

Recursion Theory I

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Abstract

This document presents the formalization of introductory material from recursion theory — definitions and basic properties of primitive recursive functions, Cantor pairing function and computably enumerable sets (including a proof of existence of a one-complete computably enumerable set and a proof of the Rice's theorem).

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1 Cantor pairing function

```
theory CPair
imports Main
begin
```

We introduce a particular coding *c-pair* from ordered pairs of natural numbers to natural numbers. See [1] and the Isabelle documentation for more information.

1.1 Pairing function

definition

```
sf :: nat ⇒ nat where
sf-def: sf x = x * (x+1) div 2
```

definition

```
c-pair :: nat ⇒ nat ⇒ nat where
c-pair x y = sf (x+y) + x
```

lemma *sf-at-0*: *sf* 0 = 0 *<proof>*

lemma *sf-at-1*: *sf* 1 = 1 *<proof>*

lemma *sf-at-Suc*: *sf* (*x*+1) = *sf* *x* + *x* + 1
<proof>

lemma *arg-le-sf*: *x* ≤ *sf* *x*
<proof>

lemma *sf-mono*: *x* ≤ *y* ⇒ *sf* *x* ≤ *sf* *y*
<proof>

lemma *sf-strict-mono*: *x* < *y* ⇒ *sf* *x* < *sf* *y*

<proof>

lemma *sf-posI*: $x > 0 \implies sf(x) > 0$
<proof>

lemma *arg-less-sf*: $x > 1 \implies x < sf(x)$
<proof>

lemma *sf-eq-arg*: $sf\ x = x \implies x \leq 1$
<proof>

lemma *sf-le-sfD*: $sf\ x \leq sf\ y \implies x \leq y$
<proof>

lemma *sf-less-sfD*: $sf\ x < sf\ y \implies x < y$
<proof>

lemma *sf-inj*: $sf\ x = sf\ y \implies x = y$
<proof>

Auxiliary lemmas

lemma *sf-aux1*: $x + y < z \implies sf(x+y) + x < sf(z)$
<proof>

lemma *sf-aux2*: $sf(z) \leq sf(x+y) + x \implies z \leq x+y$
<proof>

lemma *sf-aux3*: $sf(z) + m < sf(z+1) \implies m \leq z$
<proof>

lemma *sf-aux4*: $(s::nat) < t \implies (sf\ s) + s < sf\ t$
<proof>

Basic properties of *c_pair* function

lemma *sum-le-c-pair*: $x + y \leq c\text{-pair}\ x\ y$
<proof>

lemma *arg1-le-c-pair*: $x \leq c\text{-pair}\ x\ y$
<proof>

lemma *arg2-le-c-pair*: $y \leq c\text{-pair}\ x\ y$
<proof>

lemma *c-pair-sum-mono*: $(x1::nat) + y1 < x2 + y2 \implies c\text{-pair}\ x1\ y1 < c\text{-pair}\ x2\ y2$
<proof>

lemma *c-pair-sum-inj*: $c\text{-pair}\ x1\ y1 = c\text{-pair}\ x2\ y2 \implies x1 + y1 = x2 + y2$
<proof>

lemma *c-pair-inj*: $c\text{-pair } x1 \ y1 = c\text{-pair } x2 \ y2 \implies x1 = x2 \wedge y1 = y2$
(proof)

lemma *c-pair-inj1*: $c\text{-pair } x1 \ y1 = c\text{-pair } x2 \ y2 \implies x1 = x2$ (proof)

lemma *c-pair-inj2*: $c\text{-pair } x1 \ y1 = c\text{-pair } x2 \ y2 \implies y1 = y2$ (proof)

lemma *c-pair-strict-mono1*: $x1 < x2 \implies c\text{-pair } x1 \ y < c\text{-pair } x2 \ y$
(proof)

lemma *c-pair-mono1*: $x1 \leq x2 \implies c\text{-pair } x1 \ y \leq c\text{-pair } x2 \ y$
(proof)

lemma *c-pair-strict-mono2*: $y1 < y2 \implies c\text{-pair } x \ y1 < c\text{-pair } x \ y2$
(proof)

lemma *c-pair-mono2*: $y1 \leq y2 \implies c\text{-pair } x \ y1 \leq c\text{-pair } x \ y2$
(proof)

1.2 Inverse mapping

c-fst and *c-snd* are the functions which yield the inverse mapping to *c-pair*.

definition

c-sum :: $nat \Rightarrow nat$ **where**
c-sum $u = (LEAST \ z. \ u < sf \ (z+1))$

definition

c-fst :: $nat \Rightarrow nat$ **where**
c-fst $u = u - sf \ (c\text{-sum } u)$

definition

c-snd :: $nat \Rightarrow nat$ **where**
c-snd $u = c\text{-sum } u - c\text{-fst } u$

lemma *arg-less-sf-at-Suc-of-c-sum*: $u < sf \ ((c\text{-sum } u) + 1)$
(proof)

lemma *arg-less-sf-imp-c-sum-less-arg*: $u < sf(x) \implies c\text{-sum } u < x$
(proof)

lemma *sf-c-sum-le-arg*: $u \geq sf \ (c\text{-sum } u)$
(proof)

lemma *c-sum-le-arg*: $c\text{-sum } u \leq u$
(proof)

lemma *c-sum-of-c-pair [simp]*: $c\text{-sum } (c\text{-pair } x \ y) = x + y$
(proof)

lemma *c-fst-of-c-pair[simp]*: $c\text{-fst } (c\text{-pair } x \ y) = x$
<proof>

lemma *c-snd-of-c-pair[simp]*: $c\text{-snd } (c\text{-pair } x \ y) = y$
<proof>

lemma *c-pair-at-0*: $c\text{-pair } 0 \ 0 = 0$ *<proof>*

lemma *c-fst-at-0*: $c\text{-fst } 0 = 0$
<proof>

lemma *c-snd-at-0*: $c\text{-snd } 0 = 0$
<proof>

lemma *sf-c-sum-plus-c-fst*: $sf(c\text{-sum } u) + c\text{-fst } u = u$
<proof>

lemma *c-fst-le-c-sum*: $c\text{-fst } u \leq c\text{-sum } u$
<proof>

lemma *c-snd-le-c-sum*: $c\text{-snd } u \leq c\text{-sum } u$ *<proof>*

lemma *c-fst-le-arg*: $c\text{-fst } u \leq u$
<proof>

lemma *c-snd-le-arg*: $c\text{-snd } u \leq u$
<proof>

lemma *c-sum-is-sum*: $c\text{-sum } u = c\text{-fst } u + c\text{-snd } u$ *<proof>*

lemma *proj-eq-imp-arg-eq*: $\llbracket c\text{-fst } u = c\text{-fst } v; c\text{-snd } u = c\text{-snd } v \rrbracket \implies u = v$
<proof>

lemma *c-pair-of-c-fst-c-snd[simp]*: $c\text{-pair } (c\text{-fst } u) \ (c\text{-snd } u) = u$
<proof>

lemma *c-sum-eq-arg*: $c\text{-sum } x = x \implies x \leq 1$
<proof>

lemma *c-sum-eq-arg-2*: $c\text{-sum } x = x \implies c\text{-fst } x = 0$
<proof>

lemma *c-fst-eq-arg*: $c\text{-fst } x = x \implies x = 0$
<proof>

lemma *c-fst-less-arg*: $x > 0 \implies c\text{-fst } x < x$
<proof>

lemma *c-snd-eq-arg*: $c\text{-snd } x = x \implies x \leq 1$
(*proof*)

lemma *c-snd-less-arg*: $x > 1 \implies c\text{-snd } x < x$
(*proof*)

end

2 Primitive recursive functions

theory *PRecFun* **imports** *CPair*
begin

This theory contains definition of the primitive recursive functions.

2.1 Basic definitions

primrec

PrimRecOp :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat})$

where

PrimRecOp *g h* 0 *x* = *g x*

| *PrimRecOp* *g h* (*Suc y*) *x* = *h y* (*PrimRecOp* *g h y x*) *x*

primrec

PrimRecOp-last :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat})$

where

PrimRecOp-last *g h* *x* 0 = *g x*

| *PrimRecOp-last* *g h* *x* (*Suc y*) = *h x* (*PrimRecOp-last* *g h x y*) *y*

primrec

PrimRecOp1 :: $\text{nat} \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat})$

where

PrimRecOp1 *a h* 0 = *a*

| *PrimRecOp1* *a h* (*Suc y*) = *h y* (*PrimRecOp1* *a h y*)

inductive-set

PrimRec1 :: $(\text{nat} \Rightarrow \text{nat})$ set **and**

PrimRec2 :: $(\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat})$ set **and**

PrimRec3 :: $(\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat})$ set

where

zero: $(\lambda x. 0) \in \text{PrimRec1}$

| *suc*: *Suc* $\in \text{PrimRec1}$

| *id1-1*: $(\lambda x. x) \in \text{PrimRec1}$

| *id2-1*: $(\lambda x y. x) \in \text{PrimRec2}$

| *id2-2*: $(\lambda x y. y) \in \text{PrimRec2}$

| *id3-1*: $(\lambda x y z. x) \in \text{PrimRec3}$

| *id3-2*: $(\lambda x y z. y) \in \text{PrimRec3}$

| *id3-3*: $(\lambda x y z. z) \in \text{PrimRec3}$

$\mid \text{comp1-1: } \llbracket f \in \text{PrimRec1}; g \in \text{PrimRec1} \rrbracket \implies (\lambda x. f (g x)) \in \text{PrimRec1}$
 $\mid \text{comp1-2: } \llbracket f \in \text{PrimRec1}; g \in \text{PrimRec2} \rrbracket \implies (\lambda x y. f (g x y)) \in \text{PrimRec2}$
 $\mid \text{comp1-3: } \llbracket f \in \text{PrimRec1}; g \in \text{PrimRec3} \rrbracket \implies (\lambda x y z. f (g x y z)) \in \text{PrimRec3}$
 $\mid \text{comp2-1: } \llbracket f \in \text{PrimRec2}; g \in \text{PrimRec1}; h \in \text{PrimRec1} \rrbracket \implies (\lambda x. f (g x) (h x)) \in \text{PrimRec1}$
 $\mid \text{comp3-1: } \llbracket f \in \text{PrimRec3}; g \in \text{PrimRec1}; h \in \text{PrimRec1}; k \in \text{PrimRec1} \rrbracket \implies (\lambda x. f (g x) (h x) (k x)) \in \text{PrimRec1}$
 $\mid \text{comp2-2: } \llbracket f \in \text{PrimRec2}; g \in \text{PrimRec2}; h \in \text{PrimRec2} \rrbracket \implies (\lambda x y. f (g x y) (h x y)) \in \text{PrimRec2}$
 $\mid \text{comp2-3: } \llbracket f \in \text{PrimRec2}; g \in \text{PrimRec3}; h \in \text{PrimRec3} \rrbracket \implies (\lambda x y z. f (g x y z) (h x y z)) \in \text{PrimRec3}$
 $\mid \text{comp3-2: } \llbracket f \in \text{PrimRec3}; g \in \text{PrimRec2}; h \in \text{PrimRec2}; k \in \text{PrimRec2} \rrbracket \implies (\lambda x y. f (g x y) (h x y) (k x y)) \in \text{PrimRec2}$
 $\mid \text{comp3-3: } \llbracket f \in \text{PrimRec3}; g \in \text{PrimRec3}; h \in \text{PrimRec3}; k \in \text{PrimRec3} \rrbracket \implies (\lambda x y z. f (g x y z) (h x y z) (k x y z)) \in \text{PrimRec3}$
 $\mid \text{prim-rec: } \llbracket g \in \text{PrimRec1}; h \in \text{PrimRec3} \rrbracket \implies \text{PrimRecOp } g h \in \text{PrimRec2}$

lemmas *pr-zero* = *PrimRec1-PrimRec2-PrimRec3.zero*

lemmas *pr-suc* = *PrimRec1-PrimRec2-PrimRec3.suc*

lemmas *pr-id1-1* = *PrimRec1-PrimRec2-PrimRec3.id1-1*

lemmas *pr-id2-1* = *PrimRec1-PrimRec2-PrimRec3.id2-1*

lemmas *pr-id2-2* = *PrimRec1-PrimRec2-PrimRec3.id2-2*

lemmas *pr-id3-1* = *PrimRec1-PrimRec2-PrimRec3.id3-1*

lemmas *pr-id3-2* = *PrimRec1-PrimRec2-PrimRec3.id3-2*

lemmas *pr-id3-3* = *PrimRec1-PrimRec2-PrimRec3.id3-3*

lemmas *pr-comp1-1* = *PrimRec1-PrimRec2-PrimRec3.comp1-1*

lemmas *pr-comp1-2* = *PrimRec1-PrimRec2-PrimRec3.comp1-2*

lemmas *pr-comp1-3* = *PrimRec1-PrimRec2-PrimRec3.comp1-3*

lemmas *pr-comp2-1* = *PrimRec1-PrimRec2-PrimRec3.comp2-1*

lemmas *pr-comp2-2* = *PrimRec1-PrimRec2-PrimRec3.comp2-2*

lemmas *pr-comp2-3* = *PrimRec1-PrimRec2-PrimRec3.comp2-3*

lemmas *pr-comp3-1* = *PrimRec1-PrimRec2-PrimRec3.comp3-1*

lemmas *pr-comp3-2* = *PrimRec1-PrimRec2-PrimRec3.comp3-2*

lemmas *pr-comp3-3* = *PrimRec1-PrimRec2-PrimRec3.comp3-3*

lemmas *pr-rec* = *PrimRec1-PrimRec2-PrimRec3.prim-rec*

$\langle ML \rangle$

named-theorems *prec*

$\langle ML \rangle$

lemmas [*prec*] = *pr-zero pr-suc pr-id1-1 pr-id2-1 pr-id2-2 pr-id3-1 pr-id3-2 pr-id3-3*

lemma *pr-swap*: $f \in \text{PrimRec2} \implies (\lambda x y. f y x) \in \text{PrimRec2}$ $\langle \text{proof} \rangle$

theorem *pr-rec-scheme*: $\llbracket g \in \text{PrimRec1}; h \in \text{PrimRec3}; \forall x. f 0 x = g x; \forall x y. f (\text{Suc } y) x = h y (f y x) x \rrbracket \implies f \in \text{PrimRec2}$

$\langle proof \rangle$

lemma *op-plus-is-pr* [prec]: $(\lambda x y. x + y) \in PrimRec2$
 $\langle proof \rangle$

lemma *op-mult-is-pr* [prec]: $(\lambda x y. x * y) \in PrimRec2$
 $\langle proof \rangle$

lemma *const-is-pr*: $(\lambda x. (n :: nat)) \in PrimRec1$
 $\langle proof \rangle$

lemma *const-is-pr-2*: $(\lambda x y. (n :: nat)) \in PrimRec2$
 $\langle proof \rangle$

lemma *const-is-pr-3*: $(\lambda x y z. (n :: nat)) \in PrimRec3$
 $\langle proof \rangle$

theorem *pr-rec-last*: $\llbracket g \in PrimRec1; h \in PrimRec3 \rrbracket \implies PrimRecOp-last\ g\ h \in PrimRec2$
 $\langle proof \rangle$

theorem *pr-rec1*: $h \in PrimRec2 \implies PrimRecOp1\ (a :: nat)\ h \in PrimRec1$
 $\langle proof \rangle$

theorem *pr-rec1-scheme*: $\llbracket h \in PrimRec2; f\ 0 = a; \forall y. f\ (Suc\ y) = h\ y\ (f\ y) \rrbracket \implies f \in PrimRec1$
 $\langle proof \rangle$

lemma *pred-is-pr*: $(\lambda x. x - (1 :: nat)) \in PrimRec1$
 $\langle proof \rangle$

lemma *op-sub-is-pr* [prec]: $(\lambda x y. x - y) \in PrimRec2$
 $\langle proof \rangle$

lemmas [prec] =
 const-is-pr [of 0] *const-is-pr-2* [of 0] *const-is-pr-3* [of 0]
 const-is-pr [of 1] *const-is-pr-2* [of 1] *const-is-pr-3* [of 1]
 const-is-pr [of 2] *const-is-pr-2* [of 2] *const-is-pr-3* [of 2]

definition

sgn1 :: $nat \Rightarrow nat$ **where**
sgn1 $x = (case\ x\ of\ 0 \Rightarrow 0 \mid Suc\ y \Rightarrow 1)$

definition

sgn2 :: $nat \Rightarrow nat$ **where**
sgn2 $x \equiv (case\ x\ of\ 0 \Rightarrow 1 \mid Suc\ y \Rightarrow 0)$

definition

abs-of-diff :: $nat \Rightarrow nat \Rightarrow nat$ **where**

$abs\text{-of-diff} = (\lambda x y. (x - y) + (y - x))$

lemma $[simp]: sgn1\ 0 = 0$ $\langle proof \rangle$

lemma $[simp]: sgn1\ (Suc\ y) = 1$ $\langle proof \rangle$

lemma $[simp]: sgn2\ 0 = 1$ $\langle proof \rangle$

lemma $[simp]: sgn2\ (Suc\ y) = 0$ $\langle proof \rangle$

lemma $[simp]: x \neq 0 \implies sgn1\ x = 1$ $\langle proof \rangle$

lemma $[simp]: x \neq 0 \implies sgn2\ x = 0$ $\langle proof \rangle$

lemma $sgn1\text{-nz-impl-arg-pos}: sgn1\ x \neq 0 \implies x > 0$ $\langle proof \rangle$

lemma $sgn1\text{-zero-impl-arg-zero}: sgn1\ x = 0 \implies x = 0$ $\langle proof \rangle$

lemma $sgn2\text{-nz-impl-arg-zero}: sgn2\ x \neq 0 \implies x = 0$ $\langle proof \rangle$

lemma $sgn2\text{-zero-impl-arg-pos}: sgn2\ x = 0 \implies x > 0$ $\langle proof \rangle$

lemma $sgn1\text{-nz-eq-arg-pos}: (sgn1\ x \neq 0) = (x > 0)$ $\langle proof \rangle$

lemma $sgn1\text{-zero-eq-arg-zero}: (sgn1\ x = 0) = (x = 0)$ $\langle proof \rangle$

lemma $sgn2\text{-nz-eq-arg-pos}: (sgn2\ x \neq 0) = (x = 0)$ $\langle proof \rangle$

lemma $sgn2\text{-zero-eq-arg-zero}: (sgn2\ x = 0) = (x > 0)$ $\langle proof \rangle$

lemma $sgn1\text{-pos-eq-one}: sgn1\ x > 0 \implies sgn1\ x = 1$ $\langle proof \rangle$

lemma $sgn2\text{-pos-eq-one}: sgn2\ x > 0 \implies sgn2\ x = 1$ $\langle proof \rangle$

lemma $sgn2\text{-eq-1-sub-arg}: sgn2 = (\lambda x. 1 - x)$
 $\langle proof \rangle$

lemma $sgn1\text{-eq-1-sub-sgn2}: sgn1 = (\lambda x. 1 - (sgn2\ x))$
 $\langle proof \rangle$

lemma $sgn2\text{-is-pr [prec]: } sgn2 \in PrimRec1$
 $\langle proof \rangle$

lemma $sgn1\text{-is-pr [prec]: } sgn1 \in PrimRec1$
 $\langle proof \rangle$

lemma $abs\text{-of-diff-is-pr [prec]: } abs\text{-of-diff} \in PrimRec2$ $\langle proof \rangle$

lemma $abs\text{-of-diff-eq}: (abs\text{-of-diff}\ x\ y = 0) = (x = y)$ $\langle proof \rangle$

lemma $sf\text{-is-pr [prec]: } sf \in PrimRec1$
 $\langle proof \rangle$

lemma $c\text{-pair-is-pr [prec]: } c\text{-pair} \in PrimRec2$
 $\langle proof \rangle$

lemma $if\text{-is-pr}: \llbracket p \in PrimRec1; q1 \in PrimRec1; q2 \in PrimRec1 \rrbracket \implies (\lambda x. \text{if } (p\ x = 0) \text{ then } (q1\ x) \text{ else } (q2\ x)) \in PrimRec1$
 $\langle proof \rangle$

lemma $if\text{-eq-is-pr [prec]: } \llbracket p1 \in PrimRec1; p2 \in PrimRec1; q1 \in PrimRec1; q2$

$\in \text{PrimRec1}] \implies (\lambda x. \text{if } (p1\ x = p2\ x) \text{ then } (q1\ x) \text{ else } (q2\ x)) \in \text{PrimRec1}$
 ⟨proof⟩

lemma *if-is-pr2* [prec]: $\llbracket p \in \text{PrimRec2}; q1 \in \text{PrimRec2}; q2 \in \text{PrimRec2} \rrbracket \implies (\lambda x\ y. \text{if } (p\ x\ y = 0) \text{ then } (q1\ x\ y) \text{ else } (q2\ x\ y)) \in \text{PrimRec2}$
 ⟨proof⟩

lemma *if-eq-is-pr2*: $\llbracket p1 \in \text{PrimRec2}; p2 \in \text{PrimRec2}; q1 \in \text{PrimRec2}; q2 \in \text{PrimRec2} \rrbracket \implies (\lambda x\ y. \text{if } (p1\ x\ y = p2\ x\ y) \text{ then } (q1\ x\ y) \text{ else } (q2\ x\ y)) \in \text{PrimRec2}$
 ⟨proof⟩

lemma *if-is-pr3* [prec]: $\llbracket p \in \text{PrimRec3}; q1 \in \text{PrimRec3}; q2 \in \text{PrimRec3} \rrbracket \implies (\lambda x\ y\ z. \text{if } (p\ x\ y\ z = 0) \text{ then } (q1\ x\ y\ z) \text{ else } (q2\ x\ y\ z)) \in \text{PrimRec3}$
 ⟨proof⟩

lemma *if-eq-is-pr3*: $\llbracket p1 \in \text{PrimRec3}; p2 \in \text{PrimRec3}; q1 \in \text{PrimRec3}; q2 \in \text{PrimRec3} \rrbracket \implies (\lambda x\ y\ z. \text{if } (p1\ x\ y\ z = p2\ x\ y\ z) \text{ then } (q1\ x\ y\ z) \text{ else } (q2\ x\ y\ z)) \in \text{PrimRec3}$
 ⟨proof⟩

⟨ML⟩

2.2 Bounded least operator

definition

b-least :: $(\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat})$ **where**
b-least $f\ x \equiv (\text{Least } (\%y. y = x \vee (y < x \wedge (f\ x\ y) \neq 0)))$

definition

b-least2 :: $(\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat})$ **where**
b-least2 $f\ x\ y \equiv (\text{Least } (\%z. z = y \vee (z < y \wedge (f\ x\ z) \neq 0)))$

lemma *b-least-aux1*: $b\text{-least } f\ x = x \vee (b\text{-least } f\ x < x \wedge (f\ x\ (b\text{-least } f\ x)) \neq 0)$
 ⟨proof⟩

lemma *b-least-le-arg*: $b\text{-least } f\ x \leq x$
 ⟨proof⟩

lemma *less-b-least-impl-zero*: $y < b\text{-least } f\ x \implies f\ x\ y = 0$
 ⟨proof⟩

lemma *nz-impl-b-least-le*: $(f\ x\ y) \neq 0 \implies (b\text{-least } f\ x) \leq y$
 ⟨proof⟩

lemma *b-least-less-impl-nz*: $b\text{-least } f\ x < x \implies f\ x\ (b\text{-least } f\ x) \neq 0$
 ⟨proof⟩

lemma *b-least-less-impl-eq*: $b\text{-least } f\ x < x \implies (b\text{-least } f\ x) = (\text{Least } (\%y. (f\ x\ y) \neq 0))$

<proof>

lemma *less-b-least-impl-zero2*: $\llbracket y < x; b\text{-least } f x = x \rrbracket \implies f x y = 0$ *<proof>*

lemma *nz-impl-b-least-less*: $\llbracket y < x; (f x y) \neq 0 \rrbracket \implies (b\text{-least } f x) < x$
<proof>

lemma *b-least-aux2*: $\llbracket y < x; (f x y) \neq 0 \rrbracket \implies (b\text{-least } f x) = (\text{Least } (\%y. (f x y) \neq 0))$
<proof>

lemma *b-least2-aux1*: $b\text{-least2 } f x y = y \vee (b\text{-least2 } f x y < y \wedge (f x (b\text{-least2 } f x y)) \neq 0)$
<proof>

lemma *b-least2-le-arg*: $b\text{-least2 } f x y \leq y$
<proof>

lemma *less-b-least2-impl-zero*: $z < b\text{-least2 } f x y \implies f x z = 0$
<proof>

lemma *nz-impl-b-least2-le*: $(f x z) \neq 0 \implies (b\text{-least2 } f x y) \leq z$
<proof>

lemma *b-least2-less-impl-nz*: $b\text{-least2 } f x y < y \implies f x (b\text{-least2 } f x y) \neq 0$
<proof>

lemma *b-least2-less-impl-eq*: $b\text{-least2 } f x y < y \implies (b\text{-least2 } f x y) = (\text{Least } (\%z. (f x z) \neq 0))$
<proof>

lemma *less-b-least2-impl-zero2*: $\llbracket z < y; b\text{-least2 } f x y = y \rrbracket \implies f x z = 0$
<proof>

lemma *nz-b-least2-impl-less*: $\llbracket z < y; (f x z) \neq 0 \rrbracket \implies (b\text{-least2 } f x y) < y$
<proof>

lemma *b-least2-less-impl-eq2*: $\llbracket z < y; (f x z) \neq 0 \rrbracket \implies (b\text{-least2 } f x y) = (\text{Least } (\%z. (f x z) \neq 0))$
<proof>

lemma *b-least2-aux2*: $b\text{-least2 } f x y < y \implies b\text{-least2 } f x (\text{Suc } y) = b\text{-least2 } f x y$
<proof>

lemma *b-least2-aux3*: $\llbracket b\text{-least2 } f x y = y; f x y \neq 0 \rrbracket \implies b\text{-least2 } f x (\text{Suc } y) = y$
<proof>

lemma *b-least2-mono*: $y1 \leq y2 \implies b\text{-least2 } f x y1 \leq b\text{-least2 } f x y2$
<proof>

lemma *b-least2-aux4*: $\llbracket b\text{-least2 } f x y = y; f x y = 0 \rrbracket \implies b\text{-least2 } f x (\text{Suc } y) = \text{Suc } y$
 $\langle \text{proof} \rangle$

lemma *b-least2-at-zero*: $b\text{-least2 } f x 0 = 0$
 $\langle \text{proof} \rangle$

theorem *pr-b-least2*: $f \in \text{PrimRec2} \implies b\text{-least2 } f \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

lemma *b-least-def1*: $b\text{-least } f = (\lambda x. b\text{-least2 } f x x)$ $\langle \text{proof} \rangle$

theorem *pr-b-least*: $f \in \text{PrimRec2} \implies b\text{-least } f \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

2.3 Examples

theorem *c-sum-as-b-least*: $c\text{-sum} = (\lambda u. b\text{-least2 } (\lambda u z. (\text{sgn1 } (\text{sf}(z+1) - u))) u (\text{Suc } u))$
 $\langle \text{proof} \rangle$

theorem *c-sum-is-pr*: $c\text{-sum} \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

theorem *c-fst-is-pr* [*prec*]: $c\text{-fst} \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

theorem *c-snd-is-pr* [*prec*]: $c\text{-snd} \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

theorem *pr-1-to-2*: $f \in \text{PrimRec1} \implies (\lambda x y. f (c\text{-pair } x y)) \in \text{PrimRec2}$ $\langle \text{proof} \rangle$

theorem *pr-2-to-1*: $f \in \text{PrimRec2} \implies (\lambda z. f (c\text{-fst } z) (c\text{-snd } z)) \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

definition *pr-conv-1-to-2* = $(\lambda f x y. f (c\text{-pair } x y))$

definition *pr-conv-1-to-3* = $(\lambda f x y z. f (c\text{-pair } (c\text{-pair } x y) z))$

definition *pr-conv-2-to-1* = $(\lambda f x. f (c\text{-fst } x) (c\text{-snd } x))$

definition *pr-conv-3-to-1* = $(\lambda f x. f (c\text{-fst } (c\text{-fst } x)) (c\text{-snd } (c\text{-fst } x)) (c\text{-snd } x))$

definition *pr-conv-3-to-2* = $(\lambda f. \text{pr-conv-1-to-2 } (\text{pr-conv-3-to-1 } f))$

definition *pr-conv-2-to-3* = $(\lambda f. \text{pr-conv-1-to-3 } (\text{pr-conv-2-to-1 } f))$

lemma [*simp*]: $\text{pr-conv-1-to-2 } (\text{pr-conv-2-to-1 } f) = f$ $\langle \text{proof} \rangle$

lemma [*simp*]: $\text{pr-conv-2-to-1 } (\text{pr-conv-1-to-2 } f) = f$ $\langle \text{proof} \rangle$

lemma [*simp*]: $\text{pr-conv-1-to-3 } (\text{pr-conv-3-to-1 } f) = f$ $\langle \text{proof} \rangle$

lemma [*simp*]: $\text{pr-conv-3-to-1 } (\text{pr-conv-1-to-3 } f) = f$ $\langle \text{proof} \rangle$

lemma [*simp*]: $\text{pr-conv-3-to-2 } (\text{pr-conv-2-to-3 } f) = f$ $\langle \text{proof} \rangle$

lemma [*simp*]: $\text{pr-conv-2-to-3 } (\text{pr-conv-3-to-2 } f) = f$ $\langle \text{proof} \rangle$

lemma *pr-conv-1-to-2-lm*: $f \in \text{PrimRec1} \implies \text{pr-conv-1-to-2 } f \in \text{PrimRec2}$ *<proof>*
lemma *pr-conv-1-to-3-lm*: $f \in \text{PrimRec1} \implies \text{pr-conv-1-to-3 } f \in \text{PrimRec3}$ *<proof>*
lemma *pr-conv-2-to-1-lm*: $f \in \text{PrimRec2} \implies \text{pr-conv-2-to-1 } f \in \text{PrimRec1}$ *<proof>*
lemma *pr-conv-3-to-1-lm*: $f \in \text{PrimRec3} \implies \text{pr-conv-3-to-1 } f \in \text{PrimRec1}$ *<proof>*
lemma *pr-conv-3-to-2-lm*: $f \in \text{PrimRec3} \implies \text{pr-conv-3-to-2 } f \in \text{PrimRec2}$
<proof>
lemma *pr-conv-2-to-3-lm*: $f \in \text{PrimRec2} \implies \text{pr-conv-2-to-3 } f \in \text{PrimRec3}$
<proof>

theorem *b-least2-scheme*: $\llbracket f \in \text{PrimRec2}; g \in \text{PrimRec1}; \forall x. h\ x\ x < g\ x; \forall x. f\ x\ (h\ x) \neq 0; \forall z\ x. z < h\ x \longrightarrow f\ x\ z = 0 \rrbracket \implies$
 $h \in \text{PrimRec1}$
<proof>

theorem *b-least2-scheme2*: $\llbracket f \in \text{PrimRec3}; g \in \text{PrimRec2}; \forall x\ y. h\ x\ y < g\ x\ y; \forall x\ y. f\ x\ y\ (h\ x\ y) \neq 0; \forall z\ x\ y. z < h\ x\ y \longrightarrow f\ x\ y\ z = 0 \rrbracket \implies$
 $h \in \text{PrimRec2}$
<proof>

theorem *div-is-pr*: $(\lambda a\ b. a\ \text{div}\ b) \in \text{PrimRec2}$
<proof>

theorem *mod-is-pr*: $(\lambda a\ b. a\ \text{mod}\ b) \in \text{PrimRec2}$
<proof>

theorem *pr-rec-last-scheme*: $\llbracket g \in \text{PrimRec1}; h \in \text{PrimRec3}; \forall x. f\ x\ 0 = g\ x; \forall x\ y. f\ x\ (\text{Suc}\ y) = h\ x\ (f\ x\ y)\ y \rrbracket \implies f \in \text{PrimRec2}$
<proof>

theorem *power-is-pr*: $(\lambda (x::\text{nat}) (n::\text{nat}). x \wedge n) \in \text{PrimRec2}$
<proof>

end

3 Primitive recursive coding of lists of natural numbers

theory *PRecList*
imports *PRecFun*
begin

We introduce a particular coding *list-to-nat* from lists of natural numbers to natural numbers.

definition

$c\text{-len} :: \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-len} = (\lambda (u::\text{nat}). (\text{sgn1}\ u) * (c\text{-fst}(u - (1::\text{nat})) + 1))$

lemma *c-len-1*: $c\text{-len } u = (\text{case } u \text{ of } 0 \Rightarrow 0 \mid \text{Suc } v \Rightarrow c\text{-fst}(v)+1)$ $\langle \text{proof} \rangle$

lemma *c-len-is-pr*: $c\text{-len} \in \text{PrimRec1}$ $\langle \text{proof} \rangle$

lemma [*simp*]: $c\text{-len } 0 = 0$ $\langle \text{proof} \rangle$

lemma *c-len-2*: $u \neq 0 \Longrightarrow c\text{-len } u = c\text{-fst}(u-(1::\text{nat}))+1$ $\langle \text{proof} \rangle$

lemma *c-len-3*: $u > 0 \Longrightarrow c\text{-len } u > 0$ $\langle \text{proof} \rangle$

lemma *c-len-4*: $c\text{-len } u = 0 \Longrightarrow u = 0$
 $\langle \text{proof} \rangle$

lemma *c-len-5*: $c\text{-len } u > 0 \Longrightarrow u > 0$
 $\langle \text{proof} \rangle$

fun *c-fold* :: $\text{nat list} \Rightarrow \text{nat}$ **where**
 c-fold [] = 0
 | *c-fold* [x] = x
 | *c-fold* (x#ls) = *c-pair* x (*c-fold* ls)

lemma *c-fold-0*: $ls \neq [] \Longrightarrow c\text{-fold } (x\#ls) = c\text{-pair } x (c\text{-fold } ls)$
 $\langle \text{proof} \rangle$

primrec

c-unfold :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat list}$

where

c-unfold 0 u = []
 | *c-unfold* (Suc k) u = (if k = 0 then [u] else ((c-fst u) # (c-unfold k (c-snd u))))

lemma *c-fold-1*: $c\text{-unfold } 1 (c\text{-fold } [x]) = [x]$ $\langle \text{proof} \rangle$

lemma *c-fold-2*: $c\text{-fold } (c\text{-unfold } 1 u) = u$ $\langle \text{proof} \rangle$

lemma *c-unfold-1*: $c\text{-unfold } 1 u = [u]$ $\langle \text{proof} \rangle$

lemma *c-unfold-2*: $c\text{-unfold } (\text{Suc } 1) u = (c\text{-fst } u) \# (c\text{-unfold } 1 (c\text{-snd } u))$ $\langle \text{proof} \rangle$

lemma *c-unfold-3*: $c\text{-unfold } (\text{Suc } 1) u = [c\text{-fst } u, c\text{-snd } u]$ $\langle \text{proof} \rangle$

lemma *c-unfold-4*: $k > 0 \Longrightarrow c\text{-unfold } (\text{Suc } k) u = (c\text{-fst } u) \# (c\text{-unfold } k (c\text{-snd } u))$ $\langle \text{proof} \rangle$

lemma *c-unfold-4-1*: $k > 0 \Longrightarrow c\text{-unfold } (\text{Suc } k) u \neq []$ $\langle \text{proof} \rangle$

lemma *two*: $(2::\text{nat}) = \text{Suc } 1$ $\langle \text{proof} \rangle$

lemma *c-unfold-5*: $c\text{-unfold } 2 u = [c\text{-fst } u, c\text{-snd } u]$ $\langle \text{proof} \rangle$

lemma *c-unfold-6*: $k > 0 \implies c\text{-unfold } k \ u \neq []$
 ⟨proof⟩

lemma *th-lm-1*: $k = 1 \implies (\forall u. c\text{-fold } (c\text{-unfold } k \ u) = u)$ ⟨proof⟩

lemma *th-lm-2*: $\llbracket k > 0; (\forall u. c\text{-fold } (c\text{-unfold } k \ u) = u) \rrbracket \implies (\forall u. c\text{-fold } (c\text{-unfold } (Suc \ k) \ u) = u)$
 ⟨proof⟩

lemma *th-lm-3*: $(\forall u. c\text{-fold } (c\text{-unfold } (Suc \ k) \ u) = u) \implies (\forall u. c\text{-fold } (c\text{-unfold } (Suc \ (Suc \ k)) \ u) = u)$
 ⟨proof⟩

theorem *th-1*: $\forall u. c\text{-fold } (c\text{-unfold } (Suc \ k) \ u) = u$
 ⟨proof⟩

theorem *th-2*: $k > 0 \implies (\forall u. c\text{-fold } (c\text{-unfold } k \ u) = u)$
 ⟨proof⟩

lemma *c-fold-3*: $c\text{-unfold } 2 \ (c\text{-fold } [x, y]) = [x, y]$ ⟨proof⟩

theorem *c-unfold-len*: $ALL \ u. \ length \ (c\text{-unfold } k \ u) = k$
 ⟨proof⟩

lemma *th-3-lm-0*: $\llbracket c\text{-unfold } (length \ ls) \ (c\text{-fold } ls) = ls; \ ls = a \ \# \ ls1; \ ls1 = aa \ \# \ list \rrbracket \implies c\text{-unfold } (length \ (x \ \# \ ls)) \ (c\text{-fold } (x \ \# \ ls)) = x \ \# \ ls$
 ⟨proof⟩

lemma *th-3-lm-1*: $\llbracket c\text{-unfold } (length \ ls) \ (c\text{-fold } ls) = ls; \ ls = a \ \# \ ls1 \rrbracket \implies c\text{-unfold } (length \ (x \ \# \ ls)) \ (c\text{-fold } (x \ \# \ ls)) = x \ \# \ ls$
 ⟨proof⟩

lemma *th-3-lm-2*: $c\text{-unfold } (length \ ls) \ (c\text{-fold } ls) = ls \implies c\text{-unfold } (length \ (x \ \# \ ls)) \ (c\text{-fold } (x \ \# \ ls)) = x \ \# \ ls$
 ⟨proof⟩

theorem *th-3*: $c\text{-unfold } (length \ ls) \ (c\text{-fold } ls) = ls$
 ⟨proof⟩

definition

list-to-nat :: $nat \ list \Rightarrow nat$ **where**
list-to-nat = $(\lambda \ ls. \ if \ ls = [] \ then \ 0 \ else \ (c\text{-pair } ((length \ ls) - 1) \ (c\text{-fold } ls)) + 1)$

definition

nat-to-list :: $nat \Rightarrow nat \ list$ **where**
nat-to-list = $(\lambda \ u. \ if \ u = 0 \ then \ [] \ else \ (c\text{-unfold } (c\text{-len } u) \ (c\text{-snd } (u - (1 :: nat))))))$

lemma *nat-to-list-of-pos*: $u > 0 \implies nat\text{-to-list } u = c\text{-unfold } (c\text{-len } u) \ (c\text{-snd } (u - (1 :: nat)))$

$\langle proof \rangle$

theorem *list-to-nat-th* [simp]: $list\text{-to-nat} (nat\text{-to-list } u) = u$
 $\langle proof \rangle$

theorem *nat-to-list-th* [simp]: $nat\text{-to-list} (list\text{-to-nat } ls) = ls$
 $\langle proof \rangle$

lemma [simp]: $list\text{-to-nat } [] = 0$ $\langle proof \rangle$

lemma [simp]: $nat\text{-to-list } 0 = []$ $\langle proof \rangle$

theorem *c-len-th-1*: $c\text{-len} (list\text{-to-nat } ls) = length\ ls$
 $\langle proof \rangle$

theorem $length (nat\text{-to-list } u) = c\text{-len } u$
 $\langle proof \rangle$

definition

$c\text{-hd} :: nat \Rightarrow nat$ **where**
 $c\text{-hd} = (\lambda u. \text{if } u=0 \text{ then } 0 \text{ else } hd (nat\text{-to-list } u))$

definition

$c\text{-tl} :: nat \Rightarrow nat$ **where**
 $c\text{-tl} = (\lambda u. list\text{-to-nat} (tl (nat\text{-to-list } u)))$

definition

$c\text{-cons} :: nat \Rightarrow nat \Rightarrow nat$ **where**
 $c\text{-cons} = (\lambda x u. list\text{-to-nat} (x \# (nat\text{-to-list } u)))$

lemma [simp]: $c\text{-hd } 0 = 0$ $\langle proof \rangle$

lemma *c-hd-aux0*: $c\text{-len } u = 1 \implies nat\text{-to-list } u = [c\text{-snd } (u-(1::nat))]$ $\langle proof \rangle$

lemma *c-hd-aux1*: $c\text{-len } u = 1 \implies c\text{-hd } u = c\text{-snd } (u-(1::nat))$
 $\langle proof \rangle$

lemma *c-hd-aux2*: $c\text{-len } u > 1 \implies c\text{-hd } u = c\text{-fst } (c\text{-snd } (u-(1::nat)))$
 $\langle proof \rangle$

lemma *c-hd-aux3*: $u > 0 \implies c\text{-hd } u = (\text{if } (c\text{-len } u) = 1 \text{ then } c\text{-snd } (u-(1::nat))$
 $\text{else } c\text{-fst } (c\text{-snd } (u-(1::nat))))$
 $\langle proof \rangle$

lemma *c-hd-aux4*: $c\text{-hd } u = (\text{if } u=0 \text{ then } 0 \text{ else } (\text{if } (c\text{-len } u) = 1 \text{ then } c\text{-snd}$
 $(u-(1::nat)) \text{ else } c\text{-fst } (c\text{-snd } (u-(1::nat))))$
 $\langle proof \rangle$

lemma *c-hd-is-pr*: $c\text{-hd} \in \text{PrimRec1}$
<proof>

lemma [*simp*]: $c\text{-tl } 0 = 0$ *<proof>*

lemma *c-tl-eq-tl*: $c\text{-tl } (\text{list-to-nat } ls) = \text{list-to-nat } (\text{tl } ls)$ *<proof>*

lemma *tl-eq-c-tl*: $\text{tl } (\text{nat-to-list } x) = \text{nat-to-list } (c\text{-tl } x)$ *<proof>*

lemma *c-tl-aux1*: $c\text{-len } u = 1 \implies c\text{-tl } u = 0$ *<proof>*

lemma *c-tl-aux2*: $c\text{-len } u > 1 \implies c\text{-tl } u = (c\text{-pair } (c\text{-len } u - (2::\text{nat})) (c\text{-snd } (c\text{-snd } (u-(1::\text{nat})))))) + 1$
<proof>

lemma *c-tl-aux3*: $c\text{-tl } u = (\text{sgn1 } ((c\text{-len } u) - 1)) * ((c\text{-pair } (c\text{-len } u - (2::\text{nat})) (c\text{-snd } (c\text{-snd } (u-(1::\text{nat})))))) + 1)$ (**is** $- = ?R$)
<proof>

lemma *c-tl-less*: $u > 0 \implies c\text{-tl } u < u$
<proof>

lemma *c-tl-le*: $c\text{-tl } u \leq u$
<proof>

theorem *c-tl-is-pr*: $c\text{-tl} \in \text{PrimRec1}$
<proof>

lemma *c-cons-aux1*: $c\text{-cons } x \ 0 = (c\text{-pair } 0 \ x) + 1$
<proof>

lemma *c-cons-aux2*: $u > 0 \implies c\text{-cons } x \ u = (c\text{-pair } (c\text{-len } u) (c\text{-pair } x (c\text{-snd } (u-(1::\text{nat})))))) + 1$
<proof>

lemma *c-cons-aux3*: $c\text{-cons} = (\lambda \ x \ u. (\text{sgn2 } u) * ((c\text{-pair } 0 \ x) + 1) + (\text{sgn1 } u) * ((c\text{-pair } (c\text{-len } u) (c\text{-pair } x (c\text{-snd } (u-(1::\text{nat})))))) + 1)$
<proof>

lemma *c-cons-pos*: $c\text{-cons } x \ u > 0$
<proof>

theorem *c-cons-is-pr*: $c\text{-cons} \in \text{PrimRec2}$
<proof>

definition

$c\text{-drop} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-drop} = \text{PrimRecOp } (\lambda \ x. \ x) (\lambda \ x \ y \ z. \ c\text{-tl } y)$

lemma *c-drop-at-0* [simp]: $c\text{-drop } 0\ x = x$ *<proof>*

lemma *c-drop-at-Suc*: $c\text{-drop } (\text{Suc } y)\ x = c\text{-tl } (c\text{-drop } y\ x)$ *<proof>*

theorem *c-drop-is-pr*: $c\text{-drop} \in \text{PrimRec2}$
<proof>

lemma *c-tl-c-drop*: $c\text{-tl } (c\text{-drop } y\ x) = c\text{-drop } y\ (c\text{-tl } x)$
<proof>

lemma *c-drop-at-Suc1*: $c\text{-drop } (\text{Suc } y)\ x = c\text{-drop } y\ (c\text{-tl } x)$
<proof>

lemma *c-drop-df*: $\forall\ ls.\ \text{drop } n\ ls = \text{nat-to-list } (c\text{-drop } n\ (\text{list-to-nat } ls))$
<proof>

definition

$c\text{-nth} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-nth} = (\lambda\ x\ n.\ c\text{-hd } (c\text{-drop } n\ x))$

lemma *c-nth-is-pr*: $c\text{-nth} \in \text{PrimRec2}$
<proof>

lemma *c-nth-at-0*: $c\text{-nth } x\ 0 = c\text{-hd } x$ *<proof>*

lemma *c-hd-c-cons* [simp]: $c\text{-hd } (c\text{-cons } x\ y) = x$
<proof>

lemma *c-tl-c-cons* [simp]: $c\text{-tl } (c\text{-cons } x\ y) = y$ *<proof>*

definition

$c\text{-f-list} :: (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-f-list} = (\lambda\ f.\$
let $g = (\%x.\ c\text{-cons } (f\ 0\ x)\ 0)$; $h = (\%a\ b\ c.\ c\text{-cons } (f\ (\text{Suc } a)\ c)\ b)$ in $\text{PrimRecOp } g\ h$)

lemma *c-f-list-at-0*: $c\text{-f-list } f\ 0\ x = c\text{-cons } (f\ 0\ x)\ 0$ *<proof>*

lemma *c-f-list-at-Suc*: $c\text{-f-list } f\ (\text{Suc } y)\ x = c\text{-cons } (f\ (\text{Suc } y)\ x)\ (c\text{-f-list } f\ y\ x)$
<proof>

lemma *c-f-list-is-pr*: $f \in \text{PrimRec2} \implies c\text{-f-list } f \in \text{PrimRec2}$
<proof>

lemma *c-f-list-to-f-0*: $f\ y\ x = c\text{-hd } (c\text{-f-list } f\ y\ x)$
<proof>

lemma *c-f-list-to-f*: $f = (\lambda\ y\ x.\ c\text{-hd } (c\text{-f-list } f\ y\ x))$
<proof>

lemma *c-f-list-f-is-pr*: $c\text{-f-list } f \in \text{PrimRec2} \implies f \in \text{PrimRec2}$
 ⟨proof⟩

lemma *c-f-list-lm-1*: $c\text{-nth } (c\text{-cons } x \ y) \ (Suc \ z) = c\text{-nth } y \ z$ ⟨proof⟩

lemma *c-f-list-lm-2*: $z < Suc \ n \implies c\text{-nth } (c\text{-f-list } f \ (Suc \ n) \ x) \ (Suc \ n - z) =$
 $c\text{-nth } (c\text{-f-list } f \ n \ x) \ (n - z)$
 ⟨proof⟩

lemma *c-f-list-nth*: $z \leq y \longrightarrow c\text{-nth } (c\text{-f-list } f \ y \ x) \ (y - z) = f \ z \ x$
 ⟨proof⟩

theorem *th-pr-rec*: $\llbracket g \in \text{PrimRec1}; h \in \text{PrimRec3}; (\forall x. (f \ 0 \ x) = (g \ x)); (\forall x \ y. (f \ (Suc \ y) \ x) = h \ y \ (f \ y \ x) \ x) \rrbracket \implies f \in \text{PrimRec2}$
 ⟨proof⟩

theorem *th-rec*: $\llbracket g \in \text{PrimRec1}; \alpha \in \text{PrimRec2}; h \in \text{PrimRec3}; (\forall x \ y. \alpha \ y \ x \leq$
 $y); (\forall x. (f \ 0 \ x) = (g \ x)); (\forall x \ y. (f \ (Suc \ y) \ x) = h \ y \ (f \ (\alpha \ y \ x) \ x) \ x) \rrbracket \implies f \in$
 PrimRec2
 ⟨proof⟩

declare *c-tl-less* [termination-simp]

fun *c-assoc-have-key* :: $nat \Rightarrow nat \Rightarrow nat$ **where**
c-assoc-have-key-df [simp del]: $c\text{-assoc-have-key } y \ x = (\text{if } y = 0 \ \text{then } 1 \ \text{else}$
 $(\text{if } c\text{-fst } (c\text{-hd } y) = x \ \text{then } 0 \ \text{else } c\text{-assoc-have-key } (c\text{-tl } y) \ x)$

lemma *c-assoc-have-key-lm-1*: $y \neq 0 \implies c\text{-assoc-have-key } y \ x = (\text{if } c\text{-fst } (c\text{-hd } y)$
 $= x \ \text{then } 0 \ \text{else } c\text{-assoc-have-key } (c\text{-tl } y) \ x)$ ⟨proof⟩

theorem *c-assoc-have-key-is-pr*: $c\text{-assoc-have-key} \in \text{PrimRec2}$
 ⟨proof⟩

fun *c-assoc-value* :: $nat \Rightarrow nat \Rightarrow nat$ **where**
c-assoc-value-df [simp del]: $c\text{-assoc-value } y \ x = (\text{if } y = 0 \ \text{then } 0 \ \text{else}$
 $(\text{if } c\text{-fst } (c\text{-hd } y) = x \ \text{then } c\text{-snd } (c\text{-hd } y) \ \text{else } c\text{-assoc-value } (c\text{-tl } y) \ x)$

lemma *c-assoc-value-lm-1*: $y \neq 0 \implies c\text{-assoc-value } y \ x = (\text{if } c\text{-fst } (c\text{-hd } y) = x$
 $\text{then } c\text{-snd } (c\text{-hd } y) \ \text{else } c\text{-assoc-value } (c\text{-tl } y) \ x)$ ⟨proof⟩

theorem *c-assoc-value-is-pr*: $c\text{-assoc-value} \in \text{PrimRec2}$
 ⟨proof⟩

lemma *c-assoc-lm-1*: $c\text{-assoc-have-key } (c\text{-cons } (c\text{-pair } x \ y) \ z) \ x = 0$
 ⟨proof⟩

lemma *c-assoc-lm-2*: $c\text{-assoc-value } (c\text{-cons } (c\text{-pair } x \ y) \ z) \ x = y$
 ⟨proof⟩

lemma *c-assoc-lm-3*: $x1 \neq x \implies c\text{-assoc-have-key } (c\text{-cons } (c\text{-pair } x y) z) x1 = c\text{-assoc-have-key } z x1$
 ⟨proof⟩

lemma *c-assoc-lm-4*: $x1 \neq x \implies c\text{-assoc-value } (c\text{-cons } (c\text{-pair } x y) z) x1 = c\text{-assoc-value } z x1$
 ⟨proof⟩

end

4 Primitive recursive functions of one variable

theory *PRecFun2*
imports *PRecFun*
begin

4.1 Alternative definition of primitive recursive functions of one variable

definition

UnaryRecOp :: $(nat \Rightarrow nat) \Rightarrow (nat \Rightarrow nat) \Rightarrow (nat \Rightarrow nat)$ **where**
UnaryRecOp = $(\lambda g h. pr\text{-conv-2-to-1 } (PrimRecOp g (pr\text{-conv-1-to-3 } h)))$

lemma *unary-rec-into-pr*: $\llbracket g \in PrimRec1; h \in PrimRec1 \rrbracket \implies UnaryRecOp g h \in PrimRec1$ ⟨proof⟩

definition

c-f-pair :: $(nat \Rightarrow nat) \Rightarrow (nat \Rightarrow nat) \Rightarrow (nat \Rightarrow nat)$ **where**
c-f-pair = $(\lambda f g x. c\text{-pair } (f x) (g x))$

lemma *c-f-pair-to-pr*: $\llbracket f \in PrimRec1; g \in PrimRec1 \rrbracket \implies c\text{-f-pair } f g \in PrimRec1$
 ⟨proof⟩

inductive-set *PrimRec1'* :: $(nat \Rightarrow nat)$ *set*

where

zero: $(\lambda x. 0) \in PrimRec1'$
 | *suc*: $Suc \in PrimRec1'$
 | *fst*: $c\text{-fst} \in PrimRec1'$
 | *snd*: $c\text{-snd} \in PrimRec1'$
 | *comp*: $\llbracket f \in PrimRec1'; g \in PrimRec1' \rrbracket \implies (\lambda x. f (g x)) \in PrimRec1'$
 | *pair*: $\llbracket f \in PrimRec1'; g \in PrimRec1' \rrbracket \implies c\text{-f-pair } f g \in PrimRec1'$
 | *un-rec*: $\llbracket f \in PrimRec1'; g \in PrimRec1' \rrbracket \implies UnaryRecOp f g \in PrimRec1'$

lemma *primrec'-into-primrec*: $f \in PrimRec1' \implies f \in PrimRec1$
 ⟨proof⟩

lemma *pr-id1-1'*: $(\lambda x. x) \in PrimRec1'$
 ⟨proof⟩

lemma *pr-id2-1'*: *pr-conv-2-to-1* $(\lambda x y. x) \in \text{PrimRec1}'$ $\langle \text{proof} \rangle$

lemma *pr-id2-2'*: *pr-conv-2-to-1* $(\lambda x y. y) \in \text{PrimRec1}'$ $\langle \text{proof} \rangle$

lemma *pr-id3-1'*: *pr-conv-3-to-1* $(\lambda x y z. x) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-id3-2'*: *pr-conv-3-to-1* $(\lambda x y z. y) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-id3-3'*: *pr-conv-3-to-1* $(\lambda x y z. z) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp2-1'*: $\llbracket \text{pr-conv-2-to-1 } f \in \text{PrimRec1}'; g \in \text{PrimRec1}'; h \in \text{PrimRec1}' \rrbracket \implies (\lambda x. f (g x) (h x)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp3-1'*: $\llbracket \text{pr-conv-3-to-1 } f \in \text{PrimRec1}'; g \in \text{PrimRec1}'; h \in \text{PrimRec1}'; k \in \text{PrimRec1}' \rrbracket \implies (\lambda x. f (g x) (h x) (k x)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp1-2'*: $\llbracket f \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } g \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-2-to-1 } (\lambda x y. f (g x y)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp1-3'*: $\llbracket f \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } g \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-3-to-1 } (\lambda x y z. f (g x y z)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp2-2'*: $\llbracket \text{pr-conv-2-to-1 } f \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } g \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } h \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-2-to-1 } (\lambda x y. f (g x y) (h x y)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp2-3'*: $\llbracket \text{pr-conv-2-to-1 } f \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } g \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } h \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-3-to-1 } (\lambda x y z. f (g x y z) (h x y z)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp3-2'*: $\llbracket \text{pr-conv-3-to-1 } f \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } g \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } h \in \text{PrimRec1}'; \text{pr-conv-2-to-1 } k \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-2-to-1 } (\lambda x y. f (g x y) (h x y) (k x y)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *pr-comp3-3'*: $\llbracket \text{pr-conv-3-to-1 } f \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } g \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } h \in \text{PrimRec1}'; \text{pr-conv-3-to-1 } k \in \text{PrimRec1}' \rrbracket \implies \text{pr-conv-3-to-1 } (\lambda x y z. f (g x y z) (h x y z) (k x y z)) \in \text{PrimRec1}'$
 $\langle \text{proof} \rangle$

lemma *lm'*: $(f1 \in \text{PrimRec1} \longrightarrow f1 \in \text{PrimRec1}') \wedge (g1 \in \text{PrimRec2} \longrightarrow \text{pr-conv-2-to-1 } g1 \in \text{PrimRec1}') \wedge (h1 \in \text{PrimRec3} \longrightarrow \text{pr-conv-3-to-1 } h1 \in \text{PrimRec1}')$
 ⟨proof⟩

theorem *pr-1-eq-1'*: $\text{PrimRec1} = \text{PrimRec1}'$
 ⟨proof⟩

4.2 The scheme datatype

datatype *PrimScheme* = *Base-zero* | *Base-suc* | *Base-fst* | *Base-snd*
 | *Comp-op PrimScheme PrimScheme*
 | *Pair-op PrimScheme PrimScheme*
 | *Rec-op PrimScheme PrimScheme*

primrec

sch-to-pr :: *PrimScheme* \Rightarrow (*nat* \Rightarrow *nat*)

where

sch-to-pr Base-zero = $(\lambda x. 0)$
 | *sch-to-pr Base-suc* = *Suc*
 | *sch-to-pr Base-fst* = *c-fst*
 | *sch-to-pr Base-snd* = *c-snd*
 | *sch-to-pr (Comp-op t1 t2)* = $(\lambda x. (\text{sch-to-pr } t1) ((\text{sch-to-pr } t2) x))$
 | *sch-to-pr (Pair-op t1 t2)* = *c-f-pair (sch-to-pr t1) (sch-to-pr t2)*
 | *sch-to-pr (Rec-op t1 t2)* = *UnaryRecOp (sch-to-pr t1) (sch-to-pr t2)*

lemma *sch-to-pr-into-pr*: $\text{sch-to-pr } sch \in \text{PrimRec1}$ ⟨proof⟩

lemma *sch-to-pr-srj*: $f \in \text{PrimRec1} \Longrightarrow (\exists sch. f = \text{sch-to-pr } sch)$
 ⟨proof⟩

definition

loc-f :: *nat* \Rightarrow *PrimScheme* \Rightarrow *PrimScheme* \Rightarrow *PrimScheme* **where**
loc-f n sch1 sch2 =

(*if n=0 then Base-zero else*
if n=1 then Base-suc else
if n=2 then Base-fst else
if n=3 then Base-snd else
if n=4 then (Comp-op sch1 sch2) else
if n=5 then (Pair-op sch1 sch2) else
if n=6 then (Rec-op sch1 sch2) else
Base-zero)

definition

mod7 :: *nat* \Rightarrow *nat* **where**
mod7 = $(\lambda x. x \text{ mod } 7)$

lemma *c-snd-snd-lt [termination-simp]*: $c\text{-snd } (c\text{-snd } (\text{Suc } (\text{Suc } x))) < \text{Suc } (\text{Suc } x)$

<proof>

lemma *c-fst-snd-lt* [*termination-simp*]: $c\text{-fst } (c\text{-snd } (Suc (Suc x))) < Suc (Suc x)$
<proof>

fun *nat-to-sch* :: $nat \Rightarrow PrimScheme$ **where**
 nat-to-sch 0 = *Base-zero*
 | *nat-to-sch* (Suc 0) = *Base-zero*
 | *nat-to-sch* x = (let u=mod7 (c-fst x); v=c-snd x; v1=c-fst v; v2 = c-snd v;
 sch1=nat-to-sch v1; sch2=nat-to-sch v2 in loc-f u sch1 sch2)

primrec *sch-to-nat* :: $PrimScheme \Rightarrow nat$ **where**
 sch-to-nat *Base-zero* = 0
 | *sch-to-nat* *Base-suc* = *c-pair* 1 0
 | *sch-to-nat* *Base-fst* = *c-pair* 2 0
 | *sch-to-nat* *Base-snd* = *c-pair* 3 0
 | *sch-to-nat* (*Comp-op* t1 t2) = *c-pair* 4 (c-pair (sch-to-nat t1) (sch-to-nat t2))
 | *sch-to-nat* (*Pair-op* t1 t2) = *c-pair* 5 (c-pair (sch-to-nat t1) (sch-to-nat t2))
 | *sch-to-nat* (*Rec-op* t1 t2) = *c-pair* 6 (c-pair (sch-to-nat t1) (sch-to-nat t2))

lemma *loc-srj-lm-1*: $nat\text{-to-sch } (Suc (Suc x)) = (let u=mod7 (c-fst (Suc (Suc x)));$
 $v=c-snd (Suc (Suc x)); v1=c-fst v; v2 = c-snd v; sch1=nat\text{-to-sch } v1; sch2=nat\text{-to-sch } v2$
 $in loc-f u sch1 sch2)$ *<proof>*

lemma *loc-srj-lm-2*: $x > 1 \implies nat\text{-to-sch } x = (let u=mod7 (c-fst x); v=c-snd x;$
 $v1=c-fst v; v2 = c-snd v; sch1=nat\text{-to-sch } v1; sch2=nat\text{-to-sch } v2$
 $in loc-f u sch1 sch2)$
<proof>

lemma *loc-srj-0*: $nat\text{-to-sch } (c\text{-pair } 1\ 0) = Base\text{-suc}$
<proof>

lemma *nat-to-sch-at-2*: $nat\text{-to-sch } 2 = Base\text{-suc}$
<proof>

lemma *loc-srj-1*: $nat\text{-to-sch } (c\text{-pair } 2\ 0) = Base\text{-fst}$
<proof>

lemma *loc-srj-2*: $nat\text{-to-sch } (c\text{-pair } 3\ 0) = Base\text{-snd}$
<proof>

lemma *loc-srj-3*: $\llbracket nat\text{-to-sch } (sch\text{-to-nat } sch1) = sch1; nat\text{-to-sch } (sch\text{-to-nat } sch2)$
 $= sch2 \rrbracket$
 $\implies nat\text{-to-sch } (c\text{-pair } 4 (c\text{-pair } (sch\text{-to-nat } sch1) (sch\text{-to-nat } sch2))) =$
Comp-op sch1 sch2
<proof>

lemma *loc-srj-3-1*: $nat\text{-to-sch } (c\text{-pair } 4 (c\text{-pair } n1\ n2)) = Comp\text{-op } (nat\text{-to-sch } n1)$
 $(nat\text{-to-sch } n2)$

<proof>

lemma *loc-srj-4*: $\llbracket \text{nat-to-sch } (\text{sch-to-nat } \text{sch1}) = \text{sch1}; \text{nat-to-sch } (\text{sch-to-nat } \text{sch2}) = \text{sch2} \rrbracket$

$\implies \text{nat-to-sch } (\text{c-pair } 5 \ (\text{c-pair } (\text{sch-to-nat } \text{sch1}) \ (\text{sch-to-nat } \text{sch2}))) = \text{Pair-op } \text{sch1 } \text{sch2}$
<proof>

lemma *loc-srj-4-1*: $\text{nat-to-sch } (\text{c-pair } 5 \ (\text{c-pair } n1 \ n2)) = \text{Pair-op } (\text{nat-to-sch } n1) \ (\text{nat-to-sch } n2)$
<proof>

lemma *loc-srj-5*: $\llbracket \text{nat-to-sch } (\text{sch-to-nat } \text{sch1}) = \text{sch1}; \text{nat-to-sch } (\text{sch-to-nat } \text{sch2}) = \text{sch2} \rrbracket$

$\implies \text{nat-to-sch } (\text{c-pair } 6 \ (\text{c-pair } (\text{sch-to-nat } \text{sch1}) \ (\text{sch-to-nat } \text{sch2}))) = \text{Rec-op } \text{sch1 } \text{sch2}$
<proof>

lemma *loc-srj-5-1*: $\text{nat-to-sch } (\text{c-pair } 6 \ (\text{c-pair } n1 \ n2)) = \text{Rec-op } (\text{nat-to-sch } n1) \ (\text{nat-to-sch } n2)$
<proof>

theorem *nat-to-sch-srj*: $\text{nat-to-sch } (\text{sch-to-nat } \text{sch}) = \text{sch}$
<proof>

4.3 Indexes of primitive recursive functions of one variables

definition

nat-to-pr :: $\text{nat} \Rightarrow (\text{nat} \Rightarrow \text{nat})$ **where**
nat-to-pr = $(\lambda x. \text{sch-to-pr } (\text{nat-to-sch } x))$

theorem *nat-to-pr-into-pr*: $\text{nat-to-pr } n \in \text{PrimRec1}$ *<proof>*

lemma *nat-to-pr-srj*: $f \in \text{PrimRec1} \implies (\exists n. f = \text{nat-to-pr } n)$
<proof>

lemma *nat-to-pr-at-0*: $\text{nat-to-pr } 0 = (\lambda x. 0)$ *<proof>*

definition

index-of-pr :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{nat}$ **where**
index-of-pr *f* = $(\text{SOME } n. f = \text{nat-to-pr } n)$

theorem *index-of-pr-is-real*: $f \in \text{PrimRec1} \implies \text{nat-to-pr } (\text{index-of-pr } f) = f$
<proof>

definition

comp-by-index :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
comp-by-index = $(\lambda n1 \ n2. \text{c-pair } 4 \ (\text{c-pair } n1 \ n2))$

definition

pair-by-index :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
pair-by-index = $(\lambda n1 n2. \text{c-pair } 5 (\text{c-pair } n1 n2))$

definition

rec-by-index :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
rec-by-index = $(\lambda n1 n2. \text{c-pair } 6 (\text{c-pair } n1 n2))$

lemma *comp-by-index-is-pr*: $\text{comp-by-index} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

lemma *comp-by-index-inj*: $\text{comp-by-index } x1 y1 = \text{comp-by-index } x2 y2 \implies x1=x2 \wedge y1=y2$
 $\langle \text{proof} \rangle$

lemma *comp-by-index-inj1*: $\text{comp-by-index } x1 y1 = \text{comp-by-index } x2 y2 \implies x1 = x2$
 $\langle \text{proof} \rangle$

lemma *comp-by-index-inj2*: $\text{comp-by-index } x1 y1 = \text{comp-by-index } x2 y2 \implies y1 = y2$
 $\langle \text{proof} \rangle$

lemma *comp-by-index-main*: $\text{nat-to-pr } (\text{comp-by-index } n1 n2) = (\lambda x. (\text{nat-to-pr } n1) ((\text{nat-to-pr } n2) x))$
 $\langle \text{proof} \rangle$

lemma *pair-by-index-is-pr*: $\text{pair-by-index} \in \text{PrimRec2}$ $\langle \text{proof} \rangle$

lemma *pair-by-index-inj*: $\text{pair-by-index } x1 y1 = \text{pair-by-index } x2 y2 \implies x1=x2 \wedge y1=y2$
 $\langle \text{proof} \rangle$

lemma *pair-by-index-inj1*: $\text{pair-by-index } x1 y1 = \text{pair-by-index } x2 y2 \implies x1 = x2$
 $\langle \text{proof} \rangle$

lemma *pair-by-index-inj2*: $\text{pair-by-index } x1 y1 = \text{pair-by-index } x2 y2 \implies y1 = y2$
 $\langle \text{proof} \rangle$

lemma *pair-by-index-main*: $\text{nat-to-pr } (\text{pair-by-index } n1 n2) = \text{c-f-pair } (\text{nat-to-pr } n1) (\text{nat-to-pr } n2)$
 $\langle \text{proof} \rangle$

lemma *nat-to-sch-of-pair-by-index [simp]*: $\text{nat-to-sch } (\text{pair-by-index } n1 n2) = \text{Pair-op } (\text{nat-to-sch } n1) (\text{nat-to-sch } n2)$
 $\langle \text{proof} \rangle$

lemma *rec-by-index-is-pr*: $\text{rec-by-index} \in \text{PrimRec2}$ $\langle \text{proof} \rangle$

lemma *rec-by-index-inj*: $\text{rec-by-index } x1 y1 = \text{rec-by-index } x2 y2 \implies x1=x2 \wedge y1=y2$
 $\langle \text{proof} \rangle$

lemma *rec-by-index-inj1*: $\text{rec-by-index } x1 \ y1 = \text{rec-by-index } x2 \ y2 \implies x1 = x2$
(proof)

lemma *rec-by-index-inj2*: $\text{rec-by-index } x1 \ y1 = \text{rec-by-index } x2 \ y2 \implies y1 = y2$
(proof)

lemma *rec-by-index-main*: $\text{nat-to-pr } (\text{rec-by-index } n1 \ n2) = \text{UnaryRecOp } (\text{nat-to-pr } n1) (\text{nat-to-pr } n2)$ (proof)

4.4 s-1-1 theorem for primitive recursive functions of one variable

definition

index-of-const :: $\text{nat} \Rightarrow \text{nat}$ **where**
index-of-const = *PrimRecOp1* 0 ($\lambda x \ y. \text{c-pair } 4 \ (\text{c-pair } 2 \ y)$)

lemma *index-of-const-is-pr*: $\text{index-of-const} \in \text{PrimRec1}$
(proof)

lemma *index-of-const-at-0*: $\text{index-of-const } 0 = 0$ (proof)

lemma *index-of-const-at-suc*: $\text{index-of-const } (\text{Suc } u) = \text{c-pair } 4 \ (\text{c-pair } 2 \ (\text{index-of-const } u))$ (proof)

lemma *index-of-const-main*: $\text{nat-to-pr } (\text{index-of-const } n) = (\lambda x. n)$ (is ?P n)
(proof)

lemma *index-of-const-lm-1*: $(\text{nat-to-pr } (\text{index-of-const } n)) \ 0 = n$ (proof)

lemma *index-of-const-inj*: $\text{index-of-const } n1 = \text{index-of-const } n2 \implies n1 = n2$
(proof)

definition *index-of-zero* = *sch-to-nat* *Base-zero*

definition *index-of-suc* = *sch-to-nat* *Base-suc*

definition *index-of-c-fst* = *sch-to-nat* *Base-fst*

definition *index-of-c-snd* = *sch-to-nat* *Base-snd*

definition *index-of-id* = *pair-by-index* *index-of-c-fst* *index-of-c-snd*

lemma *index-of-zero-main*: $\text{nat-to-pr } \text{index-of-zero} = (\lambda x. 0)$ (proof)

lemma *index-of-suc-main*: $\text{nat-to-pr } \text{index-of-suc} = \text{Suc}$
(proof)

lemma *index-of-c-fst-main*: $\text{nat-to-pr } \text{index-of-c-fst} = \text{c-fst}$ (proof)

lemma [*simp*]: $\text{nat-to-sch } \text{index-of-c-fst} = \text{Base-fst}$ (proof)

lemma *index-of-c-snd-main*: $\text{nat-to-pr } \text{index-of-c-snd} = \text{c-snd}$ (proof)

lemma *[simp]*: $\text{nat-to-sch } \text{index-of-c-snd} = \text{Base-snd} \langle \text{proof} \rangle$

lemma *index-of-id-main*: $\text{nat-to-pr } \text{index-of-id} = (\lambda x. x) \langle \text{proof} \rangle$

definition

$\text{index-of-c-pair-n} :: \text{nat} \Rightarrow \text{nat}$ **where**
 $\text{index-of-c-pair-n} = (\lambda n. \text{pair-by-index } (\text{index-of-const } n) \text{ index-of-id})$

lemma *index-of-c-pair-n-is-pr*: $\text{index-of-c-pair-n} \in \text{PrimRec1}$
 $\langle \text{proof} \rangle$

lemma *index-of-c-pair-n-main*: $\text{nat-to-pr } (\text{index-of-c-pair-n } n) = (\lambda x. \text{c-pair } n x)$
 $\langle \text{proof} \rangle$

lemma *index-of-c-pair-n-inj*: $\text{index-of-c-pair-n } x1 = \text{index-of-c-pair-n } x2 \implies x1=x2$
 $\langle \text{proof} \rangle$

definition

$s1-1 :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $s1-1 = (\lambda n x. \text{comp-by-index } n (\text{index-of-c-pair-n } x))$

lemma *s1-1-is-pr*: $s1-1 \in \text{PrimRec2} \langle \text{proof} \rangle$

theorem *s1-1-th*: $(\lambda y. (\text{nat-to-pr } n) (\text{c-pair } x y)) = \text{nat-to-pr } (s1-1 n x)$
 $\langle \text{proof} \rangle$

lemma *s1-1-inj*: $s1-1 x1 y1 = s1-1 x2 y2 \implies x1=x2 \wedge y1=y2$
 $\langle \text{proof} \rangle$

lemma *s1-1-inj1*: $s1-1 x1 y1 = s1-1 x2 y2 \implies x1=x2 \langle \text{proof} \rangle$

lemma *s1-1-inj2*: $s1-1 x1 y1 = s1-1 x2 y2 \implies y1=y2 \langle \text{proof} \rangle$

primrec

$\text{pr-index-enumerator} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$

where

$\text{pr-index-enumerator } n 0 = n$
 $| \text{pr-index-enumerator } n (\text{Suc } m) = \text{comp-by-index } \text{index-of-id } (\text{pr-index-enumerator } n m)$

theorem *pr-index-enumerator-is-pr*: $\text{pr-index-enumerator} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

lemma *pr-index-enumerator-increase1*: $\text{pr-index-enumerator } n m < \text{pr-index-enumerator } (n+1) m$
 $\langle \text{proof} \rangle$

lemma *pr-index-enumerator-increase2*: $\text{pr-index-enumerator } n m < \text{pr-index-enumerator } n (m + 1)$

<proof>

lemma *f-inc-mono*: $(\forall (x::nat). (f::nat \Rightarrow nat) x < f (x+1)) \implies \forall (x::nat) (y::nat).$
 $(x < y \implies f x < f y)$
<proof>

lemma *pr-index-enumerator-mono1*: $n1 < n2 \implies pr-index-enumerator n1 m <$
 $pr-index-enumerator n2 m$
<proof>

lemma *pr-index-enumerator-mono2*: $m1 < m2 \implies pr-index-enumerator n m1 <$
 $pr-index-enumerator n m2$
<proof>

lemma *f-mono-inj*: $\forall (x::nat) (y::nat). (x < y \implies (f::nat \Rightarrow nat) x < f y) \implies \forall$
 $(x::nat) (y::nat). (f x = f y \implies x = y)$
<proof>

theorem *pr-index-enumerator-inj1*: $pr-index-enumerator n1 m = pr-index-enumerator$
 $n2 m \implies n1 = n2$
<proof>

theorem *pr-index-enumerator-inj2*: $pr-index-enumerator n m1 = pr-index-enumerator$
 $n m2 \implies m1 = m2$
<proof>

theorem *pr-index-enumerator-main*: $nat-to-pr n = nat-to-pr (pr-index-enumerator$
 $n m)$
<proof>

end

5 Finite sets

theory *PRecFinSet*
imports *PRecFun*
begin

We introduce a particular mapping *nat-to-set* from natural numbers to finite sets of natural numbers and a particular mapping *set-to-nat* from finite sets of natural numbers to natural numbers. See [1] and [2] for more information.

definition

$c-in :: nat \Rightarrow nat \Rightarrow nat$ **where**
 $c-in = (\lambda x u. (u \text{ div } (2 \wedge x)) \text{ mod } 2)$

lemma *c-in-is-pr*: $c-in \in PrimRec2$
<proof>

definition

nat-to-set :: *nat* \Rightarrow *nat set* **where**
nat-to-set *u* \equiv $\{x. 2^{\wedge}x \leq u \wedge c\text{-in } x \ u = 1\}$

lemma *c-in-upper-bound*: $c\text{-in } x \ u = 1 \Longrightarrow 2^{\wedge}x \leq u$
 \langle *proof* \rangle

lemma *nat-to-set-upper-bound*: $x \in \text{nat-to-set } u \Longrightarrow 2^{\wedge}x \leq u$ \langle *proof* \rangle

lemma *x-lt-2-x*: $x < 2^{\wedge}x$
 \langle *proof* \rangle

lemma *nat-to-set-upper-bound1*: $x \in \text{nat-to-set } u \Longrightarrow x < u$
 \langle *proof* \rangle

lemma *nat-to-set-upper-bound2*: $\text{nat-to-set } u \subseteq \{i. i < u\}$
 \langle *proof* \rangle

lemma *nat-to-set-is-finite*: *finite* (*nat-to-set* *u*)
 \langle *proof* \rangle

lemma *x-in-u-eq*: $(x \in \text{nat-to-set } u) = (c\text{-in } x \ u = 1)$ \langle *proof* \rangle

definition

log2 :: *nat* \Rightarrow *nat* **where**
log2 = $(\lambda x. \text{Least}(\%z. x < 2^{\wedge}(z+1)))$

lemma *log2-at-0*: $\text{log2 } 0 = 0$
 \langle *proof* \rangle

lemma *log2-at-1*: $\text{log2 } 1 = 0$
 \langle *proof* \rangle

lemma *log2-le*: $x > 0 \Longrightarrow 2^{\wedge}\text{log2 } x \leq x$
 \langle *proof* \rangle

lemma *log2-gt*: $x < 2^{\wedge}(\text{log2 } x + 1)$
 \langle *proof* \rangle

lemma *x-div-x*: $x > 0 \Longrightarrow (x::\text{nat}) \text{ div } x = 1$ \langle *proof* \rangle

lemma *div-ge*: $(k::\text{nat}) \leq m \text{ div } n \Longrightarrow n*k \leq m$
 \langle *proof* \rangle

lemma *div-lt*: $m < n*k \Longrightarrow m \text{ div } n < (k::\text{nat})$
 \langle *proof* \rangle

lemma *log2-lm1*: $u > 0 \Longrightarrow u \text{ div } 2^{\wedge}(\text{log2 } u) = 1$
 \langle *proof* \rangle

lemma *log2-lm2*: $u > 0 \Longrightarrow c\text{-in } (\text{log2 } u) \ u = 1$

<proof>

lemma *log2-lm3*: $\log2\ u < x \implies c\text{-in}\ x\ u = 0$
<proof>

lemma *log2-lm4*: $c\text{-in}\ x\ u = 1 \implies x \leq \log2\ u$
<proof>

lemma *nat-to-set-lub*: $x \in \text{nat-to-set}\ u \implies x \leq \log2\ u$
<proof>

lemma *log2-lm5*: $u > 0 \implies \log2\ u \in \text{nat-to-set}\ u$
<proof>

lemma *pos-imp-ne*: $u > 0 \implies \text{nat-to-set}\ u \neq \{\}$
<proof>

lemma *empty-is-zero*: $\text{nat-to-set}\ u = \{\} \implies u = 0$
<proof>

lemma *log2-is-max*: $u > 0 \implies \log2\ u = \text{Max}\ (\text{nat-to-set}\ u)$
<proof>

lemma *zero-is-empty*: $\text{nat-to-set}\ 0 = \{\}$
<proof>

lemma *ne-imp-pos*: $\text{nat-to-set}\ u \neq \{\} \implies u > 0$
<proof>

lemma *div-mod-lm*: $y < x \implies ((u + (2::\text{nat})^\wedge x) \text{ div } (2::\text{nat})^\wedge y) \text{ mod } 2 = (u \text{ div } (2::\text{nat})^\wedge y) \text{ mod } 2$
<proof>

lemma *add-power*: $u < 2^\wedge x \implies \text{nat-to-set}\ (u + 2^\wedge x) = \text{nat-to-set}\ u \cup \{x\}$
<proof>

theorem *nat-to-set-inj*: $\text{nat-to-set}\ u = \text{nat-to-set}\ v \implies u = v$
<proof>

definition

set-to-nat :: *nat set => nat where*
set-to-nat = ($\lambda D. \text{sum } (\lambda x. 2^\wedge x) D$)

lemma *two-power-sum*: $\text{sum } (\lambda x. (2::\text{nat})^\wedge x) \{i. i < \text{Suc } m\} = (2^\wedge \text{Suc } m) - 1$
<proof>

lemma *finite-interval*: *finite* $\{i. (i::\text{nat}) < m\}$
<proof>

lemma *set-to-nat-at-empty*: $set\text{-to-nat } \{\} = 0$ *<proof>*

lemma *set-to-nat-of-interval*: $set\text{-to-nat } \{i. (i::nat) < m\} = 2^{\wedge} m - 1$
<proof>

lemma *set-to-nat-mono*: $\llbracket finite\ B; A \subseteq B \rrbracket \implies set\text{-to-nat } A \leq set\text{-to-nat } B$
<proof>

theorem *nat-to-set-srj*: $finite\ (D::nat\ set) \implies nat\text{-to-set } (set\text{-to-nat } D) = D$
<proof>

theorem *nat-to-set-srj1*: $finite\ (D::nat\ set) \implies \exists\ u. nat\text{-to-set } u = D$
<proof>

lemma *sum-of-pr-is-pr*: $g \in PrimRec1 \implies (\lambda\ n. sum\ g\ \{i. i < n\}) \in PrimRec1$
<proof>

lemma *sum-of-pr-is-pr2*: $p \in PrimRec2 \implies (\lambda\ n\ m. sum\ (\lambda\ x. p\ x\ m)\ \{i. i < n\}) \in PrimRec2$
<proof>

lemma *sum-is-pr*: $g \in PrimRec1 \implies (\lambda\ u. sum\ g\ (nat\text{-to-set } u)) \in PrimRec1$
<proof>

definition

c-card :: $nat \Rightarrow nat$ **where**
c-card = $(\lambda\ u. card\ (nat\text{-to-set } u))$

theorem *c-card-is-pr*: $c\text{-card} \in PrimRec1$
<proof>

definition

c-insert :: $nat \Rightarrow nat \Rightarrow nat$ **where**
c-insert = $(\lambda\ x\ u. if\ c\text{-in } x\ u = 1\ then\ u\ else\ u + 2^{\wedge} x)$

lemma *c-insert-is-pr*: $c\text{-insert} \in PrimRec2$
<proof>

lemma [*simp*]: $set\text{-to-nat } (nat\text{-to-set } u) = u$
<proof>

lemma *insert-lemma*: $x \notin nat\text{-to-set } u \implies set\text{-to-nat } (nat\text{-to-set } u \cup \{x\}) = u + 2^{\wedge} x$
<proof>

lemma *c-insert-df*: $c\text{-insert} = (\lambda\ x\ u. set\text{-to-nat } ((nat\text{-to-set } u) \cup \{x\}))$
<proof>

definition

$c\text{-remove} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-remove} = (\lambda x u. \text{if } c\text{-in } x \ u = 0 \text{ then } u \text{ else } u - 2^{\wedge}x)$

lemma $c\text{-remove-is-pr}$: $c\text{-remove} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

lemma remove-lemma : $x \in \text{nat-to-set } u \implies \text{set-to-nat } (\text{nat-to-set } u - \{x\}) = u - 2^{\wedge}x$
 $\langle \text{proof} \rangle$

lemma $c\text{-remove-df}$: $c\text{-remove} = (\lambda x u. \text{set-to-nat } ((\text{nat-to-set } u) - \{x\}))$
 $\langle \text{proof} \rangle$

definition

$c\text{-union} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-union} = (\lambda u v. \text{set-to-nat } (\text{nat-to-set } u \cup \text{nat-to-set } v))$

theorem $c\text{-union-is-pr}$: $c\text{-union} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

definition

$c\text{-diff} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-diff} = (\lambda u v. \text{set-to-nat } (\text{nat-to-set } u - \text{nat-to-set } v))$

theorem $c\text{-diff-is-pr}$: $c\text{-diff} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

definition

$c\text{-intersect} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**
 $c\text{-intersect} = (\lambda u v. \text{set-to-nat } (\text{nat-to-set } u \cap \text{nat-to-set } v))$

theorem $c\text{-intersect-is-pr}$: $c\text{-intersect} \in \text{PrimRec2}$
 $\langle \text{proof} \rangle$

end

6 The function which is universal for primitive recursive functions of one variable

theory PRecUnGr
imports PRecFun2 PRecList
begin

We introduce a particular function which is universal for primitive recursive functions of one variable.

definition

$g\text{-comp} :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ **where**


```

g-comp c-ls key = (
  let n = c-fst key; x = c-snd key; m = c-snd n;
  m1 = c-fst m; m2 = c-snd m in
  — We have key = <n, x>; n = <?, m>; m = <m1, m2>.
  if c-assoc-have-key c-ls (c-pair m2 x) = 0 then
    (let y = c-assoc-value c-ls (c-pair m2 x) in
     if c-assoc-have-key c-ls (c-pair m1 y) = 0 then
       (let z = c-assoc-value c-ls (c-pair m1 y) in
        c-cons (c-pair key z) c-ls)
       else c-ls
    )
  else c-ls
)

```

definition

```

g-pair :: nat ⇒ nat ⇒ nat where
g-pair c-ls key = (
  let n = c-fst key; x = c-snd key; m = c-snd n;
  m1 = c-fst m; m2 = c-snd m in
  — We have key = <n, x>; n = <?, m>; m = <m1, m2>.
  if c-assoc-have-key c-ls (c-pair m1 x) = 0 then
    (let y1 = c-assoc-value c-ls (c-pair m1 x) in
     if c-assoc-have-key c-ls (c-pair m2 x) = 0 then
       (let y2 = c-assoc-value c-ls (c-pair m2 x) in
        c-cons (c-pair key (c-pair y1 y2)) c-ls)
       else c-ls
    )
  else c-ls
)

```

definition

```

g-rec :: nat ⇒ nat ⇒ nat where
g-rec c-ls key = (
  let n = c-fst key; x = c-snd key; m = c-snd n;
  m1 = c-fst m; m2 = c-snd m; y1 = c-fst x; x1 = c-snd x in
  — We have key = <n, x>; n = <?, m>; m = <m1, m2>; x = <y1, x1>.
  if y1 = 0 then
    (
      if c-assoc-have-key c-ls (c-pair m1 x1) = 0 then
        c-cons (c-pair key (c-assoc-value c-ls (c-pair m1 x1))) c-ls
      else c-ls
    )
  else
    (
      let y2 = y1 - (1 :: nat) in
      if c-assoc-have-key c-ls (c-pair n (c-pair y2 x1)) = 0 then
        (
          let t1 = c-assoc-value c-ls (c-pair n (c-pair y2 x1)); t2 = c-pair (c-pair y2
t1) x1 in

```

```

    if c-assoc-have-key c-ls (c-pair m2 t2) = 0 then
      c-cons (c-pair key (c-assoc-value c-ls (c-pair m2 t2))) c-ls
    else c-ls
  )
else c-ls
)
)
)

```

definition

```

g-step :: nat => nat => nat where
g-step c-ls key = (
  let n = c-fst key; x = c-snd key; n1 = (c-fst n) mod 7 in
  if n1 = 0 then c-cons (c-pair key 0) c-ls else
  if n1 = 1 then c-cons (c-pair key (Suc x)) c-ls else
  if n1 = 2 then c-cons (c-pair key (c-fst x)) c-ls else
  if n1 = 3 then c-cons (c-pair key (c-snd x)) c-ls else
  if n1 = 4 then g-comp c-ls key else
  if n1 = 5 then g-pair c-ls key else
  if n1 = 6 then g-rec c-ls key else
  c-ls
)

```

definition

```

pr-gr :: nat => nat where
pr-gr-def: pr-gr = PrimRecOp1 0 (λ a b. g-step b (c-fst a))

```

lemma *pr-gr-at-0*: $pr-gr\ 0 = 0$ *<proof>*

lemma *pr-gr-at-Suc*: $pr-gr\ (Suc\ x) = g-step\ (pr-gr\ x)\ (c-fst\ x)$ *<proof>*

definition

```

univ-for-pr :: nat => nat where
univ-for-pr = pr-conv-2-to-1 nat-to-pr

```

theorem *univ-is-not-pr*: $univ-for-pr \notin PrimRec1$
<proof>

definition

```

c-is-sub-fun :: nat => (nat => nat) => bool where
c-is-sub-fun ls f <math>\iff (\forall x. c-assoc-have-key\ ls\ x = 0 \implies c-assoc-value\ ls\ x = f\ x)</math>

```

lemma *c-is-sub-fun-lm-1*: $\llbracket c-is-sub-fun\ ls\ f; c-assoc-have-key\ ls\ x = 0 \rrbracket \implies c-assoc-value\ ls\ x = f\ x$
<proof>

lemma *c-is-sub-fun-lm-2*: $c-is-sub-fun\ ls\ f \implies c-is-sub-fun\ (c-cons\ (c-pair\ x\ (f\ x))\ ls)\ f$
<proof>

lemma *mod7-lm*: $(n::nat) \bmod 7 = 0 \vee$
 $(n::nat) \bmod 7 = 1 \vee$
 $(n::nat) \bmod 7 = 2 \vee$
 $(n::nat) \bmod 7 = 3 \vee$
 $(n::nat) \bmod 7 = 4 \vee$
 $(n::nat) \bmod 7 = 5 \vee$
 $(n::nat) \bmod 7 = 6$ *<proof>*

lemma *nat-to-sch-at-pos*: $x > 0 \implies nat\text{-to}\text{-sch } x = (let\ u=(c\text{-fst } x) \bmod 7;$
 $v=c\text{-snd } x; v1=c\text{-fst } v; v2 = c\text{-snd } v; sch1=nat\text{-to}\text{-sch } v1; sch2=nat\text{-to}\text{-sch } v2$
in loc-f u sch1 sch2)
<proof>

lemma *nat-to-sch-0*: $c\text{-fst } n \bmod 7 = 0 \implies nat\text{-to}\text{-sch } n = Base\text{-zero}$
<proof>

lemma *loc-lm-1*: $c\text{-fst } n \bmod 7 \neq 0 \implies n > 0$
<proof>

lemma *loc-lm-2*: $c\text{-fst } n \bmod 7 \neq 0 \implies nat\text{-to}\text{-sch } n = (let\ u=(c\text{-fst } n) \bmod 7;$
 $v=c\text{-snd } n; v1=c\text{-fst } v; v2 = c\text{-snd } v; sch1=nat\text{-to}\text{-sch } v1; sch2=nat\text{-to}\text{-sch } v2$ *in*
 $loc\text{-f } u\ sch1\ sch2)$
<proof>

lemma *nat-to-sch-1*: $c\text{-fst } n \bmod 7 = 1 \implies nat\text{-to}\text{-sch } n = Base\text{-suc}$
<proof>

lemma *nat-to-sch-2*: $c\text{-fst } n \bmod 7 = 2 \implies nat\text{-to}\text{-sch } n = Base\text{-fst}$
<proof>

lemma *nat-to-sch-3*: $c\text{-fst } n \bmod 7 = 3 \implies nat\text{-to}\text{-sch } n = Base\text{-snd}$
<proof>

lemma *nat-to-sch-4*: $c\text{-fst } n \bmod 7 = 4 \implies nat\text{-to}\text{-sch } n = Comp\text{-op } (nat\text{-to}\text{-sch}$
 $(c\text{-fst } (c\text{-snd } n))) (nat\text{-to}\text{-sch } (c\text{-snd } (c\text{-snd } n)))$
<proof>

lemma *nat-to-sch-5*: $c\text{-fst } n \bmod 7 = 5 \implies nat\text{-to}\text{-sch } n = Pair\text{-op } (nat\text{-to}\text{-sch}$
 $(c\text{-fst } (c\text{-snd } n))) (nat\text{-to}\text{-sch } (c\text{-snd } (c\text{-snd } n)))$
<proof>

lemma *nat-to-sch-6*: $c\text{-fst } n \bmod 7 = 6 \implies nat\text{-to}\text{-sch } n = Rec\text{-op } (nat\text{-to}\text{-sch}$
 $(c\text{-fst } (c\text{-snd } n))) (nat\text{-to}\text{-sch } (c\text{-snd } (c\text{-snd } n)))$
<proof>

lemma *nat-to-pr-lm-0*: $c\text{-fst } n \bmod 7 = 0 \implies nat\text{-to}\text{-pr } n\ x = 0$
<proof>

lemma *nat-to-pr-lm-1*: $c\text{-fst } n \bmod 7 = 1 \implies \text{nat-to-pr } n \ x = \text{Suc } x$
{proof}

lemma *nat-to-pr-lm-2*: $c\text{-fst } n \bmod 7 = 2 \implies \text{nat-to-pr } n \ x = c\text{-fst } x$
{proof}

lemma *nat-to-pr-lm-3*: $c\text{-fst } n \bmod 7 = 3 \implies \text{nat-to-pr } n \ x = c\text{-snd } x$
{proof}

lemma *nat-to-pr-lm-4*: $c\text{-fst } n \bmod 7 = 4 \implies \text{nat-to-pr } n \ x = (\text{nat-to-pr } (c\text{-fst } (c\text{-snd } n)) (\text{nat-to-pr } (c\text{-snd } (c\text{-snd } n)) x))$
{proof}

lemma *nat-to-pr-lm-5*: $c\text{-fst } n \bmod 7 = 5 \implies \text{nat-to-pr } n \ x = (c\text{-f-pair } (\text{nat-to-pr } (c\text{-fst } (c\text{-snd } n)) (\text{nat-to-pr } (c\text{-snd } (c\text{-snd } n)))) x$
{proof}

lemma *nat-to-pr-lm-6*: $c\text{-fst } n \bmod 7 = 6 \implies \text{nat-to-pr } n \ x = (\text{UnaryRecOp } (\text{nat-to-pr } (c\text{-fst } (c\text{-snd } n)) (\text{nat-to-pr } (c\text{-snd } (c\text{-snd } n)))) x$
{proof}

lemma *univ-for-pr-lm-0*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 0 \implies \text{univ-for-pr } \text{key} = 0$
{proof}

lemma *univ-for-pr-lm-1*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 1 \implies \text{univ-for-pr } \text{key} = \text{Suc } (c\text{-snd } \text{key})$
{proof}

lemma *univ-for-pr-lm-2*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 2 \implies \text{univ-for-pr } \text{key} = c\text{-fst } (c\text{-snd } \text{key})$
{proof}

lemma *univ-for-pr-lm-3*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 3 \implies \text{univ-for-pr } \text{key} = c\text{-snd } (c\text{-snd } \text{key})$
{proof}

lemma *univ-for-pr-lm-4*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 4 \implies \text{univ-for-pr } \text{key} = (\text{nat-to-pr } (c\text{-fst } (c\text{-snd } (c\text{-fst } \text{key}))) (\text{nat-to-pr } (c\text{-snd } (c\text{-snd } (c\text{-fst } \text{key}))) (c\text{-snd } \text{key})))$
{proof}

lemma *univ-for-pr-lm-4-1*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 4 \implies \text{univ-for-pr } \text{key} = \text{univ-for-pr } (c\text{-pair } (c\text{-fst } (c\text{-snd } (c\text{-fst } \text{key}))) (\text{univ-for-pr } (c\text{-pair } (c\text{-snd } (c\text{-snd } (c\text{-fst } \text{key}))) (c\text{-snd } \text{key}))))$
{proof}

lemma *univ-for-pr-lm-5*: $c\text{-fst } (c\text{-fst } \text{key}) \bmod 7 = 5 \implies \text{univ-for-pr } \text{key} = c\text{-pair } (\text{univ-for-pr } (c\text{-pair } (c\text{-fst } (c\text{-snd } (c\text{-fst } \text{key}))) (c\text{-snd } \text{key}))) (\text{univ-for-pr } (c\text{-pair } (c\text{-snd } (c\text{-snd } (c\text{-fst } \text{key}))) (c\text{-snd } \text{key}))))$
{proof}

lemma *univ-for-pr-lm-6-1*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 6; c\text{-fst } (c\text{-snd } key) = 0 \rrbracket$
 $\implies univ\text{-for-pr } key = univ\text{-for-pr } (c\text{-pair } (c\text{-fst } (c\text{-snd } (c\text{-fst } key))) (c\text{-snd } (c\text{-snd } key)))$
 $\langle proof \rangle$

lemma *univ-for-pr-lm-6-2*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 6; c\text{-fst } (c\text{-snd } key) = Suc\ u \rrbracket$
 $\implies univ\text{-for-pr } key = univ\text{-for-pr } (c\text{-pair } (c\text{-snd } (c\text{-snd } (c\text{-fst } key))) (c\text{-pair } (c\text{-pair } u (univ\text{-for-pr } (c\text{-pair } (c\text{-fst } key) (c\text{-pair } u (c\text{-snd } (c\text{-snd } key)))))) (c\text{-snd } (c\text{-snd } key))))$
 $\langle proof \rangle$

lemma *univ-for-pr-lm-6-3*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 6; c\text{-fst } (c\text{-snd } key) \neq 0 \rrbracket$
 $\implies univ\text{-for-pr } key = univ\text{-for-pr } (c\text{-pair } (c\text{-snd } (c\text{-snd } (c\text{-fst } key))) (c\text{-pair } (c\text{-pair } (c\text{-fst } (c\text{-snd } key) - 1) (univ\text{-for-pr } (c\text{-pair } (c\text{-fst } key) (c\text{-pair } (c\text{-fst } (c\text{-snd } key) - 1) (c\text{-snd } (c\text{-snd } key)))))) (c\text{-snd } (c\text{-snd } key))))$
 $\langle proof \rangle$

lemma *g-comp-lm-0*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 4; c\text{-is-sub-fun } ls \text{ univ-for-pr}; g\text{-comp } ls \text{ key} \neq ls \rrbracket$
 $\implies g\text{-comp } ls \text{ key} = c\text{-cons } (c\text{-pair } key (univ\text{-for-pr } key)) \text{ } ls$
 $\langle proof \rangle$

lemma *g-comp-lm-1*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 4; c\text{-is-sub-fun } ls \text{ univ-for-pr} \rrbracket$
 $\implies c\text{-is-sub-fun } (g\text{-comp } ls \text{ key}) \text{ univ-for-pr}$
 $\langle proof \rangle$

lemma *g-pair-lm-0*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 5; c\text{-is-sub-fun } ls \text{ univ-for-pr}; g\text{-pair } ls \text{ key} \neq ls \rrbracket$
 $\implies g\text{-pair } ls \text{ key} = c\text{-cons } (c\text{-pair } key (univ\text{-for-pr } key)) \text{ } ls$
 $\langle proof \rangle$

lemma *g-pair-lm-1*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 5; c\text{-is-sub-fun } ls \text{ univ-for-pr} \rrbracket$
 $\implies c\text{-is-sub-fun } (g\text{-pair } ls \text{ key}) \text{ univ-for-pr}$
 $\langle proof \rangle$

lemma *g-rec-lm-0*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 6; c\text{-is-sub-fun } ls \text{ univ-for-pr}; g\text{-rec } ls \text{ key} \neq ls \rrbracket$
 $\implies g\text{-rec } ls \text{ key} = c\text{-cons } (c\text{-pair } key (univ\text{-for-pr } key)) \text{ } ls$
 $\langle proof \rangle$

lemma *g-rec-lm-1*: $\llbracket c\text{-fst } (c\text{-fst } key) \bmod 7 = 6; c\text{-is-sub-fun } ls \text{ univ-for-pr} \rrbracket$
 $\implies c\text{-is-sub-fun } (g\text{-rec } ls \text{ key}) \text{ univ-for-pr}$
 $\langle proof \rangle$

lemma *g-step-lm-0*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 0 \implies g\text{-step } ls \text{ key} = c\text{-cons } (c\text{-pair } key \ 0) \text{ } ls$
 $\langle proof \rangle$

lemma *g-step-lm-1*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 1 \implies g\text{-step } ls \text{ key} = c\text{-cons } (c\text{-pair } key (Suc (c\text{-snd } key))) \text{ } ls$
 $\langle proof \rangle$

lemma *g-step-lm-2*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 2 \implies g\text{-step } ls \ key = c\text{-cons } (c\text{-pair } key \ (c\text{-fst } (c\text{-snd } key))) \ ls$ *<proof>*

lemma *g-step-lm-3*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 3 \implies g\text{-step } ls \ key = c\text{-cons } (c\text{-pair } key \ (c\text{-snd } (c\text{-snd } key))) \ ls$ *<proof>*

lemma *g-step-lm-4*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 4 \implies g\text{-step } ls \ key = g\text{-comp } ls \ key$ *<proof>*

lemma *g-step-lm-5*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 5 \implies g\text{-step } ls \ key = g\text{-pair } ls \ key$ *<proof>*

lemma *g-step-lm-6*: $c\text{-fst } (c\text{-fst } key) \bmod 7 = 6 \implies g\text{-step } ls \ key = g\text{-rec } ls \ key$ *<proof>*

lemma *g-step-lm-7*: $c\text{-is-sub-fun } ls \ \text{univ-for-pr} \implies c\text{-is-sub-fun } (g\text{-step } ls \ key) \ \text{univ-for-pr}$ *<proof>*

theorem *pr-gr-1*: $c\text{-is-sub-fun } (pr\text{-gr } x) \ \text{univ-for-pr}$ *<proof>*

lemma *comp-next*: $g\text{-comp } ls \ key = ls \vee \ c\text{-tl } (g\text{-comp } ls \ key) = ls$ *<proof>*

lemma *pair-next*: $g\text{-pair } ls \ key = ls \vee \ c\text{-tl } (g\text{-pair } ls \ key) = ls$ *<proof>*

lemma *rec-next*: $g\text{-rec } ls \ key = ls \vee \ c\text{-tl } (g\text{-rec } ls \ key) = ls$ *<proof>*

lemma *step-next*: $g\text{-step } ls \ key = ls \vee \ c\text{-tl } (g\text{-step } ls \ key) = ls$ *<proof>*

lemma *lm1*: $pr\text{-gr } (Suc \ x) = pr\text{-gr } x \vee \ c\text{-tl } (pr\text{-gr } (Suc \ x)) = pr\text{-gr } x$ *<proof>*

lemma *c-assoc-have-key-pos*: $c\text{-assoc-have-key } ls \ x = 0 \implies ls > 0$ *<proof>*

lemma *lm2*: $c\text{-assoc-have-key } (c\text{-tl } ls) \ key = 0 \implies c\text{-assoc-have-key } ls \ key = 0$ *<proof>*

lemma *lm3*: $c\text{-assoc-have-key } (pr\text{-gr } x) \ key = 0 \implies c\text{-assoc-have-key } (pr\text{-gr } (Suc \ x)) \ key = 0$ *<proof>*

lemma *lm4*: $\llbracket c\text{-assoc-have-key } (pr\text{-gr } x) \ key = 0; \ 0 \leq y \rrbracket \implies c\text{-assoc-have-key } (pr\text{-gr } (x+y)) \ key = 0$ *<proof>*

lemma *lm5*: $\llbracket c\text{-assoc-have-key } (pr\text{-gr } x) \ key = 0; \ x \leq y \rrbracket \implies c\text{-assoc-have-key } (pr\text{-gr } y) \ key = 0$ *<proof>*

lemma *loc-upb-lm-1*: $n = 0 \implies (c\text{-fst } n) \bmod 7 = 0$

<proof>

lemma *loc-upb-lm-2*: $(c\text{-fst } n) \bmod 7 > 1 \implies c\text{-snd } n < n$

<proof>

lemma *loc-upb-lm-2-0*: $(c\text{-fst } n) \bmod 7 = 4 \implies c\text{-fst } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-2*: $(c\text{-fst } n) \bmod 7 = 4 \implies c\text{-snd } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-3*: $(c\text{-fst } n) \bmod 7 = 5 \implies c\text{-fst } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-4*: $(c\text{-fst } n) \bmod 7 = 5 \implies c\text{-snd } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-5*: $(c\text{-fst } n) \bmod 7 = 6 \implies c\text{-fst } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-6*: $(c\text{-fst } n) \bmod 7 = 6 \implies c\text{-snd } (c\text{-snd } n) < n$

<proof>

lemma *loc-upb-lm-2-7*: $\llbracket y2 = y1 - (1::nat); 0 < y1; x1 = c\text{-snd } x; y1 = c\text{-fst } x \rrbracket \implies c\text{-pair } y2 \ x1 < x$

<proof>

function *loc-upb* :: $nat \Rightarrow nat \Rightarrow nat$ **where**

aa: $loc\text{-upb } n \ x =$ (

let $n1 = (c\text{-fst } n) \bmod 7$ *in*

if $n1 = 0$ *then* $(c\text{-pair } (c\text{-pair } n \ x) \ 0) + 1$ *else*

if $n1 = 1$ *then* $(c\text{-pair } (c\text{-pair } n \ x) \ 0) + 1$ *else*

if $n1 = 2$ *then* $(c\text{-pair } (c\text{-pair } n \ x) \ 0) + 1$ *else*

if $n1 = 3$ *then* $(c\text{-pair } (c\text{-pair } n \ x) \ 0) + 1$ *else*

if $n1 = 4$ *then* (

let $m = c\text{-snd } n; m1 = c\text{-fst } m; m2 = c\text{-snd } m;$

$y = c\text{-assoc-value } (pr\text{-gr } (loc\text{-upb } m2 \ x)) \ (c\text{-pair } m2 \ x)$ *in*

$(c\text{-pair } (c\text{-pair } n \ x) \ (loc\text{-upb } m2 \ x + loc\text{-upb } m1 \ y)) + 1$

) *else*

if $n1 = 5$ *then* (

let $m = c\text{-snd } n; m1 = c\text{-fst } m; m2 = c\text{-snd } m$ *in*

$(c\text{-pair } (c\text{-pair } n \ x) \ (loc\text{-upb } m1 \ x + loc\text{-upb } m2 \ x)) + 1$

) *else*

if $n1 = 6$ *then* (

let $m = c\text{-snd } n; m1 = c\text{-fst } m; m2 = c\text{-snd } m; y1 = c\text{-fst } x; x1 = c\text{-snd } x$ *in*

if $y1 = 0$ *then* (

$(c\text{-pair } (c\text{-pair } n \ x) \ (loc\text{-upb } m1 \ x1)) + 1$

$$\begin{aligned}
& \text{) else (} \\
& \quad \text{let } y2 = y1 - (1 :: \text{nat}); \\
& \quad \quad t1 = \text{c-assoc-value (pr-gr (loc-upb n (c-pair y2 x1))) (c-pair n (c-pair} \\
& \text{y2 x1)); } t2 = \text{c-pair (c-pair y2 t1) x1 in} \\
& \quad \quad (\text{c-pair (c-pair n x) (loc-upb n (c-pair y2 x1) + loc-upb m2 t2)) + 1} \\
& \quad \text{)} \\
& \text{)} \\
& \text{else 0} \\
& \text{)} \\
\langle \text{proof} \rangle
\end{aligned}$$

termination

$\langle \text{proof} \rangle$

definition

$$\begin{aligned}
\text{lex-p} &:: ((\text{nat} \times \text{nat}) \times \text{nat} \times \text{nat}) \text{ set where} \\
\text{lex-p} &= ((\text{measure } (\lambda m. m)) < * \text{lex} * > (\text{measure } (\lambda n. n)))
\end{aligned}$$

lemma *wf-lex-p*: $\text{wf}(\text{lex-p})$

$\langle \text{proof} \rangle$

lemma *lex-p-eq*: $((n', x'), (n, x)) \in \text{lex-p} = (n' < n \vee n' = n \wedge x' < x)$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-0*: $\text{c-fst } n \text{ mod } 7 = 0 \implies \text{c-assoc-have-key (pr-gr (loc-upb n x)) (c-pair n x)} = 0$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-1*: $\text{c-fst } n \text{ mod } 7 = 1 \implies \text{c-assoc-have-key (pr-gr (loc-upb n x)) (c-pair n x)} = 0$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-2*: $\text{c-fst } n \text{ mod } 7 = 2 \implies \text{c-assoc-have-key (pr-gr (loc-upb n x)) (c-pair n x)} = 0$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-3*: $\text{c-fst } n \text{ mod } 7 = 3 \implies \text{c-assoc-have-key (pr-gr (loc-upb n x)) (c-pair n x)} = 0$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-4*: $\llbracket \bigwedge n' x'. ((n', x'), (n, x)) \in \text{lex-p} \implies \text{c-assoc-have-key (pr-gr (loc-upb n' x')) (c-pair n' x')} = 0; \text{c-fst } n \text{ mod } 7 = 4 \rrbracket \implies$

$$\text{c-assoc-have-key (pr-gr (loc-upb n x)) (c-pair n x)} = 0$$

$\langle \text{proof} \rangle$

lemma *loc-upb-lex-5*: $\llbracket \bigwedge n' x'. ((n', x'), (n, x)) \in \text{lex-p} \implies \text{c-assoc-have-key (pr-gr (loc-upb n' x')) (c-pair n' x')} = 0; \text{c-fst } n \text{ mod } 7 = 5 \rrbracket \implies$

$$\text{c-fst } n \text{ mod } 7 = 5 \implies$$

$$c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } n \ x)) \ (c\text{-pair } n \ x) = 0$$

$\langle proof \rangle$

lemma *loc-upb-6-z*: $\llbracket c\text{-fst } n \text{ mod } 7 = 6; c\text{-fst } x = 0 \rrbracket \implies$

$$loc\text{-upb } n \ x = c\text{-pair } (c\text{-pair } n \ x) \ (loc\text{-upb } (c\text{-fst } (c\text{-snd } n)) \ (c\text{-snd } x)) + 1 \ \langle proof \rangle$$

lemma *loc-upb-6*: $\llbracket c\text{-fst } n \text{ mod } 7 = 6; c\text{-fst } x \neq 0 \rrbracket \implies loc\text{-upb } n \ x = ($

$$let \ m = c\text{-snd } n; \ m1 = c\text{-fst } m; \ m2 = c\text{-snd } m; \ y1 = c\text{-fst}$$

$x; \ x1 = c\text{-snd } x;$

$$y2 = y1 - 1;$$

$$t1 = c\text{-assoc-value } (pr\text{-gr } (loc\text{-upb } n \ (c\text{-pair } y2 \ x1))) \ (c\text{-pair}$$

$n \ (c\text{-pair } y2 \ x1));$

$$t2 = c\text{-pair } (c\text{-pair } y2 \ t1) \ x1 \ \text{in}$$

$$c\text{-pair } (c\text{-pair } n \ x) \ (loc\text{-upb } n \ (c\text{-pair } y2 \ x1)) + (loc\text{-upb}$$

$m2 \ t2)) + 1)$

$\langle proof \rangle$

lemma *loc-upb-lex-6*: $\llbracket \bigwedge n' \ x'. \ ((n', x'), (n, x)) \in lex\text{-p} \implies c\text{-assoc-have-key } (pr\text{-gr}$

$(loc\text{-upb } n' \ x')) \ (c\text{-pair } n' \ x') = 0;$

$$c\text{-fst } n \text{ mod } 7 = 6 \rrbracket \implies$$

$$c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } n \ x)) \ (c\text{-pair } n \ x) = 0$$

$\langle proof \rangle$

lemma *wf-upb-step-0*:

$$\llbracket \bigwedge n' \ x'. \ ((n', x'), (n, x)) \in lex\text{-p} \implies c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } n' \ x'))$$

$(c\text{-pair } n' \ x') = 0 \rrbracket \implies$

$$c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } n \ x)) \ (c\text{-pair } n \ x) = 0$$

$\langle proof \rangle$

lemma *wf-upb-step*:

$$\text{assumes } A1: \bigwedge p2. \ (p2, p1) \in lex\text{-p} \implies$$

$$c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } (fst \ p2) \ (snd \ p2))) \ (c\text{-pair } (fst \ p2) \ (snd \ p2)) = 0$$

shows $c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } (fst \ p1) \ (snd \ p1))) \ (c\text{-pair } (fst \ p1) \ (snd \ p1)) = 0$

$\langle proof \rangle$

theorem *loc-upb-main*: $c\text{-assoc-have-key } (pr\text{-gr } (loc\text{-upb } n \ x)) \ (c\text{-pair } n \ x) = 0$

$\langle proof \rangle$

theorem *pr-gr-value*: $c\text{-assoc-value } (pr\text{-gr } (loc\text{-upb } n \ x)) \ (c\text{-pair } n \ x) = univ\text{-for-pr}$

$(c\text{-pair } n \ x)$

$\langle proof \rangle$

theorem *g-comp-is-pr*: $g\text{-comp} \in PrimRec2$

$\langle proof \rangle$

theorem *g-pair-is-pr*: $g\text{-pair} \in PrimRec2$

$\langle proof \rangle$

theorem *g-rec-is-pr*: $g\text{-rec} \in \text{PrimRec2}$
<proof>

theorem *g-step-is-pr*: $g\text{-step} \in \text{PrimRec2}$
<proof>

theorem *pr-gr-is-pr*: $pr\text{-gr} \in \text{PrimRec1}$
<proof>

end

7 Computationally enumerable sets of natural numbers

theory *RecEnSet*
imports *PRecList PRecFun2 PRecFinSet PRecUnGr*
begin

7.1 Basic definitions

definition
fn-to-set :: $(\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow \text{nat set}$ **where**
fn-to-set $f = \{ x. \exists y. f\ x\ y = 0 \}$

definition
ce-sets :: $(\text{nat set}) \text{ set}$ **where**
ce-sets = $\{ (fn\text{-to-set}\ p) \mid p. p \in \text{PrimRec2} \}$

7.2 Basic properties of computably enumerable sets

lemma *ce-set-lm-1*: $p \in \text{PrimRec2} \implies fn\text{-to-set}\ p \in ce\text{-sets}$ *<proof>*

lemma *ce-set-lm-2*: $\llbracket p \in \text{PrimRec2}; \forall x. (x \in A) = (\exists y. p\ x\ y = 0) \rrbracket \implies A \in ce\text{-sets}$
<proof>

lemma *ce-set-lm-3*: $A \in ce\text{-sets} \implies \exists p \in \text{PrimRec2}. A = fn\text{-to-set}\ p$
<proof>

lemma *ce-set-lm-4*: $A \in ce\text{-sets} \implies \exists p \in \text{PrimRec2}. \forall x. (x \in A) = (\exists y. p\ x\ y = 0)$
<proof>

lemma *ce-set-lm-5*: $\llbracket A \in ce\text{-sets}; p \in \text{PrimRec1} \rrbracket \implies \{ x. p\ x \in A \} \in ce\text{-sets}$
<proof>

lemma *ce-set-lm-6*: $\llbracket A \in ce\text{-sets}; A \neq \{\} \rrbracket \implies \exists q \in \text{PrimRec1}. A = \{ q\ x \mid x. x \in UNIV \}$

<proof>

lemma *ce-set-lm-7*: $\llbracket A \in \text{ce-sets}; p \in \text{PrimRec1} \rrbracket \implies \{ p\ x \mid x. x \in A \} \in \text{ce-sets}$
<proof>

theorem *ce-empty*: $\{\} \in \text{ce-sets}$
<proof>

theorem *ce-univ*: $\text{UNIV} \in \text{ce-sets}$
<proof>

theorem *ce-singleton*: $\{a\} \in \text{ce-sets}$
<proof>

theorem *ce-union*: $\llbracket A \in \text{ce-sets}; B \in \text{ce-sets} \rrbracket \implies A \cup B \in \text{ce-sets}$
<proof>

theorem *ce-intersect*: $\llbracket A \in \text{ce-sets}; B \in \text{ce-sets} \rrbracket \implies A \cap B \in \text{ce-sets}$
<proof>

7.3 Enumeration of computably enumerable sets

definition

nat-to-ce-set :: $\text{nat} \Rightarrow (\text{nat set})$ **where**
nat-to-ce-set = $(\lambda n. \text{fn-to-set } (\text{pr-conv-1-to-2 } (\text{nat-to-pr } n)))$

lemma *nat-to-ce-set-lm-1*: $\text{nat-to-ce-set } n = \{ x . \exists y. (\text{nat-to-pr } n) (\text{c-pair } x\ y) = 0 \}$
<proof>

lemma *nat-to-ce-set-into-ce*: $\text{nat-to-ce-set } n \in \text{ce-sets}$
<proof>

lemma *nat-to-ce-set-srj*: $A \in \text{ce-sets} \implies \exists n. A = \text{nat-to-ce-set } n$
<proof>

7.4 Characteristic functions

definition

chf :: $\text{nat set} \Rightarrow (\text{nat} \Rightarrow \text{nat})$ — Characteristic function **where**
chf = $(\lambda A\ x. \text{if } x \in A \text{ then } 0 \text{ else } 1)$

definition

zero-set :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{nat set}$ **where**
zero-set = $(\lambda f. \{ x. f\ x = 0 \})$

lemma *chf-lm-1* [*simp*]: $\text{zero-set } (\text{chf } A) = A$ *<proof>*

lemma *chf-lm-2*: $(x \in A) = (\text{chf } A\ x = 0)$ *<proof>*

lemma *chf-lm-3*: $(x \notin A) = (chf\ A\ x = 1)$ $\langle proof \rangle$

lemma *chf-lm-4*: $chf\ A \in PrimRec1 \implies A \in ce\ sets$
 $\langle proof \rangle$

lemma *chf-lm-5*: $finite\ A \implies chf\ A \in PrimRec1$
 $\langle proof \rangle$

theorem *ce-finite*: $finite\ A \implies A \in ce\ sets$
 $\langle proof \rangle$

7.5 Computably enumerable relations

definition

ce-set-to-rel :: $nat\ set \Rightarrow (nat * nat)\ set$ **where**
ce-set-to-rel = $(\lambda A. \{ (c\ fst\ x, c\ snd\ x) \mid x. x \in A \})$

definition

ce-rel-to-set :: $(nat * nat)\ set \Rightarrow nat\ set$ **where**
ce-rel-to-set = $(\lambda R. \{ c\ pair\ x\ y \mid x\ y. (x,y) \in R \})$

definition

ce-rels :: $((nat * nat)\ set)\ set$ **where**
ce-rels = $\{ R \mid R. ce\ rel\ to\ set\ R \in ce\ sets \}$

lemma *ce-rel-lm-1* [*simp*]: $ce\ set\ to\ rel\ (ce\ rel\ to\ set\ r) = r$
 $\langle proof \rangle$

lemma *ce-rel-lm-2* [*simp*]: $ce\ rel\ to\ set\ (ce\ set\ to\ rel\ A) = A$
 $\langle proof \rangle$

lemma *ce-rels-def1*: $ce\ rels = \{ ce\ set\ to\ rel\ A \mid A. A \in ce\ sets \}$
 $\langle proof \rangle$

lemma *ce-rel-to-set-inj*: $inj\ ce\ rel\ to\ set$
 $\langle proof \rangle$

lemma *ce-rel-to-set-surj*: $surj\ ce\ rel\ to\ set$
 $\langle proof \rangle$

lemma *ce-rel-to-set-bij*: $bij\ ce\ rel\ to\ set$
 $\langle proof \rangle$

lemma *ce-set-to-rel-inj*: $inj\ ce\ set\ to\ rel$
 $\langle proof \rangle$

lemma *ce-set-to-rel-surj*: $surj\ ce\ set\ to\ rel$
 $\langle proof \rangle$

lemma *ce-set-to-rel-bij*: *bij ce-set-to-rel*
<proof>

lemma *ce-rel-lm-3*: $A \in \text{ce-sets} \implies \text{ce-set-to-rel } A \in \text{ce-rels}$
<proof>

lemma *ce-rel-lm-4*: $\text{ce-set-to-rel } A \in \text{ce-rels} \implies A \in \text{ce-sets}$
<proof>

lemma *ce-rel-lm-5*: $(A \in \text{ce-sets}) = (\text{ce-set-to-rel } A \in \text{ce-rels})$
<proof>

lemma *ce-rel-lm-6*: $r \in \text{ce-rels} \implies \text{ce-rel-to-set } r \in \text{ce-sets}$
<proof>

lemma *ce-rel-lm-7*: $\text{ce-rel-to-set } r \in \text{ce-sets} \implies r \in \text{ce-rels}$
<proof>

lemma *ce-rel-lm-8*: $(r \in \text{ce-rels}) = (\text{ce-rel-to-set } r \in \text{ce-sets})$ *<proof>*

lemma *ce-rel-lm-9*: $(x,y) \in r \implies \text{c-pair } x \ y \in \text{ce-rel-to-set } r$ *<proof>*

lemma *ce-rel-lm-10*: $x \in A \implies (\text{c-fst } x, \text{c-snd } x) \in \text{ce-set-to-rel } A$ *<proof>*

lemma *ce-rel-lm-11*: $\text{c-pair } x \ y \in \text{ce-rel-to-set } r \implies (x,y) \in r$
<proof>

lemma *ce-rel-lm-12*: $(\text{c-pair } x \ y \in \text{ce-rel-to-set } r) = ((x,y) \in r)$
<proof>

lemma *ce-rel-lm-13*: $(x,y) \in \text{ce-set-to-rel } A \implies \text{c-pair } x \ y \in A$
<proof>

lemma *ce-rel-lm-14*: $\text{c-pair } x \ y \in A \implies (x,y) \in \text{ce-set-to-rel } A$
<proof>

lemma *ce-rel-lm-15*: $((x,y) \in \text{ce-set-to-rel } A) = (\text{c-pair } x \ y \in A)$
<proof>

lemma *ce-rel-lm-16*: $x \in \text{ce-rel-to-set } r \implies (\text{c-fst } x, \text{c-snd } x) \in r$
<proof>

lemma *ce-rel-lm-17*: $(\text{c-fst } x, \text{c-snd } x) \in \text{ce-set-to-rel } A \implies x \in A$
<proof>

lemma *ce-rel-lm-18*: $((\text{c-fst } x, \text{c-snd } x) \in \text{ce-set-to-rel } A) = (x \in A)$
<proof>

lemma *ce-rel-lm-19*: $(\text{c-fst } x, \text{c-snd } x) \in r \implies x \in \text{ce-rel-to-set } r$

<proof>

lemma *ce-rel-lm-20*: $((c\text{-fst } x, c\text{-snd } x) \in r) = (x \in ce\text{-rel-to-set } r)$
<proof>

lemma *ce-rel-lm-21*: $r \in ce\text{-rels} \implies \exists p \in PrimRec3. \forall x y. ((x,y) \in r) = (\exists u. p\ x\ y\ u = 0)$
<proof>

lemma *ce-rel-lm-22*: $r \in ce\text{-rels} \implies \exists p \in PrimRec3. r = \{ (x,y). \exists u. p\ x\ y\ u = 0 \}$
<proof>

lemma *ce-rel-lm-23*: $\llbracket p \in PrimRec3; \forall x y. ((x,y) \in r) = (\exists u. p\ x\ y\ u = 0) \rrbracket \implies r \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-24*: $\llbracket r \in ce\text{-rels}; s \in ce\text{-rels} \rrbracket \implies s\ O\ r \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-25*: $r \in ce\text{-rels} \implies r^{\wedge-1} \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-26*: $r \in ce\text{-rels} \implies Domain\ r \in ce\text{-sets}$
<proof>

lemma *ce-rel-lm-27*: $r \in ce\text{-rels} \implies Range\ r \in ce\text{-sets}$
<proof>

lemma *ce-rel-lm-28*: $r \in ce\text{-rels} \implies Field\ r \in ce\text{-sets}$
<proof>

lemma *ce-rel-lm-29*: $\llbracket A \in ce\text{-sets}; B \in ce\text{-sets} \rrbracket \implies A \times B \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-30*: $\{\} \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-31*: $UNIV \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-32*: $ce\text{-rel-to-set } (r \cup s) = (ce\text{-rel-to-set } r) \cup (ce\text{-rel-to-set } s)$
<proof>

lemma *ce-rel-lm-33*: $\llbracket r \in ce\text{-rels}; s \in ce\text{-rels} \rrbracket \implies r \cup s \in ce\text{-rels}$
<proof>

lemma *ce-rel-lm-34*: $ce\text{-rel-to-set } (r \cap s) = (ce\text{-rel-to-set } r) \cap (ce\text{-rel-to-set } s)$
<proof>

lemma *ce-rel-lm-35*: $\llbracket r \in \text{ce-rels}; s \in \text{ce-rels} \rrbracket \implies r \cap s \in \text{ce-rels}$
 ⟨proof⟩

lemma *ce-rel-lm-36*: $\text{ce-set-to-rel } (A \cup B) = (\text{ce-set-to-rel } A) \cup (\text{ce-set-to-rel } B)$
 ⟨proof⟩

lemma *ce-rel-lm-37*: $\text{ce-set-to-rel } (A \cap B) = (\text{ce-set-to-rel } A) \cap (\text{ce-set-to-rel } B)$
 ⟨proof⟩

lemma *ce-rel-lm-38*: $\llbracket r \in \text{ce-rels}; A \in \text{ce-sets} \rrbracket \implies r^{\text{``}}A \in \text{ce-sets}$
 ⟨proof⟩

7.6 Total computable functions

definition

graph :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow (\text{nat} \times \text{nat}) \text{ set}$ **where**
graph = $(\lambda f. \{ (x, f x) \mid x. x \in \text{UNIV} \})$

lemma *graph-lm-1*: $(x, y) \in \text{graph } f \implies y = f x$ ⟨proof⟩

lemma *graph-lm-2*: $y = f x \implies (x, y) \in \text{graph } f$ ⟨proof⟩

lemma *graph-lm-3*: $((x, y) \in \text{graph } f) = (y = f x)$ ⟨proof⟩

lemma *graph-lm-4*: $\text{graph } (f \circ g) = (\text{graph } g) \circ (\text{graph } f)$ ⟨proof⟩

definition

c-graph :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{nat set}$ **where**
c-graph = $(\lambda f. \{ \text{c-pair } x (f x) \mid x. x \in \text{UNIV} \})$

lemma *c-graph-lm-1*: $\text{c-pair } x y \in \text{c-graph } f \implies y = f x$
 ⟨proof⟩

lemma *c-graph-lm-2*: $y = f x \implies \text{c-pair } x y \in \text{c-graph } f$ ⟨proof⟩

lemma *c-graph-lm-3*: $(\text{c-pair } x y \in \text{c-graph } f) = (y = f x)$
 ⟨proof⟩

lemma *c-graph-lm-4*: $\text{c-graph } f = \text{ce-rel-to-set } (\text{graph } f)$ ⟨proof⟩

lemma *c-graph-lm-5*: $\text{graph } f = \text{ce-set-to-rel } (\text{c-graph } f)$ ⟨proof⟩

definition

total-recursive :: $(\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{bool}$ **where**
total-recursive = $(\lambda f. \text{graph } f \in \text{ce-rels})$

lemma *total-recursive-def1*: $\text{total-recursive} = (\lambda f. \text{c-graph } f \in \text{ce-sets})$
 ⟨proof⟩

theorem *pr-is-total-rec*: $f \in \text{PrimRec1} \implies \text{total-recursive } f$
<proof>

theorem *comp-tot-rec*: $\llbracket \text{total-recursive } f; \text{total-recursive } g \rrbracket \implies \text{total-recursive } (f \circ g)$
<proof>

lemma *univ-for-pr-tot-rec-lm*: $c\text{-graph } \text{univ-for-pr} \in \text{ce-sets}$
<proof>

theorem *univ-for-pr-tot-rec*: $\text{total-recursive } \text{univ-for-pr}$
<proof>

7.7 Computable sets, Post's theorem

definition

computable :: *nat set* \implies *bool* **where**
computable = $(\lambda A. A \in \text{ce-sets} \wedge \neg A \in \text{ce-sets})$

lemma *computable-complement-1*: $\text{computable } A \implies \text{computable } (\neg A)$
<proof>

lemma *computable-complement-2*: $\text{computable } (\neg A) \implies \text{computable } A$
<proof>

lemma *computable-complement-3*: $(\text{computable } A) = (\text{computable } (\neg A))$ *<proof>*

theorem *comp-impl-tot-rec*: $\text{computable } A \implies \text{total-recursive } (\text{chf } A)$
<proof>

theorem *tot-rec-impl-comp*: $\text{total-recursive } (\text{chf } A) \implies \text{computable } A$
<proof>

theorem *post-th-0*: $(\text{computable } A) = (\text{total-recursive } (\text{chf } A))$
<proof>

7.8 Universal computably enumerable set

definition

univ-ce :: *nat set* **where**
univ-ce = $\{ c\text{-pair } n \ x \mid n \ x. x \in \text{nat-to-ce-set } n \}$

lemma *univ-for-pr-lm*: $\text{univ-for-pr } (c\text{-pair } n \ x) = (\text{nat-to-pr } n) \ x$
<proof>

theorem *univ-is-ce*: $\text{univ-ce} \in \text{ce-sets}$
<proof>

lemma *univ-ce-lm-1*: $(c\text{-pair } n \ x \in \text{univ-ce}) = (x \in \text{nat-to-ce-set } n)$

<proof>

theorem *univ-ce-is-not-comp1*: $\neg \text{univ-ce} \notin \text{ce-sets}$
<proof>

theorem *univ-ce-is-not-comp2*: $\neg \text{total-recursive} (\text{chf univ-ce})$
<proof>

theorem *univ-ce-is-not-comp3*: $\neg \text{computable univ-ce}$
<proof>

7.9 s-1-1 theorem, one-one and many-one reducibilities

definition

index-of-r-to-l :: *nat* **where**
index-of-r-to-l =
pair-by-index
(*pair-by-index index-of-c-fst (comp-by-index index-of-c-fst index-of-c-snd)*)
(*comp-by-index index-of-c-snd index-of-c-snd*)

lemma *index-of-r-to-l-lm*: $\text{nat-to-pr } \text{index-of-r-to-l} (\text{c-pair } x (\text{c-pair } y z)) = \text{c-pair} (\text{c-pair } x y) z$
<proof>

definition

s-ce :: *nat* \Rightarrow *nat* \Rightarrow *nat* **where**
s-ce == ($\lambda e x. \text{s1-1} (\text{comp-by-index } e \text{ index-of-r-to-l}) x$)

lemma *s-ce-is-pr*: $s\text{-ce} \in \text{PrimRec2}$
<proof>

lemma *s-ce-inj*: $s\text{-ce } e1 x1 = s\text{-ce } e2 x2 \implies e1=e2 \wedge x1=x2$
<proof>

lemma *s-ce-inj1*: $s\text{-ce } e1 x = s\text{-ce } e2 x \implies e1=e2$
<proof>

lemma *s-ce-inj2*: $s\text{-ce } e x1 = s\text{-ce } e x2 \implies x1=x2$
<proof>

theorem *s1-1-th1*: $\forall n x y. ((\text{nat-to-pr } n) (\text{c-pair } x y)) = (\text{nat-to-pr } (\text{s1-1 } n x)) y$
<proof>

lemma *s-lm*: $(\text{nat-to-pr } (s\text{-ce } e x)) (\text{c-pair } y z) = (\text{nat-to-pr } e) (\text{c-pair } (\text{c-pair } x y) z)$
<proof>

theorem *s-ce-1-1-th*: $(\text{c-pair } x y \in \text{nat-to-ce-set } e) = (y \in \text{nat-to-ce-set } (s\text{-ce } e x))$
<proof>

definition

one-reducible-to-via :: (nat set) \Rightarrow (nat set) \Rightarrow (nat \Rightarrow nat) \Rightarrow bool **where**
one-reducible-to-via = (λ A B f. total-recursive f \wedge inj f \wedge (\forall x. (x \in A) = (f x \in B)))

definition

one-reducible-to :: (nat set) \Rightarrow (nat set) \Rightarrow bool **where**
one-reducible-to = (λ A B. \exists f. *one-reducible-to-via* A B f)

definition

many-reducible-to-via :: (nat set) \Rightarrow (nat set) \Rightarrow (nat \Rightarrow nat) \Rightarrow bool **where**
many-reducible-to-via = (λ A B f. total-recursive f \wedge (\forall x. (x \in A) = (f x \in B)))

definition

many-reducible-to :: (nat set) \Rightarrow (nat set) \Rightarrow bool **where**
many-reducible-to = (λ A B. \exists f. *many-reducible-to-via* A B f)

lemma *one-reducible-to-via-trans*: \llbracket *one-reducible-to-via* A B f; *one-reducible-to-via* B C g $\rrbracket \Longrightarrow$ *one-reducible-to-via* A C (g o f)
 \langle proof \rangle

lemma *one-reducible-to-trans*: \llbracket *one-reducible-to* A B; *one-reducible-to* B C $\rrbracket \Longrightarrow$ *one-reducible-to* A C
 \langle proof \rangle

lemma *one-reducible-to-via-refl*: *one-reducible-to-via* A A (λ x. x)
 \langle proof \rangle

lemma *one-reducible-to-refl*: *one-reducible-to* A A
 \langle proof \rangle

lemma *many-reducible-to-via-trans*: \llbracket *many-reducible-to-via* A B f; *many-reducible-to-via* B C g $\rrbracket \Longrightarrow$ *many-reducible-to-via* A C (g o f)
 \langle proof \rangle

lemma *many-reducible-to-trans*: \llbracket *many-reducible-to* A B; *many-reducible-to* B C $\rrbracket \Longrightarrow$ *many-reducible-to* A C
 \langle proof \rangle

lemma *one-reducibility-via-is-many*: *one-reducible-to-via* A B f \Longrightarrow *many-reducible-to-via* A B f
 \langle proof \rangle

lemma *one-reducibility-is-many*: *one-reducible-to* A B \Longrightarrow *many-reducible-to* A B
 \langle proof \rangle

lemma *many-reducible-to-via-refl*: *many-reducible-to-via* A A (λ x. x)

<proof>

lemma *many-reducible-to-refl: many-reducible-to A A*
<proof>

theorem *m-red-to-comp: [many-reducible-to A B; computable B] \implies computable A*
<proof>

lemma *many-reducible-lm-1: many-reducible-to univ-ce A \implies \neg computable A*
<proof>

lemma *one-reducible-lm-1: one-reducible-to univ-ce A \implies \neg computable A*
<proof>

lemma *one-reducible-lm-2: one-reducible-to-via (nat-to-ce-set n) univ-ce ($\lambda x. c\text{-pair } n x$)*
<proof>

lemma *one-reducible-lm-3: one-reducible-to (nat-to-ce-set n) univ-ce*
<proof>

lemma *one-reducible-lm-4: A \in ce-sets \implies one-reducible-to A univ-ce*
<proof>

7.10 One-complete sets

definition

one-complete :: nat set \implies bool **where**
one-complete = ($\lambda A. A \in ce\text{-sets} \wedge (\forall B. B \in ce\text{-sets} \longrightarrow one\text{-reducible-to } B A)$)

theorem *univ-is-complete: one-complete univ-ce*
<proof>

7.11 Index sets, Rice's theorem

definition

index-set :: nat set \implies bool **where**
index-set = ($\lambda A. \forall n m. n \in A \wedge (nat\text{-to-ce-set } n = nat\text{-to-ce-set } m) \longrightarrow m \in A$)

lemma *index-set-lm-1: [index-set A; n \in A; nat-to-ce-set n = nat-to-ce-set m] \implies m \in A*
<proof>

lemma *index-set-lm-2: index-set A \implies index-set ($\neg A$)*
<proof>

lemma *Rice-lm-1*: $\llbracket \text{index-set } A; A \neq \{\}; A \neq UNIV; \exists n \in A. \text{nat-to-ce-set } n = \{\} \rrbracket \implies \text{one-reducible-to univ-ce } (\neg A)$
<proof>

lemma *Rice-lm-2*: $\llbracket \text{index-set } A; A \neq \{\}; A \neq UNIV; n \in A; \text{nat-to-ce-set } n = \{\} \rrbracket \implies \text{one-reducible-to univ-ce } (\neg A)$
<proof>

theorem *Rice-1*: $\llbracket \text{index-set } A; A \neq \{\}; A \neq UNIV \rrbracket \implies \text{one-reducible-to univ-ce } A \vee \text{one-reducible-to univ-ce } (\neg A)$
<proof>

theorem *Rice-2*: $\llbracket \text{index-set } A; A \neq \{\}; A \neq UNIV \rrbracket \implies \neg \text{computable } A$
<proof>

theorem *Rice-3*: $\llbracket C \subseteq \text{ce-sets}; \text{computable } \{ n. \text{nat-to-ce-set } n \in C \} \rrbracket \implies C = \{\} \vee C = \text{ce-sets}$
<proof>

end

References

- [1] Rogers. *Theory of recursive functions and effective computability*. 1967.
- [2] Soare. *Recursively enumerable sets and degrees*. 1987.