

Ramsey's Theorem

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

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1 Ramsey's Theorem

```
theory Ramsey
imports Main HOL-Library.Infinite-Set
begin
```

```
declare [[simp-depth-limit = 5]]
```

1.1 Library lemmas

```
lemma infinite-inj-infinite-image: infinite Z  $\implies$  inj-on f Z  $\implies$  infinite (f ` Z)
  <proof>
```

```
lemma infinite-dom-finite-rng: [[ infinite A; finite (f ` A) ]]  $\implies$   $\exists b \in f ` A.$ 
infinite {a : A. f a = b}
  <proof>
```

```
lemma infinite-mem: infinite X  $\implies$   $\exists x. x \in X$ 
  <proof>
```

```
lemma not-empty-least: (Y::nat set)  $\neq$  {}  $\implies$   $\exists m. m \in Y \wedge (\forall m'. m' \in Y \longrightarrow$ 
m  $\leq m')$ 
  <proof>
```

1.2 Dependent Choice Variant

primrec *choice* :: ('a ⇒ bool) ⇒ ('a ⇒ 'a ⇒ bool) ⇒ nat ⇒ 'a **where**
 choice P R 0 = (SOME x. P x)
| *choice* P R (Suc n) = (let x = *choice* P R n in SOME y. P y ∧ R x y)
—

lemma *dc*:

(∀ x y z. R x y ∧ R y z ⟶ R x z)
∧ (∃ x 0. P x 0)
∧ (∀ x. P x ⟶ (∃ y. P y ∧ R x y))
⟶ (∃ f :: nat ⇒ 'b. (∀ n. P (f n)) ∧ (∀ n m. R (f n) (f (n+m+1))))

{*proof*}

1.3 Partitions

definition

part :: nat ⇒ nat ⇒ 'a set ⇒ ('a set ⇒ nat) ⇒ bool **where**
part r s Y f = (∀ X. X ⊆ Y ∧ finite X ∧ card X = r ⟶ f X < s)

lemma *part*: [| infinite YY; *part* (Suc n) s YY f; yy : YY |] ==> *part* n s (YY
– {yy}) (λu. f (insert yy u))
{*proof*}

lemma *part-subset*: *part* (Suc n) s YY f ⟹ Y ⊆ YY ⟹ *part* (Suc n) s Y f
{*proof*}

1.4 Ramsey's theorem

lemma *ramsey*:

∀ (s :: nat) (r :: nat) (YY :: 'a set) (f :: 'a set ⇒ nat).
infinite YY
∧ (∀ X. X ⊆ YY ∧ finite X ∧ card X = r ⟶ f X < s)
⟶ (∃ Y' t'.
 Y' ⊆ YY
 ∧ infinite Y'
 ∧ t' < s
 ∧ (∀ X. X ⊆ Y' ∧ finite X ∧ card X = r ⟶ f X = t'))
{*proof*}

end