Ramsey's Theorem

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

Contents

1 Ramsey's Theorem

theory *Ramsey*
imports *Main HOL−Library.Infinite-Set HOL−Library.Ramsey*

begin

Please note: this entire development has been updated and incorporated into *HOL−Library.Ramsey* above. Below, some of the results of the original development are linked to their current versions elsewhere in the Isabelle libraries.

1.1 Library lemmas

lemma *infinite-inj-infinite-image*: $\textit{infinite}\ Z \Longrightarrow \textit{inj-on}\ f\ Z \Longrightarrow \textit{infinite}\ (f\ `\ Z)$
⟨*proof*⟩

lemma *infinite-dom-finite-rng*: $[|\ \textit{infinite}\ A;\ \textit{finite}\ (f\ `\ A)\ |] ==> \exists\, b \in f\ `\ A.\ \textit{infinite}\ \{a : A.\ f\ a = b\}$
⟨*proof*⟩

lemma *infinite-mem*: $\textit{infinite}\ X \Longrightarrow \exists\, x.\ x \in X$
⟨*proof*⟩

lemma *not-empty-least*: $(Y{::}nat\ set) \neq \{\} \Longrightarrow \exists\, m.\ m \in Y \wedge (\forall\, m'.\ m' \in Y \longrightarrow m \leq m')$
  $\langle proof \rangle$

1.2 Dependent Choice Variant

lemma *dc*:
  assumes *trans*: *trans r*
    and *P0*: *P x0*
    and *Pstep*: $\bigwedge x.\ P\ x \Longrightarrow \exists\, y.\ P\ y \wedge (x,\ y) \in r$
  obtains $f :: nat \Rightarrow {'a}$ **where** $\bigwedge n.\ P\ (f\ n)$ **and** $\bigwedge n\ m.\ n < m \Longrightarrow (f\ n,\ f\ m) \in r$
  $\langle proof \rangle$

1.3 Ramsey's theorem

lemma *ramsey*: $\forall\, (s{::}nat)\ (r{::}nat)\ (YY{::}{'a}\ set)\ (f{::}{'a}\ set \Rightarrow nat).$
  $infinite\ YY$
  $\wedge\ (\forall\, X.\ X \subseteq YY \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X < s)$
  $\longrightarrow (\exists\, Y'\ t'.$
  $Y' \subseteq YY$
  $\wedge\ infinite\ Y'$
  $\wedge\ t' < s$
  $\wedge\ (\forall\, X.\ X \subseteq Y' \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X = t'))$
  $\langle proof \rangle$

end