Ramsey's Theorem

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

Contents

1 Ramsey's Theorem

theory *Ramsey*
imports *Main HOL−Library.Infinite-Set*
begin

declare [[*simp-depth-limit* = *5*]]

1.1 Library lemmas

lemma *infinite-inj-infinite-image*: *infinite* Z ==> *inj-on* f Z ==> *infinite* (f ` Z)
⟨*proof*⟩

lemma *infinite-dom-finite-rng*: [| *infinite* A; *finite* (f ` $A$) |] ==> ? $b$ : $f$ ` A. *infinite* {a : A. f a = b}
⟨*proof*⟩

lemma *infinite-mem*: *infinite* X ==> ? x. x : X
⟨*proof*⟩

lemma *not-empty-least*: (Y::*nat set*) ~= {} ==> ? m. m : Y & (! m'. m' : Y −−> m <= m')
⟨*proof*⟩

1.2 Dependent Choice Variant

—

primrec choice :: ('a => bool) => ('a => 'a => bool) => nat => 'a **where**
choice P R 0 = (SOME x. P x)
| choice P R (Suc n) = (let x = choice P R n in SOME y. P y & R x y)
—

lemma dc:
(! x y z. R x y & R y z --> R x z)
& (? x0. P x0)
& (! x. P x --> (? y. P y & R x y))
--> (? f::nat=>'b. (! n. P (f n)) & (! n m. R (f n) (f (n+m+1))))

⟨proof⟩

1.3 Partitions

definition
part :: nat => nat => 'a set => ('a set => nat) => bool **where**
part r s Y f = (! X. X <= Y & finite X & card X = r --> f X < s)

lemma part: [| infinite YY; part (Suc n) s YY f; yy : YY |] ==> part n s (YY − {yy}) (%u. f (insert yy u))
⟨proof⟩

lemma part-subset: part (Suc n) s YY f ==> Y <= YY ==> part (Suc n) s Y f
⟨proof⟩

1.4 Ramsey's theorem

lemma ramsey:
! (s::nat) (r::nat) (YY::'a set) (f::'a set => nat).
infinite YY
& (! X. X <= YY & finite X & card X = r --> f X < s)
--> (? Y' t'.
Y' <= YY
& infinite Y'
& t' < s
& (! X. X <= Y' & finite X & card X = r --> f X = t'))
⟨proof⟩

end