

Ramsey's Theorem

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Abstract

The infinite form of Ramsey's Theorem is proved following Boolos and Jeffrey, Chapter 26.

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1 Ramsey's Theorem

theory *Ramsey*

imports *Main HOL-Library.Infinite-Set HOL-Library.Ramsey*

begin

Please note: this entire development has been updated and incorporated into *HOL-Library.Ramsey* above. Below, some of the results of the original development are linked to their current versions elsewhere in the Isabelle libraries.

1.1 Library lemmas

lemma *infinite-inj-infinite-image*: $\text{infinite } Z \implies \text{inj-on } f Z \implies \text{infinite } (f \text{ ` } Z)$
using *finite-imageD* **by** *blast*

lemma *infinite-dom-finite-rng*: $[[\text{infinite } A; \text{finite } (f \text{ ` } A)]] \implies \exists b \in f \text{ ` } A.$
infinite $\{a : A. f a = b\}$
by (*simp add: pigeonhole-infinite*)

lemma *infinite-mem*: $\text{infinite } X \implies \exists x. x \in X$
using *finite-insert* **by** *fastforce*

lemma not-empty-least: $(Y::\text{nat set}) \neq \{\} \implies \exists m. m \in Y \wedge (\forall m'. m' \in Y \longrightarrow m \leq m')$

by (*meson Inf-nat-def1 bdd-below-bot cInf-lower*)

1.2 Dependent Choice Variant

lemma dc:

assumes *trans:* $\text{trans } r$

and *P0:* $P x0$

and *Pstep:* $\bigwedge x. P x \implies \exists y. P y \wedge (x, y) \in r$

obtains $f :: \text{nat} \Rightarrow 'a$ **where** $\bigwedge n. P (f n)$ **and** $\bigwedge n m. n < m \implies (f n, f m) \in r$

by (*metis P0 Pstep dependent-choice local.trans*)

1.3 Ramsey's theorem

lemma ramsey: $\forall (s::\text{nat}) (r::\text{nat}) (YY::'a \text{ set}) (f::'a \text{ set} \Rightarrow \text{nat}).$

infinite YY

$\wedge (\forall X. X \subseteq YY \wedge \text{finite } X \wedge \text{card } X = r \longrightarrow f X < s)$

$\longrightarrow (\exists Y' t'.$

$Y' \subseteq YY$

$\wedge \text{infinite } Y'$

$\wedge t' < s$

$\wedge (\forall X. X \subseteq Y' \wedge \text{finite } X \wedge \text{card } X = r \longrightarrow f X = t')$)

using *Ramsey* **by** *fastforce*

end