfold-tac ctxt = let
      (*val ctxt = put-simpset HOL-basic-ss ctxt
        addsimps sepref-prep-side-simps.get ctxt*)
    in
      CONVERSION (Id-Op.unprotect-conv ctxt)
    THEN' SELECT-GOAL (Local-Defs.unfold0-tac ctxt @{thms bind-ref-tag-def})
      (*THEN' asm-full-simp-tac ctxt*)
    end

   fun side-fallback-tac ctxt = side-unfold-tac ctxt THEN' TRADE (SELECT-GOAL
o auto-tac) ctxt

    val side-frame-tac = Sepref-Frame.frame-tac side-fallback-tac
    val side-merge-tac = Sepref-Frame.merge-tac side-fallback-tac
    fun side-constraint-tac ctxt = Sepref-Constraints.constraint-tac ctxt
   fun side-mono-tac ctxt = side-unfold-tac ctxt THEN' TRADE Pf-Mono-Prover.mono-tac
ctxt

    fun side-gen-algo-tac ctxt =
      side-unfold-tac ctxt
     THEN' resolve-tac ctxt (Named-Theorems-Rev.get ctxt @{named-theorems-rev
sepref-gen-algo-rules})

    fun side-pref-def-tac ctxt =
      side-unfold-tac ctxt THEN'
      TRADE (fn ctxt =>
        resolve-tac ctxt @{thms PREFER-tagI DEFER-tagI}
          THEN' (Sepref-Debugging.warning-tac' Obsolete PREFER/DEFER side
condition ctxt THEN' Tagged-Solver.solve-tac ctxt)
      ) ctxt
```

```
fun side-rprem-tac ctxt =
  resolve-tac ctxt @{thms RPREMI} THEN' Refine-Util.rprems-tac ctxt
  THEN' (K (smash-new-rule ctxt))

(*
  Solve side condition, or invoke hnr-tac on hn-refine goal.

  In debug mode, side-condition solvers are allowed to not completely solve
  the side condition, but must change the goal.
*)
fun side-cond-dispatch-tac dbg hnr-tac ctxt = let
  fun MK tac = if dbg then CHANGED o tac ctxt else SOLVED' (tac ctxt)

  val t-merge = MK side-merge-tac
  val t-frame = MK side-frame-tac
  val t-indep = MK Indep-Vars.indep-tac
  val t-constraint = MK side-constraint-tac
  val t-mono = MK side-mono-tac
  val t-pref-def = MK side-pref-def-tac
  val t-rprem = MK side-rprem-tac
  val t-gen-algo = side-gen-algo-tac ctxt
  val t-fallback = MK side-fallback-tac
in
  WITH-concl
    (fn @{mpat Trueprop ?t} => (case t of
          @{mpat - ∨A - ⟹t -} => t-merge
        | @{mpat - ⟹t -} => t-frame
        | @{mpat - ⟹A -} => Sepref-Debugging.warning-tac' Old-style frame
side condition ctxt THEN' (K no-tac)
        | @{mpat INDEP -} => t-indep     (* TODO: Get rid of this!? *)
        | @{mpat CONSTRAINT - -} => t-constraint
        | @{mpat mono-Heap -} => t-mono
        | @{mpat PREFER-tag -} => t-pref-def
        | @{mpat DEFER-tag -} => t-pref-def
        | @{mpat RPREM -} => t-rprem
        | @{mpat GEN-ALGO - -} => t-gen-algo
        | @{mpat hn-refine - - - - -} => hnr-tac
        | - => t-fallback
      )
    | - => K no-tac
  )
end

end

(***** Main Translation Tactic *)
local
  open Sepref-Basic STactical
```

```
(* ATTENTION: Beware of evaluation order, as some initialization operations
on
   context are expensive, and should not be repeated during proof search! *)
 in


  (* Translate combinator, yields new translation goals and side conditions
    which must be processed in order. *)
  fun trans-comb-tac ctxt = let
    val comb-rl-net = sepref-comb-rules.get ctxt
      |> Tactic.build-net

  in
    DETERM o (
      resolve-from-net-tac ctxt comb-rl-net
      ORELSE' (
        Sepref-Frame.norm-goal-pre-tac ctxt
        THEN' resolve-from-net-tac ctxt comb-rl-net
      )
    )
  end

  (* Translate operator. Only succeeds if it finds an operator rule such that
    all resulting side conditions can be solved. Takes the first such rule.

    In debug mode, it returns a sequence of the unsolved side conditions of
    each applicable rule.
  *)
  fun gen-trans-op-tac dbg ctxt = let
    val fr-rl-net = sepref-fr-rules.get ctxt |> Tactic.build-net
    val fr-rl-tac =
      resolve-from-net-tac ctxt fr-rl-net (* Try direct match *)
      ORELSE' (
        Sepref-Frame.norm-goal-pre-tac ctxt (* Normalize precondition *)
        THEN' (
          resolve-from-net-tac ctxt fr-rl-net
          ORELSE' (
              resolve-tac ctxt @{thms cons-pre-rule} (* Finally, generate a frame
condition *)
            THEN-ALL-NEW-LIST [
              SOLVED' (REPEAT-ALL-NEW-FWD (DETERM o resolve-tac ctxt
@{thms CPR-TAG-rules})),
                K all-tac,  (* Frame remains unchanged as first goal, even if fr-rl
creates side−conditions *)
              resolve-from-net-tac ctxt fr-rl-net
            ]
          )
        )
```

```
        )

        val side-tac = REPEAT-ALL-NEW-FWD (side-cond-dispatch-tac false (K
no-tac) ctxt)

      val fr-tac =
       if dbg then (* Present all possibilities with (partially resolved) side conditions
*)
          fr-rl-tac THEN-ALL-NEW-FWD (TRY o side-tac)
        else (* Choose first rule that solves all side conditions *)
            DETERM o SOLVED' (fr-rl-tac THEN-ALL-NEW-FWD (SOLVED'
side-tac))

    in
      PHASES' [
        (Align goal,Sepref-Frame.align-goal-tac, 0),
        (Frame rule,fn ctxt => resolve-tac ctxt @{thms trans-frame-rule}, 1),
        (* RECOVER-PURE goal *)
        (Recover pure,Sepref-Frame.recover-pure-tac, ~1),
        (* hn−refine goal with stripped precondition *)
        (Apply rule,K fr-tac,~1)
      ] (flag-phases-ctrl dbg) ctxt
    end

    (* Translate combinator, operator, or side condition. *)
    fun gen-trans-step-tac dbg ctxt = side-cond-dispatch-tac dbg
      (trans-comb-tac ctxt ORELSE' gen-trans-op-tac dbg ctxt)
      ctxt

    val trans-step-tac = gen-trans-step-tac false
    val trans-step-keep-tac = gen-trans-step-tac true

    fun gen-trans-tac dbg ctxt =
      PHASES' [
        (Translation steps,REPEAT-DETERM' o trans-step-tac,~1),
      (Constraint solving,fn ctxt => fn - => Sepref-Constraints.process-constraint-slot
ctxt, 0)
      ] (flag-phases-ctrl dbg) ctxt

    val trans-tac = gen-trans-tac false
    val trans-keep-tac = gen-trans-tac true


  end


  val setup = I
    #> sepref-fr-rules.setup
    #> sepref-comb-rules.setup
```

end

›

setup *Sepref-Translate.setup*

Basic Setup

lemma *hn-pass*[*sepref-fr-rules*]:
  shows *hn-refine* (*hn-ctxt P x x'*) (*return x'*) (*hn-invalid P x x'*) *P* (*PASS*$*x*)
  apply *rule* **apply** (*sep-auto simp*: *hn-ctxt-def invalidate-clone'*)
  done

lemma *hn-bind*[*sepref-comb-rules*]:
  assumes *D1*: *hn-refine* Γ *m'* Γ*1 Rh m*
  assumes *D2*:
    $\bigwedge x\ x'$. *bind-ref-tag x m* $\Longrightarrow$
      hn-refine (Γ*1* $*$ *hn-ctxt Rh x x'*) (*f' x'*) (Γ*2 x x'*) *R* (*f x*)
  assumes *IMP*: $\bigwedge x\ x'$. Γ*2 x x'* $\Longrightarrow_t$ Γ' $*$ *hn-ctxt Rx x x'*
  shows *hn-refine* Γ (*m'*$\gg=$*f'*) Γ' *R* (*Refine-Basic.bind*$*m*$($\lambda_2 x$. *f x*))
  using *assms*
  unfolding *APP-def PROTECT2-def bind-ref-tag-def*
  by (*rule hnr-bind*)

lemma *hn-RECT'*[*sepref-comb-rules*]:
  assumes *INDEP Ry INDEP Rx INDEP Rx'*
  assumes *FR*: *P* $\Longrightarrow_t$ *hn-ctxt Rx ax px* $*$ *F*
  assumes *S*: $\bigwedge$*cf af ax px*. $\llbracket$
    $\bigwedge$*ax px*. *hn-refine* (*hn-ctxt Rx ax px* $*$ *F*) (*cf px*) (*hn-ctxt Rx' ax px* $*$ *F*) *Ry*
      (*RCALL*$*af*$*ax*)$\rrbracket$
    $\Longrightarrow$ *hn-refine* (*hn-ctxt Rx ax px* $*$ *F*) (*cB cf px*) (*F' ax px*) *Ry*
        (*aB af ax*)
  assumes *FR'*: $\bigwedge$*ax px*. *F' ax px* $\Longrightarrow_t$ *hn-ctxt Rx' ax px* $*$ *F*
  assumes *M*: ($\bigwedge x$. *mono-Heap* (λf. *cB f x*))

  shows *hn-refine*
    (*P*) (*heap.fixp-fun cB px*) (*hn-ctxt Rx' ax px* $*$ *F*) *Ry*
      (*RECT*$($\lambda_2 D\ x$. *aB D x*)$*ax*)
  unfolding *APP-def PROTECT2-def*
  apply (*rule hn-refine-cons-pre*[*OF FR*])
  apply (*rule hnr-RECT*)

  apply (*rule hn-refine-cons-post*[*OF - FR'*])
  apply (*rule S*[*unfolded RCALL-def APP-def*])

apply *assumption*
apply *fact+*
done

lemma *hn-RCALL*[*sepref-comb-rules*]:
assumes *RPREM* (*hn-refine* P' *c* Q' *R* (*RCALL* \$ *a* \$ *b*))
and $P \Longrightarrow_t F * P'$
shows *hn-refine P c* $(F * Q')$ *R* (*RCALL* \$ *a* \$ *b*)
using *assms hn-refine-frame*[**where** *m=RCALL*\$*a*\$*b*]
by *simp*

definition *monadic-WHILEIT I b f s* ≡ *do* {
RECT (λ*D s. do* {
ASSERT (*I s*);
bv ← *b s*;
if bv then do {
s ← *f s*;
D s
} *else do* {*RETURN s*}
}) *s*
}

definition *heap-WHILET b f s* ≡ *do* {
heap.fixp-fun (λ*D s. do* {
bv ← *b s*;
if bv then do {
s ← *f s*;
D s
} *else do* {*return s*}
}) *s*
}

lemma *heap-WHILET-unfold*[*code*]: *heap-WHILET b f s* =
do {
bv ← *b s*;
if bv then do {
s ← *f s*;
heap-WHILET b f s
} *else*
return s
}
unfolding *heap-WHILET-def*
apply (*subst heap.mono-body-fixp*)
apply *pf-mono*
apply *simp*
**done*

lemma *WHILEIT-to-monadic*: *WHILEIT I b f s* = *monadic-WHILEIT I* (λs. *RETURN* (*b s*)) *f s*
  unfolding *WHILEIT-def monadic-WHILEIT-def*
  unfolding *WHILEI-body-def bind-ASSERT-eq-if*
  by (*simp cong*: *if-cong*)

lemma *WHILEIT-pat*[*def-pat-rules*]:
  WHILEIT\$*I* $\equiv$ *UNPROTECT* (*WHILEIT I*)
  WHILET $\equiv$ *PR-CONST* (*WHILEIT* (λ-. *True*))
  by (*simp-all add*: *WHILET-def*)

lemma *id-WHILEIT*[*id-rules*]:
  PR-CONST (*WHILEIT I*) $::_i$ *TYPE*(($'a \Rightarrow$ *bool*) $\Rightarrow$ ($'a \Rightarrow$ $'a$ *nres*) $\Rightarrow$ $'a \Rightarrow$ $'a$ *nres*)
  by *simp*

lemma *WHILE-arities*[*sepref-monadify-arity*]:

  PR-CONST (*WHILEIT I*) $\equiv$ $\lambda_2 b$ *f s*. *SP* (*PR-CONST* (*WHILEIT I*))\$($\lambda_2 s$. *b*\$*s*)\$($\lambda_2 s$. *f*\$*s*)\$*s*
  by (*simp-all add*: *WHILET-def*)

lemma *WHILEIT-comb*[*sepref-monadify-comb*]:
  PR-CONST (*WHILEIT I*)\$($\lambda_2 x$. *b x*)\$*f*\$*s* $\equiv$
    Refine-Basic.bind\$(*EVAL*\$*s*)\$($\lambda_2 s$.
      SP (*PR-CONST* (*monadic-WHILEIT I*))\$($\lambda_2 x$. (*EVAL*\$(*b x*)))\$*f*\$*s*
)
  by (*simp-all add*: *WHILEIT-to-monadic*)

lemma *hn-monadic-WHILE-aux*:
  assumes *FR*: $P \Longrightarrow_t \Gamma *$ *hn-ctxt Rs s' s*
  assumes *b-ref*: $\bigwedge s\ s'$. *I s'* $\Longrightarrow$ *hn-refine*
    ($\Gamma *$ *hn-ctxt Rs s' s*)
    (*b s*)
    (Γb *s' s*)
    (*pure bool-rel*)
    (*b' s'*)
  assumes *b-fr*: $\bigwedge s'\ s$. Γb *s' s* $\Longrightarrow_t \Gamma *$ *hn-ctxt Rs s' s*

  assumes *f-ref*: $\bigwedge s'\ s$. $[\![I\ s']\!] \Longrightarrow$ *hn-refine*
    ($\Gamma *$ *hn-ctxt Rs s' s*)
    (*f s*)
    (Γf *s' s*)
    Rs
    (*f' s'*)
  assumes *f-fr*: $\bigwedge s'\ s$. Γf *s' s* $\Longrightarrow_t \Gamma *$ *hn-ctxt* (λ- -. *true*) *s' s*

 shows *hn-refine* (*P*) (*heap-WHILET b f s*) ($\Gamma *$ *hn-invalid Rs s' s*) *Rs* (*monadic-WHILEIT*

```
I b' f' s')
  unfolding monadic-WHILEIT-def heap-WHILET-def
  apply1 (rule hn-refine-cons-pre[OF FR])
  apply weaken-hnr-post
  focus (rule hn-refine-cons-pre[OF - hnr-RECT])
    applyS (subst mult-ac(2)[of Γ]; rule entt-refl; fail)

    apply1 (rule hnr-ASSERT)
    focus (rule hnr-bind)
      focus (rule hn-refine-cons[OF - b-ref b-fr entt-refl])
        applyS (simp add: star-aci)
        applyS assumption
      solved

      focus (rule hnr-If)
        applyS (sep-auto; fail)
        focus (rule hnr-bind)
          focus (rule hn-refine-cons[OF - f-ref f-fr entt-refl])
            apply (sep-auto simp: hn-ctxt-def pure-def intro!: enttI; fail)
            apply assumption
          solved

          focus (rule hn-refine-frame)
            applyS rprems
            applyS (rule enttI; solve-entails)
          solved

          apply (sep-auto intro!: enttI; fail)
        solved
        applyF (sep-auto,rule hn-refine-frame)
          applyS (rule hnr-RETURN-pass)

          apply (rule enttI)
          apply (fr-rot-rhs 1)
          apply (fr-rot 1, rule fr-refl)
          apply (rule fr-refl)
          apply solve-entails
        solved

        apply (rule entt-refl)
      solved

      apply (rule enttI)
      applyF (rule ent-disjE)
        apply1 (sep-auto simp: hn-ctxt-def pure-def)
        apply1 (rule ent-true-drop)
        apply1 (rule ent-true-drop)
        applyS (rule ent-refl)
```

applyS (*sep-auto simp*: *hn-ctxt-def pure-def*)
solved
solved
apply *pf-mono*
solved
done

lemma *hn-monadic-WHILE-lin*[*sepref-comb-rules*]:
assumes *INDEP Rs*
assumes *FR*: $P \Longrightarrow_t \Gamma * \textit{hn-ctxt Rs } s' \ s$
assumes *b-ref*: $\bigwedge s\ s'.\ I\ s' \Longrightarrow \textit{hn-refine}$
$(\Gamma * \textit{hn-ctxt Rs } s'\ s)$
$(b\ s)$
$(\Gamma b\ s'\ s)$
$(\textit{pure bool-rel})$
$(b'\ s')$
assumes *b-fr*: $\bigwedge s'\ s.\ \textit{TERM}\ (\textit{monadic-WHILEIT},''cond'') \Longrightarrow \Gamma b\ s'\ s \Longrightarrow_t \Gamma *$ $\textit{hn-ctxt Rs } s'\ s$

assumes *f-ref*: $\bigwedge s'\ s.\ I\ s' \Longrightarrow \textit{hn-refine}$
$(\Gamma * \textit{hn-ctxt Rs } s'\ s)$
$(f\ s)$
$(\Gamma f\ s'\ s)$
Rs
$(f'\ s')$
assumes *f-fr*: $\bigwedge s'\ s.\ \textit{TERM}\ (\textit{monadic-WHILEIT},''body'') \Longrightarrow \Gamma f\ s'\ s \Longrightarrow_t \Gamma *$ $\textit{hn-ctxt}\ (\lambda\text{- -}.\ \textit{true})\ s'\ s$
shows *hn-refine*
P
$(\textit{heap-WHILET } b\ f\ s)$
$(\Gamma * \textit{hn-invalid Rs } s'\ s)$
Rs
$(\textit{PR-CONST}\ (\textit{monadic-WHILEIT } I)\$(\lambda_2 s'.\ b'\ s')\$(\lambda_2 s'.\ f'\ s')\$(s'))$
using *assms*$(2-)$
unfolding *APP-def PROTECT2-def CONSTRAINT-def PR-CONST-def*
by (*rule hn-monadic-WHILE-aux*)

lemma *monadic-WHILEIT-refine*[*refine*]:
assumes [*refine*]: $(s',s) \in R$
assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s\ ]\!] \Longrightarrow I'\ s'$
assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s;\ I'\ s'\ ]\!] \Longrightarrow b'\ s' \le \Downarrow \textit{bool-rel}\ (b\ s)$
assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s;\ I'\ s';\ \textit{nofail}\ (b\ s);\ \textit{inres}\ (b\ s)\ \textit{True}\ ]\!] \Longrightarrow$ $f'\ s' \le \Downarrow R\ (f\ s)$
shows $\textit{monadic-WHILEIT } I'\ b'\ f'\ s' \le \Downarrow R\ (\textit{monadic-WHILEIT } I\ b\ f\ s)$
unfolding *monadic-WHILEIT-def*
by (*refine-rcg bind-refine'*; *assumption?*; *auto*)

lemma *monadic-WHILEIT-refine-WHILEIT*[*refine*]:
assumes [*refine*]: $(s',s) \in R$

assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s\ ]\!] \Longrightarrow I'\ s'$
assumes [*THEN order-trans,refine-vcg*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s;\ I'\ s'\ ]\!] \Longrightarrow b'\ s' \leq SPEC\ (\lambda r.\ r = b\ s)$
assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ I\ s;\ I'\ s';\ b\ s\ ]\!] \Longrightarrow f'\ s' \leq\Downarrow R\ (f\ s)$
shows *monadic-WHILEIT* $I'\ b'\ f'\ s' \leq\Downarrow R$ (*WHILEIT* $I\ b\ f\ s$)
unfolding *WHILEIT-to-monadic*
by (*refine-vcg*; *assumption?*; *auto*)

lemma *monadic-WHILEIT-refine-WHILET*[*refine*]:
assumes [*refine*]: $(s',s) \in R$
assumes [*THEN order-trans,refine-vcg*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R\ ]\!] \Longrightarrow b'\ s' \leq SPEC\ (\lambda r.\ r = b\ s)$
assumes [*refine*]: $\bigwedge s'\ s.\ [\![\ (s',s){\in}R;\ b\ s\ ]\!] \Longrightarrow f'\ s' \leq\Downarrow R\ (f\ s)$
shows *monadic-WHILEIT* (λ-. *True*) $b'\ f'\ s' \leq\Downarrow R$ (*WHILET* $b\ f\ s$)
unfolding *WHILET-def*
by (*refine-vcg*; *assumption?*)

lemma *monadic-WHILEIT-pat*[*def-pat-rules*]:
monadic-WHILEIT\$$I \equiv$ *UNPROTECT* (*monadic-WHILEIT* I)
by *auto*

lemma *id-monadic-WHILEIT*[*id-rules*]:
PR-CONST (*monadic-WHILEIT* I) $::_i$ *TYPE*(('a ⇒ bool nres) ⇒ ('a ⇒ 'a nres) ⇒ 'a ⇒ 'a nres)
by *simp*

lemma *monadic-WHILEIT-arities*[*sepref-monadify-arity*]:
PR-CONST (*monadic-WHILEIT* I) $\equiv \lambda_2 b\ f\ s.$ *SP* (*PR-CONST* (*monadic-WHILEIT* I))\$($\lambda_2 s.\ b\$s$)\$($\lambda_2 s.\ f\$s$)\$$s$
by (*simp*)

lemma *monadic-WHILEIT-comb*[*sepref-monadify-comb*]:
PR-CONST (*monadic-WHILEIT* I)\$$b$\$$f$\$$s$ $\equiv$
  Refine-Basic.bind\$(*EVAL*\$$s$)\$($\lambda_2 s.$
    SP (*PR-CONST* (*monadic-WHILEIT* I))\$$b$\$$f$\$$s$
)
by (*simp*)

definition [*simp*]: *op-ASSERT-bind* $I\ m \equiv$ *Refine-Basic.bind* (*ASSERT* I) (λ-. m)
lemma *pat-ASSERT-bind*[*def-pat-rules*]:
Refine-Basic.bind\$(*ASSERT*\$$I$)\$(λ_2-. m) $\equiv$ *UNPROTECT* (*op-ASSERT-bind* I)\$$m$
by *simp*

term *PR-CONST* (*op-ASSERT-bind* I)
lemma *id-op-ASSERT-bind*[*id-rules*]:
PR-CONST (*op-ASSERT-bind* I) $::_i$ *TYPE*('a nres ⇒ 'a nres)
by *simp*

lemma *arity-ASSERT-bind*[*sepref-monadify-arity*]:
PR-CONST (*op-ASSERT-bind I*) $\equiv$ $\lambda_2 m$. *SP* (*PR-CONST* (*op-ASSERT-bind I*))$*m*
apply (*rule eq-reflection*)
by *auto*

lemma *hn-ASSERT-bind*[*sepref-comb-rules*]:
assumes $I \Longrightarrow$ *hn-refine* Γ *c* Γ' *R m*
shows *hn-refine* Γ *c* Γ' *R* (*PR-CONST* (*op-ASSERT-bind I*)$*m*)
using *assms*
apply (*cases I*)
apply *auto*
done

definition [*simp*]: *op-ASSUME-bind I m* $\equiv$ *Refine-Basic.bind* (*ASSUME I*) (λ-. *m*)
lemma *pat-ASSUME-bind*[*def-pat-rules*]:
Refine-Basic.bind$(*ASSUME*$*I*)$($\lambda_2$-. *m*) $\equiv$ *UNPROTECT* (*op-ASSUME-bind I*)$*m*
by *simp*

lemma *id-op-ASSUME-bind*[*id-rules*]:
PR-CONST (*op-ASSUME-bind I*) $::_i$ *TYPE*('*a nres* $\Rightarrow$ '*a nres*)
by *simp*

lemma *arity-ASSUME-bind*[*sepref-monadify-arity*]:
PR-CONST (*op-ASSUME-bind I*) $\equiv$ $\lambda_2 m$. *SP* (*PR-CONST* (*op-ASSUME-bind I*))$*m*
apply (*rule eq-reflection*)
by *auto*

lemma *hn-ASSUME-bind*[*sepref-comb-rules*]:
assumes *vassn-tag* $\Gamma \Longrightarrow I$
assumes $I \Longrightarrow$ *hn-refine* Γ *c* Γ' *R m*
shows *hn-refine* Γ *c* Γ' *R* (*PR-CONST* (*op-ASSUME-bind I*)$*m*)
apply (*rule hn-refine-preI*)
using *assms*
apply (*cases I*)
apply (*auto simp*: *vassn-tag-def*)
done

1.7.1 Import of Parametricity Theorems

lemma *pure-hn-refineI*:
assumes $Q \longrightarrow (c,a) \in R$
shows *hn-refine* ($\uparrow Q$) (*return c*) ($\uparrow Q$) (*pure R*) (*RETURN a*)
unfolding *hn-refine-def* **using** *assms*
by (*sep-auto simp*: *pure-def*)

lemma *pure-hn-refineI-no-asm*:
assumes $(c,a)\in R$
shows *hn-refine emp* (*return c*) *emp* (*pure R*) (*RETURN a*)
unfolding *hn-refine-def* **using** *assms*
by (*sep-auto simp*: *pure-def*)

lemma *import-param-0*:
$(P \Longrightarrow Q) \equiv Trueprop\ (PROTECT\ P \longrightarrow Q)$
apply (*rule*, *simp*+)+
done

lemma *import-param-1*:
$(P \Longrightarrow Q) \equiv Trueprop\ (P \longrightarrow Q)$
$(P \longrightarrow Q \longrightarrow R) \longleftrightarrow (P \wedge Q \longrightarrow R)$
$PROTECT\ (P \wedge Q) \equiv PROTECT\ P \wedge PROTECT\ Q$
$(P \wedge Q) \wedge R \equiv P \wedge Q \wedge R$
$(a,c)\in Rel \wedge PROTECT\ P \longleftrightarrow PROTECT\ P \wedge (a,c)\in Rel$
apply (*rule*, *simp*+)+
done

lemma *import-param-2*:
$Trueprop\ (PROTECT\ P \wedge Q \longrightarrow R) \equiv (P \Longrightarrow Q \longrightarrow R)$
apply (*rule*, *simp*+)+
done

lemma *import-param-3*:
$\uparrow(P \wedge Q) = \uparrow P * \uparrow Q$
$\uparrow((c,a)\in R) = hn\text{-}val\ R\ a\ c$
by (*simp-all add*: *hn-ctxt-def pure-def*)

named-theorems-rev *sepref-import-rewrite* ‹*Rewrite rules on importing parametricity theorems*›

lemma *to-import-frefD*:
assumes $(f,g)\in fref\ P\ R\ S$
shows $[\![PROTECT\ (P\ y); (x,y)\in R]\!] \Longrightarrow (f\ x,\ g\ y)\in S$
using *assms*
unfolding *fref-def*
by *auto*

lemma *add-PR-CONST*: $(c,a)\in R \Longrightarrow (c,PR\text{-}CONST\ a)\in R$ **by** *simp*

ML ‹
structure Sepref-Import-Param = struct

(∗ *TODO*: *Almost clone of Sepref-Rules.to-foparam*∗)
fun to-import-fo ctxt thm = let
val unf-thms = @{*thms*

```
    split-tupled-all prod-rel-simp uncurry-apply cnv-conj-to-meta Product-Type.split}
  in
    case Thm.concl-of thm of
      @{mpat Trueprop ((-,-) ∈ fref - - -)} =>
        (@{thm to-import-frefD} OF [thm])
        |> Thm.forall-intr-vars
        |> Local-Defs.unfold0 ctxt unf-thms
        |> Variable.gen-all ctxt
    | @{mpat Trueprop ((-,-) ∈ -)} =>
        Parametricity.fo-rule thm
    | - => raise THM(Expected parametricity or fref theorem,~1,[thm])
  end

  fun add-PR-CONST thm = case Thm.concl-of thm of
      @{mpat Trueprop ((-,-) ∈ fref - - -)} => thm (* TODO: Hack. Need clean
interfaces for fref and param rules. Also add PR-CONST to fref rules! *)
  | @{mpat Trueprop ((-,PR-CONST -) ∈ -)} => thm
  | @{mpat Trueprop ((-,?a) ∈ -)} => if is-Const a orelse is-Free a orelse is-Var a
then
      thm
    else
      thm RS @{thm add-PR-CONST}
  | - => thm


  fun import ctxt thm = let
    open Sepref-Basic
    val thm = thm
      |> Conv.fconv-rule Thm.eta-conversion
      |> add-PR-CONST
      |> Local-Defs.unfold0 ctxt @{thms import-param-0}
      |> Local-Defs.unfold0 ctxt @{thms imp-to-meta}
      |> to-import-fo ctxt
      |> Local-Defs.unfold0 ctxt @{thms import-param-1}
      |> Local-Defs.unfold0 ctxt @{thms import-param-2}

    val thm = case Thm.concl-of thm of
      @{mpat Trueprop (-⟶-)} => thm RS @{thm pure-hn-refineI}
    | - => thm RS @{thm pure-hn-refineI-no-asm}

    val thm = Local-Defs.unfold0 ctxt @{thms import-param-3} thm
     |> Conv.fconv-rule (hn-refine-concl-conv-a (K (Id-Op.protect-conv ctxt)) ctxt)

   val thm = Local-Defs.unfold0 ctxt (Named-Theorems-Rev.get ctxt @{named-theorems-rev
sepref-import-rewrite}) thm
    val thm = Sepref-Rules.add-pure-constraints-rule ctxt thm
  in
    thm
  end
```

```
  val import-attr = Scan.succeed (Thm.mixed-attribute (fn (context,thm) =>
    let
      val thm = import (Context.proof-of context) thm
      val context = Sepref-Translate.sepref-fr-rules.add-thm thm context
    in (context,thm) end
  ))

  val import-attr-rl = Scan.succeed (Thm.rule-attribute [] (fn context =>
   import (Context.proof-of context) #> Sepref-Rules.ensure-hfref (Context.proof-of
context)
  ))

  val setup = I
    #> Attrib.setup @{binding sepref-import-param} import-attr
        Sepref: Import parametricity rule
    #> Attrib.setup @{binding sepref-param} import-attr-rl
        Sepref: Transform parametricity rule to sepref rule
    #> Attrib.setup @{binding sepref-dbg-import-rl-only}
        (Scan.succeed (Thm.rule-attribute [] (import o Context.proof-of)))
        Sepref: Parametricity to hnr−rule, no conversion to hfref

end
›
```

setup *Sepref-Import-Param.setup*

1.7.2 Purity

definition *import-rel1* $R \equiv \lambda A\ c\ ci.\ \uparrow(\textit{is-pure}\ A \wedge (ci,c)\in\langle\textit{the-pure}\ A\rangle R)$
definition *import-rel2* $R \equiv \lambda A\ B\ c\ ci.\ \uparrow(\textit{is-pure}\ A \wedge \textit{is-pure}\ B \wedge (ci,c)\in\langle\textit{the-pure}\ A,\ \textit{the-pure}\ B\rangle R)$

lemma *import-rel1-pure-conv*: *import-rel1* R (*pure* A) = *pure* ($\langle A\rangle R$)
  unfolding *import-rel1-def*
  apply *simp*
  apply (*simp add*: *pure-def*)
  done

lemma *import-rel2-pure-conv*: *import-rel2* R (*pure* A) (*pure* B) = *pure* ($\langle A,B\rangle R$)
  unfolding *import-rel2-def*
  apply *simp*
  apply (*simp add*: *pure-def*)
  done

lemma *precise-pure*[*constraint-rules*]: *single-valued* $R \Longrightarrow$ *precise* (*pure* R)
  unfolding *precise-def pure-def*
  by (*auto dest*: *single-valuedD*)

lemma *precise-pure-iff-sv*: *precise* (*pure* *R*) $\longleftrightarrow$ *single-valued* *R*
  apply (*auto simp*: *precise-pure*)
  using *preciseD*[**where** *R*=*pure R* **and** *F*=*emp* **and** F'=*emp*]
  by (*sep-auto simp*: *mod-and-dist intro*: *single-valuedI*)

lemma *pure-precise-iff-sv*: $[\![$*is-pure R*$]\!]$
  $\Longrightarrow$ *precise R* $\longleftrightarrow$ *single-valued* (*the-pure R*)
  by (*auto simp*: *is-pure-conv precise-pure-iff-sv*)

lemmas [*safe-constraint-rules*] = *single-valued-Id br-sv*

end

1.8 Sepref-Definition Command

theory *Sepref-Definition*
imports *Sepref-Rules Lib/Pf-Mono-Prover Lib/Term-Synth*
keywords *sepref-definition* :: *thy-goal*
   and *sepref-thm* :: *thy-goal*
begin

1.8.1 Setup of Extraction-Tools

  declare [[*cd-patterns hn-refine - ?f - - -*]]

  lemma *heap-fixp-codegen*:
   assumes *DEF*: $f \equiv$ *heap.fixp-fun cB*
   assumes *M*: ($\bigwedge x$. *mono-Heap* (λf. *cB f x*))
   shows *f x* = *cB f x*
   unfolding *DEF*
   apply (*rule fun-cong*[*of - - x*])
   apply (*rule heap.mono-body-fixp*)
   apply *fact*
   done

  ML ‹
   structure Sepref-Extraction = struct
    val heap-extraction: *Refine-Automation.extraction* = {
      pattern = *Logic.varify-global* @{*term heap.fixp-fun x*},
      gen-thm = @{*thm heap-fixp-codegen*},
      gen-tac = (*fn ctxt* =>
       Pf-Mono-Prover.mono-tac ctxt
)
     }

val setup = *I*
(*∗#> Refine-Automation.add-extraction trivial triv-extraction∗*)
#> Refine-Automation.add-extraction heap heap-extraction

end
›

setup *Sepref-Extraction.setup*

1.8.2 Synthesis setup for sepref-definition goals

consts *UNSPEC*::*'a*

abbreviation *hfunspec*
:: (*'a* ⇒ *'b* ⇒ *assn*) ⇒ (*'a* ⇒ *'b* ⇒ *assn*)×(*'a* ⇒ *'b* ⇒ *assn*)
(‹(-$^?$)› [*1000*] *999*)
where $R^?$ ≡ *hf-pres R UNSPEC*

definition *SYNTH* :: (*'a* ⇒ *'r nres*) ⇒ ((*'ai* ⇒*'ri Heap*) × (*'a* ⇒ *'r nres*)) *set* ⇒ *bool*
where *SYNTH f R* ≡ *True*

definition [*simp*]: *CP-UNCURRY* - - ≡ *True*
definition [*simp*]: *INTRO-KD* - - ≡ *True*
definition [*simp*]: *SPEC-RES-ASSN* - - ≡ *True*

lemma [*synth-rules*]: *CP-UNCURRY f g* **by** *simp*
lemma [*synth-rules*]: *CP-UNCURRY* (*uncurry0 f*) (*uncurry0 g*) **by** *simp*
lemma [*synth-rules*]: *CP-UNCURRY f g* ⟹ *CP-UNCURRY* (*uncurry f*) (*uncurry g*) **by** *simp*

lemma [*synth-rules*]: ⟦*INTRO-KD R1 R1'*; *INTRO-KD R2 R2'*⟧ ⟹ *INTRO-KD* ($R1 *_a R2$) ($R1' *_a R2'$) **by** *simp*
lemma [*synth-rules*]: *INTRO-KD* ($R^?$) (*hf-pres R k*) **by** *simp*
lemma [*synth-rules*]: *INTRO-KD* (R^k) (R^k) **by** *simp*
lemma [*synth-rules*]: *INTRO-KD* (R^d) (R^d) **by** *simp*

lemma [*synth-rules*]: *SPEC-RES-ASSN R R* **by** *simp*
lemma [*synth-rules*]: *SPEC-RES-ASSN UNSPEC R* **by** *simp*

lemma *synth-hnrI*:
⟦*CP-UNCURRY fi f*; *INTRO-KD R R'*; *SPEC-RES-ASSN S S'*⟧ ⟹ *SYNTH-TERM* (*SYNTH f* ($[P]_a$ *R*→*S*)) ((*fi*,*SDUMMY*)∈*SDUMMY*,(*fi*,*f*)∈($[P]_a$ *R'*→*S'*))
by (*simp add*: *SYNTH-def*)

term *starts-with*

ML ‹
structure Sepref-Definition = *struct*

```
fun make-hnr-goal t ctxt = let
  val ctxt = Variable.declare-term t ctxt
  val (pat,goal) = case Term-Synth.synth-term @{thms synth-hnrI} ctxt t of
    @{mpat (?pat,?goal)} => (pat,goal) | t => raise TERM(Synthesized term
does not match,[t])
  val pat = Thm.cterm-of ctxt pat |> Refine-Automation.prepare-cd-pattern ctxt
  val goal = HOLogic.mk-Trueprop goal
in
  ((pat,goal),ctxt)
end

val cfg-prep-code = Attrib.setup-config-bool @{binding sepref-definition-prep-code}
(K true)

local
  open Refine-Util
  val flags = parse-bool-config' prep-code cfg-prep-code
  val parse-flags = parse-paren-list' flags

in
    val sd-parser = parse-flags -- Parse.binding -- Parse.opt-attribs --|
@{keyword is}
    -- Parse.term --| @{keyword ::} -- Parse.term
end

fun mk-synth-term ctxt t-raw r-raw = let
    val t = Syntax.parse-term ctxt t-raw
    val r = Syntax.parse-term ctxt r-raw
    val t = Const (@{const-name SYNTH},dummyT)$t$r
  in
    Syntax.check-term ctxt t
  end


fun sd-cmd ((((flags,name),attribs),t-raw),r-raw) lthy = let
  local
    val ctxt = Refine-Util.apply-configs flags lthy
  in
    val flag-prep-code = Config.get ctxt cfg-prep-code
  end

  val t = mk-synth-term lthy t-raw r-raw

  val ((pat,goal),ctxt) = make-hnr-goal t lthy

  fun
    after-qed [[thm]] ctxt = let
        val thm = singleton (Variable.export ctxt lthy) thm
```

```
        val (-,lthy)
          = Local-Theory.note
        ((Refine-Automation.mk-qualified (Binding.name-of name) refine-raw,[]),[thm])

            lthy;

         val ((dthm,rthm),lthy) = Refine-Automation.define-concrete-fun NONE
name attribs [] thm [pat] lthy

        val lthy = lthy
          |> flag-prep-code ? Refine-Automation.extract-recursion-eqs
            [Sepref-Extraction.heap-extraction] (Binding.name-of name) dthm

        val - = Thm.pretty-thm lthy dthm |> Pretty.string-of |> writeln
        val - = Thm.pretty-thm lthy rthm |> Pretty.string-of |> writeln
      in
        lthy
      end
    | after-qed thmss - = raise THM (After−qed: Wrong thmss structure,~1,flat
thmss)

  in
    Proof.theorem NONE after-qed [[ (goal,[]) ]] ctxt
  end



  val - = Outer-Syntax.local-theory-to-proof @{command-keyword sepref-definition}
    Synthesis of imperative program
    (sd-parser >> sd-cmd)

  val st-parser = Parse.binding −−| @{keyword is} −− Parse.term −−| @{keyword
::} −− Parse.term

  fun st-cmd ((name,t-raw),r-raw) lthy = let
    val t = mk-synth-term lthy t-raw r-raw
    val ((-,goal),ctxt) = make-hnr-goal t lthy

    fun
      after-qed [[thm]] ctxt = let
        val thm = singleton (Variable.export ctxt lthy) thm

        val - = Thm.pretty-thm lthy thm |> Pretty.string-of |> tracing

        val (-,lthy)
          = Local-Theory.note
        ((Refine-Automation.mk-qualified (Binding.name-of name) refine-raw,[]),[thm])

            lthy;
```

in
lthy
end
| *after-qed thmss - = raise THM (After−qed: Wrong thmss structure,~1,flat thmss)*

in
Proof.theorem NONE after-qed [[(goal,[])]] ctxt
end

val - = Outer-Syntax.local-theory-to-proof @{command-keyword sepref-thm}
Synthesis of imperative program: Only generate raw refinement theorem
(st-parser >> st-cmd)

end
›

end

1.9 Utilities for Interface Specifications and Implementations

theory *Sepref-Intf-Util*
imports *Sepref-Rules Sepref-Translate Lib/Term-Synth Sepref-Combinator-Setup Lib/Concl-Pres-Clarification*
keywords *sepref-decl-op* :: *thy-goal*
and *sepref-decl-impl* :: *thy-goal*
begin

1.9.1 Relation Interface Binding

definition *INTF-OF-REL* :: *('a×'b) set ⇒ 'c itself ⇒ bool*
where [*simp*]: *INTF-OF-REL R I ≡ True*

lemma *intf-of-relI*: *INTF-OF-REL (R::(-×'a) set) TYPE('a)* **by** *simp*
declare *intf-of-relI*[*synth-rules*] — Declare as fallback rule

lemma [*synth-rules*]:
INTF-OF-REL unit-rel TYPE(unit)
INTF-OF-REL nat-rel TYPE(nat)
INTF-OF-REL int-rel TYPE(int)
INTF-OF-REL bool-rel TYPE(bool)

INTF-OF-REL R TYPE('a) ⟹ INTF-OF-REL (⟨R⟩option-rel) TYPE('a option)
INTF-OF-REL R TYPE('a) ⟹ INTF-OF-REL (⟨R⟩list-rel) TYPE('a list)
INTF-OF-REL R TYPE('a) ⟹ INTF-OF-REL (⟨R⟩nres-rel) TYPE('a nres)

$\llbracket$*INTF-OF-REL R TYPE('a)*; *INTF-OF-REL S TYPE('b)*$\rrbracket \Longrightarrow$ *INTF-OF-REL* $(R \times_r S)$ *TYPE('a×'b)*
$\llbracket$*INTF-OF-REL R TYPE('a)*; *INTF-OF-REL S TYPE('b)*$\rrbracket \Longrightarrow$ *INTF-OF-REL* $(\langle R,S \rangle$*sum-rel*$)$ *TYPE('a+'b)*
$\llbracket$*INTF-OF-REL R TYPE('a)*; *INTF-OF-REL S TYPE('b)*$\rrbracket \Longrightarrow$ *INTF-OF-REL* $(R \rightarrow S)$ *TYPE('a⇒'b)*
by *simp-all*

lemma *synth-intf-of-relI*: *INTF-OF-REL R I* $\Longrightarrow$ *SYNTH-TERM R I* **by** *simp*

1.9.2 Operations with Precondition

definition *mop* :: *('a⇒bool)* ⇒ *('a⇒'b nres)* ⇒ *'a* ⇒ *'b nres*
— Package operation with precondition
where [*simp*]: *mop P f* ≡ λ*x*. *ASSERT* (*P x*) ≫ *f x*

lemma *param-op-mop-iff*:
assumes $(Q,P) \in R \rightarrow$ *bool-rel*
shows
$(f, g) \in [P]_f\ R \rightarrow \langle S \rangle$*nres-rel*
$\longleftrightarrow$
(*mop Q f*, *mop P g*) $\in R \rightarrow_f \langle S \rangle$*nres-rel*

using *assms*
by (*auto*
simp: *mop-def fref-def pw-nres-rel-iff refine-pw-simps*
dest: *fun-relD*)

lemma *param-mopI*:
assumes $(f,g) \in [P]_f\ R \rightarrow \langle S \rangle$*nres-rel*
assumes $(Q,P) \in R \rightarrow$ *bool-rel*
shows (*mop Q f*, *mop P g*) $\in R \rightarrow_f \langle S \rangle$*nres-rel*
using *assms* **by** (*simp add*: *param-op-mop-iff*)

lemma *mop-spec-rl*: *P x* $\Longrightarrow$ *mop P f x* ≤ *f x* **by** *simp*

lemma *mop-spec-rl-from-def*:
assumes *f* ≡ *mop P g*
assumes *P x*
assumes *g x* ≤ *z*
shows *f x* ≤ *z*
using *assms mop-spec-rl* **by** *simp*

lemma *mop-leof-rl-from-def*:
assumes *f* ≡ *mop P g*
assumes *P x* $\Longrightarrow$ *g x* $\leq_n$ *z*
shows *f x* $\leq_n$ *z*
using *assms*
by (*simp add*: *pw-leof-iff refine-pw-simps*)

lemma *assert-true-bind-conv*: $ASSERT\ True \gg m = m$ **by** *simp*

lemmas *mop-alt-unfolds* = *curry-def curry0-def mop-def uncurry-apply uncurry0-apply o-apply assert-true-bind-conv*

1.9.3 Constraints

lemma *add-is-pure-constraint*: $[\![PROP\ P;\ CONSTRAINT\ \textit{is-pure}\ A]\!] \Longrightarrow PROP\ P$ **.**
lemma *sepref-relpropI*: $P\ R = CONSTRAINT\ P\ R$ **by** *simp*

Purity

lemmas [*constraint-simps*] = *the-pure-pure*
definition [*constraint-abbrevs*]: $IS\text{-}PURE\ P\ R \equiv \textit{is-pure}\ R \wedge P\ (\textit{the-pure}\ R)$
lemma *IS-PURE-pureI*:
$P\ R \Longrightarrow IS\text{-}PURE\ P\ (\textit{pure}\ R)$
by (*auto simp*: *IS-PURE-def*)

lemma [*fcomp-norm-simps*]: $CONSTRAINT\ (IS\text{-}PURE\ \Phi)\ P \Longrightarrow \textit{pure}\ (\textit{the-pure}\ P) = P$
by (*simp add*: *IS-PURE-def*)

lemma [*fcomp-norm-simps*]: $CONSTRAINT\ (IS\text{-}PURE\ P)\ A \Longrightarrow P\ (\textit{the-pure}\ A)$
by (*auto simp*: *IS-PURE-def*)

lemma *handle-purity1*:
$CONSTRAINT\ (IS\text{-}PURE\ \Phi)\ A \Longrightarrow CONSTRAINT\ \Phi\ (\textit{the-pure}\ A)$
by (*auto simp*: *IS-PURE-def*)

lemma *handle-purity2*:
$CONSTRAINT\ (IS\text{-}PURE\ \Phi)\ A \Longrightarrow CONSTRAINT\ \textit{is-pure}\ A$
by (*auto simp*: *IS-PURE-def*)

1.9.4 Composition

Preconditions

definition [*simp*]: $\textit{tcomp-pre}\ Q\ T\ P \equiv \lambda a.\ Q\ a \wedge (\forall a'.\ (a', a) \in T \longrightarrow P\ a')$
definition $\textit{and-pre}\ P1\ P2 \equiv \lambda x.\ P1\ x \wedge P2\ x$
definition $\textit{imp-pre}\ P1\ P2 \equiv \lambda x.\ P1\ x \longrightarrow P2\ x$

lemma *and-pre-beta*: $PP \longrightarrow P\ x \wedge Q\ x \Longrightarrow PP \longrightarrow \textit{and-pre}\ P\ Q\ x$ **by** (*auto simp*: *and-pre-def*)
lemma *imp-pre-beta*: $PP \longrightarrow P\ x \longrightarrow Q\ x \Longrightarrow PP \longrightarrow \textit{imp-pre}\ P\ Q\ x$ **by** (*auto simp*: *imp-pre-def*)

definition *IMP-PRE P1 P2* ≡ ∀ *x*. *P1 x* ⟶ *P2 x*
lemma *IMP-PRED*: *IMP-PRE P1 P2* ⟹ *P1 x* ⟹ *P2 x* **unfolding** *IMP-PRE-def* **by** *auto*
lemma *IMP-PRE-refl*: *IMP-PRE P P* **unfolding** *IMP-PRE-def* **by** *auto*

definition *IMP-PRE-CUSTOM* ≡ *IMP-PRE*
lemma *IMP-PRE-CUSTOMD*: *IMP-PRE-CUSTOM P1 P2* ⟹ *IMP-PRE P1 P2* **by** (*simp add*: *IMP-PRE-CUSTOM-def*)
lemma *IMP-PRE-CUSTOMI*: ⟦⋀*x*. *P1 x* ⟹ *P2 x*⟧ ⟹ *IMP-PRE-CUSTOM P1 P2*
by (*simp add*: *IMP-PRE-CUSTOM-def IMP-PRE-def*)

lemma *imp-and-triv-pre*: *IMP-PRE P* (*and-pre* (λ-. *True*) *P*)
unfolding *IMP-PRE-def and-pre-def* **by** *auto*

Premises

definition *ALL-LIST A* ≡ (∀ *x*∈*set A*. *x*)
definition *IMP-LIST A B* ≡ *ALL-LIST A* ⟶ *B*

lemma *to-IMP-LISTI*:
P ⟹ *IMP-LIST* [] *P*
by (*auto simp*: *IMP-LIST-def*)

lemma *to-IMP-LIST*: (*P* ⟹ *IMP-LIST Ps Q*) ≡ *Trueprop* (*IMP-LIST* (*P*#*Ps*) *Q*)
by (*auto simp*: *IMP-LIST-def ALL-LIST-def intro*!: *equal-intr-rule*)

lemma *from-IMP-LIST*:
Trueprop (*IMP-LIST As B*) ≡ (*ALL-LIST As* ⟹ *B*)
(*ALL-LIST* [] ⟹ *B*) ≡ *Trueprop B*
(*ALL-LIST* (*A*#*As*) ⟹ *B*) ≡ (*A* ⟹ *ALL-LIST As* ⟹ *B*)
by (*auto simp*: *IMP-LIST-def ALL-LIST-def intro*!: *equal-intr-rule*)

lemma *IMP-LIST-trivial*: *IMP-LIST A B* ⟹ *IMP-LIST A B* **.**

Composition Rules

lemma *hfcomp-tcomp-pre*:
assumes *B*: $(g,h) \in [Q]_f$ *T* → ⟨*U*⟩*nres-rel*
assumes *A*: $(f,g) \in [P]_a$ *RR'* → *S*
shows $(f,h) \in$ [*tcomp-pre Q T P*]$_a$ *hrp-comp RR' T* → *hr-comp S U*
using *hfcomp*[*OF A B*] **by** *simp*

lemma *transform-pre-param*:
assumes *A*: *IMP-LIST Cns* ((*f*, *h*) ∈ [*tcomp-pre Q T P*]$_a$ *hrp-comp RR' T* → *hr-comp S U*)
assumes *P*: *IMP-LIST Cns* ((*P*,*P'*) ∈ *T* → *bool-rel*)

assumes C: *IMP-PRE* PP' (*and-pre* P' Q)
shows *IMP-LIST Cns* $((f,h) \in [PP']_a$ *hrp-comp* RR' $T \rightarrow$ *hr-comp* S $U)$
unfolding *from-IMP-LIST*
apply (*rule hfref-cons*)
apply (*rule A*[*unfolded from-IMP-LIST*])
apply *assumption*
apply (*drule IMP-PRED*[*OF C*])
using *P*[*unfolded from-IMP-LIST*] **unfolding** *and-pre-def*
apply (*auto dest*: *fun-relD*) []
by *simp-all*

lemma *hfref-mop-conv*: $((g,mop\ P\ f) \in [Q]_a\ R \rightarrow S) \longleftrightarrow (g,f) \in [\lambda x.\ P\ x \wedge Q\ x]_a\ R \rightarrow S$
apply (*simp add*: *hfref-to-ASSERT-conv*)
apply (*fo-rule arg-cong fun-cong*)+
by (*auto intro*!: *ext simp*: *pw-eq-iff refine-pw-simps*)

lemma *hfref-op-to-mop*:
assumes R: $(impl,f) \in [Q]_a\ R \rightarrow S$
assumes *DEF*: $mf \equiv mop\ P\ f$
assumes C: *IMP-PRE* PP' (*imp-pre* P Q)
shows $(impl,mf) \in [PP']_a\ R \rightarrow S$
unfolding *DEF hfref-mop-conv*
apply (*rule hfref-cons*[*OF R*])
using C
by (*auto simp*: *IMP-PRE-def imp-pre-def*)

lemma *hfref-mop-to-op*:
assumes R: $(impl,mf) \in [Q]_a\ R \rightarrow S$
assumes *DEF*: $mf \equiv mop\ P\ f$
assumes C: *IMP-PRE* PP' (*and-pre* Q P)
shows $(impl,f) \in [PP']_a\ R \rightarrow S$
using R **unfolding** *DEF hfref-mop-conv*
apply (*rule hfref-cons*)
using C
apply (*auto simp*: *and-pre-def IMP-PRE-def*)
done

Precondition Simplification

lemma *IMP-PRE-eqI*:
assumes $\bigwedge x.\ P\ x \longrightarrow Q\ x$
assumes *CNV* P P'
shows *IMP-PRE* P' Q
using *assms* **by** (*auto simp*: *IMP-PRE-def*)

lemma *simp-and1*:
assumes $Q \Longrightarrow$ *CNV* P P'
assumes $PP \longrightarrow P' \wedge Q$

shows $PP \longrightarrow P \wedge Q$
using *assms* **by** *auto*

lemma *simp-and2*:
assumes $P \Longrightarrow CNV\ Q\ Q'$
assumes $PP \longrightarrow P \wedge Q'$
shows $PP \longrightarrow P \wedge Q$
using *assms* **by** *auto*

lemma *triv-and1*: $Q \longrightarrow True \wedge Q$ **by** *blast*

lemma *simp-imp*:
assumes $P \Longrightarrow CNV\ Q\ Q'$
assumes $PP \longrightarrow Q'$
shows $PP \longrightarrow (P \longrightarrow Q)$
using *assms* **by** *auto*

lemma *CNV-split*:
assumes $CNV\ A\ A'$
assumes $CNV\ B\ B'$
shows $CNV\ (A \wedge B)\ (A' \wedge B')$
using *assms* **by** *auto*

lemma *CNV-prove*:
assumes P
shows $CNV\ P\ True$
using *assms* **by** *auto*

lemma *simp-pre-final-simp*:
assumes $CNV\ P\ P'$
shows $P' \longrightarrow P$
using *assms* **by** *auto*

lemma *auto-weaken-pre-uncurry-step'*:
assumes $PROTECT\ f\ a \equiv f'$
shows $PROTECT\ (uncurry\ f)\ (a,b) \equiv f'\ b$
using *assms*
by (*auto simp*: *curry-def dest*!: *meta-eq-to-obj-eq intro*!: *eq-reflection*)

1.9.5 Protected Constants

lemma *add-PR-CONST-to-def*: $x \equiv y \Longrightarrow PR\text{-}CONST\ x \equiv y$ **by** *simp*

1.9.6 Rule Collections

named-theorems-rev *sepref-mop-def-thms* ‹*Sepref*: *mop* − *definition theorems*›

named-theorems-rev *sepref-fref-thms* ‹*Sepref*: *fref*−*theorems*›

named-theorems *sepref-relprops-transform* ‹*Sepref*: *Simp−rules to transform relator properties*›
named-theorems *sepref-relprops* ‹*Sepref*: *Simp−rules to add CONSTRAINT−tags to relator properties*›
named-theorems *sepref-relprops-simps* ‹*Sepref*: *Simp−rules to simplify relator properties*›

Default Setup

1.9.7 ML-Level Declarations

```
ML ‹
  signature SEPREF-INTF-UTIL = sig
    (* Miscellaneous*)
    val list-filtered-subterms: (term −> 'a option) −> term −> 'a list


    (* Interface types for relations *)
    val get-intf-of-rel: Proof.context −> term −> typ

    (* Constraints *)
    (* Convert relations to pure assertions *)
    val to-assns-rl: bool −> Proof.context −> thm −> thm
    (* Recognize, summarize and simplify CONSTRAINT − premises *)
    val cleanup-constraints: Proof.context −> thm −> thm

    (* Preconditions *)
    (* Simplify precondition. Goal must be in IMP-PRE or IMP-PRE-CUSTOM
form. *)
    val simp-precond-tac: Proof.context −> tactic'


    (* Configuration options *)
    val cfg-def: bool Config.T        (* decl-op: Define constant *)
    val cfg-ismop: bool Config.T      (* decl-op: Specified term is mop *)
    val cfg-mop: bool Config.T        (* decl-op, decl-impl: Derive mop *)
   val cfg-rawgoals: bool Config.T  (* decl-op, decl-impl: Do not pre−process/solve
goals *)


    (* TODO: Make do-cmd usable from ML−level! *)

  end

  structure Sepref-Intf-Util: SEPREF-INTF-UTIL = struct

    val cfg-debug =
      Attrib.setup-config-bool @{binding sepref-debug-intf-util} (K false)

    val dbg-trace = Sepref-Debugging.dbg-trace-msg cfg-debug
```

```
val dbg-msg-tac = Sepref-Debugging.dbg-msg-tac cfg-debug


fun list-filtered-subterms f t = let
  fun r t = case f t of
    SOME a => [a]
  | NONE => (
      case t of
        t1$t2 => r t1 @ r t2
      | Abs (-,-,t) => r t
      | - => []
    )
in
  r t
end

fun get-intf-of-rel ctxt R =
  Term-Synth.synth-term @{thms synth-intf-of-relI} ctxt R
    |> fastype-of
    |> Refine-Util.dest-itselfT

local
  fun add-is-pure-constraint ctxt v thm = let
    val v = Thm.cterm-of ctxt v
  val rl = Drule.infer-instantiate′ ctxt [NONE, SOME v] @{thm add-is-pure-constraint}
  in
    thm RS rl
  end
in
  fun to-assns-rl add-pure-constr ctxt thm = let
    val orig-ctxt = ctxt

    val (thm,ctxt) = yield-singleton (apfst snd oo Variable.importT) thm ctxt

   val (R,S) = case Thm.concl-of thm of @{mpat Trueprop (-∈fref - ?R ?S)}
=> (R,S)
      | - => raise THM(to-assns-rl: expected fref−thm,~1,[thm])

    fun mk-cn-subst (fname,(iname,C,A)) =
      let
        val T′ = A −−> C −−> @{typ assn}
        val v′ = Free (fname,T′)
        val ct′ = @{mk-term the-pure ?v′} |> Thm.cterm-of ctxt
      in
        (v′,(iname,ct′))
      end

      fun relation-flt (name,Type (@{type-name set},[Type (@{type-name
prod},[C,A])])) = SOME (name,C,A)
```

```
    | relation-flt - = NONE


  val vars = []
    |> Term.add-vars R
    |> Term.add-vars S
    |> map-filter (relation-flt)
  val (names,ctxt) = Variable.variant-fixes (map (#1 #> fst) vars) ctxt

  val cn-substs = map mk-cn-subst (names ~~ vars)


  val thm = Drule.infer-instantiate ctxt (map snd cn-substs) thm

    val thm = thm |> add-pure-constr ? fold (fn (v,-) => fn thm =>
add-is-pure-constraint ctxt v thm) cn-substs

  val thm = singleton (Variable.export ctxt orig-ctxt) thm
in
  thm
end

fun cleanup-constraints ctxt thm = let
  val orig-ctxt = ctxt

   val (thm, ctxt) = yield-singleton (apfst snd oo Variable.import true) thm
ctxt

val xform-thms = Named-Theorems.get ctxt @{named-theorems sepref-relprops-transform}
val rprops-thms = Named-Theorems.get ctxt @{named-theorems sepref-relprops}
val simp-thms = Named-Theorems.get ctxt @{named-theorems sepref-relprops-simps}

  fun simp thms = Conv.fconv-rule (
          Simplifier.asm-full-rewrite
            (put-simpset HOL-basic-ss ctxt addsimps thms))

  (* Check for pure (the-pure R) − patterns *)

  local
    val (-,R,S) = case Thm.concl-of thm of
      @{mpat Trueprop (-∈hfref ?P ?R ?S)} => (P,R,S)
    | @{mpat Trueprop (-∈fref ?P ?R ?S)} => (P,R,S)
| - => raise THM(cleanup-constraints: Expected hfref or fref−theorem,~1,[thm])



    fun flt-pat @{mpat pure (the-pure ?A)} = SOME A | flt-pat - = NONE

    val purify-terms =
```

```
        (list-filtered-subterms flt-pat R @ list-filtered-subterms flt-pat S)
        |> distinct op aconv

      val thm = fold (add-is-pure-constraint ctxt) purify-terms thm
    in
      val thm = thm
    end

    val thm = thm
      |> Local-Defs.unfold0 ctxt xform-thms
      |> Local-Defs.unfold0 ctxt rprops-thms

    val insts = map (fn
          @{mpat Trueprop (CONSTRAINT - (the-pure -))} => @{thm han-
dle-purity1}
      | - => asm-rl
      ) (Thm.prems-of thm)

    val thm = (thm OF insts)
      |> Conv.fconv-rule Thm.eta-conversion
      |> simp @{thms handle-purity2}
      |> simp simp-thms

    val thm = singleton (Variable.export ctxt orig-ctxt) thm

  in
    thm
  end
end

fun simp-precond-tac ctxt = let
    fun simp-only thms = asm-full-simp-tac (put-simpset HOL-basic-ss ctxt
addsimps thms)
  val rtac = resolve-tac ctxt

  val cnv-ss = ctxt delsimps @{thms CNV-def}

  (*val uncurry-tac = SELECT-GOAL (ALLGOALS (DETERM o SOLVED′
(
    REPEAT′ (rtac @{thms auto-weaken-pre-uncurry-step′})
    THEN′ rtac @{thms auto-weaken-pre-uncurry-finish}
  )))*)

    val prove-cnv-tac = SOLVED′ (rtac @{thms CNV-prove} THEN′ SE-
LECT-GOAL (auto-tac ctxt))

  val do-cnv-tac =
    (cp-clarsimp-tac cnv-ss) THEN-ALL-NEW
    (TRY o REPEAT-ALL-NEW (match-tac ctxt @{thms CNV-split}))
```

```
    THEN-ALL-NEW (prove-cnv-tac ORELSE' rtac @{thms CNV-I})

  val final-simp-tac =
    rtac @{thms simp-pre-final-simp}
    THEN' cp-clarsimp-tac cnv-ss
    THEN' dbg-msg-tac (Sepref-Debugging.msg-subgoal final-simp-tac: Before
CNV-I) ctxt
    THEN' rtac @{thms CNV-I}
    THEN' dbg-msg-tac (Sepref-Debugging.msg-text Final−Simp done) ctxt

  (*val curry-tac = let open Conv in
    CONVERSION (Refine-Util.HOL-concl-conv (fn ctxt => arg1-conv (
    top-conv ( fn - => try-conv (rewr-conv @{thm uncurry-def})) ctxt)) ctxt)
    THEN' REPEAT' (EqSubst.eqsubst-tac ctxt [1] @{thms case-prod-eta})
    THEN' rtac @{thms CNV-I}
    end*)

  val simp-tupled-pre-tac =
      SELECT-GOAL (Local-Defs.unfold0-tac ctxt @{thms prod-casesK un-
curry0-hfref-post})
    THEN' REPEAT' (EqSubst.eqsubst-tac ctxt [1] @{thms case-prod-eta})
    THEN' rtac @{thms CNV-I}

  val unfold-and-tac = rtac @{thms and-pre-beta} THEN-ALL-NEW simp-only
@{thms split}

  val simp-and1-tac =
    rtac @{thms simp-and1} THEN' do-cnv-tac

  val simp-and2-tac =
    rtac @{thms simp-and2} THEN' do-cnv-tac

  val and-plan-tac =
    simp-and1-tac
    THEN' dbg-msg-tac (Sepref-Debugging.msg-subgoal State after and1) ctxt
    THEN' (
      rtac @{thms triv-and1}
      ORELSE'
      dbg-msg-tac (Sepref-Debugging.msg-subgoal Invoking and2 on) ctxt
      THEN' simp-and2-tac
          THEN' dbg-msg-tac (Sepref-Debugging.msg-subgoal State before fi-
nal-simp-tac) ctxt
      THEN' final-simp-tac
    )

  val unfold-imp-tac = rtac @{thms imp-pre-beta} THEN-ALL-NEW simp-only
@{thms split}
  val simp-imp1-tac =
    rtac @{thms simp-imp} THEN' do-cnv-tac
```

```
  val imp-plan-tac = simp-imp1-tac THEN' final-simp-tac

  val imp-pre-tac = APPLY-LIST [
      simp-only @{thms split-tupled-all}
      THEN' Refine-Util.instantiate-tuples-subgoal-tac ctxt
      THEN' CASES' [
        (unfold-and-tac, ALLGOALS and-plan-tac),
        (unfold-imp-tac, ALLGOALS imp-plan-tac)
      ]
    ,
      simp-tupled-pre-tac
    ]

  val imp-pre-custom-tac =
  SELECT-GOAL (Local-Defs.unfold0-tac ctxt @{thms and-pre-def}) THEN'
    TRY o SOLVED' (SELECT-GOAL (auto-tac ctxt))

in
  CASES' [
    (rtac @{thms IMP-PRE-eqI}, imp-pre-tac 1),
    (rtac @{thms IMP-PRE-CUSTOMI}, ALLGOALS imp-pre-custom-tac)
  ]
end
```

```
local
  fun inf-bn-aux name =
    case String.tokens (fn c => c = #.) name of
      [] => NONE
    | [a] => SOME (Binding.name a)
    | (-::a::-) => SOME (Binding.name a)
in
  fun infer-basename (Const (-type-constraint-,-)$t) = infer-basename t
    | infer-basename (Const (name,-)) = inf-bn-aux name
    | infer-basename (Free (name,-)) = inf-bn-aux name
    | infer-basename - = NONE
end
```

```
val cfg-mop = Attrib.setup-config-bool @{binding sepref-register-mop} (K true)
val cfg-ismop = Attrib.setup-config-bool @{binding sepref-register-ismop} (K false)
val cfg-rawgoals = Attrib.setup-config-bool @{binding sepref-register-rawgoals} (K false)
val cfg-transfer = Attrib.setup-config-bool @{binding sepref-decl-impl-transfer} (K true)
val cfg-def = Attrib.setup-config-bool @{binding sepref-register-def} (K true)
```

val cfg-register = Attrib.setup-config-bool @{binding sepref-decl-impl-register} (K true)

```
local
  open Refine-Util
  val flags =
       parse-bool-config' mop cfg-mop
    || parse-bool-config' ismop cfg-ismop
    || parse-bool-config' rawgoals cfg-rawgoals
    || parse-bool-config' def cfg-def
  val parse-flags = parse-paren-list' flags

  val parse-name = Scan.option (Parse.binding --| @{keyword :})
  val parse-relconds = Scan.optional (@{keyword where} |-- Parse.and-list1 (Scan.repeat1 Parse.prop) >> flat) []
in

  val do-parser = parse-flags -- parse-name -- Parse.term --| @{keyword ::} -- Parse.term -- parse-relconds
end


fun do-cmd ((((flags,name),opt-raw), relt-raw),relconds-raw) lthy = let
  local
    val ctxt = Refine-Util.apply-configs flags lthy
  in
    val flag-ismop = Config.get ctxt cfg-ismop
    val flag-mop = Config.get ctxt cfg-mop andalso not flag-ismop
    val flag-rawgoals = Config.get ctxt cfg-rawgoals
    val flag-def = Config.get ctxt cfg-def
  end

  open Sepref-Basic Sepref-Rules

  val relt = Syntax.parse-term lthy relt-raw
  val relconds = map (Syntax.parse-prop lthy) relconds-raw

  val - = dbg-trace lthy Parse relation and relation conditions together
  val relt = Const (@{const-name Pure.term}, dummyT) $ relt
  local
    val l = Syntax.check-props lthy (relt::relconds)
  in
    val (relt, relconds) = (hd l, tl l)
  end
  val relt = Logic.dest-term relt

  val opt-pre = Syntax.parse-term lthy opt-raw
```

val - = dbg-trace lthy Infer basename
val name = case name of
SOME name => name
| NONE => (
case infer-basename opt-pre of
NONE => (error Could not infer basename: You have to specify a basename; Binding.empty)
| SOME name => name
)

fun qname s n = Binding.qualify true (Binding.name-of n) (Binding.name s)
fun def name t-pre attribs lthy = let
val t = Syntax.check-term lthy t-pre
(|> Thm.cterm-of lthy*
|> Drule.mk-term
|> Local-Defs.unfold0 lthy @{thms PR-CONST-def}
|> Drule.dest-term
|> Thm.term-of)*

val lthy = (snd o Local-Theory.begin-nested) lthy
val ((dt,(-,thm)),lthy) = Local-Theory.define ((name,Mixfix.NoSyn),((Thm.def-binding name,@{attributes [code]}@attribs),t)) lthy;
val (lthy, lthy-old) = ‘Local-Theory.end-nested lthy
val phi = Proof-Context.export-morphism lthy-old lthy
val thm = Morphism.thm phi thm
val dt = Morphism.term phi dt

in
((dt,thm),lthy)
end

val - = dbg-trace lthy Analyze Relation
val (pre,args,res) = analyze-rel relt
val specified-pre = is-some pre
val pre = the-default (mk-triv-precond args) pre

val def-thms = @{thms PR-CONST-def}

val - = dbg-trace lthy Define op
val op-name = Binding.prefix-name (if flag-ismop then mop- else op-) name
val (def-thms,opc,lthy) =
if flag-def then let
val ((opc,op-def-thm),lthy) = def op-name opt-pre @{attributes [simp]} lthy
val opc = Refine-Util.dummify-tvars opc
val def-thms = op-def-thm::def-thms

```
      in
        (def-thms,opc,lthy)
      end
    else let
        val - = dbg-trace lthy Refine type of opt-pre to get opc
        val opc = Syntax.check-term lthy opt-pre
        val new-ctxt = Variable.declare-term opc lthy
        val opc = singleton (Variable.export-terms new-ctxt lthy) opc
          |> Refine-Util.dummify-tvars
      in
        (def-thms,opc,lthy)
      end


  (* PR-CONST Heuristics *)
  fun pr-const-heuristics basename c-pre lthy = let
    val - = dbg-trace lthy (PR-CONST heuristics ^@{make-string} c-pre)

    val c = Syntax.check-term lthy c-pre
  in
    case c of
      @{mpat PR-CONST -} => ((c-pre,false),lthy)
    | Const - => ((c-pre,false),lthy)
    | - => let
        val (f,args) = strip-comb c

        val lthy = case f of Const - => let
            val ctxt = Variable.declare-term c lthy
            val lhs = Autoref-Tagging.list-APP (f,args)
            val rhs = @{mk-term UNPROTECT ?c}
            val goal = Logic.mk-equals (lhs,rhs) |> Thm.cterm-of ctxt
            val tac =
              Local-Defs.unfold0-tac ctxt @{thms APP-def UNPROTECT-def}
              THEN ALLGOALS (simp-tac (put-simpset HOL-basic-ss ctxt))
            val thm = Goal.prove-internal ctxt [] goal (K tac)
              |> singleton (Variable.export ctxt lthy)

            val (-,lthy) = Local-Theory.note
          ((Binding.suffix-name -def-pat basename,@{attributes [def-pat-rules]}),[thm])
lthy

            val - = Thm.pretty-thm lthy thm |> Pretty.string-of |> writeln
          in
            lthy
          end
        | - => (
          Pretty.block [
              Pretty.str Complex operation pattern. Added PR-CONST but no
pattern rules:,
```

```
        Pretty.brk 1,Syntax.pretty-term lthy c]
      |> Pretty.string-of |> warning
      ; lthy)

      val c-pre = Const(@{const-name PR-CONST},dummyT)$c-pre
    in
      ((c-pre,true),lthy)
    end
  end

  val ((opc,-),lthy) = pr-const-heuristics op-name opc lthy

  (* Register *)
  val arg-intfs = map (get-intf-of-rel lthy) args
  val res-intf = get-intf-of-rel lthy res


  fun register basename c lthy = let
    val - = dbg-trace lthy Register
    open Sepref-Basic
    val c = Syntax.check-term lthy c

    val ri = case (is-nresT (body-type (fastype-of c)), is-nresT res-intf) of
      (true,false) => mk-nresT res-intf
    | (false,true) => dest-nresT res-intf
    | - => res-intf

    val iT = arg-intfs ---> ri

      val ((-,itype-thm),lthy) = Sepref-Combinator-Setup.sepref-register-single
(Binding.name-of basename) c iT lthy
    val - = Document-Output.pretty-thm lthy itype-thm |> Pretty.string-of |>
writeln

  in
    lthy
  end

  val lthy = register op-name opc lthy

  val - = dbg-trace lthy Define pre
  val pre-name = Binding.prefix-name pre- name
  val ((prec,pre-def-thm),lthy) = def pre-name pre @{attributes [simp]} lthy
  val prec = Refine-Util.dummify-tvars prec
  val def-thms = pre-def-thm::def-thms

  (* Re−integrate pre−constant into type−context of relation. TODO: This is
probably not clean/robust *)
  val pre = constrain-type-pre (fastype-of pre) prec |> Syntax.check-term lthy
```

val - = dbg-trace lthy Convert both, relation and operation to uncurried form, and add nres
val - = dbg-trace lthy Convert relation (arguments have already been separated by analyze−rel)
val res = case res of @{mpat ⟨-⟩nres-rel} => res | - => @{mk-term ⟨?res⟩nres-rel}
val relt = mk-rel (SOME pre,args,res)

val - = dbg-trace lthy Convert operation
val opcT = fastype-of (Syntax.check-term lthy opc)
val op-is-nres = Sepref-Basic.is-nresT (body-type opcT)
val (opcu, op-ar) = let
val arity = binder-types #> length
(∗ Arity of operation is number of arguments before result (which may be a fun−type!)∗)
val res-ar = arity (Relators.rel-absT res |> not op-is-nres ? dest-nresT)

val op-ar = arity opcT − res-ar

val - = op-ar = length args orelse
raise TERM(Operation/relation arity mismatch: ^ string-of-int op-ar ^ vs ^ string-of-int (length args),[opc,relt])

(∗ Add RETURN o...o if necessary∗)
val opc =
if op-is-nres then opc
else mk-compN-pre op-ar (Const(@{const-name Refine-Basic.RETURN},dummyT)) opc

(∗ Add uncurry if necessary ∗)
val opc = mk-uncurryN-pre op-ar opc
in
(opc, op-ar)
end

(∗ Build mop−variant ∗)
val declare-mop = (specified-pre orelse not op-is-nres) andalso flag-mop

val (mop-data,lthy) = if declare-mop then let
val - = dbg-trace lthy mop definition
val mop-rhs = Const(@{const-name mop},dummyT) $ prec $ opcu
|> mk-curryN-pre op-ar
val mop-name = Binding.prefix-name mop- name
val ((mopc,mop-def-thm),lthy) = def mop-name mop-rhs [] lthy
val mopc = Refine-Util.dummify-tvars mopc

val ((mopc,added-pr-const),lthy) = pr-const-heuristics mop-name mopc

```
lthy

        val mop-def-thm' = if added-pr-const then
            mop-def-thm RS @{thm add-PR-CONST-to-def}
          else mop-def-thm

      val (-,lthy) = Local-Theory.note ((Binding.empty, @{attributes [sepref-mop-def-thms]}),[mop-def-thm'])
lthy

        val - = dbg-trace lthy mop alternative definition
        val alt-unfolds = @{thms mop-alt-unfolds}
          |> not specified-pre ? curry op :: pre-def-thm

        val mop-alt-thm = Local-Defs.unfold0 lthy alt-unfolds mop-def-thm
          |> Refine-Util.shift-lambda-leftN op-ar
      val (-,lthy) = Local-Theory.note ((Binding.suffix-name -alt mop-name,@{attributes
[simp]}),[mop-alt-thm]) lthy

        val - = dbg-trace lthy mop: register
        val lthy = register mop-name mopc lthy

        val - = dbg-trace lthy mop: vcg theorem
        local
          val Ts = map Relators.rel-absT args
          val ctxt = Variable.declare-thm mop-def-thm lthy
          val ctxt = fold Variable.declare-typ Ts ctxt
          val (x,ctxt) = Refine-Util.fix-left-tuple-from-Ts x Ts ctxt

          val mop-def-thm = mop-def-thm
            |> Local-Defs.unfold0 ctxt @{thms curry-shl}

          fun prep-thm thm = (thm OF [mop-def-thm])
            |> Drule.infer-instantiate' ctxt [SOME (Thm.cterm-of ctxt x)]
                |> Local-Defs.unfold0 ctxt @{thms uncurry-apply uncurry0-apply
o-apply}
            |> Local-Defs.unfold0 ctxt (def-thms @
                @{thms Product-Type.split HOL.True-implies-equals})
            |> singleton (Variable.export ctxt lthy)

       val thms = map prep-thm @{thms mop-spec-rl-from-def mop-leof-rl-from-def}


        in
            val (-,lthy) = Local-Theory.note ((qname vcg mop-name,@{attributes
[refine-vcg]}),thms) lthy
        end

      in
        (SOME (mop-name,mopc,mop-def-thm),lthy)
```

```
    end
  else (NONE,lthy)


  val - = dbg-trace lthy Build Parametricity Theorem
  val param-t = mk-pair-in-pre opcu opcu relt
    |> Syntax.check-term lthy
    |> HOLogic.mk-Trueprop
    |> curry Logic.list-implies relconds

  val - = dbg-trace lthy Build Parametricity Theorem for Precondition
  val param-pre-t =
    let
      val pre-relt = Relators.mk-fun-rel (Relators.list-prodrel-left args) @{term
bool-rel}

      val param-pre-t = mk-pair-in-pre prec prec pre-relt
        |> Syntax.check-term lthy
        |> HOLogic.mk-Trueprop
        |> curry Logic.list-implies relconds
    in
      param-pre-t
    end


  val - = dbg-trace lthy Build goals
  val goals = [[ (param-t, []), (param-pre-t, []) ]]

  fun after-qed [[p-thm, pp-thm]] - (*ctxt*) =
    let
      val - = dbg-trace lthy after-qed
          val p-thm' = p-thm |> not specified-pre ? Local-Defs.unfold0 lthy
[pre-def-thm]

          val (-,lthy) = Local-Theory.note ((qname fref op-name,@{attributes
[sepref-fref-thms]}), [p-thm']) lthy
        val (-,lthy) = Local-Theory.note ((qname param pre-name,@{attributes
[param]}), [pp-thm]) lthy

      val p'-unfolds = pre-def-thm :: @{thms True-implies-equals}
  val (-,lthy) = Local-Theory.note ((qname fref' op-name,[]), [Local-Defs.unfold0
lthy p'-unfolds p-thm]) lthy


      val lthy = case mop-data of NONE => lthy |
        SOME (mop-name,mopc,mop-def-thm) => let
          val - = dbg-trace lthy Build and prove mop−stuff
          (* mop − parametricity theorem: (uncurry^n mopc,uncurry^n mopc) ∈
args →_f res *)
```

```
val mopcu = mk-uncurryN-pre op-ar mopc
val param-mop-t = mk-pair-in-pre mopcu mopcu (mk-rel (NONE,args,res))
  |> Syntax.check-term lthy
  |> HOLogic.mk-Trueprop
  |> curry Logic.list-implies relconds

val ctxt = Proof-Context.augment param-mop-t lthy

val tac = let
 val p-thm = Local-Defs.unfold0 ctxt @{thms PR-CONST-def} p-thm
in
 Local-Defs.unfold0-tac ctxt (mop-def-thm :: @{thms PR-CONST-def
uncurry-curry-id uncurry-curry0-id})
  THEN FIRSTGOAL (
    dbg-msg-tac (Sepref-Debugging.msg-subgoal Mop−param thm goal
after unfolding) ctxt THEN'
    resolve-tac ctxt @{thms param-mopI}
      THEN' SOLVED' (resolve-tac ctxt [p-thm] THEN-ALL-NEW
assume-tac ctxt)
      THEN' SOLVED' (resolve-tac ctxt [pp-thm] THEN-ALL-NEW
assume-tac ctxt)
  )
end

      val pm-thm = Goal.prove-internal lthy [] (Thm.cterm-of ctxt
param-mop-t) (K tac)
  |> singleton (Variable.export ctxt lthy)

val (-,lthy) = Local-Theory.note ((qname fref mop-name,@{attributes
[sepref-fref-thms]}), [pm-thm]) lthy
      val (-,lthy) = Local-Theory.note ((qname fref' mop-name,[]),
[Local-Defs.unfold0 lthy p'-unfolds pm-thm]) lthy


in
  lthy
end



in
  lthy
end
| after-qed thmss - = raise THM (After−qed: Wrong thmss structure,~1,flat
thmss)

fun std-tac ctxt = let
val ptac = REPEAT-ALL-NEW-FWD (Parametricity.net-tac (Parametricity.get-dflt
ctxt) ctxt)
```

(* Massage simpset a bit *)
val ctxt = ctxt
|> Context-Position.set-visible false
|> Context.proof-map (Thm.attribute-declaration Clasimp.iff-del @{thm pair-in-Id-conv})

in
if flag-rawgoals then
all-tac
else
Local-Defs.unfold0-tac ctxt def-thms THEN ALLGOALS (
TRY o SOLVED' (
TRY o resolve-tac ctxt @{thms frefI}
THEN' TRY o REPEAT-ALL-NEW (ematch-tac ctxt @{thms prod-relE})
THEN' simp-tac (put-simpset HOL-basic-ss ctxt addsimps @{thms split uncurry-apply uncurry0-apply})
THEN' (
SOLVED' (ptac THEN-ALL-NEW asm-full-simp-tac ctxt)
ORELSE' SOLVED' (cp-clarsimp-tac ctxt THEN-ALL-NEW-FWD ptac THEN-ALL-NEW SELECT-GOAL (auto-tac ctxt))
)
)
)

end

val rf-std = Proof.refine (Method.Basic (fn ctxt => SIMPLE-METHOD (std-tac ctxt)))
#> Seq.the-result do-cmd: Standard proof tactic returned empty result sequence

in
Proof.theorem NONE after-qed goals lthy
|> rf-std
end

val - = Outer-Syntax.local-theory-to-proof @{command-keyword sepref-decl-op}
(do-parser >> do-cmd)

local

fun unfold-PR-CONST-tac ctxt = SELECT-GOAL (Local-Defs.unfold0-tac ctxt @{thms PR-CONST-def})

fun transfer-precond-rl ctxt t R = let

```
(*val tfrees = Term.add-tfreesT (fastype-of t) []
 val t' = map-types (map-type-tfree (fn x => if member op= tfrees x then
dummyT else TFree x)) t
*) (* TODO: Brute force approach, that may generalize too much! *)
val t' = map-types (K dummyT) t

val goal = Sepref-Basic.mk-pair-in-pre t t' R
  |> Syntax.check-term ctxt
  |> Thm.cterm-of ctxt

    val thm = Drule.infer-instantiate' ctxt [NONE,SOME goal] @{thm
IMP-LIST-trivial}

in
  thm
end



(* Generate a hnr−thm for mop given one for op *)
fun generate-mop-thm ctxt op-thm = let
  val orig-ctxt = ctxt

    val (op-thm, ctxt) = yield-singleton (apfst snd oo Variable.import true)
op-thm ctxt

  (* Convert mop-def-thms to form uncurry^n f ≡ mop P g *)
  val mop-def-thms = Named-Theorems-Rev.get ctxt @{named-theorems-rev
sepref-mop-def-thms}
    |> map (Local-Defs.unfold0 ctxt @{thms curry-shl})

  fun fail-hnr-tac - - = raise THM(Invalid hnr−theorem,~1,[op-thm])
  fun fail-mop-def-tac i st = let
    val g = nth (Thm.prems-of st) (i−1)
  in
    raise TERM(Found no matching mop−definition,[g])
  end

  (* Tactic to solve preconditions of hfref-op-to-mop *)
  val tac = APPLY-LIST [
    resolve-tac ctxt [op-thm] ORELSE' fail-hnr-tac,
    ((*unfold-PR-CONST-tac ctxt THEN'*) resolve-tac ctxt mop-def-thms)
ORELSE' fail-mop-def-tac,
      simp-precond-tac ctxt ORELSE' Sepref-Debugging.error-tac' precond
simplification failed ctxt
  ] 1

  (* Do synthesis *)
  val st = @{thm hfref-op-to-mop}
  val st = Goal.protect (Thm.nprems-of st) st
```

```
    val mop-thm = tac st |> Seq.hd |> Goal.conclude

    val mop-thm = singleton (Variable.export ctxt orig-ctxt) mop-thm
      |> Sepref-Rules.norm-fcomp-rule orig-ctxt
  in mop-thm end

  (* Generate a hnr−thm for op given one for mop *)
  fun generate-op-thm ctxt mop-thm = let (* TODO: Almost−clone of generate-mop-thm *)
    val orig-ctxt = ctxt

    val (mop-thm, ctxt) = yield-singleton (apfst snd oo Variable.import true) mop-thm ctxt

    (* Convert mop-def-thms to form uncurry^n f ≡ mop P g *)
    val mop-def-thms = Named-Theorems-Rev.get ctxt @{named-theorems-rev sepref-mop-def-thms}
      |> map (Local-Defs.unfold0 ctxt @{thms curry-shl})

    fun fail-hnr-tac - - = raise THM(Invalid hnr−theorem,~1,[mop-thm])
    fun fail-mop-def-tac i st = let
      val g = nth (Thm.prems-of st) (i−1)
    in
      raise TERM(Found no matching mop−definition,[g])
    end

    (* Tactic to solve preconditions of hfref-mop-to-op *)
    val tac = APPLY-LIST [
      resolve-tac ctxt [mop-thm] ORELSE' fail-hnr-tac,
      ((*unfold-PR-CONST-tac ctxt THEN'*) resolve-tac ctxt mop-def-thms) ORELSE' fail-mop-def-tac,
        simp-precond-tac ctxt ORELSE' Sepref-Debugging.error-tac' precond simplification failed ctxt
    ] 1

    (* Do synthesis *)
    val st = @{thm hfref-mop-to-op}
    val st = Goal.protect (Thm.nprems-of st) st
    val op-thm = tac st |> Seq.hd |> Goal.conclude

    val op-thm = singleton (Variable.export ctxt orig-ctxt) op-thm
      |> Sepref-Rules.norm-fcomp-rule orig-ctxt
  in op-thm end



  fun chk-result ctxt thm = let
    val (-,R,S) = case Thm.concl-of thm of
      @{mpat Trueprop (-∈hfref ?P ?R ?S)} => (P,R,S)
```

| - => *raise THM*(*chk-result*: *Expected hfref−theorem*,~1,[*thm*])

fun err t = *let*
  val ts = *Syntax.pretty-term ctxt t* |> *Pretty.string-of*
in
  raise THM (*chk-result*: *Invalid pattern left in assertions*: ^ *ts*,~1,[*thm*])
end
fun check-invalid (*t as* @{*mpat hr-comp - -*}) = *err t*
  | *check-invalid* (*t as* @{*mpat hrp-comp - -*}) = *err t*
  | *check-invalid* (*t as* @{*mpat pure* (*the-pure -*)}) = *err t*
  | *check-invalid* (*t as* @{*mpat - O -*}) = *err t*
  | *check-invalid* - = *false*

val - = *exists-subterm check-invalid R*
val - = *exists-subterm check-invalid S*
in
  ()
end

fun to-IMP-LIST ctxt thm =
  (*thm RS* @{*thm to-IMP-LISTI*}) |> *Local-Defs.unfold0 ctxt* @{*thms to-IMP-LIST*}

fun from-IMP-LIST ctxt thm = *thm* |> *Local-Defs.unfold0 ctxt* @{*thms from-IMP-LIST*}

in

local
  open Refine-Util
  val flags =
      parse-bool-config′ mop cfg-mop
    || *parse-bool-config′ ismop cfg-ismop*
    || *parse-bool-config′ transfer cfg-transfer*
    || *parse-bool-config′ rawgoals cfg-rawgoals*
    || *parse-bool-config′ register cfg-register*
  val parse-flags = *parse-paren-list′ flags*

  val parse-precond = *Scan.option* (@{*keyword* [} |−− *Parse.term* −−| @{*keyword*]})

  val parse-fref-thm = *Scan.option* (@{*keyword uses*} |−− *Parse.thm*)

in
val di-parser = *parse-flags* −− *Scan.optional* (*Parse.binding* −−| @{*keyword* :}) *Binding.empty* −− *parse-precond* −− *Parse.thm* −− *parse-fref-thm*
end

```
fun di-cmd ((((flags,name), precond-raw), i-thm-raw), p-thm-raw) lthy = let
  val i-thm = singleton (Attrib.eval-thms lthy) i-thm-raw
  val p-thm = map-option (singleton (Attrib.eval-thms lthy)) p-thm-raw

  local
    val ctxt = Refine-Util.apply-configs flags lthy
  in
    val flag-mop = Config.get ctxt cfg-mop
    val flag-ismop = Config.get ctxt cfg-ismop
    val flag-rawgoals = Config.get ctxt cfg-rawgoals
    val flag-transfer = Config.get ctxt cfg-transfer
    val flag-register = Config.get ctxt cfg-register
  end

  val fr-attribs = if flag-register then @{attributes [sepref-fr-rules]} else []


  val ctxt = lthy

  (* Compose with fref−theorem *)
  val - = dbg-trace lthy Compose with fref

  local
    val hf-tcomp-pre = @{thm hfcomp-tcomp-pre} OF [asm-rl,i-thm]
    fun compose p-thm = let
      val p-thm = p-thm |> to-assns-rl false lthy
    in
      hf-tcomp-pre OF [p-thm]
    end

  in
    val thm = case p-thm of
      SOME p-thm => compose p-thm
    | NONE => let
          val p-thms = Named-Theorems-Rev.get ctxt @{named-theorems-rev
seprefe-fref-thms}

          fun err () = let
        val prem-s = nth (Thm.prems-of hf-tcomp-pre) 0 |> Syntax.pretty-term
ctxt |> Pretty.string-of
          in
            error (Found no fref−theorem matching ^ prem-s)
          end

        in
          case get-first (try compose) p-thms of
            NONE => err ()
          | SOME thm => thm
```

```
        end
      end

       val (thm,ctxt) = yield-singleton (apfst snd oo Variable.import true) thm
ctxt

      val - = dbg-trace lthy Transfer Precond
      val thm = to-IMP-LIST ctxt thm
      val thm = thm RS @{thm transform-pre-param}

      local
        val (pre,R,pp-name,pp-type) = case Thm.prems-of thm of
            [@{mpat Trueprop (IMP-LIST - ((?pre,-)∈?R))}, @{mpat Trueprop
(IMP-PRE (mpaq-STRUCT (mpaq-Var ?pp-name ?pp-type)) -)}] => (pre,R,pp-name,pp-type)
            | - => raise THM(di-cmd: Cannot recognize first prems of trans-
form-pre-param: , ~1,[thm])

      in
         val thm = if flag-transfer then thm OF [transfer-precond-rl ctxt pre R]
else thm

        val thm = case precond-raw of
          NONE => thm
        | SOME precond-raw => let
            val precond = Syntax.parse-term ctxt precond-raw
              |> Sepref-Basic.constrain-type-pre pp-type
              |> Syntax.check-term ctxt
              |> Thm.cterm-of ctxt

            val thm = Drule.infer-instantiate ctxt [(pp-name,precond)] thm
            val thm = thm OF [asm-rl,@{thm IMP-PRE-CUSTOMD}]
          in
            thm
          end

      end

      val - = dbg-trace lthy Build goals
      val goals = [map (fn x => (x,[])) (Thm.prems-of thm)]

      fun after-qed thmss - = let
        val - = dbg-trace lthy After QED
        val prems-thms = hd thmss

        val thm = thm OF prems-thms

        val thm = from-IMP-LIST ctxt thm

        (* Two rounds of cleanup−constraints, norm-fcomp *)
```

```
val - = dbg-trace lthy Cleanup
val thm = thm
  |> cleanup-constraints ctxt
  |> Sepref-Rules.norm-fcomp-rule ctxt
  |> cleanup-constraints ctxt
  |> Sepref-Rules.norm-fcomp-rule ctxt

val thm = thm
  |> singleton (Variable.export ctxt lthy)
  |> zero-var-indexes

val - = dbg-trace lthy Check Result
val - = chk-result lthy thm


  fun qname suffix = if Binding.is-empty name then name else Binding.suffix-name suffix name

val thm-name = if flag-ismop then qname -hnr-mop else qname -hnr
val (-,lthy) = Local-Theory.note ((thm-name,fr-attribs),[thm]) lthy

val - = Thm.pretty-thm lthy thm |> Pretty.string-of |> writeln

(* Create mop theorem from op−theorem *)
val cr-mop-thm = flag-mop andalso not flag-ismop
val lthy =
  if cr-mop-thm then
    let
      val - = dbg-trace lthy Create mop−thm
      val mop-thm = thm
        |> generate-mop-thm lthy
        |> zero-var-indexes

 val (-,lthy) = Local-Theory.note ((qname -hnr-mop,fr-attribs),[mop-thm]) lthy
      val - = Thm.pretty-thm lthy mop-thm |> Pretty.string-of |> writeln
    in lthy end
  else lthy

(* Create op theorem from mop−theorem *)
val cr-op-thm = flag-ismop
val lthy =
  if cr-op-thm then
    let
      val - = dbg-trace lthy Create op−thm
      val op-thm = thm
        |> generate-op-thm lthy
        |> zero-var-indexes
```

```
            val (-,lthy) = Local-Theory.note ((qname -hnr,fr-attribs),[op-thm])
lthy
            val - = Thm.pretty-thm lthy op-thm |> Pretty.string-of |> writeln
          in lthy end
        else lthy


    in
      lthy
    end

    fun std-tac ctxt = let
      val ptac = REPEAT-ALL-NEW-FWD (
          Parametricity.net-tac (Parametricity.get-dflt ctxt) ctxt ORELSE'
assume-tac ctxt
        )
    in
      if flag-rawgoals orelse not flag-transfer then
        all-tac
      else
        APPLY-LIST [
       SELECT-GOAL (Local-Defs.unfold0-tac ctxt @{thms from-IMP-LIST})
THEN' TRY o SOLVED' ptac,
          simp-precond-tac ctxt
        ] 1

    end

    val rf-std = Proof.refine (Method.Basic (fn ctxt => SIMPLE-METHOD
(std-tac ctxt)))
        #> Seq.the-result di-cmd: Standard proof tactic returned empty result
sequence

  in
    Proof.theorem NONE after-qed goals ctxt
    |> rf-std

  end

  val - = Outer-Syntax.local-theory-to-proof @{command-keyword sepref-decl-impl}
    (di-parser >> di-cmd)
  end

end
›
```

1.9.8 Obsolete Manual Specification Helpers

lemma *vcg-of-RETURN-np*:

assumes $f \equiv RETURN\ r$
shows $SPEC\ (\lambda x.\ x{=}r) \leq m \Longrightarrow f \leq m$
and $SPEC\ (\lambda x.\ x{=}r) \leq_n m \Longrightarrow f \leq_n m$
using *assms*
by (*auto simp*: *pw-le-iff pw-leof-iff*)

lemma *vcg-of-RETURN*:
assumes $f \equiv do\ \{\ ASSERT\ \Phi;\ RETURN\ r\ \}$
shows $[\![\Phi;\ SPEC\ (\lambda x.\ x{=}r) \leq m]\!] \Longrightarrow f \leq m$
and $[\![\Phi \Longrightarrow SPEC\ (\lambda x.\ x{=}r) \leq_n m]\!] \Longrightarrow f \leq_n m$
using *assms*
by (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps*)

lemma *vcg-of-SPEC*:
assumes $f \equiv do\ \{\ ASSERT\ pre;\ SPEC\ post\ \}$
shows $[\![pre;\ SPEC\ post \leq m]\!] \Longrightarrow f \leq m$
and $[\![pre \Longrightarrow SPEC\ post \leq_n m]\!] \Longrightarrow f \leq_n m$
using *assms*
by (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps*)

lemma *vcg-of-SPEC-np*:
assumes $f \equiv SPEC\ post$
shows $SPEC\ post \leq m \Longrightarrow f \leq m$
and $SPEC\ post \leq_n m \Longrightarrow f \leq_n m$
using *assms*
by *auto*

lemma *mk-mop-rl1*:
assumes $\bigwedge x.\ mf\ x \equiv ASSERT\ (P\ x) \gg RETURN\ (f\ x)$
shows $(RETURN\ o\ f,\ mf) \in Id \rightarrow \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl2*:
assumes $\bigwedge x\ y.\ mf\ x\ y \equiv ASSERT\ (P\ x\ y) \gg RETURN\ (f\ x\ y)$
shows $(RETURN\ oo\ f,\ mf) \in Id \rightarrow Id \rightarrow \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl3*:
assumes $\bigwedge x\ y\ z.\ mf\ x\ y\ z \equiv ASSERT\ (P\ x\ y\ z) \gg RETURN\ (f\ x\ y\ z)$
shows $(RETURN\ ooo\ f,\ mf) \in Id \rightarrow Id \rightarrow Id \rightarrow \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl0-np*:
assumes $mf \equiv RETURN\ f$
shows $(RETURN\ f,\ mf) \in \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl1-np*:
assumes $\bigwedge x.\ mf\ x \equiv RETURN\ (f\ x)$
shows $(RETURN\ o\ f,\ mf) \in Id \to \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl2-np*:
assumes $\bigwedge x\ y.\ mf\ x\ y \equiv RETURN\ (f\ x\ y)$
shows $(RETURN\ oo\ f,\ mf) \in Id \to Id \to \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-mop-rl3-np*:
assumes $\bigwedge x\ y\ z.\ mf\ x\ y\ z \equiv RETURN\ (f\ x\ y\ z)$
shows $(RETURN\ ooo\ f,\ mf) \in Id \to Id \to Id \to \langle Id\rangle nres\text{-}rel$
unfolding *assms*[*abs-def*]
by (*auto intro*!: *nres-relI simp*: *pw-le-iff refine-pw-simps*)

lemma *mk-op-rl0-np*:
assumes $mf \equiv RETURN\ f$
shows $(uncurry0\ mf,\ uncurry0\ (RETURN\ f)) \in unit\text{-}rel \to_f \langle Id\rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl1*:
assumes $\bigwedge x.\ mf\ x \equiv ASSERT\ (P\ x) \gg RETURN\ (f\ x)$
shows $(mf,\ RETURN\ o\ f) \in [P]_f\ Id \to \langle Id\rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl1-np*:
assumes $\bigwedge x.\ mf\ x \equiv RETURN\ (f\ x)$
shows $(mf,\ (RETURN\ o\ f)) \in Id \to_f \langle Id\rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl2*:
assumes $\bigwedge x\ y.\ mf\ x\ y \equiv ASSERT\ (P\ x\ y) \gg RETURN\ (f\ x\ y)$

shows $(uncurry\ mf,\ uncurry\ (RETURN\ oo\ f)) \in [uncurry\ P]_f\ Id \times_r Id \rightarrow \langle Id \rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl2-np*:
assumes $\bigwedge x\ y.\ mf\ x\ y \equiv RETURN\ (f\ x\ y)$
shows $(uncurry\ mf,\ uncurry\ (RETURN\ oo\ f)) \in Id \times_r Id \rightarrow_f \langle Id \rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl3*:
assumes $\bigwedge x\ y\ z.\ mf\ x\ y\ z \equiv ASSERT\ (P\ x\ y\ z) \gg RETURN\ (f\ x\ y\ z)$
shows $(uncurry2\ mf,\ uncurry2\ (RETURN\ ooo\ f)) \in [uncurry2\ P]_f\ (Id \times_r Id) \times_r Id \rightarrow \langle Id \rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

lemma *mk-op-rl3-np*:
assumes $\bigwedge x\ y\ z.\ mf\ x\ y\ z \equiv RETURN\ (f\ x\ y\ z)$
shows $(uncurry2\ mf,\ uncurry2\ (RETURN\ ooo\ f)) \in (Id \times_r Id) \times_r Id \rightarrow_f \langle Id \rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
apply (*auto simp*: *assms*)
done

end

1.10 Sepref Tool

theory *Sepref-Tool*
imports *Sepref-Translate Sepref-Definition Sepref-Combinator-Setup Sepref-Intf-Util*
begin

In this theory, we set up the sepref tool.

1.10.1 Sepref Method

lemma *CONS-init*:
assumes $hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ a$

assumes $\Gamma' \Longrightarrow_t \Gamma c'$
assumes $\bigwedge a\ c.\ hn\text{-}ctxt\ R\ a\ c \Longrightarrow_t hn\text{-}ctxt\ Rc\ a\ c$
shows *hn-refine* Γ *c* $\Gamma c'$ *Rc a*
apply (*rule hn-refine-cons*)
apply (*rule entt-refl*)
apply (*rule assms*[*unfolded hn-ctxt-def*])+
done

lemma *ID-init*: $\llbracket ID\ a\ a'\ TYPE('T);\ hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ a' \rrbracket$
$\Longrightarrow hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ a$ **by** *simp*

lemma *TRANS-init*: $\llbracket\ hn\text{-}refine\ \Gamma\ c\ \Gamma'\ R\ a;\ CNV\ c\ c'\ \rrbracket$
$\Longrightarrow hn\text{-}refine\ \Gamma\ c'\ \Gamma'\ R\ a$
by *simp*

lemma *infer-post-triv*: $P \Longrightarrow_t P$ **by** (*rule entt-refl*)

ML ‹
```
structure Sepref = struct
  structure sepref-preproc-simps = Named-Thms (
    val name = @{binding sepref-preproc}
    val description = Sepref: Preprocessor simplifications
  )

  structure sepref-opt-simps = Named-Thms (
    val name = @{binding sepref-opt-simps}
    val description = Sepref: Post−Translation optimizations, phase 1
  )

  structure sepref-opt-simps2 = Named-Thms (
    val name = @{binding sepref-opt-simps2}
    val description = Sepref: Post−Translation optimizations, phase 2
  )

  fun cons-init-tac ctxt = Sepref-Frame.weaken-post-tac ctxt THEN' resolve-tac
ctxt @{thms CONS-init}
  fun cons-solve-tac dbg ctxt = let
    val dbgSOLVED' = if dbg then I else SOLVED'
  in
    dbgSOLVED' (
      resolve-tac ctxt @{thms infer-post-triv}
      ORELSE' Sepref-Translate.side-frame-tac ctxt
    )
  end

  fun preproc-tac ctxt = let
    val ctxt = put-simpset HOL-basic-ss ctxt
    val ctxt = ctxt addsimps (sepref-preproc-simps.get ctxt)
  in
```

```
  Sepref-Rules.prepare-hfref-synth-tac ctxt THEN'
  Simplifier.simp-tac ctxt
end

fun id-tac ctxt =
  resolve-tac ctxt @{thms ID-init}
  THEN' CONVERSION Thm.eta-conversion
  THEN' DETERM o Id-Op.id-tac Id-Op.Normal ctxt

fun id-init-tac ctxt =
  resolve-tac ctxt @{thms ID-init}
  THEN' CONVERSION Thm.eta-conversion
  THEN' Id-Op.id-tac Id-Op.Init ctxt

fun id-step-tac ctxt =
  Id-Op.id-tac Id-Op.Step ctxt

fun id-solve-tac ctxt =
  Id-Op.id-tac Id-Op.Solve ctxt

(*fun id-param-tac ctxt = CONVERSION (Refine-Util.HOL-concl-conv
  (K (Sepref-Param.id-param-conv ctxt)) ctxt)*)

fun monadify-tac ctxt = Sepref-Monadify.monadify-tac ctxt

(*fun lin-ana-tac ctxt = Sepref-Lin-Ana.lin-ana-tac ctxt*)

fun trans-tac ctxt = Sepref-Translate.trans-tac ctxt

fun opt-tac ctxt = let
  val opt1-ss = put-simpset HOL-basic-ss ctxt
    addsimps sepref-opt-simps.get ctxt
    addsimprocs [@{simproc HOL.let-simp}]
  |> Simplifier.add-cong @{thm SP-cong}
  |> Simplifier.add-cong @{thm PR-CONST-cong}

  val unsp-ss = put-simpset HOL-basic-ss ctxt addsimps @{thms SP-def}

  val opt2-ss = put-simpset HOL-basic-ss ctxt
    addsimps sepref-opt-simps2.get ctxt
    addsimprocs [@{simproc HOL.let-simp}]

in
  simp-tac opt1-ss THEN' simp-tac unsp-ss THEN'
  simp-tac opt2-ss THEN' simp-tac unsp-ss THEN'
  CONVERSION Thm.eta-conversion THEN'
  resolve-tac ctxt @{thms CNV-I}
end
```

```
  fun sepref-tac dbg ctxt =
    (K Sepref-Constraints.ensure-slot-tac)
    THEN'
    Sepref-Basic.PHASES'
      [
        (preproc,preproc-tac,0),
        (cons-init,cons-init-tac,2),
        (id,id-tac,0),
        (monadify,monadify-tac false,0),
        (opt-init,fn ctxt => resolve-tac ctxt @{thms TRANS-init},1),
        (trans,trans-tac,~1),
        (opt,opt-tac,~1),
        (cons-solve1,cons-solve-tac false,~1),
        (cons-solve2,cons-solve-tac false,~1),
          (constraints,fn ctxt => K (Sepref-Constraints.solve-constraint-slot ctxt
THEN Sepref-Constraints.remove-slot-tac),~1)
      ] (Sepref-Basic.flag-phases-ctrl dbg) ctxt

  val setup = I
    #> sepref-preproc-simps.setup
    #> sepref-opt-simps.setup
    #> sepref-opt-simps2.setup
 end
›
```

setup *Sepref.setup*

method-setup *sepref* = ‹*Scan.succeed (fn ctxt =>*
SIMPLE-METHOD (DETERM (SOLVED' (IF-EXGOAL (
Sepref.sepref-tac false ctxt
)) 1)))›
‹*Automatic refinement to Imperative/HOL*›

method-setup *sepref-dbg-keep* = ‹*Scan.succeed (fn ctxt => let*
(∗val ctxt = Config.put Id-Op.cfg-id-debug true ctxt∗)
in
SIMPLE-METHOD (IF-EXGOAL (Sepref.sepref-tac true ctxt) 1)
end)›
‹*Automatic refinement to Imperative/HOL, debug mode*›

Default Optimizer Setup

lemma *return-bind-eq-let*: *do* { *x←return v*; *f x* } = *do* { *let x=v*; *f x* } **by** *simp*
lemmas [*sepref-opt-simps*] = *return-bind-eq-let bind-return bind-bind id-def*

We allow the synthesized function to contain tagged function applications. This is important to avoid higher-order unification problems when synthesizing generic algorithms, for example the to-list algorithm for foreach-loops.

lemmas [*sepref-opt-simps*] = *Autoref-Tagging.APP-def*

Revert case-pulling done by monadify

lemma *case-prod-return-opt*[*sepref-opt-simps*]:
  case-prod (*λa b. return* (*f a b*)) *p* = *return* (*case-prod f p*)
  by (*simp split*: *prod.split*)

lemma *case-option-return-opt*[*sepref-opt-simps*]:
  case-option (*return fn*) (*λs. return* (*fs s*)) *v* = *return* (*case-option fn fs v*)
  by (*simp split*: *option.split*)

lemma *case-list-return*[*sepref-opt-simps*]:
  case-list (*return fn*) (*λx xs. return* (*fc x xs*)) *l* = *return* (*case-list fn fc l*)
  by (*simp split*: *list.split*)

lemma *if-return*[*sepref-opt-simps*]:
  If b (*return t*) (*return e*) = *return* (*If b t e*) **by** *simp*

In some cases, pushing in the returns is more convenient

lemma *case-prod-opt2*[*sepref-opt-simps2*]:
  (*λx. return* (*case x of* (*a*,*b*) ⇒ *f a b*))
  = (*λ*(*a*,*b*). *return* (*f a b*))
  by *auto*

1.10.2 Debugging Methods

ML ‹
 fun SIMPLE-METHOD-NOPARAM′ tac = *Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD′*
(*IF-EXGOAL* (*tac ctxt*)))
 fun SIMPLE-METHOD-NOPARAM tac = *Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD*
(*tac ctxt*))
›
method-setup *sepref-dbg-preproc* = ‹*SIMPLE-METHOD-NOPARAM′* (*fn ctxt*
=> *K* (*Sepref-Constraints.ensure-slot-tac*) *THEN′ Sepref.preproc-tac ctxt*)›
  ‹*Sepref debug*: *Preprocessing phase*›

method-setup *sepref-dbg-cons-init* = ‹*SIMPLE-METHOD-NOPARAM′ Sepref.cons-init-tac*›
  ‹*Sepref debug*: *Initialize consequence reasoning*›
method-setup *sepref-dbg-id* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref.id-tac*)›
  ‹*Sepref debug*: *Identify operations phase*›
method-setup *sepref-dbg-id-keep* = ‹*SIMPLE-METHOD-NOPARAM′* (*Config.put*
Id-Op.cfg-id-debug true #> *Sepref.id-tac*)›
  ‹*Sepref debug*: *Identify operations phase. Debug mode*, *keep intermediate subgoals*
on failure.›
method-setup *sepref-dbg-monadify* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref.monadify-tac*
false)›
  ‹*Sepref debug*: *Monadify phase*›
method-setup *sepref-dbg-monadify-keep* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref.monadify-tac*
true)›
  ‹*Sepref debug*: *Monadify phase*›

method-setup *sepref-dbg-monadify-arity* = ‹*SIMPLE-METHOD-NOPARAM'* (*Sepref-Monadify.arity-tac*)›
‹*Sepref debug*: *Monadify phase*: *Arity phase*›
method-setup *sepref-dbg-monadify-comb* = ‹*SIMPLE-METHOD-NOPARAM'* (*Sepref-Monadify.comb-tac*)›
‹*Sepref debug*: *Monadify phase*: *Comb phase*›
method-setup *sepref-dbg-monadify-check-EVAL* = ‹*SIMPLE-METHOD-NOPARAM'*
(*K* (*CONCL-COND'* (*not o Sepref-Monadify.contains-eval*)))›
‹*Sepref debug*: *Monadify phase*: *check-EVAL phase*›
method-setup *sepref-dbg-monadify-mark-params* = ‹*SIMPLE-METHOD-NOPARAM'*
(*Sepref-Monadify.mark-params-tac*)›
‹*Sepref debug*: *Monadify phase*: *mark-params phase*›
method-setup *sepref-dbg-monadify-dup* = ‹*SIMPLE-METHOD-NOPARAM'* (*Sepref-Monadify.dup-tac*)›
‹*Sepref debug*: *Monadify phase*: *dup phase*›
method-setup *sepref-dbg-monadify-remove-pass* = ‹*SIMPLE-METHOD-NOPARAM'*
(*Sepref-Monadify.remove-pass-tac*)›
‹*Sepref debug*: *Monadify phase*: *remove-pass phase*›

method-setup *sepref-dbg-opt-init* = ‹*SIMPLE-METHOD-NOPARAM'* (*fn ctxt*
=> *resolve-tac ctxt* @{*thms TRANS-init*})›
‹*Sepref debug*: *Translation phase initialization*›
method-setup *sepref-dbg-trans* = ‹*SIMPLE-METHOD-NOPARAM' Sepref.trans-tac*›
‹*Sepref debug*: *Translation phase*›
method-setup *sepref-dbg-opt* = ‹*SIMPLE-METHOD-NOPARAM'* (*fn ctxt* =>
Sepref.opt-tac ctxt
THEN' CONVERSION Thm.eta-conversion
THEN' TRY o resolve-tac ctxt @{*thms CNV-I*}
)›
‹*Sepref debug*: *Optimization phase*›
method-setup *sepref-dbg-cons-solve* = ‹*SIMPLE-METHOD-NOPARAM'* (*Sepref.cons-solve-tac*
false)›
‹*Sepref debug*: *Solve post−consequences*›
method-setup *sepref-dbg-cons-solve-keep* = ‹*SIMPLE-METHOD-NOPARAM'* (*Sepref.cons-solve-tac*
true)›
‹*Sepref debug*: *Solve post−consequences*, *keep intermediate results*›

method-setup *sepref-dbg-constraints* = ‹*SIMPLE-METHOD-NOPARAM'* (*fn ctxt*
=> *IF-EXGOAL* (*K* (
Sepref-Constraints.solve-constraint-slot ctxt
THEN Sepref-Constraints.remove-slot-tac
)))›
‹*Sepref debug*: *Solve accumulated constraints*›

method-setup *sepref-dbg-id-init* = ‹*SIMPLE-METHOD-NOPARAM' Sepref.id-init-tac*›
‹*Sepref debug*: *Initialize operation identification phase*›
method-setup *sepref-dbg-id-step* = ‹*SIMPLE-METHOD-NOPARAM' Sepref.id-step-tac*›
‹*Sepref debug*: *Single step operation identification phase*›
method-setup *sepref-dbg-id-solve* = ‹*SIMPLE-METHOD-NOPARAM' Sepref.id-solve-tac*›

‹*Sepref debug*: *Complete current operation identification goal*›

method-setup *sepref-dbg-trans-keep* = ‹*SIMPLE-METHOD-NOPARAM′ Sepref-Translate.trans-keep-tac*›
‹*Sepref debug*: *Translation phase*, *stop at failed subgoal*›

method-setup *sepref-dbg-trans-step* = ‹*SIMPLE-METHOD-NOPARAM′ Sepref-Translate.trans-step-tac*›
‹*Sepref debug*: *Translation step*›

method-setup *sepref-dbg-trans-step-keep* = ‹*SIMPLE-METHOD-NOPARAM′ Sepref-Translate.trans-step-ke*
‹*Sepref debug*: *Translation step*, *keep unsolved subgoals*›

method-setup *sepref-dbg-side* = ‹*SIMPLE-METHOD-NOPARAM′* (*fn ctxt* => *REPEAT-ALL-NEW-FWD* (*Sepref-Translate.side-cond-dispatch-tac false* (*K no-tac*) *ctxt*))›
method-setup *sepref-dbg-side-unfold* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref-Translate.side-unfold-tac*)
method-setup *sepref-dbg-side-keep* = ‹*SIMPLE-METHOD-NOPARAM′* (*fn ctxt* => *REPEAT-ALL-NEW-FWD* (*Sepref-Translate.side-cond-dispatch-tac true* (*K no-tac*) *ctxt*))›

method-setup *sepref-dbg-prepare-frame* = ‹*SIMPLE-METHOD-NOPARAM′ Sepref-Frame.prepare-frame-ta*
‹*Sepref debug*: *Prepare frame inference*›

method-setup *sepref-dbg-frame* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref-Frame.frame-tac* (*Sepref-Translate.side-fallback-tac*))›
‹*Sepref debug*: *Frame inference*›

method-setup *sepref-dbg-merge* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref-Frame.merge-tac* (*Sepref-Translate.side-fallback-tac*))›
‹*Sepref debug*: *Frame inference*, *merge*›

method-setup *sepref-dbg-frame-step* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref-Frame.frame-step-tac* (*Sepref-Translate.side-fallback-tac*) *false*)›
‹*Sepref debug*: *Frame inference*, *single−step*›

method-setup *sepref-dbg-frame-step-keep* = ‹*SIMPLE-METHOD-NOPARAM′* (*Sepref-Frame.frame-step-tac* (*Sepref-Translate.side-fallback-tac*) *true*)›
‹*Sepref debug*: *Frame inference*, *single−step*, *keep partially solved side conditions*›

1.10.3 Utilities

Manual hfref-proofs

method-setup *sepref-to-hnr* = ‹*SIMPLE-METHOD-NOPARAM′* (*fn ctxt* =>
Sepref.preproc-tac ctxt THEN′ Sepref-Frame.weaken-post-tac ctxt)›
‹*Sepref*: *Convert to hnr−goal and weaken postcondition*›

method-setup *sepref-to-hoare* = ‹
let
fun sepref-to-hoare-tac ctxt = *let*
val ss = *put-simpset HOL-basic-ss ctxt*

addsimps @{*thms hn-ctxt-def pure-def*}

in
Sepref.preproc-tac ctxt
THEN' Sepref-Frame.weaken-post-tac ctxt
THEN' resolve-tac ctxt @{*thms hn-refineI*}
THEN' asm-full-simp-tac ss
end
in
SIMPLE-METHOD-NOPARAM' sepref-to-hoare-tac
end
› ‹*Sepref*: *Convert to hoare−triple*›

Copying of Parameters

lemma *fold-COPY*: $x = COPY\ x$ **by** *simp*

sepref-register *COPY*

Copy is treated as normal operator, and one can just declare rules for it!

lemma *hnr-pure-COPY*[*sepref-fr-rules*]:
$CONSTRAINT\ is\text{-}pure\ R \Longrightarrow (return,\ RETURN\ o\ COPY) \in R^k \rightarrow_a R$
by (*sep-auto simp*: *is-pure-conv pure-def intro*!: *hfrefI hn-refineI*)

Short-Circuit Boolean Evaluation

Convert boolean operators to short-circuiting. When applied before monadify, this will generate a short-circuit execution.

lemma *short-circuit-conv*:
$(a \wedge b) \longleftrightarrow (if\ a\ then\ b\ else\ False)$
$(a \vee b) \longleftrightarrow (if\ a\ then\ True\ else\ b)$
$(a \longrightarrow b) \longleftrightarrow (if\ a\ then\ b\ else\ True)$
by *auto*

Eliminating higher-order

lemma *ho-prod-move*[*sepref-preproc*]: $case\text{-}prod\ (\lambda a\ b\ x.\ f\ x\ a\ b) = (\lambda p\ x.\ case\text{-}prod\ (f\ x)\ p)$
by (*auto intro*!: *ext*)

declare *o-apply*[*sepref-preproc*]

Precision Proofs

We provide a method that tries to extract equalities from an assumption of the form $- \models P1 * \ldots * Pn \wedge_A P1' * \ldots * Pn'$, if it find a precision rule for Pi and Pi'. The precision rules are extracted from the constraint rules.

TODO: Extracting the precision rules from the constraint rules is not a clean solution. It might be better to collect precision rules separately, and feed them into the constraint solver.

definition *prec-spec* $h\ \Gamma\ \Gamma' \equiv h \models \Gamma * \mathit{true} \wedge_A \Gamma' * \mathit{true}$
lemma *prec-specI*: $h \models \Gamma \wedge_A \Gamma' \Longrightarrow$ *prec-spec* $h\ \Gamma\ \Gamma'$
  unfolding *prec-spec-def*
  by (*auto simp*: *mod-and-dist mod-star-trueI*)

lemma *prec-split1-aux*: $A{*}B{*}\mathit{true} \Longrightarrow_A A{*}\mathit{true}$
  apply (*fr-rot 2*, *fr-rot-rhs 1*)
  apply (*rule ent-star-mono*)
  by *simp-all*

lemma *prec-split2-aux*: $A{*}B{*}\mathit{true} \Longrightarrow_A B{*}\mathit{true}$
  apply (*fr-rot 1*, *fr-rot-rhs 1*)
  apply (*rule ent-star-mono*)
  by *simp-all*

lemma *prec-spec-splitE*:
  assumes *prec-spec* h $(A{*}B)$ $(C{*}D)$
  obtains *prec-spec* $h\ A\ C$ *prec-spec* $h\ B\ D$
  apply (*thin-tac* $[\![$-;-$]\!] \Longrightarrow$ -)
  apply (*rule that*)
  using *assms*
  apply –
  unfolding *prec-spec-def*
  apply (*erule entailsD*[*rotated*])
  apply (*rule ent-conjI*)
  apply (*rule ent-conjE1*)
  apply (*rule prec-split1-aux*)
  apply (*rule ent-conjE2*)
  apply (*rule prec-split1-aux*)

  apply (*erule entailsD*[*rotated*])
  apply (*rule ent-conjI*)
  apply (*rule ent-conjE1*)
  apply (*rule prec-split2-aux*)
  apply (*rule ent-conjE2*)
  apply (*rule prec-split2-aux*)
  done

lemma *prec-specD*:
  assumes *precise R*
  assumes *prec-spec* h $(R\ a\ p)$ $(R\ a'\ p)$
  shows $a{=}a'$
  using *assms* **unfolding** *precise-def prec-spec-def CONSTRAINT-def* **by** *blast*

ML ‹
  fun prec-extract-eqs-tac ctxt = *let*

```
    fun is-precise thm = case Thm.concl-of thm of
      @{mpat Trueprop (precise -)} => true
    | - => false

    val thms = Sepref-Constraints.get-constraint-rules ctxt
      @ Sepref-Constraints.get-safe-constraint-rules ctxt
    val thms = thms
      |> filter is-precise
    val thms = @{thms snga-prec sngr-prec} @ thms
    val thms = map (fn thm => thm RS @{thm prec-specD}) thms

    val thin-prec-spec-rls = @{thms thin-rl[Pure.of prec-spec a b c for a b c]}

    val tac =
      forward-tac ctxt @{thms prec-specI}
      THEN' REPEAT-ALL-NEW (ematch-tac ctxt @{thms prec-spec-splitE})
      THEN' REPEAT o (dresolve-tac ctxt thms)
      THEN' REPEAT o (eresolve-tac ctxt thin-prec-spec-rls )
  in tac end
›
```

method-setup *prec-extract-eqs* = ‹*SIMPLE-METHOD-NOPARAM' prec-extract-eqs-tac*›
‹*Extract equalities from - |= - & - assumption, using precision rules*›

Combinator Rules

lemma *split-merge*: $[\![A \vee_A B \Longrightarrow_t X; X \vee_A C \Longrightarrow_t D]\!] \Longrightarrow (A \vee_A B \vee_A C \Longrightarrow_t D)$
proof −
  assume *a1*: $X \vee_A C \Longrightarrow_t D$
  assume $A \vee_A B \Longrightarrow_t X$
  then have $A \vee_A B \Longrightarrow_A D * true$
    using *a1* **by** (*meson ent-disjI1-direct ent-frame-fwd enttD entt-def-true*)
  then show *?thesis*
    using *a1* **by** (*metis* (*no-types*) *Assertions.ent-disjI2 ent-disjE enttD enttI semigroup.assoc sup.semigroup-axioms*)
qed

ML ‹
```
  fun prep-comb-rule thm = let
    fun mrg t = case Logic.strip-assums-concl t of
      @{mpat Trueprop (- ∨A - ∨A - ⟹t -)} => (@{thm split-merge},true)
    | @{mpat Trueprop (hn-refine - - ?G - -)} => (
        if not (is-Var (head-of G)) then (@{thm hn-refine-cons-post}, true)
        else (asm-rl,false)
      )
    | - => (asm-rl,false)
```

val inst = *Thm.prems-of thm* |> *map mrg*
in
if exists snd inst then
prep-comb-rule (*thm OF* (*map fst inst*))
else
thm |> *zero-var-indexes*
end
›

attribute-setup *sepref-prep-comb-rule* = ‹*Scan.succeed* (*Thm.rule-attribute* [] (*K prep-comb-rule*))›
‹*Preprocess combinator rule*: *Split merge*−*rules and add missing frame rules*›

end

Chapter 2

Basic Setup

This chapter contains the basic setup of the Sepref tool.

2.1 HOL Setup

theory *Sepref-HOL-Bindings*
imports *Sepref-Tool*
begin

2.1.1 Assertion Annotation

Annotate an assertion to a term. The term must then be refined with this assertion.

definition *ASSN-ANNOT* :: $('a \Rightarrow 'ai \Rightarrow assn) \Rightarrow 'a \Rightarrow 'a$ **where** [*simp*]: *ASSN-ANNOT* $A\ x \equiv x$
context fixes A :: $'a \Rightarrow 'ai \Rightarrow assn$ **begin**
 sepref-register *PR-CONST* (*ASSN-ANNOT* A)
 lemma [*def-pat-rules*]: *ASSN-ANNOT*\$$A \equiv$ *UNPROTECT* (*ASSN-ANNOT* A)
by *simp*
 lemma [*sepref-fr-rules*]: (*return o* $(\lambda x.\ x)$, *RETURN o PR-CONST* (*ASSN-ANNOT* A)) $\in A^d \rightarrow_a A$
  by *sepref-to-hoare sep-auto*
end

lemma *annotate-assn*: $x \equiv$ *ASSN-ANNOT* $A\ x$ **by** *simp*

2.1.2 Shortcuts

abbreviation (*input*) *nat-assn* $\equiv$ (*id-assn*::*nat* $\Rightarrow$ -)
abbreviation (*input*) *int-assn* $\equiv$ (*id-assn*::*int* $\Rightarrow$ -)
abbreviation (*input*) *bool-assn* $\equiv$ (*id-assn*::*bool* $\Rightarrow$ -)

2.1.3 Identity Relations

definition *IS-ID R* $\equiv$ *R=Id*
definition *IS-BELOW-ID R* $\equiv$ *R⊆Id*

lemma [*safe-constraint-rules*]:
IS-ID Id
IS-ID R1 $\Longrightarrow$ *IS-ID R2* $\Longrightarrow$ *IS-ID* (*R1* $\rightarrow$ *R2*)
IS-ID R $\Longrightarrow$ *IS-ID* (⟨*R*⟩*option-rel*)
IS-ID R $\Longrightarrow$ *IS-ID* (⟨*R*⟩*list-rel*)
IS-ID R1 $\Longrightarrow$ *IS-ID R2* $\Longrightarrow$ *IS-ID* (*R1* $\times_r$ *R2*)
IS-ID R1 $\Longrightarrow$ *IS-ID R2* $\Longrightarrow$ *IS-ID* (⟨*R1*,*R2*⟩*sum-rel*)
by (*auto simp*: *IS-ID-def*)

lemma [*safe-constraint-rules*]:
IS-BELOW-ID Id
IS-BELOW-ID R $\Longrightarrow$ *IS-BELOW-ID* (⟨*R*⟩*option-rel*)
IS-BELOW-ID R1 $\Longrightarrow$ *IS-BELOW-ID R2* $\Longrightarrow$ *IS-BELOW-ID* (*R1* $\times_r$ *R2*)
IS-BELOW-ID R1 $\Longrightarrow$ *IS-BELOW-ID R2* $\Longrightarrow$ *IS-BELOW-ID* (⟨*R1*,*R2*⟩*sum-rel*)
by (*auto simp*: *IS-ID-def IS-BELOW-ID-def option-rel-def sum-rel-def list-rel-def*)

lemma *IS-BELOW-ID-fun-rel-aux*: *R1⊇Id* $\Longrightarrow$ *IS-BELOW-ID R2* $\Longrightarrow$ *IS-BELOW-ID* (*R1* $\rightarrow$ *R2*)
by (*auto simp*: *IS-BELOW-ID-def dest*: *fun-relD*)

corollary *IS-BELOW-ID-fun-rel*[*safe-constraint-rules*]:
IS-ID R1 $\Longrightarrow$ *IS-BELOW-ID R2* $\Longrightarrow$ *IS-BELOW-ID* (*R1* $\rightarrow$ *R2*)
using *IS-BELOW-ID-fun-rel-aux*[*of Id R2*]
by (*auto simp*: *IS-ID-def*)

lemma *IS-BELOW-ID-list-rel*[*safe-constraint-rules*]:
IS-BELOW-ID R $\Longrightarrow$ *IS-BELOW-ID* (⟨*R*⟩*list-rel*)
unfolding *IS-BELOW-ID-def*
proof *safe*
fix *l l'*
assume *A*: *R⊆Id*
assume (*l*,*l'*)∈⟨*R*⟩*list-rel*
thus *l*=*l'*
apply *induction*
using *A* **by** *auto*
qed

lemma *IS-ID-imp-BELOW-ID*[*constraint-rules*]:
IS-ID R $\Longrightarrow$ *IS-BELOW-ID R*
by (*auto simp*: *IS-ID-def IS-BELOW-ID-def*)

2.1.4 Inverse Relation

lemma *inv-fun-rel-eq*[*simp*]: $(A \rightarrow B)^{-1} = A^{-1} \rightarrow B^{-1}$

by (*auto dest*: *fun-relD*)

lemma *inv-option-rel-eq*[*simp*]: $(\langle K\rangle \mathit{option\text{-}rel})^{-1} = \langle K^{-1}\rangle \mathit{option\text{-}rel}$
by (*auto simp*: *option-rel-def*)

lemma *inv-prod-rel-eq*[*simp*]: $(P \times_r Q)^{-1} = P^{-1} \times_r Q^{-1}$
by (*auto*)

lemma *inv-sum-rel-eq*[*simp*]: $(\langle P,Q\rangle \mathit{sum\text{-}rel})^{-1} = \langle P^{-1},Q^{-1}\rangle \mathit{sum\text{-}rel}$
by (*auto simp*: *sum-rel-def*)

lemma *inv-list-rel-eq*[*simp*]: $(\langle R\rangle \mathit{list\text{-}rel})^{-1} = \langle R^{-1}\rangle \mathit{list\text{-}rel}$
unfolding *list-rel-def*
apply *safe*
apply (*subst list.rel-flip*[*symmetric*])
apply (*simp add*: *conversep-iff* [*abs-def*])
apply (*subst list.rel-flip*[*symmetric*])
apply (*simp add*: *conversep-iff* [*abs-def*])
done

lemmas [*constraint-simps*] =
Relation.converse-Id
inv-fun-rel-eq
inv-option-rel-eq
inv-prod-rel-eq
inv-sum-rel-eq
inv-list-rel-eq

2.1.5 Single Valued and Total Relations

definition *IS-LEFT-UNIQUE R* ≡ *single-valued* (R^{-1})
definition *IS-LEFT-TOTAL R* ≡ *Domain R* = *UNIV*
definition *IS-RIGHT-TOTAL R* ≡ *Range R* = *UNIV*
abbreviation (*input*) *IS-RIGHT-UNIQUE* ≡ *single-valued*

lemmas *IS-RIGHT-UNIQUED* = *single-valuedD*
lemma *IS-LEFT-UNIQUED*: ⟦*IS-LEFT-UNIQUE r*; $(y, x) \in r$; $(z, x) \in r$⟧ $\Longrightarrow$ $y = z$
by (*auto simp*: *IS-LEFT-UNIQUE-def dest*: *single-valuedD*)

lemma *prop2p*:
IS-LEFT-UNIQUE R = *left-unique* (*rel2p R*)
IS-RIGHT-UNIQUE R = *right-unique* (*rel2p R*)
right-unique (*rel2p* (R^{-1})) = *left-unique* (*rel2p R*)
IS-LEFT-TOTAL R = *left-total* (*rel2p R*)
IS-RIGHT-TOTAL R = *right-total* (*rel2p R*)
by (*auto*
simp: *IS-LEFT-UNIQUE-def left-unique-def single-valued-def*
simp: *right-unique-def*

simp: *IS-LEFT-TOTAL-def left-total-def*
simp: *IS-RIGHT-TOTAL-def right-total-def*
simp: *rel2p-def*
)

lemma *p2prop*:
left-unique P = *IS-LEFT-UNIQUE* (*p2rel P*)
right-unique P = *IS-RIGHT-UNIQUE* (*p2rel P*)
left-total P = *IS-LEFT-TOTAL* (*p2rel P*)
right-total P = *IS-RIGHT-TOTAL* (*p2rel P*)
bi-unique P $\longleftrightarrow$ *left-unique P* $\wedge$ *right-unique P*
by (*auto*
simp: *IS-LEFT-UNIQUE-def left-unique-def single-valued-def*
simp: *right-unique-def bi-unique-alt-def*
simp: *IS-LEFT-TOTAL-def left-total-def*
simp: *IS-RIGHT-TOTAL-def right-total-def*
simp: *p2rel-def*
)

lemmas [*safe-constraint-rules*] =
single-valued-Id
prod-rel-sv
list-rel-sv
option-rel-sv
sum-rel-sv

lemma [*safe-constraint-rules*]:
IS-LEFT-UNIQUE Id
IS-LEFT-UNIQUE R1 $\Longrightarrow$ *IS-LEFT-UNIQUE R2* $\Longrightarrow$ *IS-LEFT-UNIQUE* ($R1 \times_r R2$)
IS-LEFT-UNIQUE R1 $\Longrightarrow$ *IS-LEFT-UNIQUE R2* $\Longrightarrow$ *IS-LEFT-UNIQUE* ($\langle R1,R2\rangle$*sum-rel*)
IS-LEFT-UNIQUE R $\Longrightarrow$ *IS-LEFT-UNIQUE* ($\langle R\rangle$*option-rel*)
IS-LEFT-UNIQUE R $\Longrightarrow$ *IS-LEFT-UNIQUE* ($\langle R\rangle$*list-rel*)
by (*auto simp*: *IS-LEFT-UNIQUE-def prod-rel-sv sum-rel-sv option-rel-sv list-rel-sv*)

lemma *IS-LEFT-TOTAL-alt*: *IS-LEFT-TOTAL R* $\longleftrightarrow$ ($\forall x.\ \exists y.\ (x,y)\in R$)
by (*auto simp*: *IS-LEFT-TOTAL-def*)

lemma *IS-RIGHT-TOTAL-alt*: *IS-RIGHT-TOTAL R* $\longleftrightarrow$ ($\forall x.\ \exists y.\ (y,x)\in R$)
by (*auto simp*: *IS-RIGHT-TOTAL-def*)

lemma [*safe-constraint-rules*]:
IS-LEFT-TOTAL Id
IS-LEFT-TOTAL R1 $\Longrightarrow$ *IS-LEFT-TOTAL R2* $\Longrightarrow$ *IS-LEFT-TOTAL* ($R1 \times_r R2$)
IS-LEFT-TOTAL R1 $\Longrightarrow$ *IS-LEFT-TOTAL R2* $\Longrightarrow$ *IS-LEFT-TOTAL* ($\langle R1,R2\rangle$*sum-rel*)
IS-LEFT-TOTAL R $\Longrightarrow$ *IS-LEFT-TOTAL* ($\langle R\rangle$*option-rel*)
apply (*auto simp*: *IS-LEFT-TOTAL-alt sum-rel-def option-rel-def list-rel-def*)
apply (*rename-tac x*; *case-tac x*; *auto*)
apply (*rename-tac x*; *case-tac x*; *auto*)
done

lemma [*safe-constraint-rules*]: *IS-LEFT-TOTAL R* $\Longrightarrow$ *IS-LEFT-TOTAL* ($\langle R\rangle$*list-rel*)
  unfolding *IS-LEFT-TOTAL-alt*
proof *safe*
  assume *A*: $\forall x.\exists y.\ (x,y)\in R$
  fix *l*
  show $\exists l'.\ (l,l')\in\langle R\rangle$*list-rel*
    apply (*induction l*)
    using *A*
    by (*auto simp*: *list-rel-split-right-iff*)
qed

lemma [*safe-constraint-rules*]:
  IS-RIGHT-TOTAL Id
  IS-RIGHT-TOTAL R1 $\Longrightarrow$ *IS-RIGHT-TOTAL R2* $\Longrightarrow$ *IS-RIGHT-TOTAL* (*R1* $\times_r$ *R2*)
  IS-RIGHT-TOTAL R1 $\Longrightarrow$ *IS-RIGHT-TOTAL R2* $\Longrightarrow$ *IS-RIGHT-TOTAL* ($\langle R1,R2\rangle$*sum-rel*)
  IS-RIGHT-TOTAL R $\Longrightarrow$ *IS-RIGHT-TOTAL* ($\langle R\rangle$*option-rel*)
  apply (*auto simp*: *IS-RIGHT-TOTAL-alt sum-rel-def option-rel-def*) []
  apply (*auto simp*: *IS-RIGHT-TOTAL-alt sum-rel-def option-rel-def*) []
  apply (*auto simp*: *IS-RIGHT-TOTAL-alt sum-rel-def option-rel-def*) []
  apply (*rename-tac x*; *case-tac x*; *auto*)
  apply (*clarsimp simp*: *IS-RIGHT-TOTAL-alt option-rel-def*)
  apply (*rename-tac x*; *case-tac x*; *auto*)
  done

lemma [*safe-constraint-rules*]: *IS-RIGHT-TOTAL R* $\Longrightarrow$ *IS-RIGHT-TOTAL* ($\langle R\rangle$*list-rel*)
  unfolding *IS-RIGHT-TOTAL-alt*
proof *safe*
  assume *A*: $\forall x.\exists y.\ (y,x)\in R$
  fix *l*
  show $\exists l'.\ (l',l)\in\langle R\rangle$*list-rel*
    apply (*induction l*)
    using *A*
    by (*auto simp*: *list-rel-split-left-iff*)
qed

lemma [*constraint-simps*]:
  IS-LEFT-TOTAL (R^{-1}) $\longleftrightarrow$ *IS-RIGHT-TOTAL R*
  IS-RIGHT-TOTAL (R^{-1}) $\longleftrightarrow$ *IS-LEFT-TOTAL R*
  IS-LEFT-UNIQUE (R^{-1}) $\longleftrightarrow$ *IS-RIGHT-UNIQUE R*
  IS-RIGHT-UNIQUE (R^{-1}) $\longleftrightarrow$ *IS-LEFT-UNIQUE R*
  by (*auto simp*: *IS-RIGHT-TOTAL-alt IS-LEFT-TOTAL-alt IS-LEFT-UNIQUE-def*)

lemma [*safe-constraint-rules*]:
  IS-RIGHT-UNIQUE A $\Longrightarrow$ *IS-RIGHT-TOTAL B* $\Longrightarrow$ *IS-RIGHT-TOTAL* ($A\rightarrow B$)
  IS-RIGHT-TOTAL A $\Longrightarrow$ *IS-RIGHT-UNIQUE B* $\Longrightarrow$ *IS-RIGHT-UNIQUE* ($A\rightarrow B$)
  IS-LEFT-UNIQUE A $\Longrightarrow$ *IS-LEFT-TOTAL B* $\Longrightarrow$ *IS-LEFT-TOTAL* ($A\rightarrow B$)
  IS-LEFT-TOTAL A $\Longrightarrow$ *IS-LEFT-UNIQUE B* $\Longrightarrow$ *IS-LEFT-UNIQUE* ($A\rightarrow B$)
  apply (*simp-all add*: *prop2p rel2p*)

apply (*blast intro*!: *transfer-raw*)+
done

lemma [*constraint-rules*]:
IS-BELOW-ID R $\Longrightarrow$ *IS-RIGHT-UNIQUE R*
IS-BELOW-ID R $\Longrightarrow$ *IS-LEFT-UNIQUE R*
IS-ID R $\Longrightarrow$ *IS-RIGHT-TOTAL R*
IS-ID R $\Longrightarrow$ *IS-LEFT-TOTAL R*
by (*auto simp*: *IS-BELOW-ID-def IS-ID-def IS-LEFT-UNIQUE-def IS-RIGHT-TOTAL-def IS-LEFT-TOTAL-def*
intro: *single-valuedI*)

thm *constraint-rules*

Additional Parametricity Lemmas

lemma *param-distinct*[*param*]: $[\![$*IS-LEFT-UNIQUE A*; *IS-RIGHT-UNIQUE A*$]\!] \Longrightarrow$ (*distinct*, *distinct*) $\in$ $\langle A\rangle$*list-rel* $\rightarrow$ *bool-rel*
apply (*fold rel2p-def*)
apply (*simp add*: *rel2p*)
apply (*rule distinct-transfer*)
apply (*simp add*: *p2prop*)
done

lemma *param-Image*[*param*]:
assumes *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A*
shows ((‘‘), (‘‘)) $\in$ $\langle A\times_r B\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle B\rangle$*set-rel*
apply (*clarsimp simp*: *set-rel-def*; *intro conjI*)
apply (*fastforce dest*: *IS-RIGHT-UNIQUED*[*OF assms*(*2*)])
apply (*fastforce dest*: *IS-LEFT-UNIQUED*[*OF assms*(*1*)])
done

lemma *pres-eq-iff-svb*: ((=),(=))$\in$*K*$\rightarrow$*K*$\rightarrow$*bool-rel* $\longleftrightarrow$ (*single-valued K* $\wedge$ *single-valued* (K^{-1}))
apply (*safe intro*!: *single-valuedI*)
apply (*metis* (*full-types*) *IdD fun-relD1*)
apply (*metis* (*full-types*) *IdD fun-relD1*)
by (*auto dest*: *single-valuedD*)

definition *IS-PRES-EQ R* $\equiv$ ((=), (=))$\in$*R*$\rightarrow$*R*$\rightarrow$*bool-rel*
lemma [*constraint-rules*]: $[\![$*single-valued R*; *single-valued* (R^{-1})$]\!] \Longrightarrow$ *IS-PRES-EQ R*
by (*simp add*: *pres-eq-iff-svb IS-PRES-EQ-def*)

2.1.6 Bounded Assertions

definition *b-rel R P* $\equiv$ *R* $\cap$ *UNIV*$\times$*Collect P*
definition *b-assn A P* $\equiv$ $\lambda x\ y.\ A\ x\ y * \uparrow(P\ x)$

lemma *b-assn-pure-conv*[*constraint-simps*]: *b-assn* (*pure R*) *P* = *pure* (*b-rel R P*)
by (*auto intro*!: *ext simp*: *b-rel-def b-assn-def pure-def*)
lemmas [*sepref-import-rewrite*, *sepref-frame-normrel-eqs*, *fcomp-norm-unfold*]
= *b-assn-pure-conv*[*symmetric*]

lemma *b-rel-nesting*[*simp*]:
b-rel (*b-rel R P1*) *P2* = *b-rel R* (λx. *P1 x* $\wedge$ *P2 x*)
by (*auto simp*: *b-rel-def*)
lemma *b-rel-triv*[*simp*]:
b-rel R (λ-. *True*) = *R*
by (*auto simp*: *b-rel-def*)
lemma *b-assn-nesting*[*simp*]:
b-assn (*b-assn A P1*) *P2* = *b-assn A* (λx. *P1 x* $\wedge$ *P2 x*)
by (*auto simp*: *b-assn-def pure-def intro*!: *ext*)
lemma *b-assn-triv*[*simp*]:
b-assn A (λ-. *True*) = *A*
by (*auto simp*: *b-assn-def pure-def intro*!: *ext*)

lemmas [*simp*,*constraint-simps*,*sepref-import-rewrite*, *sepref-frame-normrel-eqs*, *fcomp-norm-unfold*]
= *b-rel-nesting b-assn-nesting*

lemma *b-rel-simp*[*simp*]: $(x,y) \in$ *b-rel R P* $\longleftrightarrow$ $(x,y) \in R \wedge P\ y$
by (*auto simp*: *b-rel-def*)

lemma *b-assn-simp*[*simp*]: *b-assn A P x y* = $A\ x\ y * \uparrow(P\ x)$
by (*auto simp*: *b-assn-def*)

lemma *b-rel-Range*[*simp*]: *Range* (*b-rel R P*) = *Range R* $\cap$ *Collect P* **by** *auto*
lemma *b-assn-rdom*[*simp*]: *rdomp* (*b-assn R P*) *x* $\longleftrightarrow$ *rdomp R x* $\wedge$ *P x*
by (*auto simp*: *rdomp-def*)

lemma *b-rel-below-id*[*constraint-rules*]:
IS-BELOW-ID R $\Longrightarrow$ *IS-BELOW-ID* (*b-rel R P*)
by (*auto simp*: *IS-BELOW-ID-def*)

lemma *b-rel-left-unique*[*constraint-rules*]:
IS-LEFT-UNIQUE R $\Longrightarrow$ *IS-LEFT-UNIQUE* (*b-rel R P*)
by (*auto simp*: *IS-LEFT-UNIQUE-def single-valued-def*)

lemma *b-rel-right-unique*[*constraint-rules*]:
IS-RIGHT-UNIQUE R $\Longrightarrow$ *IS-RIGHT-UNIQUE* (*b-rel R P*)
by (*auto simp*: *single-valued-def*)

— Registered as safe rule, although may loose information in the odd case that purity depends condition.
lemma *b-assn-is-pure*[*safe-constraint-rules*]:
is-pure A $\Longrightarrow$ *is-pure* (*b-assn A P*)
by (*auto simp*: *is-pure-conv b-assn-pure-conv*)

— Most general form
lemma *b-assn-subtyping-match*[*sepref-frame-match-rules*]:
  assumes *hn-ctxt* (*b-assn* A P) x y $\Longrightarrow_t$ *hn-ctxt* A' x y
  assumes $[\![$*vassn-tag* (*hn-ctxt* A x y); *vassn-tag* (*hn-ctxt* A' x y); P $x]\!] \Longrightarrow P'$ x
  shows *hn-ctxt* (*b-assn* A P) x y $\Longrightarrow_t$ *hn-ctxt* (*b-assn* A' P') x y
  using *assms*
  unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
  by (*fastforce simp*: *vassn-tag-def mod-star-conv*)

— Simplified forms:
lemma *b-assn-subtyping-match-eqA*[*sepref-frame-match-rules*]:
  assumes $[\![$*vassn-tag* (*hn-ctxt* A x y); P $x]\!] \Longrightarrow P'$ x
  shows *hn-ctxt* (*b-assn* A P) x y $\Longrightarrow_t$ *hn-ctxt* (*b-assn* A P') x y
  apply (*rule b-assn-subtyping-match*)
  subgoal
    unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
    by (*fastforce simp*: *vassn-tag-def mod-star-conv*)
  subgoal
    using *assms* **.**
  done

lemma *b-assn-subtyping-match-tR*[*sepref-frame-match-rules*]:
  assumes $[\![P$ $x]\!] \Longrightarrow$ *hn-ctxt* A x y $\Longrightarrow_t$ *hn-ctxt* A' x y
  shows *hn-ctxt* (*b-assn* A P) x y $\Longrightarrow_t$ *hn-ctxt* A' x y
  using *assms*
  unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
  by (*fastforce simp*: *vassn-tag-def mod-star-conv*)

lemma *b-assn-subtyping-match-tL*[*sepref-frame-match-rules*]:
  assumes *hn-ctxt* A x y $\Longrightarrow_t$ *hn-ctxt* A' x y
  assumes $[\![$*vassn-tag* (*hn-ctxt* A x y)$]\!] \Longrightarrow P'$ x
  shows *hn-ctxt* A x y $\Longrightarrow_t$ *hn-ctxt* (*b-assn* A' P') x y
  using *assms*
  unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
  by (*fastforce simp*: *vassn-tag-def mod-star-conv*)

lemma *b-assn-subtyping-match-eqA-tR*[*sepref-frame-match-rules*]:
  hn-ctxt (*b-assn* A P) x y $\Longrightarrow_t$ *hn-ctxt* A x y
  unfolding *hn-ctxt-def b-assn-def*
  by (*sep-auto intro*!: *enttI*)

lemma *b-assn-subtyping-match-eqA-tL*[*sepref-frame-match-rules*]:
  assumes $[\![$*vassn-tag* (*hn-ctxt* A x y)$]\!] \Longrightarrow P'$ x
  shows *hn-ctxt* A x y $\Longrightarrow_t$ *hn-ctxt* (*b-assn* A P') x y
  using *assms*
  unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
  by (*fastforce simp*: *vassn-tag-def mod-star-conv*)

— General form
lemma *b-rel-subtyping-merge*[*sepref-frame-merge-rules*]:
assumes $hn\text{-}ctxt\ A\ x\ y \vee_A hn\text{-}ctxt\ A'\ x\ y \Longrightarrow_t hn\text{-}ctxt\ Am\ x\ y$
shows $hn\text{-}ctxt\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}ctxt\ (b\text{-}assn\ A'\ P')\ x\ y \Longrightarrow_t hn\text{-}ctxt\ (b\text{-}assn\ Am\ (\lambda x.\ P\ x \vee P'\ x))\ x\ y$
using *assms*
unfolding *hn-ctxt-def b-assn-def entailst-def entails-def*
by (*fastforce simp*: *vassn-tag-def*)

— Simplified forms
lemma *b-rel-subtyping-merge-eqA*[*sepref-frame-merge-rules*]:
shows $hn\text{-}ctxt\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}ctxt\ (b\text{-}assn\ A\ P')\ x\ y \Longrightarrow_t hn\text{-}ctxt\ (b\text{-}assn\ A\ (\lambda x.\ P\ x \vee P'\ x))\ x\ y$
apply (*rule b-rel-subtyping-merge*)
by *simp*

lemma *b-rel-subtyping-merge-tL*[*sepref-frame-merge-rules*]:
assumes $hn\text{-}ctxt\ A\ x\ y \vee_A hn\text{-}ctxt\ A'\ x\ y \Longrightarrow_t hn\text{-}ctxt\ Am\ x\ y$
shows $hn\text{-}ctxt\ A\ x\ y \vee_A hn\text{-}ctxt\ (b\text{-}assn\ A'\ P')\ x\ y \Longrightarrow_t hn\text{-}ctxt\ Am\ x\ y$
using *b-rel-subtyping-merge*[*of* $A\ x\ y\ A'\ Am\ \lambda\text{-}.\ True\ P'$, *simplified*] *assms* **.**

lemma *b-rel-subtyping-merge-tR*[*sepref-frame-merge-rules*]:
assumes $hn\text{-}ctxt\ A\ x\ y \vee_A hn\text{-}ctxt\ A'\ x\ y \Longrightarrow_t hn\text{-}ctxt\ Am\ x\ y$
shows $hn\text{-}ctxt\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}ctxt\ A'\ x\ y \Longrightarrow_t hn\text{-}ctxt\ Am\ x\ y$
using *b-rel-subtyping-merge*[*of* $A\ x\ y\ A'\ Am\ P\ \lambda\text{-}.\ True$, *simplified*] *assms* **.**

lemma *b-rel-subtyping-merge-eqA-tL*[*sepref-frame-merge-rules*]:
shows $hn\text{-}ctxt\ A\ x\ y \vee_A hn\text{-}ctxt\ (b\text{-}assn\ A\ P')\ x\ y \Longrightarrow_t hn\text{-}ctxt\ A\ x\ y$
using *b-rel-subtyping-merge-eqA*[*of* $A\ \lambda\text{-}.\ True\ x\ y\ P'$, *simplified*] **.**

lemma *b-rel-subtyping-merge-eqA-tR*[*sepref-frame-merge-rules*]:
shows $hn\text{-}ctxt\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}ctxt\ A\ x\ y \Longrightarrow_t hn\text{-}ctxt\ A\ x\ y$
using *b-rel-subtyping-merge-eqA*[*of* $A\ P\ x\ y\ \lambda\text{-}.\ True$, *simplified*] **.**

lemma *b-assn-invalid-merge1*: $hn\text{-}invalid\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}invalid\ (b\text{-}assn\ A\ P')\ x\ y$
$\Longrightarrow_t hn\text{-}invalid\ (b\text{-}assn\ A\ (\lambda x.\ P\ x \vee P'\ x))\ x\ y$
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)

lemma *b-assn-invalid-merge2*: $hn\text{-}invalid\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}invalid\ A\ x\ y$
$\Longrightarrow_t hn\text{-}invalid\ A\ x\ y$
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)
lemma *b-assn-invalid-merge3*: $hn\text{-}invalid\ A\ x\ y \vee_A hn\text{-}invalid\ (b\text{-}assn\ A\ P)\ x\ y$
$\Longrightarrow_t hn\text{-}invalid\ A\ x\ y$
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)

lemma *b-assn-invalid-merge4*: $hn\text{-}invalid\ (b\text{-}assn\ A\ P)\ x\ y \vee_A hn\text{-}ctxt\ (b\text{-}assn\ A$

P') x y
$\Longrightarrow_t$ *hn-invalid* (*b-assn* A ($\lambda x.$ P x $\vee$ P' x)) x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)
lemma *b-assn-invalid-merge5*: *hn-ctxt* (*b-assn* A P') x y $\vee_A$ *hn-invalid* (*b-assn* A P) x y
$\Longrightarrow_t$ *hn-invalid* (*b-assn* A ($\lambda x.$ P x $\vee$ P' x)) x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)

lemma *b-assn-invalid-merge6*: *hn-invalid* (*b-assn* A P) x y $\vee_A$ *hn-ctxt* A x y
$\Longrightarrow_t$ *hn-invalid* A x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)
lemma *b-assn-invalid-merge7*: *hn-ctxt* A x y $\vee_A$ *hn-invalid* (*b-assn* A P) x y
$\Longrightarrow_t$ *hn-invalid* A x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)

lemma *b-assn-invalid-merge8*: *hn-ctxt* (*b-assn* A P) x y $\vee_A$ *hn-invalid* A x y
$\Longrightarrow_t$ *hn-invalid* A x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)
lemma *b-assn-invalid-merge9*: *hn-invalid* A x y $\vee_A$ *hn-ctxt* (*b-assn* A P) x y
$\Longrightarrow_t$ *hn-invalid* A x y
by (*sep-auto simp*: *hn-ctxt-def invalid-assn-def entailst-def*)

lemmas *b-assn-invalid-merge*[*sepref-frame-merge-rules*] =
b-assn-invalid-merge1
b-assn-invalid-merge2
b-assn-invalid-merge3
b-assn-invalid-merge4
b-assn-invalid-merge5
b-assn-invalid-merge6
b-assn-invalid-merge7
b-assn-invalid-merge8
b-assn-invalid-merge9

abbreviation *nbn-rel* :: *nat* $\Rightarrow$ (*nat* $\times$ *nat*) *set*
— Natural numbers with upper bound.
where *nbn-rel* $n \equiv$ *b-rel nat-rel* (λx::*nat*. $x<n$)

abbreviation *nbn-assn* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ *assn*
— Natural numbers with upper bound.
where *nbn-assn* $n \equiv$ *b-assn nat-assn* (λx::*nat*. $x<n$)

2.1.7 Tool Setup

lemmas [*sepref-relprops*] =
sepref-relpropI[*of IS-LEFT-UNIQUE*]
sepref-relpropI[*of IS-RIGHT-UNIQUE*]
sepref-relpropI[*of IS-LEFT-TOTAL*]
sepref-relpropI[*of IS-RIGHT-TOTAL*]
sepref-relpropI[*of is-pure*]
sepref-relpropI[*of IS-PURE* Φ **for** Φ]
sepref-relpropI[*of IS-ID*]
sepref-relpropI[*of IS-BELOW-ID*]

lemma [*sepref-relprops-simps*]:
CONSTRAINT (*IS-PURE IS-ID*) *A* $\Longrightarrow$ *CONSTRAINT* (*IS-PURE IS-BELOW-ID*) *A*
CONSTRAINT (*IS-PURE IS-ID*) *A* $\Longrightarrow$ *CONSTRAINT* (*IS-PURE IS-LEFT-TOTAL*) *A*
CONSTRAINT (*IS-PURE IS-ID*) *A* $\Longrightarrow$ *CONSTRAINT* (*IS-PURE IS-RIGHT-TOTAL*) *A*
CONSTRAINT (*IS-PURE IS-BELOW-ID*) *A* $\Longrightarrow$ *CONSTRAINT* (*IS-PURE IS-LEFT-UNIQUE*) *A*
CONSTRAINT (*IS-PURE IS-BELOW-ID*) *A* $\Longrightarrow$ *CONSTRAINT* (*IS-PURE IS-RIGHT-UNIQUE*) *A*
by (*auto*
simp: *IS-ID-def IS-BELOW-ID-def IS-PURE-def IS-LEFT-UNIQUE-def*
simp: *IS-LEFT-TOTAL-def IS-RIGHT-TOTAL-def*
simp: *single-valued-below-Id*)

declare *True-implies-equals*[*sepref-relprops-simps*]

lemma [*sepref-relprops-transform*]: *single-valued* (R^{-1}) = *IS-LEFT-UNIQUE R*
by (*auto simp*: *IS-LEFT-UNIQUE-def*)

2.1.8 HOL Combinators

lemma *hn-if*[*sepref-comb-rules*]:
assumes *P*: $\Gamma \Longrightarrow_t \Gamma 1 * \textit{hn-val bool-rel } a\ a'$
assumes *RT*: $a \Longrightarrow \textit{hn-refine } (\Gamma 1 * \textit{hn-val bool-rel } a\ a')\ b'\ \Gamma 2b\ R\ b$
assumes *RE*: $\neg a \Longrightarrow \textit{hn-refine } (\Gamma 1 * \textit{hn-val bool-rel } a\ a')\ c'\ \Gamma 2c\ R\ c$
assumes *IMP*: $\textit{TERM If} \Longrightarrow \Gamma 2b \vee_A \Gamma 2c \Longrightarrow_t \Gamma'$
shows $\textit{hn-refine } \Gamma\ (\textit{if } a' \textit{ then } b' \textit{ else } c')\ \Gamma'\ R\ (\textit{If}\$a\$b\$c)$
using *P RT RE IMP*[*OF TERMI*]
unfolding *APP-def PROTECT2-def*
by (*rule hnr-If*)

lemmas [*sepref-opt-simps*] = *if-True if-False*

lemma *hn-let*[*sepref-comb-rules*]:

assumes P: $\Gamma \Longrightarrow_t \Gamma 1 * \textit{hn-ctxt } R\ v\ v'$
assumes R: $\bigwedge x\ x'.\ x{=}v \Longrightarrow \textit{hn-refine } (\Gamma 1 * \textit{hn-ctxt } R\ x\ x')\ (f'\ x')$
$(\Gamma'\ x\ x')\ R2\ (f\ x)$
assumes F: $\bigwedge x\ x'.\ \Gamma'\ x\ x' \Longrightarrow_t \Gamma 2 * \textit{hn-ctxt } R'\ x\ x'$
shows
$\textit{hn-refine } \Gamma\ (\textit{Let } v'\ f')\ (\Gamma 2 * \textit{hn-ctxt } R'\ v\ v')\ R2\ (\textit{Let}\$v\$(\lambda_2 x.\ f\ x))$
apply (*rule hn-refine-cons*[*OF P - F entt-refl*])
apply (*simp*)
apply (*rule R*)
by *simp*

2.1.9 Basic HOL types

lemma *hnr-default*[*sepref-import-param*]: (*default*,*default*)∈*Id* **by** *simp*

lemma *unit-hnr*[*sepref-import-param*]: ((),())∈*unit-rel* **by** *auto*

lemmas [*sepref-import-param*] =
param-bool
param-nat1
param-int

lemmas [*id-rules*] =
itypeI[*Pure.of 0 TYPE* (*nat*)]
itypeI[*Pure.of 0 TYPE* (*int*)]
itypeI[*Pure.of 1 TYPE* (*nat*)]
itypeI[*Pure.of 1 TYPE* (*int*)]
itypeI[*Pure.of numeral TYPE* (*num* ⇒ *nat*)]
itypeI[*Pure.of numeral TYPE* (*num* ⇒ *int*)]
itype-self[*of num.One*]
itype-self[*of num.Bit0*]
itype-self[*of num.Bit1*]

lemma *param-min-nat*[*param*,*sepref-import-param*]: (*min*,*min*)∈*nat-rel* → *nat-rel* → *nat-rel* **by** *auto*
lemma *param-max-nat*[*param*,*sepref-import-param*]: (*max*,*max*)∈*nat-rel* → *nat-rel* → *nat-rel* **by** *auto*

lemma *param-min-int*[*param*,*sepref-import-param*]: (*min*,*min*)∈*int-rel* → *int-rel* → *int-rel* **by** *auto*
lemma *param-max-int*[*param*,*sepref-import-param*]: (*max*,*max*)∈*int-rel* → *int-rel* → *int-rel* **by** *auto*

lemma *uminus-hnr*[*sepref-import-param*]: (*uminus*,*uminus*)∈*int-rel* → *int-rel* **by** *auto*

lemma *nat-param*[*param*,*sepref-import-param*]: (*nat*,*nat*) ∈ *int-rel* → *nat-rel* **by** *auto*
lemma *int-param*[*param*,*sepref-import-param*]: (*int*,*int*) ∈ *nat-rel* → *int-rel* **by**

auto

2.1.10 Product

lemmas [*sepref-import-rewrite*, *sepref-frame-normrel-eqs*, *fcomp-norm-unfold*] = *prod-assn-pure-conv*[*symmetric*]

lemma *prod-assn-precise*[*constraint-rules*]:
precise P1 $\Longrightarrow$ *precise P2* $\Longrightarrow$ *precise* (*prod-assn P1 P2*)
apply *rule*
apply (*clarsimp simp*: *prod-assn-def star-assoc*)
apply *safe*
apply (*erule* (*1*) *prec-frame*) **apply** *frame-inference+*
apply (*erule* (*1*) *prec-frame*) **apply** *frame-inference+*
done

lemma
precise P1 $\Longrightarrow$ *precise P2* $\Longrightarrow$ *precise* (*prod-assn P1 P2*) — Original proof
apply *rule*
apply (*clarsimp simp*: *prod-assn-def*)
proof (*rule conjI*)
fix $F\ F'\ h\ as\ a\ b\ a'\ b'\ ap\ bp$
assume *P1*: *precise P1* **and** *P2*: *precise P2*
assume *F*: $(h, as) \models P1\ a\ ap * P2\ b\ bp * F \wedge_A P1\ a'\ ap * P2\ b'\ bp * F'$

from *F* **have** $(h, as) \models P1\ a\ ap * (P2\ b\ bp * F) \wedge_A P1\ a'\ ap * (P2\ b'\ bp * F')$
by (*simp only*: *mult.assoc*)
with *preciseD*[*OF P1*] **show** $a=a'$ **.**
from *F* **have** $(h, as) \models P2\ b\ bp * (P1\ a\ ap * F) \wedge_A P2\ b'\ bp * (P1\ a'\ ap * F')$
by (*simp only*: *mult.assoc*[**where** *'a=assn*] *mult.commute*[**where** *'a=assn*] *mult.left-commute*[**where** *'a=assn*])
with *preciseD*[*OF P2*] **show** $b=b'$ **.**
qed

lemma *intf-of-prod-assn*[*intf-of-assn*]:
assumes *intf-of-assn A TYPE*(*'a*) *intf-of-assn B TYPE*(*'b*)
shows *intf-of-assn* (*prod-assn A B*) *TYPE*(*'a* $*$ *'b*)
by *simp*

lemma *pure-prod*[*constraint-rules*]:
assumes *P1*: *is-pure P1* **and** *P2*: *is-pure P2*
shows *is-pure* (*prod-assn P1 P2*)
proof −
from *P1* **obtain** *P1'* **where** *P1'*: $\bigwedge x\ x'.\ P1\ x\ x' = \uparrow(P1'\ x\ x')$
using *is-pureE* **by** *blast*
from *P2* **obtain** *P2'* **where** *P2'*: $\bigwedge x\ x'.\ P2\ x\ x' = \uparrow(P2'\ x\ x')$
using *is-pureE* **by** *blast*

show *?thesis* **proof**
fix $x\ x'$
show *prod-assn P1 P2* $x\ x' =$
$\uparrow$ (*case* $(x,\ x')$ *of* $((a1,\ a2),\ c1,\ c2) \Rightarrow P1'\ a1\ c1 \wedge P2'\ a2\ c2$)
unfolding *prod-assn-def*
apply (*simp add*: *P1′ P2′ split*: *prod.split*)
done
qed
qed

lemma *prod-frame-match*[*sepref-frame-match-rules*]:
assumes *hn-ctxt A* (*fst x*) (*fst y*) $\Longrightarrow_t$ *hn-ctxt A′* (*fst x*) (*fst y*)
assumes *hn-ctxt B* (*snd x*) (*snd y*) $\Longrightarrow_t$ *hn-ctxt B′* (*snd x*) (*snd y*)
shows *hn-ctxt* (*prod-assn A B*) *x y* $\Longrightarrow_t$ *hn-ctxt* (*prod-assn A′ B′*) *x y*
apply (*cases x*; *cases y*; *simp*)
apply (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
using *assms* **apply** (*auto simp*: *hn-ctxt-def*)
done

lemma *prod-frame-merge*[*sepref-frame-merge-rules*]:
assumes *hn-ctxt A* (*fst x*) (*fst y*) $\vee_A$ *hn-ctxt A′* (*fst x*) (*fst y*) $\Longrightarrow_t$ *hn-ctxt Am* (*fst x*) (*fst y*)
assumes *hn-ctxt B* (*snd x*) (*snd y*) $\vee_A$ *hn-ctxt B′* (*snd x*) (*snd y*) $\Longrightarrow_t$ *hn-ctxt Bm* (*snd x*) (*snd y*)
shows *hn-ctxt* (*prod-assn A B*) *x y* $\vee_A$ *hn-ctxt* (*prod-assn A′ B′*) *x y* $\Longrightarrow_t$ *hn-ctxt* (*prod-assn Am Bm*) *x y*
by (*blast intro*: *entt-disjE prod-frame-match*
entt-disjD1[*OF assms*(*1*)] *entt-disjD2*[*OF assms*(*1*)]
entt-disjD1[*OF assms*(*2*)] *entt-disjD2*[*OF assms*(*2*)])

lemma *entt-invalid-prod*: *hn-invalid* (*prod-assn A B*) *p p′* $\Longrightarrow_t$ *hn-ctxt* (*prod-assn* (*invalid-assn A*) (*invalid-assn B*)) *p p′*
apply (*simp add*: *hn-ctxt-def invalid-assn-def*[*abs-def*])
apply (*rule enttI*)
apply *clarsimp*
apply (*cases p*; *cases p′*; *auto simp*: *mod-star-conv pure-def*)
done

lemmas *invalid-prod-merge*[*sepref-frame-merge-rules*] = *gen-merge-cons*[*OF entt-invalid-prod*]

lemma *prod-assn-ctxt*: *prod-assn A1 A2 x y* = *z* $\Longrightarrow$ *hn-ctxt* (*prod-assn A1 A2*) *x y* = *z*
by (*simp add*: *hn-ctxt-def*)

lemma *hn-case-prod′*[*sepref-prep-comb-rule*,*sepref-comb-rules*]:
assumes *FR*: $\Gamma \Longrightarrow_t$*hn-ctxt* (*prod-assn P1 P2*) *p′ p* $*$ $\Gamma 1$
assumes *Pair*: $\bigwedge a1\ a2\ a1'\ a2'.\ [\![p'=(a1',a2')]\!]$
$\Longrightarrow$ *hn-refine* (*hn-ctxt P1 a1′ a1* $*$ *hn-ctxt P2 a2′ a2* $*$ $\Gamma 1$ $*$ *hn-invalid*

```
(prod-assn P1 P2) p′ p) (f a1 a2)
        (hn-ctxt P1′ a1′ a1 ∗ hn-ctxt P2′ a2′ a2 ∗ hn-ctxt XX1 p′ p ∗ Γ1′) R (f′
a1′ a2′)
  shows hn-refine Γ (case-prod f p) (hn-ctxt (prod-assn P1′ P2′) p′ p ∗ Γ1′)
    R (case-prod$(λ2a b. f′ a b)$p′) (is ?G Γ)
    apply1 (rule hn-refine-cons-pre[OF FR])
    apply1 extract-hnr-invalids
   apply1 (cases p; cases p′; simp add: prod-assn-pair-conv[THEN prod-assn-ctxt])
    apply (rule hn-refine-cons[OF - Pair - entt-refl])
    applyS (simp add: hn-ctxt-def)
    applyS simp
    applyS (simp add: hn-ctxt-def entt-fr-refl entt-fr-drop)
    done

lemma hn-case-prod-old:
  assumes P: Γ⟹tΓ1 ∗ hn-ctxt (prod-assn P1 P2) p′ p
  assumes R: ⋀a1 a2 a1′ a2′. ⟦p′=(a1′,a2′)⟧
     ⟹ hn-refine (Γ1 ∗ hn-ctxt P1 a1′ a1 ∗ hn-ctxt P2 a2′ a2 ∗ hn-invalid
(prod-assn P1 P2) p′ p) (f a1 a2)
        (Γh a1 a1′ a2 a2′) R (f′ a1′ a2′)
  assumes M: ⋀a1 a1′ a2 a2′. Γh a1 a1′ a2 a2′
   ⟹t Γ′ ∗ hn-ctxt P1′ a1′ a1 ∗ hn-ctxt P2′ a2′ a2 ∗ hn-ctxt Pxx p′ p
  shows hn-refine Γ (case-prod f p) (Γ′ ∗ hn-ctxt (prod-assn P1′ P2′) p′ p)
    R (case-prod$(λ2a b. f′ a b)$p′)
  apply1 (cases p; cases p′; simp)
  apply1 (rule hn-refine-cons-pre[OF P])
  apply (rule hn-refine-preI)
  apply (simp add: hn-ctxt-def assn-aci)
  apply (rule hn-refine-cons[OF - R])
  apply1 (rule enttI)
  applyS (sep-auto simp add: hn-ctxt-def invalid-assn-def mod-star-conv)

  applyS simp
  apply1 (rule entt-trans[OF M])
  applyS (sep-auto intro!: enttI simp: hn-ctxt-def)

  applyS simp
  done

lemma hn-Pair[sepref-fr-rules]: hn-refine
  (hn-ctxt P1 x1 x1′ ∗ hn-ctxt P2 x2 x2′)
  (return (x1′,x2′))
  (hn-invalid P1 x1 x1′ ∗ hn-invalid P2 x2 x2′)
  (prod-assn P1 P2)
  (RETURN$(Pair$x1$x2))
  unfolding hn-refine-def
  apply (sep-auto simp: hn-ctxt-def prod-assn-def)
  apply (rule ent-frame-fwd[OF invalidate-clone′[of P1]], frame-inference)
  apply (rule ent-frame-fwd[OF invalidate-clone′[of P2]], frame-inference)
```

apply *sep-auto*
done

lemma *fst-hnr*[*sepref-fr-rules*]: (*return o fst*,*RETURN o fst*) $\in$ (*prod-assn A B*)^d $\rightarrow_a$ *A*
by *sepref-to-hoare sep-auto*
lemma *snd-hnr*[*sepref-fr-rules*]: (*return o snd*,*RETURN o snd*) $\in$ (*prod-assn A B*)^d $\rightarrow_a$ *B*
by *sepref-to-hoare sep-auto*

lemmas [*constraint-simps*] = *prod-assn-pure-conv*
lemmas [*sepref-import-param*] = *param-prod-swap*

lemma *rdomp-prodD*[*dest!*]: *rdomp* (*prod-assn A B*) (*a*,*b*) $\Longrightarrow$ *rdomp A a* $\wedge$ *rdomp B b*
unfolding *rdomp-def prod-assn-def*
by (*sep-auto simp*: *mod-star-conv*)

2.1.11 Option

fun *option-assn* :: (*'a* $\Rightarrow$ *'c* $\Rightarrow$ *assn*) $\Rightarrow$ *'a option* $\Rightarrow$ *'c option* $\Rightarrow$ *assn* **where**
option-assn P None None = *emp*
| *option-assn P* (*Some a*) (*Some c*) = *P a c*
| *option-assn - - -* = *false*

lemma *option-assn-simps*[*simp*]:
option-assn P None v' = $\uparrow$(*v'*=*None*)
option-assn P v None = $\uparrow$(*v*=*None*)
apply (*cases v'*, *simp-all*)
apply (*cases v*, *simp-all*)
done

lemma *option-assn-alt-def*: *option-assn R a b* =
(*case* (*a*,*b*) *of* (*Some x*, *Some y*) $\Rightarrow$ *R x y*
| (*None*,*None*) $\Rightarrow$ *emp*
| - $\Rightarrow$ *false*)
by (*auto split*: *option.split*)

lemma *option-assn-pure-conv*[*constraint-simps*]: *option-assn* (*pure R*) = *pure* ($\langle R\rangle$*option-rel*)
apply (*intro ext*)
apply (*rename-tac a c*)
apply (*case-tac* (*pure R*,*a*,*c*) *rule*: *option-assn.cases*)
by (*auto simp*: *pure-def*)

lemmas [*sepref-import-rewrite*, *sepref-frame-normrel-eqs*, *fcomp-norm-unfold*] = *option-assn-pure-conv*[*symmetric*]

lemma *hr-comp-option-conv*[*simp*, *fcomp-norm-unfold*]:
hr-comp (*option-assn* R) (⟨R'⟩*option-rel*)
= *option-assn* (*hr-comp* R R')
unfolding *hr-comp-def*[*abs-def*]
apply (*intro ext ent-iffI*)
apply *solve-entails*
apply (*case-tac* (R,b,c) *rule*: *option-assn.cases*)
apply *clarsimp-all*

apply (*sep-auto simp*: *option-assn-alt-def split*: *option.splits*)
apply (*clarsimp simp*: *option-assn-alt-def split*: *option.splits*; *safe*)
apply (*sep-auto split*: *option.splits*)
apply (*intro ent-ex-preI*)
apply (*rule ent-ex-postI*)
apply (*sep-auto split*: *option.splits*)
done

lemma *option-assn-precise*[*safe-constraint-rules*]:
assumes *precise* P
shows *precise* (*option-assn* P)
proof
fix a a' p h F F'
assume A: $h \models$ *option-assn* P a $p * F \wedge_A$ *option-assn* P a' $p * F'$
thus a=a' **proof** (*cases* (P,a,p) *rule*: *option-assn.cases*)
case (*2* - *av pv*) **hence** [*simp*]: a=*Some av* p=*Some pv* **by** *simp-all*

from A **obtain** av' **where** [*simp*]: a'=*Some* av' **by** (*cases* a', *simp-all*)

from A **have** $h \models P$ *av pv* $* F \wedge_A P$ av' *pv* $* F'$ **by** *simp*
with ‹*precise* P› **have** *av*=av' **by** (*rule preciseD*)
thus *?thesis* **by** *simp*
qed *simp-all*
qed

lemma *pure-option*[*safe-constraint-rules*]:
assumes P: *is-pure* P
shows *is-pure* (*option-assn* P)
proof −
from P **obtain** P' **where** P': $\bigwedge x\ x'.\ P\ x\ x' = \uparrow(P'\ x\ x')$
using *is-pureE* **by** *blast*

show *?thesis* **proof**
fix x x'
show *option-assn* P x x' =
↑ (*case* (x, x') *of*
(*None*,*None*) ⇒ *True* | (*Some* v, *Some* v') ⇒ P' v v' | - ⇒ *False*
)
apply (*simp add*: P' *split*: *prod.split option.split*)

done
qed
qed

lemma *hn-ctxt-option*: *option-assn* $A\ x\ y = z \Longrightarrow$ *hn-ctxt* (*option-assn* A) $x\ y = z$
by (*simp add*: *hn-ctxt-def*)

lemma *hn-case-option*[*sepref-prep-comb-rule*, *sepref-comb-rules*]:
fixes $p\ p'\ P$
defines [*simp*]: *INVE* $\equiv$ *hn-invalid* (*option-assn* P) $p\ p'$
assumes *FR*: $\Gamma \Longrightarrow_t$ *hn-ctxt* (*option-assn* P) $p\ p' * F$
assumes *Rn*: p=*None* $\Longrightarrow$ *hn-refine* (*hn-ctxt* (*option-assn* P) $p\ p' * F$) *f1′* (*hn-ctxt XX1* $p\ p' * \Gamma 1'$) R *f1*
assumes *Rs*: $\bigwedge x\ x'$. $[\![$ p=*Some* x; p'=*Some* x' $]\!] \Longrightarrow$
hn-refine (*hn-ctxt* $P\ x\ x'$ $*$ *INVE* $* F$) (*f2′* x') (*hn-ctxt* $P'\ x\ x'$ $*$ *hn-ctxt XX2* $p\ p' * \Gamma 2'$) R (*f2* x)
assumes *MERGE1*: $\Gamma 1' \vee_A \Gamma 2' \Longrightarrow_t \Gamma'$
shows *hn-refine* Γ (*case-option f1′ f2′* p') (*hn-ctxt* (*option-assn* P') $p\ p' * \Gamma'$) R (*case-option*\$*f1*\$($\lambda_2 x$. *f2* x)\$$p$)
apply (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *extract-hnr-invalids*
apply (*cases p*; *cases p′*; *simp add*: *option-assn.simps*[*THEN hn-ctxt-option*])
subgoal
apply (*rule hn-refine-cons*[*OF - Rn - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)

apply (*subst mult.commute*, *rule entt-fr-drop*)
apply (*rule entt-trans*[*OF - MERGE1*])
apply (*simp add*: *ent-disjI1′ ent-disjI2′*)
done

subgoal
apply (*rule hn-refine-cons*[*OF - Rs - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
applyS (*simp add*: *hn-ctxt-def*)
apply1 (*rule entt-trans*[*OF - MERGE1*])
applyS (*simp add*: *hn-ctxt-def*)
done
done

lemma *hn-None*[*sepref-fr-rules*]:
hn-refine emp (*return None*) *emp* (*option-assn* P) (*RETURN*\$*None*)
by *rule sep-auto*

lemma *hn-Some*[*sepref-fr-rules*]: *hn-refine*
(*hn-ctxt* $P\ v\ v'$)
(*return* (*Some* v'))

($hn\text{-}invalid\ P\ v\ v'$)
($option\text{-}assn\ P$)
($RETURN\$(Some\$v)$)
by *rule* (*sep-auto simp*: *hn-ctxt-def invalidate-clone'*)

definition *imp-option-eq eq a b* ≡ *case* (*a*,*b*) *of*
(*None*,*None*) ⇒ *return True*
| (*Some a*, *Some b*) ⇒ *eq a b*
| - ⇒ *return False*

lemma *option-assn-eq*[*sepref-comb-rules*]:
fixes *a b* :: *'a option*
assumes *F1*: $\Gamma \Longrightarrow_t$ *hn-ctxt* (*option-assn P*) *a a'* $*$ *hn-ctxt* (*option-assn P*) *b b'* $*$ $\Gamma 1$
assumes *EQ*: $\bigwedge va\ va'\ vb\ vb'$. *hn-refine*
(*hn-ctxt P va va'* $*$ *hn-ctxt P vb vb'* $*$ $\Gamma 1$)
($eq'\ va'\ vb'$)
($\Gamma'\ va\ va'\ vb\ vb'$)
bool-assn
($RETURN\$((=)\ \$va\$vb)$)
assumes *F2*:
$\bigwedge va\ va'\ vb\ vb'$.
$\Gamma'\ va\ va'\ vb\ vb' \Longrightarrow_t$ *hn-ctxt P va va'* $*$ *hn-ctxt P vb vb'* $*$ $\Gamma 1$
shows *hn-refine*
Γ
(*imp-option-eq eq' a' b'*)
(*hn-ctxt* (*option-assn P*) *a a'* $*$ *hn-ctxt* (*option-assn P*) *b b'* $*$ $\Gamma 1$)
bool-assn
($RETURN\$((=)\ \$a\$b)$)
apply (*rule hn-refine-cons-pre*[*OF F1*])
unfolding *imp-option-eq-def*
apply *rule*
apply (*simp split*: *option.split add*: *hn-ctxt-def*, *intro impI conjI*)

apply (*sep-auto split*: *option.split simp*: *hn-ctxt-def pure-def*)
apply (*cases a*, (*sep-auto split*: *option.split simp*: *hn-ctxt-def pure-def*)+)[]
apply (*cases a*, (*sep-auto split*: *option.split simp*: *hn-ctxt-def pure-def*)+)[]
apply (*cases b*, (*sep-auto split*: *option.split simp*: *hn-ctxt-def pure-def*)+)[]
apply (*rule cons-post-rule*)
apply (*rule hn-refineD*[*OF EQ*[*unfolded hn-ctxt-def*]])
apply *simp*
apply (*rule ent-frame-fwd*[*OF F2*[*THEN enttD*,*unfolded hn-ctxt-def*]])
apply (*fr-rot 2*)
apply (*fr-rot-rhs 1*)
apply (*rule fr-refl*)
apply (*rule ent-refl*)
apply (*sep-auto simp*: *pure-def*)
done

lemma [*pat-rules*]:
(=) \$*a*\$*None* ≡ *is-None*\$*a*
(=) \$*None*\$*a* ≡ *is-None*\$*a*
apply (*rule eq-reflection*, *simp split*: *option.split*)+
done

lemma *hn-is-None*[*sepref-fr-rules*]: *hn-refine*
(*hn-ctxt* (*option-assn P*) *a a′*)
(*return* (*is-None a′*))
(*hn-ctxt* (*option-assn P*) *a a′*)
bool-assn
(*RETURN*\$(*is-None*\$*a*))
apply *rule*
apply (*sep-auto split*: *option.split simp*: *hn-ctxt-def pure-def*)
done

lemma (**in** −) *sepref-the-complete*[*sepref-fr-rules*]:
assumes *x*≠*None*
shows *hn-refine*
(*hn-ctxt* (*option-assn R*) *x xi*)
(*return* (*the xi*))
(*hn-invalid* (*option-assn R*) *x xi*)
(*R*)
(*RETURN*\$(*the*\$*x*))
using *assms*
apply (*cases x*)
apply *simp*
apply (*cases xi*)
apply (*simp add*: *hn-ctxt-def*)
apply *rule*
apply (*sep-auto simp*: *hn-ctxt-def invalidate-clone′ vassn-tagI invalid-assn-const*)
done

lemma (**in** −) *sepref-the-id*:
assumes *CONSTRAINT* (*IS-PURE IS-ID*) *R*
shows *hn-refine*
(*hn-ctxt* (*option-assn R*) *x xi*)
(*return* (*the xi*))
(*hn-ctxt* (*option-assn R*) *x xi*)
(*R*)
(*RETURN*\$(*the*\$*x*))
using *assms*
apply (*clarsimp simp*: *IS-PURE-def IS-ID-def hn-ctxt-def is-pure-conv*)
apply (*cases x*)
apply *simp*
apply (*cases xi*)
apply (*simp add*: *hn-ctxt-def invalid-assn-def*)

apply *rule* **apply** (*sep-auto simp*: *pure-def*)
apply *rule* **apply** (*sep-auto*)
apply (*simp add*: *option-assn-pure-conv*)
apply *rule* **apply** (*sep-auto simp*: *pure-def invalid-assn-def*)
done

2.1.12 Lists

fun *list-assn* :: ($'a \Rightarrow 'c \Rightarrow assn$) $\Rightarrow$ $'a$ *list* $\Rightarrow$ $'c$ *list* $\Rightarrow$ *assn* **where**
list-assn P [] [] = *emp*
| *list-assn P* ($a\#as$) ($c\#cs$) = *P a c* $*$ *list-assn P as cs*
| *list-assn* - - - = *false*

lemma *list-assn-aux-simps*[*simp*]:
list-assn P [] l' = ($\uparrow$(l'=[]))
list-assn P l [] = ($\uparrow$(l=[]))
unfolding *hn-ctxt-def*
apply (*cases* l')
apply *simp*
apply *simp*
apply (*cases* l)
apply *simp*
apply *simp*
done

lemma *list-assn-aux-append*[*simp*]:
length l1=*length l1'* $\Longrightarrow$
list-assn P (*l1*@*l2*) (*l1'*@*l2'*)
= *list-assn P l1 l1'* $*$ *list-assn P l2 l2'*
apply (*induct rule*: *list-induct2*)
apply *simp*
apply (*simp add*: *star-assoc*)
done

lemma *list-assn-aux-ineq-len*: *length l* $\neq$ *length li* $\Longrightarrow$ *list-assn A l li* = *false*
proof (*induction l arbitrary*: *li*)
case (*Cons x l li*) **thus** *?case* **by** (*cases li*; *auto*)
qed *simp*

lemma *list-assn-aux-append2*[*simp*]:
assumes *length l2*=*length l2'*
shows *list-assn P* (*l1*@*l2*) (*l1'*@*l2'*)
= *list-assn P l1 l1'* $*$ *list-assn P l2 l2'*
apply (*cases length l1* = *length l1'*)
apply (*erule list-assn-aux-append*)
apply (*simp add*: *list-assn-aux-ineq-len assms*)
done

lemma *list-assn-pure-conv*[*constraint-simps*]: *list-assn* (*pure R*) = *pure* ($\langle R\rangle$*list-rel*)

proof (*intro ext*)
  fix *l li*
  show *list-assn* (*pure R*) *l li* = *pure* (⟨*R*⟩*list-rel*) *l li*
    apply (*induction pure R l li rule*: *list-assn.induct*)
    by (*auto simp*: *pure-def*)
qed

lemmas [*sepref-import-rewrite*, *sepref-frame-normrel-eqs*, *fcomp-norm-unfold*] = *list-assn-pure-conv*[*symmetric*]

lemma *list-assn-simps*[*simp*]:
  hn-ctxt (*list-assn P*) [] *l'* = (↑(*l'*=[]))
  hn-ctxt (*list-assn P*) *l* [] = (↑(*l*=[]))
  hn-ctxt (*list-assn P*) [] [] = *emp*
  hn-ctxt (*list-assn P*) (*a#as*) (*c#cs*) = *hn-ctxt P a c* ∗ *hn-ctxt* (*list-assn P*) *as cs*
  hn-ctxt (*list-assn P*) (*a#as*) [] = *false*
  hn-ctxt (*list-assn P*) [] (*c#cs*) = *false*
  unfolding *hn-ctxt-def*
  apply (*cases l'*)
  apply *simp*
  apply *simp*
  apply (*cases l*)
  apply *simp*
  apply *simp*
  apply *simp-all*
  done

lemma *list-assn-precise*[*constraint-rules*]: *precise P* ⟹ *precise* (*list-assn P*)
proof
  fix *l1 l2 l h F1 F2*
  assume *P*: *precise P*
  assume *h*⊨*list-assn P l1 l* ∗ *F1* $\wedge_A$ *list-assn P l2 l* ∗ *F2*
  thus *l1*=*l2*
  proof (*induct l arbitrary*: *l1 l2 F1 F2*)
    case *Nil* **thus** *?case* **by** *simp*
  next
    case (*Cons a ls*)
    from *Cons* **obtain** *a1 ls1* **where** [*simp*]: *l1*=*a1#ls1*
      by (*cases l1*, *simp*)
    from *Cons* **obtain** *a2 ls2* **where** [*simp*]: *l2*=*a2#ls2*
      by (*cases l2*, *simp*)

    from *Cons.prems* **have** *M*:
      h ⊨ *P a1 a* ∗ *list-assn P ls1 ls* ∗ *F1*
        $\wedge_A$ *P a2 a* ∗ *list-assn P ls2 ls* ∗ *F2* **by** *simp*
    have *a1*=*a2*
      apply (*rule preciseD*[*OF P*, **where** *a*=*a1* **and** *a'*=*a2* **and** *p*=*a*
        and *F*= *list-assn P ls1 ls* ∗ *F1*

and F'=*list-assn* P *ls2* *ls* $*$ *F2*
])
using M
by (*simp add*: *star-assoc*)

moreover have *ls1*=*ls2*
apply (*rule Cons.hyps*[**where** *?F1.0*=P *a1* a $*$ *F1* **and** *?F2.0*=P *a2* a $*$ *F2*])
using M
by (*simp only*: *star-aci*)
ultimately show *?case* **by** *simp*
qed
qed
lemma *list-assn-pure*[*constraint-rules*]:
assumes P: *is-pure* P
shows *is-pure* (*list-assn* P)
proof −
from P **obtain** P' **where** *P-eq*: $\bigwedge x\ x'.\ P\ x\ x' = \uparrow(P'\ x\ x')$
by (*rule is-pureE*) *blast*

{
fix $l\ l'$
have *list-assn* $P\ l\ l' = \uparrow$(*list-all2* $P'\ l\ l'$)
by (*induct* $P{\equiv}P\ l\ l'$ *rule*: *list-assn.induct*)
(*simp-all add*: *P-eq*)
} thus *?thesis* **by** *rule*
qed

lemma *list-assn-mono*:
$[\![\bigwedge x\ x'.\ P\ x\ x' \Longrightarrow_A P'\ x\ x']\!] \Longrightarrow$ *list-assn* $P\ l\ l' \Longrightarrow_A$ *list-assn* $P'\ l\ l'$
unfolding *hn-ctxt-def*
apply (*induct* $P\ l\ l'$ *rule*: *list-assn.induct*)
by (*auto intro*: *ent-star-mono*)

lemma *list-assn-monot*:
$[\![\bigwedge x\ x'.\ P\ x\ x' \Longrightarrow_t P'\ x\ x']\!] \Longrightarrow$ *list-assn* $P\ l\ l' \Longrightarrow_t$ *list-assn* $P'\ l\ l'$
unfolding *hn-ctxt-def*
apply (*induct* $P\ l\ l'$ *rule*: *list-assn.induct*)
by (*auto intro*: *entt-star-mono*)

lemma *list-match-cong*[*sepref-frame-match-rules*]:
$[\![\bigwedge x\ x'.\ [\![x{\in}set\ l;\ x'{\in}set\ l']\!] \Longrightarrow$ *hn-ctxt* $A\ x\ x' \Longrightarrow_t$ *hn-ctxt* $A'\ x\ x'\ ]\!] \Longrightarrow$ *hn-ctxt* (*list-assn* A) $l\ l' \Longrightarrow_t$ *hn-ctxt* (*list-assn* A') $l\ l'$
unfolding *hn-ctxt-def*
by (*induct* $A\ l\ l'$ *rule*: *list-assn.induct*) (*simp-all add*: *entt-star-mono*)

lemma *list-merge-cong*[*sepref-frame-merge-rules*]:
assumes $\bigwedge x\ x'.\ [\![x{\in}set\ l;\ x'{\in}set\ l']\!] \Longrightarrow$ *hn-ctxt* $A\ x\ x' \vee_A$ *hn-ctxt* $A'\ x\ x' \Longrightarrow_t$ *hn-ctxt* $Am\ x\ x'$
shows *hn-ctxt* (*list-assn* A) $l\ l' \vee_A$ *hn-ctxt* (*list-assn* A') $l\ l' \Longrightarrow_t$ *hn-ctxt* (*list-assn*

Am) *l l'*
apply (*blast intro*: *entt-disjE list-match-cong entt-disjD1*[*OF assms*] *entt-disjD2*[*OF assms*])
done

lemma *invalid-list-split*:
invalid-assn (*list-assn A*) (*x#xs*) (*y#ys*) $\Longrightarrow_t$ *invalid-assn A x y* $*$ *invalid-assn* (*list-assn A*) *xs ys*
by (*fastforce simp*: *invalid-assn-def intro*!: *enttI simp*: *mod-star-conv*)

lemma *entt-invalid-list*: *hn-invalid* (*list-assn A*) *l l'* $\Longrightarrow_t$ *hn-ctxt* (*list-assn* (*invalid-assn A*)) *l l'*
apply (*induct A l l' rule*: *list-assn.induct*)
applyS *simp*

subgoal
apply1 (*simp add*: *hn-ctxt-def cong del*: *invalid-assn-cong*)
apply1 (*rule entt-trans*[*OF invalid-list-split*])
apply (*rule entt-star-mono*)
applyS *simp*

apply (*rule entt-trans*)
applyS *assumption*
applyS *simp*
done

applyS (*simp add*: *hn-ctxt-def invalid-assn-def*)
applyS (*simp add*: *hn-ctxt-def invalid-assn-def*)
done

lemmas *invalid-list-merge*[*sepref-frame-merge-rules*] = *gen-merge-cons*[*OF entt-invalid-list*]

lemma *list-assn-comp*[*fcomp-norm-unfold*]: *hr-comp* (*list-assn A*) ($\langle B\rangle$*list-rel*) = *list-assn* (*hr-comp A B*)
proof (*intro ext*)
{ fix *x l y m*
have *hr-comp* (*list-assn A*) ($\langle B\rangle$*list-rel*) (*x # l*) (*y # m*) =
hr-comp A B x y $*$ *hr-comp* (*list-assn A*) ($\langle B\rangle$*list-rel*) *l m*
by (*sep-auto*
simp: *hr-comp-def list-rel-split-left-iff*
intro!: *ent-ex-preI ent-iffI*)
} note *aux* = *this*

fix *l li*
show *hr-comp* (*list-assn A*) ($\langle B\rangle$*list-rel*) *l li* = *list-assn* (*hr-comp A B*) *l li*
apply (*induction l arbitrary*: *li*; *case-tac li*; *intro ent-iffI*)
apply (*sep-auto simp*: *hr-comp-def*; *fail*)+
by (*simp-all add*: *aux*)

qed

lemma *hn-ctxt-eq*: $A\ x\ y = z \Longrightarrow$ *hn-ctxt* $A\ x\ y = z$ **by** (*simp add*: *hn-ctxt-def*)

lemmas *hn-ctxt-list* = *hn-ctxt-eq*[*of list-assn A* **for** *A*]

lemma *hn-case-list*[*sepref-prep-comb-rule*, *sepref-comb-rules*]:
fixes $p\ p'\ P$
defines [*simp*]: *INVE* $\equiv$ *hn-invalid* (*list-assn P*) $p\ p'$
assumes *FR*: $\Gamma \Longrightarrow_t$ *hn-ctxt* (*list-assn P*) $p\ p' * F$
assumes *Rn*: $p=[] \Longrightarrow$ *hn-refine* (*hn-ctxt* (*list-assn P*) $p\ p' * F$) *f1'* (*hn-ctxt XX1* $p\ p' * \Gamma 1'$) *R f1*
assumes *Rs*: $\bigwedge x\ l\ x'\ l'.$ $[\![\ p=x\#l;\ p'=x'\#l'\ ]\!] \Longrightarrow$
hn-refine (*hn-ctxt* $P\ x\ x' *$ *hn-ctxt* (*list-assn P*) $l\ l' *$ *INVE* $* F$) (*f2'* $x'\ l'$) (*hn-ctxt P1'* $x\ x' *$ *hn-ctxt* (*list-assn P2'*) $l\ l' *$ *hn-ctxt XX2* $p\ p' * \Gamma 2'$) *R* (*f2* $x\ l$)
assumes *MERGE1*[*unfolded hn-ctxt-def*]: $\bigwedge x\ x'.$ *hn-ctxt P1'* $x\ x' \vee_A$ *hn-ctxt P2'* $x\ x' \Longrightarrow_t$ *hn-ctxt* $P'\ x\ x'$
assumes *MERGE2*: $\Gamma 1' \vee_A \Gamma 2' \Longrightarrow_t \Gamma'$
shows *hn-refine* Γ (*case-list f1' f2'* p') (*hn-ctxt* (*list-assn* P') $p\ p' * \Gamma'$) *R* (*case-list*\$*f1*\$($\lambda_2 x\ l.$ *f2* $x\ l$)\$$p$)
apply (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *extract-hnr-invalids*
apply (*cases p*; *cases p'*; *simp add*: *list-assn.simps*[*THEN hn-ctxt-list*])
subgoal
apply (*rule hn-refine-cons*[*OF - Rn - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)

apply (*subst mult.commute*, *rule entt-fr-drop*)
apply (*rule entt-trans*[*OF - MERGE2*])
apply (*simp add*: *ent-disjI1' ent-disjI2'*)
done

subgoal
apply (*rule hn-refine-cons*[*OF - Rs - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
apply (*rule entt-star-mono*)

apply1 (*simp add*: *hn-ctxt-def*)
apply1 (*rule entt-trans*[*OF - MERGE1*])
applyS (*simp*)

apply1 (*simp add*: *hn-ctxt-def*)
apply (*rule list-assn-monot*)
apply1 (*rule entt-trans*[*OF - MERGE1*])
applyS (*simp*)

apply1 (*rule entt-trans*[*OF - MERGE2*])

applyS (*simp*)
done
done

lemma *hn-Nil*[*sepref-fr-rules*]:
hn-refine emp (*return* []) *emp* (*list-assn P*) (*RETURN*$[])
unfolding *hn-refine-def*
by *sep-auto*

lemma *hn-Cons*[*sepref-fr-rules*]: *hn-refine* (*hn-ctxt P x x'* ∗ *hn-ctxt* (*list-assn P*) *xs xs'*)
(*return* (*x'#xs'*)) (*hn-invalid P x x'* ∗ *hn-invalid* (*list-assn P*) *xs xs'*) (*list-assn P*)
(*RETURN*$((#) $*x*$*xs*))
unfolding *hn-refine-def*
apply (*sep-auto simp*: *hn-ctxt-def*)
apply (*rule ent-frame-fwd*[*OF invalidate-clone'*[*of P*]], *frame-inference*)
apply (*rule ent-frame-fwd*[*OF invalidate-clone'*[*of list-assn P*]], *frame-inference*)
apply *solve-entails*
done

lemma *list-assn-aux-len*:
list-assn P l l' = *list-assn P l l'* ∗ ↑(*length l* = *length l'*)
apply (*induct P*≡*P l l' rule*: *list-assn.induct*)
apply *simp-all*
subgoal for *a as c cs*
by (*erule-tac t*=*list-assn P as cs* **in** *subst*[*OF sym*]) *simp*
done

lemma *list-assn-aux-eqlen-simp*:
vassn-tag (*list-assn P l l'*) ⟹ *length l'* = *length l*
h ⊨ (*list-assn P l l'*) ⟹ *length l'* = *length l*
apply (*subst* (*asm*) *list-assn-aux-len*; *auto simp*: *vassn-tag-def*)+
done

lemma *hn-append*[*sepref-fr-rules*]: *hn-refine* (*hn-ctxt* (*list-assn P*) *l1 l1'* ∗ *hn-ctxt* (*list-assn P*) *l2 l2'*)
(*return* (*l1'@l2'*)) (*hn-invalid* (*list-assn P*) *l1 l1'* ∗ *hn-invalid* (*list-assn P*) *l2 l2'*) (*list-assn P*)
(*RETURN*$((@) $*l1*$*l2*))
apply *rule*
apply (*sep-auto simp*: *hn-ctxt-def*)
apply (*subst list-assn-aux-len*)
apply (*sep-auto*)
apply (*rule ent-frame-fwd*[*OF invalidate-clone'*[*of list-assn P*]], *frame-inference*)
apply (*rule ent-frame-fwd*[*OF invalidate-clone'*[*of list-assn P*]], *frame-inference*)
apply *solve-entails*
done

lemma *list-assn-aux-cons-conv1*:
  list-assn R $(a\#l)$ $m = (\exists_A b\ m'.\ R\ a\ b * \textit{list-assn}\ R\ l\ m' * \uparrow(m=b\#m'))$
  apply (*cases m*)
  apply *sep-auto*
  apply (*sep-auto intro*!: *ent-iffI*)
  done
lemma *list-assn-aux-cons-conv2*:
  list-assn R l $(b\#m) = (\exists_A a\ l'.\ R\ a\ b * \textit{list-assn}\ R\ l'\ m * \uparrow(l=a\#l'))$
  apply (*cases l*)
  apply *sep-auto*
  apply (*sep-auto intro*!: *ent-iffI*)
  done
lemmas *list-assn-aux-cons-conv* = *list-assn-aux-cons-conv1 list-assn-aux-cons-conv2*

lemma *list-assn-aux-append-conv1*:
  list-assn R $(l1@l2)$ $m = (\exists_A m1\ m2.\ \textit{list-assn}\ R\ l1\ m1 * \textit{list-assn}\ R\ l2\ m2 * \uparrow(m=m1@m2))$
  apply (*induction l1 arbitrary*: *m*)
  apply (*sep-auto intro*!: *ent-iffI*)
  apply (*sep-auto intro*!: *ent-iffI simp*: *list-assn-aux-cons-conv*)
  done
lemma *list-assn-aux-append-conv2*:
  list-assn R l $(m1@m2) = (\exists_A l1\ l2.\ \textit{list-assn}\ R\ l1\ m1 * \textit{list-assn}\ R\ l2\ m2 * \uparrow(l=l1@l2))$
  apply (*induction m1 arbitrary*: *l*)
  apply (*sep-auto intro*!: *ent-iffI*)
  apply (*sep-auto intro*!: *ent-iffI simp*: *list-assn-aux-cons-conv*)
  done
lemmas *list-assn-aux-append-conv* = *list-assn-aux-append-conv1 list-assn-aux-append-conv2*

declare *param-upt*[*sepref-import-param*]

2.1.13 Sum-Type

fun *sum-assn* :: $('ai \Rightarrow 'a \Rightarrow \textit{assn}) \Rightarrow ('bi \Rightarrow 'b \Rightarrow \textit{assn}) \Rightarrow ('ai+'bi) \Rightarrow ('a+'b) \Rightarrow \textit{assn}$ **where**
  sum-assn A B (*Inl ai*) (*Inl a*) = *A ai a*
| *sum-assn A B* (*Inr bi*) (*Inr b*) = *B bi b*
| *sum-assn A B* - - = *false*

notation *sum-assn* (**infixr** ‹$+_a$› *67*)

lemma *sum-assn-pure*[*safe-constraint-rules*]: ⟦*is-pure A*; *is-pure B*⟧ $\Longrightarrow$ *is-pure* (*sum-assn A B*)
  apply (*auto simp*: *is-pure-iff-pure-assn*)
  apply (*rename-tac x x'*)
  apply (*case-tac x*; *case-tac x'*; *simp add*: *pure-def*)
  done

lemma *sum-assn-id*[*simp*]: *sum-assn id-assn id-assn* = *id-assn*
  apply (*intro ext*)
  subgoal for *x y* **by** (*cases x*; *cases y*; *simp add*: *pure-def*)
  done

lemma *sum-assn-pure-conv*[*simp*]: *sum-assn* (*pure A*) (*pure B*) = *pure* (⟨*A*,*B*⟩*sum-rel*)
  apply (*intro ext*)
  subgoal for *a b* **by** (*cases a*; *cases b*; *auto simp*: *pure-def*)
  done

lemma *sum-match-cong*[*sepref-frame-match-rules*]:
  ⟦
    $\bigwedge$*x y*. ⟦*e* = *Inl x*; *e*′=*Inl y*⟧ ⟹ *hn-ctxt A x y* $\Longrightarrow_t$ *hn-ctxt A*′ *x y*;
    $\bigwedge$*x y*. ⟦*e* = *Inr x*; *e*′=*Inr y*⟧ ⟹ *hn-ctxt B x y* $\Longrightarrow_t$ *hn-ctxt B*′ *x y*
  ⟧ ⟹ *hn-ctxt* (*sum-assn A B*) *e e*′ $\Longrightarrow_t$ *hn-ctxt* (*sum-assn A*′ *B*′) *e e*′
  by (*cases e*; *cases e*′; *simp add*: *hn-ctxt-def entt-star-mono*)

lemma *enum-merge-cong*[*sepref-frame-merge-rules*]:
  assumes $\bigwedge$*x y*. ⟦*e*=*Inl x*; *e*′=*Inl y*⟧ ⟹ *hn-ctxt A x y* $\vee_A$ *hn-ctxt A*′ *x y* $\Longrightarrow_t$ *hn-ctxt Am x y*
  assumes $\bigwedge$*x y*. ⟦*e*=*Inr x*; *e*′=*Inr y*⟧ ⟹ *hn-ctxt B x y* $\vee_A$ *hn-ctxt B*′ *x y* $\Longrightarrow_t$ *hn-ctxt Bm x y*
  shows *hn-ctxt* (*sum-assn A B*) *e e*′ $\vee_A$ *hn-ctxt* (*sum-assn A*′ *B*′) *e e*′ $\Longrightarrow_t$ *hn-ctxt* (*sum-assn Am Bm*) *e e*′
  apply (*rule entt-disjE*)
  apply (*rule sum-match-cong*)
  apply (*rule entt-disjD1*[*OF assms*(*1*)]; *simp*)
  apply (*rule entt-disjD1*[*OF assms*(*2*)]; *simp*)

  apply (*rule sum-match-cong*)
  apply (*rule entt-disjD2*[*OF assms*(*1*)]; *simp*)
  apply (*rule entt-disjD2*[*OF assms*(*2*)]; *simp*)
  done

lemma *entt-invalid-sum*: *hn-invalid* (*sum-assn A B*) *e e*′ $\Longrightarrow_t$ *hn-ctxt* (*sum-assn* (*invalid-assn A*) (*invalid-assn B*)) *e e*′
  apply (*simp add*: *hn-ctxt-def invalid-assn-def*[*abs-def*])
  apply (*rule enttI*)
  apply *clarsimp*
  apply (*cases e*; *cases e*′; *auto simp*: *mod-star-conv pure-def*)
  done

lemmas *invalid-sum-merge*[*sepref-frame-merge-rules*] = *gen-merge-cons*[*OF entt-invalid-sum*]

sepref-register *Inr Inl*

lemma [*sepref-fr-rules*]: (*return o Inl*,*RETURN o Inl*) ∈ $A^d \rightarrow_a$ *sum-assn A B*

by *sepref-to-hoare sep-auto*
lemma [*sepref-fr-rules*]: $(return\ o\ Inr, RETURN\ o\ Inr) \in B^d \rightarrow_a sum\text{-}assn\ A\ B$
by *sepref-to-hoare sep-auto*

sepref-register *case-sum*

In the monadify phase, this eta-expands to make visible all required arguments

lemma [*sepref-monadify-arity*]: $case\text{-}sum \equiv \lambda_2 f1\ f2\ x.\ SP\ case\text{-}sum\$(\lambda_2 x.\ f1\$x)\$(\lambda_2 x.\ f2\$x)\$x$
by *simp*

This determines an evaluation order for the first-order operands

lemma [*sepref-monadify-comb*]: $case\text{-}sum\$f1\$f2\$x \equiv (\gg=)\ \$(EVAL\$x)\$(\lambda_2 x.\ SP\ case\text{-}sum\$f1\$f2\$x)$ **by** *simp*

This enables translation of the case-distinction in a non-monadic context.

lemma [*sepref-monadify-comb*]: $EVAL\$(case\text{-}sum\$(\lambda_2 x.\ f1\ x)\$(\lambda_2 x.\ f2\ x)\$x)$
$\equiv (\gg=)\ \$(EVAL\$x)\$(\lambda_2 x.\ SP\ case\text{-}sum\$(\lambda_2 x.\ EVAL\ \$\ f1\ x)\$(\lambda_2 x.\ EVAL\ \$\ f2\ x)\$x)$
apply (*rule eq-reflection*)
by (*simp split*: *sum.splits*)

Auxiliary lemma, to lift simp-rule over *hn-ctxt*

lemma *sum-assn-ctxt*: $sum\text{-}assn\ A\ B\ x\ y = z \Longrightarrow hn\text{-}ctxt\ (sum\text{-}assn\ A\ B)\ x\ y = z$
by (*simp add*: *hn-ctxt-def*)

The cases lemma first extracts the refinement for the datatype from the precondition. Next, it generate proof obligations to refine the functions for every case. Finally the postconditions of the refinement are merged.

Note that we handle the destructed values separately, to allow reconstruction of the original datatype after the case-expression.

Moreover, we provide (invalidated) versions of the original compound value to the cases, which allows access to pure compound values from inside the case.

lemma *sum-cases-hnr*:
fixes $A\ B\ e\ e'$
defines [*simp*]: $INVe \equiv hn\text{-}invalid\ (sum\text{-}assn\ A\ B)\ e\ e'$
assumes FR: $\Gamma \Longrightarrow_t hn\text{-}ctxt\ (sum\text{-}assn\ A\ B)\ e\ e' * F$
assumes $E1$: $\bigwedge x1\ x1a.\ [\![e = Inl\ x1;\ e' = Inl\ x1a]\!] \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ A\ x1\ x1a * INVe * F)\ (f1'\ x1a)\ (hn\text{-}ctxt\ A'\ x1\ x1a * hn\text{-}ctxt\ XX1\ e\ e' * \Gamma 1')\ R\ (f1\ x1)$
assumes $E2$: $\bigwedge x2\ x2a.\ [\![e = Inr\ x2;\ e' = Inr\ x2a]\!] \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ B\ x2\ x2a * INVe * F)\ (f2'\ x2a)\ (hn\text{-}ctxt\ B'\ x2\ x2a * hn\text{-}ctxt\ XX2\ e\ e' * \Gamma 2')\ R\ (f2\ x2)$
assumes $MERGE$[*unfolded hn-ctxt-def*]: $\Gamma 1' \vee_A \Gamma 2' \Longrightarrow_t \Gamma'$
shows $hn\text{-}refine\ \Gamma\ (case\text{-}sum\ f1'\ f2'\ e')\ (hn\text{-}ctxt\ (sum\text{-}assn\ A'\ B')\ e\ e' * \Gamma')\ R\ (case\text{-}sum\$(\lambda_2 x.\ f1\ x)\$(\lambda_2 x.\ f2\ x)\$e)$

apply (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *extract-hnr-invalids*
apply (*cases e*; *cases e'*; *simp add*: *sum-assn.simps*[*THEN sum-assn-ctxt*])
subgoal
apply (*rule hn-refine-cons*[*OF - E1 - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*) — Match precondition for case, get *enum-assn* from assumption generated by *extract-hnr-invalids*
apply (*rule entt-star-mono*) — Split postcondition into pairs for compounds and frame, drop *hn-ctxt XX*
apply1 (*rule entt-fr-drop*)
applyS (*simp add*: *hn-ctxt-def entt-disjI1' entt-disjI2'*)
apply1 (*rule entt-trans*[*OF - MERGE*])
applyS (*simp add*: *entt-disjI1' entt-disjI2'*)
done
subgoal
apply (*rule hn-refine-cons*[*OF - E2 - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
applyS (*simp add*: *hn-ctxt-def entt-disjI1' entt-disjI2'*)
apply1 (*rule entt-trans*[*OF - MERGE*])
applyS (*simp add*: *entt-disjI1' entt-disjI2'*)
done
done

After some more preprocessing (adding extra frame-rules for non-atomic postconditions, and splitting the merge-terms into binary merges), this rule can be registered

lemmas [*sepref-comb-rules*] = *sum-cases-hnr*[*sepref-prep-comb-rule*]

sepref-register *isl projl projr*
lemma *isl-hnr*[*sepref-fr-rules*]: (*return o isl*,*RETURN o isl*) $\in$ (*sum-assn A B*)^k $\rightarrow_a$ *bool-assn*
apply *sepref-to-hoare*
subgoal for *a b* **by** (*cases a*; *cases b*; *sep-auto*)
done

lemma *projl-hnr*[*sepref-fr-rules*]: (*return o projl*,*RETURN o projl*) $\in$ [*isl*]$_a$ (*sum-assn A B*)^d $\rightarrow$ *A*
apply *sepref-to-hoare*
subgoal for *a b* **by** (*cases a*; *cases b*; *sep-auto*)
done

lemma *projr-hnr*[*sepref-fr-rules*]: (*return o projr*,*RETURN o projr*) $\in$ [*Not o isl*]$_a$ (*sum-assn A B*)^d $\rightarrow$ *B*
apply *sepref-to-hoare*
subgoal for *a b* **by** (*cases a*; *cases b*; *sep-auto*)
done

2.1.14 String Literals

sepref-register *PR-CONST String.empty-literal*

lemma *empty-literal-hnr* [*sepref-import-param*]:
  (*String.empty-literal*, *PR-CONST String.empty-literal*) $\in$ *Id*
  by *simp*

lemma *empty-literal-pat* [*def-pat-rules*]:
  String.empty-literal $\equiv$ *UNPROTECT String.empty-literal*
  by *simp*

context
  fixes *b0 b1 b2 b3 b4 b5 b6* :: *bool*
  and *s* :: *String.literal*
begin

sepref-register *PR-CONST* (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*)

lemma *Literal-hnr* [*sepref-import-param*]:
  (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*,
    PR-CONST (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*)) $\in$ *Id*
  by *simp*

end

lemma *Literal-pat* [*def-pat-rules*]:
  String.Literal \$ *b0* \$ *b1* \$ *b2* \$ *b3* \$ *b4* \$ *b5* \$ *b6* \$ *s* $\equiv$
    UNPROTECT (*String.Literal* \$ *b0* \$ *b1* \$ *b2* \$ *b3* \$ *b4* \$ *b5* \$ *b6* \$ *s*)
  by *simp*

end

2.2 Setup for Foreach Combinator

theory *Sepref-Foreach*
imports *Sepref-HOL-Bindings Lib/Pf-Add HOL−Library.Rewrite*
begin

2.2.1 Foreach Loops

Monadic Version of Foreach

In a first step, we define a version of foreach where the continuation condition is also monadic, and show that it is equal to the standard version for continuation conditions of the form $\lambda x.\ RETURN\ (c\ x)$

definition *FOREACH-inv xs* Φ *s* $\equiv$
  case s of $(it, \sigma) \Rightarrow \exists\, xs'.\ xs = xs'\ @\ it \wedge \Phi\ (set\ it)\ \sigma$

definition *monadic-FOREACH R Φ S c f σ0* ≡ *do* {
  ASSERT (*finite S*);
  xs0 ← *it-to-sorted-list R S*;
  (-,*σ*) ← *RECT* (*λW* (*xs*,*σ*). *do* {
    ASSERT (*FOREACH-inv xs0* Φ (*xs*,*σ*));
    if xs≠[] *then do* {
      b ← *c σ*;
      if b then
        FOREACH-body f (*xs*,*σ*) ≫= *W*
      else
        RETURN (*xs*,*σ*)
    } *else RETURN* (*xs*,*σ*)
  }) (*xs0*,*σ0*);
  RETURN σ
}

lemma *FOREACH-oci-to-monadic*:
  FOREACHoci R Φ S c f σ0 = *monadic-FOREACH R Φ S* (*λσ*. *RETURN* (*c σ*)) *f σ0*
  unfolding *FOREACHoci-def monadic-FOREACH-def WHILEIT-def WHILEI-body-def*
  unfolding *it-to-sorted-list-def FOREACH-cond-def FOREACH-inv-def*
  apply *simp*
  apply (*fo-rule arg-cong*[*THEN cong*] | *rule refl ext*)+
  apply (*simp split*: *prod.split*)
  apply (*rule refl*)+
  done

Next, we define a characterization w.r.t. *nfoldli*

definition *monadic-nfoldli l c f s* ≡ *RECT* (*λD* (*l*,*s*). *case l of*
    [] ⇒ *RETURN s*
  | *x*#*ls* ⇒ *do* {
      b ← *c s*;
      if b then do { *s'*←*f x s*; *D* (*ls*,*s'*)} *else RETURN s*
    }
) (*l*,*s*)

lemma *monadic-nfoldli-eq*:
  monadic-nfoldli l c f s = (
    case l of
      [] ⇒ *RETURN s*
    | *x*#*ls* ⇒ *do* {
        b←*c s*;
        if b then f x s ≫= *monadic-nfoldli ls c f else RETURN s*
      }
)
  apply (*subst monadic-nfoldli-def*)
  apply (*subst RECT-unfold*)
  apply (*tagged-solver*)
  apply (*subst monadic-nfoldli-def*[*symmetric*])

apply *simp*
done

lemma *monadic-nfoldli-simp*[*simp*]:
monadic-nfoldli [] *c f s* = *RETURN s*
monadic-nfoldli (*x*#*ls*) *c f s* = *do* {
b←*c s*;
if b then f x s ≫= *monadic-nfoldli ls c f else RETURN s*
}
apply (*subst monadic-nfoldli-eq*, *simp*)
apply (*subst monadic-nfoldli-eq*, *simp*)
done

lemma *nfoldli-to-monadic*:
nfoldli l c f = *monadic-nfoldli l* (λ*x*. *RETURN* (*c x*)) *f*
apply (*induct l*)
apply *auto*
done

definition *nfoldli-alt l c f s* ≡ *RECT* (λ*D* (*l*,*s*). *case l of*
[] ⇒ *RETURN s*
| *x*#*ls* ⇒ *do* {
let b = *c s*;
if b then do { *s*′←*f x s*; *D* (*ls*,*s*′)} *else RETURN s*
}
) (*l*,*s*)

lemma *nfoldli-alt-eq*:
nfoldli-alt l c f s = (
case l of
[] ⇒ *RETURN s*
| *x*#*ls* ⇒ *do* {*let b*=*c s*; *if b then f x s* ≫= *nfoldli-alt ls c f else RETURN s*}
)
apply (*subst nfoldli-alt-def*)
apply (*subst RECT-unfold*)
apply (*tagged-solver*)
apply (*subst nfoldli-alt-def*[*symmetric*])
apply *simp*
done

lemma *nfoldli-alt-simp*[*simp*]:
nfoldli-alt [] *c f s* = *RETURN s*
nfoldli-alt (*x*#*ls*) *c f s* = *do* {
let b = *c s*;
if b then f x s ≫= *nfoldli-alt ls c f else RETURN s*
}
apply (*subst nfoldli-alt-eq*, *simp*)
apply (*subst nfoldli-alt-eq*, *simp*)
done

lemma *nfoldli-alt*:
$(nfoldli{::}'a\ list \Rightarrow ('b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow 'b\ nres) \Rightarrow 'b \Rightarrow 'b\ nres)$
$= nfoldli\text{-}alt$
proof (*intro ext*)
fix $l{::}'a\ list$ **and** $c{::}'b \Rightarrow bool$ **and** $f{::}'a \Rightarrow 'b \Rightarrow 'b\ nres$ **and** $s :: 'b$
have $nfoldli\ l\ c\ f = nfoldli\text{-}alt\ l\ c\ f$
by (*induct l*) *auto*
thus $nfoldli\ l\ c\ f\ s = nfoldli\text{-}alt\ l\ c\ f\ s$ **by** *simp*
qed

lemma *monadic-nfoldli-rec*:
$monadic\text{-}nfoldli\ x'\ c\ f\ \sigma$
$\le\Downarrow Id\ (REC_T$
$(\lambda W\ (xs, \sigma).$
$ASSERT\ (FOREACH\text{-}inv\ xs0\ I\ (xs, \sigma)) \gg=$
$(\lambda\text{-}.\ if\ xs = []\ then\ RETURN\ (xs, \sigma)$
$else\ c\ \sigma \gg=$
$(\lambda b.\ if\ b\ then\ FOREACH\text{-}body\ f\ (xs, \sigma) \gg= W$
$else\ RETURN\ (xs, \sigma))))$
$(x', \sigma) \gg=$
$(\lambda(\text{-}, y).\ RETURN\ y))$
apply (*induct* x' *arbitrary*: σ)

apply (*subst RECT-unfold*, *refine-mono*)
apply (*simp*)
apply (*rule le-ASSERTI*)
apply *simp*

apply (*subst RECT-unfold*, *refine-mono*)
apply (*subst monadic-nfoldli-simp*)
apply (*simp del*: *conc-Id cong*: *if-cong*)
apply *refine-rcg*
apply *simp*
apply (*clarsimp simp add*: *FOREACH-body-def*)
apply (*rule-tac R=br* (*Pair* x') (λ-. *True*) **in** *intro-prgR*)
apply (*simp add*: *pw-le-iff refine-pw-simps br-def*)

apply (*rule order-trans*)
apply *rprems*
apply (*simp add*: *br-def*)
done

lemma *monadic-nfoldli-arities*[*sepref-monadify-arity*]:
$monadic\text{-}nfoldli \equiv \lambda_2 s\ c\ f\ \sigma.\ SP\ (monadic\text{-}nfoldli)\$s\$(\lambda_2 x.\ c\$x)\$(\lambda_2 x\ \sigma.\ f\$x\$\sigma)\$\sigma$
by (*simp-all*)

lemma *monadic-nfoldli-comb*[*sepref-monadify-comb*]:

$\bigwedge s\ c\ f\ \sigma.\ (\textit{monadic-nfoldli})\$s\$c\$f\$\sigma \equiv$
  Refine-Basic.bind$(\textit{EVAL}\$s)\$(\lambda_2 s.\ \textit{Refine-Basic.bind}\$(\textit{EVAL}\$\sigma)\$(\lambda_2 \sigma.$
    SP $(\textit{monadic-nfoldli})\$s\$c\$f\$\sigma$
  ))
 by (*simp-all*)

lemma *list-rel-congD*:
 assumes *A*: $(li,l)\in\langle S\rangle \textit{list-rel}$
 shows $(li,l)\in\langle S\cap(\textit{set li}\times \textit{set l})\rangle \textit{list-rel}$
proof –
 {
  fix *Si0 S0*
  assume *set li* ⊆ *Si0 set l* ⊆ *S0*
  with *A* **have** $(li,l)\in\langle S\cap(\textit{Si0}\times \textit{S0})\rangle \textit{list-rel}$
   by (*induction rule*: *list-rel-induct*) *auto*
 } **from** *this*[*OF order-refl order-refl*] **show** *?thesis* **.**
qed

lemma *monadic-nfoldli-refine*[*refine*]:
 assumes *L*: $(li,\ l) \in \langle S\rangle \textit{list-rel}$
  and [*simp*]: $(si,\ s) \in R$
  and *CR*[*refine*]: $\bigwedge si\ s.\ (si,s)\in R \Longrightarrow ci\ si \le\Downarrow \textit{bool-rel}\ (c\ s)$
  and [*refine*]: $\bigwedge xi\ x\ si\ s.\ [\![\ (xi,x)\in S;\ x\in \textit{set}\ l;\ (si,s)\in R;\ \textit{inres}\ (c\ s)\ \textit{True}\ ]\!] \Longrightarrow fi$
$xi\ si \le \Downarrow R\ (f\ x\ s)$
 shows *monadic-nfoldli li ci fi si* ≤ ⇓ *R* (*monadic-nfoldli l c f s*)

 supply *RELATESI*[*of* $S\cap(\textit{set li}\times\textit{set l})$, *refine-dref-RELATES*]
 supply *RELATESI*[*of R*, *refine-dref-RELATES*]
 unfolding *monadic-nfoldli-def*
 apply (*refine-rcg bind-refine′*)
 apply *refine-dref-type*
 apply (*vc-solve simp*: *list-rel-congD*[*OF L*])
 done

lemma *monadic-FOREACH-itsl*:
 fixes *R I tsl*
 shows
  do { $l \leftarrow$ *it-to-sorted-list R s*; *monadic-nfoldli l c f σ* }
  ≤ *monadic-FOREACH R I s c f σ*
  apply (*rule refine-IdD*)
  unfolding *monadic-FOREACH-def it-to-sorted-list-def*
  apply (*refine-rcg*)
  apply *simp*
  apply (*rule monadic-nfoldli-rec*[*simplified*])
  done

lemma *FOREACHoci-itsl*:
 fixes *R I tsl*

shows
$do\ \{\ l \leftarrow it\text{-}to\text{-}sorted\text{-}list\ R\ s;\ nfoldli\ l\ c\ f\ \sigma\ \}$
$\leq FOREACHoci\ R\ I\ s\ c\ f\ \sigma$
apply (*rule refine-IdD*)
unfolding *FOREACHoci-def it-to-sorted-list-def*
apply *refine-rcg*
apply *simp*
apply (*rule nfoldli-while*)
done

lemma [*def-pat-rules*]:
$FOREACHc \equiv PR\text{-}CONST\ (FOREACHoci\ (\lambda\text{-}\ \text{-}.\ True)\ (\lambda\text{-}\ \text{-}.\ True))$
$FOREACHci\$I \equiv PR\text{-}CONST\ (FOREACHoci\ (\lambda\text{-}\ \text{-}.\ True)\ I)$
$FOREACHi\$I \equiv \lambda_2 s.\ PR\text{-}CONST\ (FOREACHoci\ (\lambda\text{-}\ \text{-}.\ True)\ I)\$s\$(\lambda_2 x.\ True)$
$FOREACH \equiv FOREACHi\$(\lambda_2\text{-}\ \text{-}.\ True)$
by (*simp-all add*:
FOREACHci-def FOREACHi-def[*abs-def*] *FOREACHc-def FOREACH-def*[*abs-def*])

term *FOREACHoci R I*
lemma *id-FOREACHoci*[*id-rules*]: $PR\text{-}CONST\ (FOREACHoci\ R\ I) ::_i$
$TYPE('c\ set \Rightarrow ('d \Rightarrow bool) \Rightarrow ('c \Rightarrow 'd \Rightarrow 'd\ nres) \Rightarrow 'd \Rightarrow 'd\ nres)$
by *simp*

We set up the monadify-phase such that all FOREACH-loops get rewritten to the monadic version of FOREACH

lemma *FOREACH-arities*[*sepref-monadify-arity*]:

$PR\text{-}CONST\ (FOREACHoci\ R\ I) \equiv \lambda_2 s\ c\ f\ \sigma.\ SP\ (PR\text{-}CONST\ (FOREACHoci\ R\ I))\$s\$(\lambda_2 x.\ c\$x)\$(\lambda_2 x\ \sigma.\ f\$x\$\sigma)\$\sigma$
by (*simp-all*)

lemma *FOREACHoci-comb*[*sepref-monadify-comb*]:
$\bigwedge s\ c\ f\ \sigma.\ (PR\text{-}CONST\ (FOREACHoci\ R\ I))\$s\$(\lambda_2 x.\ c\ x)\$f\$\sigma \equiv$
$Refine\text{-}Basic.bind\$(EVAL\$s)\$(\lambda_2 s.\ Refine\text{-}Basic.bind\$(EVAL\$\sigma)\$(\lambda_2 \sigma.$
$SP\ (PR\text{-}CONST\ (monadic\text{-}FOREACH\ R\ I))\$s\$(\lambda_2 x.\ (EVAL\$(c\ x)))\$f\$\sigma$
$))$
by (*simp-all add*: *FOREACH-oci-to-monadic*)

Imperative Version of nfoldli

We define an imperative version of *nfoldli*. It is the equivalent to the monadic version in the nres-monad

definition *imp-nfoldli l c f s* $\equiv$ *heap.fixp-fun* $(\lambda D\ (l,s).\ case\ l\ of$
$[] \Rightarrow return\ s$
$|\ x\#ls \Rightarrow do\ \{$
$b \leftarrow c\ s;$
$if\ b\ then\ do\ \{\ s' \leftarrow f\ x\ s;\ D\ (ls,s')\}\ else\ return\ s$
$\}$

) (*l*,*s*)

declare *imp-nfoldli-def*[*code del*]

lemma *imp-nfoldli-simps*[*simp*,*code*]:
imp-nfoldli [] *c f s* = *return s*
imp-nfoldli (*x*#*ls*) *c f s* = (*do* {
b ← *c s*;
if b then do {
s'←*f x s*;
imp-nfoldli ls c f s'
} *else return s*
})
apply −
unfolding *imp-nfoldli-def*
apply (*subst heap.mono-body-fixp*)
apply *pf-mono*
apply *simp*
apply (*subst heap.mono-body-fixp*)
apply *pf-mono*
apply *simp*
done

lemma *monadic-nfoldli-refine-aux*:
assumes *c-ref*: $\bigwedge s\ s'$. *hn-refine*
(Γ ∗ *hn-ctxt Rs s' s*)
(*c s*)
(Γ ∗ *hn-ctxt Rs s' s*)
bool-assn
(*c' s'*)
assumes *f-ref*: $\bigwedge x\ x'\ s\ s'$. *hn-refine*
(Γ ∗ *hn-ctxt Rl x' x* ∗ *hn-ctxt Rs s' s*)
(*f x s*)
(Γ ∗ *hn-invalid Rl x' x* ∗ *hn-invalid Rs s' s*) *Rs*
(*f' x' s'*)

shows *hn-refine*
(Γ ∗ *hn-ctxt* (*list-assn Rl*) *l' l* ∗ *hn-ctxt Rs s' s*)
(*imp-nfoldli l c f s*)
(Γ ∗ *hn-invalid* (*list-assn Rl*) *l' l* ∗ *hn-invalid Rs s' s*) *Rs*
(*monadic-nfoldli l' c' f' s'*)
applyF (*induct p*≡*Rl l' l*
arbitrary: *s s'*
rule: *list-assn.induct*)

applyF *simp*
apply (*rule hn-refine-cons-post*)

```
apply (rule hn-refine-frame[OF hnr-RETURN-pass])
apply (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)
apply (simp add: hn-ctxt-def ent-true-drop invalid-assn-const)
solved

apply1 weaken-hnr-post
apply1 (simp only: imp-nfoldli-simps monadic-nfoldli-simp)
applyF (rule hnr-bind)
  apply1 (rule hn-refine-frame[OF c-ref])
  applyS (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)

  applyF (rule hnr-If)
    applyS (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)
    applyF (rule hnr-bind)
      apply1 (rule hn-refine-frame[OF f-ref])
      apply1 (simp add: assn-assoc)

      apply1 (rule ent-imp-entt)
      apply1 (fr-rot 1, rule fr-refl)
      apply1 (fr-rot 2, rule fr-refl)
      apply1 (fr-rot 1, rule fr-refl)
      applyS (rule ent-refl)

      applyF (rule hn-refine-frame)
        applyS rprems

        apply1 (simp add: assn-assoc)
        apply1 (rule ent-imp-entt)
        apply (rule fr-refl)
        apply1 (fr-rot 3, rule fr-refl)
        apply1 (fr-rot 3, rule fr-refl)
        applyS (rule ent-refl)
      solved

      apply simp

      applyS (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)
    solved

    apply1 (rule hn-refine-frame[OF hnr-RETURN-pass])
    applyS (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)

    apply1 (simp add: assn-assoc)
    applyS (tactic ‹Sepref-Frame.merge-tac (K (K no-tac)) @{context} 1›)
  solved

  apply (rule enttI)
  apply (fr-rot-rhs 1)
  apply (fr-rot 3, rule fr-refl)
```

applyS (*fr-rot 3*, *rule ent-star-mono*[*rotated*]; *sep-auto simp*: *hn-ctxt-def*)
solved

applyS (*simp add*: *hn-ctxt-def invalid-assn-def*)

applyS (*rule*, *sep-auto*)
solved
done

lemma *hn-monadic-nfoldli*:
assumes *FR*: $P \Longrightarrow_t \Gamma * \textit{hn-ctxt}\ (\textit{list-assn}\ Rl)\ l'\ l * \textit{hn-ctxt}\ Rs\ s'\ s$
assumes *c-ref*: $\bigwedge s\ s'.\ \textit{hn-refine}$
$(\Gamma * \textit{hn-ctxt}\ Rs\ s'\ s)$
$(c\ s)$
$(\Gamma * \textit{hn-ctxt}\ Rs\ s'\ s)$
bool-assn
$(c'\$s')$
assumes *f-ref*: $\bigwedge x\ x'\ s\ s'.\ \textit{hn-refine}$
$(\Gamma * \textit{hn-ctxt}\ Rl\ x'\ x * \textit{hn-ctxt}\ Rs\ s'\ s)$
$(f\ x\ s)$
$(\Gamma * \textit{hn-invalid}\ Rl\ x'\ x * \textit{hn-invalid}\ Rs\ s'\ s)\ Rs$
$(f'\$x'\$s')$
shows *hn-refine*
P
$(\textit{imp-nfoldli}\ l\ c\ f\ s)$
$(\Gamma * \textit{hn-invalid}\ (\textit{list-assn}\ Rl)\ l'\ l * \textit{hn-invalid}\ Rs\ s'\ s)$
Rs
$(\textit{monadic-nfoldli}\$l'\$c'\$f'\$s')$

apply (*rule hn-refine-cons-pre*[*OF FR*])
unfolding *APP-def*
apply (*rule monadic-nfoldli-refine-aux*)
apply (*rule c-ref*[*unfolded APP-def*])
apply (*rule f-ref*[*unfolded APP-def*])
done

definition
$\textit{imp-foreach} :: ('b \Rightarrow 'c\ \textit{list Heap}) \Rightarrow 'b \Rightarrow ('a \Rightarrow \textit{bool Heap}) \Rightarrow ('c \Rightarrow 'a \Rightarrow 'a\ \textit{Heap}) \Rightarrow 'a \Rightarrow 'a\ \textit{Heap}$
where
$\textit{imp-foreach tsl s c f}\ \sigma \equiv \textit{do}\ \{\ l \leftarrow \textit{tsl s};\ \textit{imp-nfoldli}\ l\ c\ f\ \sigma\}$

lemma *heap-fixp-mono*[*partial-function-mono*]:
assumes [*partial-function-mono*]:
$\bigwedge x\ d.\ \textit{mono-Heap}\ (\lambda xa.\ B\ x\ xa\ d)$
$\bigwedge Z\ xa.\ \textit{mono-Heap}\ (\lambda a.\ B\ a\ Z\ xa)$
shows $\textit{mono-Heap}\ (\lambda x.\ \textit{heap.fixp-fun}\ (\lambda D\ \sigma.\ B\ x\ D\ \sigma)\ \sigma)$
apply *rule*

apply (*rule ccpo.fixp-mono*[*OF heap.ccpo*, *THEN fun-ordD*])
apply (*rule mono-fun-fun-cnv*, *erule thin-rl*, *pf-mono*)+
apply (*rule fun-ordI*)
apply (*erule monotoneD*[*of fun-ord Heap-ord Heap-ord*, *rotated*])
apply *pf-mono*
done

lemma *imp-nfoldli-mono*[*partial-function-mono*]:
assumes [*partial-function-mono*]: $\bigwedge x\ \sigma.$ *mono-Heap* ($\lambda fa.$ *f fa x* σ)
shows *mono-Heap* ($\lambda x.$ *imp-nfoldli l c* (*f x*) σ)
unfolding *imp-nfoldli-def*
by *pf-mono*

lemma *imp-foreach-mono*[*partial-function-mono*]:
assumes [*partial-function-mono*]: $\bigwedge x\ \sigma.$ *mono-Heap* ($\lambda fa.$ *f fa x* σ)
shows *mono-Heap* ($\lambda x.$ *imp-foreach tsl l c* (*f x*) σ)
unfolding *imp-foreach-def*
by *pf-mono*

lemmas [*sepref-opt-simps*] = *imp-foreach-def*

definition
IS-TO-SORTED-LIST Ω *Rs Rk tsl* $\equiv$ (*tsl*,*it-to-sorted-list* Ω) $\in$ $(Rs)^k \rightarrow_a$ *list-assn Rk*

lemma *IS-TO-SORTED-LISTI*:
assumes (*tsl*,*PR-CONST* (*it-to-sorted-list* Ω)) $\in$ $(Rs)^k \rightarrow_a$ *list-assn Rk*
shows *IS-TO-SORTED-LIST* Ω *Rs Rk tsl*
using *assms* **unfolding** *IS-TO-SORTED-LIST-def PR-CONST-def* **.**

lemma *hn-monadic-FOREACH*[*sepref-comb-rules*]:
assumes *INDEP Rk INDEP Rs INDEP R*σ
assumes *FR*: $P \Longrightarrow_t \Gamma * $ *hn-ctxt Rs s' s* $*$ *hn-ctxt R*σ σ' σ
assumes *STL*: *GEN-ALGO tsl* (*IS-TO-SORTED-LIST ordR Rs Rk*)
assumes *c-ref*: $\bigwedge\sigma\ \sigma'.$ *hn-refine*
(Γ $*$ *hn-ctxt Rs s' s* $*$ *hn-ctxt R*σ σ' σ)
(*c* σ)
(Γc σ' σ)
bool-assn
(c' σ')
assumes *C-FR*:
$\bigwedge\sigma'\ \sigma.$ *TERM monadic-FOREACH* $\Longrightarrow$
Γc σ' σ $\Longrightarrow_t$ Γ $*$ *hn-ctxt Rs s' s* $*$ *hn-ctxt R*σ σ' σ

assumes *f-ref*: $\bigwedge x'\ x\ \sigma'\ \sigma.$ *hn-refine*
(Γ $*$ *hn-ctxt Rs s' s* $*$ *hn-ctxt Rk x' x* $*$ *hn-ctxt R*σ σ' σ)
(*f x* σ)
(Γf x' x σ' σ) *R*σ

$(f' \; x' \; \sigma')$

assumes *F-FR*: $\bigwedge x' \; x \; \sigma' \; \sigma$. *TERM monadic-FOREACH* $\Longrightarrow \Gamma f \; x' \; x \; \sigma' \; \sigma \Longrightarrow_t$
$\Gamma * $ *hn-ctxt Rs* $s' \; s * $ *hn-ctxt Pfx* $x' \; x * $ *hn-ctxt Pf*$\sigma \; \sigma' \; \sigma$

shows *hn-refine*
P
(*imp-foreach tsl s c f* σ)
$(\Gamma * $ *hn-ctxt Rs* $s' \; s * $ *hn-invalid R*$\sigma \; \sigma' \; \sigma)$
Rσ
((*PR-CONST* (*monadic-FOREACH ordR I*))
\$$s'$\$$(\lambda_2\sigma'. \; c' \; \sigma')$\$$(\lambda_2 x' \; \sigma'. \; f' \; x' \; \sigma')$\$$\sigma'$
)

proof –
from *STL* **have** *STL*: (*tsl*,*it-to-sorted-list ordR*) $\in (Rs)^k \rightarrow_a$ *list-assn Rk*
unfolding *GEN-ALGO-def IS-TO-SORTED-LIST-def* **by** *simp*

show *?thesis*
apply (*rule hn-refine-cons-pre*[*OF FR*])
apply *weaken-hnr-post*
unfolding *APP-def PROTECT2-def PR-CONST-def imp-foreach-def*
apply (*rule hn-refine-ref*[*OF monadic-FOREACH-itsl*])
apply (*rule hn-refine-guessI*)
apply (*rule hnr-bind*)
apply (*rule hn-refine-frame*)
apply (*rule STL*[*to-hnr*, *unfolded APP-def*])
apply (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
apply (*rule hn-monadic-nfoldli*[*unfolded APP-def*])
apply (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
apply (*rule hn-refine-cons-post*)
apply (*rule c-ref*[*unfolded APP-def*])
apply (*rule C-FR*)
apply (*rule TERMI*)
apply *weaken-hnr-post*
apply (*rule hn-refine-cons-post*)
apply (*rule f-ref*[*unfolded APP-def*])
apply (*rule entt-trans*[*OF F-FR*])
apply (*rule TERMI*)
applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)

apply *simp*
done

qed

lemma *monadic-nfoldli-assert-aux*:
assumes *set* $l \subseteq S$
shows *monadic-nfoldli l c* ($\lambda x \; s$. *ASSERT* $(x \in S) \gg f \; x \; s)$ $s =$ *monadic-nfoldli l c f s*

using *assms*
apply (*induction l arbitrary*: *s*)
apply (*auto simp*: *pw-eq-iff refine-pw-simps*)
done

lemmas *monadic-nfoldli-assert* = *monadic-nfoldli-assert-aux*[*OF order-refl*]

lemma *nfoldli-arities*[*sepref-monadify-arity*]:
$nfoldli \equiv \lambda_2 s\ c\ f\ \sigma.\ SP\ (nfoldli)\$s\$(\lambda_2 x.\ c\$x)\$(\lambda_2 x\ \sigma.\ f\$x\$\sigma)\$\sigma$
by (*simp-all*)

lemma *nfoldli-comb*[*sepref-monadify-comb*]:
$\bigwedge s\ c\ f\ \sigma.\ (nfoldli)\$s\$(\lambda_2 x.\ c\ x)\$f\$\sigma \equiv$
$Refine\text{-}Basic.bind\$(EVAL\$s)\$(\lambda_2 s.\ Refine\text{-}Basic.bind\$(EVAL\$\sigma)\$(\lambda_2 \sigma.$
$SP\ (monadic\text{-}nfoldli)\$s\$(\lambda_2 x.\ (EVAL\$(c\ x)))\$f\$\sigma$
$))$
by (*simp-all add*: *nfoldli-to-monadic*)

lemma *monadic-nfoldli-refine-aux′*:
assumes *SS*: $set\ l' \subseteq S$
assumes *c-ref*: $\bigwedge s\ s'.\ hn\text{-}refine$
$(\Gamma * hn\text{-}ctxt\ Rs\ s'\ s)$
$(c\ s)$
$(\Gamma * hn\text{-}ctxt\ Rs\ s'\ s)$
bool-assn
$(c'\ s')$
assumes *f-ref*: $\bigwedge x\ x'\ s\ s'.\ [\![x' \in S]\!] \Longrightarrow hn\text{-}refine$
$(\Gamma * hn\text{-}ctxt\ Rl\ x'\ x * hn\text{-}ctxt\ Rs\ s'\ s)$
$(f\ x\ s)$
$(\Gamma * hn\text{-}ctxt\ Rl'\ x'\ x * hn\text{-}invalid\ Rs\ s'\ s)\ Rs$
$(f'\ x'\ s')$

assumes *merge*[*sepref-frame-merge-rules*]: $\bigwedge x\ x'.\ hn\text{-}ctxt\ Rl'\ x'\ x \vee_A hn\text{-}ctxt\ Rl\ x'\ x \Longrightarrow_t hn\text{-}ctxt\ Rl''\ x'\ x$
notes [*sepref-frame-merge-rules*] = *merge-sat2*[*OF merge*]

shows *hn-refine*
$(\Gamma * hn\text{-}ctxt\ (list\text{-}assn\ Rl)\ l'\ l * hn\text{-}ctxt\ Rs\ s'\ s)$
$(imp\text{-}nfoldli\ l\ c\ f\ s)$
$(\Gamma * hn\text{-}ctxt\ (list\text{-}assn\ Rl'')\ l'\ l * hn\text{-}invalid\ Rs\ s'\ s)\ Rs$
$(monadic\text{-}nfoldli\ l'\ c'\ f'\ s')$

apply1 (*subst monadic-nfoldli-assert-aux*[*OF SS*,*symmetric*])

applyF (*induct p≡Rl l′ l*
  arbitrary: *s s′*
  rule: *list-assn.induct*)

applyF *simp*
apply (*rule hn-refine-cons-post*)
apply (*rule hn-refine-frame*[*OF hnr-RETURN-pass*])
apply (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
apply (*simp add*: *hn-ctxt-def ent-true-drop*)
solved

apply (*simp only*: *imp-nfoldli-simps monadic-nfoldli-simp*)
apply (*rule hnr-bind*)
apply (*rule hn-refine-frame*[*OF c-ref*])
apply (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)

apply (*rule hnr-If*)
apply (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
apply (*simp only*: *nres-monad-laws*)
apply (*rule hnr-ASSERT*)
apply (*rule hnr-bind*)
apply (*rule hn-refine-frame*[*OF f-ref*])
apply *assumption*
apply (*simp add*: *assn-aci*)
apply (*rule ent-imp-entt*)
apply (*fr-rot-rhs 1*)
apply (*fr-rot 2*)
apply (*rule fr-refl*)
apply (*rule fr-refl*)
apply (*rule fr-refl*)
apply (*rule ent-refl*)

applyF (*rule hn-refine-frame*)
  applyS *rprems*

  focus
    apply (*simp add*: *assn-aci*)
    apply (*rule ent-imp-entt*)

    apply (*fr-rot-rhs 1*, *rule fr-refl*)
    apply (*fr-rot 2*, *rule fr-refl*)
    apply (*fr-rot 1*, *rule fr-refl*)
    apply (*rule ent-refl*)
  solved
solved

focus (*simp add*: *assn-assoc*)
  apply (*rule ent-imp-entt*)
  apply (*rule fr-refl*)

```
    apply (rule ent-refl)
  solved

  apply1 (rule hn-refine-frame[OF hnr-RETURN-pass])
  applyS (tactic ‹Sepref-Frame.frame-tac (K (K no-tac)) @{context} 1›)

  apply1 (simp add: assn-assoc)
  applyS (tactic ‹Sepref-Frame.merge-tac (K (K no-tac)) @{context} 1›)

  apply simp
  apply (rule ent-imp-entt)
  apply solve-entails
  apply (rule, sep-auto)
  apply (rule, sep-auto)
  solved
  done

lemma hn-monadic-nfoldli-rl'[sepref-comb-rules]:
  assumes INDEP Rk INDEP Rσ
  assumes FR: P ⟹t Γ ∗ hn-ctxt (list-assn Rk) s' s ∗ hn-ctxt Rσ σ' σ
  assumes c-ref: ⋀σ σ'. hn-refine
    (Γ ∗ hn-ctxt Rσ σ' σ)
    (c σ)
    (Γc σ' σ)
    bool-assn
    (c' σ')
  assumes C-FR:
    ⋀σ' σ. TERM monadic-nfoldli ⟹
      Γc σ' σ ⟹t Γ ∗ hn-ctxt Rσ σ' σ

  assumes f-ref: ⋀x' x σ' σ. ⟦x'∈set s'⟧ ⟹ hn-refine
    (Γ ∗ hn-ctxt Rk x' x ∗ hn-ctxt Rσ σ' σ)
    (f x σ)
    (Γf x' x σ' σ) Rσ
    (f' x' σ')
  assumes F-FR: ⋀x' x σ' σ. TERM monadic-nfoldli ⟹ Γf x' x σ' σ ⟹t
    Γ ∗ hn-ctxt Rk' x' x ∗ hn-ctxt Pfσ σ' σ

  assumes MERGE: ⋀x x'. hn-ctxt Rk' x' x ∨A hn-ctxt Rk x' x ⟹t hn-ctxt Rk''
x' x

  shows hn-refine
    P
    (imp-nfoldli s c f σ)
    (Γ ∗ hn-ctxt (list-assn Rk'') s' s ∗ hn-invalid Rσ σ' σ)
    Rσ
    ((monadic-nfoldli)
      $s'$(λ2σ'. c' σ')$(λ2x' σ'. f' x' σ')$σ'
    )
```

unfolding *APP-def PROTECT2-def PR-CONST-def*
apply1 (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *weaken-hnr-post*
applyF (*rule hn-refine-cons*[*rotated*])
applyF (*rule monadic-nfoldli-refine-aux'*[*OF order-refl*])
focus
apply (*rule hn-refine-cons-post*)
applyS (*rule c-ref*)
apply1 (*rule entt-trans*[*OF C-FR*[*OF TERMI*]])
applyS (*rule entt-refl*)
solved

apply1 *weaken-hnr-post*
applyF (*rule hn-refine-cons-post*)
applyS (*rule f-ref*; *simp*)

apply1 (*rule entt-trans*[*OF F-FR*[*OF TERMI*]])
applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
solved

apply (*rule MERGE*)
solved

applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
applyS (*tactic* ‹*Sepref-Frame.frame-tac* (*K* (*K no-tac*)) @{*context*} *1*›)
solved
done

lemma *nfoldli-assert*:
assumes *set l* $\subseteq$ *S*
shows *nfoldli l c* (λ *x s. ASSERT* (*x* $\in$ *S*) $\gg$ *f x s*) *s* = *nfoldli l c f s*
using *assms* **by** (*induction l arbitrary*: *s*) (*auto simp*: *pw-eq-iff refine-pw-simps*)

lemmas *nfoldli-assert'* = *nfoldli-assert*[*OF order.refl*]

lemma *fold-eq-nfoldli*:
RETURN (*fold f l s*) = *nfoldli l* (λ-. *True*) (λ*x s. RETURN* (*f x s*)) *s*
apply (*induction l arbitrary*: *s*) **apply** (*auto*) **done**

lemma *fold-eq-nfoldli-assert*:
RETURN (*fold f l s*) = *nfoldli l* (λ-. *True*) (λ*x s. ASSERT* (*x*$\in$*set l*) $\gg$ *RETURN* (*f x s*)) *s*
by (*simp add*: *nfoldli-assert' fold-eq-nfoldli*)

lemma *fold-arity*[*sepref-monadify-arity*]: *fold* $\equiv$ λ_2*f l s. SP fold*\$(λ_2*x s. f*\$*x*\$*s*)\$*l*\$*s*
by *auto*

lemma *monadify-plain-fold*[*sepref-monadify-comb*]:

$EVAL\$(fold\$(\lambda_2 x\ s.\ f\ x\ s)\$l\$s) \equiv (\gg=)\$(EVAL\$l)\$(\lambda_2 l.\ (\gg=)\$(EVAL\$s)\$(\lambda_2 s.\ nfoldli\$l\$(\lambda_2\text{-}.\ True)\$(\lambda_2 x\ s.\ EVAL\$(f\ x\ s))\$s))$
by (*simp add*: *fold-eq-nfoldli*)

lemma *monadify-plain-fold-old-rl*:
$EVAL\$(fold\$(\lambda_2 x\ s.\ f\ x\ s)\$l\$s) \equiv (\gg=)\$(EVAL\$l)\$(\lambda_2 l.\ (\gg=)\$(EVAL\$s)\$(\lambda_2 s.\ nfoldli\$l\$(\lambda_2\text{-}.\ True)\$(\lambda_2 x\ s.\ PR\text{-}CONST\ (op\text{-}ASSERT\text{-}bind\ (x \in set\ l))\$(EVAL\$(f\ x\ s)))\$s))$
by (*simp add*: *fold-eq-nfoldli-assert*)

foldli

lemma *foldli-eq-nfoldli*:
$RETURN\ (foldli\ l\ c\ f\ s) = nfoldli\ l\ c\ (\lambda x\ s.\ RETURN\ (f\ x\ s))\ s$
by (*induction l arbitrary*: *s*) *auto*

lemma *foldli-arities*[*sepref-monadify-arity*]:
$foldli \equiv \lambda_2 s\ c\ f\ \sigma.\ SP\ (foldli)\$s\$(\lambda_2 x.\ c\$x)\$(\lambda_2 x\ \sigma.\ f\$x\$\sigma)\$\sigma$
by (*simp-all*)

lemma *monadify-plain-foldli*[*sepref-monadify-comb*]:
$EVAL\$(foldli\$l\$c\$(\lambda_2 x\ s.\ f\ x\ s)\$s) \equiv$
$(\gg=)\$(EVAL\$l)\$$
$(\lambda_2 l.\ (\gg=)\$(EVAL\$s)\$$
$(\lambda_2 s.\ nfoldli\$l\$c\$(\lambda_2 x\ s.\ (EVAL\$(f\ x\ s)))\$s))$
by (*simp add*: *foldli-eq-nfoldli*)

Deforestation

lemma *nfoldli-filter-deforestation*:
$nfoldli\ (filter\ P\ xs)\ c\ f\ s = nfoldli\ xs\ c\ (\lambda x\ s.\ if\ P\ x\ then\ f\ x\ s\ else\ RETURN\ s)\ s$
apply (*induction xs arbitrary*: *s*)
by (*auto simp*: *pw-eq-iff refine-pw-simps*)

lemma *extend-list-of-filtered-set*:
assumes [*simp*, *intro*!]: *finite S*
and *A*: $distinct\ xs'\ set\ xs' = \{x \in S.\ P\ x\}$
obtains *xs* **where** $xs' = filter\ P\ xs\ distinct\ xs\ set\ xs = S$
proof −
obtain *xs2* **where** $\{x \in S.\ \neg P\ x\} = set\ xs2\ distinct\ xs2$
using *finite-distinct-list*[**where** $A = \{x \in S.\ \neg P\ x\}$] **by** *auto*
with *A* **have** $xs' = filter\ P\ (xs'@xs2)\ distinct\ (xs'@xs2)\ set\ (xs'@xs2) = S$
by (*auto simp*: *filter-empty-conv*)
from *that*[*OF this*] **show** *?thesis* **.**
qed

lemma *FOREACHc-filter-deforestation*:
assumes *FIN*[*simp*, *intro*!]: *finite S*

shows (*FOREACHc* {x∈*S*. *P x*} *c f s*)
= *FOREACHc S c* (λ*x s*. *if P x then f x s else RETURN s*) *s*
unfolding *FOREACHc-def FOREACHci-def FOREACHoci-by-LIST-FOREACH LIST-FOREACH′-eq*
LIST-FOREACH′-def it-to-sorted-list-def
subgoal
proof (*induction rule*: *antisym*[*consumes 0*, *case-names 1 2*])
case *1*
then show *?case*
apply (*rule le-ASSERTI*)
apply (*rule ASSERT-leI*, *simp*)
apply (*rule intro-spec-refine*[**where** *R*=*Id*, *simplified*]; *clarsimp*)
apply (*rule extend-list-of-filtered-set*[*OF FIN - sym*], *assumption*, *assumption*)
subgoal for *xs′ xs*
apply (*rule rhs-step-bind-SPEC*[**where** *R*=*Id* **and** *x′*=*xs*, *simplified*])
applyS *simp*
applyS (*simp add*: *nfoldli-filter-deforestation*)
done
done
next
case *2*
then show *?case*
apply (*rule le-ASSERTI*)
apply (*rule ASSERT-leI*, (*simp*; *fail*))
apply (*rule intro-spec-refine*[**where** *R*=*Id*, *simplified*]; *clarsimp*)
subgoal for *xs*
apply (*rule rhs-step-bind-SPEC*[**where** *R*=*Id* **and** *x′*=*filter P xs*, *simplified*])
apply *simp*
apply (*simp add*: *nfoldli-filter-deforestation*)
done
done
qed
done

lemma *FOREACHc-filter-deforestation2*:
assumes [*simp*]: *distinct xs*
shows (*FOREACHc* (*set* (*filter P xs*)) *c f s*)
= *FOREACHc* (*set xs*) *c* (λ*x s*. *if P x then f x s else RETURN s*) *s*
using *FOREACHc-filter-deforestation*[*of set xs*, *simplified*, *folded set-filter*]
.

2.2.2 For Loops

partial-function (*heap*) *imp-for* :: *nat* ⇒ *nat* ⇒ (*′a* ⇒ *bool Heap*) ⇒ (*nat* ⇒ *′a* ⇒ *′a Heap*) ⇒ *′a* ⇒ *′a Heap* **where**
imp-for i u c f s = (*if i* ≥ *u then return s else do* {*ctn* <− *c s*; *if ctn then f i s* ⪢= *imp-for* (*i* + *1*) *u c f else return s*})

declare *imp-for.simps*[*code*]

lemma [*simp*]:
$i \geq u \Longrightarrow$ *imp-for i u c f s* = *return s*
$i < u \Longrightarrow$ *imp-for i u c f s* = *do* {*ctn* <− *c s*; *if ctn then f i s* ≫= *imp-for* (*i* + *1*) *u c f else return s*}
by (*auto simp*: *imp-for.simps*)

lemma *imp-nfoldli-deforest*[*sepref-opt-simps*]:
imp-nfoldli [*l*..<*u*] *c* = *imp-for l u c*
apply (*intro ext*)
subgoal for *f s*
apply (*induction u* − *l arbitrary*: *l u s*)
apply (*simp add*: *upt-conv-Cons*; *fail*)
apply (*simp add*: *upt-conv-Cons*)
apply (*fo-rule arg-cong*)
by (*auto cong*: *if-cong*)
done

partial-function (*heap*) *imp-for′* :: *nat* ⇒ *nat* ⇒ (*nat* ⇒ ′*a* ⇒ ′*a Heap*) ⇒ ′*a* ⇒ ′*a Heap* **where**
imp-for′ i u f s = (*if i* ≥ *u then return s else f i s* ≫= *imp-for′* (*i* + *1*) *u f*)

declare *imp-for′.simps*[*code*]

lemma [*simp*]:
$i \geq u \Longrightarrow$ *imp-for′ i u f s* = *return s*
$i < u \Longrightarrow$ *imp-for′ i u f s* = *f i s* ≫= *imp-for′* (*i* + *1*) *u f*
by (*auto simp*: *imp-for′.simps*)

lemma *imp-for-imp-for′*[*sepref-opt-simps*]:
imp-for i u (λ -. *return True*) = *imp-for′ i u*
apply (*intro ext*)
subgoal for *f s*
apply (*induction u* − *i arbitrary*: *i u s*)
apply (*simp*; *fail*)
apply *simp*
apply (*fo-rule arg-cong*)
by *auto*
done

partial-function (*heap*) *imp-for-down* :: *nat* ⇒ *nat* ⇒ (′*a* ⇒ *bool Heap*) ⇒ (*nat* ⇒ ′*a* ⇒ ′*a Heap*) ⇒ ′*a* ⇒ ′*a Heap* **where**
imp-for-down l i c f s = *do* {
let i = *i* − *1*;
ctn ← *c s*;
if ctn then do {
s ← *f i s*;
if i>*l then imp-for-down l i c f s else return s*
} *else return s*

```
  }

declare imp-for-down.simps[code]

lemma imp-nfoldli-deforest-down[sepref-opt-simps]:
  imp-nfoldli (rev [l..<u]) c =
    (λf s. if u≤l then return s else imp-for-down l u c f s)
proof (intro ext)
  fix f s
  show imp-nfoldli (rev [l..<u]) c f s =
        (if l ≥ u then return s else imp-for-down l u c f s)
  proof cases
    assume l≥u thus ?thesis by auto
  next
    assume ¬(l≥u) hence l<u by auto
    thus ?thesis
      apply simp
    proof (induction u − l arbitrary: u s)
      case 0 thus ?case by auto
    next
      case (Suc u')
      from Suc.prems Suc.hyps(2) have [simp]: rev [l..<u] = (u−1)#rev [l..<u−1]
          apply simp
          apply (subst upt-Suc-append[symmetric])
          apply auto
          done
        show ?case using Suc.hyps(1)[of u−1] Suc.hyps(2) Suc.prems
          apply (subst imp-for-down.simps)
          apply (cases l < u − Suc 0)
          apply (auto simp: Let-def cong: if-cong)
          done
      qed
    qed
  qed

context begin

private fun imp-for-down-induction-scheme :: nat ⇒ nat ⇒ unit where
  imp-for-down-induction-scheme l i = (
    let i=i−1 in
    if i>l then
      imp-for-down-induction-scheme l i
    else ()
  )

partial-function (heap) imp-for-down' :: nat ⇒ nat ⇒ (nat ⇒ 'a ⇒ 'a Heap) ⇒
'a ⇒ 'a Heap where
  imp-for-down' l i f s = do {
    let i = i − 1;
```

    s ← *f i s*;
    if i>l then imp-for-down′ l i f s else return s
  }
declare *imp-for-down′.simps*[*code*]

lemma *imp-for-down-no-cond*[*sepref-opt-simps*]:
  imp-for-down l u (λ-. *return True*) = *imp-for-down′ l u*
  apply (*induction l u rule*: *imp-for-down-induction-scheme.induct*)
  apply (*intro ext*)
  apply (*subst imp-for-down.simps*)
  apply (*subst imp-for-down′.simps*)
  apply (*simp cong*: *if-cong*)
  done

end

lemma *imp-for′-rule*:
  assumes *LESS*: $l \le u$
  assumes *PRE*: $P \Longrightarrow_A I\ l\ s$
  assumes *STEP*: $\bigwedge i\ s.\ [\![\ l \le i;\ i < u\ ]\!] \Longrightarrow$ <*I i s*> *f i s* <*I* (*i*+*1*)>
  shows <*P*> *imp-for′ l u f s* <*I u*>
  apply (*rule Hoare-Triple.cons-pre-rule*[*OF PRE*])
  using *LESS*
proof (*induction arbitrary*: *s rule*: *inc-induct*)
  case *base* **thus** *?case* **by** *sep-auto*
next
  case (*step k*)
  show *?case* **using** *step.hyps*
    by (*sep-auto heap*: *STEP step.IH*)
qed

This lemma is used to manually convert a fold to a loop over indices.

lemma *fold-idx-conv*: *fold f l s* = *fold* (λ*i*. *f* (*l*!*i*)) [*0*..<*length l*] *s*
proof (*induction l arbitrary*: *s rule*: *rev-induct*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*snoc x l*)
  { fix *x s*
   have *fold* (λ*a*. *f* ((*l* @ [*x*]) ! *a*)) [*0*..<*length l*] *s* = *fold* (λ*a*. *f* (*l* ! *a*)) [*0*..<*length l*] *s*
      by (*rule fold-cong*) (*simp-all add*: *nth-append*)
  }
  with *snoc* **show** *?case* **by** *simp*
qed

end

2.3 Ad-Hoc Solutions

theory *Sepref-Improper*
imports
Sepref-Tool
Sepref-HOL-Bindings

Sepref-Foreach
Sepref-Intf-Util
begin

This theory provides some ad-hoc solutions to practical problems, that, however, still need a more robust/clean solution

2.3.1 Pure Higher-Order Functions

Ad-Hoc way to support pure higher-order arguments

definition *pho-apply* :: $('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where** [*code-unfold*,*simp*]: *pho-apply* $f\ x = f\ x$
sepref-register *pho-apply*

lemmas *fold-pho-apply* = *pho-apply-def*[*symmetric*]

lemma *pure-fun-refine*[*sepref-fr-rules*]: *hn-refine*
(*hn-val* $(A \rightarrow B)$ *f fi* $*$ *hn-val* *A x xi*)
(*return* (*pho-apply*\$*fi*\$*xi*))
(*hn-val* $(A \rightarrow B)$ *f fi* $*$ *hn-val* *A x xi*)
(*pure B*)
(*RETURN*\$(*pho-apply*\$*f*\$*x*))
by (*sep-auto intro*!: *hn-refineI simp*: *pure-def hn-ctxt-def dest*: *fun-relD*)

end
theory *Sepref*
imports
Sepref-Tool
Sepref-HOL-Bindings

Sepref-Foreach
Sepref-Intf-Util
Separation-Logic-Imperative-HOL.Default-Insts
Sepref-Improper
begin

end

Chapter 3

The Imperative Isabelle Collection Framework

The Imperative Isabelle Collection Framework provides efficient imperative implementations of collection data structures.

3.1 Set Interface

theory *IICF-Set*
imports *../../Sepref*
begin

3.1.1 Operations

definition [*simp*]: *op-set-is-empty s* $\equiv$ *s*={}
lemma *op-set-is-empty-param*[*param*]: (*op-set-is-empty*,*op-set-is-empty*)$\in\langle A\rangle$*set-rel* $\rightarrow$ *bool-rel* **by** *auto*

definition *op-set-copy* :: *'a set* $\Rightarrow$ *'a set* **where** [*simp*]: *op-set-copy s* $\equiv$ *s*

context
  notes [*simp*] = *IS-LEFT-UNIQUE-def*
begin

sepref-decl-op (*no-def*) *set-copy*: *op-set-copy* :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* **where** *A* = *Id* **.**
sepref-decl-op *set-empty*: {} :: $\langle A\rangle$*set-rel* **.**
sepref-decl-op (*no-def*) *set-is-empty*: *op-set-is-empty* :: $\langle A\rangle$*set-rel* $\rightarrow$ *bool-rel* **.**
sepref-decl-op *set-member*: ($\in$) :: *A* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ *bool-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-insert*: *Set.insert* :: *A* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* **where** *IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-delete*: $\lambda x\ s.\ s - \{x\}$ :: *A* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel*

where *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-union*: $(\cup)$ :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* **.**
sepref-decl-op *set-inter*: $(\cap)$ :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-diff*: $(-)$ ::- *set* $\Rightarrow$ - :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-subseteq*: $(\subseteq)$ :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ *bool-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**
sepref-decl-op *set-subset*: $(\subset)$ :: $\langle A\rangle$*set-rel* $\rightarrow$ $\langle A\rangle$*set-rel* $\rightarrow$ *bool-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**

sepref-decl-op *set-pick*: *RES* :: $[\lambda s.\ s\neq\{\}]_f$ $\langle K\rangle$*set-rel* $\rightarrow$ K **by** *auto*

sepref-decl-op *set-size*: (*card*) :: $\langle A\rangle$*set-rel* $\rightarrow$ *nat-rel* **where** *IS-LEFT-UNIQUE A IS-RIGHT-UNIQUE A* **.**

end

3.1.2 Patterns

lemma *pat-set*[*def-pat-rules*]:
  $\{\}$ $\equiv$ *op-set-empty*
  $(\in)$ $\equiv$ *op-set-member*
  Set.insert $\equiv$ *op-set-insert*
  $(\cup)$ $\equiv$ *op-set-union*
  $(\cap)$ $\equiv$ *op-set-inter*
  $(-)$ $\equiv$ *op-set-diff*
  $(\subseteq)$ $\equiv$ *op-set-subseteq*
  $(\subset)$ $\equiv$ *op-set-subset*
  by (*auto intro*!: *eq-reflection*)

lemma *pat-set2*[*pat-rules*]:
  $(=)$ \$*s*\$$\{\}$ $\equiv$ *op-set-is-empty*\$*s*
  $(=)$ \$$\{\}$\$*s* $\equiv$ *op-set-is-empty*\$*s*

  $(-)$ \$*s*\$(*Set.insert*\$*x*\$$\{\}$) $\equiv$ *op-set-delete*\$*x*\$*s*
  SPEC\$($\lambda_2 x.$ $(\in)$ \$*x*\$*s*) $\equiv$ *op-set-pick s*
  RES\$*s* $\equiv$ *op-set-pick s*
  by (*auto intro*!: *eq-reflection*)

locale *set-custom-empty* =
  fixes *empty* **and** *op-custom-empty* :: *'a set*
  assumes *op-custom-empty-def*: *op-custom-empty* = *op-set-empty*
begin
  sepref-register *op-custom-empty* :: *'ax set*

lemma *fold-custom-empty*:
  {} = *op-custom-empty*
  op-set-empty = *op-custom-empty*
  mop-set-empty = *RETURN op-custom-empty*
  unfolding *op-custom-empty-def* **by** *simp-all*
end

end

3.2 Sets by Lists that Own their Elements

theory *IICF-List-SetO*
imports *../Intf/IICF-Set*
begin

Minimal implementation, only supporting a few operations

definition *lso-assn A* ≡ *hr-comp* (*list-assn A*) (*br set* (λ-. *True*))
lemmas [*fcomp-norm-unfold*] = *lso-assn-def*[*symmetric*]
lemma *lso-is-pure*[*safe-constraint-rules*]: *is-pure A* ⟹ *is-pure* (*lso-assn A*)
  unfolding *lso-assn-def* **by** *safe-constraint*

lemma *lso-empty-aref*: (*uncurry0* (*RETURN* []), *uncurry0* (*RETURN op-set-empty*))

  ∈ *unit-rel* $\to_f$ ⟨*br set* (λ-. *True*)⟩*nres-rel*
  by (*auto simp*: *in-br-conv intro*!: *frefI nres-relI*)

lemma *lso-ins-aref*: (*uncurry* (*RETURN oo* ((#))), *uncurry* (*RETURN oo op-set-insert*))
  ∈ *Id* $\times_r$ *br set* (λ-. *True*) $\to_f$ ⟨*br set* (λ-. *True*)⟩*nres-rel*
  by (*auto simp*: *in-br-conv intro*!: *frefI nres-relI*)

sepref-decl-impl (*no-register*) *lso-empty*: *hn-Nil*[*to-hfref*] **uses** *lso-empty-aref* **.**

definition [*simp*]: *op-lso-empty* ≡ *op-set-empty*
lemma *lso-fold-custom-empty*:
  {} = *op-lso-empty*
  op-set-empty = *op-lso-empty*
  by *auto*
lemmas [*sepref-fr-rules*] = *lso-empty-hnr*[*folded op-lso-empty-def*]

sepref-decl-impl *lso-insert*: *hn-Cons*[*to-hfref*] **uses** *lso-ins-aref* **.**

thm *hn-Cons*[*FCOMP lso-ins-aref*]

definition [*simp*]: *op-lso-bex P S* ≡ ∃ *x*∈*S*. *P x*
lemma *fold-lso-bex*: *Bex* ≡ λ*s P*. *op-lso-bex P s* **by** *auto*

definition [*simp*]: *mop-lso-bex P S* ≡ *ASSERT* (∀ *x*∈*S*. ∃ *y*. *P x* = *RETURN y*) ≫ *RETURN* (∃ *x*∈*S*. *P x* = *RETURN True*)

lemma *op-mop-lso-bex*: *RETURN* (*op-lso-bex P S*) = *mop-lso-bex* (*RETURN o P*) *S* **by** *simp*

sepref-register *op-lso-bex*
lemma *lso-bex-arity*[*sepref-monadify-arity*]:
op-lso-bex ≡ λ_2*P s*. *SP op-lso-bex*\$(λ_2*x*. *P*\$*x*)\$*s* **by** (*auto intro*!: *eq-reflection ext*)
lemma *op-lso-bex-monadify*[*sepref-monadify-comb*]:
EVAL\$(*op-lso-bex*\$(λ_2*x*. *P x*)\$*s*) ≡ (⤜) \$(*EVAL*\$*s*)\$(λ_2*s*. *mop-lso-bex*\$(λ_2*x*. *EVAL* \$ *P x*)\$*s*) **by** *simp*

definition *lso-abex P l* ≡ *nfoldli l* (*Not*) (λ*x* -. *P x*) *False*
lemma *lso-abex-to-set*: *lso-abex P l* ≤ *mop-lso-bex P* (*set l*)
proof −
{ **fix** *b*
have *nfoldli l* (*Not*) (λ*x* -. *P x*) *b* ≤ *ASSERT* (∀ *x*∈*set l*. ∃ *y*. *P x* = *RETURN y*) ≫ *RETURN* ((∃ *x*∈*set l*. *P x* = *RETURN True*) ∨ *b*)
apply (*induction l arbitrary*: *b*)
applyS *simp*
applyS (*clarsimp simp add*: *pw-le-iff refine-pw-simps*; *blast*)
done
} **from** *this*[*of False*] **show** *?thesis* **by** (*simp add*: *lso-abex-def*)
qed

locale *lso-bex-impl-loc* =
fixes *Pi* **and** *P* :: ′*a* ⇒ *bool nres*
fixes *li* :: ′*c list* **and** *l* :: ′*a list*
fixes *A* :: ′*a* ⇒ ′*c* ⇒ *assn*
fixes *F* :: *assn*

assumes *Prl*: ⋀*x xi*. ⟦*x*∈*set l*⟧ ⟹ *hn-refine* (*F* ∗ *hn-ctxt A x xi*) (*Pi xi*) (*F* ∗ *hn-ctxt A x xi*) *bool-assn* (*P x*)
begin
sepref-register *l*
sepref-register *P*

lemma [*sepref-comb-rules*]:
assumes Γ $\Longrightarrow_t$ *F*′ ∗ *F* ∗ *hn-ctxt A x xi*
assumes *x*∈*set l*
shows *hn-refine* Γ (*Pi xi*) (*F*′ ∗ *F* ∗ *hn-ctxt A x xi*) *bool-assn* (*P*\$*x*)
using *hn-refine-frame*[*OF Prl*[*OF assms*(*2*)], *of* Γ *F*′] *assms*(*1*)
by (*simp add*: *assn-assoc*)

schematic-goal *lso-bex-impl*:
hn-refine (*hn-ctxt* (*list-assn* *A*) *l* *li* ∗ *F*) (*?c*) (*F* ∗ *hn-ctxt* (*list-assn* *A*) *l* *li*) *bool-assn* (*lso-abex* *P* *l*)
unfolding *lso-abex-def*[*abs-def*]
by *sepref*
end
concrete-definition *lso-bex-impl* **uses** *lso-bex-impl-loc.lso-bex-impl*

lemma *hn-lso-bex*[*sepref-prep-comb-rule*,*sepref-comb-rules*]:
assumes *FR*: $\Gamma \Longrightarrow_t$ *hn-ctxt* (*lso-assn* *A*) *s* *li* ∗ *F*
assumes *Prl*: $\bigwedge x\ xi.\ [\![x{\in}s]\!] \Longrightarrow$ *hn-refine* (*F* ∗ *hn-ctxt* *A* *x* *xi*) (*Pi* *xi*) (*F* ∗ *hn-ctxt* *A* *x* *xi*) *bool-assn* (*P* *x*)
notes [*simp* *del*] = *mop-lso-bex-def*
shows *hn-refine* Γ (*lso-bex-impl* *Pi* *li*) (*F* ∗ *hn-ctxt* (*lso-assn* *A*) *s* *li*) *bool-assn* (*mop-lso-bex*\$($\lambda_2 x$. *P* *x*)\$*s*)
apply (*rule* *hn-refine-cons-pre*[*OF* *FR*])
apply (*clarsimp* *simp*: *hn-ctxt-def* *lso-assn-def* *hr-comp-def* *in-br-conv* *hnr-pre-ex-conv*)
apply (*rule* *hn-refine-preI*)
apply (*drule* *mod-starD*; *clarsimp*)
apply (*rule* *hn-refine-ref*[*OF* *lso-abex-to-set*])
proof −
fix *l* **assume** [*simp*]: *s*=*set* *l*

from *Prl* **have** *Prl'*: $\bigwedge x\ xi.\ [\![x{\in}set\ l]\!] \Longrightarrow$ *hn-refine* (*F* ∗ *hn-ctxt* *A* *x* *xi*) (*Pi* *xi*) (*F* ∗ *hn-ctxt* *A* *x* *xi*) *bool-assn* (*P* *x*)
by *simp*

show *hn-refine* (*list-assn* *A* *l* *li* ∗ *F*) (*lso-bex-impl* *Pi* *li*) ($\exists_A$*ba*. *F* ∗ *list-assn* *A* *ba* *li* ∗ ↑ (*set* *l* = *set* *ba*)) *bool-assn*
(*lso-abex* *P* *l*)
apply (*rule* *hn-refine-cons*[*OF* - *lso-bex-impl.refine*])
applyS (*simp* *add*: *hn-ctxt-def*; *rule* *entt-refl*)
apply1 *unfold-locales* **apply1** (*rule* *Prl'*) **applyS** *simp*
applyS (*sep-auto* *intro*!: *enttI* *simp*: *hn-ctxt-def*)
applyS (*rule* *entt-refl*)
done
qed

end

3.3 Multiset Interface

theory *IICF-Multiset*
imports *../../Sepref*
begin

3.3.1 Additions to Multiset Theory

lemma *rel-mset-Plus-gen*:

assumes *rel-mset A m m'*
assumes *rel-mset A n n'*
shows *rel-mset A* (*m*+*n*) (*m'*+*n'*)
using *assms*
by *induction* (*auto simp*: *algebra-simps dest*: *rel-mset-Plus*)

lemma *rel-mset-single*:
assumes *A x y*
shows *rel-mset A* {#*x*#} {#*y*#}
unfolding *rel-mset-def*
apply (*rule exI*[**where** *x*=[*x*]])
apply (*rule exI*[**where** *x*=[*y*]])
using *assms* **by** *auto*

lemma *rel-mset-Minus*:
assumes *BIU*: *bi-unique A*
shows ⟦ *rel-mset A m n*; *A x y* ⟧ ⟹ *rel-mset A* (*m*−{#*x*#}) (*n*−{#*y*#})
unfolding *rel-mset-def*
proof *clarsimp*
fix *ml nl*
assume *A*: *A x y*
assume *R*: *list-all2 A ml nl*
show ∃ *ml'*. *mset ml'* = *mset ml* − {#*x*#} ∧
(∃ *nl'*. *mset nl'* = *mset nl* − {#*y*#} ∧ *list-all2 A ml' nl'*)
proof (*cases x*∈*set ml*)
case *False*
have *y* ∉ *set nl* **using** *A R*
apply (*auto simp*: *in-set-conv-decomp list-all2-append2 list-all2-Cons2*)
using *False BIU*[*unfolded bi-unique-alt-def*]
apply (*auto dest*: *left-uniqueD*)
done
with *False R* **show** *?thesis* **by** (*auto simp*: *diff-single-trivial in-multiset-in-set*)
next
case *True*
then obtain *ml1 ml2* **where** [*simp*]: *ml*=*ml1*@*x*#*ml2* **by** (*auto simp*: *in-set-conv-decomp*)
then obtain *nl1 nl2* **where** [*simp*]: *nl*=*nl1*@*y*#*nl2*
and *LA*: *list-all2 A ml1 nl1 list-all2 A ml2 nl2*
using *A R*
apply (*auto simp*: *in-set-conv-decomp list-all2-append1 list-all2-Cons1*)
using *BIU*[*unfolded bi-unique-alt-def*]
apply (*auto dest*: *right-uniqueD*)
done
have
mset (*ml1*@*ml2*) = *mset ml* − {#*x*#}
mset (*nl1*@*nl2*) = *mset nl* − {#*y*#}
using *R*
by (*auto simp*: *algebra-simps add-implies-diff union-assoc*)
moreover have *list-all2 A* (*ml1*@*ml2*) (*nl1*@*nl2*)
by (*rule list-all2-appendI*) *fact*+

ultimately show *?thesis* **by** *blast*
qed
qed

lemma *rel-mset-Minus-gen*:
assumes *BIU*: *bi-unique A*
assumes *rel-mset A m m′*
assumes *rel-mset A n n′*
shows *rel-mset A* (*m*−*n*) (*m′*−*n′*)
using *assms*(*3*,*2*)
apply (*induction R≡A - - rule*: *rel-mset-induct*)
apply (*auto dest*: *rel-mset-Minus*[*OF BIU*] *simp*: *algebra-simps*)
done

lemma *pcr-count*:
assumes *bi-unique A*
shows *rel-fun* (*rel-mset A*) (*rel-fun A* (=)) *count count*
apply (*intro rel-funI*)
unfolding *rel-mset-def*
apply *clarsimp*
subgoal for *x y xs ys*
apply (*rotate-tac*,*induction xs ys rule*: *list-all2-induct*)
using *assms*
by (*auto simp*: *bi-unique-alt-def left-uniqueD right-uniqueD*)
done

3.3.2 Parametricity Setup

definition [*to-relAPP*]: *mset-rel A* ≡ *p2rel* (*rel-mset* (*rel2p A*))

lemma *rel2p-mset*[*rel2p*]: *rel2p* (⟨*A*⟩*mset-rel*) = *rel-mset* (*rel2p A*)
by (*simp add*: *mset-rel-def*)

lemma *p2re-mset*[*p2rel*]: *p2rel* (*rel-mset A*) = ⟨*p2rel A*⟩*mset-rel*
by (*simp add*: *mset-rel-def*)

lemma *mset-rel-empty*[*simp*]:
(*a*,{#})∈⟨*A*⟩*mset-rel* ⟷ *a*={#}
({#},*b*)∈⟨*A*⟩*mset-rel* ⟷ *b*={#}
by (*auto simp*: *mset-rel-def p2rel-def rel-mset-def*)

lemma *param-mset-empty*[*param*]: ({#},{#}) ∈ ⟨*A*⟩*mset-rel*
by *simp*

lemma *param-mset-Plus*[*param*]: ((+),(+))∈⟨*A*⟩*mset-rel* → ⟨*A*⟩*mset-rel* → ⟨*A*⟩*mset-rel*

apply (*rule rel2pD*)
apply (*simp add*: *rel2p*)

apply (*intro rel-funI*)
by (*rule rel-mset-Plus-gen*)

lemma *param-mset-add*[*param*]: (*add-mset*, *add-mset*) $\in A \to \langle A\rangle$*mset-rel* $\to \langle A\rangle$*mset-rel*
apply (*rule rel2pD*)
apply (*simp add*: *rel2p*)
apply (*intro rel-funI*)
by (*rule rel-mset-Plus*)

lemma *param-mset-minus*[*param*]: $[\![$*single-valued A*; *single-valued* $(A^{-1})]\!]$
$\Longrightarrow ((-), (-)) \in \langle A\rangle$*mset-rel* $\to \langle A\rangle$*mset-rel* $\to \langle A\rangle$*mset-rel*
apply (*rule rel2pD*)
apply (*simp add*: *rel2p*)
apply (*intro rel-funI*)
apply (*rule rel-mset-Minus-gen*)
subgoal apply (*unfold IS-LEFT-UNIQUE-def*[*symmetric*])
by (*simp add*: *prop2p bi-unique-alt-def*)
apply (*simp*; *fail*)
apply (*simp*; *fail*)
done

lemma *param-count*[*param*]: $[\![$*single-valued A*; *single-valued* $(A^{-1})]\!] \Longrightarrow$ (*count*,*count*)$\in\langle A\rangle$*mset-rel* $\to A \to$ *nat-rel*
apply (*rule rel2pD*)
apply (*simp add*: *prop2p rel2p*)
apply (*rule pcr-count*)
apply (*simp add*: *bi-unique-alt-def*)
done

lemma *param-set-mset*[*param*]:
shows (*set-mset*, *set-mset*) $\in \langle A\rangle$*mset-rel* $\to \langle A\rangle$*set-rel*
apply (*rule rel2pD*; *simp add*: *rel2p*)
by (*rule multiset.set-transfer*)

definition [*simp*]: *mset-is-empty m* $\equiv$ *m* = {#}

lemma *mset-is-empty-param*[*param*]: (*mset-is-empty*,*mset-is-empty*) $\in \langle A\rangle$*mset-rel* $\to$ *bool-rel*
unfolding *mset-rel-def mset-is-empty-def*[*abs-def*]
by (*auto simp*: *p2rel-def rel-mset-def intro*: *nres-relI*)

3.3.3 Operations

sepref-decl-op *mset-empty*: {#} :: $\langle A\rangle$*mset-rel* **.**

sepref-decl-op *mset-is-empty*: $\lambda m.\ m$={#} :: $\langle A\rangle$*mset-rel* $\to$ *bool-rel*

unfolding *mset-is-empty-def*[*symmetric*]
apply (*rule frefI*)
by *parametricity*

sepref-decl-op *mset-insert*: *add-mset* :: $A \rightarrow \langle A\rangle$*mset-rel* $\rightarrow \langle A\rangle$*mset-rel* **.**

sepref-decl-op *mset-delete*: $\lambda x\, m.\ m - \{\#x\#\}$ :: $A \rightarrow \langle A\rangle$*mset-rel* $\rightarrow \langle A\rangle$*mset-rel*
where *single-valued A single-valued* (A^{-1}) **.**

sepref-decl-op *mset-plus*: (+)::- *multiset* $\Rightarrow$ - :: $\langle A\rangle$*mset-rel* $\rightarrow \langle A\rangle$*mset-rel* $\rightarrow$ $\langle A\rangle$*mset-rel* **.**
sepref-decl-op *mset-minus*: $(-)$::- *multiset* $\Rightarrow$ - :: $\langle A\rangle$*mset-rel* $\rightarrow \langle A\rangle$*mset-rel* $\rightarrow$ $\langle A\rangle$*mset-rel*
where *single-valued A single-valued* (A^{-1}) **.**

sepref-decl-op *mset-contains*: $(\in\#)$ :: $A \rightarrow \langle A\rangle$*mset-rel* $\rightarrow$ *bool-rel*
where *single-valued A single-valued* (A^{-1}) **.**

sepref-decl-op *mset-count*: $\lambda x\ y.$ *count y x* :: $A \rightarrow \langle A\rangle$*mset-rel* $\rightarrow$ *nat-rel*
where *single-valued A single-valued* (A^{-1}) **.**

sepref-decl-op *mset-pick*: $\lambda m.$ *SPEC* $(\lambda(x,m').\ m = \{\#x\#\} + m')$ ::
$[\lambda m.\ m \neq \{\#\}]_f$ $\langle A\rangle$*mset-rel* $\rightarrow A \times_r \langle A\rangle$*mset-rel*
unfolding *mset-is-empty-def*[*symmetric*]
apply (*intro frefI nres-relI*)
apply (*refine-vcg SPEC-refine*)
apply1 (*rule ccontr*; *clarsimp*)
applyS (*metis msed-rel-invL rel2p-def rel2p-mset union-ac*(*2*))
applyS *parametricity*
done

3.3.4 Patterns

lemma [*def-pat-rules*]:
$\{\#\} \equiv$ *op-mset-empty*
add-mset $\equiv$ *op-mset-insert*
(=) \$*b*\$$\{\#\}$ $\equiv$ *op-mset-is-empty*\$*b*
(=) \$$\{\#\}$\$*b* $\equiv$ *op-mset-is-empty*\$*b*
(+) \$*a*\$*b* $\equiv$ *op-mset-plus*\$*a*\$*b*
$(-)$ \$*a*\$*b* $\equiv$ *op-mset-minus*\$*a*\$*b*
by (*auto intro*!: *eq-reflection simp*: *algebra-simps*)

lemma [*def-pat-rules*]:
(+) \$*b*\$(*add-mset*\$*x*\$$\{\#\}$) $\equiv$ *op-mset-insert*\$*x*\$*b*
(+) \$(*add-mset*\$*x*\$$\{\#\}$)\$*b* $\equiv$ *op-mset-insert*\$*x*\$*b*
$(-)$ \$*b*\$(*add-mset*\$*x*\$$\{\#\}$) $\equiv$ *op-mset-delete*\$*x*\$*b*

(<) 0(*count*$*a*$*x*) ≡ *op-mset-contains*$*x*$*a*
(∈) $*x*$(*set-mset*$*a*) ≡ *op-mset-contains*$*x*$*a*
by (*auto intro*!: *eq-reflection simp*: *algebra-simps*)

locale *mset-custom-empty* =
fixes *rel empty* **and** *op-custom-empty* :: *'a multiset*
assumes *customize-hnr-aux*: (*uncurry0 empty*,*uncurry0* (*RETURN* (*op-mset-empty*::*'a multiset*))) ∈ $unit\text{-}assn^k \rightarrow_a rel$
assumes *op-custom-empty-def*: *op-custom-empty* = *op-mset-empty*
begin
sepref-register *op-custom-empty* :: *'ax multiset*

lemma *fold-custom-empty*:
{#} = *op-custom-empty*
op-mset-empty = *op-custom-empty*
mop-mset-empty = *RETURN op-custom-empty*
unfolding *op-custom-empty-def* **by** *simp-all*

lemmas *custom-hnr*[*sepref-fr-rules*] = *customize-hnr-aux*[*folded op-custom-empty-def*]
end

end

3.4 Priority Bag Interface

theory *IICF-Prio-Bag*
imports *IICF-Multiset*
begin

3.4.1 Operations

We prove quite general parametricity lemmas, but restrict them to relations below identity when we register the operations.

This restriction, although not strictly necessary, makes usage of the tool much simpler, as we do not need to handle different prio-functions for abstract and concrete types.

context
fixes *prio*:: *'a* ⇒ *'b*::*linorder*
begin
definition *mop-prio-pop-min b* = *ASSERT* (*b*≠{#}) ≫ *SPEC* (λ(*v*,*b'*).
v ∈# *b*
∧ *b'*=*b* − {#*v*#}
∧ (∀ *v'*∈*set-mset b*. *prio v* ≤ *prio v'*))

definition *mop-prio-peek-min b* ≡ *ASSERT* (*b*≠{#}) ≫ *SPEC* (λ*v*.
v ∈# *b*

$\wedge$ ($\forall$ v'∈*set-mset b. prio v* $\leq$ *prio v'*))

end

lemma *param-mop-prio-pop-min*[*param*]:
assumes [*param*]: (*prio'*,*prio*) ∈ *A* → *B*
assumes [*param*]: ((≤),(≤)) ∈ *B* → *B* → *bool-rel*
shows (*mop-prio-pop-min prio'*,*mop-prio-pop-min prio*) ∈ ⟨*A*⟩*mset-rel* → ⟨*A* $\times_r$ ⟨*A*⟩*mset-rel*⟩*nres-rel*
unfolding *mop-prio-pop-min-def*[*abs-def*]
apply (*clarsimp simp*: *mop-prio-pop-min-def nres-rel-def pw-le-iff refine-pw-simps*)
apply (*safe*; *simp*)
proof *goal-cases*
case (*1 m n x*)

assume (*m*,*n*)∈⟨*A*⟩*mset-rel*
and *x*∈#*m*
and *P'*: ∀ *x'*∈*set-mset m. prio' x* ≤ *prio' x'*
hence *R*: *rel-mset* (*rel2p A*) *m n* **by** (*simp add*: *mset-rel-def p2rel-def*)
from *multi-member-split*[*OF* ‹*x*∈#*m*›] **obtain** *m'* **where** [*simp*]: *m*=*m'*+{#*x*#} **by** *auto*

from *msed-rel-invL*[*OF R*[*simplified*]] **obtain** *n' y* **where**
[*simp*]: *n*=*n'*+{#*y*#} **and** [*param*, *simp*]: (*x*,*y*)∈*A* **and** *R'*: (*m'*,*n'*)∈⟨*A*⟩*mset-rel*
by (*auto simp*: *rel2p-def mset-rel-def p2rel-def*)
have ∀ *y'*∈*set-mset n. prio y* ≤ *prio y'*
proof
fix *y'* **assume** *y'*∈*set-mset n*
then obtain *x'* **where** [*param*]: (*x'*,*y'*)∈*A* **and** *x'*∈*set-mset m*
using *R*
by (*metis insert-DiffM msed-rel-invR rel2pD union-single-eq-member*)
with *P'* **have** *prio' x* ≤ *prio' x'* **by** *blast*
moreover have (*prio' x* ≤ *prio' x'*, *prio y* ≤ *prio y'*) ∈ *bool-rel*
by *parametricity*
ultimately show *prio y* ≤ *prio y'* **by** *simp*
qed
thus
∃ *a*. (*x*, *a*) ∈ *A* ∧ (*m* − {#*x*#}, *n* − {#*a*#}) ∈ ⟨*A*⟩*mset-rel* ∧ *a* ∈# *n* ∧ (∀ *v'*∈*set-mset n. prio a* ≤ *prio v'*)
using *R'* **by** (*auto intro*!: *exI*[**where** *x*=*n'*] *exI*[**where** *x*=*y*])
qed

lemma *param-mop-prio-peek-min*[*param*]:
assumes [*param*]: (*prio'*,*prio*) ∈ *A* → *B*
assumes [*param*]: ((≤),(≤)) ∈ *B* → *B* → *bool-rel*
shows (*mop-prio-peek-min prio'*,*mop-prio-peek-min prio*) ∈ ⟨*A*⟩*mset-rel* → ⟨*A*⟩*nres-rel*
unfolding *mop-prio-peek-min-def*[*abs-def*]
apply (*clarsimp*

simp: *mop-prio-pop-min-def nres-rel-def pw-le-iff refine-pw-simps*
)
apply (*safe*; *simp?*)
proof −
fix *m n x*
assume $(m,n) \in \langle A\rangle mset\text{-}rel$
and $x \in\# m$
and P': $\forall x' \in set\text{-}mset\ m.\ prio'\ x \le prio'\ x'$
hence *R*: *rel-mset* (*rel2p A*) *m n* **by** (*simp add*: *mset-rel-def p2rel-def*)
from *multi-member-split*[*OF* ‹$x \in\# m$›] **obtain** m' **where** [*simp*]: $m=m'+\{\#x\#\}$
by *auto*

from *msed-rel-invL*[*OF R*[*simplified*]] **obtain** n' *y* **where**
[*simp*]: $n=n'+\{\#y\#\}$ **and** [*param, simp*]: $(x,y) \in A$ **and** R': $(m',n') \in \langle A\rangle mset\text{-}rel$
by (*auto simp*: *rel2p-def mset-rel-def p2rel-def*)

have $\forall y' \in set\text{-}mset\ n.\ prio\ y \le prio\ y'$
proof
fix y' **assume** $y' \in set\text{-}mset\ n$
then obtain x' **where** [*param*]: $(x',y') \in A$ **and** $x' \in set\text{-}mset\ m$
using *R*
by (*metis msed-rel-invR mset-contains-eq rel2pD union-mset-add-mset-left*
union-single-eq-member)
with P' **have** $prio'\ x \le prio'\ x'$ **by** *blast*
moreover have $(prio'\ x \le prio'\ x',\ prio\ y \le prio\ y') \in bool\text{-}rel$
by *parametricity*
ultimately show $prio\ y \le prio\ y'$ **by** *simp*
qed
thus $\exists y.\ (x,\ y) \in A \wedge y \in\# n \wedge (\forall v' \in set\text{-}mset\ n.\ prio\ y \le prio\ v')$
using R' **by** (*auto intro*!: *exI*[**where** $x=y$])
qed

context fixes *prio* :: $'a \Rightarrow\ 'b::linorder$ **and** *A* :: $('a \times 'a)$ *set* **begin**
sepref-decl-op (*no-def*,*no-mop*) *prio-pop-min*:
PR-CONST (*mop-prio-pop-min prio*) :: $\langle A\rangle mset\text{-}rel \to_f \langle A \times_r \langle A\rangle mset\text{-}rel\rangle nres\text{-}rel$
where *IS-BELOW-ID A*
proof *goal-cases*
case *1*
hence [*param*]: $(prio,prio) \in A \to Id$
by (*auto simp*: *IS-BELOW-ID-def*)
show *?case*
apply (*rule fref-ncI*)
apply *parametricity*
by *auto*
qed

sepref-decl-op (*no-def*,*no-mop*) *prio-peek-min*:
PR-CONST (*mop-prio-peek-min prio*) :: $\langle A\rangle$*mset-rel* $\rightarrow_f$ $\langle A\rangle$*nres-rel*
where *IS-BELOW-ID A*
proof *goal-cases*
case *1*
hence [*param*]: (*prio*,*prio*)∈*A* → *Id*
by (*auto simp*: *IS-BELOW-ID-def*)
show *?case*
apply (*rule fref-ncI*)
apply *parametricity*
by *auto*
qed
end

3.4.2 Patterns

lemma [*def-pat-rules*]:
mop-prio-pop-min$prio ≡ *UNPROTECT* (*mop-prio-pop-min prio*)
mop-prio-peek-min$prio ≡ *UNPROTECT* (*mop-prio-peek-min prio*)
by *auto*

end

3.5 Multisets by Lists

theory *IICF-List-Mset*
imports ../*Intf*/*IICF-Multiset*
begin

3.5.1 Abstract Operations

definition *list-mset-rel* ≡ *br mset* (λ-. *True*)

lemma *lms-empty-aref*: ([],*op-mset-empty*) ∈ *list-mset-rel*
unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

lemma *lms-is-empty-aref*: (*is-Nil*,*op-mset-is-empty*) ∈ *list-mset-rel* → *bool-rel*
unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv split*: *list.splits*)

lemma *lms-insert-aref*: ((#), *op-mset-insert*) ∈ *Id* → *list-mset-rel* → *list-mset-rel*
unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

lemma *lms-union-aref*: ((@), *op-mset-plus*) ∈ *list-mset-rel* → *list-mset-rel* → *list-mset-rel*
unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

lemma *lms-pick-aref*: $(\lambda x\#l \Rightarrow RETURN\ (x,l),\ mop\text{-}mset\text{-}pick) \in list\text{-}mset\text{-}rel \rightarrow \langle Id \times_r list\text{-}mset\text{-}rel\rangle nres\text{-}rel$
unfolding *list-mset-rel-def mop-mset-pick-alt*[*abs-def*]
apply1 (*refine-vcg nres-relI fun-relI*)
apply1 (*clarsimp simp*: *in-br-conv neq-Nil-conv*)
apply1 (*refine-vcg RETURN-SPEC-refine*)
applyS (*clarsimp simp*: *in-br-conv algebra-simps*)
done

definition *list-contains* $x\ l \equiv\ list\text{-}ex\ ((=)\ x)\ l$
lemma *lms-contains-aref*: $(list\text{-}contains,\ op\text{-}mset\text{-}contains) \in Id \rightarrow list\text{-}mset\text{-}rel \rightarrow bool\text{-}rel$
unfolding *list-mset-rel-def list-contains-def*[*abs-def*]
by (*auto simp*: *in-br-conv list-ex-iff in-multiset-in-set*)

fun *list-remove1* :: $'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
list-remove1 x [] = []
| *list-remove1* x $(y\#ys)$ = (*if* $x=y$ *then* ys *else* $y\#list\text{-}remove1\ x\ ys$)

lemma *mset-list-remove1*[*simp*]: $mset\ (list\text{-}remove1\ x\ l) = mset\ l - \{\#x\#\}$
apply (*induction l*)
applyS *simp*
by (*clarsimp simp*: *algebra-simps*)

lemma *lms-remove-aref*: $(list\text{-}remove1,\ op\text{-}mset\text{-}delete) \in Id \rightarrow list\text{-}mset\text{-}rel \rightarrow list\text{-}mset\text{-}rel$
unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

fun *list-count* :: $'a \Rightarrow 'a\ list \Rightarrow nat$ **where**
list-count - [] = *0*
| *list-count* x $(y\#ys)$ = (*if* $x=y$ *then* $1 + list\text{-}count\ x\ ys$ *else* $list\text{-}count\ x\ ys$)

lemma *mset-list-count*[*simp*]: $list\text{-}count\ x\ ys = count\ (mset\ ys)\ x$
by (*induction ys*) *auto*

lemma *lms-count-aref*: $(list\text{-}count,\ op\text{-}mset\text{-}count) \in Id \rightarrow list\text{-}mset\text{-}rel \rightarrow nat\text{-}rel$

unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

definition *list-remove-all* :: $'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
$list\text{-}remove\text{-}all\ xs\ ys \equiv fold\ list\text{-}remove1\ ys\ xs$
lemma *list-remove-all-mset*[*simp*]: $mset\ (list\text{-}remove\text{-}all\ xs\ ys) = mset\ xs - mset\ ys$
unfolding *list-remove-all-def*
by (*induction ys arbitrary*: *xs*) (*auto simp*: *algebra-simps*)

lemma *lms-minus-aref*: $(list\text{-}remove\text{-}all,op\text{-}mset\text{-}minus) \in list\text{-}mset\text{-}rel \rightarrow list\text{-}mset\text{-}rel \rightarrow list\text{-}mset\text{-}rel$

unfolding *list-mset-rel-def* **by** (*auto simp*: *in-br-conv*)

3.5.2 Declaration of Implementations

definition *list-mset-assn A* ≡ *pure* (*list-mset-rel O* ⟨*the-pure A*⟩*mset-rel*)
declare *list-mset-assn-def*[*symmetric*,*fcomp-norm-unfold*]
lemma [*safe-constraint-rules*]: *is-pure* (*list-mset-assn A*) **unfolding** *list-mset-assn-def* **by** *simp*

sepref-decl-impl (*no-register*) *lms-empty*: *lms-empty-aref*[*sepref-param*] **.**

definition [*simp*]: *op-list-mset-empty* ≡ *op-mset-empty*
lemma *lms-fold-custom-empty*:
{#} = *op-list-mset-empty*
op-mset-empty = *op-list-mset-empty*
by *auto*
sepref-register *op-list-mset-empty*
lemmas [*sepref-fr-rules*] = *lms-empty-hnr*[*folded op-list-mset-empty-def*]

sepref-decl-impl *lms-is-empty*: *lms-is-empty-aref*[*sepref-param*] **.**
sepref-decl-impl *lms-insert*: *lms-insert-aref*[*sepref-param*] **.**
sepref-decl-impl *lms-union*: *lms-union-aref*[*sepref-param*] **.**

— Some extra work is required for nondetermistic ops
lemma *lms-pick-aref'*:
($\lambda x\#l \Rightarrow$ *return* (x,l), *mop-mset-pick*) ∈ (*pure list-mset-rel*)^k $\rightarrow_a$ *prod-assn id-assn* (*pure list-mset-rel*)
apply (*simp only*: *prod-assn-pure-conv*)
apply *sepref-to-hoare*
apply (*sep-auto simp*: *refine-pw-simps list-mset-rel-def in-br-conv algebra-simps eintros del*: *exI*)
done
sepref-decl-impl (*ismop*) *lms-pick*: *lms-pick-aref'* **.**
sepref-decl-impl *lms-contains*: *lms-contains-aref*[*sepref-param*] **.**
sepref-decl-impl *lms-remove*: *lms-remove-aref*[*sepref-param*] **.**
sepref-decl-impl *lms-count*: *lms-count-aref*[*sepref-param*] **.**
sepref-decl-impl *lms-minus*: *lms-minus-aref*[*sepref-param*] **.**

end
theory *IICF-List-MsetO*
imports ../*Intf*/*IICF-Multiset*
begin

definition *lmso-assn A* ≡ *hr-comp* (*list-assn A*) (*br mset* (λ-. *True*))
lemmas [*fcomp-norm-unfold*] = *lmso-assn-def*[*symmetric*]

lemma *lmso-is-pure*[*safe-constraint-rules*]: *is-pure A* $\Longrightarrow$ *is-pure* (*lmso-assn A*)
unfolding *lmso-assn-def* **by** *safe-constraint*

lemma *lmso-empty-aref*: (*uncurry0* (*RETURN* []), *uncurry0* (*RETURN op-mset-empty*)) $\in$ *unit-rel* $\rightarrow_f$ $\langle$*br mset* (λ-. *True*)$\rangle$*nres-rel*
by (*auto intro*!: *frefI nres-relI simp*: *in-br-conv*)

lemma *lmso-is-empty-aref*: (*RETURN o List.null*, *RETURN o op-mset-is-empty*) $\in$ *br mset* (λ-. *True*) $\rightarrow_f$ $\langle$*bool-rel*$\rangle$*nres-rel*
by (*auto intro*!: *frefI nres-relI simp*: *in-br-conv List.null-def split*: *list.split*)

lemma *lmso-insert-aref*: (*uncurry* (*RETURN oo* (#)), *uncurry* (*RETURN oo op-mset-insert*)) $\in$ (*Id* $\times_r$ *br mset* (λ-. *True*)) $\rightarrow_f$ $\langle$*br mset* (λ-. *True*)$\rangle$*nres-rel*
by (*auto intro*!: *frefI nres-relI simp*: *in-br-conv*)

definition [*simp*]: *hd-tl l* $\equiv$ (*hd l*, *tl l*)

lemma *hd-tl-opt*[*sepref-opt-simps*]: *hd-tl l* = (*case l of* (*x#xs*) $\Rightarrow$ (*x*,*xs*) | - $\Rightarrow$ *CODE-ABORT* (λ-. (*hd l*, *tl l*)))
by (*auto split*: *list.split*)

lemma *lmso-pick-aref*: (*RETURN o hd-tl*,*op-mset-pick*) $\in$ [λ*m*. *m*$\neq${#}]$_f$ *br mset* (λ-. *True*) $\rightarrow$ $\langle$*Id* $\times_r$ *br mset* (λ-. *True*)$\rangle$*nres-rel*
by (*auto intro*!: *frefI nres-relI simp*: *in-br-conv pw-le-iff refine-pw-simps neq-Nil-conv algebra-simps*)

lemma *hd-tl-hnr*: (*return o hd-tl*,*RETURN o hd-tl*) $\in$ [λ*l*. $\neg$*is-Nil l*]$_a$ (*list-assn A*)^d $\rightarrow$ *prod-assn A* (*list-assn A*)
apply *sepref-to-hoare*
subgoal for *l li* **by** (*cases l*; *cases li*; *sep-auto*)
done

sepref-decl-impl (*no-register*) *lmso-empty*: *hn-Nil*[*to-hfref*] **uses** *lmso-empty-aref*
.
definition [*simp*]: *op-lmso-empty* $\equiv$ *op-mset-empty*
sepref-register *op-lmso-empty*
lemma *lmso-fold-custom-empty*:
{#} = *op-lmso-empty*
op-mset-empty = *op-lmso-empty*
mop-mset-empty = *RETURN op-lmso-empty*
by *auto*
lemmas [*sepref-fr-rules*] = *lmso-empty-hnr*[*folded op-lmso-empty-def*]

lemma *list-null-hnr*: (*return o List.null*,*RETURN o List.null*) ∈ (*list-assn A*)^k $\rightarrow_a$ *bool-assn*
  apply *sepref-to-hoare*
  subgoal for *l li* **by** (*cases l*; *cases li*; *sep-auto simp*: *List.null-def*)
  done

sepref-decl-impl *lmso-is-empty*: *list-null-hnr* **uses** *lmso-is-empty-aref* **.**

sepref-decl-impl *lmso-insert*: *hn-Cons*[*to-hfref*] **uses** *lmso-insert-aref* **.**

context notes [*simp*] = *in-br-conv* **and** [*split*] = *list.splits* **begin**

Dummy lemma, to exloit *sepref-decl-impl* automation without parametricity stuff.

private lemma *op-mset-pick-dummy-param*: (*op-mset-pick*, *op-mset-pick*) ∈ *Id* $\rightarrow_f$ ⟨*Id*⟩*nres-rel*
  by (*auto intro*!: *frefI nres-relI*)

sepref-decl-impl *lmso-pick*: *hd-tl-hnr*[*FCOMP lmso-pick-aref*] **uses** *op-mset-pick-dummy-param* **by** *simp*
end

end
theory *IICF-List*
imports
  ../../Sepref
  List−Index.List-Index
begin

lemma *param-index*[*param*]:
  ⟦*single-valued A*; *single-valued* (A^{-1})⟧ ⟹ (*index*,*index*) ∈ ⟨*A*⟩*list-rel* → *A* → *nat-rel*
  unfolding *index-def*[*abs-def*] *find-index-def*
  apply (*subgoal-tac* (((=), (=)) ∈ *A* → *A* → *bool-rel*))
  apply *parametricity*
  by (*simp add*: *pres-eq-iff-svb*)

3.5.3 Swap two elements of a list, by index

definition *swap l i j* ≡ *l*[*i* := *l*!*j*, *j*:=*l*!*i*]
lemma *swap-nth*[*simp*]: ⟦*i* < *length l*; *j*<*length l*; *k*<*length l*⟧ ⟹
  swap l i j!*k* = (
    if k=*i then l*!*j*
    else if k=*j then l*!*i*
    else l!*k*
)
  unfolding *swap-def*

by *auto*

lemma *swap-set*[*simp*]: $\llbracket$ *i* < *length l*; *j*<*length l* $\rrbracket$ $\Longrightarrow$ *set* (*swap l i j*) = *set l*
unfolding *swap-def*
by *auto*

lemma *swap-multiset*[*simp*]: $\llbracket$ *i* < *length l*; *j*<*length l* $\rrbracket$ $\Longrightarrow$ *mset* (*swap l i j*) = *mset l*
unfolding *swap-def*
by (*auto simp*: *mset-swap*)

lemma *swap-length*[*simp*]: *length* (*swap l i j*) = *length l*
unfolding *swap-def*
by *auto*

lemma *swap-same*[*simp*]: *swap l i i* = *l*
unfolding *swap-def* **by** *auto*

lemma *distinct-swap*[*simp*]:
$\llbracket$*i*<*length l*; *j*<*length l*$\rrbracket$ $\Longrightarrow$ *distinct* (*swap l i j*) = *distinct l*
unfolding *swap-def*
by *auto*

lemma *map-swap*: $\llbracket$*i*<*length l*; *j*<*length l*$\rrbracket$
$\Longrightarrow$ *map f* (*swap l i j*) = *swap* (*map f l*) *i j*
unfolding *swap-def*
by (*auto simp add*: *map-update*)

lemma *swap-param*[*param*]: $\llbracket$ *i*<*length l*; *j*<*length l*; (*l'*,*l*)∈⟨*A*⟩*list-rel*; (*i'*,*i*)∈*nat-rel*; (*j'*,*j*)∈*nat-rel*$\rrbracket$
$\Longrightarrow$ (*swap l' i' j'*, *swap l i j*)∈⟨*A*⟩*list-rel*
unfolding *swap-def*
by *parametricity*

lemma *swap-param-fref*: (*uncurry2 swap*,*uncurry2 swap*) ∈
[λ((*l*,*i*),*j*). *i*<*length l* ∧ *j*<*length l*]$_f$ (⟨*A*⟩*list-rel* $\times_r$ *nat-rel*) $\times_r$ *nat-rel* → ⟨*A*⟩*list-rel*
apply *rule* **apply** *clarsimp*
unfolding *swap-def*
apply *parametricity*
by *simp-all*

lemma *param-list-null*[*param*]: (*List.null*,*List.null*) ∈ ⟨*A*⟩*list-rel* → *bool-rel*
proof −
have *1*: *List.null* = (λ[] ⇒ *True* | - ⇒ *False*)
apply (*rule ext*) **subgoal for** *l* **by** (*cases l*) (*auto simp*: *List.null-def*)
done
show *?thesis* **unfolding** *1* **by** *parametricity*
qed

3.5.4 Operations

sepref-decl-op *list-empty*: $[]$ :: $\langle A\rangle$*list-rel* **.**
context notes [*simp*] = *eq-Nil-null* **begin**
  sepref-decl-op *list-is-empty*: $\lambda l.\ l{=}[]$ :: $\langle A\rangle$*list-rel* $\rightarrow_f$ *bool-rel* **.**
end
sepref-decl-op *list-replicate*: *replicate* :: *nat-rel* $\rightarrow$ *A* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
definition *op-list-copy* :: *'a list* $\Rightarrow$ *'a list* **where** [*simp*]: *op-list-copy l* $\equiv$ *l*
sepref-decl-op (*no-def*) *list-copy*: *op-list-copy* :: $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-prepend*: (#) :: *A* $\rightarrow$ $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-append*: $\lambda xs\ x.\ xs@[x]$ :: $\langle A\rangle$*list-rel* $\rightarrow$ *A* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-concat*: (@) :: $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-length*: *length* :: $\langle A\rangle$*list-rel* $\rightarrow$ *nat-rel* **.**
sepref-decl-op *list-get*: *nth* :: $[\lambda(l,i).\ i{<}length\ l]_f$ $\langle A\rangle$*list-rel* $\times_r$ *nat-rel* $\rightarrow$ *A* **.**
sepref-decl-op *list-set*: *list-update* :: $[\lambda((l,i),\text{-}).\ i{<}length\ l]_f$ ($\langle A\rangle$*list-rel* $\times_r$ *nat-rel*) $\times_r$ *A* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
context notes [*simp*] = *eq-Nil-null* **begin**
  sepref-decl-op *list-hd*: *hd* :: $[\lambda l.\ l{\neq}[]]_f$ $\langle A\rangle$*list-rel* $\rightarrow$ *A* **.**
  sepref-decl-op *list-tl*: *tl* :: $[\lambda l.\ l{\neq}[]]_f$ $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
  sepref-decl-op *list-last*: *last* :: $[\lambda l.\ l{\neq}[]]_f$ $\langle A\rangle$*list-rel* $\rightarrow$ *A* **.**
  sepref-decl-op *list-butlast*: *butlast* :: $[\lambda l.\ l{\neq}[]]_f$ $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
end
sepref-decl-op *list-contains*: $\lambda x\ l.\ x{\in}set\ l$ :: *A* $\rightarrow$ $\langle A\rangle$*list-rel* $\rightarrow$ *bool-rel*
  where *single-valued A single-valued* (A^{-1}) **.**
sepref-decl-op *list-swap*: *swap* :: $[\lambda((l,i),j).\ i{<}length\ l \land j{<}length\ l]_f$ (($\langle A\rangle$*list-rel* $\times_r$ *nat-rel*) $\times_r$ *nat-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-rotate1*: *rotate1* :: $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-rev*: *rev* :: $\langle A\rangle$*list-rel* $\rightarrow$ $\langle A\rangle$*list-rel* **.**
sepref-decl-op *list-index*: *index* :: $\langle A\rangle$*list-rel* $\rightarrow$ *A* $\rightarrow$ *nat-rel*
  where *single-valued A single-valued* (A^{-1}) **.**

3.5.5 Patterns

lemma [*def-pat-rules*]:
  [] $\equiv$ *op-list-empty*
  (=) \$*l*\$[] $\equiv$ *op-list-is-empty*\$*l*
  (=) \$[]\$*l* $\equiv$ *op-list-is-empty*\$*l*
  replicate\$*n*\$*v* $\equiv$ *op-list-replicate*\$*n*\$*v*
  Cons\$*x*\$*xs* $\equiv$ *op-list-prepend*\$*x*\$*xs*
  (@) \$*xs*\$(*Cons*\$*x*\$[]) $\equiv$ *op-list-append*\$*xs*\$*x*
  (@) \$*xs*\$*ys* $\equiv$ *op-list-concat*\$*xs*\$*ys*
  op-list-concat\$*xs*\$(*Cons*\$*x*\$[]) $\equiv$ *op-list-append*\$*xs*\$*x*
  length\$*xs* $\equiv$ *op-list-length*\$*xs*
  nth\$*l*\$*i* $\equiv$ *op-list-get*\$*l*\$*i*
  list-update\$*l*\$*i*\$*x* $\equiv$ *op-list-set*\$*l*\$*i*\$*x*
  hd\$*l* $\equiv$ *op-list-hd*\$*l*
  hd\$*l* $\equiv$ *op-list-hd*\$*l*
  tl\$*l* $\equiv$ *op-list-tl*\$*l*
  tl\$*l* $\equiv$ *op-list-tl*\$*l*
  last\$*l* $\equiv$ *op-list-last*\$*l*

butlast$*l* ≡ *op-list-butlast*$*l*
(∈) $*x*$(*set*$*l*) ≡ *op-list-contains*$*x*$*l*
swap$*l*$*i*$*j* ≡ *op-list-swap*$*l*$*i*$*j*
rotate1$*l* ≡ *op-list-rotate1*$*l*
rev$*l* ≡ *op-list-rev*$*l*
index$*l*$*x* ≡ *op-list-index*$*l*$*x*
by (*auto intro*!: *eq-reflection*)

Standard preconditions are preserved by list-relation. These lemmas are used for simplification of preconditions after composition.

lemma *list-rel-pres-neq-nil*[*fcomp-prenorm-simps*]: $(x',x) \in \langle A \rangle$*list-rel* ⟹ $x' \neq [] \longleftrightarrow x \neq []$ **by** *auto*
lemma *list-rel-pres-length*[*fcomp-prenorm-simps*]: $(x',x) \in \langle A \rangle$*list-rel* ⟹ *length* x' = *length* x **by** (*rule list-rel-imp-same-length*)

locale *list-custom-empty* =
fixes *rel empty* **and** *op-custom-empty* :: *'a list*
assumes *customize-hnr-aux*: (*uncurry0 empty*,*uncurry0* (*RETURN* (*op-list-empty*::*'a list*))) ∈ *unit-assn*^{k} $\rightarrow_a$ *rel*
assumes *op-custom-empty-def*: *op-custom-empty* = *op-list-empty*
begin
sepref-register *op-custom-empty* :: *'c list*

lemma *fold-custom-empty*:
[] = *op-custom-empty*
op-list-empty = *op-custom-empty*
mop-list-empty = *RETURN op-custom-empty*
unfolding *op-custom-empty-def* **by** *simp-all*

lemmas *custom-hnr*[*sepref-fr-rules*] = *customize-hnr-aux*[*folded op-custom-empty-def*]
end

lemma *gen-mop-list-swap*: *mop-list-swap l i j* = *do* {
xi ← *mop-list-get l i*;
xj ← *mop-list-get l j*;
l ← *mop-list-set l i xj*;
l ← *mop-list-set l j xi*;
RETURN l
}
unfolding *mop-list-swap-def*
by (*auto simp*: *pw-eq-iff refine-pw-simps swap-def*)

end

3.6 Heap Implementation On Lists

theory *IICF-Abs-Heap*

imports
HOL−Library.Multiset
../../../Sepref
List−Index.List-Index
../../Intf/IICF-List
../../Intf/IICF-Prio-Bag
begin

We define Min-Heaps, which implement multisets of prioritized values. The operations are: empty heap, emptiness check, insert an element, remove a minimum priority element.

3.6.1 Basic Definitions

type-synonym $'a\ heap = 'a\ list$

locale *heapstruct* =
  fixes $prio :: 'a \Rightarrow 'b::linorder$
begin
  definition $valid :: 'a\ heap \Rightarrow nat \Rightarrow bool$
    where $valid\ h\ i \equiv i>0 \wedge i\leq length\ h$

  abbreviation $\alpha :: 'a\ heap \Rightarrow 'a\ multiset$ **where** $\alpha \equiv mset$

  lemma *valid-empty*[*simp*]: $\neg valid\ []\ i$ **by** (*auto simp*: *valid-def*)
  lemma *valid0*[*simp*]: $\neg valid\ h\ 0$ **by** (*auto simp*: *valid-def*)
  lemma *valid-glen*[*simp*]: $i>length\ h \Longrightarrow \neg valid\ h\ i$ **by** (*auto simp*: *valid-def*)

  lemma *valid-len*[*simp*]: $h\neq[] \Longrightarrow valid\ h\ (length\ h)$ **by** (*auto simp*: *valid-def*)

  lemma *validI*: $0<i \Longrightarrow i\leq length\ h \Longrightarrow valid\ h\ i$
    by (*auto simp*: *valid-def*)

  definition $val\text{-}of :: 'a\ heap \Rightarrow nat \Rightarrow 'a$ **where** $val\text{-}of\ l\ i \equiv l!(i-1)$
  abbreviation $prio\text{-}of :: 'a\ heap \Rightarrow nat \Rightarrow 'b$ **where**
    $prio\text{-}of\ l\ i \equiv prio\ (val\text{-}of\ l\ i)$

Navigating the tree

  definition $parent :: nat \Rightarrow nat$ **where** $parent\ i \equiv i\ div\ 2$
  definition $left :: nat \Rightarrow nat$ **where** $left\ i \equiv 2*i$
  definition $right :: nat \Rightarrow nat$ **where** $right\ i \equiv 2*i + 1$

  abbreviation $has\text{-}parent\ h\ i \equiv valid\ h\ (parent\ i)$
  abbreviation $has\text{-}left\ h\ i \equiv valid\ h\ (left\ i)$
  abbreviation $has\text{-}right\ h\ i \equiv valid\ h\ (right\ i)$

  abbreviation $vparent\ h\ i == val\text{-}of\ h\ (parent\ i)$

abbreviation *vleft h i* == *val-of h* (*left i*)
abbreviation *vright h i* == *val-of h* (*right i*)

abbreviation *pparent h i* == *prio-of h* (*parent i*)
abbreviation *pleft h i* == *prio-of h* (*left i*)
abbreviation *pright h i* == *prio-of h* (*right i*)

lemma *parent-left-id*[*simp*]: *parent* (*left i*) = *i*
unfolding *parent-def left-def*
by *auto*

lemma *parent-right-id*[*simp*]: *parent* (*right i*) = *i*
unfolding *parent-def right-def*
by *auto*

lemma *child-of-parentD*:
has-parent l i $\Longrightarrow$ *left* (*parent i*) = *i* $\vee$ *right* (*parent i*) = *i*
unfolding *parent-def left-def right-def valid-def*
by *auto*

lemma *rc-imp-lc*: $[\![$*valid h i*; *has-right h i*$]\!]$ $\Longrightarrow$ *has-left h i*
by (*auto simp*: *valid-def left-def right-def*)

lemma *plr-corner-cases*[*simp*]:
assumes *0*<*i*
shows
i$\neq$*parent i*
i$\neq$*left i*
i$\neq$*right i*
parent i $\neq$ *i*
left i $\neq$ *i*
right i $\neq$ *i*
using *assms*
by (*auto simp*: *parent-def left-def right-def*)

lemma *i-eq-parent-conv*[*simp*]: *i*=*parent i* $\longleftrightarrow$ *i*=*0*
by (*auto simp*: *parent-def*)

Heap Property

The heap property states, that every node's priority is greater or equal to its parent's priority

definition *heap-invar* :: *'a heap* $\Rightarrow$ *bool*
where *heap-invar l*
$\equiv$ $\forall$ *i*. *valid l i* $\longrightarrow$ *has-parent l i* $\longrightarrow$ *pparent l i* $\leq$ *prio-of l i*

definition *heap-rel1* $\equiv$ *br* α *heap-invar*

```
lemma heap-invar-empty[simp]: heap-invar []
  by (auto simp: heap-invar-def)

function heap-induction-scheme :: nat ⇒ unit where
  heap-induction-scheme i = (
    if i>1 then heap-induction-scheme (parent i) else ())
  by pat-completeness auto

termination
  apply (relation less-than)
  apply (auto simp: parent-def)
  done

lemma
  heap-parent-le: ⟦heap-invar l; valid l i; has-parent l i⟧
    ⟹ pparent l i ≤ prio-of l i
  unfolding heap-invar-def
  by auto

lemma heap-min-prop:
  assumes H: heap-invar h
  assumes V: valid h i
  shows prio-of h (Suc 0) ≤ prio-of h i
proof (cases i>1)
  case False with V show ?thesis
    by (auto simp: valid-def intro: Suc-lessI)
next
  case True
  from V have i≤length h valid h (Suc 0) by (auto simp: valid-def)
  with True show ?thesis
    apply (induction i rule: heap-induction-scheme.induct)
    apply (rename-tac i)
    apply (case-tac parent i = Suc 0)
    apply (rule order-trans[rotated])
    apply (rule heap-parent-le[OF H])
    apply (auto simp: valid-def) [3]

    apply (rule order-trans)
    apply (rprems)
    apply (auto simp: parent-def) [4]
    apply (rule heap-parent-le[OF H])
    apply (auto simp: valid-def parent-def)
    done
qed
```

Obviously, the heap property can also be stated in terms of children, i.e., each node's priority is smaller or equal to it's children's priority.

```
definition children-ge h p i ≡
  (has-left h i ⟶ p ≤ pleft h i)
```

$\wedge$ (*has-right h i* $\longrightarrow$ $p \leq$ *pright h i*)

definition *heap-invar′ h* $\equiv$ $\forall$ *i. valid h i* $\longrightarrow$ *children-ge h* (*prio-of h i*) *i*

lemma *heap-eq-heap′*:
  shows *heap-invar h* $\longleftrightarrow$ *heap-invar′ h*
  unfolding *heap-invar-def*
  unfolding *heap-invar′-def children-ge-def*
  apply *rule*
  apply *auto* []
  apply *clarsimp*
  apply (*frule child-of-parentD*)
  apply *auto* []
  done

3.6.2 Basic Operations

The basic operations are the only operations that directly modify the underlying data structure.

Val-Of

abbreviation (*input*) *val-of-pre l i* $\equiv$ *valid l i*
definition *val-of-op* :: *′a heap* $\Rightarrow$ *nat* $\Rightarrow$ *′a nres*
  where *val-of-op l i* $\equiv$ *ASSERT* (*i*>*0*) $\gg$ *mop-list-get l* (*i*−*1*)
lemma *val-of-correct*[*refine-vcg*]:
  val-of-pre l i $\Longrightarrow$ *val-of-op l i* $\leq$ *SPEC* (λ*r. r* = *val-of l i*)
  unfolding *val-of-op-def val-of-def valid-def*
  by *refine-vcg auto*

abbreviation (*input*) *prio-of-pre* $\equiv$ *val-of-pre*
definition *prio-of-op l i* $\equiv$ *do* {*v* $\leftarrow$ *val-of-op l i*; *RETURN* (*prio v*)}
lemma *prio-of-op-correct*[*refine-vcg*]:
  prio-of-pre l i $\Longrightarrow$ *prio-of-op l i* $\leq$ *SPEC* (λ*r. r* = *prio-of l i*)
  unfolding *prio-of-op-def*
  apply *refine-vcg* **by** *simp*

Update

abbreviation *update-pre h i v* $\equiv$ *valid h i*
definition *update* :: *′a heap* $\Rightarrow$ *nat* $\Rightarrow$ *′a* $\Rightarrow$ *′a heap*
  where *update h i v* $\equiv$ *h*[*i* − *1* := *v*]
definition *update-op* :: *′a heap* $\Rightarrow$ *nat* $\Rightarrow$ *′a* $\Rightarrow$ *′a heap nres*
  where *update-op h i v* $\equiv$ *ASSERT* (*i*>*0*) $\gg$ *mop-list-set h* (*i*−*1*) *v*
lemma *update-correct*[*refine-vcg*]:
  update-pre h i v $\Longrightarrow$ *update-op h i v* $\leq$ *SPEC*(λ*r. r* = *update h i v*)
  unfolding *update-op-def update-def valid-def* **by** *refine-vcg auto*

lemma *update-valid*[*simp*]: *valid* (*update h i v*) *j* $\longleftrightarrow$ *valid h j*

by (*auto simp*: *update-def valid-def*)

lemma *val-of-update*[*simp*]: $\llbracket$*update-pre h i v*; *valid h j*$\rrbracket$ $\Longrightarrow$ *val-of* (*update h i v*) *j* = (
if i=j then v else val-of h j)
unfolding *update-def val-of-def*
by (*auto simp*: *nth-list-update valid-def*)

lemma *length-update*[*simp*]: *length* (*update l i v*) = *length l*
by (*auto simp*: *update-def*)

Exchange

Exchange two elements

definition *exch* :: *'a heap* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ *'a heap* **where**
exch l i j $\equiv$ *swap l* (*i* − *1*) (*j* − *1*)
abbreviation *exch-pre l i j* $\equiv$ *valid l i* $\wedge$ *valid l j*

definition *exch-op* :: *'a list* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ *'a list nres*
where *exch-op l i j* $\equiv$ *do* {
ASSERT (*i*>*0* $\wedge$ *j*>*0*);
l $\leftarrow$ *mop-list-swap l* (*i* − *1*) (*j* − *1*);
RETURN l
}

lemma *exch-op-alt*: *exch-op l i j* = *do* {
vi $\leftarrow$ *val-of-op l i*;
vj $\leftarrow$ *val-of-op l j*;
l $\leftarrow$ *update-op l i vj*;
l $\leftarrow$ *update-op l j vi*;
RETURN l }
by (*auto simp*: *exch-op-def swap-def val-of-op-def update-op-def pw-eq-iff refine-pw-simps*)

lemma *exch-op-correct*[*refine-vcg*]:
exch-pre l i j $\Longrightarrow$ *exch-op l i j* $\leq$ *SPEC* (λ*r*. *r* = *exch l i j*)
unfolding *exch-op-def*
apply *refine-vcg*
apply (*auto simp*: *exch-def valid-def*)
done

lemma *valid-exch*[*simp*]: *valid* (*exch l i j*) *k* = *valid l k*
unfolding *exch-def* **by** (*auto simp*: *valid-def*)

lemma *val-of-exch*[*simp*]: $\llbracket$*valid l i*; *valid l j*; *valid l k*$\rrbracket$ $\Longrightarrow$
val-of (*exch l i j*) *k* = (
if k=i then val-of l j
else if k=j then val-of l i
else val-of l k

)
unfolding *exch-def val-of-def valid-def*
by (*auto*)

lemma *exch-eq*[*simp*]: *exch h i i* = *h*
by (*auto simp*: *exch-def*)

lemma α*-exch*[*simp*]: $[\![$*valid l i*; *valid l j*$]\!]$
$\Longrightarrow \alpha$ (*exch l i j*) = α *l*
unfolding *exch-def valid-def*
by (*auto*)

lemma *length-exch*[*simp*]: *length* (*exch l i j*) = *length l*
by (*auto simp*: *exch-def*)

Butlast

Remove last element

abbreviation *butlast-pre l* ≡ *l*≠[]
definition *butlast-op* :: *'a heap* ⇒ *'a heap nres*
where *butlast-op l* ≡ *mop-list-butlast l*
lemma *butlast-op-correct*[*refine-vcg*]:
butlast-pre l $\Longrightarrow$ *butlast-op l* ≤ *SPEC* (λ*r*. *r* = *butlast l*)
unfolding *butlast-op-def* **by** (*refine-vcg*; *auto*)

lemma *valid-butlast-conv*[*simp*]: *valid* (*butlast h*) *i* ⟷ *valid h i* ∧ *i* < *length h*
by (*auto simp*: *valid-def*)

lemma *valid-butlast*: *valid* (*butlast h*) *i* $\Longrightarrow$ *valid h i*
by (*cases h rule*: *rev-cases*) (*auto simp*: *valid-def*)

lemma *val-of-butlast*[*simp*]: $[\![$*valid h i*; *i*<*length h*$]\!]$
$\Longrightarrow$ *val-of* (*butlast h*) *i* = *val-of h i*
by (*auto simp*: *valid-def val-of-def nth-butlast*)

lemma *val-of-butlast'*[*simp*]:
valid (*butlast h*) *i* $\Longrightarrow$ *val-of* (*butlast h*) *i* = *val-of h i*
by (*cases h rule*: *rev-cases*) (*auto simp*: *valid-def val-of-def nth-append*)

lemma α*-butlast*[*simp*]: $[\![$ *length h* ≠ *0* $]\!]$
$\Longrightarrow \alpha$ (*butlast h*) = α *h* − {# *val-of h* (*length h*)#}
apply (*cases h rule*: *rev-cases*)
apply (*auto simp*: *val-of-def*)
done

lemma *heap-invar-butlast*[*simp*]: *heap-invar h* $\Longrightarrow$ *heap-invar* (*butlast h*)
apply (*cases h* = [])
apply *simp*
apply (*auto simp*: *heap-invar-def dest*: *valid-butlast*)

done

Append

definition *append-op* :: $'a$ *heap* ⇒ $'a$ ⇒ $'a$ *heap nres*
where *append-op l v* ≡ *mop-list-append l v*
lemma *append-op-correct*[*refine-vcg*]:
append-op l v ≤ *SPEC* (λ*r*. *r* = *l*@[*v*])
unfolding *append-op-def* **by** (*refine-vcg*; *auto*)

lemma *valid-append*[*simp*]: *valid* (*l*@[*v*]) *i* ⟷ *valid l i* ∨ *i* = *length l* + *1*
by (*auto simp*: *valid-def*)

lemma *val-of-append*[*simp*]: *valid* (*l*@[*v*]) *i* ⟹
val-of (*l*@[*v*]) *i* = (*if valid l i then val-of l i else v*)
unfolding *valid-def val-of-def* **by** (*auto simp*: *nth-append*)

lemma α*-append*[*simp*]: α (*l*@[*v*]) = α *l* + {#*v*#}
by *auto*

3.6.3 Auxiliary operations

The auxiliary operations do not have a corresponding abstract operation, but are to restore the heap property after modification.

Swim

This invariant expresses that the heap has a single defect, which can be repaired by swimming up

definition *swim-invar* :: $'a$ *heap* ⇒ *nat* ⇒ *bool*
where *swim-invar h i* ≡
valid h i
∧ (∀*j*. *valid h j* ∧ *has-parent h j* ∧ *j*≠*i* ⟶ *pparent h j* ≤ *prio-of h j*)
∧ (*has-parent h i* ⟶
(∀*j*. *valid h j* ∧ *has-parent h j* ∧ *parent j* = *i*
⟶ *pparent h i* ≤ *prio-of h j*))

Move up an element that is too small, until it fits

definition *swim-op* :: $'a$ *heap* ⇒ *nat* ⇒ $'a$ *heap nres* **where**
swim-op h i ≡ *do* {
RECT (λ*swim* (*h*,*i*). *do* {
ASSERT (*valid h i* ∧ *swim-invar h i*);
if has-parent h i then do {
ppi ← *prio-of-op h* (*parent i*);
pi ← *prio-of-op h i*;
if (¬*ppi* ≤ *pi*) *then do* {
h ← *exch-op h i* (*parent i*);

swim (*h*, *parent i*)
} *else*
RETURN h
} *else*
RETURN h
}) (*h*,*i*)
}

lemma *swim-invar-valid*: *swim-invar h i* $\Longrightarrow$ *valid h i*
unfolding *swim-invar-def* **by** *simp*

lemma *swim-invar-exit1*: $\neg$*has-parent h i* $\Longrightarrow$ *swim-invar h i* $\Longrightarrow$ *heap-invar h*
unfolding *heap-invar-def swim-invar-def* **by** *auto*

lemma *swim-invar-exit2*: *pparent h i* $\leq$ *prio-of h i* $\Longrightarrow$ *swim-invar h i* $\Longrightarrow$ *heap-invar h*
unfolding *heap-invar-def swim-invar-def* **by** *auto*

lemma *swim-invar-pres*:
assumes *HPI*: *has-parent h i*
assumes *VIOLATED*: *pparent h i* $>$ *prio-of h i*
and *INV*: *swim-invar h i*
defines $h' \equiv$ *exch h i* (*parent i*)
shows *swim-invar* h' (*parent i*)
unfolding *swim-invar-def*
apply *safe*
apply (*simp add*: h'*-def HPI*)

using *HPI VIOLATED INV*
unfolding *swim-invar-def* h'*-def*
apply *auto* []

using *HPI VIOLATED INV*
unfolding *swim-invar-def* h'*-def*
apply *auto*
by (*metis order-trans*)

lemma *swim-invar-decr*:
assumes *INV*: *heap-invar h*
assumes *V*: *valid h i*
assumes *DECR*: *prio v* $\leq$ *prio-of h i*
shows *swim-invar* (*update h i v*) *i*
using *INV V DECR*
apply (*auto simp*: *swim-invar-def heap-invar-def intro*: *dual-order.trans*)
done

lemma *swim-op-correct*[*refine-vcg*]:
[[*swim-invar h i*]] $\Longrightarrow$

swim-op h i $\leq$ *SPEC* ($\lambda h'$. α $h' = \alpha$ h $\wedge$ *heap-invar* h' $\wedge$ *length* h' = *length* h)
unfolding *swim-op-def*
using [[*goals-limit* = *1*]]
apply (*refine-vcg* *RECT-rule*[**where**
pre=λ(*hh*,*i*).
swim-invar hh i
$\wedge$ α *hh* = α *h*
$\wedge$ *length hh* = *length h* **and**
V = *inv-image less-than snd*
])
apply (*auto*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto simp*: *swim-invar-valid*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto*) []

apply *rprems*
apply (*auto simp*: *swim-invar-pres*) []
apply (*auto simp*: *parent-def valid-def*) []

apply (*auto*) []
apply (*auto simp*: *swim-invar-exit2*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto simp*: *swim-invar-exit1*) []
apply (*auto*) []
done

Sink

Move down an element that is too big, until it fits in

definition *sink-op* :: *'a heap* $\Rightarrow$ *nat* $\Rightarrow$ *'a heap nres* **where**
sink-op h i $\equiv$ *do* {
RECT (λ*sink* (*h*,*i*). *do* {
ASSERT (*valid h i*);
if has-right h i then do {
ASSERT (*has-left h i*);
lp $\leftarrow$ *prio-of-op h* (*left i*);
rp $\leftarrow$ *prio-of-op h* (*right i*);
p $\leftarrow$ *prio-of-op h i*;
if (*lp* < *p* $\wedge$ *rp* $\geq$ *lp*) *then do* {
h $\leftarrow$ *exch-op h i* (*left i*);
sink (*h*,*left i*)
} *else if* (*rp*<*lp* $\wedge$ *rp* < *p*) *then do* {
h $\leftarrow$ *exch-op h i* (*right i*);

sink (*h*,*right i*)
} *else*
RETURN h
} *else if* (*has-left h i*) *then do* {
lp ← *prio-of-op h* (*left i*);
p ← *prio-of-op h i*;
if (*lp* < *p*) *then do* {
h ← *exch-op h i* (*left i*);
sink (*h*,*left i*)
} *else*
RETURN h

} *else*
RETURN h
}) (*h*,*i*)
}

This invariant expresses that the heap has a single defect, which can be repaired by sinking

definition *sink-invar l i* ≡
valid l i
∧ (∀ *j*. *valid l j* ∧ *j*≠*i* ⟶ *children-ge l* (*prio-of l j*) *j*)
∧ (*has-parent l i* ⟶ *children-ge l* (*pparent l i*) *i*)

lemma *sink-invar-valid*: *sink-invar l i* ⟹ *valid l i*
unfolding *sink-invar-def* **by** *auto*

lemma *sink-invar-exit*: ⟦*sink-invar l i*; *children-ge l* (*prio-of l i*) *i*⟧
⟹ *heap-invar′ l*
unfolding *heap-invar′-def sink-invar-def*
by *auto*

lemma *sink-aux1*: ¬ (*2*∗*i* ≤ *length h*) ⟹ ¬*has-left h i* ∧ ¬*has-right h i*
unfolding *valid-def left-def right-def* **by** *auto*

lemma *sink-invar-pres1*:
assumes *sink-invar h i*
assumes *has-left h i has-right h i*
assumes *prio-of h i* ≥ *pleft h i*
assumes *pleft h i* ≥ *pright h i*
shows *sink-invar* (*exch h i* (*right i*)) (*right i*)
using *assms*
unfolding *sink-invar-def*
apply *auto*
apply (*auto simp*: *children-ge-def*)
done

lemma *sink-invar-pres2*:
assumes *sink-invar h i*

assumes *has-left h i has-right h i*
assumes *prio-of h i* $\geq$ *pleft h i*
assumes *pleft h i* $\leq$ *pright h i*
shows *sink-invar* (*exch h i* (*left i*)) (*left i*)
using *assms*
unfolding *sink-invar-def*
apply *auto*
apply (*auto simp*: *children-ge-def*)
done

lemma *sink-invar-pres3*:
assumes *sink-invar h i*
assumes *has-left h i has-right h i*
assumes *prio-of h i* $\geq$ *pright h i*
assumes *pleft h i* $\leq$ *pright h i*
shows *sink-invar* (*exch h i* (*left i*)) (*left i*)
using *assms*
unfolding *sink-invar-def*
apply *auto*
apply (*auto simp*: *children-ge-def*)
done

lemma *sink-invar-pres4*:
assumes *sink-invar h i*
assumes *has-left h i has-right h i*
assumes *prio-of h i* $\geq$ *pright h i*
assumes *pleft h i* $\geq$ *pright h i*
shows *sink-invar* (*exch h i* (*right i*)) (*right i*)
using *assms*
unfolding *sink-invar-def*
apply *auto*
apply (*auto simp*: *children-ge-def*)
done

lemma *sink-invar-pres5*:
assumes *sink-invar h i*
assumes *has-left h i* ¬*has-right h i*
assumes *prio-of h i* $\geq$ *pleft h i*
shows *sink-invar* (*exch h i* (*left i*)) (*left i*)
using *assms*
unfolding *sink-invar-def*
apply *auto*
apply (*auto simp*: *children-ge-def*)
done

lemmas *sink-invar-pres* =
sink-invar-pres1
sink-invar-pres2
sink-invar-pres3

sink-invar-pres4
sink-invar-pres5

lemma *sink-invar-incr*:
assumes *INV*: *heap-invar h*
assumes *V*: *valid h i*
assumes *INCR*: *prio v* $\geq$ *prio-of h i*
shows *sink-invar* (*update h i v*) *i*
using *INV V INCR*
apply (*auto simp*: *sink-invar-def*)
apply (*auto simp*: *children-ge-def heap-invar-def*) []
apply (*auto simp*: *children-ge-def heap-invar-def intro*: *order-trans*) []
apply (*frule spec*[**where** *x*=*left i*])
apply *auto* []
apply (*frule spec*[**where** *x*=*right i*])
apply *auto* []
done

lemma *sink-op-correct*[*refine-vcg*]:
⟦*sink-invar h i*⟧ $\Longrightarrow$
sink-op h i $\leq$ *SPEC* ($\lambda h'.\ \alpha\ h' = \alpha\ h \wedge$ *heap-invar h'* $\wedge$ *length h'* = *length h*)

unfolding *sink-op-def heap-eq-heap'*
using [[*goals-limit* = *1*]]

apply (*refine-vcg* *RECT-rule*[**where**
pre=λ(*hh*,*i*). *sink-invar hh i* $\wedge$ α *hh* = α *h* $\wedge$ *length hh* = *length h* **and**
V = *measure* (λ(*l*,*i*). *length l* − *i*)
])

apply (*auto*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto*) []
apply (*auto simp*: *sink-invar-valid*) []
apply (*auto simp*: *valid-def left-def right-def*) []

apply *rprems*
apply (*auto intro*: *sink-invar-pres*) []
apply (*auto simp*: *valid-def left-def right-def*) []

apply *rprems*
apply (*auto intro*: *sink-invar-pres*) []
apply (*auto simp*: *valid-def left-def right-def*) []

apply (*auto*) []

apply *clarsimp*
apply (*rule sink-invar-exit*, *assumption*) []
apply (*auto simp*: *children-ge-def*) []

apply (*auto*) []

apply *rprems*
apply (*auto intro*: *sink-invar-pres*) []
apply (*auto simp*: *valid-def left-def right-def*) []

apply (*auto*) []

apply *clarsimp*
apply (*rule sink-invar-exit*, *assumption*) []
apply (*auto simp*: *children-ge-def*) []

apply (*auto*) []

apply (*auto*) []

apply *clarsimp*
apply (*rule sink-invar-exit*, *assumption*) []
apply (*auto simp*: *children-ge-def*) []

apply (*auto*) []
done

lemma *sink-op-swim-rule*:
$swim\text{-}invar\ h\ i \Longrightarrow sink\text{-}op\ h\ i \leq SPEC\ (\lambda h'.\ h'{=}h)$
apply (*frule swim-invar-valid*)
unfolding *sink-op-def*
apply (*subst RECT-unfold*, *refine-mono*)
apply (*fold sink-op-def*)
apply *refine-vcg*
apply (*simp-all*)
apply (*auto simp add*: *valid-def left-def right-def dest*: *swim-invar-valid*) []
apply (*auto simp*: *swim-invar-def*) []
apply (*auto simp*: *swim-invar-def*) []
apply (*auto simp*: *swim-invar-def*) []
apply (*auto simp*: *swim-invar-def*) []
apply (*auto simp*: *swim-invar-def*) []
apply (*auto simp*: *swim-invar-def*) []
done

definition *sink-op-opt*
— Sink operation as presented in Sedgewick et al. Algs4 reference implementation
where
$sink\text{-}op\text{-}opt\ h\ k \equiv RECT\ (\lambda D\ (h,k).\ do\ \{$

```
  ASSERT (k>0 ∧ k≤length h);
  let len = length h;
  if (2*k ≤ len) then do {
    let j = 2*k;
    pj ← prio-of-op h j;

    j ← (
      if j<len then do {
        psj ← prio-of-op h (Suc j);
        if pj>psj then RETURN (j+1) else RETURN j
      } else RETURN j);

    pj ← prio-of-op h j;
    pk ← prio-of-op h k;
    if (pk > pj) then do {
      h ← exch-op h k j;
      D (h,j)
    } else
      RETURN h
  } else RETURN h
}) (h,k)
```

lemma *sink-op-opt-eq*: *sink-op-opt h k = sink-op h k*
unfolding *sink-op-opt-def sink-op-def*
apply (*fo-rule arg-cong fun-cong*)+
apply (*intro ext*)
unfolding *sink-op-def*[*symmetric*]
apply (*simp cong*: *if-cong split del*: *if-split add*: *Let-def*)

apply (*auto simp*: *valid-def left-def right-def prio-of-op-def val-of-op-def val-of-def less-imp-diff-less ASSERT-same-eq-conv nz-le-conv-less*) []
done

Repair

Repair a local defect in the heap. This can be done by swimming and sinking. Note that, depending on the defect, only one of the operations will change the heap. Moreover, note that we do not need repair to implement the heap operations. However, it is required for heapmaps.

definition *repair-op h i* ≡ *do* {
```
  h ← sink-op h i;
  h ← swim-op h i;
  RETURN h
}
```

lemma *update-sink-swim-cases*:
assumes *heap-invar h*
assumes *valid h i*

obtains *swim-invar* (*update h i v*) *i* | *sink-invar* (*update h i v*) *i*
apply (*cases rule*: *linear*[*of prio v prio-of h i*, *THEN disjE*])
apply (*blast dest*: *swim-invar-decr*[*OF assms*])
apply (*blast dest*: *sink-invar-incr*[*OF assms*])
done

lemma *heap-invar-imp-swim-invar*: $\llbracket$*heap-invar h*; *valid h i*$\rrbracket \Longrightarrow$ *swim-invar h i*
unfolding *heap-invar-def swim-invar-def*
by (*auto intro*: *order-trans*)

lemma *repair-correct*[*refine-vcg*]:
assumes *heap-invar h* **and** *valid h i*
shows *repair-op* (*update h i v*) *i* $\leq$ *SPEC* ($\lambda h'$.
heap-invar $h' \wedge \alpha\ h' = \alpha$ (*update h i v*) $\wedge$ *length* h' = *length h*)
apply (*rule update-sink-swim-cases*[*of h i v*, *OF assms*])
unfolding *repair-op-def*
apply (*refine-vcg sink-op-swim-rule*)
apply *auto* [*4*]
apply (*refine-vcg*)
using *assms*(*2*)
apply (*auto intro*: *heap-invar-imp-swim-invar simp*: *valid-def*) []
apply *auto* [*3*]
done

3.6.4 Operations

Empty

abbreviation (*input*) *empty* :: *'a heap* — The empty heap
where *empty* $\equiv$ []
definition *empty-op* :: *'a heap nres*
where *empty-op* $\equiv$ *mop-list-empty*
lemma *empty-op-correct*[*refine-vcg*]:
empty-op $\leq$ *SPEC* ($\lambda r.\ \alpha\ r = \{\#\} \wedge$ *heap-invar r*)
unfolding *empty-op-def* **apply** *refine-vcg* **by** *auto*

Emptiness check

definition *is-empty-op* :: *'a heap* $\Rightarrow$ *bool nres* — Check for emptiness
where *is-empty-op h* $\equiv$ *do* {*ASSERT* (*heap-invar h*); *let l*=*length h*; *RETURN* (*l*=*0*)}
lemma *is-empty-op-correct*[*refine-vcg*]:
heap-invar h $\Longrightarrow$ *is-empty-op h* $\leq$ *SPEC* ($\lambda r.\ r \longleftrightarrow \alpha\ h = \{\#\}$)
unfolding *is-empty-op-def*
apply *refine-vcg* **by** *auto*

Insert

definition *insert-op* :: $'a \Rightarrow 'a$ *heap* $\Rightarrow 'a$ *heap nres* — Insert element
  where *insert-op v h* $\equiv$ *do* {
    ASSERT (*heap-invar h*);
    $h \leftarrow$ *append-op h v*;
    let l = *length h*;
    $h \leftarrow$ *swim-op h l*;
    RETURN h
  }

lemma *swim-invar-insert*: *heap-invar l* $\Longrightarrow$ *swim-invar* (l@[x]) (*Suc* (*length l*))
  unfolding *swim-invar-def heap-invar-def valid-def parent-def val-of-def*
  by (*fastforce simp*: *nth-append*)

lemma
(*insert-op*,*RETURN oo op-mset-insert*) $\in$ *Id* $\rightarrow$ *heap-rel1* $\rightarrow$ $\langle$*heap-rel1*$\rangle$*nres-rel*
  unfolding *insert-op-def*[*abs-def*] *heap-rel1-def o-def*
  by *refine-vcg* (*auto simp*: *swim-invar-insert in-br-conv*)

lemma *insert-op-correct*:
  heap-invar h $\Longrightarrow$ *insert-op v h* $\leq$ *SPEC* ($\lambda h'$. *heap-invar* h' $\wedge$ α h' = α h + {#v#})
  unfolding *insert-op-def*
  by (*refine-vcg*) (*auto simp*: *swim-invar-insert*)
lemmas [*refine-vcg*] = *insert-op-correct*

Pop minimum element

definition *pop-min-op* :: $'a$ *heap* $\Rightarrow$ ($'a \times 'a$ *heap*) *nres* **where**
  pop-min-op h $\equiv$ *do* {
    ASSERT (*heap-invar h*);
    ASSERT (*valid h 1*);
    $m \leftarrow$ *val-of-op h 1*;
    let l = *length h*;
    $h \leftarrow$ *exch-op h 1 l*;
    $h \leftarrow$ *butlast-op h*;

    if ($l \neq 1$) *then do* {
      $h \leftarrow$ *sink-op h 1*;
      RETURN (m,h)
    } *else RETURN* (m,h)
  }

lemma *left-not-one*[*simp*]: *left j* $\neq$ *Suc 0*
  by (*auto simp*: *left-def*)

lemma *right-one-conv*[*simp*]: *right j* = *Suc 0* $\longleftrightarrow$ j=0
  by (*auto simp*: *right-def*)

lemma *parent-one-conv*[*simp*]: *parent* (*Suc 0*) = *0*
  by (*auto simp*: *parent-def*)

lemma *sink-invar-init*:
  assumes *I*: *heap-invar h*
  assumes *NE*: *length h* > *1*
  shows *sink-invar* (*butlast* (*exch h* (*Suc 0*) (*length h*))) (*Suc 0*)
proof −
  from *NE* **have** *V*: *valid h* (*Suc 0*) *valid h* (*length h*)
    apply −
    apply (*auto simp*: *valid-def neq-Nil-conv*) []
    by (*cases h*) (*auto simp*: *valid-def*)

  show *?thesis* **using** *I*
    unfolding *heap-eq-heap′ heap-invar′-def sink-invar-def*
    apply (*intro impI conjI allI*)
    using *NE* **apply** (*auto simp*: *V valid-butlast-conv*) []
    apply (*auto simp add*: *children-ge-def V NE valid-butlast-conv*) []
    apply (*auto simp add*: *children-ge-def V NE valid-butlast-conv*) []
    done
qed

lemma *in-set-conv-val*: $v \in set\ h \longleftrightarrow (\exists i.\ valid\ h\ i \wedge v = val\text{-}of\ h\ i)$
  apply (*rule iffI*)
  apply (*clarsimp simp add*: *valid-def val-of-def in-set-conv-nth*)
  apply (*rule-tac x*=*Suc i* **in** *exI*; *auto*)
  apply (*clarsimp simp add*: *valid-def val-of-def in-set-conv-nth*)
  apply (*rule-tac x*=*i* − *Suc 0* **in** *exI*; *auto*)
  done

lemma *pop-min-op-correct*:
  assumes $heap\text{-}invar\ h\ \alpha\ h \neq \{\#\}$
  shows $pop\text{-}min\text{-}op\ h \leq SPEC\ (\lambda(v,h').\ heap\text{-}invar\ h' \wedge$
    $v \in\#\ \alpha\ h \wedge \alpha\ h' = \alpha\ h - \{\#v\#\} \wedge (\forall v' \in set\text{-}mset\ (\alpha\ h).\ prio\ v \leq prio\ v'))$
proof −
  note [*simp del*] = *length-greater-0-conv*
  note *LG* = *length-greater-0-conv*[*symmetric*]

  from *assms* **show** *?thesis*
    unfolding *pop-min-op-def*
    apply *refine-vcg*
    apply (*simp-all add*: *sink-invar-init LG*)
    apply (*auto simp*: *valid-def*) []
    apply (*cases h*; *auto simp*: *val-of-def*) []
    apply (*auto simp*: *in-set-conv-val simp*: *heap-min-prop*) []
    apply *auto* []
    apply (*cases h*; *auto simp*: *val-of-def*) []

apply *auto* []
apply (*cases h*; *auto simp*: *val-of-def*) []
done
qed

lemmas [*refine-vcg*] = *pop-min-op-correct*

Peek minimum element

definition *peek-min-op* :: *'a heap* ⇒ *'a nres* **where**
peek-min-op h ≡ *do* {
ASSERT (*heap-invar h*);
ASSERT (*valid h 1*);
val-of-op h 1
}

lemma *peek-min-op-correct*:
assumes *heap-invar h* α *h* ≠ {#}
shows *peek-min-op h* ≤ *SPEC* (λv.
$v \in\# \alpha\ h \wedge (\forall v' \in \mathit{set\text{-}mset}\ (\alpha\ h).\ \mathit{prio}\ v \leq \mathit{prio}\ v')$)
unfolding *peek-min-op-def*
apply *refine-vcg*
using *assms*
apply *clarsimp-all*
apply (*auto simp*: *valid-def*) []
apply (*cases h*; *auto simp*: *val-of-def*) []
apply (*auto simp*: *in-set-conv-val simp*: *heap-min-prop*) []
done

lemmas *peek-min-op-correct'*[*refine-vcg*] = *peek-min-op-correct*

3.6.5 Operations as Relator-Style Refinement

lemma *empty-op-refine*: (*empty-op*,*RETURN op-mset-empty*)∈⟨*heap-rel1*⟩*nres-rel*
apply (*rule nres-relI*)
apply (*rule order-trans*)
apply (*rule empty-op-correct*)
apply (*auto simp*: *heap-rel1-def br-def pw-le-iff refine-pw-simps*)
done

lemma *is-empty-op-refine*: (*is-empty-op*,*RETURN o op-mset-is-empty*) ∈ *heap-rel1* → ⟨*bool-rel*⟩*nres-rel*
apply (*intro nres-relI fun-relI*; *simp*)
apply *refine-vcg*
apply (*auto simp*: *heap-rel1-def br-def*)
done

lemma *insert-op-refine*: (*insert-op*,*RETURN oo op-mset-insert*) ∈ *Id* → *heap-rel1* → ⟨*heap-rel1*⟩*nres-rel*
apply (*intro nres-relI fun-relI*; *simp*)

apply (*refine-vcg RETURN-as-SPEC-refine*)
apply (*auto simp*: *heap-rel1-def br-def pw-le-iff refine-pw-simps*)
done

lemma *pop-min-op-refine*:
(*pop-min-op*, *PR-CONST* (*mop-prio-pop-min prio*)) $\in$ *heap-rel1* $\rightarrow$ $\langle$*Id* $\times_r$ *heap-rel1*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *mop-prio-pop-min-def PR-CONST-def*
apply (*refine-vcg SPEC-refine*)
apply (*auto simp*: *heap-rel1-def br-def*)
done

lemma *peek-min-op-refine*:
(*peek-min-op*, *PR-CONST* (*mop-prio-peek-min prio*)) $\in$ *heap-rel1* $\rightarrow$ $\langle$*Id*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *mop-prio-peek-min-def PR-CONST-def*
apply (*refine-vcg RES-refine*)
apply (*auto simp*: *heap-rel1-def br-def*)
done

end

end
theory *IICF-HOL-List*
imports *../Intf/IICF-List*
begin

context
begin

private lemma *id-take-nth-drop-rl*:
assumes $i{<}length\ l$
assumes $\bigwedge l1\ x\ l2.\ [\![l{=}l1@x\#l2;\ i = length\ l1\ ]\!] \Longrightarrow P\ (l1@x\#l2)$
shows *P l*
apply (*subst id-take-nth-drop*[*OF assms*(*1*)])
apply (*rule assms*(*2*))
apply (*subst id-take-nth-drop*[*OF assms*(*1*)])
apply *simp*
apply (*simp add*: *assms*(*1*))
done

private lemma *list-set-entails-aux*:
shows $list\text{-}assn\ A\ l\ li * A\ x\ xi \Longrightarrow_A list\text{-}assn\ A\ (l[i := x])\ (li[i := xi]) * true$
apply (*rule entails-preI*)
apply (*clarsimp*)
apply (*cases* $i < length\ l$; *cases* $i < length\ li$; (*sep-auto dest*!: *list-assn-aux-eqlen-simp*;*fail*)*?*)

apply (*erule id-take-nth-drop-rl*)
apply (*erule id-take-nth-drop-rl*)
apply (*sep-auto simp add*: *list-update-append*)
done

private lemma *list-set-hd-tl-aux*:
$a \neq [] \Longrightarrow$ *list-assn R a c* $\Longrightarrow_A$ *R* (*hd a*) (*hd c*) $*$ *true*
$a \neq [] \Longrightarrow$ *list-assn R a c* $\Longrightarrow_A$ *list-assn R* (*tl a*) (*tl c*) $*$ *true*
by (*cases c*; *cases a*; *sep-auto*; *fail*)+

private lemma *list-set-last-butlast-aux*:
$a \neq [] \Longrightarrow$ *list-assn R a c* $\Longrightarrow_A$ *R* (*last a*) (*last c*) $*$ *true*
$a \neq [] \Longrightarrow$ *list-assn R a c* $\Longrightarrow_A$ *list-assn R* (*butlast a*) (*butlast c*) $*$ *true*
by (*cases c rule*: *rev-cases*; *cases a rule*: *rev-cases*; *sep-auto*; *fail*)+

private lemma *swap-decomp-simp*[*simp*]:
swap (*l1* @ *x* # *c21'* @ *xa* # *l2a*) (*length l1*) (*Suc* (*length l1* + *length c21'*)) = *l1*@*xa*#*c21'*@*x*#*l2a*
swap (*l1* @ *x* # *c21'* @ *xa* # *l2a*) (*Suc* (*length l1* + *length c21'*)) (*length l1*) = *l1*@*xa*#*c21'*@*x*#*l2a*
by (*auto simp*: *swap-def list-update-append nth-append*)

private lemma *list-swap-aux*: $[\![i <$ *length l*; $j <$ *length l*$]\!] \Longrightarrow$ *list-assn A l li* $\Longrightarrow_A$ *list-assn A* (*swap l i j*) (*swap li i j*) $*$ *true*
apply (*subst list-assn-aux-len*; *clarsimp*)
apply (*cases i=j*; (*sep-auto*;*fail*)?)
apply (*rule id-take-nth-drop-rl*[**where** *l=l* **and** *i=i*]; *simp*)
apply (*rule id-take-nth-drop-rl*[**where** *l=l* **and** *i=j*]; *simp*)
apply (*erule list-match-lel-lel*; *simp*)
apply (*split-list-according li l*; *sep-auto*)
apply (*split-list-according li l*; *sep-auto*)
done

private lemma *list-rotate1-aux*: *list-assn A a c* $\Longrightarrow_A$ *list-assn A* (*rotate1 a*) (*rotate1 c*) $*$ *true*
by (*cases a*; *cases c*; *sep-auto*)

private lemma *list-rev-aux*: *list-assn A a c* $\Longrightarrow_A$ *list-assn A* (*rev a*) (*rev c*) $*$ *true*
apply (*subst list-assn-aux-len*; *clarsimp*)
apply (*induction rule*: *list-induct2*)
apply *sep-auto*
apply *sep-auto*
apply (*erule ent-frame-fwd*, *frame-inference*)
apply *sep-auto*
done

lemma *mod-starE*:
assumes $h \models A{*}B$
obtains *h1 h2* **where** $h1 \models A$ $h2 \models B$

using *assms* by (*auto simp*: *mod-star-conv*)

private lemma *CONSTRAINT-is-pureE*:
assumes *CONSTRAINT is-pure A*
obtains *R* **where** *A=pure R*
using *assms* **by** (*auto simp*: *is-pure-conv*)

private method *solve-dbg* =
((*elim CONSTRAINT-is-pureE*; (*simp only*: *list-assn-pure-conv the-pure-pure*)?)?;
sep-auto
simp: *pure-def hn-ctxt-def invalid-assn-def list-assn-aux-eqlen-simp*
intro!: *hn-refineI*[*THEN hn-refine-preI*] *hfrefI*
elim!: *mod-starE*
intro: *list-set-entails-aux list-set-hd-tl-aux list-set-last-butlast-aux*
list-swap-aux list-rotate1-aux list-rev-aux
;
((*rule entails-preI*; *sep-auto simp*: *list-assn-aux-eqlen-simp* | (*parametricity*; *simp*; *fail*))?)
)

private method *solve* = *solve-dbg*; *fail*

lemma *HOL-list-empty-hnr-aux*: (*uncurry0* (*return op-list-empty*), *uncurry0* (*RETURN op-list-empty*)) $\in$ *unit-assn*^{k} $\rightarrow_a$ (*list-assn A*) **by** *solve*
lemma *HOL-list-is-empty-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-is-empty*, *RETURN* $\circ$ *op-list-is-empty*) $\in$ (*list-assn A*)^k $\rightarrow_a$ *bool-assn* **by** *solve*
lemma *HOL-list-prepend-hnr*[*sepref-fr-rules*]: (*uncurry* (*return* $\circ\circ$ *op-list-prepend*), *uncurry* (*RETURN* $\circ\circ$ *op-list-prepend*)) $\in$ $A^d *_a$ (*list-assn A*)^d $\rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-append-hnr*[*sepref-fr-rules*]: (*uncurry* (*return* $\circ\circ$ *op-list-append*), *uncurry* (*RETURN* $\circ\circ$ *op-list-append*)) $\in$ (*list-assn A*)^d $*_a$ $A^d \rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-concat-hnr*[*sepref-fr-rules*]: (*uncurry* (*return* $\circ\circ$ *op-list-concat*), *uncurry* (*RETURN* $\circ\circ$ *op-list-concat*)) $\in$ (*list-assn A*)^d $*_a$ (*list-assn A*)^d $\rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-length-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-length*, *RETURN* $\circ$ *op-list-length*) $\in$ (*list-assn A*)^k $\rightarrow_a$ *nat-assn* **by** *solve*
lemma *HOL-list-set-hnr*[*sepref-fr-rules*]: (*uncurry2* (*return* $\circ\circ\circ$ *op-list-set*), *uncurry2* (*RETURN* $\circ\circ\circ$ *op-list-set*)) $\in$ (*list-assn A*)^d $*_a$ *nat-assn*^{k} $*_a$ $A^d \rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-hd-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-hd*, *RETURN* $\circ$ *op-list-hd*) $\in [\lambda y.\ y \neq []]_a$ (*list-assn R*)^d $\rightarrow R$ **by** *solve*
lemma *HOL-list-tl-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-tl*, *RETURN* $\circ$ *op-list-tl*) $\in [\lambda y.\ y \neq []]_a$ (*list-assn A*)^d $\rightarrow$ *list-assn A* **by** *solve*
lemma *HOL-list-last-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-last*, *RETURN* $\circ$ *op-list-last*) $\in [\lambda y.\ y \neq []]_a$ (*list-assn R*)^d $\rightarrow R$ **by** *solve*
lemma *HOL-list-butlast-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-butlast*, *RETURN* $\circ$

op-list-butlast) $\in$ $[\lambda y.\ y \neq []]_a$ (*list-assn A*)^d $\rightarrow$ *list-assn A* **by** *solve*
lemma *HOL-list-swap-hnr*[*sepref-fr-rules*]: (*uncurry2* (*return* $\circ\circ\circ$ *op-list-swap*), *uncurry2* (*RETURN* $\circ\circ\circ$ *op-list-swap*))
$\in$ $[\lambda((a,\ b),\ ba).\ b < length\ a \wedge ba < length\ a]_a$ (*list-assn A*)^d $*_a$ *nat-assn*^{k} $*_a$ *nat-assn*^{k} $\rightarrow$ *list-assn A* **by** *solve*
lemma *HOL-list-rotate1-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-rotate1*, *RETURN* $\circ$ *op-list-rotate1*) $\in$ (*list-assn A*)^d $\rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-rev-hnr*[*sepref-fr-rules*]: (*return* $\circ$ *op-list-rev*, *RETURN* $\circ$ *op-list-rev*) $\in$ (*list-assn A*)^d $\rightarrow_a$ *list-assn A* **by** *solve*

lemma *HOL-list-replicate-hnr*[*sepref-fr-rules*]: *CONSTRAINT is-pure A* $\Longrightarrow$ (*uncurry* (*return* $\circ\circ$ *op-list-replicate*), *uncurry* (*RETURN* $\circ\circ$ *op-list-replicate*)) $\in$ *nat-assn*^{k} $*_a$ A^k $\rightarrow_a$ *list-assn A* **by** *solve*
lemma *HOL-list-get-hnr*[*sepref-fr-rules*]: *CONSTRAINT is-pure A* $\Longrightarrow$ (*uncurry* (*return* $\circ\circ$ *op-list-get*), *uncurry* (*RETURN* $\circ\circ$ *op-list-get*)) $\in$ $[\lambda(a,\ b).\ b < length\ a]_a$ (*list-assn A*)^k $*_a$ *nat-assn*^{k} $\rightarrow$ *A* **by** *solve*

private lemma *bool-by-paramE*: $[\![\ a;\ (a,b) \in Id\ ]\!] \Longrightarrow b$ **by** *simp*
private lemma *bool-by-paramE'*: $[\![\ a;\ (b,a) \in Id\ ]\!] \Longrightarrow b$ **by** *simp*

lemma *HOL-list-contains-hnr*[*sepref-fr-rules*]: $[\![$*CONSTRAINT is-pure A*; *single-valued* (*the-pure A*); *single-valued* ((*the-pure A*)$^{-1}$)$]\!]$
$\Longrightarrow$ (*uncurry* (*return* $\circ\circ$ *op-list-contains*), *uncurry* (*RETURN* $\circ\circ$ *op-list-contains*)) $\in$ A^k $*_a$ (*list-assn A*)^k $\rightarrow_a$ *bool-assn*
  apply *solve-dbg*
  apply (*erule bool-by-paramE*[**where** *a*=-$\in$*set* -]) **apply** *parametricity*
  apply (*erule bool-by-paramE'*[**where** *a*=-$\in$*set* -]) **apply** *parametricity*
  done

lemmas *HOL-list-empty-hnr-mop* = *HOL-list-empty-hnr-aux*[*FCOMP mk-mop-rl0-np*[*OF mop-list-empty-alt*]]
lemmas *HOL-list-is-empty-hnr-mop*[*sepref-fr-rules*] = *HOL-list-is-empty-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-is-empty-alt*]]
lemmas *HOL-list-prepend-hnr-mop*[*sepref-fr-rules*] = *HOL-list-prepend-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-prepend-alt*]]
lemmas *HOL-list-append-hnr-mop*[*sepref-fr-rules*] = *HOL-list-append-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-append-alt*]]
lemmas *HOL-list-concat-hnr-mop*[*sepref-fr-rules*] = *HOL-list-concat-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-concat-alt*]]
lemmas *HOL-list-length-hnr-mop*[*sepref-fr-rules*] = *HOL-list-length-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-length-alt*]]
lemmas *HOL-list-set-hnr-mop*[*sepref-fr-rules*] = *HOL-list-set-hnr*[*FCOMP mk-mop-rl3*[*OF mop-list-set-alt*]]
lemmas *HOL-list-hd-hnr-mop*[*sepref-fr-rules*] = *HOL-list-hd-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-hd-alt*]]
lemmas *HOL-list-tl-hnr-mop*[*sepref-fr-rules*] = *HOL-list-tl-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-tl-alt*]]

lemmas *HOL-list-last-hnr-mop*[*sepref-fr-rules*] = *HOL-list-last-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-last-alt*]]
lemmas *HOL-list-butlast-hnr-mop*[*sepref-fr-rules*] = *HOL-list-butlast-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-butlast-alt*]]
lemmas *HOL-list-swap-hnr-mop*[*sepref-fr-rules*] = *HOL-list-swap-hnr*[*FCOMP mk-mop-rl3*[*OF mop-list-swap-alt*]]
lemmas *HOL-list-rotate1-hnr-mop*[*sepref-fr-rules*] = *HOL-list-rotate1-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-rotate1-alt*]]
lemmas *HOL-list-rev-hnr-mop*[*sepref-fr-rules*] = *HOL-list-rev-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-rev-alt*]]
lemmas *HOL-list-replicate-hnr-mop*[*sepref-fr-rules*] = *HOL-list-replicate-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-replicate-alt*]]
lemmas *HOL-list-get-hnr-mop*[*sepref-fr-rules*] = *HOL-list-get-hnr*[*FCOMP mk-mop-rl2*[*OF mop-list-get-alt*]]
lemmas *HOL-list-contains-hnr-mop*[*sepref-fr-rules*] = *HOL-list-contains-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-contains-alt*]]

lemmas *HOL-list-empty-hnr* = *HOL-list-empty-hnr-aux HOL-list-empty-hnr-mop*

end

definition [*simp*]: *op-HOL-list-empty* ≡ *op-list-empty*
interpretation *HOL-list*: *list-custom-empty list-assn A return* [] *op-HOL-list-empty*
  apply *unfold-locales*
  apply (*sep-auto intro*!: *hfrefI hn-refineI*)
  by *simp*

schematic-goal
  notes [*sepref-fr-rules*] = *HOL-list-empty-hnr*
  shows
  hn-refine (*emp*) (*?c*::*?′c Heap*) *?Γ′ ?R* (*do* {
    x ← *RETURN* [*1*,*2*,*3*::*nat*];
    let x2 = *op-list-append x 5*;
    ASSERT (*length x* = *4*);
    let x = *op-list-swap x 1 2*;
    x ← *mop-list-swap x 1 2*;
    RETURN (*x*@*x*)
  })
    by *sepref*

end
theory *IICF-Array-List*
imports
  ../Intf/IICF-List
  Separation-Logic-Imperative-HOL.Array-Blit
begin

  type-synonym *′a array-list* = *′a Heap.array* × *nat*

definition *is-array-list l* $\equiv \lambda(a,n).\ \exists_A l'.\ a \mapsto_a l' * \uparrow(n \leq$ *length* $l' \wedge l =$ *take n* $l' \wedge$ *length* $l' > 0)$

lemma *is-array-list-prec*[*safe-constraint-rules*]: *precise is-array-list*
  unfolding *is-array-list-def*[*abs-def*]
  apply(*rule preciseI*)
  apply(*simp split*: *prod.splits*)
 using *preciseD snga-prec* **by** *fastforce*

definition *initial-capacity* ≡ *16*::*nat*
definition *minimum-capacity* ≡ *16*::*nat*

definition *arl-empty* ≡ *do* {
  a ← *Array.new initial-capacity default*;
  return (*a*,*0*)
}

definition *arl-empty-sz init-cap* ≡ *do* {
  a ← *Array.new* (*max init-cap minimum-capacity*) *default*;
  return (*a*,*0*)
}

definition *arl-append* ≡ λ(*a*,*n*) *x*. *do* {
  len ← *Array.len a*;

  if n<*len then do* {
    a ← *Array.upd n x a*;
    return (*a*,*n*+*1*)
  } *else do* {
    let newcap = *2* ∗ *len*;
    a ← *array-grow a newcap default*;
    a ← *Array.upd n x a*;
    return (*a*,*n*+*1*)
  }
}

definition *arl-copy* ≡ λ(*a*,*n*). *do* {
  a ← *array-copy a*;
  return (*a*,*n*)
}

definition *arl-length* :: *'a*::*heap array-list* ⇒ *nat Heap* **where**
  arl-length ≡ λ(*a*,*n*). *return* (*n*)

definition *arl-is-empty* :: *'a*::*heap array-list* ⇒ *bool Heap* **where**
  arl-is-empty ≡ λ(*a*,*n*). *return* (*n*=*0*)

definition *arl-last* :: *'a*::*heap array-list* ⇒ *'a Heap* **where**

arl-last ≡ λ(*a*,*n*). *do* {
  Array.nth a (*n* − *1*)
}

definition *arl-butlast* :: *'a*::*heap array-list* ⇒ *'a array-list Heap* **where**
arl-butlast ≡ λ(*a*,*n*). *do* {
  let n = *n* − *1*;
  len ← *Array.len a*;
  if (*n*∗*4* < *len* ∧ *n*∗*2*≥*minimum-capacity*) *then do* {
    a ← *array-shrink a* (*n*∗*2*);
    return (*a*,*n*)
  } *else*
    return (*a*,*n*)
}

definition *arl-get* :: *'a*::*heap array-list* ⇒ *nat* ⇒ *'a Heap* **where**
arl-get ≡ λ(*a*,*n*) *i*. *Array.nth a i*

definition *arl-set* :: *'a*::*heap array-list* ⇒ *nat* ⇒ *'a* ⇒ *'a array-list Heap* **where**
arl-set ≡ λ(*a*,*n*) *i x*. *do* { *a* ← *Array.upd i x a*; *return* (*a*,*n*)}

lemma *arl-empty-rule*[*sep-heap-rules*]: < *emp* > *arl-empty* <*is-array-list* []>
by (*sep-auto simp*: *arl-empty-def is-array-list-def initial-capacity-def*)

lemma *arl-empty-sz-rule*[*sep-heap-rules*]: < *emp* > *arl-empty-sz N* <*is-array-list* []>
by (*sep-auto simp*: *arl-empty-sz-def is-array-list-def minimum-capacity-def*)

lemma *arl-copy-rule*[*sep-heap-rules*]: < *is-array-list l a* > *arl-copy a* <λ*r*. *is-array-list l a* ∗ *is-array-list l r*>
by (*sep-auto simp*: *arl-copy-def is-array-list-def*)

lemma *arl-append-rule*[*sep-heap-rules*]:
< *is-array-list l a* >
  arl-append a x
<λ*a*. *is-array-list* (*l*@[*x*]) *a* >$_t$
by (*sep-auto*
  simp: *arl-append-def is-array-list-def take-update-last neq-Nil-conv*
  split: *prod.splits nat.split*)

lemma *arl-length-rule*[*sep-heap-rules*]:
<*is-array-list l a*>
  arl-length a
<λ*r*. *is-array-list l a* ∗ ↑(*r*=*length l*)>
by (*sep-auto simp*: *arl-length-def is-array-list-def*)

lemma *arl-is-empty-rule*[*sep-heap-rules*]:
<*is-array-list l a*>

arl-is-empty a
<λr. *is-array-list l a* $*$ $\uparrow$($r\longleftrightarrow$(l=[]))>
by (*sep-auto simp*: *arl-is-empty-def is-array-list-def*)

lemma *arl-last-rule*[*sep-heap-rules*]:
$l\neq$[] $\Longrightarrow$
<*is-array-list l a*>
arl-last a
<λr. *is-array-list l a* $*$ $\uparrow$(r=*last l*)>
by (*sep-auto simp*: *arl-last-def is-array-list-def last-take-nth-conv*)

lemma *arl-butlast-rule*[*sep-heap-rules*]:
$l\neq$[] $\Longrightarrow$
<*is-array-list l a*>
arl-butlast a
<*is-array-list* (*butlast l*)>$_t$
proof −
assume [*simp*]: $l\neq$[]

have [*simp*]: $\bigwedge x$. *min* (x−*Suc 0*) ((x−*Suc 0*)$*$*2*) = x−*Suc 0* **by** *auto*

show *?thesis*
by (*sep-auto*
split: *prod.splits*
simp: *arl-butlast-def is-array-list-def butlast-take minimum-capacity-def*)
qed

lemma *arl-get-rule*[*sep-heap-rules*]:
i<*length l* $\Longrightarrow$
<*is-array-list l a*>
arl-get a i
<λr. *is-array-list l a* $*$ $\uparrow$(r=l!i)>
by (*sep-auto simp*: *arl-get-def is-array-list-def split*: *prod.split*)

lemma *arl-set-rule*[*sep-heap-rules*]:
i<*length l* $\Longrightarrow$
<*is-array-list l a*>
arl-set a i x
<*is-array-list* (l[i:=x])>
by (*sep-auto simp*: *arl-set-def is-array-list-def split*: *prod.split*)

definition *arl-assn A* $\equiv$ *hr-comp is-array-list* ($\langle$*the-pure A*$\rangle$*list-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure arl-assn A* **for** *A*]

lemma *arl-assn-comp*: *is-pure A* $\Longrightarrow$ *hr-comp* (*arl-assn A*) ($\langle B\rangle$*list-rel*) = *arl-assn* (*hr-comp A B*)
unfolding *arl-assn-def*
by (*auto simp*: *hr-comp-the-pure hr-comp-assoc list-rel-compp*)

lemma *arl-assn-comp'*: *hr-comp* (*arl-assn id-assn*) (⟨*B*⟩*list-rel*) = *arl-assn* (*pure B*)
  by (*simp add*: *arl-assn-comp*)

context
  notes [*fcomp-norm-unfold*] = *arl-assn-def*[*symmetric*] *arl-assn-comp'*
  notes [*intro*!] = *hfrefI hn-refineI*[*THEN hn-refine-preI*]
  notes [*simp*] = *pure-def hn-ctxt-def invalid-assn-def*
begin

lemma *arl-empty-hnr-aux*: (*uncurry0 arl-empty*,*uncurry0* (*RETURN op-list-empty*)) $\in$ *unit-assn*^{k} $\rightarrow_a$ *is-array-list*
  by *sep-auto*
sepref-decl-impl (*no-register*) *arl-empty*: *arl-empty-hnr-aux* **.**

lemma *arl-empty-sz-hnr-aux*: (*uncurry0* (*arl-empty-sz N*),*uncurry0* (*RETURN op-list-empty*)) $\in$ *unit-assn*^{k} $\rightarrow_a$ *is-array-list*
  by *sep-auto*
sepref-decl-impl (*no-register*) *arl-empty-sz*: *arl-empty-sz-hnr-aux* **.**

definition *op-arl-empty* $\equiv$ *op-list-empty*
definition *op-arl-empty-sz* (*N*::*nat*) $\equiv$ *op-list-empty*

lemma *arl-copy-hnr-aux*: (*arl-copy*,*RETURN o op-list-copy*) $\in$ *is-array-list*^{k} $\rightarrow_a$ *is-array-list*
  by *sep-auto*
sepref-decl-impl *arl-copy*: *arl-copy-hnr-aux* **.**

lemma *arl-append-hnr-aux*: (*uncurry arl-append*,*uncurry* (*RETURN oo op-list-append*)) $\in$ (*is-array-list*^{d} $*_a$ *id-assn*^{k}) $\rightarrow_a$ *is-array-list*
  by *sep-auto*
sepref-decl-impl *arl-append*: *arl-append-hnr-aux* **.**

lemma *arl-length-hnr-aux*: (*arl-length*,*RETURN o op-list-length*) $\in$ *is-array-list*^{k} $\rightarrow_a$ *nat-assn*
  by *sep-auto*
sepref-decl-impl *arl-length*: *arl-length-hnr-aux* **.**

lemma *arl-is-empty-hnr-aux*: (*arl-is-empty*,*RETURN o op-list-is-empty*) $\in$ *is-array-list*^{k} $\rightarrow_a$ *bool-assn*
  by *sep-auto*
sepref-decl-impl *arl-is-empty*: *arl-is-empty-hnr-aux* **.**

lemma *arl-last-hnr-aux*: (*arl-last*,*RETURN o op-list-last*) $\in$ [*pre-list-last*]$_a$ *is-array-list*^{k} $\rightarrow$ *id-assn*
  by *sep-auto*
sepref-decl-impl *arl-last*: *arl-last-hnr-aux* **.**

lemma *arl-butlast-hnr-aux*: (*arl-butlast*,*RETURN o op-list-butlast*) ∈ [*pre-list-butlast*]$_a$ *is-array-list*^{d} → *is-array-list*
by *sep-auto*
sepref-decl-impl *arl-butlast*: *arl-butlast-hnr-aux* **.**

lemma *arl-get-hnr-aux*: (*uncurry arl-get*,*uncurry* (*RETURN oo op-list-get*)) ∈ [λ(*l*,*i*). *i*<*length l*]$_a$ (*is-array-list*^{k} $*_a$ *nat-assn*^{k}) → *id-assn*
by *sep-auto*
sepref-decl-impl *arl-get*: *arl-get-hnr-aux* **.**

lemma *arl-set-hnr-aux*: (*uncurry2 arl-set*,*uncurry2* (*RETURN ooo op-list-set*)) ∈ [λ((*l*,*i*),-). *i*<*length l*]$_a$ (*is-array-list*^{d} $*_a$ *nat-assn*^{k} $*_a$ *id-assn*^{k}) → *is-array-list*
by *sep-auto*
sepref-decl-impl *arl-set*: *arl-set-hnr-aux* **.**

sepref-definition *arl-swap* **is** *uncurry2 mop-list-swap* :: ((*arl-assn id-assn*)^d $*_a$ *nat-assn*^{k} $*_a$ *nat-assn*^{k} $\rightarrow_a$ *arl-assn id-assn*)
unfolding *gen-mop-list-swap*[*abs-def*]
by *sepref*
sepref-decl-impl (*ismop*) *arl-swap*: *arl-swap.refine* **.**
end

interpretation *arl*: *list-custom-empty arl-assn A arl-empty op-arl-empty*
apply *unfold-locales*
apply (*rule arl-empty-hnr*)
by (*auto simp*: *op-arl-empty-def*)

lemma [*def-pat-rules*]: *op-arl-empty-sz*$*N* ≡ *UNPROTECT* (*op-arl-empty-sz N*)
by *simp*
interpretation *arl-sz*: *list-custom-empty arl-assn A arl-empty-sz N PR-CONST* (*op-arl-empty-sz N*)
apply *unfold-locales*
apply (*rule arl-empty-sz-hnr*)
by (*auto simp*: *op-arl-empty-sz-def*)

end

3.7 Implementation of Heaps with Arrays

theory *IICF-Impl-Heap*
imports
IICF-Abs-Heap
../IICF-HOL-List
../IICF-Array-List
HOL−Library.Rewrite
begin

We implement the heap data structure by an array. The implementation is automatically synthesized by the Sepref-tool.

3.7.1 Setup of the Sepref-Tool

context
fixes *prio* :: $'a$::{*heap*,*default*} ⇒ $'b$::*linorder*
begin
interpretation *heapstruct prio* **.**
definition *heap-rel A* ≡ *hr-comp* (*hr-comp* (*arl-assn id-assn*) *heap-rel1*) (⟨*the-pure A*⟩*mset-rel*)
end

locale *heapstruct-impl* =
fixes *prio* :: $'a$::{*heap*,*default*} ⇒ $'b$::*linorder*
begin
sublocale *heapstruct prio* **.**

abbreviation *rel* ≡ *arl-assn id-assn*

sepref-register *prio*
lemma [*sepref-import-param*]: (*prio*,*prio*) ∈ *Id* → *Id* **by** *simp*

lemma [*sepref-import-param*]:
((≤), (≤)::$'b$ ⇒ -) ∈ *Id* → *Id* → *bool-rel*
((<), (<)::$'b$ ⇒ -) ∈ *Id* → *Id* → *bool-rel*
by *simp-all*

sepref-register
update-op
val-of-op
PR-CONST prio-of-op
exch-op
valid
length::$'a$ *list* ⇒ -
append-op
butlast-op

PR-CONST sink-op
PR-CONST swim-op
PR-CONST repair-op

lemma [*def-pat-rules*]:
heapstruct.prio-of-op\$*prio* ≡ *PR-CONST prio-of-op*
heapstruct.sink-op\$*prio* ≡ *PR-CONST sink-op*
heapstruct.swim-op\$*prio* ≡ *PR-CONST swim-op*

heapstruct.repair-op$prio ≡ *PR-CONST repair-op*
by *simp-all*

end

context
fixes *prio* :: *'a*::{*heap*,*default*} ⇒ *'b*::*linorder*
begin

interpretation *heapstruct-impl prio* **.**

3.7.2 Synthesis of operations

Note that we have to repeat some boilerplate per operation. It is future work to add more automation here.

sepref-definition *update-impl* **is** *uncurry2 update-op* :: $rel^d *_a nat\text{-}assn^k *_a id\text{-}assn^k \rightarrow_a rel$
unfolding *update-op-def*[*abs-def*]
by *sepref*
lemmas [*sepref-fr-rules*] = *update-impl.refine*

sepref-definition *val-of-impl* **is** *uncurry val-of-op* :: $rel^k *_a nat\text{-}assn^k \rightarrow_a id\text{-}assn$
unfolding *val-of-op-def*[*abs-def*]
by *sepref*
lemmas [*sepref-fr-rules*] = *val-of-impl.refine*

sepref-definition *exch-impl* **is** *uncurry2 exch-op* :: $rel^d *_a nat\text{-}assn^k *_a nat\text{-}assn^k \rightarrow_a rel$
unfolding *exch-op-def*[*abs-def*]
by *sepref*
lemmas [*sepref-fr-rules*] = *exch-impl.refine*

sepref-definition *valid-impl* **is** *uncurry* (*RETURN oo valid*) :: $rel^k *_a nat\text{-}assn^k \rightarrow_a bool\text{-}assn$
unfolding *valid-def*[*abs-def*]
by *sepref*
lemmas [*sepref-fr-rules*] = *valid-impl.refine*

sepref-definition *prio-of-impl* **is** *uncurry* (*PR-CONST prio-of-op*) :: $rel^k *_a nat\text{-}assn^k \rightarrow_a id\text{-}assn$
unfolding *prio-of-op-def*[*abs-def*] *PR-CONST-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *prio-of-impl.refine*

sepref-definition *swim-impl* **is** *uncurry* (*PR-CONST swim-op*) :: $rel^d *_a nat\text{-}assn^k \rightarrow_a rel$
unfolding *swim-op-def*[*abs-def*] *parent-def PR-CONST-def*
by *sepref*

lemmas [*sepref-fr-rules*] = *swim-impl.refine*

sepref-definition *sink-impl* **is** *uncurry* (*PR-CONST sink-op*) :: $rel^d *_a nat\text{-}assn^k$
$\rightarrow_a rel$
unfolding *sink-op-opt-def*[*abs-def*] *sink-op-opt-eq*[*symmetric*,*abs-def*] *PR-CONST-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *sink-impl.refine*

lemmas [*fcomp-norm-unfold*] = *heap-rel-def*[*symmetric*]

sepref-definition *empty-impl* **is** *uncurry0 empty-op* :: $unit\text{-}assn^k \rightarrow_a rel$
unfolding *empty-op-def arl.fold-custom-empty*
by *sepref*

sepref-decl-impl (*no-register*) *heap-empty*: *empty-impl.refine*[*FCOMP empty-op-refine*]
.

sepref-definition *is-empty-impl* **is** *is-empty-op* :: $rel^k \rightarrow_a bool\text{-}assn$
unfolding *is-empty-op-def*[*abs-def*]
by *sepref*

sepref-decl-impl *heap-is-empty*: *is-empty-impl.refine*[*FCOMP is-empty-op-refine*]
.

sepref-definition *insert-impl* **is** *uncurry insert-op* :: $id\text{-}assn^k *_a rel^d \rightarrow_a rel$
unfolding *insert-op-def*[*abs-def*] *append-op-def*
by *sepref*
sepref-decl-impl *heap-insert*: *insert-impl.refine*[*FCOMP insert-op-refine*] .

sepref-definition *pop-min-impl* **is** *pop-min-op* :: $rel^d \rightarrow_a prod\text{-}assn\ id\text{-}assn\ rel$
unfolding *pop-min-op-def*[*abs-def*] *butlast-op-def*
by *sepref*

sepref-decl-impl (*no-mop*) *heap-pop-min*: *pop-min-impl.refine*[*FCOMP pop-min-op-refine*]
.

sepref-definition *peek-min-impl* **is** *peek-min-op* :: $rel^k \rightarrow_a id\text{-}assn$
unfolding *peek-min-op-def*[*abs-def*]
by *sepref*
sepref-decl-impl (*no-mop*) *heap-peek-min*: *peek-min-impl.refine*[*FCOMP peek-min-op-refine*]
.

end

definition [*simp*]: *heap-custom-empty* ≡ *op-mset-empty*
interpretation *heap*: *mset-custom-empty*
heap-rel prio A empty-impl heap-custom-empty **for** *prio A*
apply *unfold-locales*
apply (*rule heap-empty-hnr*)

by *simp*

3.7.3 Regression Test

export-code *empty-impl is-empty-impl insert-impl pop-min-impl peek-min-impl* **checking** *SML*

definition *sort-by-prio prio l* ≡ *do* {
  q ← *nfoldli l* (λ-. *True*) (λ*x q*. *mop-mset-insert x q*) *heap-custom-empty*;
  (*l*,*q*) ← *WHILET* (λ(*l*,*q*). ¬*op-mset-is-empty q*) (λ(*l*,*q*). *do* {
    (*x*,*q*) ← *mop-prio-pop-min prio q*;
    RETURN (*l*@[*x*],*q*)
  }) (*op-arl-empty*,*q*);
  RETURN l
}

context fixes *prio*:: ′*a*::{*default*,*heap*} ⇒ ′*b*::*linorder* **begin**
sepref-definition *sort-impl* **is**
  sort-by-prio prio :: (*list-assn* (*id-assn*::′*a*::{*default*,*heap*} ⇒ -))^k $\rightarrow_a$ *arl-assn id-assn*
  unfolding *sort-by-prio-def*[*abs-def*]
  by *sepref*
end
definition *sort-impl-nat* ≡ *sort-impl* (*id*::*nat*⇒*nat*)

export-code *sort-impl* **checking** *SML*

ML ‹
  @{*code sort-impl-nat*} (*map* @{*code nat-of-integer*} [*4*,*1*,*7*,*2*,*3*,*9*,*8*,*62*]) ()
›

hide-const *sort-impl sort-impl-nat*
hide-fact *sort-impl-def sort-impl-nat-def sort-impl.refine*

end

3.8 Map Interface

theory *IICF-Map*
imports *../../Sepref*
begin

3.8.1 Parametricity for Maps

definition [*to-relAPP*]: *map-rel K V* ≡ (*K* → ⟨*V*⟩*option-rel*)
  ∩ { (*mi*,*m*). *dom mi* ⊆ *Domain K* ∧ *dom m* ⊆ *Range K* }

lemma *bi-total-map-rel-eq*:
$[\![$*IS-RIGHT-TOTAL K*; *IS-LEFT-TOTAL K*$]\!] \Longrightarrow \langle K,V\rangle$*map-rel* $= K \rightarrow \langle V\rangle$*option-rel*
unfolding *map-rel-def IS-RIGHT-TOTAL-def IS-LEFT-TOTAL-def*
by (*auto dest*: *fun-relD*)

lemma *map-rel-Id*[*simp*]: $\langle Id,Id\rangle$*map-rel* = *Id*
unfolding *map-rel-def* **by** *auto*

lemma *map-rel-empty1-simp*[*simp*]:
(*Map.empty*,*m*)$\in\langle K,V\rangle$*map-rel* $\longleftrightarrow$ *m*=*Map.empty*
apply (*auto simp*: *map-rel-def*)
by (*meson RangeE domIff option-rel-simp*(*1*) *subsetCE tagged-fun-relD-none*)

lemma *map-rel-empty2-simp*[*simp*]:
(*m*,*Map.empty*)$\in\langle K,V\rangle$*map-rel* $\longleftrightarrow$ *m*=*Map.empty*
apply (*auto simp*: *map-rel-def*)
by (*meson Domain.cases domIff fun-relD2 option-rel-simp*(*2*) *subset-eq*)

lemma *map-rel-obtain1*:
assumes *1*: (*m*,*n*)$\in\langle K,V\rangle$*map-rel*
assumes *2*: *n l* = *Some w*
obtains *k v* **where** *m k* = *Some v* (*k*,*l*)$\in$*K* (*v*,*w*)$\in$*V*
using *1* **unfolding** *map-rel-def*
proof *clarsimp*
assume *R*: (*m*, *n*) $\in K \rightarrow \langle V\rangle$*option-rel*
assume *dom n* $\subseteq$ *Range K*
with *2* **obtain** *k* **where** (*k*,*l*)$\in$*K* **by** *auto*
moreover from *fun-relD*[*OF R this*] **have** (*m k*, *n l*) $\in\langle V\rangle$*option-rel* **.**
with *2* **obtain** *v* **where** *m k* = *Some v* (*v*,*w*)$\in$*V* **by** (*cases m k*; *auto*)
ultimately show *thesis* **by** − (*rule that*)
qed

lemma *map-rel-obtain2*:
assumes *1*: (*m*,*n*)$\in\langle K,V\rangle$*map-rel*
assumes *2*: *m k* = *Some v*
obtains *l w* **where** *n l* = *Some w* (*k*,*l*)$\in$*K* (*v*,*w*)$\in$*V*
using *1* **unfolding** *map-rel-def*
proof *clarsimp*
assume *R*: (*m*, *n*) $\in K \rightarrow \langle V\rangle$*option-rel*
assume *dom m* $\subseteq$ *Domain K*
with *2* **obtain** *l* **where** (*k*,*l*)$\in$*K* **by** *auto*
moreover from *fun-relD*[*OF R this*] **have** (*m k*, *n l*) $\in\langle V\rangle$*option-rel* **.**
with *2* **obtain** *w* **where** *n l* = *Some w* (*v*,*w*)$\in$*V* **by** (*cases n l*; *auto*)
ultimately show *thesis* **by** − (*rule that*)
qed

lemma *param-dom*[*param*]: (*dom*,*dom*)$\in\langle K,V\rangle$*map-rel* $\rightarrow\langle K\rangle$*set-rel*
apply (*clarsimp simp*: *set-rel-def*; *safe*)
apply (*erule* (*1*) *map-rel-obtain2*; *auto*)

apply (*erule* (*1*) *map-rel-obtain1*; *auto*)
done

3.8.2 Interface Type

sepref-decl-intf ($'k$,$'v$) *i-map* **is** $'k \rightharpoonup 'v$

lemma [*synth-rules*]: ⟦*INTF-OF-REL K TYPE*($'k$); *INTF-OF-REL V TYPE*($'v$)⟧

$\Longrightarrow$ *INTF-OF-REL* (⟨*K*,*V*⟩*map-rel*) *TYPE*(($'k$,$'v$) *i-map*) **by** *simp*

3.8.3 Operations

sepref-decl-op *map-empty*: *Map.empty* :: ⟨*K*,*V*⟩*map-rel* **.**

sepref-decl-op *map-is-empty*: (=) *Map.empty* :: ⟨*K*,*V*⟩*map-rel* → *bool-rel*
apply (*rule fref-ncI*)
apply *parametricity*
apply (*rule fun-relI*; *auto*)
done

sepref-decl-op *map-update*: λ*k v m*. *m*(*k*↦*v*) :: *K* → *V* → ⟨*K*,*V*⟩*map-rel* → ⟨*K*,*V*⟩*map-rel*
where *single-valued K single-valued* (K^{-1})
apply (*rule fref-ncI*)
apply *parametricity*
unfolding *map-rel-def*
apply (*intro fun-relI*)
apply (*elim IntE*; *rule IntI*)
apply (*intro fun-relI*)
apply *parametricity*
apply (*simp add*: *pres-eq-iff-svb*)
apply *auto*
done

sepref-decl-op *map-delete*: λ*k m*. *fun-upd m k None* :: *K* → ⟨*K*,*V*⟩*map-rel* → ⟨*K*,*V*⟩*map-rel*
where *single-valued K single-valued* (K^{-1})
apply (*rule fref-ncI*)
apply *parametricity*
unfolding *map-rel-def*
apply (*intro fun-relI*)
apply (*elim IntE*; *rule IntI*)
apply (*intro fun-relI*)
apply *parametricity*
apply (*simp add*: *pres-eq-iff-svb*)
apply *auto*
done

sepref-decl-op *map-lookup*: λ*k* (*m*::$'k \rightharpoonup 'v$). *m k* :: *K* → ⟨*K*,*V*⟩*map-rel* → ⟨*V*⟩*option-rel*

apply (*rule fref-ncI*)
apply *parametricity*
unfolding *map-rel-def*
apply (*intro fun-relI*)
apply (*elim IntE*)
apply *parametricity*
done

lemma *in-dom-alt*: $k{\in}dom\ m \longleftrightarrow \neg is\text{-}None\ (m\ k)$ **by** (*auto split*: *option.split*)

sepref-decl-op *map-contains-key*: $\lambda k\ m.\ k{\in}dom\ m :: K \rightarrow \langle K,V\rangle map\text{-}rel \rightarrow bool\text{-}rel$
unfolding *in-dom-alt*
apply (*rule fref-ncI*)
apply *parametricity*
unfolding *map-rel-def*
apply (*elim IntE*)
apply *parametricity*
done

3.8.4 Patterns

lemma *pat-map-empty*[*pat-rules*]: λ_2-. *None* ≡ *op-map-empty* **by** *simp*

lemma *pat-map-is-empty*[*pat-rules*]:
(=) \$*m*\$(λ_2-. *None*) ≡ *op-map-is-empty*\$*m*
(=) \$(λ_2-. *None*)\$*m* ≡ *op-map-is-empty*\$*m*
(=) \$(*dom*\$*m*)\${} ≡ *op-map-is-empty*\$*m*
(=) \${}\$(*dom*\$*m*) ≡ *op-map-is-empty*\$*m*
unfolding *atomize-eq*
by (*auto dest*: *sym*)

lemma *pat-map-update*[*pat-rules*]:
fun-upd\$*m*\$*k*\$(*Some*\$*v*) ≡ *op-map-update*\$'*k*\$'*v*\$'*m*
by *simp*
lemma *pat-map-lookup*[*pat-rules*]: *m*\$*k* ≡ *op-map-lookup*\$'*k*\$'*m*
by *simp*

lemma *op-map-delete-pat*[*pat-rules*]:
(|\`) \$ *m* \$ (*uminus* \$ (*insert* \$ *k* \$ {})) ≡ *op-map-delete*\$'*k*\$'*m*
fun-upd\$*m*\$*k*\$*None* ≡ *op-map-delete*\$'*k*\$'*m*
by (*simp-all add*: *map-upd-eq-restrict*)

lemma *op-map-contains-key*[*pat-rules*]:
(∈) \$ *k* \$ (*dom*\$*m*) ≡ *op-map-contains-key*\$'*k*\$'*m*
Not\$((=) \$(*m*\$*k*)\$*None*) ≡ *op-map-contains-key*\$'*k*\$'*m*
by (*auto intro*!: *eq-reflection*)

3.8.5 Parametricity

locale *map-custom-empty* =
  fixes *op-custom-empty* :: $'k \rightharpoonup 'v$
  assumes *op-custom-empty-def*: *op-custom-empty* = *op-map-empty*
begin
  sepref-register *op-custom-empty* :: $('kx,'vx)$ *i-map*

  lemma *fold-custom-empty*:
    Map.empty = *op-custom-empty*
    op-map-empty = *op-custom-empty*
    mop-map-empty = *RETURN op-custom-empty*
    unfolding *op-custom-empty-def* **by** *simp-all*
end

end

3.9 Priority Maps

theory *IICF-Prio-Map*
imports *IICF-Map*
begin

This interface inherits from maps, and adds some operations

  lemma *uncurry-fun-rel-conv*:
    $(uncurry\ f,\ uncurry\ g) \in A \times_r B \to R \longleftrightarrow (f,g) \in A \to B \to R$
    by (*auto simp*: *uncurry-def dest*!: *fun-relD intro*: *prod-relI*)

  lemma *uncurry0-fun-rel-conv*:
    $(uncurry0\ f,\ uncurry0\ g) \in unit\text{-}rel \to R \longleftrightarrow (f,g) \in R$
    by (*auto dest*!: *fun-relD*)

  lemma *RETURN-rel-conv0*: $(RETURN\ f,\ RETURN\ g) \in \langle A\rangle nres\text{-}rel \longleftrightarrow (f,g) \in A$
    by (*auto simp*: *nres-rel-def*)

  lemma *RETURN-rel-conv1*: $(RETURN\ o\ f,\ RETURN\ o\ g) \in A \to \langle B\rangle nres\text{-}rel \longleftrightarrow (f,g) \in A \to B$
    by (*auto simp*: *nres-rel-def dest*!: *fun-relD*)

  lemma *RETURN-rel-conv2*: $(RETURN\ oo\ f,\ RETURN\ oo\ g) \in A \to B \to \langle R\rangle nres\text{-}rel \longleftrightarrow (f,g) \in A \to B \to R$
    by (*auto simp*: *nres-rel-def dest*!: *fun-relD*)

  lemma *RETURN-rel-conv3*: $(RETURN\ ooo\ f,\ RETURN\ ooo\ g) \in A \to B \to C \to \langle R\rangle nres\text{-}rel \longleftrightarrow (f,g) \in A \to B \to C \to R$
    by (*auto simp*: *nres-rel-def dest*!: *fun-relD*)

  lemmas *fref2param-unfold* =
    uncurry-fun-rel-conv uncurry0-fun-rel-conv

RETURN-rel-conv0 RETURN-rel-conv1 RETURN-rel-conv2 RETURN-rel-conv3

lemmas *param-op-map-update*[*param*] = *op-map-update.fref*[*THEN fref-ncD*, *unfolded fref2param-unfold*]
lemmas *param-op-map-delete*[*param*] = *op-map-delete.fref*[*THEN fref-ncD*, *unfolded fref2param-unfold*]
lemmas *param-op-map-is-empty*[*param*] = *op-map-is-empty.fref*[*THEN fref-ncD*, *unfolded fref2param-unfold*]

3.9.1 Additional Operations

sepref-decl-op *map-update-new*: *op-map-update* :: $[\lambda((k,v),m).\ k \notin dom\ m]_f\ (K \times_r V) \times_r \langle K,V\rangle map\text{-}rel \to \langle K,V\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) **.**

sepref-decl-op *map-update-ex*: *op-map-update* :: $[\lambda((k,v),m).\ k \in dom\ m]_f\ (K \times_r V) \times_r \langle K,V\rangle map\text{-}rel \to \langle K,V\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) **.**

sepref-decl-op *map-delete-ex*: *op-map-delete* :: $[\lambda(k,m).\ k \in dom\ m]_f\ K \times_r \langle K,V\rangle map\text{-}rel \to \langle K,V\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) **.**

context
fixes *prio* :: $'v \Rightarrow\ 'p$::*linorder*
begin
sepref-decl-op *pm-decrease-key*: *op-map-update*
:: $[\lambda((k,v),m).\ k \in dom\ m \wedge prio\ v \le prio\ (the\ (m\ k))]_f\ (K \times_r V) \times_r \langle K,V\rangle map\text{-}rel \to \langle K,(V::('v \times 'v)\ set)\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) *IS-BELOW-ID V*
proof *goal-cases*
case *1*
have [*param*]: $((\le),(\le)) \in Id \to Id \to bool\text{-}rel$ **by** *simp*
from *1* **show** *?case*
apply (*parametricity add*: *param-and-cong1*)
apply (*auto simp*: *IS-BELOW-ID-def map-rel-def dest*!: *fun-relD*)
done
qed

sepref-decl-op *pm-increase-key*: *op-map-update*
:: $[\lambda((k,v),m).\ k \in dom\ m \wedge prio\ v \ge prio\ (the\ (m\ k))]_f\ (K \times_r V) \times_r \langle K,V\rangle map\text{-}rel \to \langle K,(V::('v \times 'v)\ set)\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) *IS-BELOW-ID V*
proof *goal-cases*
case *1*
have [*param*]: $((\le),(\le)) \in Id \to Id \to bool\text{-}rel$ **by** *simp*
from *1* **show** *?case*

apply (*parametricity add*: *param-and-cong1*)
apply (*auto simp*: *IS-BELOW-ID-def map-rel-def dest*!: *fun-relD*)
done
qed

lemma *IS-BELOW-ID-D*: $(a,b) \in R \Longrightarrow$ *IS-BELOW-ID* $R \Longrightarrow a=b$ **by** (*auto simp*: *IS-BELOW-ID-def*)

sepref-decl-op *pm-peek-min*: $\lambda m.$ *SPEC* $(\lambda(k,v).$
$m\ k = Some\ v \wedge (\forall k'\ v'.\ m\ k' = Some\ v' \longrightarrow prio\ v \leq prio\ v'))$
$:: [Not\ o\ op\text{-}map\text{-}is\text{-}empty]_f\ \langle K,V\rangle map\text{-}rel \rightarrow K \times_r (V::('v \times 'v)\ set)$
where *IS-BELOW-ID V*
apply (*rule frefI*)
apply (*intro nres-relI*)
apply (*clarsimp simp*: *pw-le-iff refine-pw-simps*)
apply (*rule map-rel-obtain2*, *assumption*, *assumption*)
apply1 (*intro exI conjI allI impI*; *assumption?*)
proof −
fix $x\ y\ k'\ v'\ b\ w$
assume $(x,\ y) \in \langle K,\ V\rangle map\text{-}rel\ y\ k' = Some\ v'$
then obtain $k\ v$ **where** $(k,k') \in K\ (v,v') \in V\ x\ k = Some\ v$
by (*rule map-rel-obtain1*)

assume *IS-BELOW-ID V* $(b,\ w) \in V$
with ‹$(v,v') \in V$› **have** [*simp*]: $b=w\ v=v'$ **by** (*auto simp*: *IS-BELOW-ID-def*)

assume $\forall k'\ v'.\ x\ k' = Some\ v' \longrightarrow prio\ b \leq prio\ v'$
with ‹$x\ k = Some\ v$› **show** $prio\ w \leq prio\ v'$
by *auto*
qed

sepref-decl-op *pm-pop-min*: $\lambda m.$ *SPEC* $(\lambda((k,v),m').$
$m\ k = Some\ v$
$\wedge\ m' = op\text{-}map\text{-}delete\ k\ m$
$\wedge\ (\forall k'\ v'.\ m\ k' = Some\ v' \longrightarrow prio\ v \leq prio\ v')$
$) :: [Not\ o\ op\text{-}map\text{-}is\text{-}empty]_f\ \langle K,V\rangle map\text{-}rel \rightarrow (K \times_r (V::('v \times 'v)\ set)) \times_r \langle K,V\rangle map\text{-}rel$
where *single-valued K single-valued* (K^{-1}) *IS-BELOW-ID V*
apply (*rule frefI*)
apply (*intro nres-relI*)
apply (*clarsimp simp*: *pw-le-iff refine-pw-simps simp del*: *op-map-delete-def*)
apply (*rule map-rel-obtain2*, *assumption*, *assumption*)
apply (*intro exI conjI allI impI*; *assumption?*)
applyS *parametricity*
proof −
fix $x\ y\ k'\ v'\ b\ w$
assume $(x,\ y) \in \langle K,\ V\rangle map\text{-}rel\ y\ k' = Some\ v'$
then obtain $k\ v$ **where** $(k,k') \in K\ (v,v') \in V\ x\ k = Some\ v$
by (*rule map-rel-obtain1*)

assume *IS-BELOW-ID V* $(b,\ w) \in V$
with ‹$(v,v')\in V$› **have** [*simp*]: $b=w$ $v=v'$ **by** (*auto simp*: *IS-BELOW-ID-def*)

assume $\forall k'\ v'.\ x\ k' = Some\ v' \longrightarrow prio\ b \le prio\ v'$
with ‹$x\ k = Some\ v$› **show** $prio\ w \le prio\ v'$
by *auto*
qed
end

end

3.10 Priority Maps implemented with List and Map

theory *IICF-Abs-Heapmap*
imports *IICF-Abs-Heap HOL−Library.Rewrite ../../Intf/IICF-Prio-Map*
begin

type-synonym $('k,'v)\ ahm = 'k\ list \times ('k \rightharpoonup 'v)$

3.10.1 Basic Setup

First, we define a mapping to list-based heaps

definition *hmr-α* :: $('k,'v)\ ahm \Rightarrow 'v\ heap$ **where**
hmr-α $\equiv \lambda(pq,m).\ map\ (the\ o\ m)\ pq$

definition *hmr-invar* $\equiv \lambda(pq,m).\ distinct\ pq \wedge dom\ m = set\ pq$

definition *hmr-rel* $\equiv$ *br hmr-α hmr-invar*

lemmas *hmr-rel-defs* = *hmr-rel-def br-def hmr-α-def hmr-invar-def*

lemma *hmr-empty-invar*[*simp*]: *hmr-invar* ([],*Map.empty*)
by (*auto simp*: *hmr-invar-def*)

locale *hmstruct* = *h*: *heapstruct prio* **for** *prio* :: $'v \Rightarrow 'b$::*linorder*
begin

Next, we define a mapping to priority maps.

definition *heapmap-α* :: $('k,'v)\ ahm \Rightarrow ('k \rightharpoonup 'v)$ **where**
heapmap-α $\equiv \lambda(pq,m).\ m$

definition *heapmap-invar* :: $('k,'v)\ ahm \Rightarrow bool$ **where**
heapmap-invar $\equiv \lambda hm.$ *hmr-invar hm* $\wedge$ *h.heap-invar* (*hmr-α hm*)

definition *heapmap-rel* $\equiv$ *br heapmap-α heapmap-invar*

lemmas *heapmap-rel-defs* = *heapmap-rel-def br-def heapmap-α-def heapmap-invar-def*

lemma [*refine-dref-RELATES*]: *RELATES hmr-rel* **by** (*simp add*: *RELATES-def*)

lemma *h-heap-invarI*[*simp*]: *heapmap-invar hm* $\Longrightarrow$ *h.heap-invar* (*hmr-α hm*)

by (*simp add*: *heapmap-invar-def*)

lemma *hmr-invarI*[*simp*]: *heapmap-invar hm* $\Longrightarrow$ *hmr-invar hm*
unfolding *heapmap-invar-def* **by** *blast*

lemma *set-hmr-α*[*simp*]: *hmr-invar hm* $\Longrightarrow$ *set* (*hmr-α hm*) = *ran* (*heapmap-α hm*)
apply (*clarsimp simp*: *hmr-α-def hmr-invar-def heapmap-α-def eq-commute*[*of dom - set -*] *ran-def*)
apply *force*
done

lemma *in-h-hmr-α-conv*[*simp*]: *hmr-invar hm* $\Longrightarrow$ $x \in\#$ *h.α* (*hmr-α hm*) $\longleftrightarrow$ $x \in$ *ran* (*heapmap-α hm*)
by (*force simp*: *hmr-α-def hmr-invar-def heapmap-α-def in-multiset-in-set ran-is-image*)

3.10.2 Basic Operations

In this section, we define the basic operations on heapmaps, and their relations to heaps and maps.

Length

Length of the list that represents the heap

definition *hm-length* :: (*'k*,*'v*) *ahm* ⇒ *nat* **where**
hm-length ≡ λ(*pq*,-). *length pq*

lemma *hm-length-refine*: (*hm-length*, *length*) ∈ *hmr-rel* → *nat-rel*
apply (*intro fun-relI*)
unfolding *hm-length-def*
by (*auto simp*: *hmr-rel-defs*)

lemma *hm-length-hmr-α*[*simp*]: *length* (*hmr-α hm*) = *hm-length hm*
by (*auto simp*: *hm-length-def hmr-α-def split*: *prod.splits*)

lemmas [*refine*] = *hm-length-refine*[*param-fo*]

Valid

Check whether index is valid

definition *hm-valid hm i* $\equiv$ *i>0* $\wedge$ *i*$\leq$ *hm-length hm*

lemma *hm-valid-refine*: (*hm-valid*,*h.valid*)$\in$*hmr-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *bool-rel*
apply (*intro fun-relI*)
unfolding *hm-valid-def h.valid-def*
by (*parametricity add*: *hm-length-refine*)

lemma *hm-valid-hmr-*α[*simp*]: *h.valid* (*hmr-*α *hm*) = *hm-valid hm*
by (*intro ext*) (*auto simp*: *h.valid-def hm-valid-def*)

Key-Of

definition *hm-key-of* :: ('*k*,'*v*) *ahm* $\Rightarrow$ *nat* $\Rightarrow$ '*k* **where**
hm-key-of $\equiv$ λ(*pq*,*m*) *i*. *pq*!(*i* − *1*)

definition *hm-key-of-op* :: ('*k*,'*v*) *ahm* $\Rightarrow$ *nat* $\Rightarrow$ '*k nres* **where**
hm-key-of-op $\equiv$ λ(*pq*,*m*) *i*. *ASSERT* (*i>0*) $\gg$ *mop-list-get pq* (*i* − *1*)

lemma *hm-key-of-op-unfold*:
shows *hm-key-of-op hm i* = *ASSERT* (*hm-valid hm i*) $\gg$ *RETURN* (*hm-key-of hm i*)
unfolding *hm-valid-def hm-length-def hm-key-of-op-def hm-key-of-def*
by (*auto split*: *prod.splits simp*: *pw-eq-iff refine-pw-simps*)

lemma *val-of-hmr-*α[*simp*]: *hm-valid hm i* $\Longrightarrow$ *h.val-of* (*hmr-*α *hm*) *i*
= *the* (*heapmap-*α *hm* (*hm-key-of hm i*))
by (*auto*
simp: *hmr-*α*-def h.val-of-def heapmap-*α*-def hm-key-of-def hm-valid-def hm-length-def*
split: *prod.splits*)

lemma *hm-*α*-key-ex*[*simp*]:
$[\![$*hmr-invar hm*; *hm-valid hm i*$]\!]$ $\Longrightarrow$ (*heapmap-*α *hm* (*hm-key-of hm i*) $\neq$ *None*)
unfolding *heapmap-invar-def hmr-invar-def hm-valid-def heapmap-*α*-def*
hm-key-of-def hm-length-def
by (*auto split*: *prod.splits*)

Lookup

abbreviation (*input*) *hm-lookup* **where** *hm-lookup* $\equiv$ *heapmap-*α

definition *hm-the-lookup-op hm k* $\equiv$
ASSERT (*heapmap-*α *hm k* $\neq$ *None* $\wedge$ *hmr-invar hm*)
$\gg$ *RETURN* (*the* (*heapmap-*α *hm k*))

Exchange

Exchange two indices

definition *hm-exch-op* ≡ λ(*pq*,*m*) *i j*. *do* {
  ASSERT (*hm-valid* (*pq*,*m*) *i*);
  ASSERT (*hm-valid* (*pq*,*m*) *j*);
  ASSERT (*hmr-invar* (*pq*,*m*));
  pq ← *mop-list-swap pq* (*i* − *1*) (*j* − *1*);
  RETURN (*pq*,*m*)
}

lemma *hm-exch-op-invar*: *hm-exch-op hm i j* $\leq_n$ *SPEC hmr-invar*
  unfolding *hm-exch-op-def h.exch-op-def h.val-of-op-def h.update-op-def*
  apply *simp*
  apply *refine-vcg*
  apply (*auto simp*: *hm-valid-def map-swap hm-length-def hmr-rel-defs*)
  done

lemma *hm-exch-op-refine*: (*hm-exch-op*,*h.exch-op*) ∈ *hmr-rel* → *nat-rel* → *nat-rel* → ⟨*hmr-rel*⟩*nres-rel*
  apply (*intro fun-relI nres-relI*)
  unfolding *hm-exch-op-def h.exch-op-def h.val-of-op-def h.update-op-def*
  apply *simp*
  apply *refine-vcg*
  apply (*auto simp*: *hm-valid-def map-swap hm-length-def hmr-rel-defs*)
  done

lemmas *hm-exch-op-refine*′[*refine*] = *hm-exch-op-refine*[*param-fo*, *THEN nres-relD*]

definition *hm-exch* :: (′*k*,′*v*) *ahm* ⇒ *nat* ⇒ *nat* ⇒ (′*k*,′*v*) *ahm*
  where *hm-exch* ≡ λ(*pq*,*m*) *i j*. (*swap pq* (*i*−*1*) (*j*−*1*),*m*)

lemma *hm-exch-op-α-correct*: *hm-exch-op hm i j* $\leq_n$ *SPEC* (λ*hm*′.
  hm-valid hm i ∧ *hm-valid hm j* ∧ *hm*′=*hm-exch hm i j*
)
  unfolding *hm-exch-op-def*
  apply *refine-vcg*
  apply (*vc-solve simp*: *hm-valid-def hm-length-def heapmap-α-def solve*: *asm-rl*)
  apply (*auto simp add*: *hm-key-of-def hm-exch-def swap-def*) []
  done

lemma *hm-exch-α*[*simp*]: *heapmap-α* (*hm-exch hm i j*) = (*heapmap-α hm*)
  by (*auto simp*: *heapmap-α-def hm-exch-def split*: *prod.splits*)
lemma *hm-exch-valid*[*simp*]: *hm-valid* (*hm-exch hm i j*) = *hm-valid hm*
  by (*intro ext*) (*auto simp*: *hm-valid-def hm-length-def hm-exch-def split*: *prod.splits*)
lemma *hm-exch-length*[*simp*]: *hm-length* (*hm-exch hm i j*) = *hm-length hm*
  by (*auto simp*: *hm-length-def hm-exch-def split*: *prod.splits*)

lemma *hm-exch-same*[*simp*]: *hm-exch hm i i* = *hm*
by (*auto simp*: *hm-exch-def split*: *prod.splits*)

lemma *hm-key-of-exch-conv*[*simp*]:
⟦*hm-valid hm i*; *hm-valid hm j*; *hm-valid hm k*⟧ ⟹
hm-key-of (*hm-exch hm i j*) *k* = (
if k=i then hm-key-of hm j
else if k=j then hm-key-of hm i
else hm-key-of hm k
)
unfolding *hm-exch-def hm-valid-def hm-length-def hm-key-of-def*
by (*auto split*: *prod.splits*)

lemma *hm-key-of-exch-matching*[*simp*]:
⟦*hm-valid hm i*; *hm-valid hm j*⟧ ⟹ *hm-key-of* (*hm-exch hm i j*) *i* = *hm-key-of hm j*
⟦*hm-valid hm i*; *hm-valid hm j*⟧ ⟹ *hm-key-of* (*hm-exch hm i j*) *j* = *hm-key-of hm i*
by *simp-all*

Index

Obtaining the index of a key

definition *hm-index* ≡ λ(*pq*,*m*) *k*. *index pq k* + *1*

lemma *hm-index-valid*[*simp*]: ⟦*hmr-invar hm*; *heapmap-α hm k* ≠ *None*⟧ ⟹ *hm-valid hm* (*hm-index hm k*)
by (*force simp*: *hm-valid-def heapmap-α-def hmr-invar-def hm-index-def hm-length-def Suc-le-eq*)

lemma *hm-index-key-of*[*simp*]: ⟦*hmr-invar hm*; *heapmap-α hm k* ≠ *None*⟧ ⟹ *hm-key-of hm* (*hm-index hm k*) = *k*
by (*force*
simp: *hm-valid-def heapmap-α-def hmr-invar-def hm-index-def hm-length-def hm-key-of-def Suc-le-eq*)

definition *hm-index-op* ≡ λ(*pq*,*m*) *k*.
do {
ASSERT (*hmr-invar* (*pq*,*m*) ∧ *heapmap-α* (*pq*,*m*) *k* ≠ *None*);
i ← *mop-list-index pq k*;
RETURN (*i*+*1*)
}

lemma *hm-index-op-correct*:
assumes *hmr-invar hm*
assumes *heapmap-α hm k* ≠ *None*

shows *hm-index-op hm k* $\leq$ *SPEC* $(\lambda r.\ r=$ *hm-index hm k*)
using *assms* **unfolding** *hm-index-op-def*
apply *refine-vcg*
apply (*auto simp*: *heapmap-α-def hmr-invar-def hm-index-def index-nth-id*)
done
lemmas [*refine-vcg*] = *hm-index-op-correct*

Update

Updating the heap at an index

definition *hm-update-op* :: $('k,'v)$ *ahm* $\Rightarrow$ *nat* $\Rightarrow$ $'v \Rightarrow ('k,'v)$ *ahm nres* **where**
hm-update-op $\equiv \lambda(pq,m)\ i\ v.\ do\ \{$
ASSERT (*hm-valid* (pq,m) *i* $\wedge$ *hmr-invar* (pq,m));
$k \leftarrow$ *mop-list-get pq* $(i - 1)$;
RETURN $(pq,\ m(k \mapsto v))$
}

lemma *hm-update-op-invar*: *hm-update-op hm k v* $\leq_n$ *SPEC hmr-invar*
unfolding *hm-update-op-def h.update-op-def*
apply *refine-vcg*
by (*auto simp*: *hmr-rel-defs map-distinct-upd-conv hm-valid-def hm-length-def*)

lemma *hm-update-op-refine*: (*hm-update-op*, *h.update-op*) $\in$ *hmr-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *Id* $\rightarrow$ $\langle$*hmr-rel*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-update-op-def h.update-op-def mop-list-get-alt mop-list-set-alt*
apply *refine-vcg*
apply (*auto simp*: *hmr-rel-defs map-distinct-upd-conv hm-valid-def hm-length-def*)
done

lemmas [*refine*] = *hm-update-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-update-op-α-correct*:
assumes *hmr-invar hm*
assumes *heapmap-α hm k* $\neq$ *None*
shows *hm-update-op hm* (*hm-index hm k*) *v* $\leq_n$ *SPEC* $(\lambda hm'.$ *heapmap-α hm'* = (*heapmap-α hm*)$(k \mapsto v)$)
using *assms*
unfolding *hm-update-op-def*
apply *refine-vcg*
apply (*force simp*: *heapmap-rel-defs hmr-rel-defs hm-index-def*)
done

Butlast

Remove last element

definition *hm-butlast-op* :: $('k,'v)$ *ahm* $\Rightarrow ('k,'v)$ *ahm nres* **where**
hm-butlast-op $\equiv \lambda(pq,m).\ do\ \{$

$ASSERT$ $(hmr\text{-}invar\ (pq,m))$;
$k \leftarrow mop\text{-}list\text{-}get\ pq\ (length\ pq - 1)$;
$pq \leftarrow mop\text{-}list\text{-}butlast\ pq$;
$let\ m = m(k{:=}None)$;
$RETURN\ (pq,m)$
}

lemma *hm-butlast-op-refine*: $(hm\text{-}butlast\text{-}op,\ h.butlast\text{-}op) \in hmr\text{-}rel \rightarrow \langle hmr\text{-}rel\rangle nres\text{-}rel$
supply [*simp del*] = *map-upd-eq-restrict*
apply (*intro fun-relI nres-relI*)
unfolding *hm-butlast-op-def h.butlast-op-def*
apply *simp*
apply *refine-vcg*
apply (*clarsimp-all simp*: *hmr-rel-defs map-butlast distinct-butlast*)
apply (*auto simp*: *neq-Nil-rev-conv*) []
done

lemmas [*refine*] = *hm-butlast-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-butlast-op-α-correct*: $hm\text{-}butlast\text{-}op\ hm \leq_n SPEC$ (
$\lambda hm'.\ heapmap\text{-}\alpha\ hm' = (heapmap\text{-}\alpha\ hm)(\ hm\text{-}key\text{-}of\ hm\ (hm\text{-}length\ hm) :=$
$None$))
unfolding *hm-butlast-op-def*
apply *refine-vcg*
apply (*auto simp*: *heapmap-α-def hm-key-of-def hm-length-def*)
done

Append

Append new element at end of heap

definition *hm-append-op* :: $('k,'v)\ ahm \Rightarrow 'k \Rightarrow 'v \Rightarrow ('k,'v)\ ahm\ nres$
where $hm\text{-}append\text{-}op \equiv \lambda(pq,m)\ k\ v.\ do$ {
$ASSERT\ (k \notin dom\ m)$;
$ASSERT\ (hmr\text{-}invar\ (pq,m))$;
$pq \leftarrow mop\text{-}list\text{-}append\ pq\ k$;
$let\ m = m\ (k \mapsto v)$;
$RETURN\ (pq,m)$
}

lemma *hm-append-op-invar*: $hm\text{-}append\text{-}op\ hm\ k\ v \leq_n SPEC\ hmr\text{-}invar$
unfolding *hm-append-op-def h.append-op-def*
apply *refine-vcg*
unfolding *heapmap-α-def hmr-rel-defs*
apply *auto*
done

lemma *hm-append-op-refine*: $[\![\ heapmap\text{-}\alpha\ hm\ k = None;\ (hm,h) \in hmr\text{-}rel\ ]\!]$
$\Longrightarrow (hm\text{-}append\text{-}op\ hm\ k\ v,\ h.append\text{-}op\ h\ v) \in \langle hmr\text{-}rel\rangle nres\text{-}rel$
apply (*intro fun-relI nres-relI*)

unfolding *hm-append-op-def h.append-op-def*
apply *refine-vcg*
unfolding *heapmap-α-def hmr-rel-defs*
apply *auto*
done

lemmas *hm-append-op-refine'*[*refine*] = *hm-append-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-append-op-α-correct*:
$hm\text{-}append\text{-}op\ hm\ k\ v \le_n SPEC\ (\lambda hm'.\ heapmap\text{-}\alpha\ hm' = (heapmap\text{-}\alpha\ hm)(k \mapsto v))$
unfolding *hm-append-op-def*
apply *refine-vcg*
by (*auto simp*: *heapmap-α-def*)

3.10.3 Auxiliary Operations

Auxiliary operations on heapmaps, which are derived from the basic operations, but do not correspond to operations of the priority map interface

We start with some setup

lemma *heapmap-hmr-relI*: $(hm,h) \in heapmap\text{-}rel \Longrightarrow (hm, hmr\text{-}\alpha\ hm) \in hmr\text{-}rel$

by (*auto simp*: *heapmap-rel-defs hmr-rel-defs*)

lemma *heapmap-hmr-relI'*: $heapmap\text{-}invar\ hm \Longrightarrow (hm, hmr\text{-}\alpha\ hm) \in hmr\text{-}rel$

by (*auto simp*: *heapmap-rel-defs hmr-rel-defs*)

The basic principle how we prove correctness of our operations: Invariant preservation is shown by relating the operations to operations on heaps. Then, only correctness on the abstraction remains to be shown, assuming the operation does not fail.

lemma *heapmap-nres-relI'*:
assumes $hm \le \Downarrow hmr\text{-}rel\ h'$
assumes $h' \le SPEC\ (h.heap\text{-}invar)$
assumes $hm \le_n SPEC\ (\lambda hm'.\ RETURN\ (heapmap\text{-}\alpha\ hm') \le h)$
shows $hm \le \Downarrow heapmap\text{-}rel\ h$
using *assms*
unfolding *heapmap-rel-defs hmr-rel-def*
by (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps*)

lemma *heapmap-nres-relI''*:
assumes $hm \le \Downarrow hmr\text{-}rel\ h'$
assumes $h' \le SPEC\ \Phi$
assumes $\bigwedge h'.\ \Phi\ h' \Longrightarrow h.heap\text{-}invar\ h'$
assumes $hm \le_n SPEC\ (\lambda hm'.\ RETURN\ (heapmap\text{-}\alpha\ hm') \le h)$

shows $hm \leq \Downarrow$*heapmap-rel h*
apply (*rule heapmap-nres-relI'*)
apply *fact*
apply (*rule order-trans*, *fact*)
apply (*clarsimp*; *fact*)
apply *fact*
done

Val-of

Indexing into the heap

definition *hm-val-of-op* :: ('k,'v) *ahm* ⇒ *nat* ⇒ 'v *nres* **where**
hm-val-of-op ≡ λ*hm i. do* {
k ← *hm-key-of-op hm i*;
v ← *hm-the-lookup-op hm k*;
RETURN v
}

lemma *hm-val-of-op-refine*: (*hm-val-of-op*,*h.val-of-op*) ∈ (*hmr-rel* → *nat-rel* → ⟨*Id*⟩*nres-rel*)
apply (*intro fun-relI nres-relI*)
unfolding *hm-val-of-op-def h.val-of-op-def*
hm-key-of-op-def hm-key-of-def hm-valid-def hm-length-def
hm-the-lookup-op-def
apply *clarsimp*
apply (*rule refine-IdD*)
apply *refine-vcg*
apply (*auto simp*: *hmr-rel-defs heapmap-α-def*)
done

lemmas [*refine*] = *hm-val-of-op-refine*[*param-fo*, *THEN nres-relD*]

Prio-of

Priority of key

definition *hm-prio-of-op h i* ≡ *do* {*v* ← *hm-val-of-op h i*; *RETURN* (*prio v*)}

lemma *hm-prio-of-op-refine*: (*hm-prio-of-op*, *h.prio-of-op*) ∈ *hmr-rel* → *nat-rel* → ⟨*Id*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-prio-of-op-def h.prio-of-op-def*
apply *refine-rcg*
by *auto*

lemmas *hm-prio-of-op-refine'*[*refine*] = *hm-prio-of-op-refine*[*param-fo*, *THEN nres-relD*]

Swim

definition *hm-swim-op* :: $('k,'v)$ *ahm* $\Rightarrow$ *nat* $\Rightarrow$ $('k,'v)$ *ahm nres* **where**
hm-swim-op h i $\equiv$ *do* {
  RECT (λ*swim* (*h*,*i*). *do* {
    ASSERT (*hm-valid h i* $\wedge$ *h.swim-invar* (*hmr-α h*) *i*);
    if hm-valid h (*h.parent i*) *then do* {
      ppi $\leftarrow$ *hm-prio-of-op h* (*h.parent i*);
      pi $\leftarrow$ *hm-prio-of-op h i*;
      if ($\neg$*ppi* $\leq$ *pi*) *then do* {
        h $\leftarrow$ *hm-exch-op h i* (*h.parent i*);
        swim (*h*, *h.parent i*)
      } *else*
        RETURN h
    } *else*
      RETURN h
  }) (*h*,*i*)
}

lemma *hm-swim-op-refine*: (*hm-swim-op*, *h.swim-op*) $\in$ *hmr-rel* $\rightarrow$ *nat-rel* $\rightarrow$ $\langle$*hmr-rel*$\rangle$*nres-rel*
  apply (*intro fun-relI nres-relI*)
  unfolding *hm-swim-op-def h.swim-op-def*
  apply *refine-rcg*
  apply *refine-dref-type*
  apply (*clarsimp-all simp*: *hm-valid-refine*[*param-fo*, *THEN IdD*])
  apply (*simp add*: *hmr-rel-def in-br-conv*)
  done

lemmas *hm-swim-op-refine'*[*refine*] = *hm-swim-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-swim-op-nofail-imp-valid*:
  nofail (*hm-swim-op hm i*) $\Longrightarrow$ *hm-valid hm i* $\wedge$ *h.swim-invar* (*hmr-α hm*) *i*
  unfolding *hm-swim-op-def*
  apply (*subst* (*asm*) *RECT-unfold*, *refine-mono*)
  by (*auto simp*: *refine-pw-simps*)

lemma *hm-swim-op-α-correct*: *hm-swim-op hm i* $\leq_n$ *SPEC* (λ*hm'*. *heapmap-α hm'* = *heapmap-α hm*)
  apply (*rule leof-add-nofailI*)
  apply (*drule hm-swim-op-nofail-imp-valid*)
  unfolding *hm-swim-op-def*
  apply (*rule RECT-rule-leof*[**where**
    pre=λ(*hm'*,*i*). *hm-valid hm' i* $\wedge$ *heapmap-α hm'* = *heapmap-α hm*
    and *V* = *inv-image less-than snd*
    ])
  apply *simp*
  apply *simp*

unfolding *hm-prio-of-op-def hm-val-of-op-def*
hm-exch-op-def hm-key-of-op-def hm-the-lookup-op-def
apply (*refine-vcg*)
apply (*vc-solve simp add*: *hm-valid-def hm-length-def*)
apply *rprems*
apply (*vc-solve simp*: *heapmap-α-def h.parent-def*)
done

Sink

definition *hm-sink-op*
where
hm-sink-op h k ≡ *RECT* (λ*D* (*h*,*k*). *do* {
ASSERT (*k*>*0* ∧ *k*≤*hm-length h*);
let len = *hm-length h*;
if (*2*∗*k* ≤ *len*) *then do* {
let j = *2*∗*k*;
pj ← *hm-prio-of-op h j*;

j ← (
if j<*len then do* {
psj ← *hm-prio-of-op h* (*Suc j*);
if pj>*psj then RETURN* (*j*+*1*) *else RETURN j*
} *else RETURN j*);

pj ← *hm-prio-of-op h j*;
pk ← *hm-prio-of-op h k*;
if (*pk* > *pj*) *then do* {
h ← *hm-exch-op h k j*;
D (*h*,*j*)
} *else*
RETURN h
} *else RETURN h*
}) (*h*,*k*)

lemma *hm-sink-op-refine*: (*hm-sink-op*, *h.sink-op*) ∈ *hmr-rel* → *nat-rel* → ⟨*hmr-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-sink-op-def h.sink-op-opt-eq*[*symmetric*] *h.sink-op-opt-def*
apply *refine-rcg*
apply *refine-dref-type*

unfolding *hmr-rel-def heapmap-rel-def*
apply (*clarsimp-all simp*: *in-br-conv*)
done

lemmas *hm-sink-op-refine'*[*refine*] = *hm-sink-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-sink-op-nofail-imp-valid*: *nofail* (*hm-sink-op hm i*) ⟹ *hm-valid hm* *i*

unfolding *hm-sink-op-def*
apply (*subst* (*asm*) *RECT-unfold*, *refine-mono*)
by (*auto simp*: *refine-pw-simps hm-valid-def*)

lemma *hm-sink-op-α-correct*: *hm-sink-op hm i* $\leq_n$ *SPEC* ($\lambda hm'$. *heapmap-α* hm' = *heapmap-α hm*)
apply (*rule leof-add-nofailI*)
apply (*drule hm-sink-op-nofail-imp-valid*)
unfolding *hm-sink-op-def*
apply (*rule RECT-rule-leof*[**where**
pre=$\lambda(hm',i)$. *hm-valid* hm' *i* ∧ *heapmap-α* hm' = *heapmap-α hm* ∧ *hm-length* hm' = *hm-length hm*
and *V* = *measure* ($\lambda(hm',i)$. *hm-length* hm' − *i*)
])
apply *simp*
apply *simp*

unfolding *hm-prio-of-op-def hm-val-of-op-def hm-exch-op-def*
hm-key-of-op-def hm-the-lookup-op-def
apply (*refine-vcg*)
apply (*vc-solve simp add*: *hm-valid-def hm-length-def*)
apply *rprems*
apply (*vc-solve simp*: *heapmap-α-def h.parent-def split*: *prod.splits*)
apply (*auto*)
done

Repair

definition *hm-repair-op hm i* ≡ *do* {
hm ← *hm-sink-op hm i*;
hm ← *hm-swim-op hm i*;
RETURN hm
}

lemma *hm-repair-op-refine*: (*hm-repair-op*, *h.repair-op*) ∈ *hmr-rel* → *nat-rel* → ⟨*hmr-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-repair-op-def h.repair-op-def*
by *refine-rcg*

lemmas *hm-repair-op-refine′*[*refine*] = *hm-repair-op-refine*[*param-fo*, *THEN nres-relD*]

lemma *hm-repair-op-α-correct*: *hm-repair-op hm i* $\leq_n$ *SPEC* ($\lambda hm'$. *heapmap-α* hm' = *heapmap-α hm*)
unfolding *hm-repair-op-def*
apply (*refine-vcg*
hm-swim-op-α-correct[*THEN leof-trans*]

hm-sink-op-α-correct[*THEN leof-trans*])
by *auto*

3.10.4 Operations

In this section, we define the operations that implement the priority-map interface

Empty

definition *hm-empty-op* :: ($'k$,$'v$) *ahm nres*
where *hm-empty-op* $\equiv$ *RETURN* ([],*Map.empty*)

lemma *hm-empty-aref*: (*hm-empty-op*,*RETURN op-map-empty*) $\in$ $\langle$*heapmap-rel*$\rangle$*nres-rel*

unfolding *hm-empty-op-def*
by (*auto simp*: *heapmap-rel-defs hmr-rel-defs intro*: *nres-relI*)

Insert

definition *hm-insert-op* :: $'k \Rightarrow 'v \Rightarrow ('k,'v)$ *ahm* $\Rightarrow$ ($'k$,$'v$) *ahm nres* **where**
hm-insert-op $\equiv$ $\lambda k\ v\ h.$ *do* {
ASSERT (*h.heap-invar* (*hmr-α h*));
h $\leftarrow$ *hm-append-op h k v*;
let l = *hm-length h*;
h $\leftarrow$ *hm-swim-op h l*;
RETURN h
}

lemma *hm-insert-op-refine*[*refine*]: $[\![$ *heapmap-α hm k* = *None*; (*hm*,*h*)$\in$*hmr-rel*
$]\!] \Longrightarrow$
hm-insert-op k v hm $\leq$ $\Downarrow$*hmr-rel* (*h.insert-op v h*)
unfolding *hm-insert-op-def h.insert-op-def*
apply *refine-rcg*
by (*auto simp*: *hmr-rel-def br-def*)

lemma *hm-insert-op-aref*:
(*hm-insert-op*,*mop-map-update-new*) $\in$ *Id* $\rightarrow$ *Id* $\rightarrow$ *heapmap-rel* $\rightarrow$ $\langle$*heapmap-rel*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *mop-map-update-new-alt*
apply (*rule ASSERT-refine-right*)
apply (*rule heapmap-nres-relI''*[*OF hm-insert-op-refine h.insert-op-correct*])
apply (*unfold heapmap-rel-def in-br-conv*; *clarsimp*)
apply (*erule heapmap-hmr-relI*)
apply (*unfold heapmap-rel-def in-br-conv*; *clarsimp*)
apply (*unfold heapmap-rel-def in-br-conv*; *clarsimp*)
unfolding *hm-insert-op-def*
apply (*refine-vcg*
hm-append-op-α-correct[*THEN leof-trans*]

$hm\text{-}swim\text{-}op\text{-}\alpha\text{-}correct[THEN\ leof\text{-}trans])$
apply (*unfold heapmap-rel-def in-br-conv*; *clarsimp*)
done

Is-Empty

lemma *hmr-α-empty-iff* [*simp*]:
hmr-invar hm $\Longrightarrow$ *hmr-α hm* = [] $\longleftrightarrow$ *heapmap-α hm* = *Map.empty*
by (*auto*
simp: *hmr-α-def heapmap-invar-def heapmap-α-def hmr-invar-def*
split: *prod.split*)

definition *hm-is-empty-op* :: ('*k*,'*v*) *ahm* ⇒ *bool nres* **where**
hm-is-empty-op ≡ λ*hm*. *do* {
ASSERT (*hmr-invar hm*);
let l = *hm-length hm*;
RETURN (*l*=*0*)
}

lemma *hm-is-empty-op-refine*: (*hm-is-empty-op*, *h.is-empty-op*) ∈ *hmr-rel* → ⟨*bool-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-is-empty-op-def h.is-empty-op-def*
apply *refine-rcg*
apply (*auto simp*: *hmr-rel-defs*) []
apply (*parametricity add*: *hm-length-refine*)
done

lemma *hm-is-empty-op-aref*: (*hm-is-empty-op*, *RETURN o op-map-is-empty*) ∈ *heapmap-rel* → ⟨*bool-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-is-empty-op-def*
apply *refine-vcg*
apply (*auto simp*: *hmr-rel-defs heapmap-rel-defs hm-length-def*)
done

Lookup

definition *hm-lookup-op* :: '*k* ⇒ ('*k*,'*v*) *ahm* ⇒ '*v option nres*
where *hm-lookup-op* ≡ λ*k hm*. *ASSERT* (*heapmap-invar hm*) ≫ *RETURN* (*hm-lookup hm k*)

lemma *hm-lookup-op-aref*: (*hm-lookup-op*,*RETURN oo op-map-lookup*) ∈ *Id* → *heapmap-rel* → ⟨⟨*Id*⟩*option-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-lookup-op-def heapmap-rel-def in-br-conv*
apply *refine-vcg*
apply *simp-all*
done

Contains-Key

definition *hm-contains-key-op* ≡ λk (pq,m). *ASSERT* (*heapmap-invar* (pq,m)) ≫ *RETURN* (k∈*dom* m)

lemma *hm-contains-key-op-aref*: (*hm-contains-key-op*,*RETURN oo op-map-contains-key*) ∈ *Id* → *heapmap-rel* → ⟨*bool-rel*⟩*nres-rel*
  apply (*intro fun-relI nres-relI*)
  unfolding *hm-contains-key-op-def heapmap-rel-defs*
  apply *refine-vcg*
  by (*auto*)

Decrease-Key

definition *hm-decrease-key-op* ≡ λk v *hm*. *do* {
  ASSERT (*heapmap-invar hm*);
  ASSERT (*heapmap-α hm k* ≠ *None* ∧ *prio v* ≤ *prio* (*the* (*heapmap-α hm k*)));
  i ← *hm-index-op hm k*;
  hm ← *hm-update-op hm i v*;
  hm-swim-op hm i
}

definition (**in** *heapstruct*) *decrease-key-op i v h* ≡ *do* {
  ASSERT (*valid h i* ∧ *prio v* ≤ *prio-of h i*);
  h ← *update-op h i v*;
  swim-op h i
}

lemma (**in** *heapstruct*) *decrease-key-op-invar*:
  ⟦*heap-invar h*; *valid h i*; *prio v* ≤ *prio-of h i*⟧ ⟹ *decrease-key-op i v h* ≤ *SPEC heap-invar*
  unfolding *decrease-key-op-def*
  apply *refine-vcg*
  by (*auto simp*: *swim-invar-decr*)

lemma *index-op-inline-refine*:
  assumes *heapmap-invar hm*
  assumes *heapmap-α hm k* ≠ *None*
  assumes *f* (*hm-index hm k*) ≤ *m*
  shows *do* {*i* ← *hm-index-op hm k*; *f i*} ≤ *m*
  using *hm-index-op-correct*[*of hm k*] *assms*
  by (*auto simp*: *pw-le-iff refine-pw-simps*)

lemma *hm-decrease-key-op-refine*:
  ⟦(*hm*,*h*)∈*hmr-rel*; (*hm*,*m*)∈*heapmap-rel*; *m k* = *Some v'*⟧
  ⟹ *hm-decrease-key-op k v hm* ≤⇓*hmr-rel* (*h.decrease-key-op* (*hm-index hm k*) *v h*)
  unfolding *hm-decrease-key-op-def h.decrease-key-op-def*

apply (*refine-rcg index-op-inline-refine*)
unfolding *hmr-rel-def heapmap-rel-def in-br-conv*
apply (*clarsimp-all*)
done

lemma *hm-index-op-inline-leof*:
assumes f (*hm-index hm k*) $\leq_n$ *m*
shows *do* $\{i \leftarrow$ *hm-index-op hm k*; *f i*$\}$ $\leq_n$ *m*
using *hm-index-op-correct*[*of hm k*] *assms* **unfolding** *hm-index-op-def*
by (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps split*: *prod.splits*)

lemma *hm-decrease-key-op-*α*-correct*:
heapmap-invar hm $\Longrightarrow$ *hm-decrease-key-op k v hm* $\leq_n$ *SPEC* ($\lambda hm'$. *heapmap-*α hm' = (*heapmap-*α *hm*)($k \mapsto v$))
unfolding *hm-decrease-key-op-def*
apply (*refine-vcg*
hm-update-op-α*-correct*[*THEN leof-trans*]
hm-swim-op-α*-correct*[*THEN leof-trans*]
hm-index-op-inline-leof
)
apply *simp-all*
done

lemma *hm-decrease-key-op-aref*:
(*hm-decrease-key-op*, *PR-CONST* (*mop-pm-decrease-key prio*)) $\in$ *Id* $\rightarrow$ *Id* $\rightarrow$ *heapmap-rel* $\rightarrow$ $\langle$*heapmap-rel*$\rangle$*nres-rel*
unfolding *PR-CONST-def*
apply (*intro fun-relI nres-relI*)
apply (*frule heapmap-hmr-relI*)
unfolding *mop-pm-decrease-key-alt*
apply (*rule ASSERT-refine-right*; *clarsimp*)
apply (*rule heapmap-nres-relI'*)
apply (*rule hm-decrease-key-op-refine*; *assumption*)
unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
apply (*rule h.decrease-key-op-invar*; *simp*; *fail*)
apply (*refine-vcg hm-decrease-key-op-*α*-correct*[*THEN leof-trans*]; *simp*; *fail*)
done

Increase-Key

definition *hm-increase-key-op* $\equiv$ $\lambda k\ v\ hm$. *do* {
ASSERT (*heapmap-invar hm*);
ASSERT (*heapmap-*α *hm k* $\neq$ *None* $\wedge$ *prio v* $\geq$ *prio* (*the* (*heapmap-*α *hm k*)));
$i \leftarrow$ *hm-index-op hm k*;
hm $\leftarrow$ *hm-update-op hm i v*;
hm-sink-op hm i
}

definition (**in** *heapstruct*) *increase-key-op i v h* ≡ *do* {
ASSERT (*valid h i* ∧ *prio v* ≥ *prio-of h i*);
h ← *update-op h i v*;
sink-op h i
}

lemma (**in** *heapstruct*) *increase-key-op-invar*:
⟦*heap-invar h*; *valid h i*; *prio v* ≥ *prio-of h i*⟧ ⟹ *increase-key-op i v h* ≤ *SPEC heap-invar*
unfolding *increase-key-op-def*
apply *refine-vcg*
by (*auto simp*: *sink-invar-incr*)

lemma *hm-increase-key-op-refine*:
⟦(*hm*,*h*)∈*hmr-rel*; (*hm*,*m*)∈*heapmap-rel*; *m k* = *Some v'*⟧
⟹ *hm-increase-key-op k v hm* ≤⇓*hmr-rel* (*h.increase-key-op* (*hm-index hm k*) *v h*)
unfolding *hm-increase-key-op-def h.increase-key-op-def*

apply (*refine-rcg index-op-inline-refine*)
unfolding *hmr-rel-def heapmap-rel-def in-br-conv*
apply (*clarsimp-all*)
done

lemma *hm-increase-key-op-α-correct*:
heapmap-invar hm ⟹ *hm-increase-key-op k v hm* $\leq_n$ *SPEC* (λ*hm'*. *heapmap-α hm'* = (*heapmap-α hm*)(*k*↦*v*))
unfolding *hm-increase-key-op-def*
apply (*refine-vcg*
hm-update-op-α-correct[*THEN leof-trans*]
hm-sink-op-α-correct[*THEN leof-trans*]
hm-index-op-inline-leof)
apply *simp-all*
done

lemma *hm-increase-key-op-aref*:
(*hm-increase-key-op*, *PR-CONST* (*mop-pm-increase-key prio*)) ∈ *Id* → *Id* → *heapmap-rel* → ⟨*heapmap-rel*⟩*nres-rel*
unfolding *PR-CONST-def*
apply (*intro fun-relI nres-relI*)
apply (*frule heapmap-hmr-relI*)
unfolding *mop-pm-increase-key-alt*
apply (*rule ASSERT-refine-right*; *clarsimp*)
apply (*rule heapmap-nres-relI'*)
apply (*rule hm-increase-key-op-refine*; *assumption*)
unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
apply (*rule h.increase-key-op-invar*; *simp*; *fail*)
apply (*refine-vcg hm-increase-key-op-α-correct*[*THEN leof-trans*]; *simp*)
done

Change-Key

definition *hm-change-key-op* ≡ λ*k v hm. do* {
  ASSERT (*heapmap-invar hm*);
  ASSERT (*heapmap-α hm k* ≠ *None*);
  i ← *hm-index-op hm k*;
  hm ← *hm-update-op hm i v*;
  hm-repair-op hm i
}

definition (**in** *heapstruct*) *change-key-op i v h* ≡ *do* {
  ASSERT (*valid h i*);
  h ← *update-op h i v*;
  repair-op h i
}

lemma (**in** *heapstruct*) *change-key-op-invar*:
  ⟦*heap-invar h*; *valid h i*⟧ ⟹ *change-key-op i v h* ≤ *SPEC heap-invar*
  unfolding *change-key-op-def*
  apply (*refine-vcg*)
  apply *hypsubst*
  apply *refine-vcg*
  by (*auto simp*: *sink-invar-incr*)

lemma *hm-change-key-op-refine*:
  ⟦(*hm*,*h*)∈*hmr-rel*; (*hm*,*m*)∈*heapmap-rel*; *m k* = *Some v'*⟧
    ⟹ *hm-change-key-op k v hm* ≤⇓*hmr-rel* (*h.change-key-op* (*hm-index hm k*) *v h*)
  unfolding *hm-change-key-op-def h.change-key-op-def*

  apply (*refine-rcg index-op-inline-refine*)
  unfolding *hmr-rel-def heapmap-rel-def in-br-conv*
  apply (*clarsimp-all*)
  done

lemma *hm-change-key-op-α-correct*:
  heapmap-invar hm ⟹ *hm-change-key-op k v hm* $\leq_n$ *SPEC* (λ*hm'*. *heapmap-α hm'* = (*heapmap-α hm*)(*k*↦*v*))
  unfolding *hm-change-key-op-def*
  apply (*refine-vcg*
    hm-update-op-α-correct[*THEN leof-trans*]
    hm-repair-op-α-correct[*THEN leof-trans*]
    hm-index-op-inline-leof)
  unfolding *heapmap-rel-def in-br-conv*
  apply *simp*
  apply *simp*
  done

lemma *hm-change-key-op-aref*:
    (*hm-change-key-op*, *mop-map-update-ex*) ∈ *Id* → *Id* → *heapmap-rel* →

⟨*heapmap-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
apply (*frule heapmap-hmr-relI*)
unfolding *mop-map-update-ex-alt*
apply (*rule ASSERT-refine-right*; *clarsimp*)
apply (*rule heapmap-nres-relI'*)
apply (*rule hm-change-key-op-refine*; *assumption*)
unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
apply (*rule h.change-key-op-invar*; *simp*; *fail*)
apply ((*refine-vcg hm-change-key-op-α-correct*[*THEN leof-trans*]; *simp*))
done

Set

Realized as generic algorithm!

lemma (**in** −) *op-pm-set-gen-impl*: *RETURN ooo op-map-update* = (*λk v m. do* {
c ← *RETURN* (*op-map-contains-key k m*);
if c then
mop-map-update-ex k v m
else
mop-map-update-new k v m
})
apply (*intro ext*)
unfolding *op-map-contains-key-def mop-map-update-ex-def mop-map-update-new-def*
by *simp*

definition *hm-set-op k v hm* ≡ *do* {
c ← *hm-contains-key-op k hm*;
if c then
hm-change-key-op k v hm
else
hm-insert-op k v hm
}

lemma *hm-set-op-aref*:
(*hm-set-op*, *RETURN ooo op-map-update*) ∈ *Id* → *Id* → *heapmap-rel* → ⟨*heapmap-rel*⟩*nres-rel*
unfolding *op-pm-set-gen-impl*
apply (*intro fun-relI nres-relI*)
unfolding *hm-set-op-def o-def*
apply (*refine-rcg*
hm-contains-key-op-aref[*param-fo*, *unfolded o-def*, *THEN nres-relD*]
hm-change-key-op-aref[*param-fo*, *THEN nres-relD*]
hm-insert-op-aref[*param-fo*, *THEN nres-relD*]
)
by *auto*

Pop-Min

definition *hm-pop-min-op* :: $('k,'v)$ *ahm* $\Rightarrow$ $(('k\times'v) \times ('k,'v)$ *ahm*$)$ *nres* **where**
hm-pop-min-op hm $\equiv$ *do* {
ASSERT (*heapmap-invar hm*);
ASSERT (*hm-valid hm 1*);
k $\leftarrow$ *hm-key-of-op hm 1*;
v $\leftarrow$ *hm-the-lookup-op hm k*;
let l = *hm-length hm*;
hm $\leftarrow$ *hm-exch-op hm 1 l*;
hm $\leftarrow$ *hm-butlast-op hm*;

if ($l\neq 1$) *then do* {
hm $\leftarrow$ *hm-sink-op hm 1*;
RETURN ((*k*,*v*),*hm*)
} *else RETURN* ((*k*,*v*),*hm*)
}

lemma *hm-pop-min-op-refine*:
(*hm-pop-min-op*, *h.pop-min-op*) $\in$ *hmr-rel* $\rightarrow$ $\langle$*UNIV* $\times_r$ *hmr-rel*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-pop-min-op-def h.pop-min-op-def*

unfolding *ignore-snd-refine-conv hm-the-lookup-op-def hm-key-of-op-unfold*
apply (*simp cong*: *if-cong add*: *Let-def*)
apply (*simp add*: *unused-bind-conv h.val-of-op-def refine-pw-simps*)

apply *refine-rcg*
unfolding *hmr-rel-def in-br-conv*
apply (*unfold heapmap-invar-def*;*simp*)
apply (*auto simp*: *in-br-conv*)
done

We demonstrate two different approaches for proving correctness here. The first approach uses the relation to plain heaps only to establish the invariant.

The second approach also uses the relation to heaps to establish correctness of the result.

The first approach seems to be more robust against badly set up simpsets, which may be the case in early stages of development.

Assuming a working simpset, the second approach may be less work, and the proof may look more elegant.

First approach Transfer heapmin-property to heapmap-domain

lemma *heapmap-min-prop*:
assumes *INV*: *heapmap-invar hm*
assumes *V'*: *heapmap-α hm k* = *Some v'*

assumes *NE*: *hm-valid hm* (*Suc 0*)
shows *prio* (*the* (*heapmap-α hm* (*hm-key-of hm* (*Suc 0*)))) $\leq$ *prio v′*
proof −
— Transform into the domain of heaps
obtain *pq m* **where** [*simp*]: *hm*=(*pq*,*m*) **by** (*cases hm*)

from *NE* **have** [*simp*]: *pq*≠[] **by** (*auto simp*: *hm-valid-def hm-length-def*)

have *CNV-LHS*: *prio* (*the* (*heapmap-α hm* (*hm-key-of hm* (*Suc 0*))))
= *h.prio-of* (*hmr-α hm*) (*Suc 0*)
by (*auto simp*: *heapmap-α-def hm-key-of-def hmr-α-def h.val-of-def*)

from *INV* **have** *INV′*: *h.heap-invar* (*hmr-α hm*)
unfolding *heapmap-invar-def* **by** *auto*

from *V′ INV* **obtain** *i* **where** *IDX*: *h.valid* (*hmr-α hm*) *i*
and *CNV-RHS*: *prio v′* = *h.prio-of* (*hmr-α hm*) *i*
apply (*clarsimp simp*: *heapmap-α-def heapmap-invar-def hmr-invar-def hmr-α-def*
h.valid-def h.val-of-def)
by (*metis* (*no-types*, *opaque-lifting*) *Suc-leI comp-apply diff-Suc-Suc*
diff-zero domI index-less-size-conv neq0-conv nth-index nth-map
old.nat.distinct(*2*) *option.sel*)

from *h.heap-min-prop*[*OF INV′ IDX*] **show** *?thesis*
unfolding *CNV-LHS CNV-RHS* **.**
qed

With the above lemma, the correctness proof is straightforward

lemma *hm-pop-min-α-correct*: *hm-pop-min-op hm* $\leq_n$ *SPEC* (λ((*k*,*v*),*hm′*).
heapmap-α hm k = *Some v*
∧ *heapmap-α hm′* = (*heapmap-α hm*)(*k*:=*None*)
∧ (∀ *k′ v′*. *heapmap-α hm k′* = *Some v′* ⟶ *prio v* $\leq$ *prio v′*))
unfolding *hm-pop-min-op-def hm-key-of-op-unfold hm-the-lookup-op-def*
apply (*refine-vcg*
hm-exch-op-α-correct[*THEN leof-trans*]
hm-butlast-op-α-correct[*THEN leof-trans*]
hm-sink-op-α-correct[*THEN leof-trans*]
)
apply (*auto simp*: *heapmap-min-prop*)
done

lemma *heapmap-nres-rel-prodI*:
assumes *hmx* $\leq$ ⇓(*UNIV* $\times_r$ *hmr-rel*) *h′x*
assumes *h′x* $\leq$ *SPEC* (λ(-,*h′*). *h.heap-invar h′*)
assumes *hmx* $\leq_n$ *SPEC* (λ(*r*,*hm′*). *RETURN* (*r*,*heapmap-α hm′*) $\leq$ ⇓(*R*$\times_r$*Id*) *hx*)
shows *hmx* $\leq$ ⇓(*R*$\times_r$*heapmap-rel*) *hx*
using *assms*

unfolding *heapmap-rel-def hmr-rel-def br-def heapmap-invar-def*
apply (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps*; *blast*)
done

lemma *hm-pop-min-op-aref*: (*hm-pop-min-op*, *PR-CONST* (*mop-pm-pop-min prio*)) $\in$ *heapmap-rel* $\rightarrow$ $\langle(Id\times_r Id)\times_r heapmap\text{-}rel\rangle nres\text{-}rel$
unfolding *PR-CONST-def*
apply (*intro fun-relI nres-relI*)
apply (*frule heapmap-hmr-relI*)
unfolding *mop-pm-pop-min-alt*
apply (*intro ASSERT-refine-right*)
apply (*rule heapmap-nres-rel-prodI*)
apply (*rule hm-pop-min-op-refine*[*param-fo*, *THEN nres-relD*]; *assumption*)
unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
apply (*refine-vcg*; *simp*)
apply (*refine-vcg hm-pop-min-α-correct*[*THEN leof-trans*]; *simp split*: *prod.splits*)
done

Second approach **definition** *hm-kv-of-op hm i* $\equiv$ *do* {
ASSERT (*hm-valid hm i* $\wedge$ *hmr-invar hm*);
k $\leftarrow$ *hm-key-of-op hm i*;
v $\leftarrow$ *hm-the-lookup-op hm k*;
RETURN (*k*, *v*)
}

definition *kvi-rel hm i* $\equiv$ $\{((k,v),v) \mid k\ v.\ hm\text{-}key\text{-}of\ hm\ i = k\}$

lemma *hm-kv-op-refine*[*refine*]:
assumes (*hm*,*h*)$\in$*hmr-rel*
shows *hm-kv-of-op hm i* $\leq$ $\Downarrow$(*kvi-rel hm i*) (*h.val-of-op h i*)
unfolding *hm-kv-of-op-def h.val-of-op-def kvi-rel-def*
hm-key-of-op-unfold hm-the-lookup-op-def
apply *simp*
apply *refine-vcg*
using *assms*
by (*auto*
simp: *hm-valid-def hm-length-def hmr-rel-defs heapmap-α-def hm-key-of-def*
split: *prod.splits*)

definition *hm-pop-min-op'* :: (*'k*,*'v*) *ahm* $\Rightarrow$ ((*'k*$\times$*'v*) $\times$ (*'k*,*'v*) *ahm*) *nres* **where**
hm-pop-min-op' hm $\equiv$ *do* {
ASSERT (*heapmap-invar hm*);
ASSERT (*hm-valid hm 1*);
kv $\leftarrow$ *hm-kv-of-op hm 1*;
let l = *hm-length hm*;
hm $\leftarrow$ *hm-exch-op hm 1 l*;
hm $\leftarrow$ *hm-butlast-op hm*;

if (*l*≠*1*) *then do* {
  hm ← *hm-sink-op hm 1*;
  RETURN (*kv*,*hm*)
} *else RETURN* (*kv*,*hm*)
}

lemma *hm-pop-min-op-refine′*:
⟦ (*hm*,*h*)∈*hmr-rel* ⟧ ⟹ *hm-pop-min-op′ hm* ≤ ⇓(*kvi-rel hm 1* $\times_r$ *hmr-rel*) (*h.pop-min-op h*)
unfolding *hm-pop-min-op′-def h.pop-min-op-def*

unfolding *ignore-snd-refine-conv*

apply *refine-rcg*
unfolding *hmr-rel-def heapmap-rel-def*
apply (*unfold heapmap-invar-def*; *simp add*: *in-br-conv*)
apply (*simp-all add*: *in-br-conv*)
done

lemma *heapmap-nres-rel-prodI′*:
assumes *hmx* ≤ ⇓(*S* $\times_r$ *hmr-rel*) *h′x*
assumes *h′x* ≤ *SPEC* Φ
assumes ⋀*h′ r*. Φ (*r*,*h′*) ⟹ *h.heap-invar h′*
assumes *hmx* $\leq_n$ *SPEC* (λ(*r*,*hm′*). (∃ *r′*. (*r*,*r′*)∈*S* ∧ Φ (*r′*,*hmr-α hm′*)) ∧ *hmr-invar hm′* ⟶ *RETURN* (*r*,*heapmap-α hm′*) ≤ ⇓(*R*$\times_r$*Id*) *hx*)
shows *hmx* ≤ ⇓(*R*$\times_r$*heapmap-rel*) *hx*
using *assms*
unfolding *heapmap-rel-def hmr-rel-def heapmap-invar-def*
apply (*auto*
  simp: *pw-le-iff pw-leof-iff refine-pw-simps in-br-conv*
)
by *meson*

lemma *ex-in-kvi-rel-conv*:
(∃ *r′*. (*r*,*r′*)∈*kvi-rel hm i* ∧ Φ *r′*) ⟷ (*fst r* = *hm-key-of hm i* ∧ Φ (*snd r*))
unfolding *kvi-rel-def*
apply (*cases r*)
apply *auto*
done

lemma *hm-pop-min-aref′*: (*hm-pop-min-op′*, *mop-pm-pop-min prio*) ∈ *heapmap-rel* → ⟨(*Id*$\times_r$*Id*) $\times_r$ *heapmap-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
apply (*frule heapmap-hmr-relI*)
unfolding *mop-pm-pop-min-alt*

apply (*intro ASSERT-refine-right*)
apply (*rule heapmap-nres-rel-prodI'*)
  apply (*erule hm-pop-min-op-refine'*)

  apply (*unfold heapmap-rel-def hmr-rel-def in-br-conv*) []
  apply (*rule h.pop-min-op-correct*)
  apply *simp*
  apply *simp*

  apply *simp*

  apply (*clarsimp simp*: *ex-in-kvi-rel-conv split*: *prod.splits*)
  unfolding *hm-pop-min-op'-def hm-kv-of-op-def hm-key-of-op-unfold*
    hm-the-lookup-op-def
  apply (*refine-vcg*
    hm-exch-op-α*-correct*[*THEN leof-trans*]
    hm-butlast-op-α*-correct*[*THEN leof-trans*]
    hm-sink-op-α*-correct*[*THEN leof-trans*]
)
  unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
  apply (*auto intro*: *ranI*)
done

Remove

definition *hm-remove-op k hm* $\equiv$ *do* {
  ASSERT (*heapmap-invar hm*);
  ASSERT ($k \in$ *dom* (*heapmap-*α *hm*));
  $i \leftarrow$ *hm-index-op hm k*;
  let l = *hm-length hm*;
  hm $\leftarrow$ *hm-exch-op hm i l*;
  hm $\leftarrow$ *hm-butlast-op hm*;
  if $i \neq l$ *then*
    hm-repair-op hm i
  else
    RETURN hm
}

definition (**in** *heapstruct*) *remove-op i h* $\equiv$ *do* {
  ASSERT (*heap-invar h*);
  ASSERT (*valid h i*);
  let l = *length h*;
  $h \leftarrow$ *exch-op h i l*;
  $h \leftarrow$ *butlast-op h*;
  if $i \neq l$ *then*
    repair-op h i
  else
    RETURN h
}

lemma (**in** −) *swap-empty-iff*[*iff*]: *swap l i j* = [] ⟷ *l*=[]
by (*auto simp*: *swap-def*)

lemma (**in** *heapstruct*)
butlast-exch-last: *butlast* (*exch h i* (*length h*)) = *update* (*butlast h*) *i* (*last h*)
unfolding *exch-def update-def*
apply (*cases h rule*: *rev-cases*)
apply (*auto simp*: *swap-def butlast-list-update*)
done

lemma (**in** *heapstruct*) *remove-op-invar*:
⟦ *heap-invar h*; *valid h i* ⟧ ⟹ *remove-op i h* ≤ *SPEC heap-invar*
unfolding *remove-op-def*
apply *refine-vcg*
apply (*auto simp*: *valid-def*) []
apply (*auto simp*: *valid-def exch-def*) []
apply (*simp add*: *butlast-exch-last*)
apply *refine-vcg*
apply *auto* []
apply *auto* []
apply (*auto simp*: *valid-def*) []
apply *auto* []
apply *auto* []
done

lemma *hm-remove-op-refine*[*refine*]:
⟦ (*hm*,*m*)∈*heapmap-rel*; (*hm*,*h*)∈*hmr-rel*; *heapmap-α hm k* ≠ *None*⟧ ⟹
hm-remove-op k hm ≤ ⇓*hmr-rel* (*h.remove-op* (*hm-index hm k*) *h*)
unfolding *hm-remove-op-def h.remove-op-def heapmap-rel-def*

apply (*refine-rcg index-op-inline-refine*)
unfolding *hmr-rel-def*
apply (*auto simp*: *in-br-conv*)
done

lemma *hm-remove-op-α-correct*:
hm-remove-op k hm $\leq_n$ *SPEC* (λ*hm'*. *heapmap-α hm'* = (*heapmap-α hm*)(*k*:=*None*))

unfolding *hm-remove-op-def*
apply (*refine-vcg*
hm-exch-op-α-correct[*THEN leof-trans*]
hm-butlast-op-α-correct[*THEN leof-trans*]
hm-repair-op-α-correct[*THEN leof-trans*]
hm-index-op-inline-leof
)
apply (*auto*; *fail*)

apply *clarsimp*

apply (*rewrite at hm-index - k = hm-length -* **in** *asm eq-commute*)
apply (*auto*; *fail*)
done

lemma *hm-remove-op-aref*:
(*hm-remove-op*,*mop-map-delete-ex*) ∈ *Id* → *heapmap-rel* → ⟨*heapmap-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *mop-map-delete-ex-alt*
apply (*rule ASSERT-refine-right*)
apply (*frule heapmap-hmr-relI*)
apply (*rule heapmap-nres-relI'*)
apply (*rule hm-remove-op-refine*; *assumption?*)
apply (*unfold heapmap-rel-def in-br-conv*; *auto*)

unfolding *heapmap-rel-def hmr-rel-def in-br-conv*
apply (*refine-vcg h.remove-op-invar*; *clarsimp*; *fail*)
apply (*refine-vcg hm-remove-op-α-correct*[*THEN leof-trans*]; *simp*; *fail*)
done

Peek-Min

definition *hm-peek-min-op* :: (*'k*,*'v*) *ahm* ⇒ (*'k*×*'v*) *nres* **where**
hm-peek-min-op hm ≡ *hm-kv-of-op hm 1*

lemma *hm-peek-min-op-aref*:
(*hm-peek-min-op*, *PR-CONST* (*mop-pm-peek-min prio*)) ∈ *heapmap-rel* → ⟨*Id*$\times_r$*Id*⟩*nres-rel*
unfolding *PR-CONST-def*
apply (*intro fun-relI nres-relI*)
proof −
fix *hm* **and** *m* :: *'k* ⇀ *'v*
assume *A*: (*hm*,*m*)∈*heapmap-rel*

from *A* **have** [*simp*]: *h.heap-invar* (*hmr-α hm*) *hmr-invar hm m*=*heapmap-α hm*
unfolding *heapmap-rel-def in-br-conv heapmap-invar-def*
by *simp-all*

have *hm-peek-min-op hm* ≤ ⇓ (*kvi-rel hm 1*) (*h.peek-min-op* (*hmr-α hm*))
unfolding *hm-peek-min-op-def h.peek-min-op-def*
apply (*refine-rcg hm-kv-op-refine*)
using *A*
apply (*simp add*: *heapmap-hmr-relI*)
done
also have ⟦*hmr-α hm* ≠ []⟧ ⟹ (*h.peek-min-op* (*hmr-α hm*))
≤ *SPEC* (λ*v*. *v*∈*ran* (*heapmap-α hm*) ∧ (∀ *v'*∈*ran* (*heapmap-α hm*). *prio v* ≤ *prio v'*))
apply *refine-vcg*
by *simp-all*

finally show *hm-peek-min-op hm* $\leq \Downarrow$ (*Id* $\times_r$ *Id*) (*mop-pm-peek-min prio m*)

unfolding *mop-pm-peek-min-alt*
apply (*simp add*: *pw-le-iff refine-pw-simps hm-peek-min-op-def hm-kv-of-op-def*

hm-key-of-op-unfold hm-the-lookup-op-def)
apply (*fastforce simp*: *kvi-rel-def ran-def*)
done

qed

end

end

3.11 Plain Arrays Implementing List Interface

theory *IICF-Array*
imports ../*Intf*/*IICF-List*
begin

Lists of fixed length are directly implemented with arrays.

definition *is-array l p* $\equiv$ $p \mapsto_a l$

lemma *is-array-precise*[*safe-constraint-rules*]: *precise is-array*
apply *rule*
unfolding *is-array-def*
apply *prec-extract-eqs*
by *simp*

definition *array-assn* **where** *array-assn A* $\equiv$ *hr-comp is-array* ($\langle$*the-pure A*$\rangle$*list-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure array-assn A* **for** *A*]

definition [*simp*,*code-unfold*]: *heap-array-empty* $\equiv$ *Array*.*of-list* []
definition [*simp*,*code-unfold*]: *heap-array-set p i v* $\equiv$ *Array*.*upd i v p*

context
notes [*fcomp-norm-unfold*] = *array-assn-def*[*symmetric*]
notes [*intro*!] = *hfrefI hn-refineI*[*THEN hn-refine-preI*]
notes [*simp*] = *pure-def hn-ctxt-def is-array-def invalid-assn-def*
begin

lemma *array-empty-hnr-aux*: (*uncurry0 heap-array-empty*,*uncurry0* (*RETURN op-list-empty*)) $\in$ *unit-assn*^{k} $\rightarrow_a$ *is-array*
by *sep-auto*
sepref-decl-impl (*no-register*) *array-empty*: *array-empty-hnr-aux* **.**

lemma *array-replicate-hnr-aux*:
(*uncurry Array.new*, *uncurry* (*RETURN oo op-list-replicate*))
$\in$ *nat-assn*^{k} $*_a$ *id-assn*^{k} $\rightarrow_a$ *is-array*
by (*sep-auto*)
sepref-decl-impl (*no-register*) *array-replicate*: *array-replicate-hnr-aux* **.**

definition [*simp*]: *op-array-replicate* $\equiv$ *op-list-replicate*
sepref-register *op-array-replicate*
lemma *array-fold-custom-replicate*:
replicate = *op-array-replicate*
op-list-replicate = *op-array-replicate*
mop-list-replicate = *RETURN oo op-array-replicate*
by (*auto simp*: *op-array-replicate-def intro*!: *ext*)
lemmas *array-replicate-custom-hnr*[*sepref-fr-rules*] = *array-replicate-hnr*[*unfolded array-fold-custom-replicate*]

lemma *array-of-list-hnr-aux*: (*Array.of-list*,*RETURN o op-list-copy*) $\in$ (*list-assn id-assn*)^k $\rightarrow_a$ *is-array*
unfolding *list-assn-pure-conv*
by (*sep-auto*)
sepref-decl-impl (*no-register*) *array-of-list*: *array-of-list-hnr-aux* **.**

definition [*simp*]: *op-array-of-list* $\equiv$ *op-list-copy*
sepref-register *op-array-of-list*
lemma *array-fold-custom-of-list*:
l = *op-array-of-list l*
op-list-copy = *op-array-of-list*
mop-list-copy = *RETURN o op-array-of-list*
by (*auto intro*!: *ext*)
lemmas *array-of-list-custom-hnr*[*sepref-fr-rules*] = *array-of-list-hnr*[*folded op-array-of-list-def*]

lemma *array-copy-hnr-aux*: (*array-copy*, *RETURN o op-list-copy*) $\in$ *is-array*^{k} $\rightarrow_a$ *is-array*
by *sep-auto*
sepref-decl-impl *array-copy*: *array-copy-hnr-aux* **.**

lemma *array-get-hnr-aux*: (*uncurry Array.nth*,*uncurry* (*RETURN oo op-list-get*)) $\in$ [λ(*l*,*i*). *i*<*length l*]$_a$ *is-array*^{k} $*_a$ *nat-assn*^{k} $\rightarrow$ *id-assn*
by *sep-auto*
sepref-decl-impl *array-get*: *array-get-hnr-aux* **.**

lemma *array-set-hnr-aux*: (*uncurry2 heap-array-set*,*uncurry2* (*RETURN ooo op-list-set*)) $\in$ [λ((*l*,*i*),-). *i*<*length l*]$_a$ *is-array*^{d} $*_a$ *nat-assn*^{k} $*_a$ *id-assn*^{k} $\rightarrow$ *is-array*

by *sep-auto*
sepref-decl-impl *array-set*: *array-set-hnr-aux* **.**

lemma *array-length-hnr-aux*: (*Array.len*,*RETURN o op-list-length*) $\in$ *is-array*^{k} $\rightarrow_a$ *nat-assn*
by *sep-auto*
sepref-decl-impl *array-length*: *array-length-hnr-aux* **.**

end

definition [*simp*]: *op-array-empty* $\equiv$ *op-list-empty*
interpretation *array*: *list-custom-empty array-assn A heap-array-empty op-array-empty*
apply *unfold-locales*
apply (*rule array-empty-hnr*[*simplified pre-list-empty-def*])
by (*auto*)

end
theory *IICF-MS-Array-List*
imports
../Intf/IICF-List
Separation-Logic-Imperative-HOL.Array-Blit
Separation-Logic-Imperative-HOL.Default-Insts
begin

type-synonym *'a ms-array-list* = *'a Heap.array* $\times$ *nat*

definition *is-ms-array-list ms l* $\equiv$ $\lambda(a,n).\ \exists_A l'.\ a \mapsto_a l' * \uparrow(n \leq \mathit{length}\ l' \wedge l = \mathit{take}\ n\ l' \wedge ms{=}\mathit{length}\ l')$

lemma *is-ms-array-list-prec*[*safe-constraint-rules*]: *precise* (*is-ms-array-list ms*)
unfolding *is-ms-array-list-def*[*abs-def*]
apply(*rule preciseI*)
apply(*simp split*: *prod.splits*)
using *preciseD snga-prec* **by** *fastforce*

definition *marl-empty-sz maxsize* $\equiv$ *do* {
a $\leftarrow$ *Array.new maxsize default*;
return (*a*,*0*)
}

definition *marl-append* $\equiv$ $\lambda(a,n)$ *x*. *do* {
a $\leftarrow$ *Array.upd n x a*;
return (*a*,*n*+*1*)
}

definition *marl-length* :: *'a*::*heap ms-array-list* $\Rightarrow$ *nat Heap* **where**
marl-length $\equiv$ $\lambda(a,n)$. *return* (*n*)

definition *marl-is-empty* :: *'a*::*heap ms-array-list* $\Rightarrow$ *bool Heap* **where**
marl-is-empty $\equiv$ $\lambda(a,n)$. *return* (*n*=*0*)

definition *marl-last* :: *'a::heap ms-array-list* ⇒ *'a Heap* **where**
marl-last ≡ λ(*a*,*n*). *do* {
Array.nth a (*n* − *1*)
}

definition *marl-butlast* :: *'a::heap ms-array-list* ⇒ *'a ms-array-list Heap* **where**
marl-butlast ≡ λ(*a*,*n*). *do* {
return (*a*,*n* − *1*)
}

definition *marl-get* :: *'a::heap ms-array-list* ⇒ *nat* ⇒ *'a Heap* **where**
marl-get ≡ λ(*a*,*n*) *i*. *Array.nth a i*

definition *marl-set* :: *'a::heap ms-array-list* ⇒ *nat* ⇒ *'a* ⇒ *'a ms-array-list Heap*
where
marl-set ≡ λ(*a*,*n*) *i x*. *do* { *a* ← *Array.upd i x a*; *return* (*a*,*n*)}

lemma *marl-empty-sz-rule*[*sep-heap-rules*]: < *emp* > *marl-empty-sz N* <*is-ms-array-list*
N []>
by (*sep-auto simp*: *marl-empty-sz-def is-ms-array-list-def*)

lemma *marl-append-rule*[*sep-heap-rules*]: *length l* < *N* ⟹
< *is-ms-array-list N l a* >
marl-append a x
<λ*a*. *is-ms-array-list N* (*l*@[*x*]) *a* >$_t$
by (*sep-auto*
simp: *marl-append-def is-ms-array-list-def take-update-last*
split: *prod.splits*)

lemma *marl-length-rule*[*sep-heap-rules*]:
<*is-ms-array-list N l a*>
marl-length a
<λ*r*. *is-ms-array-list N l a* ∗ ↑(*r*=*length l*)>
by (*sep-auto simp*: *marl-length-def is-ms-array-list-def*)

lemma *marl-is-empty-rule*[*sep-heap-rules*]:
<*is-ms-array-list N l a*>
marl-is-empty a
<λ*r*. *is-ms-array-list N l a* ∗ ↑(*r*⟷(*l*=[]))>
by (*sep-auto simp*: *marl-is-empty-def is-ms-array-list-def*)

lemma *marl-last-rule*[*sep-heap-rules*]:
l≠[] ⟹
<*is-ms-array-list N l a*>
marl-last a
<λ*r*. *is-ms-array-list N l a* ∗ ↑(*r*=*last l*)>
by (*sep-auto simp*: *marl-last-def is-ms-array-list-def last-take-nth-conv*)

lemma *marl-butlast-rule*[*sep-heap-rules*]:
l≠[] ⟹
<*is-ms-array-list N l a*>
marl-butlast a
<*is-ms-array-list N* (*butlast l*)>$_t$
by (*sep-auto*
split: *prod.splits*
simp: *marl-butlast-def is-ms-array-list-def butlast-take*)

lemma *marl-get-rule*[*sep-heap-rules*]:
i<*length l* ⟹
<*is-ms-array-list N l a*>
marl-get a i
<λ*r*. *is-ms-array-list N l a* ∗ ↑(*r*=*l*!*i*)>
by (*sep-auto simp*: *marl-get-def is-ms-array-list-def split*: *prod.split*)

lemma *marl-set-rule*[*sep-heap-rules*]:
i<*length l* ⟹
<*is-ms-array-list N l a*>
marl-set a i x
<*is-ms-array-list N* (*l*[*i*:=*x*])>
by (*sep-auto simp*: *marl-set-def is-ms-array-list-def split*: *prod.split*)

definition *marl-assn N A* ≡ *hr-comp* (*is-ms-array-list N*) (⟨*the-pure A*⟩*list-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure marl-assn N A* **for** *N A*]

context
notes [*fcomp-norm-unfold*] = *marl-assn-def*[*symmetric*]
notes [*intro*!] = *hfrefI hn-refineI*[*THEN hn-refine-preI*]
notes [*simp*] = *pure-def hn-ctxt-def invalid-assn-def*
begin

definition [*simp*]: *op-marl-empty-sz* (*N*::*nat*) ≡ *op-list-empty*
context fixes *N* :: *nat* **begin**
sepref-register *PR-CONST* (*op-marl-empty-sz N*)
end

lemma [*def-pat-rules*]: *op-marl-empty-sz*$*N* ≡ *UNPROTECT* (*op-marl-empty-sz N*) **by** *simp*

lemma *marl-fold-custom-empty-sz*:
op-list-empty = *op-marl-empty-sz N*
mop-list-empty = *RETURN* (*op-marl-empty-sz N*)
[] = *op-marl-empty-sz N*
by *auto*

lemma *marl-empty-hnr-aux*: (*uncurry0* (*marl-empty-sz N*), *uncurry0* (*RETURN op-list-empty*)) ∈ *unit-assn*^{k} $\rightarrow_a$ *is-ms-array-list N*

by *sep-auto*
lemmas *marl-empty-hnr* = *marl-empty-hnr-aux*[*FCOMP op-list-empty.fref*[*of the-pure A* **for** *A*]]
lemmas *marl-empty-hnr-mop* = *marl-empty-hnr*[*FCOMP mk-mop-rl0-np*[*OF mop-list-empty-alt*]]

lemma *marl-empty-sz-hnr*[*sepref-fr-rules*]:
(*uncurry0* (*marl-empty-sz N*), *uncurry0* (*RETURN* (*PR-CONST* (*op-marl-empty-sz N*)))) $\in$ *unit-assn*^{k} $\rightarrow_a$ *marl-assn N A*
using *marl-empty-hnr*
by *simp*

lemma *marl-append-hnr-aux*: (*uncurry marl-append*,*uncurry* (*RETURN oo op-list-append*)) $\in$ [λ(*l*,-). *length l*<*N*]$_a$ ((*is-ms-array-list N*)^d $*_a$ *id-assn*^{k}) $\rightarrow$ *is-ms-array-list N*
by *sep-auto*
lemmas *marl-append-hnr*[*sepref-fr-rules*] = *marl-append-hnr-aux*[*FCOMP op-list-append.fref*]
lemmas *marl-append-hnr-mop*[*sepref-fr-rules*] = *marl-append-hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-append-alt*]]

lemma *marl-length-hnr-aux*: (*marl-length*,*RETURN o op-list-length*) $\in$ (*is-ms-array-list N*)^k $\rightarrow_a$ *nat-assn*
by *sep-auto*
lemmas *marl-length-hnr*[*sepref-fr-rules*] = *marl-length-hnr-aux*[*FCOMP op-list-length.fref*[*of the-pure A* **for** *A*]]
lemmas *marl-length-hnr-mop*[*sepref-fr-rules*] = *marl-length-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-length-alt*]]

lemma *marl-is-empty-hnr-aux*: (*marl-is-empty*,*RETURN o op-list-is-empty*) $\in$ (*is-ms-array-list N*)^k $\rightarrow_a$ *bool-assn*
by *sep-auto*
lemmas *marl-is-empty-hnr*[*sepref-fr-rules*] = *marl-is-empty-hnr-aux*[*FCOMP op-list-is-empty.fref*[*of the-pure A* **for** *A*]]
lemmas *marl-is-empty-hnr-mop*[*sepref-fr-rules*] = *marl-is-empty-hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-is-empty-alt*]]

lemma *marl-last-hnr-aux*: (*marl-last*,*RETURN o op-list-last*) $\in$ [λx. $x \neq []$]$_a$ (*is-ms-array-list N*)^k $\rightarrow$ *id-assn*
by *sep-auto*
lemmas *marl-last-hnr*[*sepref-fr-rules*] = *marl-last-hnr-aux*[*FCOMP op-list-last.fref*]
lemmas *marl-last-hnr-mop*[*sepref-fr-rules*] = *marl-last-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-last-alt*]]

lemma *marl-butlast-hnr-aux*: (*marl-butlast*,*RETURN o op-list-butlast*) $\in$ [λx. $x \neq []$]$_a$ (*is-ms-array-list N*)^d $\rightarrow$ (*is-ms-array-list N*)
by *sep-auto*
lemmas *marl-butlast-hnr*[*sepref-fr-rules*] = *marl-butlast-hnr-aux*[*FCOMP op-list-butlast.fref*[*of the-pure A* **for** *A*]]
lemmas *marl-butlast-hnr-mop*[*sepref-fr-rules*] = *marl-butlast-hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-butlast-alt*]]

lemma *marl-get-hnr-aux*: (*uncurry marl-get*,*uncurry* (*RETURN oo op-list-get*))
$\in$ [λ(*l*,*i*). *i*<*length l*]$_a$ ((*is-ms-array-list N*)^k $*_a$ *nat-assn*^{k}) $\rightarrow$ *id-assn*
by *sep-auto*
lemmas *marl-get-hnr*[*sepref-fr-rules*] = *marl-get-hnr-aux*[*FCOMP op-list-get.fref*]
lemmas *marl-get-hnr-mop*[*sepref-fr-rules*] = *marl-get-hnr*[*FCOMP mk-mop-rl2*[*OF mop-list-get-alt*]]

lemma *marl-set-hnr-aux*: (*uncurry2 marl-set*,*uncurry2* (*RETURN ooo op-list-set*))
$\in$ [λ((*l*,*i*),-). *i*<*length l*]$_a$ ((*is-ms-array-list N*)^d $*_a$ *nat-assn*^{k} $*_a$ *id-assn*^{k}) $\rightarrow$ (*is-ms-array-list N*)
by *sep-auto*
lemmas *marl-set-hnr*[*sepref-fr-rules*] = *marl-set-hnr-aux*[*FCOMP op-list-set.fref*]
lemmas *marl-set-hnr-mop*[*sepref-fr-rules*] = *marl-set-hnr*[*FCOMP mk-mop-rl3*[*OF mop-list-set-alt*]]

end

context
fixes *N* :: *nat*
assumes *N-sz*: *N*>*10*
begin

schematic-goal *hn-refine* (*emp*) (*?c*::*?'c Heap*) *?Γ' ?R* (*do* {
let x = *op-marl-empty-sz N*;
RETURN (*x*@[*1*::*nat*])
})
using *N-sz*
by *sepref*

end

schematic-goal *hn-refine* (*emp*) (*?c*::*?'c Heap*) *?Γ' ?R* (*do* {
let x = *op-list-empty*;
RETURN (*x*@[*1*::*nat*])
})
apply (*subst marl-fold-custom-empty-sz*[**where** *N*=*10*])
apply *sepref*
done

end
theory *IICF-Indexed-Array-List*
imports
HOL−Library.Rewrite
../Intf/IICF-List
List−Index.List-Index
IICF-Array
IICF-MS-Array-List
begin

We implement distinct lists of natural numbers in the range {*0*..<*N*} by a length counter and two arrays of size N. The first array stores the list, and the second array stores the positions of the elements in the list, or N if the element is not in the list.

This allows for an efficient index query.

The implementation is done in two steps: First, we use a list and a fixed size list for the index mapping. Second, we refine the lists to arrays.

```
type-synonym aial = nat list × nat list

locale ial-invar = fixes
      maxsize :: nat
  and  l :: nat list
  and qp :: nat list
  assumes maxsize-eq[simp]: maxsize = length qp
  assumes l-distinct[simp]: distinct l
  assumes l-set: set l ⊆ {0..<length qp}
  assumes qp-def: ∀ k<length qp. qp!k = (if k∈set l then List-Index.index l k else
length qp)
begin
  lemma l-len: length l ≤ length qp
  proof −
    from card-mono[OF - l-set] have card (set l) ≤ length qp by auto
    with distinct-card[OF l-distinct] show ?thesis by simp
  qed

  lemma idx-len[simp]: i<length l ⟹ l!i < length qp
    using l-set
    by (metis atLeastLessThan-iff nth-mem psubsetD psubsetI)

  lemma l-set-simp[simp]: k∈set l ⟹ k < length qp
    by (auto dest: subsetD[OF l-set])

  lemma qpk-idx: k<length qp ⟹ qp ! k < length l ⟷ k ∈ set l
  proof (rule iffI)
    assume A: k<length qp
    {
      assume qp!k < length l
      hence qp!k < length qp using l-len by simp
      with spec[OF qp-def, of k] A show k∈set l
        by (auto split: if-split-asm)
    }
    {
      assume k∈set l
      thus qp!k<length l
        using qp-def by (auto split: if-split-asm) []
    }
  qed

  lemma lqpk[simp]: k ∈ set l ⟹ l ! (qp ! k) = k
```

using *spec*[*OF qp-def*, *of k*] **by** *auto*

lemma ⟦*i*<*length l*; *j*<*length l*; *l*!*i*=*l*!*j*⟧ ⟹ *i*=*j*
by (*simp add*: *nth-eq-iff-index-eq*)

lemmas *index-swap*[*simp*] = *index-swap-if-distinct*[*folded swap-def*, *OF l-distinct*]

lemma *swap-invar*:
assumes *i*<*length l j*<*length l*
shows *ial-invar* (*length qp*) (*swap l i j*) (*qp*[*l* ! *j* := *i*, *l* ! *i* := *j*])
using *assms*
apply *unfold-locales*
apply *auto* []
apply *auto* []
apply *auto* []
apply (*auto simp*: *simp*: *nth-list-update nth-eq-iff-index-eq index-nth-id*) []
using *qp-def* **apply** *auto* [*2*]
done

end

definition *ial-rel1 maxsize* ≡ *br fst* (*uncurry* (*ial-invar maxsize*))

definition *ial-assn2* :: *nat* ⇒ *nat list* ∗ *nat list* ⇒ - **where**
ial-assn2 maxsize ≡ *prod-assn* (*marl-assn maxsize nat-assn*) (*array-assn nat-assn*)

definition *ial-assn maxsize A* ≡ *hr-comp* (*hr-comp* (*ial-assn2 maxsize*) (*ial-rel1 maxsize*)) (⟨*the-pure A*⟩*list-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure ial-assn maxsize A* **for** *maxsize A*]

3.11.1 Empty

definition *op-ial-empty-sz* :: *nat* ⇒ ′*a list*
where [*simp*]: *op-ial-empty-sz ms* ≡ *op-list-empty*

lemma [*def-pat-rules*]: *op-ial-empty-sz*$*maxsize* ≡ *UNPROTECT* (*op-ial-empty-sz maxsize*)
by *simp*

context fixes *maxsize* :: *nat* **begin**
sepref-register *PR-CONST* (*op-ial-empty-sz maxsize*)
end

context
fixes *maxsize* :: *nat*

notes [*fcomp-norm-unfold*] = *ial-assn-def* [*symmetric*]
notes [*simp*] = *hn-ctxt-def pure-def*
begin

definition *aial-empty* ≡ *do* {
let l = *op-marl-empty-sz maxsize*;
let qp = *op-array-replicate maxsize maxsize*;
RETURN (*l*,*qp*)
}

lemma *aial-empty-impl*: (*aial-empty*,*RETURN op-list-empty*) ∈ ⟨*ial-rel1 maxsize*⟩*nres-rel*
unfolding *aial-empty-def*
apply (*refine-vcg nres-relI*)
apply (*clarsimp simp*: *ial-rel1-def br-def*)
apply *unfold-locales*
apply *auto*
done

context
notes [*id-rules*] = *itypeI*[*Pure.of maxsize TYPE*(*nat*)]
notes [*sepref-import-param*] = *IdI*[*of maxsize*]
begin
sepref-definition *ial-empty* **is** *uncurry0 aial-empty* :: $unit\text{-}assn^k \rightarrow_a$ *ial-assn2 maxsize*
unfolding *aial-empty-def ial-assn2-def*
using [[*id-debug*]]
by *sepref*
end

sepref-decl-impl (*no-register*) *ial-empty*: *ial-empty.refine*[*FCOMP aial-empty-impl*]
.
lemma *ial-empty-sz-hnr*[*sepref-fr-rules*]:
(*uncurry0 local.ial-empty*, *uncurry0* (*RETURN* (*PR-CONST* (*op-ial-empty-sz maxsize*)))) ∈ $unit\text{-}assn^k \rightarrow_a$ *ial-assn maxsize A*
using *ial-empty-hnr*[*of A*] **by** *simp*

3.11.2 Swap

definition *aial-swap* ≡ λ(*l*,*qp*) *i j*. *do* {
vi ← *mop-list-get l i*;
vj ← *mop-list-get l j*;
l ← *mop-list-set l i vj*;
l ← *mop-list-set l j vi*;
qp ← *mop-list-set qp vj i*;
qp ← *mop-list-set qp vi j*;
RETURN (*l*,*qp*)
}

lemma *in-ial-rel1-conv*:
$((pq,\ qp),\ l) \in$ *ial-rel1 ms* $\longleftrightarrow$ *pq=l* $\wedge$ *ial-invar ms l qp*
by (*auto simp*: *ial-rel1-def in-br-conv*)

lemma *aial-swap-impl*:
(*aial-swap*,*mop-list-swap*) $\in$ *ial-rel1 maxsize* $\rightarrow$ *nat-rel* $\rightarrow$ *nat-rel* $\rightarrow$ $\langle$*ial-rel1 maxsize*$\rangle$*nres-rel*
proof (*intro fun-relI nres-relI*; *clarsimp simp*: *in-ial-rel1-conv*; *refine-vcg*; *clarsimp*)
fix *l qp i j*
assume [*simp*]: *i<length l j<length l* **and** *ial-invar maxsize l qp*
then interpret *ial-invar maxsize l qp* **by** *simp*

show *aial-swap* (*l*, *qp*) *i j* $\leq$ *SPEC* ($\lambda c.$ (*c*, *swap l i j*) $\in$ *ial-rel1 maxsize*)
unfolding *aial-swap-def*
apply *refine-vcg*
apply (*vc-solve simp add*: *in-ial-rel1-conv swap-def*[*symmetric*] *swap-invar*)
done
qed

sepref-definition *ial-swap* **is**
uncurry2 aial-swap :: (*ial-assn2 maxsize*)^d $*_a$ *nat-assn*^{k} $*_a$ *nat-assn*^{k} $\rightarrow_a$ *ial-assn2 maxsize*
unfolding *aial-swap-def ial-assn2-def*
by *sepref*

sepref-decl-impl (*ismop*) *test*: *ial-swap.refine*[*FCOMP aial-swap-impl*]
uses *mop-list-swap.fref* **.**

3.11.3 Length

definition *aial-length* :: *aial* $\Rightarrow$ *nat nres*
where *aial-length* $\equiv$ λ(*l*,-). *RETURN* (*op-list-length l*)

lemma *aial-length-impl*: (*aial-length*, *mop-list-length*) $\in$ *ial-rel1 maxsize* $\rightarrow$ $\langle$*nat-rel*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *ial-rel1-def in-br-conv aial-length-def*
by *auto*

sepref-definition *ial-length* **is** *aial-length* :: (*ial-assn2 maxsize*)^k $\rightarrow_a$ *nat-assn*
unfolding *aial-length-def ial-assn2-def*
by *sepref*

sepref-decl-impl (*ismop*) *ial-length*: *ial-length.refine*[*FCOMP aial-length-impl*]
.

3.11.4 Index

definition *aial-index* :: *aial* ⇒ *nat* ⇒ *nat nres* **where**
aial-index ≡ λ(*l*,*qp*) *k*. *do* {
ASSERT (*k*∈*set l*);
i ← *mop-list-get qp k*;
RETURN i
}

lemma *aial-index-impl*:
(*uncurry aial-index*, *uncurry mop-list-index*) ∈
[λ(*l*,*k*). *k*∈*set l*]$_f$ *ial-rel1 maxsize* $\times_r$ *nat-rel* → ⟨*nat-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI frefI*)
unfolding *ial-rel1-def*
proof (*clarsimp simp*: *in-br-conv*)
fix *l qp k*
assume *ial-invar maxsize l qp*
then interpret *ial-invar maxsize l qp* **.**

assume *k*∈*set l*
then show *aial-index* (*l*,*qp*) *k* ≤ *RETURN* (*index l k*)
unfolding *aial-index-def*
apply (*refine-vcg*)
by (*auto simp*: *qp-def*)
qed

sepref-definition *ial-index* **is** *uncurry aial-index* :: (*ial-assn2 maxsize*)^k $*_a$ *nat-assn*^{k} $\rightarrow_a$ *nat-assn*
unfolding *aial-index-def ial-assn2-def*
by *sepref*

sepref-decl-impl (*ismop*) *ial-index*: *ial-index.refine*[*FCOMP aial-index-impl*] **.**

3.11.5 Butlast

definition *aial-butlast* :: *aial* ⇒ *aial nres* **where**
aial-butlast ≡ λ(*l*,*qp*). *do* {
ASSERT (*l*≠[]);
len ← *mop-list-length l*;
k ← *mop-list-get l* (*len* − *1*);
l ← *mop-list-butlast l*;
qp ← *mop-list-set qp k* (*length qp*);
RETURN (*l*,*qp*)
}

lemma *aial-butlast-refine*: (*aial-butlast*, *mop-list-butlast*) ∈ *ial-rel1 maxsize* → ⟨*ial-rel1 maxsize*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *ial-rel1-def*
proof (*clarsimp simp*: *in-br-conv simp del*: *mop-list-butlast-alt*)

fix *l qp*
assume *ial-invar maxsize l qp*
then interpret *ial-invar maxsize l qp* **.**

{
assume *A*: $l \neq []$
have *ial-invar* (*length qp*) (*butlast l*) (*qp*[*l* ! (*length l* − *Suc 0*) := *length qp*])
apply *standard*
apply *clarsimp-all*
apply (*auto simp*: *distinct-butlast*) []
using *l-set* **apply** (*auto dest*: *in-set-butlastD*) []
using *qp-def A l-distinct*
apply (*auto simp*: *nth-list-update neq-Nil-rev-conv index-append simp del*: *l-distinct*)
done
} **note** *aux1=this*

show *aial-butlast* (*l*, *qp*) $\leq \Downarrow$ (*br fst* (*uncurry* (*ial-invar maxsize*))) (*mop-list-butlast l*)
unfolding *aial-butlast-def mop-list-butlast-alt*
apply *refine-vcg*
apply (*clarsimp-all simp*: *in-br-conv aux1*)
done
qed

sepref-definition *ial-butlast* **is** *aial-butlast* :: $(ial\text{-}assn2\ maxsize)^d \rightarrow_a ial\text{-}assn2\ maxsize$
unfolding *aial-butlast-def ial-assn2-def* **by** *sepref*

sepref-decl-impl (*ismop*) *ial-butlast*: *ial-butlast.refine*[*FCOMP aial-butlast-refine*] **.**

3.11.6 Append

definition *aial-append* :: *aial* ⇒ *nat* ⇒ *aial nres* **where**
aial-append ≡ λ(*l*,*qp*) *k*. *do* {
ASSERT (*k*<*length qp* ∧ *k*∉*set l* ∧ *length l* < *length qp*);
len ← *mop-list-length l*;
l ← *mop-list-append l k*;
qp ← *mop-list-set qp k len*;
RETURN (*l*,*qp*)
}

lemma *aial-append-refine*:
(*uncurry aial-append*,*uncurry mop-list-append*) ∈
$[\lambda(l,k).\ k<maxsize \wedge k \notin set\ l]_f\ ial\text{-}rel1\ maxsize \times_r nat\text{-}rel \rightarrow \langle ial\text{-}rel1\ maxsize\rangle nres\text{-}rel$
apply (*intro frefI nres-relI*)
unfolding *ial-rel1-def*

proof (*clarsimp simp*: *in-br-conv*)
  fix *l qp k*
  assume *KLM*: *k<maxsize* **and** *KNL*: *k∉set l*
  assume *ial-invar maxsize l qp*
  then interpret *ial-invar maxsize l qp* **.**

  from *KLM* **have** *KLL*: *k<length qp* **by** *simp*

  note *distinct-card*[*OF l-distinct*, *symmetric*]
  also from *KNL l-set* **have** *set l* ⊆ {*0..<k*} ∪ {*Suc k..<length qp*}
    by (*auto simp*: *nat-less-le*)
  from *card-mono*[*OF - this*] **have** *card* (*set l*) ≤ *card* ...
    by *simp*
  also note *card-Un-le*
  also have *card* {*0..<k*} + *card* {*Suc k..<length qp*} = *k* + (*length qp* − *Suc k*)
    by *simp*
  also have ... < *length qp* **using** *KLL* **by** *simp*
  finally have *LLEN*: *length l* < *length qp* **.**

  have *aux1*[*simp*]: *ial-invar* (*length qp*) (*l* @ [*k*]) (*qp*[*k* := *length l*])
    apply *standard*
    apply (*clarsimp-all simp*: *KNL KLL*)
    using *KLL* **apply** (*auto simp*: *Suc-le-eq LLEN*) []
    apply (*auto simp*: *index-append KNL nth-list-update′*)
    apply (*simp add*: *qp-def*)
    apply (*simp add*: *qp-def*)
    done

  show *aial-append* (*l*, *qp*) *k* ≤ ⇓ (*br fst* (*uncurry* (*ial-invar maxsize*))) (*RETURN* (*l*@[*k*]))
    unfolding *aial-append-def mop-list-append-def*
    apply *refine-vcg*
    apply (*clarsimp-all simp*: *in-br-conv KLL KNL LLEN*)
    done
qed

private lemma *aial-append-impl-aux*: ((*l*, *qp*), *l′*) ∈ *ial-rel1 maxsize* ⟹ *l′*=*l* ∧ *maxsize* = *length qp*
  unfolding *ial-rel1-def*
  by (*clarsimp simp*: *in-br-conv ial-invar.maxsize-eq*[*symmetric*])

context
  notes [*dest*!] = *aial-append-impl-aux*
begin

  sepref-definition *ial-append* **is**
    uncurry aial-append :: [λ(*lqp*,-). *lqp*∈*Domain* (*ial-rel1 maxsize*)]$_a$ (*ial-assn2 maxsize*)^d $*_a$ *nat-assn*^{k} → *ial-assn2 maxsize*

unfolding *aial-append-def ial-assn2-def*
by *sepref*
end

lemma $(\lambda b.\ b<maxsize,\ X) \in A \to bool\text{-}rel$
apply *auto*
oops

context begin

private lemma *append-fref'*: $[\![IS\text{-}BELOW\text{-}ID\ R]\!]$
$\Longrightarrow (uncurry\ mop\text{-}list\text{-}append,\ uncurry\ mop\text{-}list\text{-}append) \in \langle R\rangle list\text{-}rel \times_r R$
$\to_f \langle\langle R\rangle list\text{-}rel\rangle nres\text{-}rel$
by (*rule mop-list-append.fref*)

sepref-decl-impl (*ismop*) *ial-append*: *ial-append.refine*[*FCOMP aial-append-refine*]
uses *append-fref'*
unfolding *IS-BELOW-ID-def*
apply (*parametricity*; *auto simp*: *single-valued-below-Id*)
done
end

3.11.7 Get

definition *aial-get* :: $aial \Rightarrow nat \Rightarrow nat\ nres$ **where**
$aial\text{-}get \equiv \lambda(l,qp)\ i.\ mop\text{-}list\text{-}get\ l\ i$

lemma *aial-get-refine*: $(aial\text{-}get,mop\text{-}list\text{-}get) \in ial\text{-}rel1\ maxsize \to nat\text{-}rel \to$
$\langle nat\text{-}rel\rangle nres\text{-}rel$
apply (*intro fun-relI nres-relI*)
unfolding *aial-get-def ial-rel1-def mop-list-get-def in-br-conv*
apply *refine-vcg*
apply *clarsimp-all*
done

sepref-definition *ial-get* **is** *uncurry aial-get* :: $(ial\text{-}assn2\ maxsize)^k *_a nat\text{-}assn^k$
$\to_a nat\text{-}assn$
unfolding *aial-get-def ial-assn2-def* **by** *sepref*

sepref-decl-impl (*ismop*) *ial-get*: *ial-get.refine*[*FCOMP aial-get-refine*] **.**

3.11.8 Contains

definition *aial-contains* :: $nat \Rightarrow aial \Rightarrow bool\ nres$ **where**
aial-contains $\equiv \lambda k\ (l,qp).\ do\ \{$
if $k<maxsize$ *then do* {
$i \leftarrow mop\text{-}list\text{-}get\ qp\ k;$
RETURN $(i<maxsize)$
} *else RETURN False*

}

lemma *aial-contains-refine*: (*uncurry aial-contains*,*uncurry mop-list-contains*)
$\in$ (*nat-rel* $\times_r$ *ial-rel1 maxsize*) $\to_f$ $\langle$*bool-rel*$\rangle$*nres-rel*
apply (*intro frefI nres-relI*)
unfolding *ial-rel1-def*
proof (*clarsimp simp*: *in-br-conv*)
fix *l qp k*

assume *ial-invar maxsize l qp*
then interpret *ial-invar maxsize l qp* **.**

show *aial-contains k* (*l*, *qp*) $\leq$ *RETURN* (*k*$\in$*set l*)
unfolding *aial-contains-def*
apply *refine-vcg*
by (*auto simp*: *l-len qp-def split*: *if-split-asm*)
qed

context
notes [*id-rules*] = *itypeI*[*Pure.of maxsize TYPE*(*nat*)]
notes [*sepref-import-param*] = *IdI*[*of maxsize*]
begin
sepref-definition *ial-contains* **is** *uncurry aial-contains* :: *nat-assn*^{k} $*_a$ (*ial-assn2 maxsize*)^k $\to_a$ *bool-assn*
unfolding *aial-contains-def ial-assn2-def* **by** *sepref*
end

sepref-decl-impl (*ismop*) *ial-contains*: *ial-contains.refine*[*FCOMP aial-contains-refine*]
.
end

interpretation *ial-sz*: *list-custom-empty ial-assn N A ial-empty N PR-CONST* (*op-ial-empty-sz N*)
apply *unfold-locales*
apply (*rule ial-empty-sz-hnr* [*unfolded op-ial-empty-sz-def PR-CONST-def*])
by *simp*

end

3.12 Implementation of Heaps by Arrays

theory *IICF-Impl-Heapmap*
imports *IICF-Abs-Heapmap ../IICF-Indexed-Array-List*
begin

Some setup to circumvent the really inefficient implementation of division in the code generator, which has to consider several cases for negative divisors and dividends.

definition [*code-unfold*]:
  efficient-nat-div2 n
    $\equiv$ *nat-of-integer* (*fst* (*Code-Numeral.divmod-abs* (*integer-of-nat n*) *2*))

lemma *efficient-nat-div2*[*simp*]: *efficient-nat-div2 n* = *n div 2*
  by (*simp add*: *efficient-nat-div2-def nat-of-integer.rep-eq*)

  type-synonym *'v hma* = *nat list* $\times$ (*'v list*)
  sepref-decl-intf *'v i-hma* **is** *nat list* $\times$ (*nat* $\rightharpoonup$ *'v*)

  locale *hmstruct-impl* = *hmstruct prio* **for** *prio* :: *'v*::*heap* $\Rightarrow$ *'p*::*linorder*
  begin
    lemma *param-prio*: (*prio*,*prio*) $\in$ *Id* $\to$ *Id* **by** *simp*
    lemmas [*sepref-import-param*] = *param-prio*
    sepref-register *prio*
  end

  context
    fixes *maxsize* :: *nat*
    fixes *prio* :: *'v*::*heap* $\Rightarrow$ *'p*::*linorder*
     notes [*map-type-eqs*] = *map-type-eqI*[*Pure.of TYPE*((*nat*,*'v*) *ahm*) *TYPE*(*'v i-hma*)]
  begin

    interpretation *hmstruct* **.**
    interpretation *hmstruct-impl* **.**

    definition *hm-impl1-α* $\equiv$ λ(*pq*,*ml*).
      (*pq*,λ*k*. *if k*$\in$*set pq then Some* (*ml*!*k*) *else None*)

    definition *hm-impl1-invar* $\equiv$ λ(*pq*,*ml*).
        hmr-invar (*hm-impl1-α* (*pq*,*ml*))
      $\wedge$ *set pq* $\subseteq$ {*0*..<*maxsize*}
      $\wedge$ ((*pq*=[] $\wedge$ *ml*=[]) $\vee$ (*length ml* = *maxsize*))

    definition *hm-impl1-weak-invar* $\equiv$ λ(*pq*,*ml*).
        set pq $\subseteq$ {*0*..<*maxsize*}
      $\wedge$ ((*pq*=[] $\wedge$ *ml*=[]) $\vee$ (*length ml* = *maxsize*))

    definition *hm-impl1-rel* $\equiv$ *br hm-impl1-α hm-impl1-invar*
    definition *hm-weak-impl'-rel* $\equiv$ *br hm-impl1-α hm-impl1-weak-invar*

    lemmas *hm-impl1-rel-defs* =
      hm-impl1-rel-def hm-weak-impl'-rel-def hm-impl1-weak-invar-def hm-impl1-invar-def hm-impl1-α-def in-br-conv

lemma *hm-impl-α-fst-eq*:
$(x1,\ x2) = hm\text{-}impl1\text{-}\alpha\ (x1a,\ x2a) \Longrightarrow x1 = x1a$
unfolding *hm-impl1-α-def* **by** (*auto split*: *if-split-asm*)

term *hm-empty-op*
definition *hm-empty-op′* :: *′v hma nres*
where *hm-empty-op′* ≡ *do* {
let pq = *op-ial-empty-sz maxsize*;
let ml = *op-list-empty*;
RETURN (*pq*,*ml*)
}

lemma *hm-empty-op′-refine*: (*hm-empty-op′*, *hm-empty-op*) ∈ ⟨*hm-impl1-rel*⟩*nres-rel*

apply (*intro fun-relI nres-relI*)
unfolding *hm-empty-op′-def hm-empty-op-def hm-impl1-rel-defs*
by (*auto simp*: *in-br-conv*)

definition *hm-length′* :: *′v hma* ⇒ *nat* **where** *hm-length′* ≡ λ(*pq*,*ml*). *length pq*

lemma *hm-length′-refine*: (*RETURN o hm-length′*,*RETURN o hm-length*) ∈ *hm-impl1-rel* → ⟨*nat-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-length′-def hm-length-def hm-impl1-rel-defs*
by (*auto*)

term *hm-key-of-op*
definition *hm-key-of-op′* ≡ λ(*pq*,*ml*) *i*. *ASSERT* (*i*>*0*) ≫ *mop-list-get pq* (*i* − *1*)
lemma *hm-key-of-op′-refine*: (*hm-key-of-op′*, *hm-key-of-op*) ∈ *hm-impl1-rel* → *nat-rel* → ⟨*nat-rel*⟩*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-key-of-op′-def hm-key-of-op-def hm-impl1-rel-defs*
by (*auto*)

term *hm-lookup*
definition *hm-lookup-op′* ≡ λ(*pq*,*ml*) *k*. *do* {
if (*k*<*maxsize*) *then do* { — TODO: This check can be eliminated, but this will complicate refinement of keys in basic ops
let c = *op-list-contains k pq*;
if c then do {
v ← *mop-list-get ml k*;
RETURN (*Some v*)
} *else RETURN None*
} *else RETURN None*

}

lemma *hm-lookup-op′-refine*: (*uncurry hm-lookup-op′*, *uncurry* (*RETURN oo hm-lookup*))
∈ (*hm-impl1-rel* $\times_r$ *nat-rel*) $\rightarrow_f$ ⟨⟨*Id*⟩*option-rel*⟩*nres-rel*
apply (*intro frefI nres-relI*)
unfolding *hm-lookup-op-def hm-lookup-op′-def o-def uncurry-def*
apply *refine-vcg*
apply (*auto simp*: *hm-impl1-rel-defs heapmap-α-def hmr-invar-def*)
done

term *hm-contains-key-op*
definition *hm-contains-key-op′* ≡ λ*k* (*pq*,*ml*). *do* {
if (*k*<*maxsize*) *then do* { — TODO: This check can be eliminated, but this will complicate refinement of keys in basic ops
RETURN (*op-list-contains k pq*)
} *else RETURN False*
}

lemma *hm-contains-key-op′-refine*: (*uncurry hm-contains-key-op′*, *uncurry hm-contains-key-op*)

∈ (*nat-rel* $\times_r$ *hm-impl1-rel*) $\rightarrow_f$ ⟨*bool-rel*⟩*nres-rel*
apply (*intro frefI nres-relI*)
unfolding *hm-contains-key-op-def hm-contains-key-op′-def o-def uncurry-def PR-CONST-def*
apply *refine-vcg*
apply (*auto simp*: *hm-impl1-rel-defs heapmap-α-def hmr-invar-def*)
done

term *hm-valid*

definition *hm-exch-op′* ≡ λ(*pq*,*ml*) *i j*. *do* {
ASSERT (*hm-valid* (*hm-impl1-α* (*pq*,*ml*)) *i*);
ASSERT (*hm-valid* (*hm-impl1-α* (*pq*,*ml*)) *j*);
pq ← *mop-list-swap pq* (*i* − *1*) (*j* − *1*);
RETURN (*pq*,*ml*)
}

lemma *hm-impl1-relI*:
assumes *hmr-invar b*
assumes (*a*,*b*)∈*hm-weak-impl′-rel*
shows (*a*,*b*)∈*hm-impl1-rel*
using *assms*
unfolding *hmr-rel-def hm-impl1-rel-def hm-weak-impl′-rel-def in-br-conv hm-impl1-weak-invar-def hm-impl1-invar-def*
by *auto*

lemma *hm-impl1-nres-relI*:

assumes $b \leq_n$ *SPEC hmr-invar*
assumes $(a,b) \in \langle$*hm-weak-impl′-rel*$\rangle$*nres-rel*
shows $(a,b) \in \langle$*hm-impl1-rel*$\rangle$*nres-rel*
using *assms hm-impl1-relI*
apply (*auto simp*: *pw-le-iff pw-leof-iff refine-pw-simps in-br-conv nres-rel-def*)
apply *blast*
done

lemma *hm-exch-op′-refine*: (*hm-exch-op′*, *hm-exch-op*) $\in$ *hm-impl1-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *nat-rel* $\rightarrow$ $\langle$*hm-impl1-rel*$\rangle$*nres-rel*
apply (*intro fun-relI hm-impl1-nres-relI*[*OF hm-exch-op-invar*])
unfolding *hm-exch-op′-def hm-exch-op-def*
apply (*auto simp*: *pw-le-iff refine-pw-simps nres-rel-def*
hm-impl1-rel-def in-br-conv split: *prod.splits*)
apply (*auto simp*: *hm-impl1-*α*-def*)
unfolding *hm-impl1-rel-defs*
apply *auto*
done

term *hm-index-op*

definition *hm-index-op′* $\equiv$ λ(*pq*,*ml*) *k*.
do {
ASSERT (*hm-impl1-invar* (*pq*,*ml*) $\wedge$ *heapmap-*α (*hm-impl1-*α (*pq*,*ml*)) *k* $\neq$ *None* $\wedge$ *k*$\in$*set pq*);
i $\leftarrow$ *mop-list-index pq k*;
RETURN (*i*+*1*)
}
lemma *hm-index-op′-refine*: (*hm-index-op′*,*hm-index-op*)
$\in$ *hm-impl1-rel* $\rightarrow$ *nat-rel* $\rightarrow$ $\langle$*nat-rel*$\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-index-op′-def hm-index-op-def hm-impl1-rel-defs*
apply (*auto simp*: *pw-le-iff refine-pw-simps heapmap-*α*-def split*: *if-split-asm*)
done

definition *hm-update-op′* **where**
hm-update-op′ $\equiv$ λ(*pq*,*ml*) *i v*. *do* {
ASSERT (*hm-valid* (*hm-impl1-*α (*pq*,*ml*)) *i* $\wedge$ *hm-impl1-invar* (*pq*,*ml*));
k $\leftarrow$ *mop-list-get pq* (*i* $-$ *1*);
ml $\leftarrow$ *mop-list-set ml k v*;
RETURN (*pq*, *ml*)
}
lemma *hm-update-op′-refine*: (*hm-update-op′*, *hm-update-op*) $\in$ *hm-impl1-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *Id* $\rightarrow$ $\langle$*hm-impl1-rel*$\rangle$*nres-rel*
apply (*intro fun-relI hm-impl1-nres-relI*[*OF hm-update-op-invar*])
unfolding *hm-update-op′-def hm-update-op-def*
apply (*auto simp*: *pw-le-iff refine-pw-simps nres-rel-def*

hm-impl1-rel-def in-br-conv split: *prod.splits*)
apply (*auto simp*: *hm-impl1-α-def*)
unfolding *hm-impl1-rel-defs*
apply (*auto simp*: *subset-code*(*1*))
done

term *hm-butlast-op*

lemma *hm-butlast-op-invar*: *hm-butlast-op hm* $\leq_n$ *SPEC hmr-invar*
unfolding *hm-butlast-op-def h.butlast-op-def*
apply *refine-vcg*
apply (*clarsimp-all simp*: *hmr-rel-defs map-butlast distinct-butlast*)
apply *safe*

apply (*auto simp*: *in-set-conv-nth nth-butlast*) []
apply (*metis Suc-pred len-greater-imp-nonempty length-greater-0-conv less-antisym*)

apply (*auto dest*: *in-set-butlastD*) []

apply (*metis One-nat-def append-butlast-last-id distinct-butlast last-conv-nth not-distinct-conv-prefix*)
done

definition *hm-butlast-op′* **where**
hm-butlast-op′ ≡ λ(*pq*,*ml*). *do* {
ASSERT (*hmr-invar* (*hm-impl1-α* (*pq*,*ml*)));
pq ← *mop-list-butlast pq*;
RETURN (*pq*,*ml*)
}

lemma *set-butlast-distinct-conv*:
⟦*distinct l*⟧ ⟹ *set* (*butlast l*) = *set l* − {*last l*}
by (*cases l rule*: *rev-cases*; *auto*)

lemma *hm-butlast-op′-refine*: (*hm-butlast-op′*, *hm-butlast-op*) ∈ *hm-impl1-rel* → ⟨*hm-impl1-rel*⟩*nres-rel*
apply (*intro fun-relI hm-impl1-nres-relI*[*OF hm-butlast-op-invar*])
unfolding *hm-butlast-op′-def hm-butlast-op-def*
apply (*auto simp*: *pw-le-iff refine-pw-simps nres-rel-def*
hm-impl1-rel-def in-br-conv split: *prod.splits*)
apply (*auto simp*: *hm-impl1-α-def*)
unfolding *hm-impl1-rel-defs*
apply (*auto simp*: *restrict-map-def*) []

defer

apply (*auto dest*: *in-set-butlastD*) []

apply (*auto intro*!: *ext*
simp: *hmr-invar-def set-butlast-distinct-conv last-conv-nth*
dest: *in-set-butlastD*) []
done

definition *hm-append-op'*
where *hm-append-op'* $\equiv$ λ(*pq*,*ml*) *k v*. *do* {
ASSERT ($k \notin$ *set pq* $\wedge$ *k*<*maxsize*);
ASSERT (*hm-impl1-invar* (*pq*,*ml*));
pq $\leftarrow$ *mop-list-append pq k*;
ml $\leftarrow$ (*if length ml* = *0 then mop-list-replicate maxsize v else RETURN ml*);
ml $\leftarrow$ *mop-list-set ml k v*;
RETURN (*pq*,*ml*)
}

lemma *hm-append-op'-refine*: (*uncurry2 hm-append-op'*, *uncurry2 hm-append-op*)

$\in$ [λ((*hm*,*k*),*v*). *k*<*maxsize*]$_f$ (*hm-impl1-rel* $\times_r$ *nat-rel*) $\times_r$ *Id* $\rightarrow$ $\langle$*hm-impl1-rel*$\rangle$*nres-rel*

apply (*intro frefI hm-impl1-nres-relI*[*OF hm-append-op-invar*])
unfolding *hm-append-op'-def hm-append-op-def*
apply (*auto simp*: *pw-le-iff refine-pw-simps nres-rel-def*
hm-impl1-rel-def in-br-conv split: *prod.splits*)
unfolding *hm-impl1-rel-defs*
apply (*auto simp*: *restrict-map-def hmr-invar-def split*: *prod.splits if-split-asm*)

done

definition *hm-impl2-rel* $\equiv$ *prod-assn* (*ial-assn maxsize id-assn*) (*array-assn id-assn*)
definition *hm-impl-rel* $\equiv$ *hr-comp hm-impl2-rel hm-impl1-rel*

lemmas [*fcomp-norm-unfold*] = *hm-impl-rel-def*[*symmetric*]

3.12.1 Implement Basic Operations

lemma *param-parent*: (*efficient-nat-div2*,*h.parent*) $\in$ *Id* $\rightarrow$ *Id*
by (*intro fun-relI*) (*simp add*: *h.parent-def*)
lemmas [*sepref-import-param*] = *param-parent*
sepref-register *h.parent*

lemma *param-left*: (*h.left*,*h.left*) $\in$ *Id* $\rightarrow$ *Id* **by** *simp*
lemmas [*sepref-import-param*] = *param-left*
sepref-register *h.left*

lemma *param-right*: (*h.right*,*h.right*) $\in$ *Id* $\rightarrow$ *Id* **by** *simp*
lemmas [*sepref-import-param*] = *param-right*
sepref-register *h.right*

abbreviation (*input*) *prio-rel* ≡ (*Id*::(′*p*×′*p*) *set*)

lemma *param-prio-le*: ((≤), (≤)) ∈ *prio-rel* → *prio-rel* → *bool-rel* **by** *simp*
lemmas [*sepref-import-param*] = *param-prio-le*

lemma *param-prio-lt*: ((<), (<)) ∈ *prio-rel* → *prio-rel* → *bool-rel* **by** *simp*
lemmas [*sepref-import-param*] = *param-prio-lt*

abbreviation *I-HM-UNF* ≡ *TYPE*(*nat list* × ′*v list*)

sepref-definition *hm-length-impl* **is** *RETURN o hm-length*′ :: $\textit{hm-impl2-rel}^k \rightarrow_a \textit{nat-assn}$
unfolding *hm-length*′*-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-length-impl.refine*[*FCOMP hm-length*′*-refine*]
sepref-register *hm-length*::(*nat*,′*v*) *ahm* ⇒ -

sepref-definition *hm-key-of-op-impl* **is** *uncurry hm-key-of-op*′ :: $\textit{hm-impl2-rel}^k *_a \textit{nat-assn}^k \rightarrow_a \textit{nat-assn}$
unfolding *hm-key-of-op*′*-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-key-of-op-impl.refine*[*FCOMP hm-key-of-op*′*-refine*]
sepref-register *hm-key-of-op*::(*nat*,′*v*) *ahm* ⇒ -

context
notes [*id-rules*] = *itypeI*[*Pure.of maxsize TYPE*(*nat*)]
notes [*sepref-import-param*] = *IdI*[*of maxsize*]
begin

sepref-definition *hm-lookup-impl* **is** *uncurry hm-lookup-op*′ :: $(\textit{hm-impl2-rel}^k *_a \textit{nat-assn}^k \rightarrow_a \textit{option-assn id-assn})$
unfolding *hm-lookup-op*′*-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] =
hm-lookup-impl.refine[*FCOMP hm-lookup-op*′*-refine*]
sepref-register *hm-lookup*::(*nat*,′*v*) *ahm* ⇒ -

sepref-definition *hm-exch-op-impl* **is** *uncurry2 hm-exch-op*′ :: $\textit{hm-impl2-rel}^d *_a \textit{nat-assn}^k *_a \textit{nat-assn}^k \rightarrow_a \textit{hm-impl2-rel}$
unfolding *hm-exch-op*′*-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-exch-op-impl.refine*[*FCOMP hm-exch-op*′*-refine*]
sepref-register *hm-exch-op*::(*nat*,′*v*) *ahm* ⇒ -

sepref-definition *hm-index-op-impl* **is** *uncurry hm-index-op*′ :: $\textit{hm-impl2-rel}^k *_a \textit{id-assn}^k \rightarrow_a \textit{id-assn}$
unfolding *hm-index-op*′*-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-index-op-impl.refine*[*FCOMP hm-index-op*′*-refine*]

sepref-register *hm-index-op*::$(nat,'v)$ *ahm* $\Rightarrow$ -

sepref-definition *hm-update-op-impl* **is** *uncurry2 hm-update-op'* :: *hm-impl2-rel*$^d *_a$ *id-assn*$^k *_a$ *id-assn*^{k} $\rightarrow_a$ *hm-impl2-rel*
unfolding *hm-update-op'-def hm-impl2-rel-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-update-op-impl.refine*[*FCOMP hm-update-op'-refine*]
sepref-register *hm-update-op*::$(nat,'v)$ *ahm* $\Rightarrow$ -

sepref-definition *hm-butlast-op-impl* **is** *hm-butlast-op'* :: *hm-impl2-rel*^{d} $\rightarrow_a$ *hm-impl2-rel*
unfolding *hm-butlast-op'-def hm-impl2-rel-def* **by** *sepref*
lemmas [*sepref-fr-rules*] = *hm-butlast-op-impl.refine*[*FCOMP hm-butlast-op'-refine*]
sepref-register *hm-butlast-op*::$(nat,'v)$ *ahm* $\Rightarrow$ -

sepref-definition *hm-append-op-impl* **is** *uncurry2 hm-append-op'* :: *hm-impl2-rel*^{d} $*_a$ *id-assn*^{k} $*_a$ *id-assn*^{k} $\rightarrow_a$ *hm-impl2-rel*
unfolding *hm-append-op'-def hm-impl2-rel-def*
apply (*rewrite array-fold-custom-replicate*)
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-append-op-impl.refine*[*FCOMP hm-append-op'-refine*]
sepref-register *hm-append-op*::$(nat,'v)$ *ahm* $\Rightarrow$ -

3.12.2 Auxiliary Operations

lemmas [*intf-of-assn*] = *intf-of-assnI*[**where** *R=hm-impl-rel* :: $(nat,'v)$ *ahm* $\Rightarrow$ - **and** *'a='v i-hma*]

sepref-definition *hm-valid-impl* **is** *uncurry* (*RETURN oo hm-valid*) :: *hm-impl-rel*$^k *_a$ *nat-assn*^{k} $\rightarrow_a$ *bool-assn*
unfolding *hm-valid-def*[*abs-def*]
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-valid-impl.refine*
sepref-register *hm-valid*::$(nat,'v)$ *ahm* $\Rightarrow$ -

definition *hm-the-lookup-op' hm k* $\equiv$ *do* {
let (*pq,ml*) = *hm*;
ASSERT (*heapmap-*α (*hm-impl1-*α *hm*) $k \neq$ *None* $\wedge$ *hm-impl1-invar hm*);
$v \leftarrow$ *mop-list-get ml k*;
RETURN v
}
lemma *hm-the-lookup-op'-refine*:
(*hm-the-lookup-op'*, *hm-the-lookup-op*) $\in$ *hm-impl1-rel* $\rightarrow$ *nat-rel* $\rightarrow$ $\langle Id\rangle$*nres-rel*
apply (*intro fun-relI nres-relI*)
unfolding *hm-the-lookup-op'-def hm-the-lookup-op-def*
apply *refine-vcg*

apply (*auto simp*: *hm-impl1-rel-defs heapmap-α-def hmr-invar-def split*: *if-split-asm*)
done

sepref-definition *hm-the-lookup-op-impl* **is** *uncurry hm-the-lookup-op'* :: *hm-impl2-rel*^{k}$*_a$*id-assn*^{k} $\rightarrow_a$ *id-assn*
unfolding *hm-the-lookup-op'-def*[*abs-def*] *hm-impl2-rel-def*
by *sepref*
lemmas *hm-the-lookup-op-impl*[*sepref-fr-rules*] = *hm-the-lookup-op-impl.refine*[*FCOMP hm-the-lookup-op'-refine*]
sepref-register *hm-the-lookup-op*::(*nat*,'*v*) *ahm* $\Rightarrow$ -

sepref-definition *hm-val-of-op-impl* **is** *uncurry hm-val-of-op* :: *hm-impl-rel*^{k}$*_a$*id-assn*^{k} $\rightarrow_a$ *id-assn*
unfolding *hm-val-of-op-def* **by** *sepref*
lemmas [*sepref-fr-rules*] = *hm-val-of-op-impl.refine*
sepref-register *hm-val-of-op*::(*nat*,'*v*) *ahm* $\Rightarrow$ -

sepref-definition *hm-prio-of-op-impl* **is** *uncurry* (*PR-CONST hm-prio-of-op*) :: *hm-impl-rel*^{k}$*_a$*id-assn*^{k} $\rightarrow_a$ *id-assn*
unfolding *hm-prio-of-op-def*[*abs-def*] *PR-CONST-def* **by** *sepref*
lemmas [*sepref-fr-rules*] = *hm-prio-of-op-impl.refine*
sepref-register *PR-CONST hm-prio-of-op*::(*nat*,'*v*) *ahm* $\Rightarrow$ -
lemma [*def-pat-rules*]: *hmstruct.hm-prio-of-op*\$*prio* $\equiv$ *PR-CONST hm-prio-of-op*
by *simp*

No code theorem preparation, as we define optimized version later

sepref-definition (*no-prep-code*) *hm-swim-op-impl* **is** *uncurry* (*PR-CONST hm-swim-op*) :: *hm-impl-rel*^{d}$*_a$*nat-assn*^{k} $\rightarrow_a$ *hm-impl-rel*
unfolding *hm-swim-op-def*[*abs-def*] *PR-CONST-def*
using [[*goals-limit* = *1*]]
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-swim-op-impl.refine*
sepref-register *PR-CONST hm-swim-op*::(*nat*,'*v*) *ahm* $\Rightarrow$ -
lemma [*def-pat-rules*]: *hmstruct.hm-swim-op*\$*prio* $\equiv$ *PR-CONST hm-swim-op* **by** *simp*

No code theorem preparation, as we define optimized version later

sepref-definition (*no-prep-code*) *hm-sink-op-impl* **is** *uncurry* (*PR-CONST hm-sink-op*) :: *hm-impl-rel*^{d}$*_a$*nat-assn*^{k} $\rightarrow_a$ *hm-impl-rel*
unfolding *hm-sink-op-def*[*abs-def*] *PR-CONST-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-sink-op-impl.refine*
sepref-register *PR-CONST hm-sink-op*::(*nat*,'*v*) *ahm* $\Rightarrow$ -
lemma [*def-pat-rules*]: *hmstruct.hm-sink-op*\$*prio* $\equiv$ *PR-CONST hm-sink-op* **by** *simp*

sepref-definition *hm-repair-op-impl* **is** *uncurry* (*PR-CONST hm-repair-op*) :: *hm-impl-rel*^{d}$*_a$*nat-assn*^{k} $\rightarrow_a$ *hm-impl-rel*

unfolding *hm-repair-op-def*[*abs-def*] *PR-CONST-def*
by *sepref*
lemmas [*sepref-fr-rules*] = *hm-repair-op-impl.refine*
sepref-register *PR-CONST hm-repair-op*::(*nat*,′*v*) *ahm* ⇒ -
lemma [*def-pat-rules*]: *hmstruct.hm-repair-op*$*prio* ≡ *PR-CONST hm-repair-op*
by *simp*

3.12.3 Interface Operations

definition *hm-rel-np* **where**
hm-rel-np ≡ *hr-comp hm-impl-rel heapmap-rel*
lemmas [*fcomp-norm-unfold*] = *hm-rel-np-def*[*symmetric*]

definition *hm-rel* **where**
hm-rel K V ≡ *hr-comp hm-rel-np* (⟨*the-pure K*,*the-pure V*⟩*map-rel*)
lemmas [*fcomp-norm-unfold*] = *hm-rel-def*[*symmetric*]

lemmas [*intf-of-assn*] = *intf-of-assnI*[**where** *R*=*hm-rel K V* **and** ′*a*=(′*kk*,′*vv*) *i-map* **for** *K V*]

lemma *hm-rel-id-conv*: *hm-rel id-assn id-assn* = *hm-rel-np*
— Used for generic algorithms: Unfold with this, then let decl-impl compose with *map-rel* again.
unfolding *hm-rel-def* **by** *simp*

Synthesis

definition *op-hm-empty-sz* :: *nat* ⇒ ′*kk*⇀′*vv*
where [*simp*]: *op-hm-empty-sz sz* ≡ *op-map-empty*
sepref-register *PR-CONST* (*op-hm-empty-sz maxsize*) :: (′*k*,′*v*) *i-map*
lemma [*def-pat-rules*]: *op-hm-empty-sz*$*maxsize* ≡ *UNPROTECT* (*op-hm-empty-sz maxsize*) **by** *simp*

lemma *hm-fold-custom-empty-sz*:
op-map-empty = *op-hm-empty-sz sz*
Map.empty = *op-hm-empty-sz sz*
by *auto*

sepref-definition *hm-empty-op-impl* **is** *uncurry0 hm-empty-op*′ :: *unit-assn*^{k} $\rightarrow_a$ *hm-impl2-rel*
unfolding *hm-empty-op*′*-def hm-impl2-rel-def array.fold-custom-empty*
by *sepref*

sepref-definition *hm-insert-op-impl* **is** *uncurry2 hm-insert-op* :: [λ((*k*,-),-). *k*<*maxsize*]$_a$ *id-assn*^{k}$*_a$*id-assn*^{k}$*_a$*hm-impl-rel*^{d} → *hm-impl-rel*
unfolding *hm-insert-op-def*
by *sepref*

sepref-definition *hm-is-empty-op-impl* **is** *hm-is-empty-op* :: *hm-impl-rel*^{k} $\rightarrow_a$ *bool-assn*

unfolding *hm-is-empty-op-def*
by *sepref*

sepref-definition *hm-lookup-op-impl* **is** *uncurry hm-lookup-op* :: $id\text{-}assn^k *_a hm\text{-}impl\text{-}rel^k \rightarrow_a option\text{-}assn\ id\text{-}assn$
unfolding *hm-lookup-op-def* **by** *sepref*

sepref-definition *hm-contains-key-impl* **is** *uncurry hm-contains-key-op'* :: $id\text{-}assn^k *_a hm\text{-}impl2\text{-}rel^k \rightarrow_a bool\text{-}assn$
unfolding *hm-contains-key-op'-def hm-impl2-rel-def*
by *sepref*

sepref-definition *hm-decrease-key-op-impl* **is** *uncurry2 hm-decrease-key-op* :: $id\text{-}assn^k *_a id\text{-}assn^k *_a hm\text{-}impl\text{-}rel^d \rightarrow_a hm\text{-}impl\text{-}rel$
unfolding *hm-decrease-key-op-def* **by** *sepref*

sepref-definition *hm-increase-key-op-impl* **is** *uncurry2 hm-increase-key-op* :: $id\text{-}assn^k *_a id\text{-}assn^k *_a hm\text{-}impl\text{-}rel^d \rightarrow_a hm\text{-}impl\text{-}rel$
unfolding *hm-increase-key-op-def* **by** *sepref*

sepref-definition *hm-change-key-op-impl* **is** *uncurry2 hm-change-key-op* :: $id\text{-}assn^k *_a id\text{-}assn^k *_a hm\text{-}impl\text{-}rel^d \rightarrow_a hm\text{-}impl\text{-}rel$
unfolding *hm-change-key-op-def* **by** *sepref*

sepref-definition *hm-pop-min-op-impl* **is** *hm-pop-min-op* :: $hm\text{-}impl\text{-}rel^d \rightarrow_a prod\text{-}assn\ (prod\text{-}assn\ nat\text{-}assn\ id\text{-}assn)\ hm\text{-}impl\text{-}rel$
unfolding *hm-pop-min-op-def[abs-def]*
by *sepref*

sepref-definition *hm-remove-op-impl* **is** *uncurry hm-remove-op* :: $id\text{-}assn^k *_a hm\text{-}impl\text{-}rel^d \rightarrow_a hm\text{-}impl\text{-}rel$
unfolding *hm-remove-op-def[abs-def]* **by** *sepref*

sepref-definition *hm-peek-min-op-impl* **is** *hm-peek-min-op* :: $hm\text{-}impl\text{-}rel^k \rightarrow_a prod\text{-}assn\ nat\text{-}assn\ id\text{-}assn$
unfolding *hm-peek-min-op-def[abs-def] hm-kv-of-op-def*
by *sepref*

Setup of Refinements

sepref-decl-impl (*no-register*) *hm-empty*:
hm-empty-op-impl.refine[FCOMP hm-empty-op'-refine, FCOMP hm-empty-aref]
.

context fixes K **assumes** *IS-BELOW-ID K* **begin**
lemmas *mop-map-update-new-fref' = mop-map-update-new.fref[of K]*
lemmas *op-map-update-fref' = op-map-update.fref[of K]*
end

sepref-decl-impl (*ismop*) *hm-insert*: *hm-insert-op-impl.refine*[*FCOMP hm-insert-op-aref*]
uses *mop-map-update-new-fref'*
unfolding *IS-BELOW-ID-def*
apply (*parametricity*; *auto simp*: *single-valued-below-Id*)
done

sepref-decl-impl *hm-is-empty*: *hm-is-empty-op-impl.refine*[*FCOMP hm-is-empty-op-aref*]
.
sepref-decl-impl *hm-lookup*: *hm-lookup-op-impl.refine*[*FCOMP hm-lookup-op-aref*]
.

sepref-decl-impl *hm-contains-key*:
hm-contains-key-impl.refine[*FCOMP hm-contains-key-op'-refine*, *FCOMP hm-contains-key-op-aref*]
.

sepref-decl-impl (*ismop*) *hm-decrease-key*: *hm-decrease-key-op-impl.refine*[*FCOMP hm-decrease-key-op-aref*] **.**
sepref-decl-impl (*ismop*) *hm-increase-key*: *hm-increase-key-op-impl.refine*[*FCOMP hm-increase-key-op-aref*] **.**
sepref-decl-impl (*ismop*) *hm-change-key*: *hm-change-key-op-impl.refine*[*FCOMP hm-change-key-op-aref*] **.**

sepref-decl-impl (*ismop*) *hm-remove*: *hm-remove-op-impl.refine*[*FCOMP hm-remove-op-aref*]
.

sepref-decl-impl (*ismop*) *hm-pop-min*: *hm-pop-min-op-impl.refine*[*FCOMP hm-pop-min-op-aref*]
.
sepref-decl-impl (*ismop*) *hm-peek-min*: *hm-peek-min-op-impl.refine*[*FCOMP hm-peek-min-op-aref*]
.

— Realized as generic algorithm. Note that we use *id-assn* for the elements.
sepref-definition *hm-upd-op-impl* **is** *uncurry2* (*RETURN ooo op-map-update*) :: $[\lambda((k,-),-).\ k<maxsize]_a\ id\text{-}assn^k *_a id\text{-}assn^k *_a (hm\text{-}rel\ id\text{-}assn\ id\text{-}assn)^d \rightarrow hm\text{-}rel\ id\text{-}assn\ id\text{-}assn$
unfolding *op-pm-set-gen-impl* **by** *sepref*

sepref-decl-impl *hm-upd-op-impl.refine*[*unfolded hm-rel-id-conv*] **uses** *op-map-update-fref'*
unfolding *IS-BELOW-ID-def*
apply (*parametricity*; *auto simp*: *single-valued-below-Id*)
done

end
end

interpretation *hm*: *map-custom-empty PR-CONST* (*op-hm-empty-sz maxsize*)
apply *unfold-locales* **by** *simp*

lemma *op-hm-empty-sz-hnr*[*sepref-fr-rules*]:
(*uncurry0* (*hm-empty-op-impl maxsize*), *uncurry0* (*RETURN* (*PR-CONST* (*op-hm-empty-sz*

maxsize)))) $\in$ *unit-assn*^{k} $\rightarrow_a$ *hm-rel maxsize prio K V*
using *hm-empty-hnr* **by** *simp*

3.12.4 Manual fine-tuning of code-lemmas

context
notes [*simp del*] = *CNV-def efficient-nat-div2*
begin

lemma *nested-case-bind*:
(*case p of* (*a*,*b*) $\Rightarrow$ *bind* (*case a of* (*a1*,*a2*) $\Rightarrow$ *m a b a1 a2*) (*f a b*))
= (*case p of* ((*a1*,*a2*),*b*) $\Rightarrow$ *bind* (*m* (*a1*,*a2*) *b a1 a2*) (*f* (*a1*,*a2*) *b*))
(*case p of* (*a*,*b*) $\Rightarrow$ *bind* (*case b of* (*b1*,*b2*) $\Rightarrow$ *m a b b1 b2*) (*f a b*))
= (*case p of* (*a*,*b1*,*b2*) $\Rightarrow$ *bind* (*m a* (*b1*,*b2*) *b1 b2*) (*f a* (*b1*,*b2*)))
by (*simp-all split*: *prod.splits*)

lemma *it-case*: (*case p of* (*a*,*b*) $\Rightarrow$ *f p a b*) = (*case p of* (*a*,*b*) $\Rightarrow$ *f* (*a*,*b*) *a b*)
by (*auto split*: *prod.split*)

lemma *c2l*: (*case p of* (*a*,*b*) $\Rightarrow$ *bind* (*m a b*) (*f a b*)) =
do { *let* (*a*,*b*) = *p*; *bind* (*m a b*) (*f a b*)} **by** *simp*

lemma *bind-Let*: *do* { *x* $\leftarrow$ *do* { *let y* = *v*; (*f y* :: *'a Heap*)}; *g x* } = *do* { *let y*=*v*; *x* $\leftarrow$ *f y*; *g x* } **by** *auto*
lemma *bind-case*: *do* { *x* $\leftarrow$ (*case y of* (*a*,*b*) $\Rightarrow$ *f a b*); (*g x* :: *'a Heap*) } = *do* { *let* (*a*,*b*) = *y*; *x* $\leftarrow$ *f a b*; *g x* }
by (*auto split*: *prod.splits*)

lemma *bind-case-mvup*: *do* { *x* $\leftarrow$ *f*; *case y of* (*a*,*b*) $\Rightarrow$ *g a b x* }
= *do* { *let* (*a*,*b*) = *y*; *x* $\leftarrow$ *f*; (*g a b x* :: *'a Heap*) }
by (*auto split*: *prod.splits*)

lemma *if-case-mvup*: (*if b then case p of* (*x1*,*x2*) $\Rightarrow$ *f x1 x2 else e*)
= (*case p of* (*x1*,*x2*) $\Rightarrow$ *if b then f x1 x2 else e*) **by** *auto*

lemma *nested-case*: (*case p of* (*a*,*b*) $\Rightarrow$ (*case p of* (*c*,*d*) $\Rightarrow$ *f a b c d*)) =
(*case p of* (*a*,*b*) $\Rightarrow$ *f a b a b*)
by (*auto split*: *prod.split*)

lemma *split-prod-bound*: (λ*p*. *f p*) = (λ(*a*,*b*). *f* (*a*,*b*)) **by** *auto*

lemma *bpc-conv*: *do* { (*a*,*b*) $\leftarrow$ (*m*::(-$*$-) *Heap*); *f a b* } = *do* {
ab $\leftarrow$ (*m*);
f (*fst ab*) (*snd ab*)
}
apply (*subst* (*2*) *split-prod-bound*)
by *simp*

lemma *it-case-pp*: (*case p of* ((*p1*,*p2*)) $\Rightarrow$ *case p of* ((*p1'*,*p2'*)) $\Rightarrow$ *f p1 p2 p1'*

p2′)
  = (*case p of* ((*p1*,*p2*)) ⇒ *f p1 p2 p1 p2*)
  by (*auto split*: *prod.split*)

lemma *it-case-ppp*: (*case p of* ((*p1*,*p2*),*p3*) ⇒ *case p of* ((*p1′*,*p2′*),*p3′*) ⇒ *f p1 p2 p3 p1′ p2′ p3′*)
  = (*case p of* ((*p1*,*p2*),*p3*) ⇒ *f p1 p2 p3 p1 p2 p3*)
  by (*auto split*: *prod.split*)

lemma *it-case-pppp*: (*case a1 of*
      (((*a*, *b*), *c*), *d*) ⇒
        case a1 of
        (((*a′*, *b′*), *c′*), *d′*) ⇒ *f a b c d a′ b′ c′ d′*) =
    (*case a1 of*
      (((*a*, *b*), *c*), *d*) ⇒ *f a b c d a b c d*)
  by (*auto split*: *prod.splits*)

private lemmas *inlines* = *hm-append-op-impl-def ial-append-def marl-length-def marl-append-def hm-length-impl-def ial-length-def hm-valid-impl-def hm-prio-of-op-impl-def hm-val-of-op-impl-def hm-key-of-op-impl-def ial-get-def hm-the-lookup-op-impl-def heap-array-set-def marl-get-def it-case-ppp it-case-pppp bind-case bind-case-mvup nested-case if-case-mvup it-case-pp*

schematic-goal [*code*]: *hm-insert-op-impl maxsize prio hm k v* = *?f*
  unfolding *hm-insert-op-impl-def*
  apply (*rule CNV-eqD*)
  apply (*simp add*: *inlines* *cong*: *if-cong*)
  by (*rule CNV-I*)

schematic-goal *hm-swim-op-impl prio hm i* ≡ *?f*
  unfolding *hm-swim-op-impl-def*
  apply (*rule eq-reflection*)
  apply (*rule CNV-eqD*)
  apply (*simp add*: *inlines efficient-nat-div2*
    cong: *if-cong*)
  by (*rule CNV-I*)

lemma *hm-swim-op-impl-code*[*code*]: *hm-swim-op-impl prio hm i* ≡ *ccpo.fixp* (*fun-lub Heap-lub*) (*fun-ord Heap-ord*)
    (λ*cf* (*a1*, *a2*).
        case a1 of
        ((*a1b*, *a2b*), *a2a*) ⇒
          case a1b of
          (*a*, *b*) ⇒ *do* {
            let d2 = *efficient-nat-div2 a2*;

```
          if 0 < d2 ∧ d2 ≤ b
          then do {
                x ← (case a1b of (a, n) ⇒ Array.nth a) (d2 − Suc 0);
                x ← Array.nth a2a x;
                xa ← (case a1b of (a, n) ⇒ Array.nth a) (a2 − Suc 0);
                xa ← Array.nth a2a xa;
                if prio x ≤ prio xa then return a1
                else do {
                      x'g ← hm-exch-op-impl a1 a2 (d2);
                      cf (x'g, d2)
                    }
              }
          else return a1
        })
    (hm, i)
  unfolding hm-swim-op-impl-def
  apply (rule eq-reflection)
  apply (simp add: inlines efficient-nat-div2 Let-def
    cong: if-cong)
  done

prepare-code-thms hm-swim-op-impl-code

schematic-goal hm-sink-opt-impl-code[code]: hm-sink-op-impl prio hm i ≡ ?f
  unfolding hm-sink-op-impl-def
  apply (rule eq-reflection)
  apply (rule CNV-eqD)
  apply (simp add: inlines
    cong: if-cong)
  by (rule CNV-I)

prepare-code-thms hm-sink-opt-impl-code

export-code hm-swim-op-impl in SML-imp module-name Test


schematic-goal hm-change-key-opt-impl-code[code]:
  hm-change-key-op-impl prio k v hm ≡ ?f
  unfolding hm-change-key-op-impl-def
  apply (rule eq-reflection)
  apply (rule CNV-eqD)
  apply (simp add: inlines hm-contains-key-impl-def ial-contains-def
    hm-change-key-op-impl-def hm-index-op-impl-def hm-update-op-impl-def
    ial-index-def
    cong: if-cong split: prod.splits)
  oops


schematic-goal hm-change-key-opt-impl-code[code]:
```

hm-change-key-op-impl prio k v hm ≡ *case hm of* (((*a*, *b*), *ba*), *x2*) ⇒
  (*do* {
    x ← *Array.nth ba k*;
    xa ← *Array.nth a x*;
    xa ← *Array.upd xa v x2*;
    hm-repair-op-impl prio (((*a*, *b*), *ba*), *xa*) (*Suc x*)
  })
unfolding *hm-change-key-op-impl-def*
apply (*rule eq-reflection*)
apply (*simp add*: *inlines hm-contains-key-impl-def ial-contains-def*
  hm-change-key-op-impl-def hm-index-op-impl-def hm-update-op-impl-def
  ial-index-def
  cong: *if-cong split*: *prod.splits*)
done

schematic-goal *hm-set-opt-impl-code*[*code*]: *hm-upd-op-impl maxsize prio hm k v* ≡ *?f*
unfolding *hm-upd-op-impl-def*
apply (*rule eq-reflection*)
apply (*rule CNV-eqD*)
apply (*simp add*: *inlines hm-contains-key-impl-def ial-contains-def*
  hm-change-key-op-impl-def hm-index-op-impl-def hm-update-op-impl-def
  ial-index-def
  cong: *if-cong*)
by (*rule CNV-I*)

schematic-goal *hm-pop-min-opt-impl-code*[*code*]: *hm-pop-min-op-impl prio hm* ≡ *?f*
unfolding *hm-pop-min-op-impl-def*
apply (*rule eq-reflection*)
apply (*rule CNV-eqD*)
apply (*simp add*: *inlines hm-contains-key-impl-def ial-contains-def*
  hm-change-key-op-impl-def hm-index-op-impl-def hm-update-op-impl-def
  hm-butlast-op-impl-def ial-butlast-def
  ial-index-def
  cong: *if-cong*)
by (*rule CNV-I*)

end

export-code
hm-empty-op-impl
hm-insert-op-impl
hm-is-empty-op-impl
hm-lookup-op-impl
hm-contains-key-impl
hm-decrease-key-op-impl
hm-increase-key-op-impl

hm-change-key-op-impl
hm-upd-op-impl
hm-pop-min-op-impl
hm-remove-op-impl
hm-peek-min-op-impl
checking *SML-imp*

end

3.13 Matrices

theory *IICF-Matrix*
imports *../../Sepref*
begin

3.13.1 Relator and Interface

definition [*to-relAPP*]: *mtx-rel A* $\equiv$ *nat-rel* $\times_r$ *nat-rel* $\rightarrow$ *A*

lemma *mtx-rel-id*[*simp*]: $\langle Id \rangle$*mtx-rel* = *Id* **unfolding** *mtx-rel-def* **by** *auto*

type-synonym *'a mtx* = *nat*$\times$*nat* $\Rightarrow$ *'a*
sepref-decl-intf *'a i-mtx* **is** *nat*$\times$*nat* $\Rightarrow$ *'a*

lemma [*synth-rules*]: *INTF-OF-REL A TYPE*(*'a*) $\Longrightarrow$ *INTF-OF-REL* ($\langle A \rangle$*mtx-rel*) *TYPE*(*'a i-mtx*)
by *simp*

3.13.2 Operations

definition *op-mtx-new* :: *'a mtx* $\Rightarrow$ *'a mtx* **where** [*simp*]: *op-mtx-new c* $\equiv$ *c*

sepref-decl-op (*no-def*) *mtx-new*: *op-mtx-new* :: (*nat-rel*$\times_r$*nat-rel* $\rightarrow$ *A*) $\rightarrow$ $\langle A \rangle$*mtx-rel*
apply (*rule fref-ncI*) **unfolding** *op-mtx-new-def*[*abs-def*] *mtx-rel-def*
by *parametricity*

lemma *mtx-init-adhoc-frame-match-rule*[*sepref-frame-match-rules*]:
hn-val (*nat-rel*$\times_r$*nat-rel* $\rightarrow$ *A*) *x y* $\Longrightarrow_t$ *hn-val* (*nat-rel*$\times_r$*nat-rel* $\rightarrow$ *the-pure* (*pure A*)) *x y*
by *simp*

definition *op-mtx-copy* :: *'a mtx* $\Rightarrow$ *'a mtx* **where** [*simp*]: *op-mtx-copy c* $\equiv$ *c*

sepref-decl-op (*no-def*) *mtx-copy*: *op-mtx-copy* :: $\langle A \rangle$*mtx-rel* $\rightarrow$ $\langle A \rangle$*mtx-rel* **.**

sepref-decl-op *mtx-get*: $\lambda(c::'a\ mtx)\ ij.\ c\ ij :: \langle A\rangle mtx\text{-}rel \to (nat\text{-}rel \times_r nat\text{-}rel) \to A$
apply (*rule fref-ncI*) **unfolding** *mtx-rel-def*
by *parametricity*

sepref-decl-op *mtx-set*: *fun-upd*$::'a\ mtx \Rightarrow$ - $:: \langle A\rangle mtx\text{-}rel \to (nat\text{-}rel \times_r nat\text{-}rel) \to A \to \langle A\rangle mtx\text{-}rel$
apply (*rule fref-ncI*)
unfolding *mtx-rel-def*
proof *goal-cases* **case** *1*
have [*param*]: $((=), (=)) \in nat\text{-}rel \times_r nat\text{-}rel \to nat\text{-}rel \times_r nat\text{-}rel \to bool\text{-}rel$ **by** *simp*
show *?case* **by** *parametricity*
qed

definition *mtx-nonzero* :: - $mtx \Rightarrow (nat \times nat)$ *set* **where** *mtx-nonzero* $m \equiv \{(i,j).\ m\ (i,j) \neq 0\}$

sepref-decl-op *mtx-nonzero*: *mtx-nonzero* $:: \langle A\rangle mtx\text{-}rel \to \langle nat\text{-}rel \times_r nat\text{-}rel\rangle set\text{-}rel$
where *IS-ID* $(A::(\text{-} \times (\text{-}::zero))\ set)$
proof *goal-cases*
case *1*
assume *IS-ID A*
hence *U*: $A = Id$ **by** (*simp only*: *IS-ID-def*)
have [*param*]: $((=),(=)) \in A \to A \to bool\text{-}rel$ **using** *U* **by** *simp*
show *?case*
apply (*rule fref-ncI*)
unfolding *mtx-rel-def*
apply *parametricity*
unfolding *U* **by** *simp-all*
qed

3.13.3 Patterns

lemma *pat-amtx-get*: $c\$e \equiv op\text{-}mtx\text{-}get\$'c\$'e$ **by** *simp*
lemma *pat-amtx-set*: $fun\text{-}upd\$c\$e\$v \equiv op\text{-}mtx\text{-}set\$'c\$'e\$'v$ **by** *simp*

lemmas *amtx-pats*[*pat-rules*] = *pat-amtx-get pat-amtx-set*

3.13.4 Pointwise Operations

Auxiliary Definitions and Lemmas

locale *pointwise-op* =
fixes $f :: 'p \Rightarrow 's \Rightarrow 's$
fixes $q :: 's \Rightarrow 'p \Rightarrow 'a$
assumes *upd-indep1*[*simp*, *intro*]: $p \neq p' \Longrightarrow q\ (f\ p\ s)\ p' = q\ s\ p'$
assumes *upd-indep2*[*simp*, *intro*]: $p \neq p' \Longrightarrow q\ (f\ p\ (f\ p'\ s))\ p = q\ (f\ p\ s)\ p$
begin
lemma *pointwise-upd-fold*: *distinct ps* $\Longrightarrow$

q (*fold f ps s*) p = (*if p∈set ps then q* (*f p s*) *p else q s p*)
by (*induction ps arbitrary*: *s*) *auto*

end

lemma *pointwise-fun-fold*:
fixes $f :: 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)$
fixes $s :: 'a \Rightarrow 'b$
assumes *indep1*: $\bigwedge x\ x'\ s.\ x \neq x' \Longrightarrow f\ x\ s\ x' = s\ x'$
assumes *indep2*: $\bigwedge x\ x'\ s.\ x \neq x' \Longrightarrow f\ x\ (f\ x'\ s)\ x = f\ x\ s\ x$
assumes [*simp*]: *distinct xs*
shows *fold f xs s x* = (*if* $x \in$ *set xs then f x s x else s x*)
proof −
interpret *pointwise-op f* $\lambda s.\ s$
by *unfold-locales fact+*

show *?thesis*
using *pointwise-upd-fold*[*of xs s x*]
by *auto*
qed

lemma *list-prod-divmod-eq*: *List.product* [*0..<M*] [*0..<N*] = *map* ($\lambda i.$ (*i div N*, *i mod N*)) [*0..<N∗M*]
proof −
have [*simp*]: $i < m*n \Longrightarrow$ (*i::nat*) *div n* $<$ *m* **for** *i m n*
by (*metis mult.commute div-eq-0-iff div-mult2-eq gr-implies-not-zero mult-not-zero*)

have [*simp*]: $i<N*M \Longrightarrow N>0 \wedge M>0$ **for** *i*
by (*cases N*; *cases M*; *auto*)

show *?thesis*
by (*rule nth-equalityI*) (*auto simp add*: *product-nth algebra-simps*)
qed

lemma *nfoldli-prod-divmod-conv*:
nfoldli (*List.product* [*0..<N*] [*0..<M*]) *ctd* ($\lambda(i,j).$ *f i j*) = *nfoldli* [*0..<N∗M*] *ctd* ($\lambda i.$ *f* (*i div M*) (*i mod M*))
apply (*intro ext*)
apply (*subst list-prod-divmod-eq*)
apply (*simp add*: *nfoldli-map*)
apply (*fo-rule cong*)+
apply (*auto simp*: *algebra-simps*)
done

lemma *nfoldli-prod-divmod-conv'*:
nfoldli [*0..<M*] *ctd* ($\lambda i.$ *nfoldli* [*0..<N*] *ctd* (*f i*)) = *nfoldli* [*0..<N∗M*] *ctd* ($\lambda i.$ *f* (*i div N*) (*i mod N*))
apply (*intro ext*)

apply (*subst nfoldli-nfoldli-prod-conv*)
by (*simp add*: *nfoldli-prod-divmod-conv algebra-simps*)

lemma *foldli-prod-divmod-conv'*:
foldli [*0*..<*M*] *ctd* (λ*i*. *foldli* [*0*..<*N*] *ctd* (*f i*)) = *foldli* [*0*..<*N*∗*M*] *ctd* (λ*i*. *f* (*i div N*) (*i mod N*))
(**is** *?lhs*=*?rhs*)
proof −
have *RETURN* (*?lhs s*) = *RETURN* (*?rhs s*) **for** *s*
apply (*subst foldli-eq-nfoldli*)+
apply (*subst nfoldli-prod-divmod-conv'*)
..
thus *?thesis* **by** *auto*
qed

lemma *fold-prod-divmod-conv'*: *fold* (λ*i*. *fold* (*f i*) [*0*..<*N*]) [*0*..<*M*] = *fold* (λ*i*. *f* (*i div N*) (*i mod N*)) [*0*..<*N*∗*M*]
using *foldli-prod-divmod-conv'*[*of M* λ-. *True N f*, *THEN fun-cong*]
apply (*intro ext*)
apply (*simp add*: *foldli-foldl foldl-conv-fold*)
done

lemma *mtx-nonzero-cases*[*consumes 0*, *case-names nonzero zero*]:
obtains (*i*,*j*)∈*mtx-nonzero m* | *m* (*i*,*j*) = *0*
by (*auto simp*: *mtx-nonzero-def*)

Unary Pointwise

definition *mtx-pointwise-unop* :: (*nat*×*nat* ⇒ *'a* ⇒ *'a*) ⇒ *'a mtx* ⇒ *'a mtx* **where**
mtx-pointwise-unop f m ≡ λ(*i*,*j*). *f* (*i*,*j*) (*m*(*i*,*j*))

context fixes *f* :: *nat*×*nat* ⇒ *'a* ⇒ *'a* **begin**
sepref-register *PR-CONST* (*mtx-pointwise-unop f*) :: *'a i-mtx* ⇒ *'a i-mtx*
lemma [*def-pat-rules*]: *mtx-pointwise-unop*$*f* ≡ *UNPROTECT* (*mtx-pointwise-unop f*) **by** *simp*
end

locale *mtx-pointwise-unop-loc* =
fixes *N* :: *nat* **and** *M* :: *nat*
fixes *f* :: (*nat*×*nat*) ⇒ *'a*::{*zero*} ⇒ *'a*
assumes *pres-zero*[*simp*]: ⟦ *i*≥*N* ∨ *j*≥*M* ⟧ ⟹ *f* (*i*,*j*) *0* = *0*
begin
definition *opr-fold-impl* ≡ *fold* (λ*i*. *fold* (λ*j m*. *m*((*i*,*j*) := *f* (*i*,*j*) (*m*(*i*,*j*)))) [*0*..<*M*]) [*0*..<*N*]

lemma *opr-fold-impl-eq*:

assumes *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}
shows *mtx-pointwise-unop f m* = *opr-fold-impl m*
apply (*rule ext*)
unfolding *opr-fold-impl-def*
apply (*simp add*: *fold-fold-prod-conv*)
apply (*subst pointwise-fun-fold*)
apply (*auto simp*: *mtx-pointwise-unop-def distinct-product*) [*3*]
apply *clarsimp*
subgoal for *a b*
apply (*cases a b m rule*: *mtx-nonzero-cases*)
using *assms*
apply (*auto simp*: *mtx-pointwise-unop-def*)
done
done

lemma *opr-fold-impl-refine*: (*opr-fold-impl*, *mtx-pointwise-unop f*) ∈ [λ*m*. *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}]$_f$ *Id* → *Id*
apply (*rule frefI*)
using *opr-fold-impl-eq*
by *auto*

end

locale *mtx-pointwise-unop-gen-impl* = *mtx-pointwise-unop-loc* +
fixes *assn* :: *'a mtx* ⇒ *'i* ⇒ *assn*
fixes *A* :: *'a* ⇒ *'ai* ⇒ *assn*
fixes *get-impl* :: *'i* ⇒ *nat*×*nat* ⇒ *'ai Heap*
fixes *set-impl* :: *'i* ⇒ *nat*×*nat* ⇒ *'ai* ⇒ *'i Heap*
fixes *fi* :: *nat*×*nat* ⇒ *'ai* ⇒ *'ai Heap*
assumes *assn-range*: *rdomp assn m* ⟹ *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}
assumes *get-impl-hnr*:
(*uncurry get-impl*,*uncurry* (*RETURN oo op-mtx-get*)) ∈ $assn^k$ $*_a$ (*prod-assn* (*nbn-assn N*) (*nbn-assn M*))^k $\rightarrow_a$ *A*
assumes *set-impl-hnr*:
(*uncurry2 set-impl*,*uncurry2* (*RETURN ooo op-mtx-set*)) ∈ $assn^d$ $*_a$ (*prod-assn* (*nbn-assn N*) (*nbn-assn M*))^k $*_a$ A^k $\rightarrow_a$ *assn*
assumes *fi-hnr*:
(*uncurry fi*,*uncurry* (*RETURN oo f*)) ∈ (*prod-assn nat-assn nat-assn*)^k $*_a$ A^k $\rightarrow_a$ *A*
begin

lemma *this-loc*: *mtx-pointwise-unop-gen-impl N M f assn A get-impl set-impl fi*
by *unfold-locales*

context
notes [[*sepref-register-adhoc f N M*]]
notes [*intf-of-assn*] = *intf-of-assnI*[**where** *R*=*assn* **and** *'a*=*'a i-mtx*]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *get-impl-hnr set-impl-hnr fi-hnr*

begin
sepref-thm *opr-fold-impl1* **is** *RETURN o opr-fold-impl* :: $assn^d \rightarrow_a assn$
unfolding *opr-fold-impl-def*
supply [[*goals-limit* = *1*]]
by *sepref*
end

concrete-definition (**in** −) *mtx-pointwise-unnop-fold-impl1* **uses** *mtx-pointwise-unop-gen-impl.opr-fold-im*
prepare-code-thms (**in** −) *mtx-pointwise-unnop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]: (*mtx-pointwise-unnop-fold-impl1 N M get-impl set-impl fi*, *RETURN* ∘ *PR-CONST* (*mtx-pointwise-unop f*)) ∈ $assn^d \rightarrow_a assn$
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre'*[*OF - mtx-pointwise-unnop-fold-impl1.refine*[*OF this-loc*,*FCOMP opr-fold-impl-refine*]])
by (*simp add*: *assn-range*)

end

Binary Pointwise

definition *mtx-pointwise-binop* :: (*'a* ⇒ *'a* ⇒ *'a*) ⇒ *'a mtx* ⇒ *'a mtx* ⇒ *'a mtx*
where
mtx-pointwise-binop f m n ≡ λ(*i*,*j*). *f* (*m*(*i*,*j*)) (*n*(*i*,*j*))
context fixes *f* :: *'a* ⇒ *'a* ⇒ *'a* **begin**
sepref-register *PR-CONST* (*mtx-pointwise-binop f*) :: *'a i-mtx* ⇒ *'a i-mtx* ⇒ *'a i-mtx*
lemma [*def-pat-rules*]: *mtx-pointwise-binop$f* ≡ *UNPROTECT* (*mtx-pointwise-binop f*) **by** *simp*
end

locale *mtx-pointwise-binop-loc* =
fixes *N* :: *nat* **and** *M* :: *nat*
fixes *f* :: *'a*::{*zero*} ⇒ *'a* ⇒ *'a*
assumes *pres-zero*[*simp*]: *f 0 0* = *0*
begin
definition *opr-fold-impl m n* ≡ *fold* (λ*i*. *fold* (λ*j m*. *m*((*i*,*j*) := *f* (*m*(*i*,*j*)) (*n*(*i*,*j*)))) [*0*..<*M*]) [*0*..<*N*] *m*

lemma *opr-fold-impl-eq*:
assumes *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}
assumes *mtx-nonzero n* ⊆ {*0*..<*N*}×{*0*..<*M*}
shows *mtx-pointwise-binop f m n* = *opr-fold-impl m n*
apply (*rule ext*)
unfolding *opr-fold-impl-def*
apply (*simp add*: *fold-fold-prod-conv*)
apply (*subst pointwise-fun-fold*)
apply (*auto simp*: *mtx-pointwise-binop-def distinct-product*) [*3*]
apply *clarsimp*

subgoal for *a b*
apply (*cases a b m rule*: *mtx-nonzero-cases*; *cases a b n rule*: *mtx-nonzero-cases*)
using *assms*
apply (*auto simp*: *mtx-pointwise-binop-def*)
done
done

lemma *opr-fold-impl-refine*: (*uncurry opr-fold-impl*, *uncurry* (*mtx-pointwise-binop f*)) $\in [\lambda(m,n).\ mtx\text{-}nonzero\ m \subseteq \{0..<N\}\times\{0..<M\} \wedge mtx\text{-}nonzero\ n \subseteq \{0..<N\}\times\{0..<M\}]_f\ Id\times_r Id \rightarrow Id$
apply (*rule frefI*)
using *opr-fold-impl-eq*
by *auto*

end

locale *mtx-pointwise-binop-gen-impl* = *mtx-pointwise-binop-loc* +
fixes $assn :: 'a\ mtx \Rightarrow 'i \Rightarrow assn$
fixes $A :: 'a \Rightarrow 'ai \Rightarrow assn$
fixes $get\text{-}impl :: 'i \Rightarrow nat\times nat \Rightarrow 'ai\ Heap$
fixes $set\text{-}impl :: 'i \Rightarrow nat\times nat \Rightarrow 'ai \Rightarrow 'i\ Heap$
fixes $fi :: 'ai \Rightarrow 'ai \Rightarrow 'ai\ Heap$
assumes *assn-range*: $rdomp\ assn\ m \Longrightarrow mtx\text{-}nonzero\ m \subseteq \{0..<N\}\times\{0..<M\}$
assumes *get-impl-hnr*:
$(uncurry\ get\text{-}impl, uncurry\ (RETURN\ oo\ op\text{-}mtx\text{-}get)) \in assn^k *_a (prod\text{-}assn\ (nbn\text{-}assn\ N)\ (nbn\text{-}assn\ M))^k \rightarrow_a A$
assumes *set-impl-hnr*:
$(uncurry2\ set\text{-}impl, uncurry2\ (RETURN\ ooo\ op\text{-}mtx\text{-}set)) \in assn^d *_a (prod\text{-}assn\ (nbn\text{-}assn\ N)\ (nbn\text{-}assn\ M))^k *_a A^k \rightarrow_a assn$
assumes *fi-hnr*:
$(uncurry\ fi, uncurry\ (RETURN\ oo\ f)) \in A^k *_a A^k \rightarrow_a A$
begin

lemma *this-loc*: *mtx-pointwise-binop-gen-impl N M f assn A get-impl set-impl fi*
by *unfold-locales*

context
notes [[*sepref-register-adhoc f N M*]]
notes [*intf-of-assn*] = *intf-of-assnI*[**where** *R*=*assn* **and** *'a*=*'a i-mtx*]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *get-impl-hnr set-impl-hnr fi-hnr*
begin
sepref-thm *opr-fold-impl1* **is** *uncurry* (*RETURN oo opr-fold-impl*) :: $assn^d *_a assn^k \rightarrow_a assn$
unfolding *opr-fold-impl-def*[*abs-def*]
by *sepref*

end

concrete-definition (**in** −) *mtx-pointwise-binop-fold-impl1*
uses *mtx-pointwise-binop-gen-impl.opr-fold-impl1.refine-raw* **is** (*uncurry ?f*,-)∈-
prepare-code-thms (**in** −) *mtx-pointwise-binop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]: (*uncurry* (*mtx-pointwise-binop-fold-impl1 N M get-impl set-impl fi*), *uncurry* (*RETURN oo PR-CONST* (*mtx-pointwise-binop f*))) $\in assn^d *_a assn^k \rightarrow_a assn$
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre′*[*OF - mtx-pointwise-binop-fold-impl1.refine*[*OF this-loc*,*FCOMP opr-fold-impl-refine*]])
apply (*auto dest*: *assn-range*)
done

end

Compare Pointwise

definition *mtx-pointwise-cmpop* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ ′*a mtx* ⇒ ′*a mtx* ⇒ *bool* **where**
mtx-pointwise-cmpop f g m n ≡ (∀ *i j*. *f* (*m*(*i*,*j*)) (*n*(*i*,*j*))) ∧ (∃ *i j*. *g* (*m*(*i*,*j*)) (*n*(*i*,*j*)))
context fixes *f g* :: ′*a* ⇒ ′*a* ⇒ *bool* **begin**
sepref-register *PR-CONST* (*mtx-pointwise-cmpop f g*) :: ′*a i-mtx* ⇒ ′*a i-mtx* ⇒ *bool*
lemma [*def-pat-rules*]: *mtx-pointwise-cmpop*$*f*$*g* ≡ *UNPROTECT* (*mtx-pointwise-cmpop f g*) **by** *simp*
end

lemma *mtx-nonzeroD*:
⟦¬*i*<*N*; *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}⟧ ⟹ *m*(*i*,*j*) = *0*
⟦¬*j*<*M*; *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}⟧ ⟹ *m*(*i*,*j*) = *0*
by (*auto simp*: *mtx-nonzero-def*)

locale *mtx-pointwise-cmpop-loc* =
fixes *N* :: *nat* **and** *M* :: *nat*
fixes *f g* :: ′*a*::{*zero*} ⇒ ′*a* ⇒ *bool*
assumes *pres-zero*[*simp*]: *f 0 0* = *True g 0 0* = *False*
begin
definition *opr-fold-impl m n* ≡ *do* {
s ← *nfoldli* (*List.product* [*0*..<*N*] [*0*..<*M*]) (λ*s*. *s*≠*2*) (λ(*i*,*j*) *s*. *do* {
if f (*m*(*i*,*j*)) (*n*(*i*,*j*)) *then*
if s=*0 then*
if g (*m*(*i*,*j*)) (*n*(*i*,*j*)) *then RETURN 1 else RETURN s*
else
RETURN s

else RETURN 2
}) (*0::nat*);
RETURN (*s=1*)
}

lemma *opr-fold-impl-eq*:
assumes *mtx-nonzero m* ⊆ {*0..<N*}×{*0..<M*}
assumes *mtx-nonzero n* ⊆ {*0..<N*}×{*0..<M*}
shows *opr-fold-impl m n* ≤ *RETURN* (*mtx-pointwise-cmpop f g m n*)
proof −
have (∀ *i*<*N*. ∀ *j*<*M*. *f* (*m* (*i*, *j*)) (*n* (*i*, *j*))) ⟹ *f* (*m* (*i*, *j*)) (*n* (*i*, *j*)) **for** *i j*
apply (*cases i*<*N*; *cases j*<*M*)
using *assms* **by** (*auto simp*: *mtx-nonzeroD*)
moreover have *g* (*m* (*i*, *j*)) (*n* (*i*, *j*)) ⟹ (∃ *i*<*N*. ∃ *j*<*M*. *g* (*m* (*i*, *j*)) (*n* (*i*, *j*))) **for** *i j*
apply (*cases i*<*N*; *cases j*<*M*)
using *assms* **by** (*auto simp*: *mtx-nonzeroD*)
ultimately have *EQ*: *mtx-pointwise-cmpop f g m n*
⟷ (∀ *i*<*N*. ∀ *j*<*M*. *f* (*m*(*i*,*j*)) (*n*(*i*,*j*))) ∧ (∃ *i*<*N*. ∃ *j*<*M*. *g* (*m*(*i*,*j*)) (*n*(*i*,*j*)))
unfolding *mtx-pointwise-cmpop-def* **by** *meson*

have *aux*: *List.product* [*0..<N*] [*0..<M*] = *l1* @ (*i*, *j*) # *l2* ⟹ *i*<*N* ∧ *j*<*M* **for** *l1 i j l2*
proof −
assume *List.product* [*0..<N*] [*0..<M*] = *l1* @ (*i*, *j*) # *l2*
hence (*i*,*j*)∈*set* (*List.product* [*0..<N*] [*0..<M*]) **by** *simp*
thus *?thesis* **by** *simp*
qed

show *?thesis*
unfolding *opr-fold-impl-def*
apply (*refine-vcg*
nfoldli-rule[**where** *I*=λ*l1* - *s*.
if s=2 then ∃ *i*<*N*. ∃ *j*<*M*. ¬*f* (*m*(*i*,*j*)) (*n*(*i*,*j*))
else (
(*s=0* ∨ *s=1*) ∧
(∀ (*i*,*j*)∈*set l1*. *f* (*m*(*i*,*j*)) (*n*(*i*,*j*))) ∧
(*s=1* ⟷ (∃ (*i*,*j*)∈*set l1*. *g* (*m*(*i*,*j*)) (*n*(*i*,*j*))))
)]
)
apply (*vc-solve dest*: *aux solve*: *asm-rl simp*: *EQ*) [*6*]
apply (*fastforce simp*: *EQ*)
done
qed

lemma *opr-fold-impl-refine*:
(*uncurry opr-fold-impl*, *uncurry* (*RETURN oo mtx-pointwise-cmpop f g*)) ∈ [λ(*m*,*n*). *mtx-nonzero m* ⊆ {*0..<N*}×{*0..<M*} ∧ *mtx-nonzero n* ⊆ {*0..<N*}×{*0..<M*}]$_f$

$Id \times_r Id \rightarrow \langle bool\text{-}rel \rangle nres\text{-}rel$
apply (*rule frefI*)
using *opr-fold-impl-eq*
by (*auto intro*: *nres-relI*)

end

locale *mtx-pointwise-cmpop-gen-impl* = *mtx-pointwise-cmpop-loc* +
fixes *assn* :: *'a mtx* ⇒ *'i* ⇒ *assn*
fixes *A* :: *'a* ⇒ *'ai* ⇒ *assn*
fixes *get-impl* :: *'i* ⇒ *nat×nat* ⇒ *'ai Heap*
fixes *fi* :: *'ai* ⇒ *'ai* ⇒ *bool Heap*
fixes *gi* :: *'ai* ⇒ *'ai* ⇒ *bool Heap*
assumes *assn-range*: *rdomp assn m* ⟹ *mtx-nonzero m* ⊆ {*0..<N*}×{*0..<M*}
assumes *get-impl-hnr*:
(*uncurry get-impl*,*uncurry* (*RETURN oo op-mtx-get*)) ∈ $assn^k *_a$ (*prod-assn* (*nbn-assn N*) (*nbn-assn M*))$^k \rightarrow_a$ *A*
assumes *fi-hnr*:
(*uncurry fi*,*uncurry* (*RETURN oo f*)) ∈ $A^k *_a A^k \rightarrow_a$ *bool-assn*
assumes *gi-hnr*:
(*uncurry gi*,*uncurry* (*RETURN oo g*)) ∈ $A^k *_a A^k \rightarrow_a$ *bool-assn*
begin

lemma *this-loc*: *mtx-pointwise-cmpop-gen-impl N M f g assn A get-impl fi gi*
by *unfold-locales*

context
notes [[*sepref-register-adhoc f g N M*]]
notes [*intf-of-assn*] = *intf-of-assnI*[**where** *R*=*assn* **and** *'a*=*'a i-mtx*]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *get-impl-hnr fi-hnr gi-hnr*
begin
sepref-thm *opr-fold-impl1* **is** *uncurry opr-fold-impl* :: $assn^d *_a assn^k \rightarrow_a$ *bool-assn*
unfolding *opr-fold-impl-def*[*abs-def*] *nfoldli-nfoldli-prod-conv*[*symmetric*]
by *sepref*

end

concrete-definition (**in** −) *mtx-pointwise-cmpop-fold-impl1*
uses *mtx-pointwise-cmpop-gen-impl.opr-fold-impl1.refine-raw* **is** (*uncurry* *?f*,-)∈-
prepare-code-thms (**in** −) *mtx-pointwise-cmpop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]: (*uncurry* (*mtx-pointwise-cmpop-fold-impl1 N M get-impl fi gi*), *uncurry* (*RETURN oo PR-CONST* (*mtx-pointwise-cmpop f g*))) ∈ $assn^d *_a assn^k \rightarrow_a$ *bool-assn*
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre'*[*OF* - *mtx-pointwise-cmpop-fold-impl1.refine*[*OF*

this-loc,*FCOMP opr-fold-impl-refine*]])
apply (*auto dest*: *assn-range*)
done

end

end

3.14 Matrices by Array (Row-Major)

theory *IICF-Array-Matrix*
imports *../Intf/IICF-Matrix Separation-Logic-Imperative-HOL.Array-Blit*
begin

definition *is-amtx N M c mtx* ≡ $\exists_A l.\ mtx \mapsto_a l * \uparrow($
length l = *N*∗*M*
∧ (∀ *i*<*N*. ∀ *j*<*M*. *l*!(*i*∗*M*+*j*) = *c* (*i*,*j*))
∧ (∀ *i j*. (*i*≥*N* ∨ *j*≥*M*) ⟶ *c* (*i*,*j*) = *0*))

lemma *is-amtx-precise*[*safe-constraint-rules*]: *precise* (*is-amtx N M*)
apply *rule*
unfolding *is-amtx-def*
apply *clarsimp*
apply *prec-extract-eqs*
apply (*rule ext*)
apply (*rename-tac x*)
apply (*case-tac x*; *simp*)
apply (*rename-tac i j*)
apply (*case-tac i*<*N*; *case-tac j*<*M*; *simp*)
done

lemma *is-amtx-bounded*:
shows *rdomp* (*is-amtx N M*) *m* ⟹ *mtx-nonzero m* ⊆ {*0*..<*N*}×{*0*..<*M*}
unfolding *rdomp-def*
apply (*clarsimp simp*: *mtx-nonzero-def is-amtx-def*)
by (*meson not-less*)

definition *mtx-tabulate N M c* ≡ *do* {
m ← *Array.new* (*N*∗*M*) *0*;
(-,-,*m*) ← *imp-for′ 0* (*N*∗*M*) (λ*k* (*i*,*j*,*m*). *do* {
Array.upd k (*c* (*i*,*j*)) *m*;
let j=*j*+*1*;
if j<*M then return* (*i*,*j*,*m*)
else return (*i*+*1*,*0*,*m*)
}) (*0*,*0*,*m*);
return m

}

definition *amtx-copy* ≡ *array-copy*

definition *amtx-dflt N M v* ≡ *Array.make* (*N*∗*M*) (λ*i*. *v*)

definition *mtx-get M mtx e* ≡ *Array.nth mtx* (*fst e* ∗ *M* + *snd e*)
definition *mtx-set M mtx e v* ≡ *Array.upd* (*fst e* ∗ *M* + *snd e*) *v mtx*

lemma *mtx-idx-valid*[*simp*]: ⟦*i* < (*N*::*nat*); *j* < *M*⟧ ⟹ *i* ∗ *M* + *j* < *N* ∗ *M*
  by (*rule mlex-bound*)

lemma *mtx-idx-unique-conv*[*simp*]:
  fixes *M* :: *nat*
  assumes *j*<*M* *j*′<*M*
  shows (*i* ∗ *M* + *j* = *i*′ ∗ *M* + *j*′) ⟷ (*i*=*i*′ ∧ *j*=*j*′)
  using *assms*
  apply *auto*
  subgoal
    by (*metis add-right-cancel div-if div-mult-self3 linorder-neqE-nat not-less0*)
  subgoal
    using ‹⟦*j* < *M*; *j*′ < *M*; *i* ∗ *M* + *j* = *i*′ ∗ *M* + *j*′⟧ ⟹ *i* = *i*′› **by** *auto*
  done

lemma *mtx-tabulate-rl*[*sep-heap-rules*]:
  assumes *NONZ*: *mtx-nonzero c* ⊆ {*0*..<*N*}×{*0*..<*M*}
  shows <*emp*> *mtx-tabulate N M c* <*IICF-Array-Matrix.is-amtx N M c*>
proof (*cases M*=*0*)
  case *True* **thus** *?thesis*
    unfolding *mtx-tabulate-def*
    using *mtx-nonzeroD*[*OF - NONZ*]
    by (*sep-auto simp*: *is-amtx-def*)
next
  case *False* **hence** *M-POS*: *0*<*M* **by** *auto*
  show *?thesis*
    unfolding *mtx-tabulate-def*
    apply (*sep-auto*
      decon:
        imp-for′*-rule*[**where**
          I=λ*k* (*i*,*j*,*mi*). ∃$_A$*m*. *mi* ↦$_a$ *m*
          ∗ ↑(*k*=*i*∗*M*+*j* ∧ *j*<*M* ∧ *k*≤*N*∗*M* ∧ *length m* = *N*∗*M*)
          ∗ ↑(∀ *i*′<*i*. ∀ *j*<*M*. *m*!(*i*′∗*M*+*j*) = *c* (*i*′,*j*))
          ∗ ↑(∀ *j*′<*j*. *m*!(*i*∗*M*+*j*′) = *c* (*i*,*j*′))
        ]
      simp: *nth-list-update M-POS dest*: *Suc-lessI*
)

unfolding *is-amtx-def*
using *mtx-nonzeroD*[*OF* - *NONZ*]
apply *sep-auto*
by (*metis add.right-neutral M-POS mtx-idx-unique-conv*)
qed

lemma *mtx-copy-rl*[*sep-heap-rules*]:
<*is-amtx N M c mtx*> *amtx-copy mtx* <λ*r*. *is-amtx N M c mtx* ∗ *is-amtx N M c r*>
by (*sep-auto simp*: *amtx-copy-def is-amtx-def*)

definition *PRES-ZERO-UNIQUE A* ≡ (*A*‘‘{*0*}={*0*} ∧ A^{-1}‘‘{*0*} = {*0*})
lemma *IS-ID-imp-PRES-ZERO-UNIQUE*[*constraint-rules*]: *IS-ID A* ⟹ *PRES-ZERO-UNIQUE A*
unfolding *IS-ID-def PRES-ZERO-UNIQUE-def* **by** *auto*

definition *op-amtx-dfltNxM* :: *nat* ⇒ *nat* ⇒ ′*a*::*zero* ⇒ *nat*×*nat*⇒′*a* **where**
[*simp*]: *op-amtx-dfltNxM N M v* ≡ λ(*i*,*j*). *if i*<*N* ∧ *j*<*M then v else 0*
context fixes *N M*::*nat* **begin**
sepref-decl-op (*no-def*) *op-amtx-dfltNxM*: *op-amtx-dfltNxM N M* :: *A* → ⟨*A*⟩*mtx-rel*
where *CONSTRAINT PRES-ZERO-UNIQUE A*
apply (*rule fref-ncI*) **unfolding** *op-amtx-dfltNxM-def*[*abs-def*] *mtx-rel-def*
apply *parametricity*
by (*auto simp add*: *PRES-ZERO-UNIQUE-def*)
end

lemma *mtx-dflt-rl*[*sep-heap-rules*]: <*emp*> *amtx-dflt N M k* <*is-amtx N M* (*op-amtx-dfltNxM N M k*)>
by (*sep-auto simp*: *amtx-dflt-def is-amtx-def*)

lemma *mtx-get-rl*[*sep-heap-rules*]: ⟦*i*<*N*; *j*<*M* ⟧ ⟹ <*is-amtx N M c mtx*> *mtx-get M mtx* (*i*,*j*) <λ*r*. *is-amtx N M c mtx* ∗ ↑(*r* = *c* (*i*,*j*))>
by (*sep-auto simp*: *mtx-get-def is-amtx-def*)

lemma *mtx-set-rl*[*sep-heap-rules*]: ⟦*i*<*N*; *j*<*M* ⟧
⟹ <*is-amtx N M c mtx*> *mtx-set M mtx* (*i*,*j*) *v* <λ*r*. *is-amtx N M* (*c*((*i*,*j*) := *v*)) *r*>
by (*sep-auto simp*: *mtx-set-def is-amtx-def nth-list-update*)

definition *amtx-assn N M A* ≡ *hr-comp* (*is-amtx N M*) (⟨*the-pure A*⟩*mtx-rel*)
lemmas [*fcomp-norm-unfold*] = *amtx-assn-def*[*symmetric*]
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure amtx-assn N M A* **for** *N M A*]

lemma [*intf-of-assn*]: *intf-of-assn A TYPE*(′*a*) ⟹ *intf-of-assn* (*amtx-assn N M A*) *TYPE*(′*a i-mtx*)
by *simp*

abbreviation *asmtx-assn N A* ≡ *amtx-assn N N A*

lemma *mtx-rel-pres-zero*:
assumes *PRES-ZERO-UNIQUE A*
assumes $(m,m') \in \langle A \rangle$*mtx-rel*
shows $m\ ij = 0 \longleftrightarrow m'\ ij = 0$
using *assms*
apply1 (*clarsimp simp*: *IS-PURE-def PRES-ZERO-UNIQUE-def is-pure-conv mtx-rel-def*)
apply (*drule fun-relD*) **applyS** (*rule IdI*[*of ij*]) **applyS** *auto*
done

lemma *amtx-assn-bounded*:
assumes *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
shows *rdomp* (*amtx-assn N M A*) $m \Longrightarrow$ *mtx-nonzero* $m \subseteq \{0..<N\} \times \{0..<M\}$
apply (*clarsimp simp*: *mtx-nonzero-def amtx-assn-def rdomp-hrcomp-conv*)
apply (*drule is-amtx-bounded*)
using *assms*
by (*fastforce simp*: *IS-PURE-def is-pure-conv mtx-rel-pres-zero*[*symmetric*] *mtx-nonzero-def*)

lemma *mtx-tabulate-aref*:
(*mtx-tabulate N M*, *RETURN o op-mtx-new*)
$\in [\lambda c.$ *mtx-nonzero* $c \subseteq \{0..<N\} \times \{0..<M\}]_a$ *id-assn*$^k \rightarrow$ *IICF-Array-Matrix.is-amtx N M*
by *sepref-to-hoare sep-auto*

lemma *mtx-copy-aref*:
(*amtx-copy*, *RETURN o op-mtx-copy*) $\in$ (*is-amtx N M*)$^k \rightarrow_a$ *is-amtx N M*
apply *rule* **apply** *rule*
apply (*sep-auto simp*: *pure-def*)
done

lemma *mtx-nonzero-bid-eq*:
assumes $R \subseteq Id$
assumes $(a,\ a') \in Id \rightarrow R$
shows *mtx-nonzero* $a =$ *mtx-nonzero* a'
using *assms*
apply (*clarsimp simp*: *mtx-nonzero-def*)
apply (*metis fun-relE2 pair-in-Id-conv subsetCE*)
done

lemma *mtx-nonzero-zu-eq*:
assumes *PRES-ZERO-UNIQUE R*
assumes $(a,\ a') \in Id \rightarrow R$
shows *mtx-nonzero* $a =$ *mtx-nonzero* a'
using *assms*
apply (*clarsimp simp*: *mtx-nonzero-def PRES-ZERO-UNIQUE-def*)
by (*metis* (*no-types*, *opaque-lifting*) *IdI Image-singleton-iff converse-iff singletonD tagged-fun-relD-none*)

lemma *op-mtx-new-fref'*:
CONSTRAINT PRES-ZERO-UNIQUE A $\Longrightarrow$ (*RETURN* $\circ$ *op-mtx-new*, *RETURN* $\circ$ *op-mtx-new*) $\in$ (*nat-rel* $\times_r$ *nat-rel* $\rightarrow$ *A*) $\rightarrow_f$ $\langle\langle A\rangle$*mtx-rel*$\rangle$*nres-rel*
by (*rule op-mtx-new.fref*)

sepref-decl-impl (*no-register*) *amtx-new-by-tab*: *mtx-tabulate-aref* **uses** *op-mtx-new-fref'*
by (*auto simp*: *mtx-nonzero-zu-eq*)

sepref-decl-impl *amtx-copy*: *mtx-copy-aref* **.**

definition [*simp*]: *op-amtx-new* (*N*::*nat*) (*M*::*nat*) $\equiv$ *op-mtx-new*
lemma *amtx-fold-custom-new*:
op-mtx-new $\equiv$ *op-amtx-new N M*
mop-mtx-new $\equiv$ λc. *RETURN* (*op-amtx-new N M c*)
by (*auto simp*: *mop-mtx-new-alt*[*abs-def*])

context fixes *N M* :: *nat* **begin**
sepref-register *PR-CONST* (*op-amtx-new N M*) :: (*nat* $\times$ *nat* $\Rightarrow$ *'a*) $\Rightarrow$ *'a i-mtx*
end

lemma *amtx-new-hnr*[*sepref-fr-rules*]:
fixes *A* :: *'a*::*zero* $\Rightarrow$ *'b*::{*zero*,*heap*} $\Rightarrow$ *assn*
shows *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A* $\Longrightarrow$
(*mtx-tabulate N M*, (*RETURN* $\circ$ *PR-CONST* (*op-amtx-new N M*)))
$\in$ [λx. *mtx-nonzero x* $\subseteq$ {*0*..<*N*} $\times$ {*0*..<*M*}]$_a$ (*pure* (*nat-rel* $\times_r$ *nat-rel* $\rightarrow$ *the-pure A*))^k $\rightarrow$ *amtx-assn N M A*
using *amtx-new-by-tab-hnr*[*of A N M*] **by** *simp*

lemma [*def-pat-rules*]: *op-amtx-new*\$*N*\$*M* $\equiv$ *UNPROTECT* (*op-amtx-new N M*) **by** *simp*

context fixes *N M* :: *nat* **notes** [*param*] = *IdI*[*of N*] *IdI*[*of M*] **begin**

lemma *mtx-dflt-aref*:
(*amtx-dflt N M*, *RETURN o PR-CONST* (*op-amtx-dfltNxM N M*)) $\in$ *id-assn*^{k} $\rightarrow_a$ *is-amtx N M*
apply *rule* **apply** *rule*
apply (*sep-auto simp*: *pure-def*)
done
sepref-decl-impl *amtx-dflt*: *mtx-dflt-aref* **.**

lemma *amtx-get-aref*:
(*uncurry* (*mtx-get M*), *uncurry* (*RETURN oo op-mtx-get*)) $\in$ [λ(-,(*i*,*j*)). *i*<*N* $\wedge$ *j*<*M*]$_a$ (*is-amtx N M*)^k $*_a$ (*prod-assn nat-assn nat-assn*)^k $\rightarrow$ *id-assn*

apply *rule* **apply** *rule*
apply (*sep-auto simp*: *pure-def*)
done
sepref-decl-impl *amtx-get*: *amtx-get-aref* **.**

lemma *amtx-set-aref*: (*uncurry2* (*mtx-set M*), *uncurry2* (*RETURN ooo op-mtx-set*))

$\in$ [λ((-,(*i*,*j*)),-). *i*<*N* $\wedge$ *j*<*M*]$_a$ (*is-amtx N M*)^d $*_a$ (*prod-assn nat-assn nat-assn*)^k $*_a$ *id-assn*^{k} $\rightarrow$ *is-amtx N M*
apply *rule* **apply** (*rule hn-refine-preI*) **apply** *rule*
apply (*sep-auto simp*: *pure-def hn-ctxt-def invalid-assn-def*)
done

sepref-decl-impl *amtx-set*: *amtx-set-aref* **.**

lemma *amtx-get-aref'*:
(*uncurry* (*mtx-get M*), *uncurry* (*RETURN oo op-mtx-get*)) $\in$ (*is-amtx N M*)^k $*_a$ (*prod-assn* (*pure* (*nbn-rel N*)) (*pure* (*nbn-rel M*)))^k $\rightarrow_a$ *id-assn*
apply *rule* **apply** *rule*
apply (*sep-auto simp*: *pure-def IS-PURE-def IS-ID-def*)
done

sepref-decl-impl *amtx-get'*: *amtx-get-aref'* **.**

lemma *amtx-set-aref'*: (*uncurry2* (*mtx-set M*), *uncurry2* (*RETURN ooo op-mtx-set*))

$\in$ (*is-amtx N M*)^d $*_a$ (*prod-assn* (*pure* (*nbn-rel N*)) (*pure* (*nbn-rel M*)))^k $*_a$ *id-assn*^{k} $\rightarrow_a$ *is-amtx N M*
apply *rule* **apply** (*rule hn-refine-preI*) **apply** *rule*
apply (*sep-auto simp*: *pure-def hn-ctxt-def invalid-assn-def IS-PURE-def IS-ID-def*)
done

sepref-decl-impl *amtx-set'*: *amtx-set-aref'* **.**

end

3.14.1 Pointwise Operations

context
fixes *M N* :: *nat*
begin
sepref-decl-op *amtx-lin-get*: λ*f i*. *op-mtx-get f* (*i div M*, *i mod M*) :: $\langle A\rangle$*mtx-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *A*
unfolding *op-mtx-get-def mtx-rel-def*
by (*rule frefI*) (*parametricity*; *simp*)

sepref-decl-op *amtx-lin-set*: λ*f i x*. *op-mtx-set f* (*i div M*, *i mod M*) *x* :: $\langle A\rangle$*mtx-rel* $\rightarrow$ *nat-rel* $\rightarrow$ *A* $\rightarrow$ $\langle A\rangle$*mtx-rel*

unfolding *op-mtx-set-def mtx-rel-def*
apply (*rule frefI*) **apply** *parametricity* **by** *simp-all*

lemma *op-amtx-lin-get-aref*: (*uncurry Array.nth*, *uncurry* (*RETURN oo PR-CONST op-amtx-lin-get*)) $\in [\lambda(\text{-},i).\ i<N*M]_a$ (*is-amtx N M*)$^k *_a$ *nat-assn*$^k \rightarrow$ *id-assn*
apply *sepref-to-hoare*
unfolding *is-amtx-def*
apply *sep-auto*
apply (*metis mult.commute div-eq-0-iff div-mult2-eq div-mult-mod-eq mod-less-divisor mult-is-0 not-less0*)
done

sepref-decl-impl *amtx-lin-get*: *op-amtx-lin-get-aref* **by** *auto*

lemma *op-amtx-lin-set-aref*: (*uncurry2* ($\lambda m\ i\ x.$ *Array.upd i x m*), *uncurry2* (*RETURN ooo PR-CONST op-amtx-lin-set*)) $\in [\lambda((\text{-},i),\text{-}).\ i<N*M]_a$ (*is-amtx N M*)$^d *_a$ *nat-assn*$^k *_a$ *id-assn*$^k \rightarrow$ *is-amtx N M*
proof −
have [*simp*]: $i < N * M \Longrightarrow \neg(M \leq i\ mod\ M)$ **for** *i*
by (*cases* $N = 0 \vee M = 0$) (*auto simp add*: *not-le*)
have [*simp*]: $i < N * M \Longrightarrow \neg(N \leq i\ div\ M)$ **for** *i*
apply (*cases* $N = 0 \vee M = 0$)
apply (*auto simp add*: *not-le*)
apply (*metis mult.commute div-eq-0-iff div-mult2-eq neq0-conv*)
done
show *?thesis*
apply *sepref-to-hoare*
unfolding *is-amtx-def*
by (*sep-auto simp*: *nth-list-update*)
qed

sepref-decl-impl *amtx-lin-set*: *op-amtx-lin-set-aref* **by** *auto*
end

lemma *amtx-fold-lin-get*: $m\ (i\ div\ M,\ i\ mod\ M) =$ *op-amtx-lin-get M m i* **by** *simp*
lemma *amtx-fold-lin-set*: $m\ ((i\ div\ M,\ i\ mod\ M) := x) =$ *op-amtx-lin-set M m i x* **by** *simp*

locale *amtx-pointwise-unop-impl* = *mtx-pointwise-unop-loc* +
fixes *A* :: $'a \Rightarrow 'ai$::{*zero*,*heap*} $\Rightarrow$ *assn*
fixes *fi* :: $nat \times nat \Rightarrow 'ai \Rightarrow 'ai$ *Heap*
assumes *fi-hnr*:
(*uncurry fi*,*uncurry* (*RETURN oo f*)) $\in$ (*prod-assn nat-assn nat-assn*)$^k *_a A^k \rightarrow_a A$
begin

lemma *this-loc*: *amtx-pointwise-unop-impl N M f A fi* **by** *unfold-locales*

context
assumes *PURE*: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
begin
context
notes [[*sepref-register-adhoc f N M*]]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *fi-hnr*
notes [*safe-constraint-rules*] = *PURE*
notes [*simp*] = *algebra-simps*
begin
sepref-thm *opr-fold-impl1* **is** *RETURN o opr-fold-impl* :: (*amtx-assn N M A*)^d $\rightarrow_a$ *amtx-assn N M A*
unfolding *opr-fold-impl-def fold-prod-divmod-conv'*
apply (*rewrite amtx-fold-lin-set*)
apply (*rewrite* **in** *f* - ¤ *amtx-fold-lin-get*)
by *sepref*
end
end
concrete-definition (**in** −) *amtx-pointwise-unnop-fold-impl1* **uses** *amtx-pointwise-unop-impl.opr-fold-impl*
prepare-code-thms (**in** −) *amtx-pointwise-unnop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]:
assumes *PURE*: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
shows (*amtx-pointwise-unnop-fold-impl1 N M fi*, *RETURN* ∘ *PR-CONST* (*mtx-pointwise-unop f*)) ∈ (*amtx-assn N M A*)^d $\rightarrow_a$ *amtx-assn N M A*
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre'*[*OF* - *amtx-pointwise-unnop-fold-impl1.refine*[*OF this-loc PURE*,*FCOMP opr-fold-impl-refine*]])
by (*simp add*: *amtx-assn-bounded*[*OF PURE*])
end

locale *amtx-pointwise-binop-impl* = *mtx-pointwise-binop-loc* +
fixes *A* :: *'a* ⇒ *'ai*::{*zero*,*heap*} ⇒ *assn*
fixes *fi* :: *'ai* ⇒ *'ai* ⇒ *'ai Heap*
assumes *fi-hnr*: (*uncurry fi*,*uncurry* (*RETURN oo f*)) ∈ $A^k *_a A^k \rightarrow_a A$
begin

lemma *this-loc*: *amtx-pointwise-binop-impl f A fi*
by *unfold-locales*

context
notes [[*sepref-register-adhoc f N M*]]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *fi-hnr*
assumes *PURE*[*safe-constraint-rules*]: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*

notes [*simp*] = *algebra-simps*
begin
sepref-thm *opr-fold-impl1* **is** *uncurry* (*RETURN oo opr-fold-impl*) :: (*amtx-assn N M A*)^d$*_a$(*amtx-assn N M A*)^k $\rightarrow_a$ *amtx-assn N M A*
unfolding *opr-fold-impl-def*[*abs-def*] *fold-prod-divmod-conv′*
apply (*rewrite amtx-fold-lin-set*)
apply (*rewrite* **in** *f* ¤ - *amtx-fold-lin-get*)
apply (*rewrite* **in** *f* - ¤ *amtx-fold-lin-get*)
by *sepref*

end

concrete-definition (**in** −) *amtx-pointwise-binop-fold-impl1* **for** *fi N M*
uses *amtx-pointwise-binop-impl.opr-fold-impl1.refine-raw* **is** (*uncurry ?f*,-)∈-
prepare-code-thms (**in** −) *amtx-pointwise-binop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]:
assumes *PURE*: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
shows (*uncurry* (*amtx-pointwise-binop-fold-impl1 fi N M*), *uncurry* (*RETURN oo PR-CONST* (*mtx-pointwise-binop f*))) ∈ (*amtx-assn N M A*)^d $*_a$ (*amtx-assn N M A*)^k $\rightarrow_a$ *amtx-assn N M A*
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre′*[*OF* - *amtx-pointwise-binop-fold-impl1.refine*[*OF this-loc PURE*,*FCOMP opr-fold-impl-refine*]])
apply (*auto dest*: *amtx-assn-bounded*[*OF PURE*])
done

end

locale *amtx-pointwise-cmpop-impl* = *mtx-pointwise-cmpop-loc* +
fixes *A* :: *′a* ⇒ *′ai*::{*zero*,*heap*} ⇒ *assn*
fixes *fi* :: *′ai* ⇒ *′ai* ⇒ *bool Heap*
fixes *gi* :: *′ai* ⇒ *′ai* ⇒ *bool Heap*
assumes *fi-hnr*:
(*uncurry fi*,*uncurry* (*RETURN oo f*)) ∈ A^k $*_a$ A^k $\rightarrow_a$ *bool-assn*
assumes *gi-hnr*:
(*uncurry gi*,*uncurry* (*RETURN oo g*)) ∈ A^k $*_a$ A^k $\rightarrow_a$ *bool-assn*
begin

lemma *this-loc*: *amtx-pointwise-cmpop-impl f g A fi gi*
by *unfold-locales*

context
notes [[*sepref-register-adhoc f g N M*]]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
notes [*sepref-fr-rules*] = *fi-hnr gi-hnr*
assumes *PURE*[*safe-constraint-rules*]: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
begin

sepref-thm *opr-fold-impl1* **is** *uncurry opr-fold-impl* :: ($amtx\text{-}assn\ N\ M\ A)^d *_a (amtx\text{-}assn\ N\ M\ A)^k \rightarrow_a bool\text{-}assn$
unfolding *opr-fold-impl-def*[*abs-def*] *nfoldli-prod-divmod-conv*
apply (*rewrite* **in** *f* ⌑ - *amtx-fold-lin-get*)
apply (*rewrite* **in** *f* - ⌑ *amtx-fold-lin-get*)
apply (*rewrite* **in** *g* ⌑ - *amtx-fold-lin-get*)
apply (*rewrite* **in** *g* - ⌑ *amtx-fold-lin-get*)
by *sepref*
end

concrete-definition (**in** −) *amtx-pointwise-cmpop-fold-impl1* **for** *N M fi gi*
uses *amtx-pointwise-cmpop-impl.opr-fold-impl1.refine-raw* **is** (*uncurry ?f*,-)∈-
prepare-code-thms (**in** −) *amtx-pointwise-cmpop-fold-impl1-def*

lemma *op-hnr*[*sepref-fr-rules*]:
assumes *PURE*: *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) *A*
shows (*uncurry* (*amtx-pointwise-cmpop-fold-impl1 N M fi gi*), *uncurry* (*RETURN oo PR-CONST* (*mtx-pointwise-cmpop f g*))) ∈ ($amtx\text{-}assn\ N\ M\ A)^d *_a (amtx\text{-}assn\ N\ M\ A)^k \rightarrow_a bool\text{-}assn$
unfolding *PR-CONST-def*
apply (*rule hfref-weaken-pre'*[*OF - amtx-pointwise-cmpop-fold-impl1.refine*[*OF this-loc PURE*,*FCOMP opr-fold-impl-refine*]])
apply (*auto dest*: *amtx-assn-bounded*[*OF PURE*])
done

end

3.14.2 Regression Test and Usage Example

context begin

To work with a matrix, the dimension should be fixed in a context

context
fixes *N M* :: *nat*
— We also register the dimension as an operation, such that we can use it like a constant
notes [[*sepref-register-adhoc N M*]]
notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
— Finally, we fix a type variable with the required type classes for matrix entries
fixes *dummy*:: *'a*::{*times*,*zero*,*heap*}
begin

First, we implement scalar multiplication with destructive update of the matrix:

private definition *scmul* :: *'a* ⇒ *'a mtx* ⇒ *'a mtx nres* **where**
scmul x m ≡ *nfoldli* [*0..<N*] (λ-. *True*) (λ*i m*.
nfoldli [*0..<M*] (λ-. *True*) (λ*j m*. *do* {
let mij = *m*(*i*,*j*);

RETURN $(m((i,j) := x * mij))$
}
) *m*
) *m*

After declaration of an implementation for multiplication, refinement is straightforward. Note that we use the fixed N in the refinement assertions.

private lemma *times-param*: $((*),(*)::'a\Rightarrow\text{-}) \in Id \rightarrow Id \rightarrow Id$ **by** *simp*

context
notes [*sepref-import-param*] = *times-param*
begin
sepref-definition *scmul-impl*
is *uncurry scmul* :: $(id\text{-}assn^k *_a (amtx\text{-}assn\ N\ M\ id\text{-}assn)^d \rightarrow_a amtx\text{-}assn\ N\ M\ id\text{-}assn)$
unfolding *scmul-def*[*abs-def*]
by *sepref*
end

Initialization with default value

private definition *init-test* ≡ *do* {
let m = *op-amtx-dfltNxM 10 5* (*0*::*nat*);
RETURN (*m*(*1*,*2*))
}
private sepref-definition *init-test-impl* **is** *uncurry0 init-test* :: $unit\text{-}assn^k \rightarrow_a nat\text{-}assn$
unfolding *init-test-def*
by *sepref*

Initialization from function diagonal is more complicated: First, we have to define the function as a new constant

qualified definition *diagonalN k* ≡ $\lambda(i,j).$ *if* $i=j \wedge j<N$ *then k else 0*

If it carries implicit parameters, we have to wrap it into a *PR-CONST* tag:

private sepref-register *PR-CONST diagonalN*
private lemma [*def-pat-rules*]: *IICF-Array-Matrix.diagonalN$N* ≡ *UNPROTECT diagonalN* **by** *simp*

Then, we have to implement the constant, where the result assertion must be for a pure function. Note that, due to technical reasons, we need the *the-pure* in the function type, and the refinement rule to be parameterized over an assertion variable (here A). Of course, you can constrain A further, e.g., *CONSTRAINT* (*IS-PURE IS-ID*) A

private lemma *diagonalN-hnr*[*sepref-fr-rules*]:
assumes *CONSTRAINT* (*IS-PURE PRES-ZERO-UNIQUE*) A

shows (*return o diagonalN*, *RETURN o* (*PR-CONST diagonalN*)) $\in A^k \rightarrow_a pure\ (nat\text{-}rel \times_r nat\text{-}rel \rightarrow the\text{-}pure\ A)$

using *assms*
apply *sepref-to-hoare*
apply (*sep-auto simp*: *diagonalN-def is-pure-conv IS-PURE-def PRES-ZERO-UNIQUE-def*
)
done

In order to discharge preconditions, we need to prove some auxiliary lemma that non-zero indexes are within range

lemma *diagonal-nonzero-ltN*[*simp*]: $(a,b) \in$ *mtx-nonzero* (*diagonalN k*) $\Longrightarrow$ $a<N \wedge b<N$
by (*auto simp*: *mtx-nonzero-def diagonalN-def split*: *if-split-asm*)

private definition *init-test2* ≡ *do* {
ASSERT (*N*>*2*); — Ensure that the coordinate (*1*,*2*) is valid
let m = *op-mtx-new* (*diagonalN* (*1*::*int*));
RETURN (*m*(*1*,*2*))
}
private sepref-definition *init-test2-impl* **is** *uncurry0 init-test2* :: $\textit{unit-assn}^k \rightarrow_a \textit{int-assn}$
unfolding *init-test2-def amtx-fold-custom-new*[*of N N*]
by *sepref*

end

export-code *scmul-impl* **in** *SML-imp*
end
hide-const *scmul-impl*

hide-const(**open**) *is-amtx*

end

3.15 Sepref Bindings for Imp/HOL Collections

theory *IICF-Sepl-Binding*
imports
Separation-Logic-Imperative-HOL.Imp-Map-Spec
Separation-Logic-Imperative-HOL.Imp-Set-Spec
Separation-Logic-Imperative-HOL.Imp-List-Spec

Separation-Logic-Imperative-HOL.Hash-Map-Impl
Separation-Logic-Imperative-HOL.Array-Map-Impl

Separation-Logic-Imperative-HOL.To-List-GA
Separation-Logic-Imperative-HOL.Hash-Set-Impl
Separation-Logic-Imperative-HOL.Array-Set-Impl

Separation-Logic-Imperative-HOL.Open-List
Separation-Logic-Imperative-HOL.Circ-List

../Intf/IICF-Map
../Intf/IICF-Set
../Intf/IICF-List

Collections.Locale-Code
begin

This theory binds collection data structures from the basic collection framework established in *AFP/Separation-Logic-Imperative-HOL* for usage with Sepref.

locale *imp-map-contains-key* = *imp-map* +
  constrains *is-map* :: $('k \rightharpoonup 'v) \Rightarrow 'm \Rightarrow assn$
  fixes *contains-key* :: $'k \Rightarrow 'm \Rightarrow bool\ Heap$
  assumes *contains-key-rule*[*sep-heap-rules*]:
    $<is\text{-}map\ m\ p>\ contains\text{-}key\ k\ p\ <\lambda r.\ is\text{-}map\ m\ p * \uparrow(r \longleftrightarrow k \in dom\ m)>_t$

locale *gen-contains-key-by-lookup* = *imp-map-lookup*
begin
  definition $contains\text{-}key\ k\ m \equiv do\ \{r \leftarrow lookup\ k\ m;\ return\ (\neg is\text{-}None\ r)\}$

  sublocale *imp-map-contains-key is-map contains-key*
    apply *unfold-locales*
    unfolding *contains-key-def*
    apply (*sep-auto split*: *option.splits*)
    done

end

locale *imp-list-tail* = *imp-list* +
  constrains *is-list* :: $'a\ list \Rightarrow 'l \Rightarrow assn$
  fixes *tail* :: $'l \Rightarrow 'l\ Heap$
  assumes *tail-rule*[*sep-heap-rules*]:
    $l \neq [] \Longrightarrow <is\text{-}list\ l\ p>\ tail\ p\ <is\text{-}list\ (tl\ l)>_t$

definition *os-head* :: $'a::heap\ os\text{-}list \Rightarrow ('a)\ Heap$ **where**
  $os\text{-}head\ p \equiv case\ p\ of$
    $None \Rightarrow raise\ STR\ ''os\text{-}Head: Empty\ list''$
  $|\ Some\ p \Rightarrow do\ \{\ m \leftarrow !p;\ return\ (val\ m)\ \}$

primrec *os-tl* :: $'a::heap\ os\text{-}list \Rightarrow ('a\ os\text{-}list)\ Heap$ **where**
  $os\text{-}tl\ None = raise\ STR\ ''os\text{-}tl: Empty\ list''$
$|\ os\text{-}tl\ (Some\ p) = do\ \{\ m \leftarrow !p;\ return\ (next\ m)\ \}$

interpretation *os*: *imp-list-head os-list os-head*
  by *unfold-locales* (*sep-auto simp*: *os-head-def neq-Nil-conv*)

interpretation *os*: *imp-list-tail os-list os-tl*
  by *unfold-locales* (*sep-auto simp*: *os-tl-def neq-Nil-conv*)

definition *cs-is-empty* :: *'a::heap cs-list* ⇒ *bool Heap* **where**
  cs-is-empty p ≡ *return* (*is-None p*)
interpretation *cs*: *imp-list-is-empty cs-list cs-is-empty*
  by *unfold-locales* (*sep-auto simp*: *cs-is-empty-def split*: *option.splits*)

definition *cs-head* :: *'a::heap cs-list* ⇒ *'a Heap* **where**
  cs-head p ≡ *case p of*
    None ⇒ *raise STR ''cs-head: Empty list''*
  | *Some p* ⇒ *do* { *n* ← !*p*; *return* (*val n*)}
interpretation *cs*: *imp-list-head cs-list cs-head*
  by *unfold-locales* (*sep-auto simp*: *neq-Nil-conv cs-head-def*)

definition *cs-tail* :: *'a::heap cs-list* ⇒ *'a cs-list Heap* **where**
  cs-tail p ≡ *do* { (-,*r*) ← *cs-pop p*; *return r* }
interpretation *cs*: *imp-list-tail cs-list cs-tail*
  by *unfold-locales* (*sep-auto simp*: *cs-tail-def*)

lemma *is-hashmap-finite*[*simp*]: $h \models$ *is-hashmap m mi* $\Longrightarrow$ *finite* (*dom m*)
  unfolding *is-hashmap-def is-hashmap'-def*
  by *auto*

lemma *is-hashset-finite*[*simp*]: $h \models$ *is-hashset s si* $\Longrightarrow$ *finite s*
  unfolding *is-hashset-def*
  by (*auto dest*: *is-hashmap-finite*)

definition *ias-is-it s a si* ≡ λ(*a'*,*i*).
  $\exists_A l.\ a \mapsto_a l * \uparrow(a'=a \wedge s=\textit{ias-of-list}\ l \wedge (i=\textit{length}\ l \wedge si=\{\} \vee i<\textit{length}\ l \wedge i\in s \wedge si=s \cap \{x.\ x\geq i\}\ ))$

context begin
private function *first-memb* **where**
  first-memb lmax a i = *do* {
    if i<lmax then do {
      x ← *Array.nth a i*;
      if x then return i else first-memb lmax a (*Suc i*)

} *else*
return *i*
}
by *pat-completeness auto*
termination by (*relation measure* (λ(*l*,-,*i*). *l*−*i*)) *auto*
declare *first-memb.simps*[*simp del*]

private lemma *first-memb-rl-aux*:
assumes *lmax* $\leq$ *length l i*$\leq$*lmax*
shows
< *a* $\mapsto_a$ *l* >
first-memb lmax a i
<λ*k*. *a*$\mapsto_a$ *l* $*$ $\uparrow$(*k*$\leq$*lmax* $\wedge$ ($\forall$ *j*. *i*$\leq$*j* $\wedge$ *j*<*k* $\longrightarrow$ $\neg$*l*!*j*) $\wedge$ *i*$\leq$*k* $\wedge$ (*k*=*lmax* $\vee$ *l*!*k*)) >
using *assms*
proof (*induction lmax a i rule*: *first-memb.induct*)
case (*1 lmax a i*)
show *?case*
apply (*subst first-memb.simps*)
using *1.prems*
apply (*sep-auto heap*: *1.IH*; ((*sep-auto*;*fail*) | *metis eq-iff not-less-eq-eq*))
done
qed

private lemma *first-memb-rl*[*sep-heap-rules*]:
assumes *lmax* $\leq$ *length l i*$\leq$*lmax*
shows < *a* $\mapsto_a$ *l* >
first-memb lmax a i
<λ*k*. *a*$\mapsto_a$ *l* $*$ $\uparrow$(*ias-of-list l* $\cap$ {*i*..<*k*} = {} $\wedge$ *i*$\leq$*k* $\wedge$ (*k*<*lmax* $\wedge$ *k*$\in$*ias-of-list l* $\vee$ *k*=*lmax*)) >
using *assms*
by (*sep-auto simp*: *ias-of-list-def heap*: *first-memb-rl-aux*)

definition *ias-it-init a* = *do* {
l $\leftarrow$ *Array.len a*;
i $\leftarrow$ *first-memb l a 0*;
return (*a*,*i*)
}

definition *ias-it-has-next* $\equiv$ λ(*a*,*i*). *do* {
l $\leftarrow$ *Array.len a*;
return (*i*<*l*)
}

definition *ias-it-next* $\equiv$ λ(*a*,*i*). *do* {
l $\leftarrow$ *Array.len a*;
i' $\leftarrow$ *first-memb l a* (*Suc i*);
return (*i*,(*a*,*i'*))
}

lemma *ias-of-list-bound*: *ias-of-list l* $\subseteq$ {*0..<length l*} **by** (*auto simp*: *ias-of-list-def*)

end

interpretation *ias*: *imp-set-iterate is-ias ias-is-it ias-it-init ias-it-has-next ias-it-next*
  apply *unfold-locales*
  unfolding *is-ias-def ias-is-it-def*
  unfolding *ias-it-init-def* **using** *ias-of-list-bound*
  apply (*sep-auto*)
  unfolding *ias-it-next-def* **using** *ias-of-list-bound*
  apply (*sep-auto*; *fastforce*)
  unfolding *ias-it-has-next-def*
  apply *sep-auto*
  apply *sep-auto*
  done

lemma *ias-of-list-finite*[*simp*, *intro*!]: *finite* (*ias-of-list l*)
  using *finite-subset*[*OF ias-of-list-bound*] **by** *auto*

lemma *is-ias-finite*[*simp*]: $h \models$ *is-ias S x* $\Longrightarrow$ *finite S*
  unfolding *is-ias-def* **by** *auto*

lemma *to-list-ga-rec-rule*:
  assumes *imp-set-iterate is-set is-it it-init it-has-next it-next*
  assumes *imp-list-prepend is-list l-prepend*
  assumes *FIN*: *finite it*
  assumes *DIS*: *distinct l set l* $\cap$ *it* = {}
  shows
  < *is-it s si it iti* $*$ *is-list l li* >
    to-list-ga-rec it-has-next it-next l-prepend iti li
  < $\lambda r. \exists_A l'.$ *is-set s si*
    $*$ *is-list* l' *r*
    $* \uparrow$(*distinct* l' $\wedge$ *set* l' = *set l* $\cup$ *it*) $>_t$
proof −
  interpret *imp-set-iterate is-set is-it it-init it-has-next it-next*
    + *imp-list-prepend is-list l-prepend*
    by *fact+*

  from *FIN DIS* **show** *?thesis*
  proof (*induction arbitrary*: *l li iti rule*: *finite-psubset-induct*)
    case (*psubset it*)
    show *?case*
      apply (*subst to-list-ga-rec.simps*)
      using *psubset.prems* **apply** (*sep-auto heap*: *psubset.IH*)
      apply (*rule ent-frame-fwd*[*OF quit-iteration*])

apply *frame-inference*
apply *solve-entails*
done
qed
qed
lemma *to-list-ga-rule*:
assumes *IT*: *imp-set-iterate is-set is-it it-init it-has-next it-next*
assumes *EM*: *imp-list-empty is-list l-empty*
assumes *PREP*: *imp-list-prepend is-list l-prepend*
assumes *FIN*: *finite s*
shows
<is-set s si>
to-list-ga it-init it-has-next it-next
l-empty l-prepend si
$<\lambda r.\ \exists_A l.\ \textit{is-set s si} * \textit{is-list l r} * \textit{true} * \uparrow(\textit{distinct l} \wedge \textit{set l} = s)>$
proof −
interpret *imp-list-empty is-list l-empty* +
imp-set-iterate is-set is-it it-init it-has-next it-next
by *fact*+

note [*sep-heap-rules*] = *to-list-ga-rec-rule*[*OF IT PREP*]
show *?thesis*
unfolding *to-list-ga-def*
by (*sep-auto simp*: *FIN*)
qed

3.15.1 Binding Locales

method *solve-sepl-binding* = (
unfold-locales;
(*unfold option-assn-pure-conv*)*?*;
sep-auto
intro!: *hfrefI hn-refineI*[*THEN hn-refine-preI*]
simp: *invalid-assn-def hn-ctxt-def pure-def*
)

Map

locale *bind-map* = *imp-map is-map* **for** *is-map* :: $('ki \rightharpoonup 'vi) \Rightarrow 'm \Rightarrow assn$
begin
definition *assn K V* ≡ *hr-comp is-map* (⟨*the-pure K*,*the-pure V*⟩*map-rel*)
lemmas [*fcomp-norm-unfold*] = *assn-def*[*symmetric*]
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure assn K V* **for** *K V*]

end

locale *bind-map-empty* = *imp-map-empty* + *bind-map*
begin
lemma *empty-hnr-aux*: (*uncurry0 empty*,*uncurry0* (*RETURN op-map-empty*))
$\in \textit{unit-assn}^k \rightarrow_a \textit{is-map}$

by *solve-sepl-binding*

sepref-decl-impl (*no-register*) *empty*: *empty-hnr-aux* **.**
end

locale *bind-map-is-empty* = *imp-map-is-empty* + *bind-map*
begin
lemma *is-empty-hnr-aux*: (*is-empty*,*RETURN o op-map-is-empty*) $\in$ *is-map*^{k} $\rightarrow_a$ *bool-assn*
by *solve-sepl-binding*

sepref-decl-impl *is-empty*: *is-empty-hnr-aux* **.**
end

locale *bind-map-update* = *imp-map-update* + *bind-map*
begin
lemma *update-hnr-aux*: (*uncurry2 update*,*uncurry2* (*RETURN ooo op-map-update*)) $\in$ *id-assn*^{k} $*_a$ *id-assn*^{k} $*_a$ *is-map*^{d} $\rightarrow_a$ *is-map*
by *solve-sepl-binding*

sepref-decl-impl *update*: *update-hnr-aux* **.**
end

locale *bind-map-delete* = *imp-map-delete* + *bind-map*
begin
lemma *delete-hnr-aux*: (*uncurry delete*,*uncurry* (*RETURN oo op-map-delete*)) $\in$ *id-assn*^{k} $*_a$ *is-map*^{d} $\rightarrow_a$ *is-map*
by *solve-sepl-binding*

sepref-decl-impl *delete*: *delete-hnr-aux* **.**
end

locale *bind-map-lookup* = *imp-map-lookup* + *bind-map*
begin
lemma *lookup-hnr-aux*: (*uncurry lookup*,*uncurry* (*RETURN oo op-map-lookup*)) $\in$ *id-assn*^{k} $*_a$ *is-map*^{k} $\rightarrow_a$ *id-assn*
by *solve-sepl-binding*

sepref-decl-impl *lookup*: *lookup-hnr-aux* **.**
end

locale *bind-map-contains-key* = *imp-map-contains-key* + *bind-map*
begin
lemma *contains-key-hnr-aux*: (*uncurry contains-key*,*uncurry* (*RETURN oo op-map-contains-key*)) $\in$ *id-assn*^{k} $*_a$ *is-map*^{k} $\rightarrow_a$ *bool-assn*

by *solve-sepl-binding*

sepref-decl-impl *contains-key*: *contains-key-hnr-aux* **.**
end

Set

locale *bind-set* = *imp-set is-set* **for** *is-set* :: (*'ai set*) ⇒ *'m* ⇒ *assn* +
fixes *A* :: *'a* ⇒ *'ai* ⇒ *assn*
begin
definition *assn* ≡ *hr-comp is-set* (⟨*the-pure A*⟩*set-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure assn*]
end

locale *bind-set-setup* = *bind-set*
begin

lemmas [*fcomp-norm-unfold*] = *assn-def*[*symmetric*]
lemma *APA*: ⟦*PROP Q*; *CONSTRAINT is-pure A*⟧ ⟹ *PROP Q* **.**
lemma *APAlu*: ⟦*PROP Q*; *CONSTRAINT* (*IS-PURE IS-LEFT-UNIQUE*) *A*⟧ ⟹ *PROP Q* **.**
lemma *APAru*: ⟦*PROP Q*; *CONSTRAINT* (*IS-PURE IS-RIGHT-UNIQUE*) *A*⟧ ⟹ *PROP Q* **.**
lemma *APAbu*: ⟦*PROP Q*; *CONSTRAINT* (*IS-PURE IS-LEFT-UNIQUE*) *A*; *CONSTRAINT* (*IS-PURE IS-RIGHT-UNIQUE*) *A*⟧ ⟹ *PROP Q* **.**

end

locale *bind-set-empty* = *imp-set-empty* + *bind-set*
begin
lemma *hnr-empty-aux*: (*uncurry0 empty*,*uncurry0* (*RETURN op-set-empty*))∈*unit-assn*^{k} $\rightarrow_a$ *is-set*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*

lemmas *hnr-op-empty* = *hnr-empty-aux*[*FCOMP op-set-empty.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-empty* = *hnr-op-empty*[*FCOMP mk-mop-rl0-np*[*OF mop-set-empty-alt*]]
end

locale *bind-set-is-empty* = *imp-set-is-empty* + *bind-set*
begin
lemma *hnr-is-empty-aux*: (*is-empty*, *RETURN o op-set-is-empty*)∈*is-set*^{k} $\rightarrow_a$ *bool-assn*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*

lemmas *hnr-op-is-empty*[*sepref-fr-rules*] = *hnr-is-empty-aux*[*THEN APA*,*FCOMP op-set-is-empty.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-is-empty*[*sepref-fr-rules*] = *hnr-op-is-empty*[*FCOMP mk-mop-rl1-np*[*OF mop-set-is-empty-alt*]]
end

locale *bind-set-member* = *imp-set-memb* + *bind-set*
begin
lemma *hnr-member-aux*: (*uncurry memb*, *uncurry* (*RETURN oo op-set-member*))∈*id-assn*^{k} $*_a$ *is-set*^{k} $\rightarrow_a$ *bool-assn*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*
lemmas *hnr-op-member*[*sepref-fr-rules*] = *hnr-member-aux*[*THEN APAbu*,*FCOMP op-set-member.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-member*[*sepref-fr-rules*] = *hnr-op-member*[*FCOMP mk-mop-rl2-np*[*OF mop-set-member-alt*]]
end

locale *bind-set-insert* = *imp-set-ins* + *bind-set*
begin
lemma *hnr-insert-aux*: (*uncurry ins*, *uncurry* (*RETURN oo op-set-insert*))∈*id-assn*^{k} $*_a$ *is-set*^{d} $\rightarrow_a$ *is-set*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*
lemmas *hnr-op-insert*[*sepref-fr-rules*] = *hnr-insert-aux*[*THEN APAru*,*FCOMP op-set-insert.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-insert*[*sepref-fr-rules*] = *hnr-op-insert*[*FCOMP mk-mop-rl2-np*[*OF mop-set-insert-alt*]]
end

locale *bind-set-delete* = *imp-set-delete* + *bind-set*
begin
lemma *hnr-delete-aux*: (*uncurry delete*, *uncurry* (*RETURN oo op-set-delete*))∈*id-assn*^{k} $*_a$ *is-set*^{d} $\rightarrow_a$ *is-set*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*
lemmas *hnr-op-delete*[*sepref-fr-rules*] = *hnr-delete-aux*[*THEN APAbu*,*FCOMP op-set-delete.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-delete*[*sepref-fr-rules*] = *hnr-op-delete*[*FCOMP mk-mop-rl2-np*[*OF mop-set-delete-alt*]]
end

locale *bind-set-size* = *imp-set-size* + *bind-set*
begin
lemma *hnr-size-aux*: (*size*, (*RETURN o op-set-size*))∈ *is-set*^{k} $\rightarrow_a$ *nat-assn*
by *solve-sepl-binding*

interpretation *bind-set-setup* **by** *standard*
lemmas *hnr-op-size*[*sepref-fr-rules*] = *hnr-size-aux*[*THEN APAbu*,*FCOMP op-set-size.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-size*[*sepref-fr-rules*] = *hnr-op-size*[*FCOMP mk-mop-rl1-np*[*OF mop-set-size-alt*]]
end

primrec *sorted-wrt′* **where**
sorted-wrt′ R [] ⟷ *True*
| *sorted-wrt′ R* (*x*#*xs*) ⟷ *list-all* (*R x*) *xs* ∧ *sorted-wrt′ R xs*

lemma *sorted-wrt′-eq*: *sorted-wrt′* = *sorted-wrt*
proof (*intro ext iffI*)
fix *R* :: *′a* ⇒ *′a* ⇒ *bool* **and** *xs* :: *′a list*
{
assume *sorted-wrt R xs*
thus *sorted-wrt′ R xs*
by (*induction xs*)(*auto simp*: *list-all-iff*)
}
{
assume *sorted-wrt′ R xs*
thus *sorted-wrt R xs*
by (*induction xs*) (*auto simp*: *list-all-iff*)
}
qed

lemma *param-sorted-wrt*[*param*]: (*sorted-wrt*, *sorted-wrt*) ∈ (*A* → *A* → *bool-rel*) → ⟨*A*⟩*list-rel* → *bool-rel*
unfolding *sorted-wrt′-eq*[*symmetric*] *sorted-wrt′-def*
by *parametricity*

lemma *obtain-list-from-setrel*:
assumes *SV*: *single-valued A*
assumes (*set l*,*s*) ∈ ⟨*A*⟩*set-rel*
obtains *m* **where** *s*=*set m* (*l*,*m*)∈⟨*A*⟩*list-rel*
using *assms*(*2*)
proof (*induction l arbitrary*: *s thesis*)
case *Nil*
show *?case*
apply (*rule Nil*(*1*)[**where** *m*=[]])
using *Nil*(*2*)
by *auto*
next
case (*Cons x l*)
obtain *s′ y* **where** *s*=*insert y s′* (*x*,*y*)∈*A* (*set l*,*s′*)∈⟨*A*⟩*set-rel*
proof −
from *Cons.prems*(*2*) **obtain** *y* **where** *X0*: *y*∈*s* (*x*,*y*)∈*A*
unfolding *set-rel-def* **by** *auto*

from *Cons.prems*(*2*) **have**
X1: ∀ *a*∈*set l*. ∃ *b*∈*s*. (*a*,*b*)∈*A* **and**
X2: ∀ *b*∈*s*. ∃ *a*∈*insert x* (*set l*). (*a*,*b*)∈*A*
unfolding *set-rel-def* **by** *auto*
show *?thesis* **proof** (*cases* ∃ *a*∈*set l*. (*a*,*y*)∈*A*)
case *True*
show *?thesis*
apply (*rule that*[*of y s*])
subgoal using *X0* **by** *auto*
subgoal by *fact*
subgoal
apply (*rule set-relI*)
subgoal using *X1* **by** *blast*
subgoal by (*metis IS-RIGHT-UNIQUED SV True X0*(*2*) *X2 insert-iff*)

done
done
next
case *False*
show *?thesis*
apply (*rule that*[*of y s*−{*y*}])
subgoal using *X0* **by** *auto*
subgoal by *fact*
subgoal
apply (*rule set-relI*)
subgoal using *False X1* **by** *fastforce*
subgoal using *IS-RIGHT-UNIQUED SV X0*(*2*) *X2* **by** *fastforce*
done
done
qed
qed
moreover from *Cons.IH*[*OF* - ‹(*set l*,*s*′)∈⟨*A*⟩*set-rel*›] **obtain** *m* **where** *s*′=*set m* (*l*,*m*)∈⟨*A*⟩*list-rel* **.**
ultimately show *thesis*
apply −
apply (*rule Cons.prems*(*1*)[*of y*#*m*])
by *auto*
qed

lemma *param-it-to-sorted-list*[*param*]: ⟦*IS-LEFT-UNIQUE A*; *IS-RIGHT-UNIQUE A*⟧ ⟹ (*it-to-sorted-list*, *it-to-sorted-list*) ∈ (*A* → *A* → *bool-rel*) → ⟨*A*⟩*set-rel* → ⟨⟨*A*⟩*list-rel*⟩*nres-rel*
unfolding *it-to-sorted-list-def*[*abs-def*]
apply (*auto simp*: *it-to-sorted-list-def pw-nres-rel-iff refine-pw-simps*)
apply (*rule obtain-list-from-setrel*; *assumption?*; *clarsimp*)
apply (*intro exI conjI*; *assumption?*)
using *param-distinct*[*param-fo*] **apply** *blast*
apply *simp*
using *param-sorted-wrt*[*param-fo*] **apply** *blast*

done

locale *bind-set-iterate* = *imp-set-iterate* + *bind-set* +
assumes *is-set-finite*: $h \models$ *is-set* S $x \Longrightarrow$ *finite* S
begin
context begin
private lemma *is-imp-set-iterate*: *imp-set-iterate is-set is-it it-init it-has-next it-next* **by** *unfold-locales*

private lemma *is-imp-list-empty*: *imp-list-empty* (*list-assn id-assn*) (*return* [])
apply *unfold-locales*
apply *solve-constraint*
apply *sep-auto*
done

private lemma *is-imp-list-prepend*: *imp-list-prepend* (*list-assn id-assn*) (*return oo List.Cons*)
apply *unfold-locales*
apply *solve-constraint*
apply (*sep-auto simp*: *pure-def*)
done

definition *to-list* $\equiv$ *to-list-ga it-init it-has-next it-next* (*return* []) (*return oo List.Cons*)
private lemmas *tl-rl* = *to-list-ga-rule*[*OF is-imp-set-iterate is-imp-list-empty is-imp-list-prepend*, *folded to-list-def*]

private lemma *to-list-sorted1*: (*to-list*,*PR-CONST* (*it-to-sorted-list* (λ- -. *True*))) $\in$ *is-set*^{k} $\rightarrow_a$ *list-assn id-assn*
unfolding *PR-CONST-def*
apply (*intro hfrefI*)
apply (*rule hn-refine-preI*)
apply (*rule hn-refineI*)
unfolding *it-to-sorted-list-def*
apply (*sep-auto intro*: *hfrefI hn-refineI intro*: *is-set-finite heap*: *tl-rl*)
done

private lemma *to-list-sorted2*: $\llbracket$
CONSTRAINT (*IS-PURE IS-LEFT-UNIQUE*) *A*;
CONSTRAINT (*IS-PURE IS-RIGHT-UNIQUE*) *A*$\rrbracket \Longrightarrow$
(*PR-CONST* (*it-to-sorted-list* (λ- -. *True*)), *PR-CONST* (*it-to-sorted-list* (λ- -. *True*))) $\in$ $\langle$*the-pure A*$\rangle$*set-rel* $\rightarrow$ $\langle\langle$*the-pure A*$\rangle$*list-rel*$\rangle$*nres-rel*
unfolding *PR-CONST-def CONSTRAINT-def IS-PURE-def*
by *clarify parametricity*

lemmas *to-list-hnr* = *to-list-sorted1*[*FCOMP to-list-sorted2*, *folded assn-def*]

lemmas *to-list-is-to-sorted-list* = *IS-TO-SORTED-LISTI*[*OF to-list-hnr*]
lemma *to-list-gen*[*sepref-gen-algo-rules*]: ⟦*CONSTRAINT* (*IS-PURE IS-LEFT-UNIQUE*) *A*; *CONSTRAINT* (*IS-PURE IS-RIGHT-UNIQUE*) *A*⟧
$\Longrightarrow$ *GEN-ALGO to-list* (*IS-TO-SORTED-LIST* (λ- -. *True*) (*bind-set.assn is-set A*) *A*)
by (*simp add*: *GEN-ALGO-def to-list-is-to-sorted-list*)

end
end

locale *bind-set-union* = *imp-set-union* + *bind-set* +
assumes *is-prime-set-finite*: $h \models$ *is-set S x* $\Longrightarrow$ *finite S*
begin
lemma *hnr-union-aux*: (*uncurry union*, *uncurry* (*RETURN oo op-set-union*))
$\in$ *is-set*^{d} $*_a$ *is-set*^{k} $\rightarrow_a$ *is-set*
apply (*sep-auto intro*!: *is-prime-set-finite*)
unfolding *invalid-assn-def pure-def pure-assn-def hfref-def*
apply (*solve-sepl-binding*) **unfolding** *entails-def*
using *is-prime-set-finite* **subgoal**
using *mod-starD* **by** *blast*
apply *sep-auto*
using *is-prime-set-finite mod-starD*
unfolding *hoare-triple-def ex-assn-def* **apply** *sep-auto*
using *mod-starD union-rule*
by (*metis assn-times-comm mult-1 pure-assn-def pure-true*)

interpretation *bind-set-setup* **by** *standard*
lemmas *hnr-op-union*[*sepref-fr-rules*] = *hnr-union-aux*[*FCOMP op-set-union.fref*[**where** *A*=*the-pure A*]]
lemmas *hnr-mop-union*[*sepref-fr-rules*] = *hnr-op-union*[*FCOMP mk-mop-rl2-np*[*OF mop-set-union-alt*]]

end

List

locale *bind-list* = *imp-list is-list* **for** *is-list* :: (′*ai list*) ⇒ ′*m* ⇒ *assn* +
fixes *A* :: ′*a* ⇒ ′*ai* ⇒ *assn*
begin

definition *assn* ≡ *hr-comp is-list* (⟨*the-pure A*⟩*list-rel*)
lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure assn*]

end

locale *bind-list-empty* = *imp-list-empty* + *bind-list*
begin

lemma *hnr-aux*: (*uncurry0 empty*,*uncurry0* (*RETURN op-list-empty*))∈(*pure unit-rel*)^k $\rightarrow_a$ *is-list*
apply *rule* **apply** *rule* **apply** (*sep-auto simp*: *pure-def*) **done**

lemmas *hnr*
= *hnr-aux*[*FCOMP op-list-empty.fref*[*of the-pure A*], *folded assn-def*]

lemmas *hnr-mop* = *hnr*[*FCOMP mk-mop-rl0-np*[*OF mop-list-empty-alt*]]
end

locale *bind-list-is-empty* = *imp-list-is-empty* + *bind-list*
begin
lemma *hnr-aux*: (*is-empty*,*RETURN o op-list-is-empty*)∈(*is-list*)^k $\rightarrow_a$ *pure bool-rel*
apply *rule* **apply** *rule* **apply** (*sep-auto simp*: *pure-def*) **done**

lemmas *hnr*[*sepref-fr-rules*]
= *hnr-aux*[*FCOMP op-list-is-empty.fref*, *of the-pure A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-is-empty-alt*]]
end

locale *bind-list-append* = *imp-list-append* + *bind-list*
begin
lemma *hnr-aux*: (*uncurry* (*swap-args2 append*),*uncurry* (*RETURN oo op-list-append*))
∈(*is-list*)^d $*_a$ (*pure Id*)^k $\rightarrow_a$ *is-list* **by** *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*]
= *hnr-aux*[*FCOMP op-list-append.fref*,*of A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-append-alt*]]
end

locale *bind-list-prepend* = *imp-list-prepend* + *bind-list*
begin
lemma *hnr-aux*: (*uncurry prepend*,*uncurry* (*RETURN oo op-list-prepend*))
∈(*pure Id*)^k $*_a$ (*is-list*)^d $\rightarrow_a$ *is-list* **by** *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*]
= *hnr-aux*[*FCOMP op-list-prepend.fref*,*of A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl2-np*[*OF mop-list-prepend-alt*]]
end

locale *bind-list-hd* = *imp-list-head* + *bind-list*
begin
lemma *hnr-aux*: (*head*,*RETURN o op-list-hd*)
∈[$\lambda l.\ l \neq []$]$_a$ (*is-list*)^d → *pure Id* **by** *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*] = *hnr-aux*[*FCOMP op-list-hd.fref*,*of A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-hd-alt*]]

end

locale *bind-list-tl* = *imp-list-tail* + *bind-list*
begin
lemma *hnr-aux*: (*tail*,*RETURN o op-list-tl*)
$\in[\lambda l.\ l\neq[]]_a$ $(\textit{is-list})^d \rightarrow$ *is-list*
by *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*] = *hnr-aux*[*FCOMP op-list-tl.fref*,*of the-pure A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl1*[*OF mop-list-tl-alt*]]
end

locale *bind-list-rotate1* = *imp-list-rotate* + *bind-list*
begin
lemma *hnr-aux*: (*rotate*,*RETURN o op-list-rotate1*)
$\in(\textit{is-list})^d \rightarrow_a$ *is-list*
by *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*] = *hnr-aux*[*FCOMP op-list-rotate1.fref*,*of the-pure A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-rotate1-alt*]]
end

locale *bind-list-rev* = *imp-list-reverse* + *bind-list*
begin
lemma *hnr-aux*: (*reverse*,*RETURN o op-list-rev*)
$\in(\textit{is-list})^d \rightarrow_a$ *is-list*
by *solve-sepl-binding*

lemmas *hnr*[*sepref-fr-rules*] = *hnr-aux*[*FCOMP op-list-rev.fref*,*of the-pure A*, *folded assn-def*]
lemmas *hnr-mop*[*sepref-fr-rules*] = *hnr*[*FCOMP mk-mop-rl1-np*[*OF mop-list-rev-alt*]]
end

3.15.2 Array Map (iam)

definition *op-iam-empty* ≡ *IICF-Map.op-map-empty*
interpretation *iam*: *bind-map-empty is-iam iam-new*
by *unfold-locales*

interpretation *iam*: *map-custom-empty op-iam-empty*
by *unfold-locales* (*simp add*: *op-iam-empty-def*)
lemmas [*sepref-fr-rules*] = *iam.empty-hnr*[*folded op-iam-empty-def*]

definition [*simp*]: *op-iam-empty-sz* (*N*::*nat*) ≡ *IICF-Map.op-map-empty*
lemma [*def-pat-rules*]: *op-iam-empty-sz*$*N* ≡ *UNPROTECT* (*op-iam-empty-sz* *N*)

by *simp*

interpretation *iam-sz*: *map-custom-empty PR-CONST* (*op-iam-empty-sz N*)
apply *unfold-locales*
apply (*simp*)
done
lemma [*sepref-fr-rules*]: (*uncurry0 iam-new*, *uncurry0* (*RETURN* (*PR-CONST* (*op-iam-empty-sz N*)))) $\in$ *unit-assn*^{k} $\rightarrow_a$ *iam.assn K V*
using *iam.empty-hnr*[*of K V*] **by** *simp*

interpretation *iam*: *bind-map-update is-iam Array-Map-Impl.iam-update*
by *unfold-locales*

interpretation *iam*: *bind-map-delete is-iam Array-Map-Impl.iam-delete*
by *unfold-locales*

interpretation *iam*: *bind-map-lookup is-iam Array-Map-Impl.iam-lookup*
by *unfold-locales*

setup *Locale-Code.open-block*
interpretation *iam*: *gen-contains-key-by-lookup is-iam Array-Map-Impl.iam-lookup*
by *unfold-locales*
setup *Locale-Code.close-block*

interpretation *iam*: *bind-map-contains-key is-iam iam.contains-key*
by *unfold-locales*

3.15.3 Array Set (ias)

definition [*simp*]: *op-ias-empty* $\equiv$ *op-set-empty*
interpretation *ias*: *bind-set-empty is-ias ias-new* **for** *A*
by *unfold-locales*

interpretation *ias*: *set-custom-empty ias-new op-ias-empty*
by *unfold-locales simp*
lemmas [*sepref-fr-rules*] = *ias.hnr-op-empty*[*folded op-ias-empty-def*]

definition [*simp*]: *op-ias-empty-sz* (*N*::*nat*) $\equiv$ *op-set-empty*
lemma [*def-pat-rules*]: *op-ias-empty-sz*\$*N* $\equiv$ *UNPROTECT* (*op-ias-empty-sz N*)
by *simp*

interpretation *ias-sz*: *bind-set-empty is-ias ias-new-sz N* **for** *N A*
by *unfold-locales*

interpretation *ias-sz*: *set-custom-empty ias-new-sz N PR-CONST* (*op-ias-empty-sz N*) **for** *A*
by *unfold-locales simp*

lemma [*sepref-fr-rules*]:
($uncurry0$ ($ias\text{-}new\text{-}sz$ N), $uncurry0$ ($RETURN$ ($PR\text{-}CONST$ ($op\text{-}ias\text{-}empty\text{-}sz$ N)))) $\in$ $unit\text{-}assn^k \rightarrow_a ias.assn\ A$
using *ias-sz.hnr-op-empty*[*of N A*] **by** *simp*

interpretation *ias*: *bind-set-member is-ias Array-Set-Impl.ias-memb* **for** *A*
by *unfold-locales*

interpretation *ias*: *bind-set-insert is-ias Array-Set-Impl.ias-ins* **for** *A*
by *unfold-locales*

interpretation *ias*: *bind-set-delete is-ias Array-Set-Impl.ias-delete* **for** *A*
by *unfold-locales*

setup *Locale-Code.open-block*
interpretation *ias*: *bind-set-iterate is-ias ias-is-it ias-it-init ias-it-has-next ias-it-next* **for** *A*
by *unfold-locales auto*
setup *Locale-Code.close-block*

3.15.4 Hash Map (hm)

interpretation *hm*: *bind-map-empty is-hashmap hm-new*
by *unfold-locales*

definition *op-hm-empty* $\equiv$ *IICF-Map.op-map-empty*
interpretation *hm*: *map-custom-empty op-hm-empty*
by *unfold-locales* (*simp add*: *op-hm-empty-def*)
lemmas [*sepref-fr-rules*] = *hm.empty-hnr*[*folded op-hm-empty-def*]

interpretation *hm*: *bind-map-is-empty is-hashmap Hash-Map.hm-isEmpty*
by *unfold-locales*

interpretation *hm*: *bind-map-update is-hashmap Hash-Map.hm-update*
by *unfold-locales*

interpretation *hm*: *bind-map-delete is-hashmap Hash-Map.hm-delete*
by *unfold-locales*

interpretation *hm*: *bind-map-lookup is-hashmap Hash-Map.hm-lookup*
by *unfold-locales*

setup *Locale-Code.open-block*
interpretation *hm*: *gen-contains-key-by-lookup is-hashmap Hash-Map.hm-lookup*
by *unfold-locales*
setup *Locale-Code.close-block*

interpretation *hm*: *bind-map-contains-key is-hashmap hm.contains-key*
by *unfold-locales*

3.15.5 Hash Set (hs)

interpretation *hs*: *bind-set-empty is-hashset hs-new* **for** *A*
by *unfold-locales*

definition *op-hs-empty* ≡ *IICF-Set.op-set-empty*
interpretation *hs*: *set-custom-empty hs-new op-hs-empty* **for** *A*
by *unfold-locales* (*simp add*: *op-hs-empty-def*)
lemmas [*sepref-fr-rules*] = *hs.hnr-op-empty*[*folded op-hs-empty-def*]

interpretation *hs*: *bind-set-is-empty is-hashset Hash-Set-Impl.hs-isEmpty* **for** *A*
by *unfold-locales*

interpretation *hs*: *bind-set-member is-hashset Hash-Set-Impl.hs-memb* **for** *A*
by *unfold-locales*

interpretation *hs*: *bind-set-insert is-hashset Hash-Set-Impl.hs-ins* **for** *A*
by *unfold-locales*

interpretation *hs*: *bind-set-delete is-hashset Hash-Set-Impl.hs-delete* **for** *A*
by *unfold-locales*

interpretation *hs*: *bind-set-size is-hashset hs-size* **for** *A*
by *unfold-locales*

setup *Locale-Code.open-block*
interpretation *hs*: *bind-set-iterate is-hashset hs-is-it hs-it-init hs-it-has-next hs-it-next* **for** *A*
by *unfold-locales simp*
setup *Locale-Code.close-block*

interpretation *hs*: *bind-set-union is-hashset hs-is-it hs-it-init hs-it-has-next hs-it-next hs-union*
for *A*
by *unfold-locales simp*

3.15.6 Open Singly Linked List (osll)

interpretation *osll*: *bind-list os-list* **for** *A* **by** *unfold-locales*
interpretation *osll-empty*: *bind-list-empty os-list os-empty* **for** *A*
by *unfold-locales*

definition *osll-empty* ≡ *op-list-empty*
interpretation *osll*: *list-custom-empty osll.assn A os-empty osll-empty*
apply *unfold-locales*
apply (*rule osll-empty.hnr*)
by (*simp add*: *osll-empty-def*)

interpretation *osll-is-empty*: *bind-list-is-empty os-list os-is-empty* **for** *A*
by *unfold-locales*

interpretation *osll-prepend*: *bind-list-prepend os-list os-prepend* **for** *A*
by *unfold-locales*

interpretation *osll-hd*: *bind-list-hd os-list os-head* **for** *A*
by *unfold-locales*

interpretation *osll-tl*: *bind-list-tl os-list os-tl* **for** *A*
by *unfold-locales*

interpretation *osll-rev*: *bind-list-rev os-list os-reverse* **for** *A*
by *unfold-locales*

3.15.7 Circular Singly Linked List (csll)

interpretation *csll*: *bind-list cs-list* **for** *A* **by** *unfold-locales*

interpretation *csll-empty*: *bind-list-empty cs-list cs-empty* **for** *A*
by *unfold-locales*

definition *csll-empty* ≡ *op-list-empty*
interpretation *csll*: *list-custom-empty csll.assn A cs-empty csll-empty*
apply *unfold-locales*
apply (*rule csll-empty.hnr*)
by (*simp add*: *csll-empty-def*)

interpretation *csll-is-empty*: *bind-list-is-empty cs-list cs-is-empty* **for** *A*
by *unfold-locales*

interpretation *csll-prepend*: *bind-list-prepend cs-list cs-prepend* **for** *A*
by *unfold-locales*

interpretation *csll-append*: *bind-list-append cs-list cs-append* **for** *A*
by *unfold-locales*

interpretation *csll-hd*: *bind-list-hd cs-list cs-head* **for** *A*
by *unfold-locales*

interpretation *csll-tl*: *bind-list-tl cs-list cs-tail* **for** *A*
by *unfold-locales*

interpretation *csll-rotate1*: *bind-list-rotate1 cs-list cs-rotate* **for** *A*
by *unfold-locales*

schematic-goal *hn-refine* (*emp*) (*?c*::*?'c Heap*) *?Γ' ?R* (*do* {
$x \leftarrow$ *mop-list-empty*;
RETURN (*1* ∈ *dom* [*1*::*nat* ↦ *True*, *2*↦*False*], {*1*,*2*::*nat*}, *1*#(*2*::*nat*)#x)
})
apply (*subst iam-sz.fold-custom-empty*[**where** *N*=*10*])

apply (*subst hs.fold-custom-empty*)
apply (*subst osll.fold-custom-empty*)
by *sepref*

end

3.16 The Imperative Isabelle Collection Framework

theory *IICF*
imports

Intf/IICF-Set
Impl/IICF-List-SetO

Intf/IICF-Multiset
Intf/IICF-Prio-Bag

Impl/IICF-List-Mset
Impl/IICF-List-MsetO

Impl/Heaps/IICF-Impl-Heap

Intf/IICF-Map
Intf/IICF-Prio-Map

Impl/Heaps/IICF-Impl-Heapmap

Intf/IICF-List

Impl/IICF-Array
Impl/IICF-HOL-List
Impl/IICF-Array-List
Impl/IICF-Indexed-Array-List
Impl/IICF-MS-Array-List

Intf/IICF-Matrix

Impl/IICF-Array-Matrix

Impl/IICF-Sepl-Binding

begin
thy-deps
end

Chapter 4

User Guides

This chapter contains the available user guides.

4.1 Quickstart Guide

theory *Sepref-Guide-Quickstart*
imports *../IICF/IICF*
begin

4.1.1 Introduction

Sepref is an Isabelle/HOL tool to semi-automatically synthesize imperative code from abstract specifications.

The synthesis works by replacing operations on abstract data by operations on concrete data, leaving the structure of the program (mostly) unchanged. Speref proves a refinement theorem, stating the relation between the abstract and generated concrete specification. The concrete specification can then be converted to executable code using the Isabelle/HOL code generator.

This quickstart guide is best appreciated in the Isabelle IDE (currently Isabelle/jedit), such that you can use cross-referencing and see intermediate proof states.

Prerequisites

Sepref is a tool for experienced Isabelle/HOL users. So, this quickstart guide assumes some familiarity with Isabelel/HOL, and will not explain standard Isabelle/HOL techniques.

Sepref is based on Imperative/HOL (*HOL−Imperative-HOL.Imperative-HOL*) and the Isabelle Refinement Framework (*Refine-Monadic.Refine-Monadic*). It makes extensive use of the Separation logic formalization for Imperative/HOL (*Separation-Logic-Imperative-HOL.Sep-Main*).

For a thorough introduction to these tools, we refer to their documentation. However, we try to explain their most basic features when we use them.

4.1.2 First Example

As a first example, let's compute a minimum value in a non-empty list, wrt. some linear order.

We start by specifying the problem:

definition *min-of-list* :: *'a::linorder list* ⇒ *'a nres* **where**
min-of-list l ≡ *ASSERT* (*l*≠[]) ≫ *SPEC* (λ*x*. ∀ *y*∈*set l*. *x*≤*y*)

This specification asserts the precondition and then specifies the valid results *x*. The ≫ operator is a bind-operator on monads.

Note that the Isabelle Refinement Framework works with a set/exception monad over the type - *nres*, where *FAIL* is the exception, and *RES X* specifies a set *X* of possible results. *SPEC* is just the predicate-version of *RES* (actually *SPEC* Φ is a syntax abbreviation for *SPEC* Φ).

Thus, *min-of-list* will fail if the list is empty, and otherwise nondeterministically return one of the minimal elements.

Abstract Algorithm

Next, we develop an abstract algorithm for the problem. A natural choice for a functional programmer is folding over the list, initializing the fold with the first element.

definition *min-of-list1* :: *'a::linorder list* ⇒ *'a nres*
where *min-of-list1 l* ≡ *ASSERT* (*l*≠[]) ≫ *RETURN* (*fold min* (*tl l*) (*hd l*))

Note that *RETURN* returns exactly one (deterministic) result.

We have to show that our implementation actually refines the specification

lemma *min-of-list1-refine*: (*min-of-list1*,*min-of-list*) ∈ *Id* → ⟨*Id*⟩*nres-rel*

This lemma has to be read as follows: If the argument given to *min-of-list1* and *min-of-list* are related by *Id* (i.e. are identical), then the result of *min-of-list1* is a refinement of the result of *min-of-list*, wrt. relation *Id*.

For an explanation, lets simplify the statement first:

apply (*clarsimp intro*!: *nres-relI*)

The - *nres* type defines the refinement ordering, which is a lifted subset ordering, with *FAIL* being the greatest element. This means, that we can assume a non-empty list during the refinement proof (otherwise, the RHS will be *FAIL*, and the statement becomes trivial)

The Isabelle Refinement Framework provides various techniques to extract verification conditions from given goals, we use the standard VCG here:

unfolding *min-of-list-def min-of-list1-def*
apply (*refine-vcg*)

The VCG leaves us with a standard HOL goal, which is easily provable

by (*auto simp*: *neq-Nil-conv Min.set-eq-fold*[*symmetric*])

A more concise proof of the same lemma omits the initial simplification, which we only inserted to explain the refinement ordering:

lemma (*min-of-list1*,*min-of-list*) $\in$ *Id* $\rightarrow$ $\langle$*Id*$\rangle$*nres-rel*
unfolding *min-of-list-def*[*abs-def*] *min-of-list1-def*[*abs-def*]
apply (*refine-vcg*)
by (*auto simp*: *neq-Nil-conv Min.set-eq-fold*[*symmetric*])

Refined Abstract Algorithm

Now, we have a nice functional implementation. However, we are interested in an imperative implementation. Ultimately, we want to implement the list by an array. Thus, we replace folding over the list by indexing into the list, and also add an index-shift to get rid of the *hd* and *tl*.

definition *min-of-list2* :: *'a*::*linorder list* $\Rightarrow$ *'a nres*
where *min-of-list2 l* $\equiv$ *ASSERT* (*l*$\neq$[]) $\gg$ *RETURN* (*fold* (λ*i*. *min* (*l*!(*i*+*1*))) [*0*..<*length l* $-$ *1*] (*l*!*0*))

Proving refinement is straightforward, using the *fold-idx-conv* lemma.

lemma *min-of-list2-refine*: (*min-of-list2*, *min-of-list1*)$\in$*Id* $\rightarrow$ $\langle$*Id*$\rangle$*nres-rel*
unfolding *min-of-list2-def*[*abs-def*] *min-of-list1-def*[*abs-def*]
apply *refine-vcg*
apply *clarsimp-all*
apply (*rewrite* **in** -=⌑ *fold-idx-conv*)
by (*auto simp*: *nth-tl hd-conv-nth*)

Imperative Algorithm

The version *min-of-list2* already looks like the desired imperative version, only that we have lists instead of arrays, and would like to replace the folding over [*0*..<*length l* $-$ *1*] by a for-loop.

This is exactly what the Sepref-tool does. The following command synthesizes an imperative version *min-of-list3* of the algorithm for natural numbers, which uses an array instead of a list:

sepref-definition *min-of-list3* **is** *min-of-list2* :: (*array-assn nat-assn*)^k $\rightarrow_a$ *nat-assn*
unfolding *min-of-list2-def*[*abs-def*]
by *sepref*

The generated constant represents an Imperative/HOL program, and is executable:

thm *min-of-list3-def*
export-code *min-of-list3* **checking** *SML-imp*

Also note that the Sepref tool applied a deforestation optimization: It recognizes a fold over $[0..<n]$, and implements it by the tail-recursive function *imp-for'*, which uses a counter instead of an intermediate list.

There are a couple of optimizations, which come in the form of two sets of simplifier rules, which are applied one after the other:

thm *sepref-opt-simps*
thm *sepref-opt-simps2*

They are just named theorem collections, e.g., *sepref-opt-simps add/del* can be used to modify them.

Moreover, a refinement theorem is generated, which states the correspondence between *min-of-list3* and *min-of-list2*:

thm *min-of-list3.refine*

It states the relations between the parameter and the result of the concrete and abstract function. The parameter is related by *array-assn id-assn*. Here, *array-assn A* relates arrays with lists, such that the elements are related *A* — in our case by *nat-assn*, which relates natural numbers to themselves. We also say that we *implement* lists of nats by arrays of nats. The result is also implemented by natural numbers.

Moreover, the parameters may be stored on the heap, and we have to indicate whether the function keeps them intact or not. Here, we use the annotation -^k (for *keep*) to indicate that the parameter is kept intact, and -^d (for *destroy*) to indicate that it is destroyed.

Overall Correctness Statement

Finally, we can use transitivity of refinement to link our implementation to the specification. The *FCOMP* attribute is able to compose refinement theorems:

theorem *min-of-list3-correct*: $(\mathit{min\text{-}of\text{-}list3}, \mathit{min\text{-}of\text{-}list}) \in (\mathit{array\text{-}assn}\ \mathit{nat\text{-}assn})^k \rightarrow_a \mathit{nat\text{-}assn}$
using *min-of-list3.refine*[*FCOMP min-of-list2-refine*, *FCOMP min-of-list1-refine*]
.

While the above statement is suited to re-use the algorithm within the sepref-framework, a more low-level correctness theorem can be stated using separation logic. This has the advantage that understanding the statement depends on less definitional overhead:

lemma $l \neq [] \Longrightarrow <\textit{array-assn nat-assn } l\ a> \textit{ min-of-list3 } a <\lambda x. \textit{ array-assn nat-assn } l\ a * \uparrow(\forall y \in \textit{set } l.\ x \leq y)>_t$

The proof of this theorem has to unfold the several layers of the Sepref framework, down to the separation logic layer. An explanation of these layers is out of scope of this quickstart guide, we just present some proof techniques that often work. In the best case, the fully automatic proof will work:

by (*sep-auto*
simp: *min-of-list-def pure-def pw-le-iff refine-pw-simps*
heap: *min-of-list3-correct*[*THEN hfrefD*, *of l a*, *THEN hn-refineD*, *simplified*])

If the automatic method does not work, here is a more explicit proof, that can be adapted for proving similar statements:

lemma $l \neq [] \Longrightarrow <\textit{array-assn nat-assn } l\ a> \textit{ min-of-list3 } a <\lambda x. \textit{ array-assn nat-assn } l\ a * \uparrow(\forall y \in \textit{set } l.\ x \leq y)>_t$
proof −

We inlined the definition of *min-of-list*. This will yield two proof obligations later, which we discharge as auxiliary lemmas here

assume [*simp*]: $l \neq []$
have [*simp*]: *nofail* (*min-of-list l*)
by (*auto simp*: *min-of-list-def refine-pw-simps*)
have *1*: $\bigwedge x.\ \textit{RETURN } x \leq \textit{min-of-list } l \Longrightarrow \forall y \in \textit{set } l.\ x \leq y$
by (*auto simp*: *min-of-list-def pw-le-iff refine-pw-simps*)

note *rl* = *min-of-list3-correct*[*THEN hfrefD*, *of l a*, *THEN hn-refineD*, *simplified*]

This should yield a Hoare-triple for *min-of-list3 a*, which can now be used to prove the desired statement via a consequence rule

show *?thesis*
apply (*rule cons-rule*[*OF - - rl*])

The preconditions should match, however, *sep-auto* is also able to discharge more complicated implications here. Be sure to simplify with *pure-def*, if you have parameters that are not stored on the heap (in our case, we don't, but include the simplification anyway.)

apply (*sep-auto simp*: *pure-def*)

The heap-parts of the postcondition should also match. The pure parts require the auxiliary statements that we proved above.

apply (*sep-auto simp*: *pure-def dest*!: *1*)
done
qed

Using the Algorithm

As an example, we now want to use our algorithm to compute the minimum value of some concrete list. In order to use an algorithm, we have to declare both, it's abstract version and its implementation to the Sepref tool.

sepref-register *min-of-list*
— This command registers the abstract version, and generates an *interface type* for it. We will explain interface types later, and only note that, by default, the interface type corresponds to the operation's HOL type.
declare *min-of-list3-correct*[*sepref-fr-rules*]
— This declares the implementation to Sepref

Now we can define the abstract version of our example algorithm. We compute the minimum value of pseudo-random lists of a given length

primrec *rand-list-aux* :: *nat* ⇒ *nat* ⇒ *nat list* **where**
rand-list-aux s 0 = []
| *rand-list-aux s* (*Suc n*) = (*let s* = (*1664525* ∗ *s* + *1013904223*) *mod 2^32 in s* # *rand-list-aux s n*)
definition *rand-list* ≡ *rand-list-aux 42*

definition *min-of-rand-list n* = *min-of-list* (*rand-list n*)

And use Sepref to synthesize a concrete version

We use a feature of Sepref to combine imperative and purely functional code, and leave the generation of the list purely functional, then copy it into an array, and invoke our algorithm. We have to declare the *rand-list* operation:

sepref-register *rand-list*
lemma [*sepref-import-param*]: (*rand-list*,*rand-list*)∈*nat-rel* → ⟨*nat-rel*⟩*list-rel* **by** *auto*

Here, we use a feature of Sepref to import parametricity theorems. Note that the parametricity theorem we provide here is trivial, as *nat-rel* is identity, and *list-rel* as well as (→) preserve identity. However, we have to specify a parametricity theorem that reflects the structure of the involved types.

Finally, we can invoke Sepref

sepref-definition *min-of-rand-list1* **is** *min-of-rand-list* :: *nat-assn*^{k} $\rightarrow_a$ *nat-assn*
unfolding *min-of-rand-list-def*[*abs-def*]

We construct a plain list, however, the implementation of *min-of-list* expects an array. We have to insert a conversion, which is conveniently done with the *rewrite* method:

apply (*rewrite* **in** *min-of-list* ⌑ *array-fold-custom-of-list*)
by *sepref*

In the generated code, we see that the pure *rand-list* function is invoked, its result is converted to an array, which is then passed to *min-of-list3*.

Note that **sepref-definition** prints the generated theorems to the output on the end of the proof. Use the output panel, or hover the mouse over the by-command to see this output.

The generated algorithm can be exported

export-code *min-of-rand-list1* **checking** *SML OCaml? Haskell? Scala*

and executed

ML-val ‹@{*code min-of-rand-list1*} (@{*code nat-of-integer*} *100*) ()›

Note that Imperative/HOL for ML generates a function from unit, and applying this function triggers execution.

4.1.3 Binary Search Example

As second example, we consider a simple binary search algorithm. We specify the abstract problem, i.e., finding an element in a sorted list.

definition *in-sorted-list x xs* ≡ *ASSERT* (*sorted xs*) ≫ *RETURN* (*x*∈*set xs*)

And give a standard iterative implementation:

```
definition in-sorted-list1-invar x xs ≡ λ(l,u,found).
    (l≤u ∧ u≤length xs)
  ∧ (found ⟶ x∈set xs)
  ∧ (¬found ⟶ (x∉set (take l xs) ∧ x∉set (drop u xs))
  )

definition in-sorted-list1 x xs ≡ do {
  let l=0;
  let u=length xs;
  (-,-,r) ← WHILEIT (in-sorted-list1-invar x xs)
    (λ(l,u,found). l<u ∧ ¬found) (λ(l,u,found). do {
      let i = (l+u) div 2;
      ASSERT (i<length xs); — Added here to help synthesis to prove precondition
      for array indexing
      let xi = xs!i;
      if x=xi then
        RETURN (l,u,True)
      else if x<xi then
        RETURN (l,i,False)
      else
        RETURN (i+1,u,False)

    }) (l,u,False);
  RETURN r
}
```

Note that we can refine certain operations only if we can prove that their preconditions are matched. For example, we can refine list indexing to array indexing only if we can prove that the index is in range. This proof has to be done during the synthesis procedure. However, such precondition proofs may be hard, in particular for automatic methods, and we have to do them anyway when proving correct our abstract implementation. Thus, it is a good idea to assert the preconditions in the abstract implementation. This way, they are immediately available during synthesis (recall, when refining an assertion, you may assume the asserted predicate ($?\Phi \Longrightarrow ?M \leq ?M'$) $\Longrightarrow ?M \leq ASSERT\ ?\Phi \gg= (\lambda\text{-}.\ ?M')$).

An alternative is to use monadic list operations that already assert their precondition. The advantage is that you cannot forget to assert the precondition, the disadvantage is that the operation is monadic, and thus, nesting it into other operations is more cumbersome. In our case, the operation would be *mop-list-get* (Look at it's simplified definition to get an impression what it does).

thm *mop-list-get-alt*

We first prove the refinement correct

context begin
private lemma *isl1-measure*: *wf* (*measure* ($\lambda(l,u,f)$. $u-l$ + (*if f then 0 else 1*)))
by *simp*

private lemma *neq-nlt-is-gt*:
  fixes *a b* :: *'a::linorder*
  shows $a\neq b \Longrightarrow \neg(a<b) \Longrightarrow a>b$ **by** *simp*

private lemma *isl1-aux1*:
  assumes *sorted xs*
  assumes *i<length xs*
  assumes *xs!i < x*
  shows $x\notin$*set* (*take i xs*)
  using *assms*
  by (*auto simp*: *take-set leD sorted-nth-mono*)

private lemma *isl1-aux2*:
  assumes $x \notin$ *set* (*take n xs*)
  shows $x\notin$*set* (*drop n xs*) $\longleftrightarrow$ $x\notin$*set xs*
  apply (*rewrite* **in** - = ¤ *append-take-drop-id*[*of n*,*symmetric*])
  using *assms*
  by (*auto simp del*: *append-take-drop-id*)

lemma *in-sorted-list1-refine*: (*in-sorted-list1*, *in-sorted-list*)$\in$*Id* $\rightarrow$ *Id* $\rightarrow$ $\langle$*Id*$\rangle$*nres-rel*
  unfolding *in-sorted-list1-def*[*abs-def*] *in-sorted-list-def*[*abs-def*]
  apply (*refine-vcg isl1-measure*)
  apply (*vc-solve simp*: *in-sorted-list1-invar-def isl1-aux1 isl1-aux2 solve*: *asm-rl*)
  apply (*auto simp*: *take-set set-drop-conv leD sorted-nth-mono*) []
  apply (*auto simp*: *take-set leD sorted-nth-mono dest*: *neq-nlt-is-gt*) []
  done

end

First, let's synthesize an implementation where the list elements are natural numbers. We will discuss later how to generalize the implementation for arbitrary types.

For technical reasons, the Sepref tool works with uncurried functions. That is, every function has exactly one argument. You can use the *uncurry* function, and we also provide abbreviations *uncurry2* up to $\lambda f.$ *uncurry2* (*uncurry2* (*uncurry* f)). If a function has no parameters, *uncurry0* adds a unit parameter.

sepref-definition *in-sorted-list2* **is** *uncurry in-sorted-list1* :: *nat-assn*^{k} $*_a$ (*array-assn nat-assn*)^k $\rightarrow_a$ *bool-assn*
  unfolding *in-sorted-list1-def*[*abs-def*]
  by *sepref*

export-code *in-sorted-list2* **checking** *SML*
lemmas *in-sorted-list2-correct* = *in-sorted-list2.refine*[*FCOMP in-sorted-list1-refine*]

4.1.4 Basic Troubleshooting

In this section, we will explain how to investigate problems with the Sepref tool. Most cases where *sepref* fails are due to some missing operations, unsolvable preconditions, or an odd setup.

Example

We start with an example. Recall the binary search algorithm. This time, we forget to assert the precondition of the indexing operation.

```
definition in-sorted-list1' x xs ≡ do {
  let l=0;
  let u=length xs;
  (-,-,r) ← WHILEIT (in-sorted-list1-invar x xs)
    (λ(l,u,found). l<u ∧ ¬found) (λ(l,u,found). do {
      let i = (l+u) div 2;
      let xi = xs!i; — It's not trivial to show that i is in range
      if x=xi then
        RETURN (l,u,True)
      else if x<xi then
        RETURN (l,i,False)
      else
        RETURN (i+1,u,False)

    }) (l,u,False);
  RETURN r
}
```

We try to synthesize the implementation. Note that **sepref-thm** behaves like **sepref-definition**, but actually defines no constant. It only generates a refinement theorem.

sepref-thm *in-sorted-list2* **is** *uncurry in-sorted-list1'* :: $\textit{nat-assn}^k *_a (\textit{array-assn nat-assn})^k \rightarrow_a \textit{bool-assn}$
unfolding *in-sorted-list1'-def*[*abs-def*]

— If *sepref* fails, you can use *sepref-dbg-keep* to get some more information.
apply *sepref-dbg-keep*
— This prints a trace of the different phases of sepref, and stops when the first phase fails. It then returns the internal proof state of the tool, which can be inspected further.
Here, the translation phase fails. The translation phase translates the control structures and operations of the abstract program to their concrete counterparts. To inspect the actual problem, we let translation run until the operation where it fails:
supply [[*goals-limit=1*]] — There will be many subgoals during translation, and printing them takes very long with Isabelle :(
apply *sepref-dbg-trans-keep*
— Things get stuck at a goal with predicate *hn-refine*. This is the internal refinement predicate, *hn-refine* Γ c Γ' R a means, that, for operands whose refinement is described by Γ, the concrete program c refines the abstract program a, such that, afterwards, the operands are described by Γ', and the results are refined by R.
Inspecting the first subgoal reveals that we got stuck on refining the abstract operation *RETURN* \$ (*op-list-get* \$ *b* \$ *xf*). Note that the (\$) is just a constant for function application, which is used to tame Isabelle's higher-order unification algorithms. You may use *unfolding APP-def*, or even *simp* to get a clearer picture of the failed goal.
If a translation step fails, it may be helpful to execute as much of the translation step as possible:
apply *sepref-dbg-trans-step-keep*
— The translation step gets stuck at proving *pre-list-get* (*b*, *xf*), which is the precondition for list indexing.
apply (*sepref-dbg-side-keep*) — If you think the side-condition should be provable, this command returns the left-over subgoals after some preprocessing and applying auto

oops

Internals of Sepref

Internally, *sepref* consists of multiple phases that are executed one after the other. Each phase comes with its own debugging method, which only executes that phase. We illustrate this by repeating the refinement of *min-of-list2*. This time, we use **sepref-thm**, which only generates a refinement theorem, but defines no constants:

sepref-thm *min-of-list3'* **is** *min-of-list2* :: $(\textit{array-assn nat-assn})^k \rightarrow_a \textit{nat-assn}$

— The *sepref-thm* or *sepref-definition* command assembles a schematic goal statement.

unfolding *min-of-list2-def* [*abs-def*]

— The preprocessing phase converts the goal into the *hn-refine*-form. Moreover, it adds interface type annotations for the parameters. (for now, the interface type is just the HOL type of the parameter, in our case, *nat list*)

apply *sepref-dbg-preproc*

— The next phase applies a consequence rule for the postcondition and result. This is mainly for technical reasons.

apply *sepref-dbg-cons-init*

— The next phase tries to identify the abstract operations, and inserts tag-constants for function application and abstraction. These tags are for technical reasons, working around Isabelle/HOL's unifier idiosyncrasies.

Operation identification assigns a single constant to each abstract operation, which is required for technical reasons. Note that there are terms in HOL, which qualify as a single operation, but consists of multiple constants, for example, $\{x\}$, which is just syntactic sugar for *insert* x $\{\}$. In our case, the operation identification phase rewrites the assertion operations followed by a bind to a single operation *op-ASSERT-bind*, and renames some operations to more canonical names.

apply *sepref-dbg-id*

— Now that it is clear which operations to execute, we have to specify an execution order. Note that HOL has no notion of execution at all. However, if we want to translate to operations that depend on a heap, we need a notion of execution order. We use the *nres*-monad's bind operation as sequencing operator, and flatten all nested operations, using left-to-right evaluation order.

apply *sepref-dbg-monadify*

— The next step just prepares the optimization phase, which will be executed on the translated program. It just applies the rule $[\![$*hn-refine* $?\Gamma$ $?c$ $?\Gamma'$ $?R$ $?a$; *CNV* $?c$ $?c'$$]\!] \Longrightarrow$ *hn-refine* $?\Gamma$ $?c'$ $?\Gamma'$ $?R$ $?a$.

apply *sepref-dbg-opt-init*

— The translation phase does the main job of translating the abstract program to the concrete one. It has rules how to translate abstract operations to concrete ones. For technical reasons, it differentiates between operations, which have only first-order arguments (e.g., *length*) and combinators, which have also higher-order arguments (e.g., *fold*).

The basic idea of translation is to repeatedly apply the translation rule for the topmost combinator/operator, and thus recursively translate the whole program. The rules may produce various types of side-conditions, which are resolved by the tool.

apply *sepref-dbg-trans*

— The next phase applies some simplification rules to optimize the translated program. It essentially simplifies first with the rules *sepref-opt-simps*, and then with *sepref-opt-simps2*.

apply *sepref-dbg-opt*

— The next two phases resolve the consequence rules introduced by the *cons-init* phase.

apply *sepref-dbg-cons-solve*

apply *sepref-dbg-cons-solve*

— The translation phase and the consequence rule solvers may postpone some side conditions on yet-unknown refinement assertions. These are solved in the last phase.

apply *sepref-dbg-constraints*
done

In the next sections, we will explain, by example, how to troubleshoot the various phases of the tool. We will focus on the phases that are most likely to fail.

Initialization

A common mistake is to forget the keep/destroy markers for the refinement assertion, or specify a refinement assertion with a non-matching type. This results in a type-error on the command

sepref-thm *test-add-2* **is** $\lambda x.\ RETURN\ (2{+}x) :: nat\text{-}assn^k \rightarrow_a nat\text{-}assn$
by *sepref*

Translation Phase

In most cases, the translation phase will fail. Let's try the following refinement:

sepref-thm *test* **is** $\lambda l.\ RETURN\ (l!1 + 2) :: (array\text{-}assn\ nat\text{-}assn)^k \rightarrow_a nat\text{-}assn$

The *sepref* method will just fail. To investigate further, we use *sepref-dbg-keep*, which executes the phases until the first one fails. It returns with the proof state before the failed phase, and, moreover, outputs a trace of the phases, such that you can easily see which phase failed.

apply *sepref-dbg-keep*

— In the trace, we see that the translation phase failed. We are presented the tool's internal goal state just before translation. If a phase fails, the usual procedure is to start the phase in debug mode, and see how far it gets. The debug mode of the translation phase stops at the first operation or combinator it cannot translate. Note, it is a good idea to limit the visible goals, as printing goals in Isabelle can be very, very slow :(

supply [[*goals-limit* = *1*]]
apply *sepref-dbg-trans-keep*

— Here, we see that translation gets stuck at *op-list-get*. This may have two reasons: Either there is no rule for this operation, or a side condition cannot be resolved. We apply a single translation step in debug mode, i.e., the translation step is applied as far as possible, leaving unsolved side conditions:

apply *sepref-dbg-trans-step-keep*

— This method reports that the "Apply rule" phase produced a wrong number of subgoals. This phase is expected to solve the goal, but left some unsolved side condition, which we are presented in the goal state. We can either guess what *pre-list-get* means and why it cannot be solved, or try to partially solve the side condition:

apply *sepref-dbg-side-keep*
— From the remaining subgoal, one can guess that there might be a problem with too short lists, where index *1* does not exist.

oops

Inserting an assertion into the abstract program solves the problem:

sepref-thm *test* **is** $\lambda l.\ ASSERT\ (length\ l > 1) \gg RETURN\ (l!1 + 2) :: (array\text{-}assn\ nat\text{-}assn)^k \rightarrow_a nat\text{-}assn$
by *sepref*

Here is an example for an unimplemented operation:

sepref-thm *test* **is** $\lambda l.\ RETURN\ (Min\ (set\ l)) :: (array\text{-}assn\ nat\text{-}assn)^k \rightarrow_a nat\text{-}assn$
supply [[*goals-limit* = *1*]]
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
— Translation stops at the *set* operation
apply *sepref-dbg-trans-step-keep*
— This tactic reports that the "Apply rule" phase failed, which means that there is no applicable rule for the *set* operation on arrays.
oops

4.1.5 The Isabelle Imperative Collection Framework (IICF)

The IICF provides a library of imperative data structures, and some management infrastructure. The main idea is to have interfaces and implementations.

An interface specifies an abstract data type (e.g., - *list*) and some operations with preconditions on it (e.g., (@) or (!) with in-range precondition).

An implementation of an interface provides a refinement assertion from the abstract data type to some concrete data type, as well as implementations for (a subset of) the interface's operations. The implementation may add some more implementation specific preconditions.

The default interfaces of the IICF are in the folder *IICF/Intf*, and the standard implementations are in *IICF/Impl*.

Map Example

Let's implement a function that maps a finite set to an initial segment of the natural numbers

definition *nat-seg-map s* ≡
$ASSERT\ (finite\ s) \gg SPEC\ (\lambda m.\ dom\ m = s \wedge ran\ m = \{0..<card\ s\})$

We implement the function by iterating over the set, and building the map

definition *nat-seg-map1 s* ≡ *do* {

ASSERT (*finite s*);
(*m*,-) ← *FOREACHi* (*λit* (*m*,*i*). *dom m* = *s*−*it* ∧ *ran m* = {*0*..<*i*} ∧ *i*=*card* (*s* − *it*))
s (*λx* (*m*,*i*). *RETURN* (*m*(*x*↦*i*),*i*+*1*)) (*Map.empty*,*0*);
RETURN m
}

lemma *nat-seg-map1-refine*: (*nat-seg-map1*, *nat-seg-map*) ∈ *Id* → ⟨*Id*⟩*nres-rel*
apply (*intro fun-relI*)
unfolding *nat-seg-map1-def*[*abs-def*] *nat-seg-map-def*[*abs-def*]
apply (*refine-vcg*)
apply (*vc-solve simp*: *it-step-insert-iff solve*: *asm-rl dest*: *domD*)
done

We use hashsets *hs.assn* and hashmaps (*hm.assn*).

sepref-definition *nat-seg-map2* **is** *nat-seg-map1* :: (*hs.assn id-assn*)^k $\rightarrow_a$ *hm.assn id-assn nat-assn*
unfolding *nat-seg-map1-def*[*abs-def*]
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
— We got stuck at *op-map-empty*. This is because Sepref is very conservative when it comes to guessing implementations. Actually, no constructor operation will be assigned a default operation, with some obvious exceptions for numbers and Booleans.
oops

Assignment of implementations to constructor operations is done by rewriting them to synonyms which are bound to a specific implementation. For hashmaps, we have *op-hm-empty*, and the rules *hm.fold-custom-empty*.

sepref-definition *nat-seg-map2* **is** *nat-seg-map1* :: (*hs.assn id-assn*)^k $\rightarrow_a$ *hm.assn id-assn nat-assn*
unfolding *nat-seg-map1-def*[*abs-def*]
— We can use the *rewrite* method for position-precise rewriting:
apply (*rewrite* **in** *FOREACHi* - - - ⌑ *hm.fold-custom-empty*)
by *sepref*

export-code *nat-seg-map2* **checking** *SML*

lemmas *nat-seg-map2-correct* = *nat-seg-map2.refine*[*FCOMP nat-seg-map1-refine*]

4.1.6 Specification of Preconditions

In this example, we will discuss how to specify precondition of operations, which are required for refinement to work. Consider the following function, which increments all members of a list by one:

definition *incr-list l* ≡ *map* ((+) *1*) *l*

We might want to implement it as follows

definition *incr-list1 l* ≡ *fold* (λ*i l. l*[*i*:=*1* + *l*!*i*]) [*0*..<*length l*] *l*

lemma *incr-list1-refine*: (*incr-list1*, *incr-list*)∈*Id* → *Id*
proof (*intro fun-relI*; *simp*)
  fix *l* :: *'a list*
  { fix *n m*
    assume *n*≤*m* **and** *length l* = *m*
    hence *fold* (λ*i l. l*[*i*:=*1*+*l*!*i*]) [*n*..<*m*] *l* = *take n l* @ *map* (((+))*1*) (*drop n l*)
      apply (*induction arbitrary*: *l rule*: *inc-induct*)
      apply *simp*
      apply (*clarsimp simp*: *upt-conv-Cons take-Suc-conv-app-nth*)
      apply (*auto simp add*: *list-eq-iff-nth-eq nth-Cons split*: *nat.split*)
      done
  }
  from *this*[*of 0 length l*] **show** *incr-list1 l* = *incr-list l*
    unfolding *incr-list-def incr-list1-def*
    by *simp*
qed

Trying to refine this reveals a problem:

sepref-thm *incr-list2* **is** *RETURN o incr-list1* :: (*array-assn nat-assn*)^d $\rightarrow_a$ *array-assn nat-assn*
  unfolding *incr-list1-def*[*abs-def*]
  apply *sepref-dbg-keep*
  apply *sepref-dbg-trans-keep*
  apply *sepref-dbg-trans-step-keep*
  apply *sepref-dbg-side-keep*
  — We get stuck at the precondition of *op-list-get*. Indeed, we cannot prove the generated precondition, as the translation process dropped any information from which we could conclude that the index is in range.
  oops

Of course, the fold loop has the invariant that the length of the list does not change, and thus, indexing is in range. We only cannot prove it during the automatic synthesis.

Here, the only solution is to do a manual refinement into the nres-monad, and adding an assertion that indexing is always in range.

We use the *nfoldli* combinator, which generalizes *fold* in two directions:

1. The function is inside the nres monad

2. There is a continuation condition. If this is not satisfied, the fold returns immediately, dropping the rest of the list.

definition *incr-list2 l* ≡ *nfoldli*
  [*0*..<*length l*]
  (λ-. *True*)
  (λ*i l. ASSERT* (*i*<*length l*) ≫ *RETURN* (*l*[*i*:=*1*+*l*!*i*]))

l

Note: Often, it is simpler to prove refinement of the abstract specification, rather than proving refinement to some intermediate specification that may have already done refinements "in the wrong direction". In our case, proving refinement of *incr-list1* would require to generalize the statement to keep track of the list-length invariant, while proving refinement of *incr-list* directly is as easy as proving the original refinement for *incr-list1*.

lemma *incr-list2-refine*: (*incr-list2*,*RETURN o incr-list*) ∈ *Id* → ⟨*Id*⟩*nres-rel*
proof (*intro nres-relI fun-relI*; *simp*)
  fix *l* :: *'a list*
  show *incr-list2 l* ≤ *RETURN* (*incr-list l*)
    unfolding *incr-list2-def incr-list-def*
    — *nfoldli* comes with an invariant proof rule. In order to use it, we have to specify the invariant manually:
    apply (*refine-vcg nfoldli-rule*[**where** *I*=λ*l1 l2 s. s* = *map* (((+))*1*) (*take* (*length l1*) *l*) @ *drop* (*length l1*) *l*])
    apply (*vc-solve*
      simp: *upt-eq-append-conv upt-eq-Cons-conv*
      simp: *nth-append list-update-append upd-conv-take-nth-drop take-Suc-conv-app-nth*
      solve: *asm-rl*
)
    done
qed

sepref-definition *incr-list3* **is** *incr-list2* :: (*array-assn nat-assn*)^d $\rightarrow_a$ *array-assn nat-assn*
  unfolding *incr-list2-def*[*abs-def*]
  by *sepref*

lemmas *incr-list3-correct* = *incr-list3.refine*[*FCOMP incr-list2-refine*]

4.1.7 Linearity and Copying

Consider the following implementation of an operation to swap to list indexes. While it is perfectly valid in a functional setting, an imperative implementation has a problem here: Once the update a index *i* is done, the old value cannot be read from index *i* any more. We try to implement the list with an array:

sepref-thm *swap-nonlinear* **is** *uncurry2* (λ*l i j. do* {
  ASSERT (*i*<*length l* ∧ *j*<*length l*);
  RETURN (*l*[*i*:=*l*!*j*, *j*:=*l*!*i*])
}) :: (*array-assn id-assn*)^d $*_a$ *nat-assn*^{k} $*_a$ *nat-assn*^{k} $\rightarrow_a$ *array-assn id-assn*
  supply [[*goals-limit* = *1*]]
  apply *sepref-dbg-keep*
  apply *sepref-dbg-trans-keep* — (1) We get stuck at an *op-list-get* operation
  apply *sepref-dbg-trans-step-keep* — (2) Further inspection reveals that the "recover pure" phase fails, and we are left with a subgoal of the form *CON-*

STRAINT is-pure (*array-assn id-assn*). Constraint side conditions are deferrable side conditions: They are produced as side-conditions, and if they cannot be solved immediately, they are deferred and processed later, latest at the end of the synthesis. However, definitely unsolvable constraints are not deferred, but halt the translation phase immediately, and this is what happened here: At (1) we can see that the refinement for the array we want to access is *hn-invalid* (*array-assn id-assn*). This means, the data structure was destroyed by some preceding operation. The *hn-invalid* only keeps a record of this fact. When translating an operation that uses an invalidated parameter, the tool tries to restore the invalidated parameter: This only works if the data structure was purely functional, i.e., not stored on the heap. This is where the *is-pure* constraint comes from. However, arrays are always stored on the heap, so this constraint is definitely unsolvable, and thus immediately rejected instead of being deferred.

Note: There are scenarios where a constraint gets deferred *before* it becomes definitely unsolvable. In these cases, you only see the problem after the translation phase, and it may be somewhat tricky to figure out the reason.

oops

The fix for our swap function is quite obvious. Using a temporary storage for the intermediate value, we write:

sepref-thm *swap-with-tmp* **is** *uncurry2* ($\lambda l\ i\ j.\ do$ {
$ASSERT$ ($i<length\ l \wedge j<length\ l$);
$let\ tmp = l!i$;
$RETURN$ ($l[i:=l!j,\ j:=tmp]$)
}) :: $(array\text{-}assn\ id\text{-}assn)^d *_a nat\text{-}assn^k *_a nat\text{-}assn^k \rightarrow_a array\text{-}assn\ id\text{-}assn$
by *sepref*

Note that also the argument must be marked as destroyed $()^d$. Otherwise, we get a similar error as above, but in a different phase:

sepref-thm *swap-with-tmp* **is** *uncurry2* ($\lambda l\ i\ j.\ do$ {
$ASSERT$ ($i<length\ l \wedge j<length\ l$);
$let\ tmp = l!i$;
$RETURN$ ($l[i:=l!j,\ j:=tmp]$)
}) :: $(array\text{-}assn\ id\text{-}assn)^k *_a nat\text{-}assn^k *_a nat\text{-}assn^k \rightarrow_a array\text{-}assn\ id\text{-}assn$
apply *sepref-dbg-keep* — We get stuck at a frame, which would require restoring an invalidated array
apply *sepref-dbg-cons-solve-keep* — Which would only work if arrays were pure
oops

If copying is really required, you have to insert it manually. Reconsider the example *incr-list* from above. This time, we want to preserve the original data (note the $()^k$ annotation):

sepref-thm *incr-list3-preserve* **is** *incr-list2* :: $(array\text{-}assn\ nat\text{-}assn)^k \rightarrow_a array\text{-}assn\ nat\text{-}assn$
unfolding *incr-list2-def*[*abs-def*]
— We explicitly insert a copy-operation on the list, before it is passed to the fold operation

apply (*rewrite* **in** *nfoldli* - - - ⌑ *op-list-copy-def*[*symmetric*])
by *sepref*

4.1.8 Nesting of Data Structures

Sepref and the IICF support nesting of data structures with some limitations:

- Only the container or its elements can be visible at the same time. For example, if you have a product of two arrays, you can either see the two arrays, or the product. An operation like *snd* would have to destroy the product, loosing the first component. Inside a case distinction, you cannot access the compound object.

  These limitations are somewhat relaxed for pure data types, which can always be restored.

- Most IICF data structures only support pure component types. Exceptions are HOL-lists, and the list-based set and multiset implementations *List-MsetO* and *List-SetO* (Here, the *O* stands for *own*, which means that the data-structure owns its elements.).

Works fine:

sepref-thm *product-ex1* **is** *uncurry0* (*do* {
  let p = (*op-array-replicate 5 True*, *op-array-replicate 2 False*);
  case p of (*a1*,*a2*) ⇒ *RETURN* (*a1*!*2*)
}) :: $\mathit{unit\text{-}assn}^k \rightarrow_a \mathit{bool\text{-}assn}$
by *sepref*

Fails: We cannot access compound type inside case distinction

sepref-thm *product-ex2* **is** *uncurry0* (*do* {
  let p = (*op-array-replicate 5 True*, *op-array-replicate 2 False*);
  case p of (*a1*,*a2*) ⇒ *RETURN* (*snd p*!*1*)
}) :: $\mathit{unit\text{-}assn}^k \rightarrow_a \mathit{bool\text{-}assn}$
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
apply *sepref-dbg-trans-step-keep*
oops

Works fine, as components of product are pure, such that product can be restored inside case.

sepref-thm *product-ex2* **is** *uncurry0* (*do* {
  let p = (*op-list-replicate 5 True*, *op-list-replicate 2 False*);
  case p of (*a1*,*a2*) ⇒ *RETURN* (*snd p*!*1*)
}) :: $\mathit{unit\text{-}assn}^k \rightarrow_a \mathit{bool\text{-}assn}$
by *sepref-dbg-keep*

Trying to create a list of arrays, first attempt:

sepref-thm *set-of-arrays-ex* **is** *uncurry0* (*RETURN* (*op-list-append* [] *op-array-empty*))
:: $unit\text{-}assn^k \rightarrow_a$ *arl-assn* (*array-assn nat-assn*)
  unfolding *arl.fold-custom-empty*
  apply *sepref-dbg-keep*
  apply *sepref-dbg-trans-keep*
  apply *sepref-dbg-trans-step-keep*
  supply [[*goals-limit* = *1*, *unify-trace-failure*]]

— Many IICF data structures, in particular the array based ones, requires the element types to be of *default*. If this is not the case, Sepref will simply find no refinement for the operations. Be aware that type-class related mistakes are hard to debug in Isabelle/HOL, above we sketched how to apply the refinement rule that is supposed to match with unifier tracing switched on. The *to-hnr* attribute is required to convert the rule from the relational form to the internal *hn-refine* form. Note that some rules are already in *hn-refine* form, and need not be converted, e.g., *hn-refine* (*hn-ctxt ?P1.0 ?x1.0 ?x1'* ∗ *hn-ctxt ?P2.0 ?x2.0 ?x2'*) (*return* (*?x1'*, *?x2'*)) (*hn-invalid ?P1.0 ?x1.0 ?x1'* ∗ *hn-invalid ?P2.0 ?x2.0 ?x2'*) (*?P1.0* $\times_a$ *?P2.0*) (*RETURN* \$ (*Pair* \$ *?x1.0* \$ *?x2.0*)).
  oops

So lets choose a circular singly linked list (csll), which does not require its elements to be of default type class

sepref-thm *set-of-arrays-ex* **is** *uncurry0* (*RETURN* (*op-list-append* [] *op-array-empty*))
:: $unit\text{-}assn^k \rightarrow_a$ *csll.assn* (*array-assn nat-assn*)
  unfolding *csll.fold-custom-empty*
  apply *sepref-dbg-keep*
  apply *sepref-dbg-trans-keep*
  apply *sepref-dbg-trans-step-keep*
— We end up with an unprovable purity-constraint: As many IICF types, csll only supports pure member types. We expect this restriction to be lifted in some future version.
  oops

Finally, there are a few data structures that already support nested element types, for example, functional lists:

sepref-thm *set-of-arrays-ex* **is** *uncurry0* (*RETURN* (*op-list-append* [] *op-array-empty*))
:: $unit\text{-}assn^k \rightarrow_a$ *list-assn* (*array-assn nat-assn*)
  unfolding *HOL-list.fold-custom-empty*
  by *sepref*

4.1.9 Fixed-Size Data Structures

For many algorithms, the required size of a data structure is already known, such that it is not necessary to use data structures with dynamic resizing.

The Sepref-tool supports such data structures, however, with some limitations.

Running Example

Assume we want to read a sequence of natural numbers in the range $\{0..<N\}$, and drop duplicate numbers. The following abstract algorithm may work:

definition *remdup l* ≡ *do* {
  (*s*,*r*) ← *nfoldli l* (λ-. *True*)
    (λ*x* (*s*,*r*). *do* {
      ASSERT (*distinct r* ∧ *set r* ⊆ *set l* ∧ *s* = *set r*); — Will be required to prove that list does not grow too long
      if x∈*s then RETURN* (*s*,*r*) *else RETURN* (*insert x s*, *r*@[*x*])
    })
    ({},[]);
  RETURN r
}

We want to use *remdup* in our abstract code, so we have to register it.

sepref-register *remdup*

The straightforward version with dynamic data-structures is:

sepref-definition *remdup1* **is** *remdup* :: (*list-assn nat-assn*)^k $\rightarrow_a$ *arl-assn nat-assn*
  unfolding *remdup-def*[*abs-def*]
  — Lets use a bit-vector for the set
  apply (*rewrite* **in** *nfoldli* - - - ⌑ *ias.fold-custom-empty*)
  — And an array-list for the list
  apply (*rewrite* **in** *nfoldli* - - - ⌑ *arl.fold-custom-empty*)
  by *sepref*

Initialization of Dynamic Data Structures

Now let's fix an upper bound for the numbers in the list. Initializations and statically sized data structures must always be fixed variables, they cannot be computed inside the refined program.

TODO: Lift this restriction at least for initialization hints that do not occur in the refinement assertions.

context fixes *N* :: *nat* **begin**

sepref-definition *remdup1-initsz* **is** *remdup* :: (*list-assn nat-assn*)^k $\rightarrow_a$ *arl-assn nat-assn*
  unfolding *remdup-def*[*abs-def*]
  — Many of the dynamic array-based data structures in the IICF can be pre-initialized to a certain size. THis initialization is only a hint, and has no abstract consequences. The list data structure will still be resized if it grows larger than the initialization size.
  apply (*rewrite* **in** *nfoldli* - - - ⌑ *ias-sz.fold-custom-empty*[*of N*])
  apply (*rewrite* **in** *nfoldli* - - - ⌑ *arl-sz.fold-custom-empty*[*of N*])
  by *sepref*

end

To get a usable function, we may add the fixed N as a parameter, effectively converting the initialization hint to a parameter, which, however, has no abstract meaning

definition *remdup-initsz* (N::*nat*) $\equiv$ *remdup*
lemma *remdup-init-hnr*:
(*uncurry remdup1-initsz*, *uncurry remdup-initsz*) $\in$ *nat-assn*^{k} $*_a$ (*list-assn nat-assn*)^k $\rightarrow_a$ *arl-assn nat-assn*
using *remdup1-initsz.refine* **unfolding** *remdup-initsz-def*[*abs-def*]
unfolding *hfref-def hn-refine-def*
by (*auto simp*: *pure-def*)

Static Data Structures

We use a locale to hide local declarations. Note: This locale will never be interpreted, otherwise all the local setup, that does not make sense outside the locale, would become visible. TODO: This is probably some abuse of locales to emulate complex private setup, including declaration of constants and lemmas.

locale *my-remdup-impl-loc* =
fixes N :: *nat*
assumes $N{>}0$ — This assumption is not necessary, but used to illustrate the general case, where the locale may have such assumptions
begin

For locale hierarchies, the following seems not to be available directly in Isabelle, however, it is useful when transferring stuff between the global theory and the locale

lemma *my-remdup-impl-loc-this*: *my-remdup-impl-loc* N **by** *unfold-locales*

Note that this will often require to use N as a usual constant, which is refined. For pure refinements, we can use the *sepref-import-param* attribute, which will convert a parametricity theorem to a rule for Sepref:

sepref-register N
lemma *N-hnr*[*sepref-import-param*]: (N,N)$\in$*nat-rel* **by** *simp*
thm *N-hnr*

Alternatively, we could directly prove the following rule, which, however, is more cumbersome:

lemma *N-hnr'*: (*uncurry0* (*return* N), *uncurry0* (*RETURN* N))$\in$*unit-assn*^{k} $\rightarrow_a$ *nat-assn*
by *sepref-to-hoare sep-auto*

Next, we use an array-list with a fixed maximum capacity. Note that the capacity is part of the refinement assertion now.

sepref-definition *remdup1-fixed* **is** *remdup* :: $(list\text{-}assn\ nat\text{-}assn)^k \rightarrow_a marl\text{-}assn$ $N\ nat\text{-}assn$
unfolding *remdup-def*[*abs-def*]
apply (*rewrite* **in** *nfoldli* - - - ⌑ *ias-sz.fold-custom-empty*[*of N*])
apply (*rewrite* **in** *nfoldli* - - - ⌑ *marl-fold-custom-empty-sz*[*of N*])
supply [[*goals-limit* = *1*]]
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
apply *sepref-dbg-trans-step-keep*
— In order to append to the array list, we have to show that the size is not yet exceeded. This may require to add some assertions on the abstract level. We already have added some assertions in the definition of *remdup*.
oops

Moreover, we add a precondition on the list

sepref-definition *remdup1-fixed* **is** *remdup* :: $[\lambda l.\ set\ l \subseteq \{0..<N\}]_a\ (list\text{-}assn\ nat\text{-}assn)^k \rightarrow marl\text{-}assn\ N\ nat\text{-}assn$
unfolding *remdup-def*[*abs-def*]
apply (*rewrite* **in** *nfoldli* - - - ⌑ *ias-sz.fold-custom-empty*[*of N*])
apply (*rewrite* **in** *nfoldli* - - - ⌑ *marl-fold-custom-empty-sz*[*of N*])
supply [[*goals-limit* = *1*]]
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
apply *sepref-dbg-trans-step-keep*
apply *sepref-dbg-side-keep*
— We can start from this subgoal to find missing lemmas
oops

We can prove the remaining subgoal, e.g., by *auto* with the following lemma declared as introduction rule:

lemma *aux1*[*intro*]: $\llbracket\ set\ l \subset \{0..<N\};\ distinct\ l\ \rrbracket \Longrightarrow length\ l < N$
apply (*simp add*: *distinct-card*[*symmetric*])
apply (*drule psubset-card-mono*[*rotated*])
apply *auto*
done

We use some standard boilerplate to define the constant globally, although being inside the locale. This is required for code-generation.

sepref-thm *remdup1-fixed* **is** *remdup* :: $[\lambda l.\ set\ l \subseteq \{0..<N\}]_a\ (list\text{-}assn\ nat\text{-}assn)^k \rightarrow marl\text{-}assn\ N\ nat\text{-}assn$
unfolding *remdup-def*[*abs-def*]
apply (*rewrite* **in** *nfoldli* - - - ⌑ *ias-sz.fold-custom-empty*[*of N*])
apply (*rewrite* **in** *nfoldli* - - - ⌑ *marl-fold-custom-empty-sz*[*of N*])
by *sepref*

concrete-definition (**in** −) *remdup1-fixed* **uses** *my-remdup-impl-loc.remdup1-fixed.refine-raw* **is** (*?f*,-)∈-
prepare-code-thms (**in** −) *remdup1-fixed-def*

lemmas *remdup1-fixed-refine*[*sepref-fr-rules*] = *remdup1-fixed.refine*[*OF my-remdup-impl-loc-this*]

The **concrete-definition** command defines the constant globally, without any locale assumptions. For this, it extracts the definition from the theorem, according to the specified pattern. Note, you have to include the uncurrying into the pattern, e.g., (*uncurry ?f*,-)∈-.

The **prepare-code-thms** command sets up code equations for recursion combinators that may have been synthesized. This is required as the code generator works with equation systems, while the heap-monad works with fixed-point combinators.

Finally, the third lemma command imports the refinement lemma back into the locale, and registers it as refinement rule for Sepref.

Now, we can refine *remdup* to *remdup1-fixed N* inside the locale. The latter is a global constant with an unconditional definition, thus code can be generated for it.

Inside the locale, we can do some more refinements:

definition *test-remdup* ≡ *do* {*l* ← *remdup* [*0..<N*]; *RETURN* (*length l*) }

Note that the abstract *test-remdup* is just an abbreviation for *test-remdup*. Whenever we want Sepref to treat a compound term like a constant, we have to wrap the term into a *PR-CONST* tag. While **sepref-register** does this automatically, the *PR-CONST* has to occur in the refinement rule.

sepref-register *test-remdup*
sepref-thm *test-remdup1* **is**
uncurry0 (*PR-CONST test-remdup*) :: $\textit{unit-assn}^k \rightarrow_a \textit{nat-assn}$
unfolding *test-remdup-def PR-CONST-def*
by *sepref*
concrete-definition (**in** −) *test-remdup1* **uses** *my-remdup-impl-loc.test-remdup1.refine-raw*
is (*uncurry0 ?f*,-)∈-
prepare-code-thms (**in** −) *test-remdup1-def*
lemmas *test-remdup1-refine*[*sepref-fr-rules*] = *test-remdup1.refine*[*of N*]

end

Outside the locale, a refinement of *my-remdup-impl-loc.test-remdup* also makes sense, however, with an extra argument *N*.

thm *test-remdup1.refine*
lemma *test-remdup1-refine-aux*: (*test-remdup1*, *my-remdup-impl-loc.test-remdup*)
∈ $[\textit{my-remdup-impl-loc}]_a\ \textit{nat-assn}^k \rightarrow \textit{nat-assn}$
using *test-remdup1.refine*
unfolding *hfref-def hn-refine-def*
by (*auto simp*: *pure-def*)

We can also write a more direct precondition, as long as it implies the locale

lemma *test-remdup1-refine*: (*test-remdup1*, *my-remdup-impl-loc.test-remdup*) ∈ [λ*N*. *N*>*0*]$_a$ *nat-assn*^{k} → *nat-assn*
  apply (*rule hfref-cons*[*OF test-remdup1-refine-aux - entt-refl entt-refl entt-refl*])
  by *unfold-locales*

export-code *test-remdup1* **checking** *SML*

We can also register the abstract constant and the refinement, to use it in further refinements

sepref-register *my-remdup-impl-loc.test-remdup*
lemmas [*sepref-fr-rules*] = *test-remdup1-refine*

Static Data Structures with Custom Element Relations

In the previous section, we have presented a refinement using an array-list without dynamic resizing. However, the argument that we actually could append to this array was quite complicated.

Another possibility is to use bounded refinement relations, i.e., a refinement relation intersected with a condition for the abstract object. In our case, *b-assn id-assn* (λ*x*. *x* < *N*) relates natural numbers less than *N* to themselves.

We will repeat the above development, using the bounded relation approach:

definition *bremdup l* ≡ *do* {
  (*s*,*r*) ← *nfoldli l* (λ-. *True*)
    (λ*x* (*s*,*r*). *do* {
      ASSERT (*distinct r* ∧ *s* = *set r*); — Less assertions than last time
      if x∈*s then RETURN* (*s*,*r*) *else RETURN* (*insert x s*, *r*@[*x*])
    })
    ({},[]);
  RETURN r
}
sepref-register *bremdup*

locale *my-bremdup-impl-loc* =
  fixes *N* :: *nat*
  assumes *N*>*0* — This assumption is not necessary, but used to illustrate the general case, where the locale may have such assumptions
begin
  lemma *my-bremdup-impl-loc-this*: *my-bremdup-impl-loc N* **by** *unfold-locales*

  sepref-register *N*
  lemma *N-hnr*[*sepref-import-param*]: (*N*,*N*)∈*nat-rel* **by** *simp*

Conceptually, what we insert in our list are elements, and these are less than *N*.

  abbreviation *elem-assn* ≡ *nbn-assn N*

lemma *aux1*[*intro*]: $[\![$ *set l* $\subset$ {*0..*<*N*}; *distinct l* $]\!]$ $\Longrightarrow$ *length l* < *N*
apply (*simp add*: *distinct-card*[*symmetric*])
apply (*drule psubset-card-mono*[*rotated*])
apply *auto*
done

sepref-thm *remdup1-fixed* **is** *remdup* :: [λ*l. set l* $\subseteq$ {*0..*<*N*}]$_a$ (*list-assn elem-assn*)^k $\rightarrow$ *marl-assn N elem-assn*
unfolding *remdup-def*[*abs-def*]
apply (*rewrite* **in** *nfoldli - - -* ⌑ *ias-sz.fold-custom-empty*[*of N*])
apply (*rewrite* **in** *nfoldli - - -* ⌑ *marl-fold-custom-empty-sz*[*of N*])
by *sepref*

concrete-definition (**in** −) *bremdup1-fixed* **uses** *my-bremdup-impl-loc.remdup1-fixed.refine-raw* **is** (*?f*,-)∈-
prepare-code-thms (**in** −) *bremdup1-fixed-def*
lemmas *remdup1-fixed-refine*[*sepref-fr-rules*] = *bremdup1-fixed.refine*[*OF my-bremdup-impl-loc-this*]

definition *test-remdup* ≡ *do* {*l* ← *remdup* [*0..*<*N*]; *RETURN* (*length l*) }
sepref-register *test-remdup*

This refinement depends on the (somewhat experimental) subtyping feature to convert from *id-assn* to *b-assn id-assn* ($\lambda x.\ x < N$), based on context information

sepref-thm *test-remdup1* **is**
uncurry0 (*PR-CONST test-remdup*) :: *unit-assn*^{k} $\rightarrow_a$ *nat-assn*
unfolding *test-remdup-def PR-CONST-def*
by *sepref*

concrete-definition (**in** −) *test-bremdup1* **uses** *my-bremdup-impl-loc.test-remdup1.refine-raw* **is** (*uncurry0 ?f*,-)∈-
prepare-code-thms (**in** −) *test-bremdup1-def*
lemmas *test-remdup1-refine*[*sepref-fr-rules*] = *test-bremdup1.refine*[*of N*]

end

lemma *test-bremdup1-refine-aux*: (*test-bremdup1*, *my-bremdup-impl-loc.test-remdup*) ∈ [*my-bremdup-impl-loc*]$_a$ *nat-assn*^{k} $\rightarrow$ *nat-assn*
using *test-bremdup1.refine*
unfolding *hfref-def hn-refine-def*
by (*auto simp*: *pure-def*)

lemma *test-bremdup1-refine*: (*test-bremdup1*, *my-bremdup-impl-loc.test-remdup*) ∈ [$\lambda N.\ N{>}0$]$_a$ *nat-assn*^{k} $\rightarrow$ *nat-assn*
apply (*rule hfref-cons*[*OF test-bremdup1-refine-aux - entt-refl entt-refl entt-refl*])
by *unfold-locales*

export-code *test-bremdup1* **checking** *SML*

We can also register the abstract constant and the refinement, to use it in further refinements

sepref-register *test-bremdup*: *my-bremdup-impl-loc.test-remdup* — Specifying a base-name for the theorems here, as default name clashes with existing names.
lemmas [*sepref-fr-rules*] = *test-bremdup1-refine*

Fixed-Value Restriction

Initialization only works with fixed values, not with dynamically computed values

sepref-definition *copy-list-to-array* **is** $\lambda l.$ *do* {
let N = *length l*; — Introduce a *let*, such that we have a single variable as size-init
let l' = *op-arl-empty-sz N*;
nfoldli l ($\lambda x.$ *True*) (λx *s. mop-list-append s x*) l'
} :: (*list-assn nat-assn*)^k $\rightarrow_a$ *arl-assn nat-assn*
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
apply *sepref-dbg-trans-step-keep*
supply [[*unify-trace-failure*, *goals-limit=1*]]

— The problem manifests itself in trying to carry an abstract variable (the argument to *op-arl-empty-sz*) to the concrete program (the second argument of *hn-refine*). However, the concrete program can only depend on the concrete variables, so unification fails.
oops

Matrix Example

We first give an example for implementing point-wise matrix operations, using some utilities from the (very prototype) matrix library.

Our matrix library uses functions *'a mtx* (which is *nat* × *nat* ⇒ *'a*) as the abstract representation. The (currently only) implementation is by arrays, mapping points at coordinates out of range to *0*.

Pointwise unary operations are those that modify every point of a matrix independently. Moreover, a zero-value must be mapped to a zero-value. As an example, we duplicate every value on the diagonal of a matrix

Abstractly, we apply the following function to every value. The first parameter are the coordinates.

definition *mtx-dup-diag-f*:: *nat*×*nat* ⇒ *'a*::{*numeral*,*times*,*mult-zero*} ⇒ *'a*
where *mtx-dup-diag-f* ≡ $\lambda(i,j)$ *x. if i=j then x*∗(*2*) *else x*

We refine this function to a heap-function, using the identity mapping for values.

context

fixes *dummy* :: *'a*::{*numeral*,*times*,*mult-zero*}
notes [[*sepref-register-adhoc PR-CONST* (*2*::*'a*)]]
— Note: The setup for numerals, like *2*, is a bit subtle in that numerals are always treated as constants, but have to be registered for any type they shall be used with. By default, they are only registered for *int* and *nat*.
notes [*sepref-import-param*] = *IdI*[*of PR-CONST* (*2*::*'a*)]
notes [*sepref-import-param*] = *IdI*[*of* (∗)::*'a*⇒-, *folded fun-rel-id-simp*]
begin

sepref-definition *mtx-dup-diag-f1* **is** *uncurry* (*RETURN oo* (*mtx-dup-diag-f*::-⇒*'a*⇒-))
:: $(prod\text{-}assn\ nat\text{-}assn\ nat\text{-}assn)^k *_a id\text{-}assn^k \rightarrow_a id\text{-}assn$
unfolding *mtx-dup-diag-f-def*
by *sepref*

end

Then, we instantiate the corresponding locale, to get an implementation for array matrices. Note that we restrict ourselves to square matrices here:

interpretation *dup-diag*: *amtx-pointwise-unop-impl N N mtx-dup-diag-f id-assn mtx-dup-diag-f1*
apply *standard*
applyS (*simp add*: *mtx-dup-diag-f-def*) []
applyS (*rule mtx-dup-diag-f1.refine*)
done

We introduce an abbreviation for the abstract operation. Note: We do not have to register it (this is done once and for all for *mtx-pointwise-unop*), nor do we have to declare a refinement rule (done by *amtx-pointwise-unop-impl*-locale)

abbreviation *mtx-dup-diag* ≡ *mtx-pointwise-unop mtx-dup-diag-f*

The operation is usable now:

sepref-thm *mtx-dup-test* **is** *λm. RETURN* (*mtx-dup-diag* (*mtx-dup-diag m*)) ::
$(asmtx\text{-}assn\ N\ int\text{-}assn)^d \rightarrow_a asmtx\text{-}assn\ N\ int\text{-}assn$
by *sepref*

Similarly, there are operations to combine to matrices, and to compare two matrices:

interpretation *pw-add*: *amtx-pointwise-binop-impl N M* (((+))::(-::*monoid-add*) ⇒ -) *id-assn return oo* ((+))
for *N M*
apply *standard*
apply *simp*
apply (*sepref-to-hoare*) **apply** *sep-auto* — Alternative to synthesize concrete operation, for simple ad-hoc refinements
done
abbreviation *mtx-add* ≡ *mtx-pointwise-binop* ((+))

sepref-thm *mtx-add-test* **is** *uncurry2* (λ*m1 m2 m3*. *RETURN* (*mtx-add m1* (*mtx-add m2 m3*)))
:: (*amtx-assn N M int-assn*)^d $*_a$ (*amtx-assn N M int-assn*)^d $*_a$ (*amtx-assn N M int-assn*)^k $\rightarrow_a$ *amtx-assn N M int-assn*
by *sepref*

A limitation here is, that the first operand is destroyed on a coarse-grained level. Although adding a matrix to itself would be valid, our tool does not support this. (However, you may use an unary operation)

sepref-thm *mtx-dup-alt-test* **is** (λ*m*. *RETURN* (*mtx-add m m*))
:: (*amtx-assn N M int-assn*)^d $\rightarrow_a$ *amtx-assn N M int-assn*
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
— We get stuck at a *COPY* goal, indicating that a matrix has to be copied.
apply *sepref-dbg-trans-step-keep*
— Which only works for pure refinements
oops

Of course, you can always copy the matrix manually:

sepref-thm *mtx-dup-alt-test* **is** (λ*m*. *RETURN* (*mtx-add* (*op-mtx-copy m*) *m*))
:: (*amtx-assn N M int-assn*)^k $\rightarrow_a$ *amtx-assn N M int-assn*
by *sepref*

A compare operation checks that all pairs of entries fulfill some property *f*, and at least one entry fullfills a property *g*.

interpretation *pw-lt*: *amtx-pointwise-cmpop-impl N M* (($\leq$)::(-::*order*) $\Rightarrow$ -) (($\neq$)::(-::*order*) $\Rightarrow$ -) *id-assn return oo* ($\leq$) *return oo* ($\neq$)
for *N M*
apply *standard*
apply *simp*
apply *simp*
apply (*sepref-to-hoare*) **apply** *sep-auto*
apply (*sepref-to-hoare*) **apply** *sep-auto*
done
abbreviation *mtx-lt* $\equiv$ *mtx-pointwise-cmpop* ($\leq$) ($\neq$)

sepref-thm *test-mtx-cmp* **is** (λ*m*. *do* { *RETURN* (*mtx-lt* (*op-amtx-dfltNxM N M 0*) *m*) }) :: (*amtx-assn N M int-assn*)^k $\rightarrow_a$ *bool-assn*
by *sepref* — Note: Better fold over single matrix (currently no locale for that), instead of creating a new matrix.

In a final example, we store some coordinates in a set, and then use the stored coordinates to access the matrix again. This illustrates how bounded relations can be used to maintain extra information, i.e., coordinates being in range

context
fixes *N M* :: *nat*
notes [[*sepref-register-adhoc N M*]]

notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of M*]
begin

We introduce an assertion for coordinates

abbreviation *co-assn* ≡ *prod-assn* (*nbn-assn N*) (*nbn-assn M*)

And one for integer matrices

abbreviation *mtx-assn* ≡ *amtx-assn N M int-assn*

definition *co-set-gen* ≡ *do* {
nfoldli [*0..<N*] (λ-. *True*) (λ*i*. *nfoldli* [*0..<M*] (λ-. *True*) (λ*j s*.
if max i j − *min i j* ≤ *1 then RETURN* (*insert* (*i,j*) *s*)
else RETURN s
)) {}
}

sepref-definition *co-set-gen1* **is** *uncurry0 co-set-gen* :: *unit-assn*^{k} $\rightarrow_a$ *hs.assn co-assn*
unfolding *co-set-gen-def*
apply (*rewrite hs.fold-custom-empty*)
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
— We run into the problem that the Sepref tool uses *nat-assn* to refine natural numbers, and only later tries to convert it to *nbn-assn*. However, at this point, the information is already lost.
oops

We can use a feature of Sepref, to annotate the desired assertion directly into the abstract program. For this, we use *annotate-assn*, which inserts the (special) constant *ASSN-ANNOT*, which is just identity, but enforces refinement with the given assertion.

sepref-definition *co-set-gen1* **is** *uncurry0* (*PR-CONST co-set-gen*) :: *unit-assn*^{k} $\rightarrow_a$ *hs.assn co-assn*
unfolding *co-set-gen-def PR-CONST-def*
apply (*rewrite hs.fold-custom-empty*)
apply (*rewrite* **in** *insert* ⌑ - *annotate-assn*[**where** *A*=*co-assn*])
— Annotate the pair as coordinate before insertion
by *sepref*
lemmas [*sepref-fr-rules*] = *co-set-gen1.refine*

sepref-register *co-set-gen*

Now we can use the entries from the set as coordinates, without any worries about them being out of range

sepref-thm *co-set-use* **is** (λ*m*. *do* {
co ← *co-set-gen*;
FOREACH co (λ(*i,j*) *m*. *RETURN* (*m*((*i,j*) := *1*))) *m*
}) :: *mtx-assn*^{d} $\rightarrow_a$ *mtx-assn*

by *sepref*

end

4.1.10 Type Classes

TBD

4.1.11 Higher-Order

TBD

4.1.12 A-Posteriori Optimizations

The theorem collection *sepref-opt-simps* and *sepref-opt-simps2* contain simplifier lemmas that are applied, in two stages, to the generated Imperative/HOL program.

This is the place where some optimizations, such as deforestation, and simplifying monad-expressions using the monad laws, take place.

thm *sepref-opt-simps*
thm *sepref-opt-simps2*

4.1.13 Short-Circuit Evaluation

Consider

sepref-thm *test-sc-eval* **is** $RETURN\ o\ (\lambda l.\ length\ l > 0 \wedge hd\ l) :: (list\text{-}assn\ bool\text{-}assn)^k \rightarrow_a bool\text{-}assn$
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
apply *sepref-dbg-trans-step-keep*
— Got stuck, as the operands of $\wedge$ are evaluated before applying the operator, i.e., *hd* is also applied to empty lists
oops

sepref-thm *test-sc-eval* **is** $RETURN\ o\ (\lambda l.\ length\ l > 0 \wedge hd\ l) :: (list\text{-}assn\ bool\text{-}assn)^k \rightarrow_a bool\text{-}assn$
unfolding *short-circuit-conv* — Enables short-circuit evaluation by rewriting $\wedge$, $\vee$, and $\longrightarrow$ to *if*-expressions
by *sepref*

end

4.2 Reference Guide

theory *Sepref-Guide-Reference*

imports *../IICF/IICF*
begin

This guide contains a short reference of the most important Sepref commands, methods, and attributes, as well as a short description of the internal working, and troubleshooting information with examples.

Note: To get an impression how to actually use the Sepref-tool, read the quickstart guide first!

4.2.1 The Sepref Method

The *sepref* method is the central method of the tool. Given a schematic goal of the form *hn-refine* Γ *?c* ?Γ' *?R f*, it tries to synthesize terms for the schematics and prove the theorem. Note that the ?Γ' and *?R* may also be fixed terms, in which case frame inference is used to match the generated assertions with the given ones. Γ must contain a description of the available refinements on the heap, the assertion for each variable must be marked with a *hn-ctxt* tag.

Alternatively, a term of the form $(?c,f)\in[P]_a\ A\rightarrow R$ is accepted, where A describes the refinement and preservation of the arguments, and R the refinement of the result. f must be in uncurried form (i.e. have exactly one argument).

We give some very basic examples here. In practice, you would almost always use the higher-level commands **sepref-definition** and **sepref-register**.

In its most primitive form, the Sepref-tool is applied like this:

schematic-goal
  notes [*id-rules*] = *itypeI*[*of x TYPE*(*nat*)] *itypeI*[*of a TYPE*(*bool list*)]
  shows *hn-refine*
    (*hn-ctxt nat-assn x xi* ∗ *hn-ctxt* (*array-assn bool-assn*) *a ai*)
    (*?c*::*?'c Heap*) ?Γ′ *?R*
    (*do* { *ASSERT* (*x*<*length a*); *RETURN* (*a*!*x*) })
  by *sepref*

The above command asks Sepref to synthesize a program, in a heap context where there is a natural number, refined by *nat-assn*, and a list of booleans, refined by *array-assn bool-assn*. The *id-rules* declarations declare the abstract variables to the operation identification heuristics, such that they are recognized as operands.

Using the alternative hfref-form, we can write:

schematic-goal (*uncurry* (*?c*), *uncurry* (λ*x a. do* {*ASSERT* (*x*<*length a*); *RETURN* (*a*!*x*)}))
  $\in$ *nat-assn*^{k} $*_a$ (*array-assn bool-assn*)^k $\rightarrow_a$ *bool-assn*
  by *sepref*

This uses the specified assertions to derive the rules for operation identification automatically. For this, it uses the assertion-interface bindings declared in *intf-of-assn*. If there is no such binding, it uses the HOL type as interface type.

thm *intf-of-assn*

The sepref-method is split into various phases, which we will explain now

Preprocessing Phase

This tactic converts a goal in *hfref* form to the more basic *hn-refine* form. It uses the theorems from *intf-of-assn* to add interface type declarations for the generated operands. The final result is massaged by rewriting with *to-hnr-post*, and then with *sepref-preproc*.

Moreover, this phase ensures that there is a constraint slot goal (see section on constraints).

The method *sepref-dbg-preproc* gives direct access to the preprocessing phase.

thm *sepref-preproc*
thm *intf-of-assn*
thm *to-hnr-post* — Note: These rules are only instantiated for up to 5 arguments. If you have functions with more arguments, you need to add corresponding theorems here!

Consequence Rule Phase

This phase rewrites *hn-invalid* - x y assertions in the postcondition to *hn-ctxt* (λ- -. *true*) x y assertions, which are trivial to discharge. Then, it applies *CONS-init*, to make postcondition and result relation schematic, and introduce (separation logic) implications to the originals, which are discharged after synthesis.

Use *sepref-dbg-cons-init* for direct access to this phase. The method *weaken-hnr-post* performs the rewriting of *hn-invalid* to λ- -. *true* postconditions, and may be useful on its own for proving combinator rules.

Operation Identification Phase

The purpose of this phase is to identify the conceptual operations in the given program. Consider, for example, a map $m::'k \Rightarrow 'v\ option$. If one writes $m(k \mapsto v)$, this is a map update. However, in Isabelle/HOL maps are encoded as functions $'k \Rightarrow 'v\ option$, and the map update is just syntactic sugar for *fun-upd* m k (*Some* v). And, likewise, map lookup is just function application.

However, the Sepref tool must be able to distinguish between maps and functions into the option type, because maps shall be refined, to e.g., hashtables, while functions into the option type shall be not. Consider, e.g., the term *Some x*. Shall *Some* be interpreted as the constructor of the option datatype, or as a map, mapping each element to itself, and perhaps be implemented with a hashtable.

Moreover, for technical reasons, the translation phase of Sepref expects each operation to be a single constant applied to its operands. This criterion is neither matched by map lookup (no constant, just application of the first to the second operand), nor map update (complex expression, involving several constants).

The operation identification phase uses a heuristics to find the conceptual types in a term (e.g., discriminate between map and function to option), and rewrite the operations to single constants (e.g. *op-map-lookup* for map lookup). The heuristics is a type-inference algorithm combined with rewriting. Note that the inferred conceptual type does not necessarily match the HOL type, nor does it have a semantic meaning, other than guiding the heuristics.

The heuristics store a set of typing rules for constants, in *id-rules*. Moreover, it stores two sets of rewrite rules, in *pat-rules* and *def-pat-rules*. A term is typed by first trying to apply a rewrite rule, and then applying standard Hindley-Milner type inference rules for application and abstraction. Constants (and free variables) are typed using the *id-rules*. If no rule for a constant exists, one is inferred from the constant's signature. This does not work for free variables, such that rules must be available for all free variables. Rewrite rules from *pat-rules* are backtracked over, while rewrite rules from *def-pat-rules* are always tried first and never backtracked over.

If typing succeeds, the result is the rewritten term.

For example, consider the type of maps. Their interface (or conceptual) type is $('k,\ 'v)$ *i-map*. The *id-rule* for map lookup is *op-map-lookup* $::_i$ $TYPE(?'a \Rightarrow (?'a,\ ?'b)\ \textit{i-map} \Rightarrow ?'b\ \textit{option})$. Moreover, there is a rule to rewrite function application to map lookup ($?m\ \$\ ?k \equiv$ *op-map-lookup* $\$'\ ?k$ $\$'\ ?m$). It can be backtracked over, such that also functions into the option type are possible.

thm *op-map-lookup.itype*
thm *pat-map-lookup*
thm *id-rules*

The operation identification phase, and all further phases, work on a tagged version of the input term, where all function applications are replaced by the tagging constant ($), and all abstractions are replaced by $\lambda x.\ (\#t\ x\#)$ (syntax: $\lambda x.\ (\#t\ x\#)$, input syntax: $\lambda x.\ (\#t\ x\#)$). This is required to tame Isabelle's higher-order unification. However, it makes tagged terms quite unreadable, and it may be helpful to *unfold APP-def PROTECT2-def* to get back the untagged form when inspecting internal states for debugging purposes.

To prevent looping, rewrite-rules can use ($\$'$) on the RHS. This is a synonym for (\$), and gets rewritten to (\$) after the operation identification phase. During the operation identification phase, it prevents infinite loops of pattern rewrite rules.

Interface type annotations can be added to the term using ($:::_i$) (syntax t $:::_i$ $TYPE('a)$).

In many cases, it is desirable to treat complex terms as a single constant, a standard example are constants defined inside locales, which may have locale parameters attached. Those terms can be wrapped into an *PR-CONST* tag, which causes them to be treated like a single constant. Such constants must always have *id-rules*, as the interface type inference from the signature does not apply here.

Troubleshooting Operation Identification

If the operation identification fails, in most cases one has forgotten to register an *id-rule* for a free variable or complex *PR-CONST* constant, or the identification rule is malformed. Note that, in practice, identification rules are registered by the **sepref-register** (see below), which catches many malformed rules, and handles *PR-CONST* tagging automatically. Another frequent source of errors here is forgetting to register a constant with a conceptual type other than its signature. In this case, operation identification gets stuck trying to unify the signature's type with the interface type, e.g., $'k \Rightarrow 'v$ *option* with $('k, 'v)$ *i-map*.

The method *sepref-dbg-id* invokes the id-phase in isolation. The method *sepref-dbg-id-keep* returns the internal state where type inference got stuck. It returns a sequence of all stuck states, which can be inspected using **back**.

The methods *sepref-dbg-id-init*,*sepref-dbg-id-step*, and *sepref-dbg-id-solve* can be used to single-step the operation identification phase. Here, solve applies single steps until the current subgoal is discharged. Be aware that application of single steps allows no automatic backtracking, such that backtracking has to be done manually.

Examples for identification errors

context
  fixes *N*::*nat*
  notes [*sepref-import-param*] = *IdI*[*of N*]
begin
  sepref-thm *N-plus-2-example* **is** *uncurry0* (*RETURN* (*N*+*2*)) :: $unit\text{-}assn^k \rightarrow_a$ *nat-assn*
    apply *sepref-dbg-keep*

apply *sepref-dbg-id-keep*
— Forgot to register n
oops

Solution: Register n, be careful not to export meaningless registrations from context!

context
notes [[*sepref-register-adhoc N*]]
begin
sepref-thm *N-plus-2-example* **is** *uncurry0* (*RETURN* (*N*+*2*)) :: *unit-assn*^{k} $\rightarrow_a$ *nat-assn* **by** *sepref*
end
end

definition *my-map* ≡ *op-map-empty*
lemmas [*sepref-fr-rules*] = *hm.empty-hnr*[*folded my-map-def*]

sepref-thm *my-map-example* **is** *uncurry0* (*RETURN* (*my-map*(*False*↦*1*))) :: *unit-assn*^{k} $\rightarrow_a$ *hm.assn bool-assn nat-assn*
apply *sepref-dbg-keep*
apply *sepref-dbg-trans-keep*
— Stuck at refinement for function update on map
oops

Solution: Register with correct interface type

sepref-register *my-map* :: ('*k*,'*v*) *i-map*
sepref-thm *my-map-example* **is** *uncurry0* (*RETURN* (*my-map*(*False*↦*1*))) :: *unit-assn*^{k} $\rightarrow_a$ *hm.assn bool-assn nat-assn*
by *sepref*

Monadify Phase

The monadify phase rewrites the program such that every operation becomes visible on the monad level, that is, nested HOL-expressions are flattened. Also combinators (e.g. if, fold, case) may get flattened, if special rules are registered for that.

Moreover, the monadify phase fixes the number of operands applied to an operation, using eta-expansion to add missing operands.

Finally, the monadify phase handles duplicate parameters to an operation, by inserting a *COPY* tag. This is necessary as our tool expects the parameters of a function to be separate, even for read-only parameters[1].

The monadify phase consists of a number of sub-phases. The method *sepref-dbg-monadify* executes the monadify phase, the method *sepref-dbg-monadify-keep*

[1] Using fractional permissions or some other more fine grained ownership model might lift this restriction in the future.

stops at a failing sub-phase and presents the internal goal state before the failing sub-phase.

Monadify: Arity

In the first sub-phase, the rules from *sepref-monadify-arity* are used to standardize the number of operands applied to a constant. The rules work by rewriting each constant to a lambda-expression with the desired number of arguments, and the using beta-reduction to account for already existing arguments. Also higher-order arguments can be enforced, for example, the rule for fold enforces three arguments, the function itself having two arguments ($fold \equiv \lambda x.\ (\#\lambda xa.\ (\#\lambda xb.\ (\#SP\ fold\ \$\ (\lambda xa.\ (\#\lambda xb.\ (\#x\ \$\ xa\ \$\ xb\#)\#))\ \$\ xa\ \$\ xb\#)\#)\#))$).

In order to prevent arity rules being applied infinitely often, the *SP* tag can be used on the RHS. It prevents anything inside from being changed, and gets removed after the arity step.

The method *sepref-dbg-monadify-arity* gives you direct access to this phase.

In the Sepref-tool, we use the terminology *operator/operation* for a function that only has first-order arguments, which are evaluated before the function is applied (e.g. (+)), and *combinator* for operations with higher-order arguments or custom evaluation orders (e.g. *fold*, *If*).

Note: In practice, most arity (and combinator) rules are declared automatically by **sepref-register** or **sepref-decl-op**. Manual declaration is only required for higher-order functions.

thm *sepref-monadify-arity*

Monadify: Combinators

The second sub-phase flattens the term. It has a rule for every function into - *nres* type, that determines the evaluation order of the arguments. First-order arguments are evaluated before an operation is applied. Higher-order arguments are treated specially, as they are evaluated during executing the (combinator) operation. The rules are in *sepref-monadify-comb*.

Evaluation of plain (non-monadic) terms is triggered by wrapping them into the *EVAL* tag. The *sepref-monadify-comb* rules may also contain rewrite-rules for the *EVAL* tag, for example to unfold plain combinators into the monad (e.g. $EVAL\ \$\ (If\ \$\ ?b\ \$\ ?t\ \$\ ?e) \equiv (\gg=)\ \$\ (EVAL\ \$\ ?b)\ \$\ (\lambda x.\ (\#If\ \$\ x\ \$\ (EVAL\ \$\ ?t)\ \$\ (EVAL\ \$\ ?e)\#))$)

$EVAL\ \$\ (case\text{-}list\ \$\ ?fn\ \$\ (\lambda x.\ (\#\lambda xa.\ (\#?fc\ x\ xa\#)\#))\ \$\ ?l) \equiv (\gg=)\ \$\ (EVAL\ \$\ ?l)\ \$\ (\lambda x.\ (\#case\text{-}list\ \$\ (EVAL\ \$\ ?fn)\ \$\ (\lambda x.\ (\#\lambda xa.\ (\#EVAL\ \$\ ?fc\ x\ xa\#)\#))\ \$\ x\#))$

$EVAL\ \$\ (case\text{-}prod\ \$\ (\lambda x.\ (\#\lambda xa.\ (\#?fp\ x\ xa\#)\#))\ \$\ ?p) \equiv (\gg=)\ \$\ (EVAL\ \$\ ?p)\ \$\ (\lambda x.\ (\#case\text{-}prod\ \$\ (\lambda x.\ (\#\lambda xa.\ (\#EVAL\ \$\ ?fp\ x\ xa\#)\#))\ \$\ x\#))$

$EVAL$ \$ ($case\text{-}option$ \$ $?fn$ \$ ($\lambda x.$ (#$?fs$ x#)) \$ $?ov$) $\equiv$ ($\gg=$) \$ ($EVAL$ \$ $?ov$) \$ ($\lambda x.$ (#$case\text{-}option$ \$ ($EVAL$ \$ $?fn$) \$ ($\lambda x.$ (#$EVAL$ \$ $?fs$ x#)) \$ x#))
$EVAL$ \$ (Let \$ $?v$ \$ ($\lambda x.$ (#$?f$ x#))) $\equiv$ ($\gg=$) \$ ($EVAL$ \$ $?v$) \$ ($\lambda x.$ (#$EVAL$ \$ $?f$ x#))). If no such rule applies, the default method is to interpret the head of the term as a function, and recursively evaluate the arguments, using left-to-right evaluation order. The head of a term inside $EVAL$ must not be an abstraction. Otherwise, the $EVAL$ tag remains in the term, and the next sub-phase detects this and fails.

The method *sepref-dbg-monadify-comb* executes the combinator-phase in isolation.

Monadify: Check-Eval

This phase just checks for remaining $EVAL$ tags in the term, and fails if there are such tags. The method *sepref-dbg-monadify-check-EVAL* gives direct access to this phase.

Remaining $EVAL$ tags indicate higher-order functions without an appropriate setup of the combinator-rules being used. For example:

definition *my-fold* $\equiv$ *fold*
sepref-thm *my-fold-test* **is** $\lambda l.$ *do* { *RETURN* (*my-fold* ($\lambda x\ y.\ x+y*2$) *l* *0*)} :: (*list-assn nat-assn*)$^k \rightarrow_a$*nat-assn*
  apply *sepref-dbg-keep*
  apply *sepref-dbg-monadify-keep*
  — An *EVAL*-tag with an abstraction remains. This is b/c the default heuristics tries to interpret the function inside the fold as a plain value argument.
  oops

Solution: Register appropriate arity and combinator-rules

lemma *my-fold-arity*[*sepref-monadify-arity*]: *my-fold* $\equiv$ $\lambda_2 f\ l\ s.$ *SP* *my-fold*\$($\lambda_2 x\ s.\ f$\$$x$\$$s$)\$$l$\$$s$ **by** *auto*

The combinator-rule rewrites to the already existing and set up combinator *nfoldli*:

lemma *monadify-plain-my-fold*[*sepref-monadify-comb*]:
  EVAL\$(*my-fold*\$($\lambda_2 x\ s.\ f\ x\ s$)\$$l$\$$s$) $\equiv$ ($\gg=$)\$(*EVAL*\$$l$)\$($\lambda_2 l.$ ($\gg=$)\$(*EVAL*\$$s$)\$($\lambda_2 s.$ *nfoldli*\$$l$\$(λ_2-. *True*)\$($\lambda_2 x\ s.$ *EVAL*\$($f\ x\ s$))\$$s$))
  by (*simp add*: *fold-eq-nfoldli my-fold-def*)

sepref-thm *my-fold-test* **is** $\lambda l.$ *do* { *RETURN* (*my-fold* ($\lambda x\ y.\ x+y*2$) *l* *0*)} :: (*list-assn nat-assn*)$^k \rightarrow_a$*nat-assn*
  by *sepref*

Monadify: Dup

The last three phases, *mark-params*, *dup*, *remove-pass* are to detect duplicate parameters, and insert *COPY* tags. The first phase, *mark-params*, adds *PASS* tags around all parameters. Parameters are bound variables and terms that have a refinement in the precondition.

The second phase detects duplicate parameters and inserts *COPY* tags to remove them. Finally, the last phase removes the *PASS* tags again.

The methods *sepref-dbg-monadify-mark-params*, *sepref-dbg-monadify-dup*, and *sepref-dbg-monadify-remove-pass* gives you access to these phases.

Monadify: Step-Through Example

We give an annotated example of the monadify phase. Note that the program utilizes a few features of monadify:

- The fold function is higher-order, and gets flattened
- The first argument to fold is eta-contracted. The missing argument is added.
- The multiplication uses the same argument twice. A copy-tag is inserted.

sepref-thm *monadify-step-thru-test* **is** $\lambda l.\ do\ \{$
$\quad let\ i = length\ l;$
$\quad RETURN\ (fold\ (\lambda x.\ (+)\ (x{*}x))\ l\ i)$
$\}\ ::\ (list\text{-}assn\ nat\text{-}assn)^k \to_a nat\text{-}assn$
apply *sepref-dbg-preproc*
apply *sepref-dbg-cons-init*
apply *sepref-dbg-id*

apply *sepref-dbg-monadify-arity* — Second operand of fold-function is added
apply *sepref-dbg-monadify-comb* — Flattened. *fold* rewritten to *monadic-nfoldli.*

apply *sepref-dbg-monadify-check-EVAL* — No *EVAL* tags left
apply *sepref-dbg-monadify-mark-params* — Parameters marked by *PASS.* Note the multiplication $x{*}x$.

apply *sepref-dbg-monadify-dup* — *COPY* tag inserted.

apply *sepref-dbg-monadify-remove-pass* — *PASS* tag removed again

apply *sepref-dbg-opt-init*
apply *sepref-dbg-trans*
apply *sepref-dbg-opt*

apply *sepref-dbg-cons-solve*
apply *sepref-dbg-cons-solve*
apply *sepref-dbg-constraints*
done

Optimization Init Phase

This phase, accessed by *sepref-dbg-opt-init*, just applies the rule $[\![$*hn-refine* $?\Gamma$ $?c$ $?\Gamma'$ $?R$ $?a$; *CNV* $?c$ $?c'$$]\!] \Longrightarrow$ *hn-refine* $?\Gamma$ $?c'$ $?\Gamma'$ $?R$ $?a$ to set up a subgoal for a-posteriori optimization

Translation Phase

The translation phase is the main phase of the Sepref tool. It performs the actual synthesis of the imperative program from the abstract one. For this, it integrates various components, among others, a frame inference tool, a semantic side-condition solver and a monotonicity prover.

The translation phase consists of two major sub-phases: Application of translation rules and solving of deferred constraints.

The method *sepref-dbg-trans* executes the translation phase, *sepref-dbg-trans-keep* executes the translation phase, presenting the internal goal state of a failed sub-phase.

The translation rule phase repeatedly applies translation steps, until the subgoal is completely solved.

The main idea of the translation phase is, that for every abstract variable x in scope, the precondition contains an assertion of the form *hn-ctxt* A x xi, indicating how this variable is implemented. Common abbreviations are *hn-val* R x xi $\equiv$ *hn-val* R x xi and *hn-invalid* A x xi $\equiv$ *hn-invalid* A x xi.

Translation: Step

A translation step applies a single synthesis step for an operator, or solves a deferred side-condition.

There are two types of translation steps: Combinator steps and operator steps. A combinator step consists of applying a rule from *sepref-comb-rules* to the goal-state. If no such rule applies, the rules are tried again after rewriting the precondition with *sepref-frame-normrel-eqs* (see frame-inference). The premises of the combinator rule become new subgoals, which are solved by subsequent steps. No backtracking is applied over combinator rules. This restriction has been introduced to make the tool more deterministic, and hence more manageable.

An operator step applies an operator rule (from *sepref-fr-rules*) with frame-inference, and then tries to solve the resulting side conditions immediately.

If not all side-conditions can be solved, it backtracks over the application of the operator rule.

Note that, currently, side conditions to operator rules cannot contain synthesis goals themselves. Again, this restriction reduces the tool's complexity by avoiding deep nesting of synthesis. However, it hinders the important feature of generic algorithms, where an operation can issue synthesis subgoals for required operations it is built from (E.g., set union can be implemented by insert and iteration). Our predecessor tool, Autoref, makes heavy use of this feature, and we consider dropping the restriction in the near future.

An operator-step itself consists of several sub-phases:

Align goal Splits the precondition into the arguments actually occurring in the operation, and the rest (called frame).

Frame rule Applies a frame rule to focus on the actual arguments. Moreover, it inserts a subgoal of the form *RECOVER-PURE* $\Gamma\ \Gamma'$, which is used to restore invalidated arguments if possible. Finally, it generates an assumption of the form *vassn-tag* Γ', which means that the precondition holds on some heap. This assumption is used to extract semantic information from the precondition during side-condition solving.

Recover pure This phase tries to recover invalidated arguments. An invalidated argument is one that has been destroyed by a previous operation. It occurs in the precondition as *hn-invalid* $A\ x\ xi$, which indicates that there exists a heap where the refinement holds. However, if the refinement assertion A does not depend on the heap (is *pure*), the invalidated argument can be recovered. The purity assumption is inserted as a constraint (see constraints), such that it can be deferred.

Apply rule This phase applies a rule from *sepref-fr-rules* to the subgoal. If there is no matching rule, matching is retried after rewriting the precondition with *sepref-frame-normrel-eqs*. If this does not succeed either, a consequence rule is used on the precondition. The implication becomes an additional side condition, which will be solved by the frame inference tool.

To avoid too much backtracking, the new precondition is massaged to have the same structure as the old one, i.e., it contains a (now schematic) refinement assertion for each operand. This excludes rules for which the frame inference would fail anyway.

If a matching rule is found, it is applied and all new subgoals are solved by the side-condition solver. If this fails, the tool backtracks over the application of the *sepref-fr-rules*-rules. Note that direct matches prevent precondition simplification, and matches after precondition simplification prevent the consequence rule to be applied.

The method *sepref-dbg-trans-step* performs a single translation step. The method *sepref-dbg-trans-step-keep* presents the internal goal state on failure. If it fails in the *apply−rule* phase, it presents the sequence of states with partially unsolved side conditions for all matching rules.

Translation: Side Conditions

The side condition solver is used to discharge goals that arise as side-conditions to the translation rules. It does a syntactic discrimination of the side condition type, and then invokes the appropriate solver. Currently, it supports the following side conditions:

Merge (-$\vee_A$- $\Longrightarrow_t$ -). These are used to merge postconditions from different branches of the program (e.g. after an if-then-else). They are solved by the frame inference tool (see section on frame inference).

Frame (- $\Longrightarrow_t$ -). Used to match up the current precondition against the precondition of the applied rule. Solved by the frame inference tool (see section on frame inference).

Independence (*INDEP* (*?R* x_1 ... x_n)). Deprecated. Used to instantiate a schematic variable such that it does not depend on any bound variables any more. Originally used to make goals more readable, we are considering of dropping this.

Constraints (*CONSTRAINT* - -) Apply solver for deferrable constraints (see section on constraints).

Monotonicity (*mono-Heap* -) Apply monotonicity solver. Monotonicity subgoals occur when translating recursion combinators. Monadic expressions are monotonic by construction, and this side-condition solver just forwards to the monotonicity prover of the partial function package, after stripping any preconditions from the subgoal, which are not supported by the case split mechanism of the monotonicity prover (as of Isabelle2016).

Prefer/Defer (*PREFER-tag* -/*DEFER-tag*). Deprecated. Invoke the tagged solver of the Autoref tool. Used historically for importing refinements from the Autoref tool, but as Sepref becomes more complete imports from Autoref are not required any more.

Resolve with Premise *RPREM* - Resolve subgoal with one of its premises. Used for translation of recursion combinators.

Generic Algorithm *GEN-ALGO* - - Triggers resolution with a rule from *sepref-gen-algo-rules*. This is a poor-man's version of generic algorithm, which is currently only used to synthesize to-list conversions for foreach-loops.

Fallback (Any pattern not matching the above, nor being a *hn-refine* goal). Unfolds the application and abstraction tagging, as well as *bind-ref-tag* tags which are inserted by several translation rules to indicate the value a variable has been bound to, and then tries to solve the goal by *auto*, after freezing schematic variables. This tactic is used to discharge semantic side conditions, e.g., in-range conditions for array indexing.

Methods: *sepref-dbg-side* to apply a side-condition solving step, *sepref-dbg-side-unfold* to apply the unfolding of application and binding tags and *sepref-dbg-side-keep* to return the internal state after failed side-condition solving.

Translation: Constraints

During the translation phase, the refinement of operands is not always known immediately, such that schematic variables may occur as refinement assertions. Side conditions on those refinement assertions cannot be discharged until the schematic variable gets instantiated.

Thus, side conditions may be tagged with *CONSTRAINT*. If the side condition solver encounters a constraint side condition, it first removes the constraint tag ($?P\ ?x \Longrightarrow CONSTRAINT\ ?P\ ?x$) and freezes all schematic variables to prevent them from accidentally getting instantiated. Then it simplifies with *constraint-simps* and tries to solve the goal using rules from *safe-constraint-rules* (no backtracking) and *constraint-rules* (with backtracking).

If solving the constraint is not successful, only the safe rules are applied, and the remaining subgoals are moved to a special *CONSTRAINT-SLOT* subgoal, that always is the last subgoal, and is initialized by the preprocessing phase of Sepref. Moving the subgoal to the constraint slot looks for Isabelle's tacticals like the subgoal has been solved. In reality, it is only deferred and must be solved later.

Constraints are used in several phases of Sepref, and all constraints are solved at the end of the translation phase, and at the end of the Sepref invocation.

Methods:

- *solve-constraint* to apply constraint solving, the *CONSTRAINT*-tag is optional.
- *safe-constraint* to apply safe rules, the *CONSTRAINT*-tag is optional.

- *print-slot* to print the contents of the constraint slot.

Translation: Merging and Frame Inference

Frame inference solves goals of the form $\Gamma \Longrightarrow_t \Gamma'$. For this, it matches *hn-ctxt* components in Γ' with those in Γ. Matching is done according to the refined variables. The matching pairs and the rest is then treated differently: The rest is resolved by repeatedly applying the rules from $?P \Longrightarrow_t ?P$

$?F \Longrightarrow_t ?F' \Longrightarrow ?F * \textit{hn-ctxt} ?A ?x ?y \Longrightarrow_t ?F' * \textit{hn-ctxt} ?A ?x ?y$

$?F \Longrightarrow_t ?F' \Longrightarrow ?F * \textit{hn-ctxt} ?A ?x ?y \Longrightarrow_t ?F'$

$?P \Longrightarrow_t \textit{emp}$. The matching pairs are resolved by repeatedly applying rules from $?P \Longrightarrow_t ?P$

$[\![?P \Longrightarrow_t ?P'; ?F \Longrightarrow_t ?F']\!] \Longrightarrow ?F * ?P \Longrightarrow_t ?F' * ?P'$

$\textit{hn-ctxt} ?R ?x ?y \Longrightarrow_t \textit{hn-invalid} ?R ?x ?y$

$\textit{hn-ctxt} ?R ?x ?y \Longrightarrow_t \textit{hn-ctxt} (\lambda\text{- -. } \textit{true}) ?x ?y$

$\textit{CONSTRAINT is-pure} ?R \Longrightarrow \textit{hn-invalid} ?R ?x ?y \Longrightarrow_t \textit{hn-ctxt} ?R ?x ?y$ and *sepref-frame-match-rules*. Any non-frame premise of these rules must be solved immediately by the side-condition's constraint or fallback tactic (see above). The tool backtracks over rules. If no rule matches (or side-conditions cannot be solved), it simplifies the goal with *sepref-frame-normrel-eqs* and tries again.

For merge rules, the theorems $?F \vee_A ?F \Longrightarrow_t ?F$

$[\![\textit{hn-ctxt} ?R1.0 ?x ?x' \vee_A \textit{hn-ctxt} ?R2.0 ?x ?x' \Longrightarrow_t \textit{hn-ctxt} ?R ?x ?x'; ?Fl \vee_A ?Fr \Longrightarrow_t ?F]\!] \Longrightarrow ?Fl * \textit{hn-ctxt} ?R1.0 ?x ?x' \vee_A ?Fr * \textit{hn-ctxt} ?R2.0 ?x ?x' \Longrightarrow_t ?F * \textit{hn-ctxt} ?R ?x ?x'$

$\textit{hn-invalid} ?R ?x ?x' \vee_A \textit{hn-ctxt} ?R ?x ?x' \Longrightarrow_t \textit{hn-invalid} ?R ?x ?x'$

$\textit{hn-ctxt} ?R ?x ?x' \vee_A \textit{hn-invalid} ?R ?x ?x' \Longrightarrow_t \textit{hn-invalid} ?R ?x ?x'$ and *sepref-frame-merge-rules* are used.

Note that a smart setup of frame and match rules together with side conditions makes the frame matcher a powerful tool for encoding structural and semantic information into relations. An example for structural information are the match rules for lists, which forward matching of list assertions to matching of the element assertions, maintaining the congruence assumption that the refined elements are actually elements of the list: $(\bigwedge x\, x'.\ [\![x \in \textit{set} ?l; x' \in \textit{set} ?l']\!] \Longrightarrow \textit{hn-ctxt} ?A\ x\ x' \Longrightarrow_t \textit{hn-ctxt} ?A'\ x\ x') \Longrightarrow \textit{hn-ctxt} (\textit{list-assn} ?A) ?l ?l' \Longrightarrow_t \textit{hn-ctxt} (\textit{list-assn} ?A') ?l ?l'$. An example for semantic information is the bounded assertion, which intersects any given assertion with a predicate on the abstract domain. The frame matcher is set up such that it can convert between bounded assertions, generating semantic side conditions to discharge implications between bounds ($[\![\textit{hn-ctxt} (\textit{b-assn} ?A ?P) ?x$

$?y \Longrightarrow_t$ *hn-ctxt* $?A'$ $?x$ $?y$; $[\![$*vassn-tag* (*hn-ctxt* $?A$ $?x$ $?y$); *vassn-tag* (*hn-ctxt* $?A'$ $?x$ $?y$); $?P$ $?x]\!] \Longrightarrow$ $?P'$ $?x]\!] \Longrightarrow$ *hn-ctxt* (*b-assn* $?A$ $?P$) $?x$ $?y \Longrightarrow_t$ *hn-ctxt* (*b-assn* $?A'$ $?P'$) $?x$ $?y$).

This is essentially a subtyping mechanism on the level of refinement assertions, which is quite useful for maintaining natural side conditions on operands. A standard example is to maintain a list of array indices: The refinement assertion for array indices is *id-assn* restricted to indices that are in range: *b-assn id-assn* ($\lambda x.\ x < N$). When inserting natural numbers into this list, one has to prove that they are actually in range (conversion from *id-assn* to $\lambda n.$ *b-assn id-assn* ($\lambda x.\ x < n$)). Elements of the list can be used as natural numbers (conversion from $\lambda n.$ *b-assn id-assn* ($\lambda x.\ x < n$) to *id-assn*). Additionally, the side condition solver can derive that the predicate holds on the abstract variable (via the *vassn-tag* inserted by the operator steps).

Translation: Annotated Example

context
  fixes *N*::*nat*
  notes [[*sepref-register-adhoc N*]]
  notes [*sepref-import-param*] = *IdI*[*of N*]
begin

This worked example utilizes the following features of the translation phase:

- We have a fold combinator, which gets translated by its combinator rule
- We add a type annotation which enforces converting the natural numbers inserted into the list being refined by *nbn-assn N*, i.e., smaller than *N*.
- We can only prove the numbers inserted into the list to be smaller than *N* because the combinator rule for *If* inserts congruence assumptions.
- By moving the elements from the list to the set, they get invalidated. However, as *nat-assn* is pure, they can be recovered later, allowing us to mark the list argument as read-only.

sepref-thm *filter-N-test* **is** $\lambda l.$ *RETURN* (*fold* (λx *s*.
  if $x<N$ *then insert* (*ASSN-ANNOT* (*nbn-assn N*) *x*) *s else s*
) *l op-hs-empty*) :: (*list-assn nat-assn*)$^k \rightarrow_a$ *hs.assn* (*nbn-assn N*)

  apply *sepref-dbg-preproc*
  apply *sepref-dbg-cons-init*
  apply *sepref-dbg-id*

apply *sepref-dbg-monadify*

apply *sepref-dbg-opt-init*

apply *sepref-dbg-trans-step* — Combinator rule for bind, generating two *hn-refine* goals, and a frame rule to separate the bound variable from the rest.
apply *sepref-dbg-trans-step* — Rule for empty hashset, solves goal
apply *sepref-dbg-trans-step* — Combinator rule for nfoldli ($[\![INDEP\ ?Rk;\ INDEP\ ?R\sigma;\ ?P \Longrightarrow_t\ ?\Gamma * hn\text{-}ctxt\ (list\text{-}assn\ ?Rk)\ ?s'\ ?s * hn\text{-}ctxt\ ?R\sigma\ ?\sigma'\ ?\sigma;\ \bigwedge\sigma\ \sigma'.\ hn\text{-}refine\ (?\Gamma * hn\text{-}ctxt\ ?R\sigma\ \sigma'\ \sigma)\ (?c\ \sigma)\ (?\Gamma c\ \sigma'\ \sigma)\ id\text{-}assn\ (?c'\ \sigma');\ \bigwedge\sigma'\ \sigma.\ TERM\ monadic\text{-}nfoldli \Longrightarrow\ ?\Gamma c\ \sigma'\ \sigma \Longrightarrow_t\ ?\Gamma * hn\text{-}ctxt\ ?R\sigma\ \sigma'\ \sigma;\ \bigwedge x'\ x\ \sigma'\ \sigma.\ x' \in set\ ?s' \Longrightarrow hn\text{-}refine\ (?\Gamma * hn\text{-}ctxt\ ?Rk\ x'\ x * hn\text{-}ctxt\ ?R\sigma\ \sigma'\ \sigma)\ (?f\ x\ \sigma)\ (?\Gamma f\ x'\ x\ \sigma'\ \sigma)\ ?R\sigma\ (?f'\ x'\ \sigma');\ \bigwedge x'\ x\ \sigma'\ \sigma.\ TERM\ monadic\text{-}nfoldli \Longrightarrow\ ?\Gamma f\ x'\ x\ \sigma'\ \sigma \Longrightarrow_t\ ?\Gamma * hn\text{-}ctxt\ ?Rk'\ x'\ x * hn\text{-}ctxt\ ?Pf\sigma\ \sigma'\ \sigma;\ \bigwedge x\ x'.\ hn\text{-}ctxt\ ?Rk'\ x'\ x \vee_A hn\text{-}ctxt\ ?Rk\ x'\ x \Longrightarrow_t hn\text{-}ctxt\ ?Rk''\ x'\ x]\!] \Longrightarrow hn\text{-}refine\ ?P\ (imp\text{-}nfoldli\ ?s\ ?c\ ?f\ ?\sigma)\ (?\Gamma * hn\text{-}ctxt\ (list\text{-}assn\ ?Rk'')\ ?s'\ ?s * hn\text{-}invalid\ ?R\sigma\ ?\sigma'\ ?\sigma)\ ?R\sigma\ (monadic\text{-}nfoldli\ \$\ ?s'\ \$\ (\lambda x.\ (\#?c'\ x\#))\ \$\ (\lambda x.\ (\#\lambda xa.\ (\#?f'\ x\ xa\#)\#))\ \$\ ?\sigma')$)
apply *sepref-dbg-trans-step* — INDEP
apply *sepref-dbg-trans-step* — INDEP
apply *sepref-dbg-trans-step* — Frame to get list and initial state
apply *sepref-dbg-trans-step* — Refinement of continuation condition
apply *sepref-dbg-trans-step* — Frame to recover state after continuation condition

— Loop body
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
— At this point, we arrived at the *nbn-rel* annotation. There is enough information to show $x'a < N$
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step*
— At this point, we have to merge the postconditions from the two if branches. *nat-rel* gets merged with *invalid-assn* (*nbn-assn n*), yielding *invalid-assn nat-assn*
apply *sepref-dbg-trans-step*
apply *sepref-dbg-trans-step* — Frame rule separating bound variable from rest

apply *sepref-dbg-trans-step* — Frame rule separating fold-state from rest
apply *sepref-dbg-trans-step* — Merging elements of list before body with elements of list after body, to get refinement for resulting list

apply *sepref-dbg-trans-step* — Frame rule from initial bind, separating bound variable from the rest

apply *sepref-dbg-opt*
apply *sepref-dbg-cons-solve* — Frame rule, recovering the invalidated list or pure elements, propagating recovery over the list structure
apply *sepref-dbg-cons-solve* — Trivial frame rule
apply *sepref-dbg-constraints*
done

end

Optimization Phase

The optimization phase simplifies the generated program, first with *sepref-opt-simps*, and then with *sepref-opt-simps2*. For simplification, the tag *CNV* is used, which is discharged with *CNV ?x ?x* after simplification.

Method *sepref-dbg-opt* gives direct access to this phase. The simplification is used to beautify the generated code. The most important simplifications collapse code that does not depend on the heap to plain expressions (using the monad laws), and apply certain deforestation optimizations.

Consider the following example:

sepref-thm *opt-example* **is** $\lambda n.$ *do* { *let r = fold* (+) [*1..<n*] *0*; *RETURN* (*n∗n+2*) }
:: $nat\text{-}assn^k \rightarrow_a nat\text{-}assn$
apply *sepref-dbg-preproc*
apply *sepref-dbg-cons-init*
apply *sepref-dbg-id*
apply *sepref-dbg-monadify*
apply *sepref-dbg-opt-init*
apply *sepref-dbg-trans*
— The generated program contains many superfluous binds, moreover, it actually generates a list and then folds over it
supply [[*show-main-goal*]]
apply *sepref-dbg-opt*
— The superfluous binds have been collapsed, and the fold over the list has been replaced by *imp-for'*, which uses a counter.
apply *sepref-dbg-cons-solve*
apply *sepref-dbg-cons-solve*
apply *sepref-dbg-constraints*
done

Cons-Solve Phases

These two phases, accessible via *sepref-dbg-cons-solve*, applies the frame inference tool to solve the two implications generated by the consequence rule phase.

Constraints Phase

This phase, accessible via *sepref-dbg-constraints*, solve the deferred constraints that are left, and then removes the *CONSTRAINT-SLOT* subgoal.

4.2.2 Refinement Rules

There are two forms of specifying refinement between an Imperative/HOL program and an abstract program in the *nres*-monad. The *hn-refine* form (also hnr-form) is the more low-level form. The term $P \Longrightarrow$ *hn-refine* Γ c Γ' R a states that, under precondition P, for a heap described by Γ, the Imperative/HOL program c produces a heap described by Γ' and the result is refined by R. Moreover, the abstract result is among the possible results of the abstract program a.

This low-level form formally enforces no restrictions on its arguments, however, there are some assumed by our tool:

- Γ must have the form *hn-ctxt* A_1 x_1 $xi_1 * \ldots *$ *hn-ctxt* A_n x_n xi_n
- Γ' must have the form *hn-ctxt* B_1 x_1 $xi_1 * \ldots *$ *hn-ctxt* B_n x_n xi_n where either $B_i = A_i$ or $B_i =$ *invalid-assn* A_i. This means that each argument to the program is either preserved or destroyed.
- R must not contain a *hn-ctxt* tag.
- a must be in protected form (($) and *PROTECT2* tags)

The high-level *hfref* form formally enforces these restrictions. Moreover, it assumes c and a to be presented as functions from exactly one argument. For constants or functions with more arguments, you may use *uncurry0* and *uncurry*. (Also available *uncurry2* to $\lambda f.$ *uncurry2* (*uncurry2* (*uncurry* f))). The general form is $PC \Longrightarrow (uncurry_x\ f,\ uncurry_x\ g) \in [P]_a\ {A_1}^{k1} *_a \ldots *_a {A_n}^{kn} \to R$, where ki is k if the argument is preserved (kept) or d is it is destroyed. PC are preconditions of the rule that do not depend on the arguments, usually restrictions on the relations. P is a predicate on the single argument of g, representing the precondition that depends on the arguments.

Optionally, g may be of the form $RETURN\ o \ldots o\ g'$, in which case the rule applies to a plain function.

If there is no precondition, there is a shorter syntax: $Args \to_a R \equiv Args \to_a R$.

For example, consider *arl-swap-hnr* [*unfolded pre-list-swap-def*]. It reads *CONSTRAINT is-pure A* $\Longrightarrow$ (*uncurry2 arl-swap*, *uncurry2* (*RETURN* $\circ\circ\circ$ *op-list-swap*)) $\in$ [$\lambda((l,\ i),\ j).\ i <$ *length* $l \wedge j <$ *length* $l]_a$ (*arl-assn* $A)^d *_a$ *id-assn*$^k *_a$ *id-assn*$^k \to$ *arl-assn A*

We have three arguments, the list and two indexes. The refinement assertion A for the list elements must be pure, and the indexes must be in range. The original list is destroyed, the indexes are kept.

thm *arl-swap-hnr*[*unfolded pre-list-swap-def*, *no-vars*]

Converting between hfref and hnr form

A subgoal in hfref form is converted to hnr form by the preprocessing phase of Sepref (see there for a description).

Theorems with hnr/hfref conclusions can be converted using *to-hfref*/*to-hnr*. This conversion is automatically done for rules registered with *sepref-fr-rules*, such that this attribute accepts both forms.

Conversion to hnr-form can be controlled by specifying *to-hnr-post* unfold-rules, which are applied after the conversion.

Note: These currently contain hard-coded rules to handle *RETURN o...o* - for up to six arguments. If you have more arguments, you need to add corresponding rules here, until this issue is fixed and the tool can produce such rules automatically.

Similarly, *to-hfref-post* is applied after conversion to hfref form.

thm *to-hnr-post*
thm *to-hfref-post*

Importing Parametricity Theorems

For pure refinements, it is sometimes simpler to specify a parametricity theorem than a hnr/hfref theorem, in particular as there is a large number of parametricity theorems readily available, in the parametricity component or Autoref, and in the Lifting/Transfer tool.

Autoref uses a set-based notation for parametricity theorems (e.g. ((@), (@)) $\in \langle A\rangle$*list-rel* $\to \langle A\rangle$*list-rel* $\to \langle A\rangle$*list-rel*), while lifting/transfer uses a predicate based notation (e.g. *rel-fun* (*list-all2 A*) (*rel-fun* (*list-all2 A*) (*list-all2 A*)) (@) (@)).

Currently, we only support the Autoref style, but provide a few lemmas that ease manual conversion from the Lifting/Transfer style.

Given a parametricity theorem, the attribute *sepref-param* converts it to a hfref theorem, the attribute *sepref-import-param* does the conversion and registers the result as operator rule. Relation variables are converted to assertion variables with an *is-pure* constraint.

The behaviour can be customized by *sepref-import-rewrite*, which contains rewrite rules applied in the last but one step of the conversion, before converting relation variables to assertion variables. These theorems can be used to convert relations to there corresponding assertions, e.g., *pure* $(\langle ?R\rangle$*list-rel*$)$ = *list-assn* (*pure* $?R$) converts a list relation to a list assertion.

For debugging purposes, the attribute *sepref-dbg-import-rl-only* converts a parametricity theorem to a hnr-theorem. This is the first step of the standard conversion, followed by a conversion to hfref form.

thm *sepref-import-rewrite*
thm *param-append* — Parametricity theorem for append
thm *param-append*[*sepref-param*] — Converted to hfref-form. *list-rel* is rewritten to *list-assn*, and the relation variable is replaced by an assertion variable and a *is-pure* constraint.
thm *param-append*[*sepref-dbg-import-rl-only*]

For re-using Lifting/Transfer style theorems, the constants *p2rel* and *rel2p* may be helpful, however, there is no automation available yet.

Usage examples can be found in, e.g., *Refine-Imperative-HOL.IICF-Multiset*, where we import parametricity lemmas for multisets from the Lifting/Transfer package.

thm *p2rel* — Simp rules to convert predicate to relational style
thm *rel2p* — Simp rules to convert relational to predicate style

4.2.3 Composition

Fref-Rules

In standard parametricity theorems as described above, one cannot specify preconditions for the parameters, e.g., *hd* is only parametric for non-empty lists.

As of Isabelle2016, the Lifting/Transfer package cannot specify such preconditions at all.

Autoref's parametricity tool can specify such preconditions by using first-order rules, (cf. $[\![?l \neq []; (?l', ?l) \in \langle ?A\rangle \mathit{list\text{-}rel}]\!] \Longrightarrow (hd\ ?l', hd\ ?l) \in ?A$). However, currently, *sepref-import-param* cannot handle these first-order rules.

Instead, Sepref supports the fref-format for parametricity rules, which resembles the hfref-format: Abstract and concrete objects are functions with exactly one parameter, uncurried if necessary. Moreover, there is an explicit precondition. The syntax is $(\mathit{uncurry}_x\ f, \mathit{uncurry}_x\ g) \in [P]_f\ (...(R_1 \times_r R_2) \times_r ...) \times_r R_n) \rightarrow R$, and without precondition, we have $(...(R_1 \times_r R_2) \times_r ...) \times_r R_n) \rightarrow_f R$.

Note the left-bracketing of the tuples, which is non-standard in Isabelle. As we currently have no syntax for a left-associative product relation, we use the right-associative syntax ($\times_r$) and explicit brackets.

The attribute *to-fref* can convert (higher-order form) parametricity theorems to the fref-form.

Composition of hfref and fref theorems

fref and hfref theorems can be composed, if the abstract function or the first theorem equals the concrete function of the second theorem. Currently, we can compose an hfref with an fref theorem, yielding a hfref theorem, and two fref-theorems, yielding an fref theorem. As we do not support refinement of heap-programs, but only refinement *into* heap programs, we cannot compose two hfref theorems.

The attribute *FCOMP* does these compositions and normalizes the result. Normalization consists of precondition simplification, and distributing composition over products, such that composition can be done argument-wise. For this, we unfold with *fcomp-norm-unfold*, and then simplify with *fcomp-norm-simps*.

The *FCOMP* attribute tries to convert its arguments to hfref/fref form, such that it also accepts hnr-rules and parametricity rules.

The standard use-case for *FCOMP* is to compose multiple refinement steps to get the final correctness theorem. Examples for this are in the quickstart guide.

Another use-case for *FCOMP* is to compose a refinement theorem of a container operation, that refines the elements by identity, with a parametricity theorem for the container operation, that adds a (pure) refinement of the elements. In practice, the high-level utilities **sepref-decl-op** and **sepref-decl-impl** are used for this purpose. Internally, they use *FCOMP*.

thm *fcomp-norm-unfold*
thm *fcomp-norm-simps*

thm *array-get-hnr-aux* — Array indexing, array elements are refined by identity
thm *op-list-get.fref* — Parametricity theorem for list indexing

thm *array-get-hnr-aux*[*FCOMP op-list-get.fref*] — Composed theorem

— Note the definition *array-assn ?A* ≡ *hr-comp is-array* (⟨*the-pure ?A*⟩*list-rel*)
context
  notes [*fcomp-norm-unfold*] = *array-assn-def*[*symmetric*]
begin
 thm *array-get-hnr-aux*[*FCOMP op-list-get.fref*] — Composed theorem, *array-assn* folded.
end

4.2.4 Registration of Interface Types

An interface type represents some conceptual type, which is encoded to a more complex type in HOL. For example, the interface type $('k, 'v)$ *i-map* represents maps, which are encoded as $'k \Rightarrow 'v$ *option* in HOL.

New interface types must be registered by the command **sepref-decl-intf**.

sepref-decl-intf $('a,'b)$ *i-my-intf* **is** $'a*'a \Rightarrow 'b$ *option*
— Declares $('a, 'b)$ *i-my-intf* as new interface type, and registers it to correspond to $'a \times 'a \Rightarrow 'b$ *option*. Note: For HOL, the interface type is just an arbitrary new type, which is not related to he corresponding HOL type.

sepref-decl-intf $('a,'b)$ *i-my-intf2* (**infix** ‹$*\rightarrow_i$› *0*) **is** $'a*'a \Rightarrow 'b$ *option*
— There is also a version that declares infix-syntax for the interface type. In this case we have $'a *\rightarrow_i 'b$. $'a \rightharpoonup 'b$ Be aware of syntax space pollution, as the syntax for interface types and HOL types is the same.

4.2.5 Registration of Abstract Operations

Registering a new abstract operation requires some amount of setup, which is automated by the *sepref-register* tool. Currently, it only works for operations, not for combinators.

The **sepref-register** command takes a list of terms and registers them as operators. Optionally, each term can have an interface type annotation.

If there is no interface type annotation, the interface type is derived from the terms HOL type, which is rewritten by the theorems from *map-type-eqs*. This rewriting is useful for bulk-setup of many constants with conceptual types different from there HOL-types. Note that the interface type must correspond to the HOL type of the registered term, otherwise, you'll get an error message.

If the term is not a single constant or variable, and does not already start with a *PR-CONST* tag, such a tag will be added, and also a pattern rule will be registered to add the tag on operator identification.

If the term has a monadic result type (- *nres*), also an arity and combinator rule for the monadify phase are generated.

There is also an attribute version *sepref-register-adhoc*. It has the same syntax, and generates the same theorems, but does not give names to the theorems. It's main application is to conveniently register fixed variables of a context. Warning: Make sure not to export such an attribute from the context, as it may become meaningless outside the context, or worse, confuse the tool.

Example for bulk-registration, utilizing type-rewriting

definition *map-op1 m n* $\equiv m(n \mapsto n+1)$
definition *map-op2 m n* $\equiv m(n \mapsto n+2)$

definition *map-op3 m n* ≡ $m(n \mapsto n+3)$
definition *map-op-to-map* $(m::'a \rightharpoonup 'b) \equiv m$

context
  notes [*map-type-eqs*] = *map-type-eqI*[*of TYPE*$('a \rightharpoonup 'b)$ *TYPE*$(('a,'b)$*i-map*)]
begin
  sepref-register *map-op1 map-op2 map-op3*
  — Registered interface types use *i-map*
  sepref-register *map-op-to-map* :: $('a \rightharpoonup 'b) \Rightarrow ('a,'b)$ *i-map*
  — Explicit type annotation is not rewritten
end

Example for insertion of *PR-CONST* tag and attribute-version

context
  fixes *N* :: *nat* **and** *D* :: *int*
  notes [[*sepref-register-adhoc N D*]]
  — In order to use *N* and *D* as operators (constant) inside this context, they have to be registered. However, issuing a *sepref-register* command inside the context would export meaningless registrations to the global theory.

  notes [*sepref-import-param*] = *IdI*[*of N*] *IdI*[*of D*]
  — For declaring refinement rules, the *sepref-import-param* attribute comes in handy here. If this is not possible, you have to work with nested contexts, proving the refinement lemmas in the first level, and declaring them as *sepref-fr-rules* on the second level.

begin
  definition *newlist* ≡ *replicate N D*

  sepref-register *newlist*
  print-theorems
  — *PR-CONST* tag is added, pattern rule is generated

  sepref-register *other-basename-newlist*: *newlist*
  print-theorems
  — The base name for the generated theorems can be overridden

  sepref-register *yet-another-basename-newlist*: *PR-CONST newlist*
  print-theorems
  — If *PR-CONST* tag is specified, no pattern rule is generated automatically

end

Example for mcomb/arity theorems

definition *select-a-one l* ≡ *SPEC* $(\lambda i.\ i<length\ l \wedge l!i = (1::nat))$

sepref-register *select-a-one*
  print-theorems
  — Arity and mcomb theorem is generated

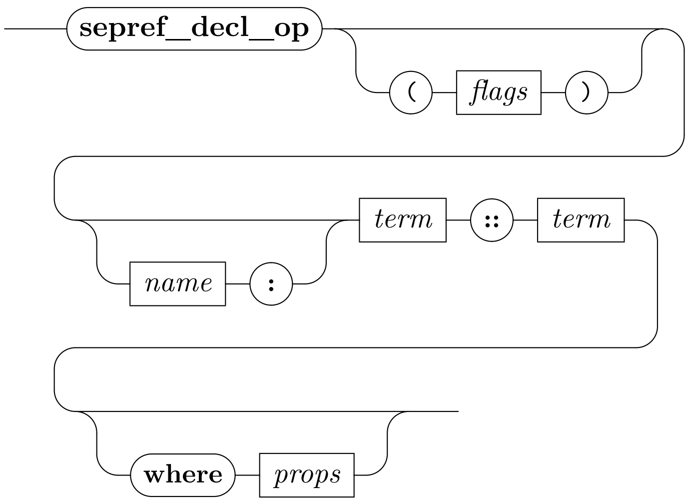
The following command fails, as the specified interface type does not correspond to the HOL type of the term: **sepref-register** *hd* :: (*nat*,*nat*) *i-map*

4.2.6 High-Level tools for Interface/Implementation Declaration

The Imperative Isabelle Collections Framework (IICF), which comes with Sepref, has a concept of interfaces, which specify a set of abstract operations for a conceptual type, and implementations, which implement these operations.

Each operation may have a natural precondition, which is established already for the abstract operation. Many operations come in a plain version, and a monadic version which asserts the precondition. Implementations may strengthen the precondition with implementation specific preconditions.

Moreover, each operation comes with a parametricity lemma. When registering an implementation, the refinement of the implementation is combined with the parametricity lemma to allow for (pure) refinements of the element types.



The command **sepref-decl-op** declares an abstract operation. It takes a term defining the operation, and a parametricity relation. It generates the monadic version from the plain version, defines constants for the operations, registers them, and tries to prove parametricity lemmas automatically. Parametricity must be proved for the operation, and for the precondition. If the automatic parametricity proofs fail, the user gets presented goals that can be proven manually.

Optionally, a basename for the operation can be specified. If none is specified, a heuristics tries to derive one from the specified term.

A list of properties (separated by space and/or *and*) can be specified, which get constraint-preconditions of the relation.

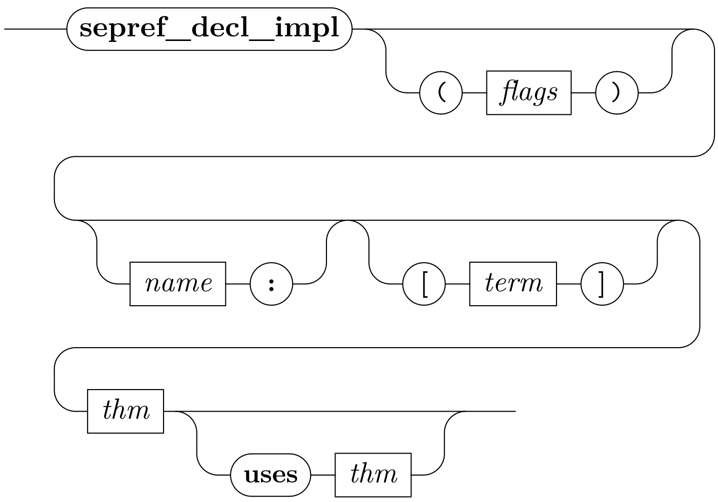
Finally, the following flags can be specified. Each flag can be prefixed by *no-* to invert its meaning:

mop (default: true) Generate monadic version of operation

ismop (default: false) Indicate that given term is the monadic version

rawgoals (default: false) Present raw goals to user, without attempting to prove them

def (default: true) Define a constant for the specified term. Otherwise, use the specified term literally.



The **sepref-decl-impl** command declares an implementation of an interface operation. It takes a refinement theorem for the implementation, and combines it with the corresponding parametricity theorem. After *uses*, one can override the parametricity theorem to be used. A heuristics is used to merge the preconditions of the refinement and parametricity theorem. This heuristics can be overridden by specifiying the desired precondition inside [...]. Finally, the user gets presented remaining subgoals that cannot be solved by the heuristics. The command accepts the following flags:

mop (default: true) Generate implementation for monadic version

ismop (default: false) Declare that the given theorems refer to the monadic version

transfer (default: true) Try to automatically transfer the implementation's precondition over the argument relation from the parametricity theorem.

rawgoals (default: false) Do not attempt to solve or simplify the goals

register (default: true) Register the generated theorems as operation rules.

4.2.7 Defining synthesized Constants

The **sepref-definition** allows one to specify a name, an abstract term and a desired refinement relation in hfref-form. It then sets up a goal that can be massaged (usually, constants are unfolded and annotations/implementation specific operations are added) and then solved by *sepref*. After the goal is solved, the command extracts the synthesized term and defines it as a constant with the specified name. Moreover, it sets up code equations for the constant, correctly handling recursion combinators. Extraction of code equations is controlled by the *prep-code* flag. Examples for this command can be found in the quickstart guide.

end

4.3 General Purpose Utilities

theory *Sepref-Guide-General-Util*
imports *../IICF/IICF*
begin

This userguide documents some of the general purpose utilities that come with the Sepref tool, but are useful in other contexts, too.

4.3.1 Methods

Resolve with Premises

The *rprems* resolves the current subgoal with one of its premises. It returns a sequence of possible resolvents. Optionally, the number of the premise to resolve with can be specified.

First-Order Resolution

The *fo-rule* applies a rule with first-order matching. It is very useful to be used with theorems like $?x = ?y \Longrightarrow ?f\ ?x = ?f\ ?y$.

notepad begin
  have $card\ \{x.\ 3{<}x \wedge x{<}(7{::}nat)\} = card\ \{x.\ 4{\le}x \wedge x{\le}(6{::}nat)\}$

apply (*fo-rule arg-cong*)
apply *auto*
done

— While the first goal could also have been solved with *rule arg-cong*[*where* $f{=}card$], things would be much more verbose for the following goal. (Such goals actually occur in practice!)

fix f :: *nat set* $\Rightarrow$ *nat set* $\Rightarrow$ *bool*
have $\bigwedge a.\ f\ \{x.\ x{*}2 + a + 3 < 10\}\ \{x.\ 3{<}x \wedge x{<}(7{::}nat)\} = f\ \{x.\ x{*}2 + a \leq 6\}\ \{x.\ 4{\leq}x \wedge x{\leq}(6{::}nat)\}$
apply (*fo-rule arg-cong fun-cong cong*)+
apply *auto*
done

end

Clarsimp all goals

clarsimp-all is a *clarsimp* on all goals. It takes the same arguments as *clarsimp*.

VCG solver

vc-solve clarsimps all subgoals. Then, it tries to apply a rule specified in the *solve*: argument, and tries to solve the result by *auto*. If the goal cannot be solved this way, it is not changed.

This method is handy to be applied after verification condition generation. If *auto* shall be tried on all subgoals, specify *solve*: *asm-rl*.

4.3.2 Structured Apply Scripts (experimental)

A few variants of the apply command, that document the subgoal structure of a proof. They are a lightweight alternative to **subgoal**, and fully support schematic variables.

applyS applies a method to the current subgoal, and fails if the subgoal is not solved.

apply1 applies a method to the current subgoal, and fails if the goal is solved or additional goals are created.

focus selects the current subgoal, and optionally applies a method.

applyF selects the current subgoal and applies a method.

solved enforces no subgoals to be left in the current selection, and unselects.

Note: The selection/unselection mechanism is a primitive version of focusing on a subgoal, realized by inserting protect-tags into the goal-state.

4.3.3 Extracting Definitions from Theorems

The **concrete-definition** can be used to extract parts of a theorem as a constant. It is documented at the place where it is defined (ctrl-click to jump there).

end

Chapter 5

Examples

This chapter contains practical examples of using the IRF and IICF. Moreover it contains some snippets that illustrate how to solve common tasks like setting up custom datatypes or higher-order combinators.

5.1 Imperative Graph Representation

theory *Sepref-Graph*
imports
  ../Sepref
  ../Sepref-ICF-Bindings
  ../IICF/IICF

begin

Graph Interface

sepref-decl-intf $'a$ *i-graph* **is** $('a \times 'a)$ *set*

definition *op-graph-succ* :: $('v \times 'v)$ *set* $\Rightarrow$ $'v$ $\Rightarrow$ $'v$ *set*
  where [*simp*]: *op-graph-succ* $E\ u \equiv E``\{u\}$
sepref-register *op-graph-succ* :: $'a$ *i-graph* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*

thm *intf-of-assnI*

lemma [*pat-rules*]: $((``))\$E\$(insert\$u\$\{\}) \equiv$ *op-graph-succ*$\$E\$u$ **by** *simp*

definition [*to-relAPP*]: *graph-rel* $A \equiv \langle A \times_r A \rangle$*set-rel*

Adjacency List Implementation

lemma *param-op-graph-succ*[*param*]:
  $[\![$*IS-LEFT-UNIQUE* A; *IS-RIGHT-UNIQUE* $A]\!] \Longrightarrow$ (*op-graph-succ*, *op-graph-succ*) $\in \langle A \rangle$*graph-rel* $\rightarrow A \rightarrow \langle A \rangle$*set-rel*
  unfolding *op-graph-succ-def*[*abs-def*] *graph-rel-def*
  by *parametricity*

context begin
private definition *graph-α1 l* ≡ { (*i*,*j*). *i*<*length l* ∧ *j*∈*l*!*i* }

private definition *graph-rel1* ≡ *br graph-α1* (λ-. *True*)

private definition *succ1 l i* ≡ *if i*<*length l then l*!*i else* {}

private lemma *succ1-refine*: (*succ1*,*op-graph-succ*) ∈ *graph-rel1* → *Id* → ⟨*Id*⟩*set-rel*
  by (*auto simp*: *graph-rel1-def graph-α1-def br-def succ1-def split*: *if-split-asm intro*!: *ext*)

private definition *assn2* ≡ *array-assn* (*pure* (⟨*Id*⟩*list-set-rel*))

definition *adjg-assn A* ≡ *hr-comp* (*hr-comp assn2 graph-rel1*) (⟨*the-pure A*⟩*graph-rel*)

context
  notes [*sepref-import-param*] = *list-set-autoref-empty*[*folded op-set-empty-def*]
  notes [*fcomp-norm-unfold*] = *adjg-assn-def*[*symmetric*]
begin
sepref-definition *succ2* **is** (*uncurry* (*RETURN oo succ1*)) :: ($assn2^k *_a id\text{-}assn^k \rightarrow_a$ *pure* (⟨*Id*⟩*list-set-rel*))
  unfolding *succ1-def*[*abs-def*] *assn2-def*
  by *sepref*

lemma *adjg-succ-hnr*[*sepref-fr-rules*]: ⟦*CONSTRAINT* (*IS-PURE IS-LEFT-UNIQUE*) *A*; *CONSTRAINT* (*IS-PURE IS-RIGHT-UNIQUE*) *A*⟧
  ⟹ (*uncurry succ2*, *uncurry* (*RETURN* ∘∘ *op-graph-succ*)) ∈ $(adjg\text{-}assn\ A)^k *_a A^k \rightarrow_a$ *pure* (⟨*the-pure A*⟩*list-set-rel*)
  using *succ2.refine*[*FCOMP succ1-refine*, *FCOMP param-op-graph-succ*, *simplified*, *of A*]
  by (*simp add*: *IS-PURE-def list-set-rel-compp*)

end

end

lemma [*intf-of-assn*]:
  intf-of-assn A (*i*::′*I itself*) ⟹ *intf-of-assn* (*adjg-assn A*) *TYPE*(′*I i-graph*) **by** *simp*

definition *cr-graph*
  :: *nat* ⇒ (*nat* × *nat*) *list* ⇒ *nat list Heap.array Heap*
where
  cr-graph numV Es ≡ *do* {
    a ← *Array.new numV* [];
    a ← *imp-nfoldli Es* (λ-. *return True*) (λ(*u*,*v*) *a*. *do* {
      l ← *Array.nth a u*;
      let l = *v*#*l*;

```
      a ← Array.upd u l a;
      return a
    }) a;
    return a
  }
```

export-code *cr-graph* **checking** *SML-imp*

end

5.2 Simple DFS Algorithm

theory *Sepref-DFS*
imports
  ../Sepref
  Sepref-Graph
begin

We define a simple DFS-algorithm, prove a simple correctness property, and do data refinement to an efficient implementation.

5.2.1 Definition

Recursive DFS-Algorithm. E is the edge relation of the graph, *vd* the node to search for, and *v0* the start node. Already explored nodes are stored in V.

context
  fixes $E :: 'v\ rel$ **and** $v0 :: 'v$ **and** $tgt :: 'v \Rightarrow bool$
begin
  definition *dfs* :: ('*v set* × *bool*) *nres* **where**

```
    dfs ≡ do {
      (V,r) ← RECT (λdfs (V,v).
        if v∈V then RETURN (V,False)
        else do {
          let V=insert v V;
          if tgt v then
            RETURN (V,True)
          else
            FOREACH_C (E``{v}) (λ(-,b). ¬b) (λv' (V,-). dfs (V,v')) (V,False)
        }
      ) ({},v0);
      RETURN (V,r)
    }
```

  definition *reachable* ≡ $\{v.\ (v0,v) \in E^*\}$

definition *dfs-spec* $\equiv$ *SPEC* $(\lambda(V,r).\ (r \longleftrightarrow reachable \cap Collect\ tgt \neq \{\}) \wedge (\neg r \longrightarrow V = reachable))$

lemma *dfs-correct*:
  assumes *fr*: *finite reachable*
  shows *dfs* $\leq$ *dfs-spec*
proof –
  have F: $\bigwedge v.\ v \in reachable \Longrightarrow finite\ (E``\{v\})$
    using *fr*
    apply (*auto simp*: *reachable-def*)
    by (*metis* (*mono-tags*) *Image-singleton Image-singleton-iff*
      finite-subset rtrancl.rtrancl-into-rtrancl subsetI)

  define *rpre* **where** $rpre = (\lambda S\ (V,v).$
      $v \in reachable$
    $\wedge\ V \subseteq reachable$
    $\wedge\ S \subseteq V$
    $\wedge\ (V \cap Collect\ tgt = \{\})$
    $\wedge\ E``(V-S) \subseteq V)$

  define *rpost* **where** $rpost = (\lambda S\ (V,v)\ (V',r).$
      $(r \longleftrightarrow V' \cap Collect\ tgt \neq \{\})$
    $\wedge\ V \subseteq V'$
    $\wedge\ v \in V'$
    $\wedge\ V' \subseteq reachable$
    $\wedge\ (\neg r \longrightarrow (E``(V'-S) \subseteq V')))$

  define *fe-inv* **where** $fe\text{-}inv = (\lambda S\ V\ v\ it\ (V',r).$
    $(r \longleftrightarrow V' \cap Collect\ tgt \neq \{\})$
  $\wedge\ insert\ v\ V \subseteq V'$
  $\wedge\ E``\{v\} - it \subseteq V'$
  $\wedge\ V' \subseteq reachable$
  $\wedge\ S \subseteq insert\ v\ V$
  $\wedge\ (\neg r \longrightarrow E``(V'-S) \subseteq V' \cup it \wedge E``(V'-insert\ v\ S) \subseteq V'))$

  have *vc-pre-initial*: *rpre* $\{\}$ $(\{\},\ v0)$
    by (*auto simp*: *rpre-def reachable-def*)

  {

    fix $S\ V\ v$
    assume *rpre* $S\ (V,v)$
      and $v \in V$
    hence *rpost* $S\ (V,v)\ (V,False)$
      unfolding *rpre-def rpost-def*
      by *auto*
  } **note** *vc-node-visited* = *this*

{

fix $S\ V\ v$
assume *tgt v*
and *rpre S* (V,v)
hence *rpost S* (V,v) (*insert v V*, *True*)
unfolding *rpre-def rpost-def*
by *auto*
} **note** *vc-node-found* = *this*

{
fix $S\ V\ v$
assume *rpre S* $(V,\ v)$
hence *finite* $(E``\{v\})$
unfolding *rpre-def* **using** *F* **by** (*auto*)
} **note** *vc-foreach-finite* = *this*

{

fix $S\ V\ v$
assume *A*: $v \notin V\ \neg tgt\ v$
and *PRE*: *rpre S* $(V,\ v)$
hence *fe-inv S V v* $(E``\{v\})$ (*insert v V*, *False*)
unfolding *fe-inv-def rpre-def* **by** (*auto*)
} **note** *vc-enter-foreach* = *this*

{

fix $S\ V\ v\ v'\ it\ V'$
assume *A*: $v \notin V\ \neg tgt\ v\ v' \in it\ it \subseteq E``\{v\}$
and *FEI*: *fe-inv S V v it* $(V',$ *False*$)$
and *PRE*: *rpre S* $(V,\ v)$

from *A* **have** $v' \in E``\{v\}$ **by** *auto*
moreover from *PRE* **have** $v \in$ *reachable* **by** (*auto simp*: *rpre-def*)
hence $E``\{v\} \subseteq$ *reachable* **by** (*auto simp*: *reachable-def*)
ultimately have [*simp*]: $v' \in$*reachable* **by** *blast*

have *rpre* (*insert v S*) $(V',\ v')$
unfolding *rpre-def*
using *FEI PRE* **by** (*auto simp*: *fe-inv-def rpre-def*) []
} **note** *vc-rec-pre* = *this*

{

fix $S\ V\ V'\ v\ v'\ it\ Vr''$
assume *fe-inv S V v it* $(V',$ *False*$)$
and *rpost* (*insert v S*) $(V',\ v')\ Vr''$

hence *fe-inv S V v* (*it* − {*v'*}) *Vr''*
unfolding *rpre-def rpost-def fe-inv-def*
by *clarsimp blast*
} **note** *vc-iterate-foreach* = *this*

{

fix *S V v V'*
assume *PRE*: *rpre S* (*V*, *v*)
assume *A*: *v* ∉ *V* ¬*tgt v*
assume *FEI*: *fe-inv S V v* {} (*V'*, *False*)
have *rpost S* (*V*, *v*) (*V'*, *False*)
unfolding *rpost-def*
using *FEI* **by** (*auto simp*: *fe-inv-def*) []
} **note** *vc-foreach-completed-imp-post* = *this*

{

fix *S V v V' it*
assume *PRE*: *rpre S* (*V*, *v*)
and *A*: *v* ∉ *V* ¬*tgt v it* ⊆ *E*‘‘{*v*}
and *FEI*: *fe-inv S V v it* (*V'*, *True*)
hence *rpost S* (*V*, *v*) (*V'*, *True*)
by (*auto simp add*: *rpre-def rpost-def fe-inv-def*) []
} **note** *vc-foreach-interrupted-imp-post* = *this*

{
fix *V r*
assume *rpost* {} ({}, *v0*) (*V*, *r*)
hence (*r* ⟷ *reachable* ∩ *Collect tgt* ≠ {}) ∧ (¬*r*⟶*V*=*reachable*)
by (*auto*
simp: *rpost-def reachable-def*
dest: *Image-closed-trancl*
intro: *rev-ImageI*)
} **note** *vc-rpost-imp-spec* = *this*

show *?thesis*
unfolding *dfs-def dfs-spec-def*
apply (*refine-rcg refine-vcg*)
apply (*rule order-trans*)

apply(*rule RECT-rule-arb*[**where**
pre=*rpre*
and *M*=λ*a x*. *SPEC* (*rpost a x*)
and *V*=*finite-psupset reachable* <∗*lex*∗> {}
])
apply *refine-mono*
apply (*blast intro*: *fr*)
apply (*rule vc-pre-initial*)

apply (*refine-rcg refine-vcg*
FOREACHc-rule'[**where** *I=fe-inv S v s* **for** *S v s*]
)
apply (*simp-all add*: *vc-node-visited vc-node-found*)

apply (*simp add*: *vc-foreach-finite*)

apply (*auto intro*: *vc-enter-foreach*) []

apply (*rule order-trans*)
apply (*rprems*)
apply (*erule* (*5*) *vc-rec-pre*)
apply (*auto simp add*: *fe-inv-def finite-psupset-def*) []
apply (*refine-rcg refine-vcg*)
apply (*simp add*: *vc-iterate-foreach*)

apply (*auto simp add*: *vc-foreach-completed-imp-post*) []

apply (*auto simp add*: *vc-foreach-interrupted-imp-post*) []

apply (*auto dest*: *vc-rpost-imp-spec*) []
done
qed

end

lemma *dfs-correct'*: (*uncurry2 dfs*, *uncurry2 dfs-spec*)
$\in [\lambda((E,s),t).\ \mathit{finite}\ (\mathit{reachable}\ E\ s)]_f\ ((\mathit{Id}\times_r\mathit{Id})\times_r\mathit{Id}) \rightarrow \langle \mathit{Id}\rangle \mathit{nres\text{-}rel}$
apply (*intro frefI nres-relI*; *clarsimp*)
by (*rule dfs-correct*)

5.2.2 Refinement to Imperative/HOL

We set up a schematic proof goal, and use the sepref-tool to synthesize the implementation.

sepref-definition *dfs-impl* **is**
uncurry2 dfs :: $(\mathit{adjg\text{-}assn}\ \mathit{nat\text{-}assn})^k *_a \mathit{nat\text{-}assn}^k *_a (\mathit{pure}\ (\mathit{nat\text{-}rel} \rightarrow \mathit{bool\text{-}rel}))^k \rightarrow_a \mathit{prod\text{-}assn}\ (\mathit{ias.assn}\ \mathit{nat\text{-}assn})\ \mathit{bool\text{-}assn}$
unfolding *dfs-def*[*abs-def*] — Unfold definition of DFS
using [[*goals-limit = 1*]]
apply (*rewrite* **in** *RECT* - (⌑,-) *ias.fold-custom-empty*) — Select impls
apply (*rewrite* **in** *if* ⌑ *then RETURN* (-,*True*) *else* - *fold-pho-apply*)
apply *sepref* — Invoke sepref-tool
done
export-code *dfs-impl* **checking** *SML-imp*
— Generate SML code with Imperative/HOL

export-code *dfs-impl* **in** *Haskell* **module-name** *DFS*

Finally, correctness is shown by combining the generated refinement theorem with the abstract correctness theorem.

lemmas *dfs-impl-correct'* = *dfs-impl.refine*[*FCOMP dfs-correct'*]

corollary *dfs-impl-correct*:
finite (*reachable E s*) $\Longrightarrow$
<*adjg-assn nat-assn E Ei*>
dfs-impl Ei s tgt
< $\lambda(Vi,r)$. $\exists_A V$. *adjg-assn nat-assn E Ei* $*$ *ias.assn nat-assn V Vi* $*$ $\uparrow((r \longleftrightarrow$ *reachable E s* $\cap$ *Collect tgt* $\neq \{\}) \wedge (\neg r \longrightarrow V{=}$*reachable E s*)) $>_t$
using *dfs-impl-correct'*[*THEN hfrefD*, *THEN hn-refineD*, *of* ((*E*,*s*),*tgt*) ((*Ei*,*s*),*tgt*), *simplified*]
apply (*rule cons-rule*[*rotated* −*1*])
apply (*sep-auto intro*!: *ent-ex-preI simp*: *dfs-spec-def pure-def*)+
done

end

5.3 Imperative Implementation of Dijkstra's Shortest Paths Algorithm

theory *Sepref-Dijkstra*
imports
../*IICF*/*IICF*
../*Sepref-ICF-Bindings*
Dijkstra-Shortest-Path.Dijkstra
Dijkstra-Shortest-Path.Test
HOL−*Library.Code-Target-Numeral*

Sepref-WGraph
begin

instantiation *infty* :: (*heap*) *heap*
begin
instance
apply *standard*
apply (*rule-tac x*=λ*Infty* $\Rightarrow$ *0* | *Num a* $\Rightarrow$ *to-nat a* + *1* **in** *exI*)
apply (*rule injI*)
apply (*auto split*: *infty.splits*)
done
end

fun *infty-assn* **where**
infty-assn A (*Num x*) (*Num y*) = *A x y*

| *infty-assn A Infty Infty* = *emp*
| *infty-assn* - - - = *false*

Connection with *infty-rel*

lemma *infty-assn-pure-conv*: *infty-assn* (*pure A*) = *pure* (⟨*A*⟩*infty-rel*)
apply (*intro ext*)
subgoal for *x y* **by** (*cases x*; *cases y*; *simp add*: *pure-def*)
done

lemmas [*sepref-import-rewrite*, *fcomp-norm-unfold*, *sepref-frame-normrel-eqs*] =
infty-assn-pure-conv[*symmetric*]
lemmas [*constraint-simps*] = *infty-assn-pure-conv*

lemma *infty-assn-pure*[*safe-constraint-rules*]: *is-pure A* ⟹ *is-pure* (*infty-assn A*)
by (*auto simp*: *is-pure-conv infty-assn-pure-conv*)

lemma *infty-assn-id*[*simp*]: *infty-assn id-assn* = *id-assn*
by (*simp add*: *infty-assn-pure-conv*)

lemma [*safe-constraint-rules*]: *IS-BELOW-ID R* ⟹ *IS-BELOW-ID* (⟨*R*⟩*infty-rel*)

by (*auto simp*: *infty-rel-def IS-BELOW-ID-def*)

sepref-register *Num Infty*

lemma *Num-hnr*[*sepref-fr-rules*]: (*return o Num*,*RETURN o Num*)∈$A^d \rightarrow_a$ *infty-assn A*
by *sepref-to-hoare sep-auto*

lemma *Infty-hnr*[*sepref-fr-rules*]: (*uncurry0* (*return Infty*),*uncurry0* (*RETURN Infty*))∈*unit-assn*$^k \rightarrow_a$ *infty-assn A*
by *sepref-to-hoare sep-auto*

sepref-register *case-infty*
lemma [*sepref-monadify-arity*]: *case-infty* ≡ λ_2*f1 f2 x*. *SP case-infty*$*f1*$(λ_2*x*. *f2*$*x*)$*x*
by *simp*
lemma [*sepref-monadify-comb*]: *case-infty*$*f1*$*f2*$*x* ≡ (⤜)$(*EVAL*$*x*)$(λ_2*x*. *SP case-infty*$*f1*$*f2*$*x*) **by** *simp*
lemma [*sepref-monadify-comb*]: *EVAL*$(*case-infty*$*f1*$($\lambda_2$*x*. *f2 x*)$*x*)
≡ (⤜)$(*EVAL*$*x*)$($\lambda_2$*x*. *SP case-infty*$(*EVAL* $ *f1*)$($\lambda_2$*x*. *EVAL* $ *f2 x*)$*x*)
apply (*rule eq-reflection*)
by (*simp split*: *infty.splits*)

lemma *infty-assn-ctxt*: *infty-assn A x y* = *z* ⟹ *hn-ctxt* (*infty-assn A*) *x y* = *z*
by (*simp add*: *hn-ctxt-def*)

lemma *infty-cases-hnr*[*sepref-prep-comb-rule*, *sepref-comb-rules*]:
fixes *A e e'*

defines [*simp*]: *INVe* ≡ *hn-invalid* (*infty-assn A*) *e e′*
assumes *FR*: Γ $\Longrightarrow_t$ *hn-ctxt* (*infty-assn A*) *e e′* ∗ *F*
assumes *Infty*: ⟦*e* = *Infty*; *e′* = *Infty*⟧ ⟹ *hn-refine* (*hn-ctxt* (*infty-assn A*) *e e′* ∗ *F*) *f1′* (*hn-ctxt XX1 e e′* ∗ Γ*1′*) *R f1*
assumes *Num*: ⋀*x1 x1a*. ⟦*e* = *Num x1*; *e′* = *Num x1a*⟧ ⟹ *hn-refine* (*hn-ctxt A x1 x1a* ∗ *INVe* ∗ *F*) (*f2′ x1a*) (*hn-ctxt A′ x1 x1a* ∗ *hn-ctxt XX2 e e′* ∗ Γ*2′*) *R* (*f2 x1*)
assumes *MERGE2*[*unfolded hn-ctxt-def*]: Γ*1′* $\vee_A$ Γ*2′* $\Longrightarrow_t$ Γ′
shows *hn-refine* Γ (*case-infty f1′ f2′ e′*) (*hn-ctxt* (*infty-assn A′*) *e e′* ∗ Γ′) *R* (*case-infty*\$*f1*\$(λ_2*x*. *f2 x*)\$*e*)
apply (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *extract-hnr-invalids*
apply (*cases e*; *cases e′*; *simp add*: *infty-assn.simps*[*THEN infty-assn-ctxt*])
subgoal
apply (*rule hn-refine-cons*[*OF - Infty - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*subst mult.commute*, *rule entt-fr-drop*)
apply (*rule entt-trans*[*OF - MERGE2*])
apply (*simp add*:)
done
subgoal
apply (*rule hn-refine-cons*[*OF - Num - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
applyS (*simp add*: *hn-ctxt-def*)
apply1 (*rule entt-trans*[*OF - MERGE2*])
applyS (*simp add*:)
done
done

lemma *hnr-val*[*sepref-fr-rules*]: (*return o Weight.val*,*RETURN o Weight.val*) ∈ [λ*x*. *x*≠*Infty*]$_a$ (*infty-assn A*)^d → *A*
apply *sepref-to-hoare*
subgoal for *x y* **by** (*cases x*; *cases y*; *sep-auto*)
done

context
fixes *A* :: ′*a*::*weight* ⇒ ′*b* ⇒ *assn*
fixes *plusi*
assumes *GA*[*unfolded GEN-ALGO-def*, *sepref-fr-rules*]: *GEN-ALGO plusi* (λ*f*. (*uncurry f*,*uncurry* (*RETURN oo* (+)))∈$A^k *_a A^k \rightarrow_a A$)
begin
sepref-thm *infty-plus-impl* **is** *uncurry* (*RETURN oo* (+)) :: ((*infty-assn A*)$^k *_a$ (*infty-assn A*)$^k \rightarrow_a$ *infty-assn A*)
unfolding *infty-plus-eq-plus*[*symmetric*] *infty-plus-def*[*abs-def*]
by *sepref*
end
concrete-definition *infty-plus-impl* **uses** *infty-plus-impl.refine-raw* **is** (*uncurry*

?impl,-)∈-
lemmas [*sepref-fr-rules*] = *infty-plus-impl.refine*

definition *infty-less* **where**
infty-less lt a b ≡ *case* (*a*,*b*) *of* (*Num a*, *Num b*) ⇒ *lt a b* | (*Num* -, *Infty*) ⇒ *True* | - ⇒ *False*

lemma *infty-less-param*[*param*]:
(*infty-less*,*infty-less*) ∈ (*R*→*R*→*bool-rel*) → ⟨*R*⟩*infty-rel* → ⟨*R*⟩*infty-rel* → *bool-rel*
unfolding *infty-less-def*[*abs-def*]
by *parametricity*

lemma *infty-less-eq-less*: *infty-less* (<) = (<)
unfolding *infty-less-def*[*abs-def*]
apply (*clarsimp intro*!: *ext*)
subgoal for *a b* **by** (*cases a*; *cases b*; *auto*)
done

context
fixes *A* :: '*a*::*weight* ⇒ '*b* ⇒ *assn*
fixes *lessi*
assumes *GA*[*unfolded GEN-ALGO-def*, *sepref-fr-rules*]: *GEN-ALGO lessi* (λ*f*. (*uncurry f*,*uncurry* (*RETURN oo* (<)))∈$A^k *_a A^k \rightarrow_a$ *bool-assn*)
begin
sepref-thm *infty-less-impl* **is** *uncurry* (*RETURN oo* (<)) :: ((*infty-assn A*)^k $*_a$ (*infty-assn A*)^k $\rightarrow_a$ *bool-assn*)
unfolding *infty-less-eq-less*[*symmetric*] *infty-less-def*[*abs-def*]
by *sepref*
end
concrete-definition *infty-less-impl* **uses** *infty-less-impl.refine-raw* **is** (*uncurry ?impl*,-)∈-
lemmas [*sepref-fr-rules*] = *infty-less-impl.refine*

lemma *param-mpath'*: (*mpath'*,*mpath'*)
∈ ⟨⟨*A*$\times_r$ *B* $\times_r$ *A*⟩*list-rel* $\times_r$ *B*⟩*option-rel* → ⟨⟨*A*$\times_r$ *B* $\times_r$ *A*⟩*list-rel*⟩*option-rel*
proof −
have *1*: *mpath'* = *map-option fst*
apply (*intro ext*, *rename-tac x*)
apply (*case-tac x*)
apply *simp*
apply (*rename-tac a*)
apply (*case-tac a*)
apply *simp*
done
show *?thesis*
unfolding *1*
by *parametricity*
qed
lemmas (**in** −) [*sepref-import-param*] = *param-mpath'*

```
lemma param-mpath-weight′:
  (mpath-weight′, mpath-weight′) ∈ ⟨⟨A×r B×r A⟩list-rel ×r B⟩option-rel → ⟨B⟩infty-rel
  by (auto elim!: option-relE simp: infty-rel-def top-infty-def)

lemmas [sepref-import-param] = param-mpath-weight′

context Dijkstra begin
 lemmas impl-aux = mdijkstra-def[unfolded mdinit-def mpop-min-def mupdate-def]

  lemma mdijkstra-correct:
    (mdijkstra, SPEC (is-shortest-path-map v0)) ∈ ⟨br αr res-invarm⟩nres-rel
  proof −
    note mdijkstra-refines
    also note dijkstra′-refines
    also note dijkstra-correct
    finally show ?thesis
      by (rule nres-relI)
  qed

end

locale Dijkstra-Impl = fixes w-dummy :: ′W::{weight,heap}
begin
```

Weights

```
  sepref-register 0::′W
  lemmas [sepref-import-param] =
    IdI[of 0::′W]

  abbreviation weight-assn ≡ id-assn :: ′W ⇒ -

  lemma w-plus-param: ((+), (+)::′W⇒-) ∈ Id → Id → Id by simp
  lemma w-less-param: ((<), (<)::′W⇒-) ∈ Id → Id → Id by simp
  lemmas [sepref-import-param] = w-plus-param w-less-param
  lemma [sepref-gen-algo-rules]:
     GEN-ALGO (return oo (+)) (λf. (uncurry f, uncurry (RETURN ∘∘ (+))) ∈
id-assn^k ∗a id-assn^k →a id-assn)
     GEN-ALGO (return oo (<)) (λf. (uncurry f, uncurry (RETURN ∘∘ (<))) ∈
id-assn^k ∗a id-assn^k →a id-assn)
    by (sep-auto simp: GEN-ALGO-def pure-def intro!: hfrefI hn-refineI)+

  lemma conv-prio-pop-min: prio-pop-min m = do {
      ASSERT (dom m ≠ {});
      ((k,v),m) ← mop-pm-pop-min id m;
      RETURN (k,v,m)
    }
    unfolding prio-pop-min-def mop-pm-pop-min-def
    by (auto simp: pw-eq-iff refine-pw-simps ran-def)
end
```

context fixes N :: *nat* **and** *w-dummy*::$'W$::{*heap*,*weight*} **begin**

interpretation *Dijkstra-Impl w-dummy* **.**

definition *drmap-assn2* $\equiv$ *IICF-Sepl-Binding.iam.assn*
(*pure* (*node-rel N*))
(*prod-assn*
(*list-assn* (*prod-assn* (*pure* (*node-rel N*)) (*prod-assn weight-assn* (*pure* (*node-rel N*)))))
weight-assn)

concrete-definition *mdijkstra′* **uses** *Dijkstra.impl-aux*

sepref-definition *dijkstra-imp* **is** *uncurry mdijkstra′*
:: $(is\text{-}graph\ N\ (Id::('W \times 'W)\ set))^k *_a (pure\ (node\text{-}rel\ N))^k \rightarrow_a drmap\text{-}assn2$
unfolding *mdijkstra′-def*
apply (*subst conv-prio-pop-min*)
apply (*rewrite* **in** *RETURN* (-,⌑) *iam.fold-custom-empty*)
apply (*rewrite hm-fold-custom-empty-sz*[*of N*])
apply (*rewrite* **in** -(- ↦ (⌑,*0*)) *HOL-list.fold-custom-empty*)
unfolding *drmap-assn2-def*
using [[*id-debug*, *goals-limit* = *1*]]
by *sepref*
export-code *dijkstra-imp* **checking** *SML-imp*
end

The main correctness theorem

thm *Dijkstra.mdijkstra-correct*

lemma *mdijkstra′-aref*: (*uncurry mdijkstra′*,*uncurry* (*SPEC oo weighted-graph.is-shortest-path-map*))
$\in [\lambda(G,v0).\ Dijkstra\ G\ v0]_f\ Id \times_r Id \rightarrow \langle br\ Dijkstra.\alpha r\ Dijkstra.res\text{-}invarm\rangle nres\text{-}rel$
using *Dijkstra.mdijkstra-correct*
by (*fastforce intro*!: *frefI simp*: *mdijkstra′.refine*[*symmetric*])

definition *drmap-assn N* $\equiv$ *hr-comp* (*drmap-assn2 N*) (*br Dijkstra.*α*r Dijkstra.res-invarm*)

context notes [*fcomp-norm-unfold*] = *drmap-assn-def*[*symmetric*] **begin**

theorem *dijkstra-imp-correct*: (*uncurry* (*dijkstra-imp N*), *uncurry* (*SPEC* ∘∘ *weighted-graph.is-shortest-path-m*
$\in [\lambda(G,\ v0).\ v0 \in nodes\ G \wedge (\forall (v,\ w,\ v') \in edges\ G.\ 0 \leq w)]_a\ (is\text{-}graph\ N\ Id)^k$
$*_a\ (node\text{-}assn\ N)^k \rightarrow drmap\text{-}assn\ N$
apply (*rule hfref-weaken-pre′*[*OF - dijkstra-imp.refine*[*FCOMP mdijkstra′-aref*]])
proof *clarsimp*
fix G :: (*nat*,$'w$::{*weight*,*heap*}) *graph* **and** *v0*
assume *v0-is-node*: $v0 \in nodes\ G$
and *nonneg-weights*: $\forall (v,\ w,\ v') \in edges\ G.\ 0 \leq w$

and *v0*<*N*
and *RDOM*: *rdomp* (*is-graph N Id*) *G*

from *RDOM* **interpret** *valid-graph G* **unfolding** *is-graph-def rdomp-def* **by** *auto*

from *RDOM* **have** [*simp*]: *finite V* **unfolding** *is-graph-def rdomp-def* **by** *auto*

from *RDOM* **have** $\forall v \in V.\ \{(w, v').\ (v, w, v') \in E\} \in$
Range ($\langle$*Id* $\times_r$ *node-rel N*$\rangle$*list-set-rel*)
by (*auto simp*: *succ-def is-graph-def rdomp-def*)
hence $\forall v \in V.$ *finite* $\{(w, v').\ (v, w, v') \in E\}$
unfolding *list-set-rel-range* **by** *simp*
hence *finite* (*Sigma V* ($\lambda v.\ \{(w, v').\ (v, w, v') \in E\}$))
by *auto*
also have $E \subseteq$ (*Sigma V* ($\lambda v.\ \{(w, v').\ (v, w, v') \in E\}$))
using *E-valid*
by *auto*
finally (*finite-subset*[*rotated*]) **have** [*simp*]: *finite E* **.**

show *Dijkstra G v0*
apply (*unfold-locales*)
unfolding *is-graph-def* **using** *v0-is-node nonneg-weights*
by *auto*
qed

end

corollary *dijkstra-imp-rule*:
<*is-graph n Id G Gi* $*$ $\uparrow$(*v0* $\in$ *nodes G* $\wedge$ ($\forall (v, w, v') \in$ *edges G*. $0 \leq w$))>
dijkstra-imp n Gi v0
<λ*mi*. (*is-graph n Id*) *G Gi*
$*$ ($\exists_A$*m*. *drmap-assn n m mi* $*$ $\uparrow$(*weighted-graph.is-shortest-path-map G v0 m*)) >$_t$
using *dijkstra-imp-correct*[*to-hnr*, *of v0 G n v0 Gi*]
unfolding *hn-refine-def*
apply (*clarsimp*)
apply (*erule cons-rule*[*rotated* -1])
apply (*sep-auto simp*: *hn-ctxt-def pure-def is-graph-def*)
apply (*sep-auto simp*: *hn-ctxt-def*)
done

end

5.4 Imperative Implementation of of Nested DFS (HPY-Improvement)

theory *Sepref-NDFS*
imports
  ../Sepref
  Collections-Examples.Nested-DFS
  Sepref-Graph
  HOL−Library.Code-Target-Numeral
begin

sepref-decl-intf *'v i-red-witness* **is** *'v list ∗ 'v*

lemma *id-red-witness*[*id-rules*]:
  red-init-witness $::_i$ *TYPE*(*'v ⇒ 'v ⇒ 'v i-red-witness option*)
  prep-wit-red $::_i$ *TYPE*(*'v ⇒ 'v i-red-witness option ⇒ 'v i-red-witness option*)
  by *simp-all*

definition
  red-witness-rel-def-internal: *red-witness-rel R* ≡ ⟨⟨*R*⟩*list-rel*,*R*⟩*prod-rel*

lemma *red-witness-rel-def*: ⟨*R*⟩*red-witness-rel* ≡ ⟨⟨*R*⟩*list-rel*,*R*⟩*prod-rel*
  unfolding *red-witness-rel-def-internal*[*abs-def*] **by** (*simp add*: *relAPP-def*)

lemma *red-witness-rel-sv*[*constraint-rules*]:
  single-valued R ⟹ *single-valued* (⟨*R*⟩*red-witness-rel*)
  unfolding *red-witness-rel-def*
  by *tagged-solver*

lemma [*sepref-fr-rules*]: *hn-refine*
  (*hn-val R u u' ∗ hn-val R v v'*)
  (*return* (*red-init-witness u' v'*))
  (*hn-val R u u' ∗ hn-val R v v'*)
  (*option-assn* (*pure* (⟨*R*⟩*red-witness-rel*)))
  (*RETURN*\$(*red-init-witness*\$*u*\$*v*))
  apply *simp*
  unfolding *red-init-witness-def*
  apply *rule*
  apply (*sep-auto simp*: *hn-ctxt-def pure-def red-witness-rel-def*)
  done

lemma [*sepref-fr-rules*]: *hn-refine*
  (*hn-val R u u' ∗ hn-ctxt* (*option-assn* (*pure* (⟨*R*⟩*red-witness-rel*))) *w w'*)
  (*return* (*prep-wit-red u' w'*))
  (*hn-val R u u' ∗ hn-ctxt* (*option-assn* (*pure* (⟨*R*⟩*red-witness-rel*))) *w w'*)
  (*option-assn* (*pure* (⟨*R*⟩*red-witness-rel*)))
  (*RETURN*\$(*prep-wit-red*\$*u*\$*w*))
  apply *rule*
  apply (*cases w*)

apply (*sep-auto simp*: *hn-ctxt-def pure-def red-witness-rel-def*)
apply (*cases w'*)
apply (*sep-auto simp*: *hn-ctxt-def pure-def red-witness-rel-def*)
apply (*sep-auto simp*: *hn-ctxt-def pure-def red-witness-rel-def*)
done

term *red-dfs*

sepref-definition *red-dfs-impl* **is**
(*uncurry2* (*uncurry red-dfs*))
:: $(adjg\text{-}assn\ nat\text{-}assn)^k *_a (ias.assn\ nat\text{-}assn)^k *_a (ias.assn\ nat\text{-}assn)^d *_a nat\text{-}assn^k \rightarrow_a UNSPEC$
unfolding *red-dfs-def*[*abs-def*]
using [[*goals-limit* = *1*]]
by *sepref*
export-code *red-dfs-impl* **checking** *SML-imp*

declare *red-dfs-impl.refine*[*sepref-fr-rules*]

sepref-register *red-dfs* :: *'a i-graph* ⇒ *'a set* ⇒ *'a set* ⇒ *'a*
⇒ (*'a set* ∗ *'a i-red-witness option*) *nres*

lemma *id-init-wit-blue*[*id-rules*]:
init-wit-blue $::_i$ *TYPE*(*'a* ⇒ *'a i-red-witness option* ⇒ *'a blue-witness*)
by *simp*

lemma *hn-blue-wit*[*sepref-import-param*]:
(*NO-CYC*,*NO-CYC*)∈*blue-wit-rel*
(*prep-wit-blue*,*prep-wit-blue*)∈*nat-rel*→*blue-wit-rel*→*blue-wit-rel*
((=),(=))∈*blue-wit-rel*→*blue-wit-rel*→*bool-rel*
by *simp-all*

lemma *hn-init-wit-blue*[*sepref-fr-rules*]: *hn-refine*
(*hn-val nat-rel v v'* ∗ *hn-ctxt* (*option-assn* (*pure* (⟨*nat-rel*⟩*red-witness-rel*))) *w w'*)
(*return* (*init-wit-blue v' w'*))
(*hn-val nat-rel v v'* ∗ *hn-ctxt* (*option-assn* (*pure* (⟨*nat-rel*⟩*red-witness-rel*))) *w w'*)
(*pure blue-wit-rel*)
(*RETURN*\$(*init-wit-blue*\$*v*\$*w*))
apply *rule*
apply (*sep-auto simp*: *hn-ctxt-def pure-def*)
apply (*case-tac w*, *sep-auto*)
apply (*case-tac w'*, *sep-auto*, *sep-auto simp*: *red-witness-rel-def*)
done

lemma *hn-extract-res*[*sepref-import-param*]:
(*extract-res*, *extract-res*) ∈ *blue-wit-rel* → *Id*
by *simp*

thm *red-dfs-impl.refine*

sepref-definition *blue-dfs-impl* **is** *uncurry2 blue-dfs* :: $((adjg\text{-}assn\ nat\text{-}assn)^k *_a (ias.assn\ nat\text{-}assn)^k *_a nat\text{-}assn^k \rightarrow_a id\text{-}assn)$
  unfolding *blue-dfs-def*[*abs-def*]
  apply (*rewrite* **in** *RECT* - ⌑ *ias.fold-custom-empty*)+
  using [[*goals-limit* = *1*]]
  by *sepref*
export-code *blue-dfs-impl* **checking** *SML-imp*

definition *blue-dfs-spec E A v0* ≡ *SPEC* (λ*r. case r of None* ⇒ ¬ *has-acc-cycle E A v0*
    | *Some* (*v*, *pc*, *pv*) ⇒ *is-acc-cycle E A v0 v pv pc*)

lemma *blue-dfs-correct′*: (*uncurry2 blue-dfs*, *uncurry2 blue-dfs-spec*) ∈ $[\lambda((E,A),v0).\ finite\ (E^* ``\{v0\})]_f\ ((Id \times_r Id) \times_r Id) \rightarrow \langle Id \rangle nres\text{-}rel$
  apply (*intro frefI nres-relI*)
  unfolding *blue-dfs-spec-def* **apply** *clarsimp*
  apply (*refine-vcg blue-dfs-correct*)
  done

lemmas *blue-dfs-impl-correct′* = *blue-dfs-impl.refine*[*FCOMP blue-dfs-correct′*]

theorem *blue-dfs-impl-correct*:
  fixes *E*
  assumes *finite* $(E^* ``\{v0\})$
  shows <*ias.assn id-assn A A-impl* ∗ *adjg-assn id-assn E succ-impl*>
      blue-dfs-impl succ-impl A-impl v0
    <λ*r. ias.assn id-assn A A-impl* ∗ *adjg-assn id-assn E succ-impl*
      ∗ ↑(
        case r of None ⇒ ¬*has-acc-cycle E A v0*
      | *Some* (*v*,*pc*,*pv*) ⇒ *is-acc-cycle E A v0 v pv pc*
)$>_t$
  using *blue-dfs-impl-correct′*[*THEN hfrefD*, *THEN hn-refineD*, *of* ((*E*,*A*),*v0*) ((*succ-impl*,*A-impl*),*v0*), *simplified*]
  apply (*rule cons-rule*[*rotated* −*1*])
  using *assms*
  by (*sep-auto simp*: *blue-dfs-spec-def pure-def*)+

We tweak the initialization vector of the outer DFS, to allow pre-initialization of the size of the array-lists. When set to the number of nodes, array-lists will never be resized during the run, which saves some time.

context
  fixes *N* :: *nat*
begin

lemma *testsuite-blue-dfs-modify*:
  $(\{\}::nat\ set,\ \{\}::nat\ set,\ \{\}::nat\ set,\ s)$
  $= (op\text{-}ias\text{-}empty\text{-}sz\ N,\ op\text{-}ias\text{-}empty\text{-}sz\ N,\ op\text{-}ias\text{-}empty\text{-}sz\ N,\ s)$
  by *simp*

sepref-definition *blue-dfs-impl-sz* **is** *uncurry2 blue-dfs* :: $((adjg\text{-}assn\ nat\text{-}assn)^k *_a (ias.assn\ nat\text{-}assn)^k *_a nat\text{-}assn^k \rightarrow_a id\text{-}assn)$
  unfolding *blue-dfs-def*[*abs-def*]
  apply (*rewrite* **in** *RECT* - ¤ *testsuite-blue-dfs-modify*)
  using [[*goals-limit* = *1*]]
  by *sepref*
export-code *blue-dfs-impl-sz* **checking** *SML-imp*

end

lemmas *blue-dfs-impl-sz-correct′* = *blue-dfs-impl-sz.refine*[*FCOMP blue-dfs-correct′*]

term *blue-dfs-impl-sz*

theorem *blue-dfs-impl-sz-correct*:
  fixes *E*
  assumes *finite* $(E^*\ ``\{v0\})$
  shows <*ias.assn id-assn A A-impl* ∗ *adjg-assn id-assn E succ-impl*>
    blue-dfs-impl-sz N succ-impl A-impl v0
   <λ*r*. *ias.assn id-assn A A-impl* ∗ *adjg-assn id-assn E succ-impl*
    ∗ ↑(
     case r of None ⇒ ¬*has-acc-cycle E A v0*
    | *Some* (*v*,*pc*,*pv*) ⇒ *is-acc-cycle E A v0 v pv pc*
)>$_t$
  using *blue-dfs-impl-sz-correct′*[*THEN hfrefD*, *THEN hn-refineD*, *of* ((*E*,*A*),*v0*) ((*succ-impl*,*A-impl*),*v0*), *simplified*]
  apply (*rule cons-rule*[*rotated* −*1*])
  using *assms*
  by (*sep-auto simp*: *blue-dfs-spec-def pure-def*)+

end

5.5 Generic Worklist Algorithm with Subsumption

theory *Worklist-Subsumption*
  imports ../*Sepref*
begin

5.5.1 Utilities

definition *take-from-set* **where**
  $take\text{-}from\text{-}set\ s = ASSERT\ (s \neq \{\}) \gg SPEC\ (\lambda\ (x,\ s').\ x \in s \land s' = s - \{x\})$

lemma *take-from-set-correct*:
  assumes $s \neq \{\}$
  shows *take-from-set* $s \leq$ *SPEC* $(\lambda\ (x,\ s').\ x \in s \wedge s' = s - \{x\})$
using *assms* **unfolding** *take-from-set-def* **by** *simp*

lemmas [*refine-vcg*] = *take-from-set-correct*[*THEN order.trans*]

definition *take-from-mset* **where**
  take-from-mset s = *ASSERT* $(s \neq \{\#\}) \gg$ *SPEC* $(\lambda\ (x,\ s').\ x \in\# s \wedge s' = s - \{\#x\#\})$

lemma *take-from-mset-correct*:
  assumes $s \neq \{\#\}$
  shows *take-from-mset* $s \leq$ *SPEC* $(\lambda\ (x,\ s').\ x \in\# s \wedge s' = s - \{\#x\#\})$
using *assms* **unfolding** *take-from-mset-def* **by** *simp*

lemmas [*refine-vcg*] = *take-from-mset-correct*[*THEN order.trans*]

lemma *set-mset-mp*: *set-mset* $m \subseteq s \Longrightarrow n <$ *count* $m\ x \Longrightarrow x \in s$
  by (*meson count-greater-zero-iff le-less-trans subsetCE zero-le*)

lemma *pred-not-lt-is-zero*: $(\neg\ n - Suc\ 0 < n) \longleftrightarrow n=0$ **by** *auto*

5.5.2 Search Spaces

A search space consists of a step relation, a start state, a final state predicate, and a subsumption preorder.

locale *Search-Space-Defs* =
  fixes $E :: 'a \Rightarrow 'a \Rightarrow bool$ — Step relation
    and $a_0 :: 'a$ — Start state
    and $F :: 'a \Rightarrow bool$ — Final states
    and *subsumes* $:: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** ‹$\preceq$› *50*) — Subsumption preorder
begin
  definition *reachable* **where**
    reachable = $E^{**}\ a_0$

  definition *F-reachable* $\equiv \exists\, a.$ *reachable* $a \wedge F\ a$

end

The set of reachable states must be finite, subsumption must be a preorder, and be compatible with steps and final states.

locale *Search-Space* = *Search-Space-Defs* +
  assumes *finite-reachable*: *finite* $\{a.$ *reachable* $a\}$

assumes *refl*[*intro*!, *simp*]: $a \preceq a$
and *trans*[*trans*]: $a \preceq b \Longrightarrow b \preceq c \Longrightarrow a \preceq c$

assumes *mono*: $a \preceq b \Longrightarrow E\ a\ a' \Longrightarrow$ *reachable* $a \Longrightarrow$ *reachable* $b \Longrightarrow \exists\ b'.\ E\ b\ b' \wedge a' \preceq b'$
and *F-mono*: $a \preceq a' \Longrightarrow F\ a \Longrightarrow F\ a'$
begin

lemma *start-reachable*[*intro*!, *simp*]:
reachable a_0
unfolding *reachable-def* **by** *simp*

lemma *step-reachable*:
assumes *reachable* a $E\ a\ a'$
shows *reachable* a'
using *assms* **unfolding** *reachable-def* **by** *simp*

lemma *finitely-branching*:
assumes *reachable* a
shows *finite* (*Collect* ($E\ a$))
by (*metis assms finite-reachable finite-subset mem-Collect-eq step-reachable subsetI*)

end

5.5.3 Worklist Algorithm

term *card*

context *Search-Space-Defs* **begin**
definition *worklist-var* = *inv-image* (*finite-psupset* (*Collect reachable*) <∗*lex*∗> *measure size*) (λ (a, b,c). (a,b))

definition *worklist-inv-frontier passed wait* =
($\forall$ $a \in$ *passed*. $\forall$ a'. $E\ a\ a' \longrightarrow$ ($\exists$ $b' \in$ *passed* $\cup$ *set-mset wait*. $a' \preceq b'$))

definition *start-subsumed passed wait* = ($\exists$ $a \in$ *passed* $\cup$ *set-mset wait*. $a_0 \preceq a$)

definition *worklist-inv* $\equiv$ λ (*passed*, *wait*, *brk*).
passed $\subseteq$ *Collect reachable* $\wedge$
(*brk* $\longrightarrow$ ($\exists$ *f*. *reachable f* $\wedge$ *F f*)) $\wedge$
($\neg$ *brk* $\longrightarrow$
worklist-inv-frontier passed wait
$\wedge$ ($\forall$ $a \in$ *passed* $\cup$ *set-mset wait*. $\neg$ $F\ a$)
$\wedge$ *start-subsumed passed wait*

$\wedge$ *set-mset wait* $\subseteq$ *Collect reachable*)

definition *add-succ-spec wait a* $\equiv$ *SPEC* (λ(*wait'*,*brk*).
if $\exists\, a'$. *E a a'* $\wedge$ *F a' then*
brk
else set-mset wait' = *set-mset wait* $\cup$ {*a'* . *E a a'*} $\wedge$ $\neg$*brk*
)

definition *worklist-algo* **where**
worklist-algo = *do*
{
if F a_0 *then RETURN True*
else do {
let passed = {};
let wait = {#a_0#};
(*passed*, *wait*, *brk*) $\leftarrow$ *WHILEIT worklist-inv* (λ (*passed*, *wait*, *brk*). $\neg$ *brk*
$\wedge$ *wait* $\neq$ {#})
(λ (*passed*, *wait*, *brk*). *do*
{
(*a*, *wait*) $\leftarrow$ *take-from-mset wait*;
ASSERT (*reachable a*);
if ($\exists$ *a'* $\in$ *passed*. *a* $\preceq$ *a'*) *then RETURN* (*passed*, *wait*, *brk*) *else*
do
{
(*wait*,*brk*) $\leftarrow$ *add-succ-spec wait a*;
let passed = *insert a passed*;
RETURN (*passed*, *wait*, *brk*)
}
}
)
(*passed*, *wait*, *False*);
RETURN brk
}
}

end

Correctness Proof

context *Search-Space* **begin**

lemma *wf-worklist-var*:
wf worklist-var
unfolding *worklist-var-def* **by** (*auto simp*: *finite-reachable*)

context
begin

private lemma *aux1*:
assumes $\forall x\in passed.\ \neg\ a \preceq x$
and *passed* $\subseteq$ *Collect reachable*
and *reachable a*
shows
((*insert a passed*, *wait'*, *brk'*),
passed, *wait*, *brk*)
$\in$ *worklist-var*
proof –
from *assms* **have** $a \notin passed$ **by** *auto*
with *assms*(*2*,*3*) **show** *?thesis*
by (*auto simp*: *worklist-inv-def worklist-var-def finite-psupset-def*)
qed

private lemma *aux2*:
assumes
$a' \in passed$
$a \preceq a'$
$a \in\#\ wait$
worklist-inv-frontier passed wait
shows *worklist-inv-frontier passed* (*wait* – {#*a*#})
using *assms* **unfolding** *worklist-inv-frontier-def*
using *trans*
apply *clarsimp*
by (*metis* (*no-types*, *lifting*) *Un-iff count-eq-zero-iff count-single mset-contains-eq mset-un-cases*)

private lemma *aux5*:
assumes
$a' \in passed$
$a \preceq a'$
$a \in\#\ wait$
start-subsumed passed wait
shows *start-subsumed passed* (*wait* – {#*a*#})
using *assms* **unfolding** *start-subsumed-def* **apply** *clarsimp*
by (*metis Un-iff insert-DiffM2 local.trans mset-right-cancel-elem*)

private lemma *aux3*:
assumes
set-mset wait $\subseteq$ *Collect reachable*
$a \in\#\ wait$
set-mset wait' = *set-mset* (*wait* – {#*a*#}) $\cup$ *Collect* (*E a*)
worklist-inv-frontier passed wait
shows *worklist-inv-frontier* (*insert a passed*) *wait'*
proof –
from *assms*(*1*,*2*) **have** *reachable a*
by (*simp add*: *subset-iff*)
with *finitely-branching* **have** [*simp*, *intro*!]: *finite* (*Collect* (*E a*)) **.**

from *assms*(*2*,*3*,*4*) **show** *?thesis* **unfolding** *worklist-inv-frontier-def*
by (*metis Un-iff insert-DiffM insert-iff local.refl mem-Collect-eq set-mset-add-mset-insert*)
qed

private lemma *aux6*:
assumes
$a \in\#$ *wait*
start-subsumed passed wait
set-mset wait' = *set-mset* (*wait* − {#*a*#}) ∪ *Collect* (*E a*)
shows *start-subsumed* (*insert a passed*) *wait'*
using *assms* **unfolding** *start-subsumed-def*
by (*metis Un-iff insert-DiffM insert-iff set-mset-add-mset-insert*)

lemma *aux4*:
assumes *worklist-inv-frontier passed* {#} *reachable x start-subsumed passed* {#}
passed ⊆ *Collect reachable*
shows $\exists\ x' \in passed.\ x \preceq x'$
proof −
from ‹*reachable x*› **have** $E^{**}\ a_0\ x$ **by** (*simp add*: *reachable-def*)
from *assms*(*3*) **obtain** *b* **where** $a_0 \preceq b$ $b \in passed$ **unfolding** *start-subsumed-def* **by** *auto*
have $\exists x'.\ \exists\ x''.\ E^{**}\ b\ x' \wedge x \preceq x' \wedge x' \preceq x'' \wedge x'' \in passed$ **if**
$E^{**}\ a\ x$ $a \preceq b$ $b \preceq b'$ $b' \in passed$
reachable a reachable b **for** *a b b'*
using *that* **proof** (*induction arbitrary*: *b b' rule*: *converse-rtranclp-induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step a a1 b b'*)
from ‹*E a a1*› ‹$a \preceq b$› ‹*reachable a*› ‹*reachable b*› **obtain** *b1* **where**
E b b1 $a1 \preceq b1$
using *mono* **by** *blast*
then obtain *b1'* **where** *E b' b1'* $b1 \preceq b1'$ **using** *assms*(*4*) *mono step.prems* **by** *blast*
with ‹$b' \in passed$› *assms*(*1*) **obtain** *b1''* **where** $b1'' \in passed$ $b1' \preceq b1''$
unfolding *worklist-inv-frontier-def* **by** *auto*
with ‹$b1 \preceq$ -› **have** $b1 \preceq b1''$ **using** *trans* **by** *blast*
with *step.IH*[*OF* ‹$a1 \preceq b1$› *this* ‹$b1'' \in passed$›] ‹*reachable a*› ‹*E a a1*› ‹*reachable b*› ‹*E b b1*›
obtain *x' x''* **where**
$E^{**}\ b1\ x'$ $x \preceq x'$ $x' \preceq x''$ $x'' \in passed$
by (*auto intro*: *step-reachable*)
moreover from ‹*E b b1*› ‹$E^{**}\ b1\ x'$› **have** $E^{**}\ b\ x'$ **by** *auto*
ultimately show *?case* **by** *auto*
qed
from *this*[*OF* ‹$E^{**}\ a_0\ x$› ‹$a_0 \preceq b$› *refl* ‹$b \in$ -›] *assms*(*4*) ‹$b \in passed$› **show** *?thesis*

by (*auto intro*: *trans*)
qed

theorem *worklist-algo-correct*:
worklist-algo $\leq$ *SPEC* (λ *brk*. *brk* $\longleftrightarrow$ *F-reachable*)
proof –
note [*simp*] = *size-Diff-submset pred-not-lt-is-zero*
note [*dest*] = *set-mset-mp*
show *?thesis*
unfolding *worklist-algo-def add-succ-spec-def F-reachable-def*
apply (*refine-vcg wf-worklist-var*)

apply (*auto*; *fail*) []

apply (*auto simp*: *worklist-inv-def worklist-inv-frontier-def start-subsumed-def*; *fail*)

apply (*simp*; *fail*)

apply (*auto simp*: *worklist-inv-def*; *fail*)

apply (*auto simp*: *worklist-inv-def aux2 aux5*
dest: *in-diffD*
split: *if-split-asm*; *fail*) []

apply (*auto simp*: *worklist-inv-def worklist-var-def intro*: *finite-subset*[*OF* - *finite-reachable*]; *fail*)

apply (*clarsimp split*: *if-split-asm*)

apply (*clarsimp simp*: *worklist-inv-def*; *blast intro*: *step-reachable*; *fail*)

apply (*auto*
simp: *worklist-inv-def step-reachable aux3 aux6 finitely-branching*
dest: *in-diffD*; *fail*)[]

apply (*auto simp*: *worklist-inv-def aux1*; *fail*)

using *F-mono* **apply** (*fastforce simp*: *worklist-inv-def dest*!: *aux4*)
done
qed

lemmas [*refine-vcg*] = *worklist-algo-correct*[*THEN order-trans*]

end — Context

end — Search Space

5.5.4 Towards an Implementation

locale *Worklist1-Defs* = *Search-Space-Defs* +
fixes *succs* :: $'a \Rightarrow 'a$ *list*

locale *Worklist1* = *Worklist1-Defs* + *Search-Space* +
assumes *succs-correct*: *reachable a* $\Longrightarrow$ *set* (*succs a*) = *Collect* (*E a*)
begin

definition *add-succ1 wait a* $\equiv$ *nfoldli* (*succs a*) (λ(-,*brk*). $\neg$*brk*) (λa (*wait*,*brk*). *if F a then RETURN* (*wait*,*True*) *else RETURN* (*wait* + {#*a*#},*False*)) (*wait*, *False*)

lemma *add-succ1-ref*[*refine*]: ⟦(*wait*,*wait'*)∈*Id*; (*a*,*a'*)∈*b-rel Id reachable*⟧ $\Longrightarrow$ *add-succ1 wait a* $\leq$ ⇓(*Id* $\times_r$ *bool-rel*) (*add-succ-spec wait' a'*)
apply *simp*
unfolding *add-succ-spec-def add-succ1-def*
apply (*refine-vcg nfoldli-rule*[**where** *I* = λ*l1* - (*wait'*,*brk*). *if brk then* $\exists$ *a'*. *E a a'* $\wedge$ *F a' else set-mset wait'* = *set-mset wait* $\cup$ *set l1* $\wedge$ *set l1* $\cap$ *Collect F* = {}])
apply (*auto*; *fail*)
using *succs-correct*[*of a*] **apply** (*auto*; *fail*)
using *succs-correct*[*of a*] **apply** (*auto*; *fail*)
apply (*auto*; *fail*)
using *succs-correct*[*of a*] **apply** (*auto*; *fail*)
done

definition *worklist-algo1* **where**
worklist-algo1 = *do*
{
if F a_0 *then RETURN True*
else do {
let passed = {};
let wait = {#a_0#};
(*passed*, *wait*, *brk*) ← *WHILEIT worklist-inv* (λ (*passed*, *wait*, *brk*). $\neg$ *brk* $\wedge$ *wait* $\neq$ {#})
(λ (*passed*, *wait*, *brk*). *do*
{
(*a*, *wait*) ← *take-from-mset wait*;
if ($\exists$ *a'* ∈ *passed*. *a* $\preceq$ *a'*) *then RETURN* (*passed*, *wait*, *brk*) *else do*
{
(*wait*,*brk*) ← *add-succ1 wait a*;
let passed = *insert a passed*;
RETURN (*passed*, *wait*, *brk*)
}
}
)

(*passed*, *wait*, *False*);
RETURN brk
}
}

lemma *worklist-algo1-ref*[*refine*]: *worklist-algo1* $\leq \Downarrow$*Id worklist-algo*
unfolding *worklist-algo1-def worklist-algo-def*
apply (*refine-rcg*)
apply *refine-dref-type*
unfolding *worklist-inv-def*
apply *auto*
done

end

end — Theory
theory *Worklist-Subsumption-Impl*
imports *../IICF/IICF Worklist-Subsumption*
begin

locale *Worklist2-Defs* = *Worklist1-Defs* +
fixes $A :: 'a \Rightarrow 'ai \Rightarrow assn$
fixes $succsi :: 'ai \Rightarrow 'ai\ list\ Heap$
fixes $a_0i :: 'ai\ Heap$
fixes $Fi :: 'ai \Rightarrow bool\ Heap$
fixes $Lei :: 'ai \Rightarrow 'ai \Rightarrow bool\ Heap$

locale *Worklist2* = *Worklist2-Defs* + *Worklist1* +

assumes [*sepref-fr-rules*]: (*uncurry0* a_0i, *uncurry0* (*RETURN* (*PR-CONST* a_0))) $\in$ *unit-assn*$^k \rightarrow_a A$
assumes [*sepref-fr-rules*]: (*Fi*,*RETURN o PR-CONST F*) $\in A^k \rightarrow_a$ *bool-assn*
assumes [*sepref-fr-rules*]: (*uncurry Lei*,*uncurry* (*RETURN oo PR-CONST* ($\preceq$))) $\in A^k *_a A^k \rightarrow_a$ *bool-assn*
assumes [*sepref-fr-rules*]: (*succsi*,*RETURN o PR-CONST succs*) $\in A^k \rightarrow_a$ *list-assn A*
begin
sepref-register *PR-CONST* a_0 *PR-CONST F PR-CONST* ($\preceq$) *PR-CONST succs*

lemma [*def-pat-rules*]:
$a_0 \equiv$ *UNPROTECT* a_0 $F \equiv$ *UNPROTECT F* $(\preceq) \equiv$ *UNPROTECT* $(\preceq)$ *succs* $\equiv$ *UNPROTECT succs*
by *simp-all*

lemma *take-from-mset-as-mop-mset-pick*: *take-from-mset* = *mop-mset-pick*
apply (*intro ext*)

unfolding *take-from-mset-def*[*abs-def*]
by (*auto simp*: *pw-eq-iff refine-pw-simps*)

lemma [*safe-constraint-rules*]: *CN-FALSE is-pure A* $\Longrightarrow$ *is-pure A* **by** *simp*

sepref-thm *worklist-algo2* **is** *uncurry0 worklist-algo1* :: *unit-assn*^{k} $\rightarrow_a$ *bool-assn*
unfolding *worklist-algo1-def add-succ1-def*
supply [[*goals-limit* = *1*]]
apply (*rewrite* **in** *Let* ¤ - *lso-fold-custom-empty*)
apply (*rewrite* **in** {#a_0#} *lmso-fold-custom-empty*)
unfolding *take-from-mset-as-mop-mset-pick fold-lso-bex*
by *sepref*

end

concrete-definition *worklist-algo2*
for *Lei* a_0i *Fi succsi*
uses *Worklist2.worklist-algo2.refine-raw* **is** (*uncurry0 ?f*,-)∈-
thm *worklist-algo2-def*

context *Worklist2* **begin**
lemma *Worklist2-this*: *Worklist2 E* a_0 *F* ($\preceq$) *succs A succsi* a_0i *Fi Lei*
by *unfold-locales*

lemma *hnr-F-reachable*: (*uncurry0* (*worklist-algo2 Lei* a_0i *Fi succsi*), *uncurry0* (*RETURN F-reachable*))
∈ *unit-assn*^{k} $\rightarrow_a$ *bool-assn*
using *worklist-algo2.refine*[*OF Worklist2-this*,
FCOMP worklist-algo1-ref[*THEN nres-relI*],
FCOMP worklist-algo-correct[*THEN Id-SPEC-refine*, *THEN nres-relI*]]
by (*simp add*: *RETURN-def*)

end

context *Worklist1* **begin**
sepref-decl-op *F-reachable* :: *bool-rel* **.**
lemma [*def-pat-rules*]: *F-reachable* ≡ *op-F-reachable* **by** *simp*

lemma *hnr-op-F-reachable*:
assumes *GEN-ALGO* a_0i (λa_0i. (*uncurry0* a_0i, *uncurry0* (*RETURN* a_0)) ∈ *unit-assn*^{k} $\rightarrow_a$ *A*)
assumes *GEN-ALGO Fi* (λ*Fi*. (*Fi*,*RETURN o F*) ∈ A^k $\rightarrow_a$ *bool-assn*)
assumes *GEN-ALGO Lei* (λ*Lei*. (*uncurry Lei*,*uncurry* (*RETURN oo* ($\preceq$))) ∈ A^k $*_a$ A^k $\rightarrow_a$ *bool-assn*)
assumes *GEN-ALGO succsi* (λ*succsi*. (*succsi*,*RETURN o succs*) ∈ A^k $\rightarrow_a$ *list-assn A*)
shows (*uncurry0* (*worklist-algo2 Lei* a_0i *Fi succsi*), *uncurry0* (*RETURN* (*PR-CONST op-F-reachable*)))

$\in$ *unit-assn*^{k} $\rightarrow_a$ *bool-assn*
proof –
from *assms* **interpret** *Worklist2 E* a_0 *F* ($\preceq$) *succs A succsi* $a_0 i$ *Fi Lei*
by (*unfold-locales*; *simp add*: *GEN-ALGO-def*)

from *hnr-F-reachable* **show** *?thesis* **by** *simp*
qed

sepref-decl-impl *hnr-op-F-reachable* **.**
end

end

5.6 Non-Recursive Algebraic Datatype

theory *Sepref-Snip-Datatype*
imports *../../IICF/IICF*
begin

We define a non-recursive datatype

datatype *'a enum* = *E1 'a* | *E2 'a* | *E3* | *E4 'a 'a* | *E5 bool 'a*

5.6.1 Refinement Assertion

fun *enum-assn* **where**
enum-assn A (*E1 x*) (*E1 x'*) = *A x x'*
| *enum-assn A* (*E2 x*) (*E2 x'*) = *A x x'*
| *enum-assn A* (*E3*) (*E3*) = *emp*
| *enum-assn A* (*E4 x y*) (*E4 x' y'*) = *A x x'* $*$ *A y y'*
| *enum-assn A* (*E5 x y*) (*E5 x' y'*) = *bool-assn x x'* $*$ *A y y'*
| *enum-assn - - -* = *false*

You might want to prove some properties

A pure-rule is required to enable recovering of invalidated data that was not stored on the heap

lemma *enum-assn-pure*[*safe-constraint-rules*]: *is-pure A* $\Longrightarrow$ *is-pure* (*enum-assn A*)
apply (*auto simp*: *is-pure-iff-pure-assn*)
apply (*rename-tac x x'*)
apply (*case-tac x*; *case-tac x'*; *simp add*: *pure-def*)
done

An identitiy rule is required to easily prove trivial refinement theorems

lemma *enum-assn-id*[*simp*]: *enum-assn id-assn* = *id-assn*
apply (*intro ext*)
subgoal for *x y* **by** (*cases x*; *cases y*; *simp add*: *pure-def*)
done

Structural rules.

Without congruence condition

lemma *enum-match-nocong*: $\llbracket\bigwedge x\ y.\ \textit{hn-ctxt}\ A\ x\ y \Longrightarrow_t \textit{hn-ctxt}\ A'\ x\ y\rrbracket \Longrightarrow \textit{hn-ctxt}$ $(\textit{enum-assn}\ A)\ e\ e' \Longrightarrow_t \textit{hn-ctxt}\ (\textit{enum-assn}\ A')\ e\ e'$
by (*cases e*; *cases e'*; *simp add*: *hn-ctxt-def entt-star-mono*)

lemma *enum-merge-nocong*:
assumes $\bigwedge x\ y.\ \textit{hn-ctxt}\ A\ x\ y \vee_A \textit{hn-ctxt}\ A'\ x\ y \Longrightarrow_A \textit{hn-ctxt}\ Am\ x\ y$
shows $\textit{hn-ctxt}\ (\textit{enum-assn}\ A)\ e\ e' \vee_A \textit{hn-ctxt}\ (\textit{enum-assn}\ A')\ e\ e' \Longrightarrow_A \textit{hn-ctxt}$ $(\textit{enum-assn}\ Am)\ e\ e'$
using *assms*
by (*cases e*; *cases e'*; *simp add*: *hn-ctxt-def ent-disj-star-mono*)

With congruence condition

lemma *enum-match-cong*[*sepref-frame-match-rules*]:
$\llbracket\bigwedge x\ y.\ \llbracket x\in\textit{set-enum}\ e;\ y\in\textit{set-enum}\ e'\rrbracket \Longrightarrow \textit{hn-ctxt}\ A\ x\ y \Longrightarrow_t \textit{hn-ctxt}\ A'\ x\ y\rrbracket$ $\Longrightarrow \textit{hn-ctxt}\ (\textit{enum-assn}\ A)\ e\ e' \Longrightarrow_t \textit{hn-ctxt}\ (\textit{enum-assn}\ A')\ e\ e'$
by (*cases e*; *cases e'*; *simp add*: *hn-ctxt-def entt-star-mono*)

lemma *enum-merge-cong*[*sepref-frame-merge-rules*]:
assumes $\bigwedge x\ y.\ \llbracket x\in\textit{set-enum}\ e;\ y\in\textit{set-enum}\ e'\rrbracket \Longrightarrow \textit{hn-ctxt}\ A\ x\ y \vee_A \textit{hn-ctxt}$ $A'\ x\ y \Longrightarrow_t \textit{hn-ctxt}\ Am\ x\ y$
shows $\textit{hn-ctxt}\ (\textit{enum-assn}\ A)\ e\ e' \vee_A \textit{hn-ctxt}\ (\textit{enum-assn}\ A')\ e\ e' \Longrightarrow_t \textit{hn-ctxt}$ $(\textit{enum-assn}\ Am)\ e\ e'$
apply (*blast intro*: *entt-disjE enum-match-cong entt-disjD1*[*OF assms*] *entt-disjD2*[*OF assms*])
done

Propagating invalid

lemma *entt-invalid-enum*: $\textit{hn-invalid}\ (\textit{enum-assn}\ A)\ e\ e' \Longrightarrow_t \textit{hn-ctxt}\ (\textit{enum-assn}$ $(\textit{invalid-assn}\ A))\ e\ e'$
apply (*simp add*: *hn-ctxt-def invalid-assn-def*[*abs-def*])
apply (*rule enttI*)
apply *clarsimp*
apply (*cases e*; *cases e'*; *auto simp*: *mod-star-conv pure-def*)
done

lemmas *invalid-enum-merge*[*sepref-frame-merge-rules*] = *gen-merge-cons*[*OF entt-invalid-enum*]

5.6.2 Constructors

Constructors need to be registered

sepref-register *E1 E2 E3 E4 E5*

Refinement rules can be proven straightforwardly on the separation logic level (method *sepref-to-hoare*)

lemma [*sepref-fr-rules*]: $(return\ o\ E1, RETURN\ o\ E1) \in A^d \rightarrow_a enum\text{-}assn\ A$
by *sepref-to-hoare sep-auto*
lemma [*sepref-fr-rules*]: $(return\ o\ E2, RETURN\ o\ E2) \in A^d \rightarrow_a enum\text{-}assn\ A$
by *sepref-to-hoare sep-auto*
lemma [*sepref-fr-rules*]: $(uncurry0\ (return\ E3), uncurry0\ (RETURN\ E3)) \in unit\text{-}assn^k \rightarrow_a enum\text{-}assn\ A$
by *sepref-to-hoare sep-auto*
lemma [*sepref-fr-rules*]: $(uncurry\ (return\ oo\ E4), uncurry\ (RETURN\ oo\ E4)) \in A^d *_a A^d \rightarrow_a enum\text{-}assn\ A$
by *sepref-to-hoare sep-auto*
lemma [*sepref-fr-rules*]: $(uncurry\ (return\ oo\ E5), uncurry\ (RETURN\ oo\ E5)) \in bool\text{-}assn^k *_a A^d \rightarrow_a enum\text{-}assn\ A$
by *sepref-to-hoare* (*sep-auto simp*: *pure-def*)

5.6.3 Destructor

There is currently no automation for destructors, so all the registration boilerplate needs to be done manually

Set ups operation identification heuristics

sepref-register *case-enum*

In the monadify phase, this eta-expands to make visible all required arguments

lemma [*sepref-monadify-arity*]: $case\text{-}enum \equiv \lambda_2 f1\ f2\ f3\ f4\ f5\ x.\ SP\ case\text{-}enum\$(\lambda_2 x.\ f1\$x)\$(\lambda_2 x.\ f2\$x)\$f3\$(\lambda_2 x\ y.\ f4\$x\$y)\$(\lambda_2 x\ y.\ f5\$x\$y)\$x$
by *simp*

This determines an evaluation order for the first-order operands

lemma [*sepref-monadify-comb*]: $case\text{-}enum\$f1\$f2\$f3\$f4\$f5\$x \equiv (\gg=)\$(EVAL\$x)\$(\lambda_2 x.\ SP\ case\text{-}enum\$f1\$f2\$f3\$f4\$f5\$x)$ **by** *simp*

This enables translation of the case-distinction in a non-monadic context.

lemma [*sepref-monadify-comb*]: $EVAL\$(case\text{-}enum\$(\lambda_2 x.\ f1\ x)\$(\lambda_2 x.\ f2\ x)\$f3\$(\lambda_2 x\ y.\ f4\ x\ y)\$(\lambda_2 x\ y.\ f5\ x\ y)\$x)$
$\equiv (\gg=)\$(EVAL\$x)\$(\lambda_2 x.\ SP\ case\text{-}enum\$(\lambda_2 x.\ EVAL\ \$\ f1\ x)\$(\lambda_2 x.\ EVAL\ \$\ f2\ x)\$(EVAL\ \$\ f3)\$(\lambda_2 x\ y.\ EVAL\ \$\ f4\ x\ y)\$(\lambda_2 x\ y.\ EVAL\ \$\ f5\ x\ y)\$x)$
apply (*rule eq-reflection*)
by (*simp split*: *enum.splits*)

Auxiliary lemma, to lift simp-rule over *hn-ctxt*

lemma *enum-assn-ctxt*: $enum\text{-}assn\ A\ x\ y = z \Longrightarrow hn\text{-}ctxt\ (enum\text{-}assn\ A)\ x\ y = z$
by (*simp add*: *hn-ctxt-def*)

The cases lemma first extracts the refinement for the datatype from the precondition. Next, it generate proof obligations to refine the functions for every case. Finally the postconditions of the refinement are merged.

Note that we handle the destructed values separately, to allow reconstruction of the original datatype after the case-expression.

Moreover, we provide (invalidated) versions of the original compound value to the cases, which allows access to pure compound values from inside the case.

lemma *enum-cases-hnr*:
fixes $A\ e\ e'$
defines [*simp*]: $INVe \equiv hn\text{-}invalid\ (enum\text{-}assn\ A)\ e\ e'$
assumes *FR*: $\Gamma \Longrightarrow_t hn\text{-}ctxt\ (enum\text{-}assn\ A)\ e\ e' * F$
assumes *E1*: $\bigwedge x1\ x1a.\ [\![e = E1\ x1;\ e' = E1\ x1a]\!] \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ A\ x1\ x1a * INVe * F)\ (f1'\ x1a)\ (hn\text{-}ctxt\ A1'\ x1\ x1a * hn\text{-}ctxt\ XX1\ e\ e' * \Gamma 1')\ R\ (f1\ x1)$
assumes *E2*: $\bigwedge x2\ x2a.\ [\![e = E2\ x2;\ e' = E2\ x2a]\!] \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ A\ x2\ x2a * INVe * F)\ (f2'\ x2a)\ (hn\text{-}ctxt\ A2'\ x2\ x2a * hn\text{-}ctxt\ XX2\ e\ e' * \Gamma 2')\ R\ (f2\ x2)$
assumes *E3*: $[\![e = E3;\ e' = E3]\!] \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ (enum\text{-}assn\ A)\ e\ e' * F)\ f3'\ (hn\text{-}ctxt\ XX3\ e\ e' * \Gamma 3')\ R\ f3$
assumes *E4*: $\bigwedge x41\ x42\ x41a\ x42a.$
$[\![e = E4\ x41\ x42;\ e' = E4\ x41a\ x42a]\!]$
$\Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ A\ x41\ x41a * hn\text{-}ctxt\ A\ x42\ x42a * INVe * F)\ (f4'\ x41a\ x42a)\ (hn\text{-}ctxt\ A4a'\ x41\ x41a * hn\text{-}ctxt\ A4b'\ x42\ x42a * hn\text{-}ctxt\ XX4\ e\ e' * \Gamma 4')\ R$
$(f4\ x41\ x42)$
assumes *E5*: $\bigwedge x51\ x52\ x51a\ x52a.$
$[\![e = E5\ x51\ x52;\ e' = E5\ x51a\ x52a]\!]$
$\Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ bool\text{-}assn\ x51\ x51a * hn\text{-}ctxt\ A\ x52\ x52a * INVe * F)\ (f5'\ x51a\ x52a)$
$(hn\text{-}ctxt\ bool\text{-}assn\ x51\ x51a * hn\text{-}ctxt\ A5'\ x52\ x52a * hn\text{-}ctxt\ XX5\ e\ e' * \Gamma 5')\ R\ (f5\ x51\ x52)$
assumes *MERGE1*[*unfolded hn-ctxt-def*]: $\bigwedge x\ x'.\ hn\text{-}ctxt\ A1'\ x\ x' \vee_A hn\text{-}ctxt\ A2'\ x\ x' \vee_A hn\text{-}ctxt\ A3'\ x\ x' \vee_A hn\text{-}ctxt\ A4a'\ x\ x' \vee_A hn\text{-}ctxt\ A4b'\ x\ x' \vee_A hn\text{-}ctxt\ A5'\ x\ x' \Longrightarrow_t hn\text{-}ctxt\ A'\ x\ x'$
assumes *MERGE2*[*unfolded hn-ctxt-def*]: $\Gamma 1' \vee_A \Gamma 2' \vee_A \Gamma 3' \vee_A \Gamma 4' \vee_A \Gamma 5' \Longrightarrow_t \Gamma'$
shows $hn\text{-}refine\ \Gamma\ (case\text{-}enum\ f1'\ f2'\ f3'\ f4'\ f5'\ e')\ (hn\text{-}ctxt\ (enum\text{-}assn\ A')\ e\ e' * \Gamma')\ R\ (case\text{-}enum\$(\lambda_2 x.\ f1\ x)\$(\lambda_2 x.\ f2\ x)\$f3\$(\lambda_2 x\ y.\ f4\ x\ y)\$(\lambda_2 x\ y.\ f5\ x\ y)\$e)$
apply (*rule hn-refine-cons-pre*[*OF FR*])
apply1 *extract-hnr-invalids*
apply (*cases e*; *cases e'*; *simp add*: *enum-assn.simps*[*THEN enum-assn-ctxt*])
subgoal
apply (*rule hn-refine-cons*[*OF - E1 - entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*) — Match precondition for case, get *enum-assn* from assumption generated by *extract-hnr-invalids*
apply (*rule entt-star-mono*) — Split postcondition into pairs for compounds and frame, drop *hn-ctxt XX*
apply1 (*rule entt-fr-drop*)
apply1 (*rule entt-trans*[*OF - MERGE1*])
applyS (*simp add*: *hn-ctxt-def entt-disjI1' entt-disjI2'*)

apply1 (*rule entt-trans*[*OF* - *MERGE2*])
applyS (*simp add*: *entt-disjI1′ entt-disjI2′*)
done
subgoal
apply (*rule hn-refine-cons*[*OF* - *E2* - *entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
apply1 (*rule entt-trans*[*OF* - *MERGE1*])
applyS (*simp add*: *hn-ctxt-def entt-disjI1′ entt-disjI2′*)
apply1 (*rule entt-trans*[*OF* - *MERGE2*])
applyS (*simp add*: *entt-disjI1′ entt-disjI2′*)
done
subgoal
apply (*rule hn-refine-cons*[*OF* - *E3* - *entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*subst mult.commute*, *rule entt-fr-drop*)
apply (*rule entt-trans*[*OF* - *MERGE2*])
apply (*simp add*: *entt-disjI1′ entt-disjI2′*)
done
subgoal
apply (*rule hn-refine-cons*[*OF* - *E4* - *entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
apply (*rule entt-star-mono*)

apply1 (*rule entt-trans*[*OF* - *MERGE1*])
applyS (*simp add*: *hn-ctxt-def entt-disjI1′ entt-disjI2′*)

apply1 (*rule entt-trans*[*OF* - *MERGE1*])
applyS (*simp add*: *hn-ctxt-def entt-disjI1′ entt-disjI2′*)

apply1 (*rule entt-trans*[*OF* - *MERGE2*])
applyS (*simp add*: *entt-disjI1′ entt-disjI2′*)
done
subgoal
apply (*rule hn-refine-cons*[*OF* - *E5* - *entt-refl*]; *assumption?*)
applyS (*simp add*: *hn-ctxt-def*)
apply (*rule entt-star-mono*)
apply1 (*rule entt-fr-drop*)
apply (*rule entt-star-mono*)

apply1 (*rule ent-imp-entt*)
applyS (*simp add*: *hn-ctxt-def*)

apply1 (*rule entt-trans*[*OF* - *MERGE1*])
applyS (*simp add*: *hn-ctxt-def entt-disjI1′ entt-disjI2′*)

apply1 (*rule entt-trans*[*OF - MERGE2*])
applyS (*simp add*: *entt-disjI1′ entt-disjI2′*)
done
done

After some more preprocessing (adding extra frame-rules for non-atomic postconditions, and splitting the merge-terms into binary merges), this rule can be registered

lemmas [*sepref-comb-rules*] = *enum-cases-hnr*[*sepref-prep-comb-rule*]

5.6.4 Regression Test

definition *test1* (*e*::*bool enum*) ≡ *RETURN e*
sepref-definition *test1-impl* **is** *test1* :: $(enum\text{-}assn\ bool\text{-}assn)^d \rightarrow_a enum\text{-}assn\ bool\text{-}assn$
unfolding *test1-def*[*abs-def*] **by** *sepref*
sepref-register *test1*
lemmas [*sepref-fr-rules*] = *test1-impl.refine*

definition *test* ≡ *do* {
let x = *E1 True*;

\- ← *case x of*
E1 a ⇒ *RETURN* (*Some a*) — Access and invalidate compound inside case
| - ⇒ *RETURN* (*Some True*);

\- ← *test1 x*; — Rely on structure being there, with valid compound

— Same thing again, with merge
\- ← *if True then*
case x of
E1 a ⇒ *RETURN* (*Some a*) — Access and invalidate compound inside case
| - ⇒ *RETURN* (*Some True*)
else RETURN None;
\- ← *test1 x*; — Rely on structure being there, with valid compound

— Now test with non-pure
let a = *op-array-replicate 4* (*3*::*nat*);
let x = *E5 False a*;

\- ← *case x of*
E1 - ⇒ *RETURN* (*0*::*nat*)
| *E2* - ⇒ *RETURN 1*
| *E3* ⇒ *RETURN 0*
| *E4* - - ⇒ *RETURN 0*
| *E5* - *a* ⇒ *mop-list-get a 0*;

— Rely on that compound still exists (it's components are only read in the case above)

case x of
E1 a ⇒ do {mop-list-set a 0 0; RETURN (0::nat)}
| E2 - ⇒ RETURN 1
| E3 ⇒ RETURN 0
| E4 - - ⇒ RETURN 0
| E5 - - ⇒ RETURN 0
}

lemmas [*safe-constraint-rules*] = *CN-FALSEI*[*of is-pure invalid-assn A* **for** *A*]

sepref-definition *foo* **is** *uncurry0 test* :: $\mathit{unit\text{-}assn}^k \rightarrow_a \mathit{nat\text{-}assn}$
unfolding *test-def*
supply [[*goals-limit=1*]]
by *sepref*

end

5.7 Snippet to Define Custom Combinators

theory *Sepref-Snip-Combinator*
imports *../../IICF/IICF*
begin

5.7.1 Definition of the Combinator

Currently, when defining new combinators, you are largely on your own. If you can show your combinator equivalent to some other, already existing, combinator, you should apply this equivalence in the monadify phase.

In this example, we show the development of a map combinator from scratch.

We set ourselves in to a context where we fix the abstract and concrete arguments of the monadic map combinator, as well as the refinement assertions, and a frame, that represents the remaining heap content, and may be read by the map-function.

context
fixes *f* :: *'a ⇒ 'b nres*
fixes *l* :: *'a list*

fixes *fi* :: *'ai ⇒ 'bi Heap*
fixes *li* :: *'ai list*

fixes *A A'* :: *'a ⇒ 'ai ⇒ assn* — Refinement for list elements before and after map-function. Different, as map function may invalidate list elements!
fixes *B* :: *'b ⇒ 'bi ⇒ assn*

fixes *F* :: *assn* — Symbolic frame, representing all heap content the map-function body may access

notes [[*sepref-register-adhoc f l*]] — Register for operation id

assumes *f-rl*: *hn-refine* (*hn-ctxt A x xi* $*$ *F*) (*fi xi*) (*hn-ctxt A′ x xi* $*$ *F*) *B* (*f*\$*x*)
— Refinement for *f*

begin

We implement our combinator using the monadic refinement framework.

definition *mmap* ≡ *RECT* (λ*mmap*.
λ[] ⇒ *RETURN* []
| *x#xs* ⇒ *do* { *x* ← *f x*; *xs* ← *mmap xs*; *RETURN* (*x#xs*) }) *l*

5.7.2 Synthesis of Implementation

In order to propagate the frame *F* during synthesis, we use a trick: We wrap the frame into a dummy refinement assertion. This way, sepref recognizes the frame just as another context element, and does correct propagation.

definition *F-assn* (*x*::*unit*) (*y*::*unit*) ≡ *F*
lemma *F-unf*: *hn-ctxt F-assn x y* = *F*
by (*auto simp*: *F-assn-def hn-ctxt-def*)

We build a combinator rule to refine *f*. We need a combinator rule here, because *f* does not only depend on its formal arguments, but also on the frame (represented as dummy argument).

lemma *f-rl′*: *hn-refine* (*hn-ctxt A x xi* $*$ *hn-ctxt* (*F-assn*) *dx dxi*) (*fi xi*) (*hn-ctxt A′ x xi* $*$ *hn-ctxt* (*F-assn*) *dx dxi*) *B* (*f*\$*x*)
unfolding *F-unf* **by** (*rule f-rl*)

Then we use the Sepref tool to synthesize an implementation of *mmap*.

schematic-goal *mmap-impl*:
notes [*sepref-comb-rules*] = *hn-refine-frame*[*OF f-rl′*]
shows *hn-refine* (*hn-ctxt* (*list-assn A*) *l li* $*$ *hn-ctxt* (*F-assn*) *dx dxi*) (*?c*::*?′c Heap*) *?Γ′ ?R mmap*
unfolding *mmap-def HOL-list.fold-custom-empty*
apply *sepref-dbg-keep*
done

We unfold the wrapped frame

lemmas *mmap-impl′* = *mmap-impl*[*unfolded F-unf*]

end

5.7.3 Setup for Sepref

Outside the context, we extract the synthesized implementation as a new constant, and set up code theorems for the fixed-point combinators.

concrete-definition *mmap-impl* **uses** *mmap-impl′*
prepare-code-thms *mmap-impl-def*

Moreover, we have to manually declare arity and monadify theorems. The arity theorem ensures that we always have a constant number of operators, and the monadify theorem determines an execution order: The list-argument is evaluated first.

lemma *mmap-arity*[*sepref-monadify-arity*]: $mmap \equiv \lambda_2 f\, l.\ SP\ mmap\$(\lambda_2 x.\ f\$x)\$l$
by *simp*
lemma *mmap-mcomb*[*sepref-monadify-comb*]: $mmap\$f\$x \equiv (\gg\!=)\$(EVAL\$x)\$(\lambda_2 x.\ SP\ mmap\$f\$x)$ **by** *simp*

We can massage the refinement theorem $(\bigwedge x\ xi.\ hn\text{-}refine\ (hn\text{-}ctxt\ ?A\ x\ xi * ?F)\ (?fi\ xi)\ (hn\text{-}ctxt\ ?A'\ x\ xi * ?F)\ ?B\ (?f\ \$\ x)) \Longrightarrow hn\text{-}refine\ (hn\text{-}ctxt\ (list\text{-}assn\ ?A)\ ?l\ ?li * ?F)\ (mmap\text{-}impl\ ?fi\ ?li)\ (hn\text{-}ctxt\ (list\text{-}assn\ ?A')\ ?l\ ?li * ?F)\ (list\text{-}assn\ ?B)\ (mmap\ ?f\ ?l)$ a bit, to get a valid combinator rule

print-statement *hn-refine-cons-pre*[*OF* - *mmap-impl.refine*, *sepref-prep-comb-rule*, *no-vars*]

lemma *mmap-comb-rl*[*sepref-comb-rules*]:
assumes $P \Longrightarrow_t hn\text{-}ctxt\ (list\text{-}assn\ A)\ l\ li * F$
— Initial frame
and $\bigwedge x\ xi.\ hn\text{-}refine\ (hn\text{-}ctxt\ A\ x\ xi * F)\ (fi\ xi)\ (Q\ x\ xi)\ B\ (f\ x)$
— Refinement of map-function
and $\bigwedge x\ xi.\ Q\ x\ xi \Longrightarrow_t hn\text{-}ctxt\ A'\ x\ xi * F$
— Recover refinement for list-element and original frame from what map-function produced
shows $hn\text{-}refine\ P\ (mmap\text{-}impl\ fi\ li)\ (hn\text{-}ctxt\ (list\text{-}assn\ A')\ l\ li * F)\ (list\text{-}assn\ B)\ (mmap\$(\lambda_2 x.\ f\ x)\$l)$
unfolding *APP-def PROTECT2-def*
using *hn-refine-cons-pre*[*OF* - *mmap-impl.refine*, *sepref-prep-comb-rule*, *of P A l li F fi Q B f A′*]
using *assms*
by *simp*

5.7.4 Example

Finally, we can test our combinator. Note how the map-function accesses the array on the heap, which is not among its arguments. This is only possible as we passed around a frame.

sepref-thm *test-mmap*
is $\lambda l.$ *do* { *let a* = *op-array-of-list* [*True*,*True*,*False*]; *mmap* ($\lambda x.$ *do* { *mop-list-get a* (*x mod 3*) }) *l* }
:: $(list\text{-}assn\ nat\text{-}assn)^k \rightarrow_a list\text{-}assn\ bool\text{-}assn$
unfolding *HOL-list.fold-custom-empty*
by *sepref*

5.7.5 Limitations

Currently, the major limitation is that combinator rules are fixed to specific data types. In our example, we did an implementation for HOL lists. We cannot come up with an alternative implementation, for, e.g., array-lists, but have to use a different abstract combinator.

One workaround is to use some generic operations, as is done for foreach-loops, which require a generic to-list operation. However, in this case, we produce unwanted intermediate lists, and would have to add complicated a-posteriori deforestation optimizations.

end

Chapter 6

Benchmarks

Contains the benchmarks of the IRF/IICF. See the README file in the benchmark folder for more information on how to run the benchmarks.

theory *Heapmap-Bench*
imports
  ../../../IICF/Impl/Heaps/IICF-Impl-Heapmap
  ../../../Sepref-ICF-Bindings
begin

context
  includes *bit-operations-syntax*
begin

definition *rrand* :: *uint32* ⇒ *uint32*
  where *rrand s* ≡ (*s* ∗ *1103515245* + *12345*) *AND 0x7FFFFFFF*

end

definition *rand* :: *uint32* ⇒ *nat* ⇒ (*uint32* ∗ *nat*) **where**
  rand s m ≡ *let*
    s = *rrand s*;
    r = *nat-of-uint32 s*;
    r = (*r* ∗ *m*) *div 0x80000000*
  in (*s*,*r*)

partial-function (*heap*) *rep* **where** *rep i N f s* = (
  if i<*N then do* {
    s ← *f s i*;
    rep (*i*+*1*) *N f s*
  } *else return s*
)

declare *rep.simps*[*code*]

term *hm-insert-op-impl*

definition *testsuite N* ≡ *do* {
  let s=0;
  let N2=efficient-nat-div2 N;
  hm ← *hm-empty-op-impl N*;

  (*hm*,*s*) ← *rep 0 N* (λ(*hm*,*s*) *i*. *do* {
    let (*s*,*v*) = *rand s N2*;
    hm ← *hm-insert-op-impl N id i v hm*;
    return (*hm*,*s*)
  }) (*hm*,*s*);

  (*hm*,*s*) ← *rep 0 N* (λ(*hm*,*s*) *i*. *do* {
    let (*s*,*v*) = *rand s N2*;
    hm ← *hm-change-key-op-impl id i v hm*;
    return (*hm*,*s*)
  }) (*hm*,*s*);

  hm ← *rep 0 N* (λ*hm i*. *do* {
    (-,*hm*) ← *hm-pop-min-op-impl id hm*;
    return hm
  }) *hm*;

  return ()
}

export-code *rep* **in** *SML-imp*

partial-function (*tailrec*) *drep* **where** *drep i N f s* = (
  if i<N then drep (*i+1*) *N f* (*f s i*)
  else s
)

declare *drep.simps*[*code*]

term *aluprioi.insert*
term *aluprioi.empty*
term *aluprioi.pop*

definition *ftestsuite N* ≡ *do* {
  let s=0;
  let N2=efficient-nat-div2 N;
  let hm= aluprioi.empty ();

  let (*hm*,*s*) = *drep 0 N* (λ(*hm*,*s*) *i*. *do* {

```
    let (s,v) = rand s N2;
    let hm = aluprioi.insert hm i v;
    (hm,s)
  }) (hm,s);

  let (hm,s) = drep 0 N (λ(hm,s) i. do {
    let (s,v) = rand s N2;
    let hm = aluprioi.insert hm i v;
    (hm,s)
  }) (hm,s);

  let hm = drep 0 N (λhm i. do {
    let (-,-,hm) = aluprioi.pop hm;
    hm
  }) hm;

  ()
}
```

export-code
  testsuite ftestsuite
  nat-of-integer integer-of-nat
  in *SML-imp* **module-name** *Heapmap*
  file ‹*heapmap-export.sml*›

end
theory *Dijkstra-Benchmark*
imports *../../../Examples/Sepref-Dijkstra*
  Dijkstra-Shortest-Path.Test
begin

definition *nat-cr-graph-imp*
  :: *nat* ⇒ (*nat* × *nat* × *nat*) *list* ⇒ *nat graph-impl Heap*
  where *nat-cr-graph-imp* ≡ *cr-graph*

concrete-definition *nat-dijkstra-imp* **uses** *dijkstra-imp-def*[**where** *'W=nat*]
prepare-code-thms *nat-dijkstra-imp-def*

lemma *nat-dijkstra-imp-eq*: *nat-dijkstra-imp* = *dijkstra-imp*
  unfolding *dijkstra-imp-def*[*abs-def*] *nat-dijkstra-imp-def*[*abs-def*]
  by *simp*

definition *nat-cr-graph-fun nn es* ≡ *hlg-from-list-nat* ([*0..<nn*], *es*)

export-code

integer-of-nat nat-of-integer

ran-graph

nat-cr-graph-fun nat-dijkstra

nat-cr-graph-imp nat-dijkstra-imp
in *SML-imp* **module-name** *Dijkstra*
file ‹*dijkstra-export.sml*›

end
theory *NDFS-Benchmark*
imports
Collections-Examples.Nested-DFS
../../../Examples/Sepref-NDFS
Separation-Logic-Imperative-HOL.From-List-GA
begin

locale *bm-fun* **begin**

schematic-goal *succ-of-list-impl*:
notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat⇀nat set* **and** *R=⟨nat-rel,R⟩dflt-rm-rel* **for** *R*]
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩list-set-rel*]

shows (*?f::?'c,succ-of-list*) ∈ *?R*
unfolding *succ-of-list-def*[*abs-def*]
apply (*autoref* (*keep-goal*))
done

concrete-definition *succ-of-list-impl* **uses** *succ-of-list-impl*

schematic-goal *acc-of-list-impl*:
notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩dflt-rs-rel* **for** *R*]

shows (*?f::?'c,acc-of-list*) ∈ *?R*
unfolding *acc-of-list-def*[*abs-def*]
apply (*autoref* (*keep-goal*))
done

concrete-definition *acc-of-list-impl* **uses** *acc-of-list-impl*

schematic-goal *red-dfs-impl-refine-aux*:

fixes *u′*::*nat* **and** *V′*::*nat set*
notes [*autoref-tyrel*] =
ty-REL[**where** *′a*=*nat set* **and** *R*=⟨*nat-rel*⟩*dflt-rs-rel*]
assumes [*autoref-rules*]:
(*u*,*u′*)∈*nat-rel*
(*V*,*V′*)∈⟨*nat-rel*⟩*dflt-rs-rel*
(*onstack*,*onstack′*)∈⟨*nat-rel*⟩*dflt-rs-rel*
(*E*,*E′*)∈⟨*nat-rel*⟩*slg-rel*
shows (*RETURN* (*?f*::*?′c*), *red-dfs E′ onstack′ V′ u′*) ∈ *?R*
apply −
unfolding *red-dfs-def*
apply (*autoref-monadic*)
done

concrete-definition *red-dfs-impl* **uses** *red-dfs-impl-refine-aux*
prepare-code-thms *red-dfs-impl-def*
declare *red-dfs-impl.refine*[*autoref-higher-order-rule*, *autoref-rules*]

schematic-goal *ndfs-impl-refine-aux*:
fixes *s*::*nat* **and** *succi*
notes [*autoref-tyrel*] =
ty-REL[**where** *′a*=*nat set* **and** *R*=⟨*nat-rel*⟩*dflt-rs-rel*]
assumes [*autoref-rules*]:
(*succi*,*E*)∈⟨*nat-rel*⟩*slg-rel*
(*Ai*,*A*)∈⟨*nat-rel*⟩*dflt-rs-rel*
notes [*autoref-rules*] = *IdI*[*of s*]
shows (*RETURN* (*?f*::*?′c*), *blue-dfs E A s*) ∈ ⟨*?R*⟩*nres-rel*
unfolding *blue-dfs-def*
apply (*autoref-monadic* (*trace*))
done

concrete-definition *fun-ndfs-impl* **for** *succi Ai s* **uses** *ndfs-impl-refine-aux*
prepare-code-thms *fun-ndfs-impl-def*

definition *fun-succ-of-list* ≡
succ-of-list-impl o map (λ(*u*,*v*). (*nat-of-integer u*, *nat-of-integer v*))

definition *fun-acc-of-list* ≡
acc-of-list-impl o map nat-of-integer

end

interpretation *fun*: *bm-fun* **.**

locale *bm-funs* **begin**

schematic-goal *succ-of-list-impl*:

notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat⇀nat set* **and** *R=⟨nat-rel,R⟩iam-map-rel* **for** *R*]
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩list-set-rel*]

shows (*?f::?'c,succ-of-list*) ∈ *?R*
unfolding *succ-of-list-def*[*abs-def*]
apply (*autoref* (*keep-goal*))
done

concrete-definition *succ-of-list-impl* **uses** *succ-of-list-impl*

schematic-goal *acc-of-list-impl*:
notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩iam-set-rel* **for** *R*]

shows (*?f::?'c,acc-of-list*) ∈ *?R*
unfolding *acc-of-list-def*[*abs-def*]
apply (*autoref* (*keep-goal*))
done

concrete-definition *acc-of-list-impl* **uses** *acc-of-list-impl*

schematic-goal *red-dfs-impl-refine-aux*:

fixes *u'::nat* **and** *V'::nat set*
notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩iam-set-rel*]
assumes [*autoref-rules*]:
(*u,u'*)∈*nat-rel*
(*V,V'*)∈⟨*nat-rel*⟩*iam-set-rel*
(*onstack,onstack'*)∈⟨*nat-rel*⟩*iam-set-rel*
(*E,E'*)∈⟨*nat-rel*⟩*slg-rel*
shows (*RETURN* (*?f::?'c*), *red-dfs E' onstack' V' u'*) ∈ *?R*
apply −
unfolding *red-dfs-def*
apply (*autoref-monadic*)
done

concrete-definition *red-dfs-impl* **uses** *red-dfs-impl-refine-aux*
prepare-code-thms *red-dfs-impl-def*
declare *red-dfs-impl.refine*[*autoref-higher-order-rule*, *autoref-rules*]

schematic-goal *ndfs-impl-refine-aux*:
fixes *s::nat* **and** *succi*
notes [*autoref-tyrel*] =
ty-REL[**where** *'a=nat set* **and** *R=⟨nat-rel⟩iam-set-rel*]
assumes [*autoref-rules*]:
(*succi,E*)∈⟨*nat-rel*⟩*slg-rel*
(*Ai,A*)∈⟨*nat-rel*⟩*iam-set-rel*

notes [*autoref-rules*] = *IdI*[*of s*]
shows (*RETURN* (*?f*::*?'c*), *blue-dfs E A s*) ∈ ⟨*?R*⟩*nres-rel*
unfolding *blue-dfs-def*
apply (*autoref-monadic* (*trace*))
done

concrete-definition *funs-ndfs-impl* **for** *succi Ai s* **uses** *ndfs-impl-refine-aux*
prepare-code-thms *funs-ndfs-impl-def*

definition *funs-succ-of-list* ≡
succ-of-list-impl o map (λ(*u*,*v*). (*nat-of-integer u*, *nat-of-integer v*))

definition *funs-acc-of-list* ≡
acc-of-list-impl o map nat-of-integer

end

interpretation *funs*: *bm-funs* **.**

definition *imp-ndfs-impl* ≡ *blue-dfs-impl*
definition *imp-ndfs-sz-impl* ≡ *blue-dfs-impl-sz*
definition *imp-acc-of-list l* ≡ *From-List-GA.ias-from-list* (*map nat-of-integer l*)
definition *imp-graph-of-list n l* ≡ *cr-graph* (*nat-of-integer n*) (*map* (*pairself nat-of-integer*) *l*)

export-code
nat-of-integer integer-of-nat
fun.fun-ndfs-impl fun.fun-succ-of-list fun.fun-acc-of-list
funs.funs-ndfs-impl funs.funs-succ-of-list funs.funs-acc-of-list
imp-ndfs-impl imp-ndfs-sz-impl imp-acc-of-list imp-graph-of-list
in *SML-imp* **module-name** *NDFS-Benchmark* **file** ‹*NDFS-Benchmark-export.sml*›

ML-val ‹*open Time*›
end