Ramsey's Theorem

Tom Ridge

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

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1 Ramsey's Theorem

theory *Ramsey*
imports *Main HOL−Library.Infinite-Set*
begin

declare [[*simp-depth-limit* = *5*]]

1.1 Library lemmas

lemma *infinite-inj-infinite-image*: $infinite\ Z \Longrightarrow inj\text{-}on\ f\ Z \Longrightarrow infinite\ (f\ `\ Z)$
⟨*proof*⟩

lemma *infinite-dom-finite-rng*: $[|\ infinite\ A;\ finite\ (f\ `\ A)\ |] ==> \exists\, b \in f\ `\ A.\ infinite\ \{a : A.\ f\ a = b\}$
⟨*proof*⟩

lemma *infinite-mem*: $infinite\ X \Longrightarrow \exists\, x.\ x \in X$
⟨*proof*⟩

lemma *not-empty-least*: $(Y::nat\ set) \neq \{\} \Longrightarrow \exists\, m.\ m \in Y \wedge (\forall\, m'.\ m' \in Y \longrightarrow m \leq m')$
⟨*proof*⟩

1.2 Dependent Choice Variant

—

primrec *choice* $:: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow nat \Rightarrow 'a$ **where**
$choice\ P\ R\ 0 = (SOME\ x.\ P\ x)$
$|\ choice\ P\ R\ (Suc\ n) = (let\ x = choice\ P\ R\ n\ in\ SOME\ y.\ P\ y \wedge R\ x\ y)$
—

lemma *dc*:
$(\forall x\ y\ z.\ R\ x\ y \wedge R\ y\ z \longrightarrow R\ x\ z)$
$\wedge\ (\exists x0.\ P\ x0)$
$\wedge\ (\forall x.\ P\ x \longrightarrow (\exists y.\ P\ y \wedge R\ x\ y))$
$\longrightarrow (\exists f{::}nat{\Rightarrow}'b.\ (\forall n.\ P\ (f\ n)) \wedge (\forall n\ m.\ R\ (f\ n)\ (f\ (n{+}m{+}1))))$

$\langle proof \rangle$

1.3 Partitions

definition
$part :: nat \Rightarrow nat \Rightarrow 'a\ set \Rightarrow ('a\ set \Rightarrow nat) \Rightarrow bool$ **where**
$part\ r\ s\ Y\ f = (\forall X.\ X \subseteq Y \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X < s)$

lemma *part*: $[|\ infinite\ YY;\ part\ (Suc\ n)\ s\ YY\ f;\ yy : YY\ |] ==> part\ n\ s\ (YY - \{yy\})\ (\lambda u.\ f\ (insert\ yy\ u))$
$\langle proof \rangle$

lemma *part-subset*: $part\ (Suc\ n)\ s\ YY\ f \Longrightarrow Y \subseteq YY \Longrightarrow part\ (Suc\ n)\ s\ Y\ f$
$\langle proof \rangle$

1.4 Ramsey's theorem

lemma *ramsey*:
$\forall (s{::}nat)\ (r{::}nat)\ (YY{::}'a\ set)\ (f{::}'a\ set \Rightarrow nat).$
$infinite\ YY$
$\wedge\ (\forall X.\ X \subseteq YY \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X < s)$
$\longrightarrow (\exists Y'\ t'.$
$Y' \subseteq YY$
$\wedge\ infinite\ Y'$
$\wedge\ t' < s$
$\wedge\ (\forall X.\ X \subseteq Y' \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X = t'))$
$\langle proof \rangle$

end