

Implementing field extensions of the form $\mathbb{Q}[\sqrt{b}]^*$

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Abstract

We apply data refinement to implement the real numbers, where we support all numbers in the field extension $\mathbb{Q}[\sqrt{b}]$, i.e., all numbers of the form $p + q\sqrt{b}$ for rational numbers p and q and some fixed natural number b . To this end, we also developed algorithms to precisely compute roots of a rational number, and to perform a factorization of natural numbers which eliminates duplicate prime factors.

Our results have been used to certify termination proofs which involve polynomial interpretations over the reals.

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1 Introduction

It has been shown that polynomial interpretations over the reals are strictly more powerful for termination proving than polynomial interpretations over the rationals. To this end, also automated termination prover started to generate such interpretations. [3, 4, 5, 7, 8]. However, for all current implementations, only reals of the form $p + q \cdot \sqrt{b}$ are generated where b is some fixed natural number and p and q may be arbitrary rationals, i.e., we get numbers within $\mathbb{Q}[\sqrt{b}]$.

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To support these termination proofs in our certifier CeTA [6], we therefore required executable functions on $\mathbb{Q}[\sqrt{b}]$, which can then be used as an implementation type for the reals. Here, we used ideas from [1, 2] to provide a sufficiently powerful partial implementations via data refinement.

2 Auxiliary lemmas which might be moved into the Isabelle distribution.

```

theory Real-Impl-Auxiliary
imports
  HOL-Computational-Algebra.Primes
begin

lemma multiplicity-prime:
  assumes p: prime (i :: nat) and ji: j ≠ i
  shows multiplicity j i = 0
  <proof>

end

```

3 Prime products

```

theory Prime-Product
imports
  Real-Impl-Auxiliary
  Sqrt-Babylonian.Sqrt-Babylonian
begin

```

Prime products are natural numbers where no prime factor occurs more than once.

```

definition prime-product
where prime-product (n :: nat) = ( $\forall$  p. prime p  $\longrightarrow$  multiplicity p n ≤ 1)

```

The main property is that whenever b_1 and b_2 are different prime products, then $p_1 + q_1\sqrt{b_1} = p_2 + q_2\sqrt{b_2}$ implies $(p_1, q_1, b_1) = (p_2, q_2, b_2)$ for all rational numbers p_1, q_1, p_2, q_2 . This is the key property to uniquely represent numbers in $\mathbb{Q}[\sqrt{b}]$ by triples. In the following we develop an algorithm to decompose any natural number n into $n = s^2 \cdot p$ for some s and prime product p .

```

function prime-product-factor-main :: nat ⇒ nat ⇒ nat ⇒ nat ⇒ nat ⇒ nat ×
nat where
  prime-product-factor-main factor-sq factor-pr limit n i =
    (if i ≤ limit ∧ i ≥ 2 then
      (if i dvd n
        then (let n' = n div i in
          (if i dvd n' then

```

```

      let n'' = n' div i in
      prime-product-factor-main (factor-sq * i) factor-pr (nat (root-nat-floor
3 n'')) n'' i
    else
      (case sqrt-nat n' of
        Cons sn - => (factor-sq * sn, factor-pr * i)
        | [] => prime-product-factor-main factor-sq (factor-pr * i) (nat
(root-nat-floor 3 n'^)) n' (Suc i)
      )
    )
  )
else
  prime-product-factor-main factor-sq factor-pr limit n (Suc i)
else
  (factor-sq, factor-pr * n) ⟨proof⟩

```

termination

⟨proof⟩

lemma *prime-product-factor-main*: **assumes** $\neg (\exists s. s * s = n)$

and $limit = nat (root-nat-floor 3 n)$

and $m = factor-sq * factor-sq * factor-pr * n$

and $prime-product-factor-main factor-sq factor-pr limit n i = (sq, p)$

and $i \geq 2$

and $\bigwedge j. j \geq 2 \implies j < i \implies \neg j \text{ dvd } n$

and $\bigwedge j. \text{prime } j \implies j < i \implies \text{multiplicity } j \text{ factor-pr} \leq 1$

and $\bigwedge j. \text{prime } j \implies j \geq i \implies \text{multiplicity } j \text{ factor-pr} = 0$

and $factor-pr > 0$

shows $m = sq * sq * p \wedge \text{prime-product } p$

⟨proof⟩

definition *prime-product-factor* :: $nat \Rightarrow nat \times nat$ **where**

prime-product-factor $n = (case \text{sqrt-nat } n \text{ of}$

$(Cons s -) \Rightarrow (s, 1)$

$| [] \Rightarrow \text{prime-product-factor-main } 1 \ 1 \ (nat (root-nat-floor 3 n)) \ n \ 2)$

lemma *prime-product-one*[*simp, intro*]: *prime-product* 1

⟨proof⟩

lemma *prime-product-factor*: **assumes** *pf*: *prime-product-factor* $n = (sq, p)$

shows $n = sq * sq * p \wedge \text{prime-product } p$

⟨proof⟩

end

4 A representation of real numbers via triples

theory *Real-Impl*

imports

Sqrt-Babylonian.Sqrt-Babylonian

begin

We represent real numbers of the form $p + q \cdot \sqrt{b}$ for $p, q \in \mathbb{Q}$, $n \in \mathbb{N}$ by triples (p, q, b) . However, we require the invariant that \sqrt{b} is irrational. Most binary operations are implemented via partial functions where the common the restriction is that the numbers b in both triples have to be identical. So, we support addition of $\sqrt{2} + \sqrt{2}$, but not $\sqrt{2} + \sqrt{3}$.

The set of natural numbers whose sqrt is irrational

definition *sqrt-irrat* = $\{ q :: \text{nat}. \neg (\exists p. p * p = \text{rat-of-nat } q) \}$

lemma *sqrt-irrat*: **assumes** *choice*: $q = 0 \vee b \in \text{sqrt-irrat}$

and *eq*: $\text{real-of-rat } p + \text{real-of-rat } q * \text{sqrt } (\text{of-nat } b) = 0$

shows $q = 0$

<proof>

To represent numbers of the form $p + q \cdot \sqrt{b}$, use mini algebraic numbers, i.e., triples (p, q, b) with irrational \sqrt{b} .

typedef *mini-alg* =

$\{(p, q, b) \mid (p :: \text{rat}) (q :: \text{rat}) (b :: \text{nat}).$

$q = 0 \vee b \in \text{sqrt-irrat}\}$

<proof>

setup-lifting *type-definition-mini-alg*

lift-definition *real-of* :: *mini-alg* \Rightarrow *real* **is**

$\lambda (p, q, b). \text{of-rat } p + \text{of-rat } q * \text{sqrt } (\text{of-nat } b)$ *<proof>*

lift-definition *ma-of-rat* :: *rat* \Rightarrow *mini-alg* **is** $\lambda x. (x, 0, 0)$ *<proof>*

lift-definition *ma-rat* :: *mini-alg* \Rightarrow *rat* **is** *fst* *<proof>*

lift-definition *ma-base* :: *mini-alg* \Rightarrow *nat* **is** *snd o snd* *<proof>*

lift-definition *ma-coeff* :: *mini-alg* \Rightarrow *rat* **is** *fst o snd* *<proof>*

lift-definition *ma-uminus* :: *mini-alg* \Rightarrow *mini-alg* **is**

$\lambda (p1, q1, b1). (- p1, - q1, b1)$ *<proof>*

lift-definition *ma-compatible* :: *mini-alg* \Rightarrow *mini-alg* \Rightarrow *bool* **is**

$\lambda (p1, q1, b1) (p2, q2, b2). q1 = 0 \vee q2 = 0 \vee b1 = b2$ *<proof>*

definition *ma-normalize* :: *rat* \times *rat* \times *nat* \Rightarrow *rat* \times *rat* \times *nat* **where**

ma-normalize $x \equiv \text{case } x \text{ of } (a, b, c) \Rightarrow \text{if } b = 0 \text{ then } (a, 0, 0) \text{ else } (a, b, c)$

lemma *ma-normalize-case[simp]*: $(\text{case } \text{ma-normalize } r \text{ of } (a, b, c) \Rightarrow \text{real-of-rat } a$

$+ \text{real-of-rat } b * \text{sqrt } (\text{of-nat } c))$

$= (\text{case } r \text{ of } (a, b, c) \Rightarrow \text{real-of-rat } a + \text{real-of-rat } b * \text{sqrt } (\text{of-nat } c))$

<proof>

lift-definition *ma-plus* :: *mini-alg* \Rightarrow *mini-alg* \Rightarrow *mini-alg* **is**

$\lambda (p1,q1,b1) (p2,q2,b2)$. if $q1 = 0$ then
 $(p1 + p2, q2, b2)$ else *ma-normalize* $(p1 + p2, q1 + q2, b1)$ \langle *proof* \rangle

lift-definition *ma-times* :: *mini-alg* \Rightarrow *mini-alg* \Rightarrow *mini-alg* **is**

$\lambda (p1,q1,b1) (p2,q2,b2)$. if $q1 = 0$ then
ma-normalize $(p1*p2, p1*q2, b2)$ else
ma-normalize $(p1*p2 + \text{of-nat } b2*q1*q2, p1*q2 + q1*p2, b1)$ \langle *proof* \rangle

lift-definition *ma-inverse* :: *mini-alg* \Rightarrow *mini-alg* **is**

$\lambda (p,q,b)$. let $d = \text{inverse } (p * p - \text{of-nat } b * q * q)$ in
ma-normalize $(p * d, - q * d, b)$ \langle *proof* \rangle

lift-definition *ma-floor* :: *mini-alg* \Rightarrow *int* **is**

$\lambda (p,q,b)$. case $(\text{quotient-of } p, \text{quotient-of } q)$ of $((z1,n1),(z2,n2)) \Rightarrow$
let $z2n1 = z2 * n1$; $z1n2 = z1 * n2$; $n12 = n1 * n2$; $\text{prod} = z2n1 * z2n1 * \text{int } b$ in
 $(z1n2 + (\text{if } z2n1 \geq 0 \text{ then } \text{sqrt-int-floor-pos } \text{prod} \text{ else } - \text{sqrt-int-ceiling-pos } \text{prod})) \text{ div } n12$ \langle *proof* \rangle

lift-definition *ma-sqrt* :: *mini-alg* \Rightarrow *mini-alg* **is**

$\lambda (p,q,b)$. let $(a,b) = \text{quotient-of } p$; $aa = \text{abs } (a * b)$ in
case $\text{sqrt-int } aa$ of $\square \Rightarrow (0, \text{inverse } (\text{of-int } b), \text{nat } aa) \mid (\text{Cons } s \text{ -}) \Rightarrow (\text{of-int } s / \text{of-int } b, 0, 0)$
 \langle *proof* \rangle

lift-definition *ma-equal* :: *mini-alg* \Rightarrow *mini-alg* \Rightarrow *bool* **is**

$\lambda (p1,q1,b1) (p2,q2,b2)$.
 $p1 = p2 \wedge q1 = q2 \wedge (q1 = 0 \vee b1 = b2)$ \langle *proof* \rangle

lift-definition *ma-ge-0* :: *mini-alg* \Rightarrow *bool* **is**

$\lambda (p,q,b)$. let $bqq = \text{of-nat } b * q * q$; $pp = p * p$ in
 $0 \leq p \wedge bqq \leq pp \vee 0 \leq q \wedge pp \leq bqq$ \langle *proof* \rangle

lift-definition *ma-is-rat* :: *mini-alg* \Rightarrow *bool* **is**

$\lambda (p,q,b)$. $q = 0$ \langle *proof* \rangle

definition *ge-0* :: *real* \Rightarrow *bool* **where** [code del]: $\text{ge-0 } x = (x \geq 0)$

lemma *ma-ge-0*: $\text{ge-0 } (\text{real-of } x) = \text{ma-ge-0 } x$
 \langle *proof* \rangle

lemma *ma-0*: $0 = \text{real-of } (\text{ma-of-rat } 0)$ \langle *proof* \rangle

lemma *ma-1*: $1 = \text{real-of } (\text{ma-of-rat } 1)$ \langle *proof* \rangle

lemma *ma-uminus*:

$- (\text{real-of } x) = \text{real-of } (\text{ma-uminus } x)$

<proof>

lemma *ma-inverse*: $\text{inverse} (\text{real-of } r) = \text{real-of} (\text{ma-inverse } r)$

<proof>

lemma *ma-sqrt-main*: $\text{ma-rat } r \geq 0 \implies \text{ma-coeff } r = 0 \implies \text{sqrt} (\text{real-of } r) = \text{real-of} (\text{ma-sqrt } r)$

<proof>

lemma *ma-sqrt*: $\text{sqrt} (\text{real-of } r) = (\text{if } \text{ma-coeff } r = 0 \text{ then } (\text{if } \text{ma-rat } r \geq 0 \text{ then } \text{real-of} (\text{ma-sqrt } r) \text{ else } - \text{real-of} (\text{ma-sqrt} (\text{ma-uminus } r)))$

$\text{else } \text{Code.abort} (\text{STR } \text{"cannot represent sqrt of irrational number"}) (\lambda -. \text{sqrt} (\text{real-of } r))$

<proof>

lemma *ma-plus*:

$(\text{real-of } r1 + \text{real-of } r2) = (\text{if } \text{ma-compatible } r1 \ r2$

$\text{then } \text{real-of} (\text{ma-plus } r1 \ r2) \text{ else}$

$\text{Code.abort} (\text{STR } \text{"different base"}) (\lambda -. \text{real-of } r1 + \text{real-of } r2)$

<proof>

lemma *ma-times*:

$(\text{real-of } r1 * \text{real-of } r2) = (\text{if } \text{ma-compatible } r1 \ r2$

$\text{then } \text{real-of} (\text{ma-times } r1 \ r2) \text{ else}$

$\text{Code.abort} (\text{STR } \text{"different base"}) (\lambda -. \text{real-of } r1 * \text{real-of } r2)$

<proof>

lemma *ma-equal*:

$\text{HOL.equal} (\text{real-of } r1) (\text{real-of } r2) = (\text{if } \text{ma-compatible } r1 \ r2$

$\text{then } \text{ma-equal } r1 \ r2 \text{ else}$

$\text{Code.abort} (\text{STR } \text{"different base"}) (\lambda -. \text{HOL.equal} (\text{real-of } r1) (\text{real-of } r2))$

<proof>

lemma *ma-floor*: $\text{floor} (\text{real-of } r) = \text{ma-floor } r$

<proof>

lemma *comparison-impl*:

$(x :: \text{real}) \leq (y :: \text{real}) = \text{ge-0} (y - x)$

$(x :: \text{real}) < (y :: \text{real}) = (x \neq y \wedge \text{ge-0} (y - x))$

<proof>

lemma *ma-of-rat*: $\text{real-of-rat } r = \text{real-of} (\text{ma-of-rat } r)$

<proof>

definition *is-rat* :: $\text{real} \Rightarrow \text{bool}$ **where**

$[\text{code-abbrev}]: \text{is-rat } x \longleftrightarrow x \in \mathbb{Q}$

lemma *ma-is-rat*: $\text{is-rat} (\text{real-of } x) = \text{ma-is-rat } x$

<proof>

definition *sqrt-real* $x = (\text{if } x \in \mathbb{Q} \wedge x \geq 0 \text{ then } (\text{if } x = 0 \text{ then } [0] \text{ else } (\text{let } sx = \text{sqrt } x \text{ in } [sx, -sx])) \text{ else } [])$

lemma *sqrt-real[simp]*: **assumes** $x: x \in \mathbb{Q}$
shows $\text{set } (\text{sqrt-real } x) = \{y . y * y = x\}$
<proof>

code-datatype *real-of*

declare *[[code drop:*
 plus :: real \Rightarrow real \Rightarrow real
 uminus :: real \Rightarrow real
 times :: real \Rightarrow real \Rightarrow real
 inverse :: real \Rightarrow real
 floor :: real \Rightarrow int
 sqrt
 HOL.equal :: real \Rightarrow real \Rightarrow bool
]]

lemma *[code]*:
 Ratreal = real-of \circ ma-of-rat
<proof>

lemmas *ma-code-eqns* *[code equation] = ma-ge-0 ma-floor ma-0 ma-1 ma-uminus*
ma-inverse ma-sqrt ma-plus ma-times ma-equal ma-is-rat
comparison-impl

lemma *[code equation]*:
 *(x :: real) / (y :: real) = x * inverse y*
 (x :: real) - (y :: real) = x + (- y)
<proof>

Some tests with small numbers. To work on larger number, one should additionally import the theories for efficient calculation on numbers

value *[101.1 * (3 * sqrt 2 + 6 * sqrt 0.5)]*
value *[606.2 * sqrt 2 + 0.001]*
value *101.1 * (3 * sqrt 2 + 6 * sqrt 0.5) = 606.2 * sqrt 2 + 0.001*
value *101.1 * (3 * sqrt 2 + 6 * sqrt 0.5) > 606.2 * sqrt 2 + 0.001*
value *(sqrt 0.1 \in \mathbb{Q} , sqrt (- 0.09) \in \mathbb{Q})*

end

5 A unique representation of real numbers via triples

theory *Real-Unique-Impl*
imports

Prime-Product
Real-Impl
Show.Show-Instances
Show.Show-Real

begin

We implement the real numbers again using triples, but now we require an additional invariant on the triples, namely that the base has to be a prime product. This has the consequence that the mapping of triples into \mathbb{R} is injective. Hence, equality on reals is now equality on triples, which can even be executed in case of different bases. Similarly, we now also allow different basis in comparisons. Ultimately, injectivity allows us to define a show-function for real numbers, which pretty prints real numbers into strings.

typedef *mini-alg-unique* =
 { *r* :: *mini-alg* . *ma-coeff* *r* = 0 \wedge *ma-base* *r* = 0 \vee *ma-coeff* *r* \neq 0 \wedge *prime-product* (*ma-base* *r*) }
 <*proof*>

setup-lifting *type-definition-mini-alg-unique*

lift-definition *real-of-u* :: *mini-alg-unique* \Rightarrow *real* **is** *real-of* <*proof*>
lift-definition *mau-floor* :: *mini-alg-unique* \Rightarrow *int* **is** *ma-floor* <*proof*>
lift-definition *mau-of-rat* :: *rat* \Rightarrow *mini-alg-unique* **is** *ma-of-rat* <*proof*>
lift-definition *mau-rat* :: *mini-alg-unique* \Rightarrow *rat* **is** *ma-rat* <*proof*>
lift-definition *mau-base* :: *mini-alg-unique* \Rightarrow *nat* **is** *ma-base* <*proof*>
lift-definition *mau-coeff* :: *mini-alg-unique* \Rightarrow *rat* **is** *ma-coeff* <*proof*>
lift-definition *mau-uminus* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* **is** *ma-uminus* <*proof*>
lift-definition *mau-compatible* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* \Rightarrow *bool* **is** *ma-compatible* <*proof*>
lift-definition *mau-ge-0* :: *mini-alg-unique* \Rightarrow *bool* **is** *ma-ge-0* <*proof*>
lift-definition *mau-inverse* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* **is** *ma-inverse* <*proof*>
lift-definition *mau-plus* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* \Rightarrow *mini-alg-unique* **is** *ma-plus* <*proof*>
lift-definition *mau-times* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* \Rightarrow *mini-alg-unique* **is** *ma-times* <*proof*>
lift-definition *ma-identity* :: *mini-alg* \Rightarrow *mini-alg* \Rightarrow *bool* **is** (=) <*proof*>
lift-definition *mau-equal* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* \Rightarrow *bool* **is** *ma-identity* <*proof*>
lift-definition *mau-is-rat* :: *mini-alg-unique* \Rightarrow *bool* **is** *ma-is-rat* <*proof*>

lemma *Ratreal-code*[*code*]:

Ratreal = *real-of-u* \circ *mau-of-rat*
 <*proof*>

lemma *mau-floor*: $\text{floor} (\text{real-of-u } r) = \text{mau-floor } r$
 ⟨*proof*⟩

lemma *mau-inverse*: $\text{inverse} (\text{real-of-u } r) = \text{real-of-u} (\text{mau-inverse } r)$
 ⟨*proof*⟩

lemma *mau-uminus*: $- (\text{real-of-u } r) = \text{real-of-u} (\text{mau-uminus } r)$
 ⟨*proof*⟩

lemma *mau-times*:
 $(\text{real-of-u } r1 * \text{real-of-u } r2) = (\text{if } \text{mau-compatible } r1 \ r2$
 $\text{then } \text{real-of-u} (\text{mau-times } r1 \ r2) \ \text{else}$
 $\text{Code.abort (STR "different base")} (\lambda -. \text{real-of-u } r1 * \text{real-of-u } r2))$
 ⟨*proof*⟩

lemma *mau-plus*:
 $(\text{real-of-u } r1 + \text{real-of-u } r2) = (\text{if } \text{mau-compatible } r1 \ r2$
 $\text{then } \text{real-of-u} (\text{mau-plus } r1 \ r2) \ \text{else}$
 $\text{Code.abort (STR "different base")} (\lambda -. \text{real-of-u } r1 + \text{real-of-u } r2))$
 ⟨*proof*⟩

lemma *real-of-u-inj[simp]*: $\text{real-of-u } x = \text{real-of-u } y \longleftrightarrow x = y$
 ⟨*proof*⟩

lift-definition *mau-sqrt* :: *mini-alg-unique* \Rightarrow *mini-alg-unique* **is**
 $\lambda \text{ ma. let } (a,b) = \text{quotient-of} (\text{ma-rat } \text{ma}); (\text{sq, fact}) = \text{prime-product-factor}$
 $(\text{nat } (\text{abs } a * b));$
 $\text{ma}' = \text{ma-of-rat} (\text{of-int } (\text{sgn}(a)) * \text{of-nat } \text{sq} / \text{of-int } b)$
 $\text{in } \text{ma-times } \text{ma}' (\text{ma-sqrt} (\text{ma-of-rat} (\text{of-nat } \text{fact})))$
 ⟨*proof*⟩

lemma *sqrt-sgn[simp]*: $\text{sqrt} (\text{of-int } (\text{sgn } a)) = \text{of-int } (\text{sgn } a)$
 ⟨*proof*⟩

lemma *mau-sqrt-main*: $\text{mau-coeff } r = 0 \implies \text{sqrt} (\text{real-of-u } r) = \text{real-of-u} (\text{mau-sqrt } r)$
 ⟨*proof*⟩

lemma *mau-sqrt*: $\text{sqrt} (\text{real-of-u } r) = (\text{if } \text{mau-coeff } r = 0 \ \text{then}$
 $\text{real-of-u} (\text{mau-sqrt } r)$
 $\text{else } \text{Code.abort (STR "cannot represent sqrt of irrational number")} (\lambda -. \text{sqrt}$
 $(\text{real-of-u } r))$
 ⟨*proof*⟩

lemma *mau-0*: $0 = \text{real-of-u} (\text{mau-of-rat } 0)$ ⟨*proof*⟩

lemma *mau-1*: $1 = \text{real-of-u} (\text{mau-of-rat } 1)$ ⟨*proof*⟩

lemma *mau-equal*:
 $\text{HOL.equal} (\text{real-of-u } r1) (\text{real-of-u } r2) = \text{mau-equal } r1 \ r2$ ⟨*proof*⟩

lemma *mau-ge-0*: $\text{ge-0} (\text{real-of-u } x) = \text{mau-ge-0 } x$ ⟨*proof*⟩

definition *real-lt* :: *real* \Rightarrow *real* \Rightarrow *bool* **where** *real-lt* = (<)

The following code equation terminates if it is started on two different inputs.

lemma *real-lt* [code equation]: *real-lt* *x y* = (let *fx* = *floor x*; *fy* = *floor y* in
 (if *fx* < *fy* then *True* else if *fx* > *fy* then *False* else *real-lt* (*x* * 1024) (*y* * 1024)))
 <proof>

For comparisons we first check for equality. Then, if the bases are compatible we can just compare the differences with 0. Otherwise, we start the recursive algorithm *real-lt* which works on arbitrary bases. In this way, we have an implementation of comparisons which can compare all representable numbers.

Note that in *Real-Impl.Real-Impl* we did not use *real-lt* as there the code-equations for equality already require identical bases.

lemma *comparison-impl*:

real-of-u x \leq *real-of-u y* \iff *real-of-u x* = *real-of-u y* \vee
 (if *mau-compatible x y* then *ge-0* (*real-of-u y* - *real-of-u x*) else *real-lt* (*real-of-u x*) (*real-of-u y*))
real-of-u x < *real-of-u y* \iff *real-of-u x* \neq *real-of-u y* \wedge
 (if *mau-compatible x y* then *ge-0* (*real-of-u y* - *real-of-u x*) else *real-lt* (*real-of-u x*) (*real-of-u y*))
 <proof>

lemma *mau-is-rat*: *is-rat* (*real-of-u x*) = *mau-is-rat x* <proof>

lift-definition *ma-show-real* :: *mini-alg* \Rightarrow *string* **is**

λ (*p,q,b*). let *sb* = *shows "sqrt("* \circ *shows b* \circ *shows ")"*;
qb = (if *q* = 1 then *sb* else if *q* = -1 then *shows "-"* \circ *sb* else *shows q* \circ
shows ""* \circ *sb*) in
 if *q* = 0 then *shows p* [] else
 if *p* = 0 then *qb* [] else
 if *q* < 0 then ((*shows p* \circ *qb*) [])
 else ((*shows p* \circ *shows "+"* \circ *qb*) []) <proof>

lift-definition *mau-show-real* :: *mini-alg-unique* \Rightarrow *string* **is** *ma-show-real* <proof>

overloading *show-real* \equiv *show-real*

begin

definition *show-real*

where *show-real x* \equiv

(if (\exists *y*. *x* = *real-of-u y*) then *mau-show-real* (*THE y*. *x* = *real-of-u y*) else
 [])

end

lemma *mau-show-real*: *show-real* (*real-of-u x*) = *mau-show-real x*
 <proof>

code-datatype *real-of-u*

```

declare [[code drop:
  plus :: real ⇒ real ⇒ real
  uminus :: real ⇒ real
  times :: real ⇒ real ⇒ real
  inverse :: real ⇒ real
  floor :: real ⇒ int
  sqrt
  HOL.equal :: real ⇒ real ⇒ bool
  ge-0
  is-rat
  less :: real ⇒ real ⇒ bool
  less-eq :: real ⇒ real ⇒ bool
]]

```

```

lemmas mau-code-eqns [code] = mau-floor mau-0 mau-1 mau-uminus mau-inverse
mau-sqrt mau-plus mau-times mau-equal mau-ge-0 mau-is-rat
mau-show-real comparison-impl

```

Some tests with small numbers. To work on larger number, one should additionally import the theories for efficient calculation on numbers

```

value [101.1 * (sqrt 18 + 6 * sqrt 0.5)]
value [324 * sqrt 7 + 0.001]
value 101.1 * (sqrt 18 + 6 * sqrt 0.5) = 324 * sqrt 7 + 0.001
value 101.1 * (sqrt 18 + 6 * sqrt 0.5) > 324 * sqrt 7 + 0.001
value show (101.1 * (sqrt 18 + 6 * sqrt 0.5))
value (sqrt 0.1 ∈ ℚ, sqrt (- 0.09) ∈ ℚ)

```

end

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