

# Ramsey's Theorem

Tom Ridge

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## Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

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## 1 Ramsey's Theorem

```
theory Ramsey
imports Main HOL-Library.Infinite-Set
begin
```

```
declare [[simp-depth-limit = 5]]
```

### 1.1 Library lemmas

```
lemma infinite-inj-infinite-image: infinite Z  $\implies$  inj-on f Z  $\implies$  infinite (f ` Z)
  <proof>
```

```
lemma infinite-dom-finite-rng: [[ infinite A; finite (f ` A) ]]  $\implies$   $\exists b \in f ` A.$ 
infinite {a : A. f a = b}
  <proof>
```

```
lemma infinite-mem: infinite X  $\implies$   $\exists x. x \in X$ 
  <proof>
```

```
lemma not-empty-least: (Y::nat set)  $\neq$  {}  $\implies$   $\exists m. m \in Y \wedge (\forall m'. m' \in Y \longrightarrow$ 
m  $\leq$  m')
  <proof>
```

## 1.2 Dependent Choice Variant

**primrec** *choice* :: ('a ⇒ bool) ⇒ ('a ⇒ 'a ⇒ bool) ⇒ nat ⇒ 'a **where**  
  *choice* P R 0 = (SOME x. P x)  
| *choice* P R (Suc n) = (let x = *choice* P R n in SOME y. P y ∧ R x y)  
—

**lemma** *dc*:

(∀ x y z. R x y ∧ R y z ⟶ R x z)  
∧ (∃ x0. P x0)  
∧ (∀ x. P x ⟶ (∃ y. P y ∧ R x y))  
⟶ (∃ f::nat⇒'b. (∀ n. P (f n)) ∧ (∀ n m. R (f n) (f (n+m+1))))  
  
{*proof*}

## 1.3 Partitions

**definition**

*part* :: nat ⇒ nat ⇒ 'a set ⇒ ('a set ⇒ nat) ⇒ bool **where**  
*part* r s Y f = (∀ X. X ⊆ Y ∧ finite X ∧ card X = r ⟶ f X < s)

**lemma** *part*: [| infinite YY; *part* (Suc n) s YY f; yy : YY |] ==> *part* n s (YY  
– {yy}) (λu. f (insert yy u))  
{*proof*}

**lemma** *part-subset*: *part* (Suc n) s YY f ⟹ Y ⊆ YY ⟹ *part* (Suc n) s Y f  
{*proof*}

## 1.4 Ramsey's theorem

**lemma** *ramsey*:

∀ (s::nat) (r::nat) (YY::'a set) (f::'a set ⇒ nat).  
infinite YY  
∧ (∀ X. X ⊆ YY ∧ finite X ∧ card X = r ⟶ f X < s)  
⟶ (∃ Y' t'.  
  Y' ⊆ YY  
  ∧ infinite Y'  
  ∧ t' < s  
  ∧ (∀ X. X ⊆ Y' ∧ finite X ∧ card X = r ⟶ f X = t'))  
{*proof*}

**end**