

# An implementation of ROBDDs for Isabelle/HOL

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## Abstract

We present a verified and executable implementation of ROBDDs in Isabelle/HOL. Our implementation relates pointer-based computation in the Heap monad to operations on an abstract definition of boolean functions. Internally, we implemented the if-then-else combinator in a recursive fashion, following the Shannon decomposition of the argument functions. The implementation mixes and adapts known techniques and is built with efficiency in mind.

## Contents

<b>1</b>	<b>Preface</b>	<b>2</b>
<b>2</b>	<b>Boolean functions</b>	<b>2</b>
2.1	Shannon decomposition . . . . .	3
<b>3</b>	<b>Binary Decision Trees</b>	<b>3</b>
<b>4</b>	<b>Option Helpers</b>	<b>13</b>
<b>5</b>	<b>Abstract ITE Implementation</b>	<b>14</b>
<b>6</b>	<b>Pointermap</b>	<b>20</b>
<b>7</b>	<b>Functional interpretation for the abstract implementation</b>	<b>23</b>
<b>8</b>	<b>Array List</b>	<b>26</b>
<b>9</b>	<b>Imparative implementation for Pointermap</b>	<b>28</b>
<b>10</b>	<b>Imparative implementation</b>	<b>29</b>
10.1	A standard library of functions . . . . .	34
10.2	Printing . . . . .	35
<b>11</b>	<b>Collapsing the levels</b>	<b>37</b>

## 1 Preface

This work is not the first to deal with BDDs in Isabelle/HOL. Ortner and Schirmer have formalized BDDs in [4] and proved the correctness of an algorithm that transforms arbitrary BDDs to ROBDDs. However, their specification does not provide efficiently executable algorithms on BDDs. Giorgino and Strecker have presented efficiently executable algorithms for ROBDDs [2] by reducing their arguments to manipulating edges of graphs. However, they have, to the best of our knowledge, not made their theory files available. Thus, no library for efficient computation on (RO)BDDs in Isabelle/HOL existed. Our work is a response to that situation.

The theoretic background of the implementation is mostly based on [1].

## 2 Boolean functions

```
theory Bool-Func
imports Main
begin
```

The end result of our implementation is verified against these functions:

```
type-synonym 'a boolfunc = ('a  $\Rightarrow$  bool)  $\Rightarrow$  bool
```

if-then-else on boolean functions.

```
definition bf-ite i t e  $\equiv$  ( $\lambda$ l. if i l then t l else e l)
```

if-then-else is interesting because we can, together with constant true and false, represent all binary boolean functions using maximally two applications of it.

```
abbreviation bf-True  $\equiv$  ( $\lambda$ l. True)
```

```
abbreviation bf-False  $\equiv$  ( $\lambda$ l. False)
```

A quick demonstration:

```
definition bf-and a b  $\equiv$  bf-ite a b bf-False
```

```
lemma (bf-and a b) as  $\longleftrightarrow$  a as  $\wedge$  b as  $\langle$ proof $\rangle$ 
```

```
definition bf-not b  $\equiv$  bf-ite b bf-False bf-True
```

```
lemma bf-not-alt: bf-not a as  $\longleftrightarrow$   $\neg$ a as  $\langle$ proof $\rangle$ 
```

For convenience, we want a few functions more:

```
definition bf-or a b  $\equiv$  bf-ite a bf-True b
```

```
definition bf-lit v  $\equiv$  ( $\lambda$ l. l v)
```

```
definition bf-if v t e  $\equiv$  bf-ite (bf-lit v) t e
```

```
lemma bf-if-alt: bf-if v t e = ( $\lambda$ l. if l v then t l else e l)  $\langle$ proof $\rangle$ 
```

**definition**  $bf\text{-}nand\ a\ b = bf\text{-}not\ (bf\text{-}and\ a\ b)$   
**definition**  $bf\text{-}nor\ a\ b = bf\text{-}not\ (bf\text{-}or\ a\ b)$   
**definition**  $bf\text{-}biimp\ a\ b = (bf\text{-}ite\ a\ b\ (bf\text{-}not\ b))$   
**lemma**  $bf\text{-}biimp\text{-}alt: bf\text{-}biimp\ a\ b = (\lambda l. a\ l \longleftrightarrow b\ l)$  *<proof>*  
**definition**  $bf\text{-}xor\ a\ b = bf\text{-}not\ (bf\text{-}biimp\ a\ b)$   
**lemma**  $bf\text{-}xor\text{-}alt: bf\text{-}xor\ a\ b = (bf\text{-}ite\ a\ (bf\text{-}not\ b)\ b)$   
*<proof>*

All of these are implemented and had their implementation verified.

**definition**  $bf\text{-}imp\ a\ b = bf\text{-}ite\ a\ b\ bf\text{-}True$   
**lemma**  $bf\text{-}imp\text{-}alt: bf\text{-}imp\ a\ b = bf\text{-}or\ (bf\text{-}not\ a)\ b$  *<proof>*

**lemma**  $[dest!,elim!]: bf\text{-}False = bf\text{-}True \implies False\ bf\text{-}True = bf\text{-}False \implies False$   
*<proof>*

**lemmas**  $[simp] = bf\text{-}and\text{-}def\ bf\text{-}or\text{-}def\ bf\text{-}nand\text{-}def\ bf\text{-}biimp\text{-}def\ bf\text{-}xor\text{-}alt\ bf\text{-}nor\text{-}def\ bf\text{-}not\text{-}def$

## 2.1 Shannon decomposition

A restriction of a boolean function on a variable is creating the boolean function that evaluates as if that variable was set to a fixed value:

**definition**  $bf\text{-}restrict\ (i::'a)\ (val::bool)\ (f::'a\ boolfunc) \equiv (\lambda v. f\ (v(i:=val)))$

Restrictions are useful, because they remove variables from the set of significant variables:

**definition**  $bf\text{-}vars\ bf = \{v. \exists as. bf\text{-}restrict\ v\ True\ bf\ as \neq bf\text{-}restrict\ v\ False\ bf\ as\}$

**lemma**  $var \notin bf\text{-}vars\ (bf\text{-}restrict\ var\ val\ ex)$   
*<proof>*

We can decompose calculating if-then-else into computing if-then-else of two triples of functions with one variable restricted to true / false. Given that the functions have finite arity, we can use this to construct a recursive definition.

**lemma**  $brace90shannon: bf\text{-}ite\ F\ G\ H\ ass =$   
 $bf\text{-}ite\ (\lambda l. l\ i)$   
 $(bf\text{-}ite\ (bf\text{-}restrict\ i\ True\ F)\ (bf\text{-}restrict\ i\ True\ G)\ (bf\text{-}restrict\ i\ True\ H))$   
 $(bf\text{-}ite\ (bf\text{-}restrict\ i\ False\ F)\ (bf\text{-}restrict\ i\ False\ G)\ (bf\text{-}restrict\ i\ False\ H))$   
 $ass$   
*<proof>*

end

## 3 Binary Decision Trees

theory *BDT*

```

imports Bool-Func
begin

```

We first define all operations and properties on binary decision trees. This has the advantage that we can use a simple, structurally defined type and the disadvantage that we cannot represent sharing.

```

datatype 'a ifex = Trueif | Falseif | IF 'a 'a ifex 'a ifex

```

The type is the same as in Boolean Expression Checkers by Nipkow [3]. Internally, Boolean Expression Checkers transforms the boolean expressions to reduced BDTs of this type. Tests like being tautology testing are then trivial.

```

fun val-ifex :: 'a ifex  $\Rightarrow$  ('a  $\Rightarrow$  bool)  $\Rightarrow$  bool where
  val-ifex Trueif s = True |
  val-ifex Falseif s = False |
  val-ifex (IF n t1 t2) s = (if s n then val-ifex t1 s else val-ifex t2 s)

```

```

fun ifex-vars :: ('a :: linorder) ifex  $\Rightarrow$  'a list where
  ifex-vars (IF v t e) = v # ifex-vars t @ ifex-vars e |
  ifex-vars Trueif = [] |
  ifex-vars Falseif = []

```

```

abbreviation ifex-var-set a  $\equiv$  set (ifex-vars a)

```

```

fun ifex-ordered :: ('a::linorder) ifex  $\Rightarrow$  bool where
  ifex-ordered (IF v t e) = (( $\forall$  tv  $\in$  (ifex-var-set t  $\cup$  ifex-var-set e). v < tv)
     $\wedge$  ifex-ordered t  $\wedge$  ifex-ordered e) |
  ifex-ordered Trueif = True |
  ifex-ordered Falseif = True

```

```

fun ifex-minimal :: ('a::linorder) ifex  $\Rightarrow$  bool where
  ifex-minimal (IF v t e)  $\longleftrightarrow$  t  $\neq$  e  $\wedge$  ifex-minimal t  $\wedge$  ifex-minimal e |
  ifex-minimal Trueif = True |
  ifex-minimal Falseif = True

```

```

abbreviation ro-ifex where ro-ifex t  $\equiv$  ifex-ordered t  $\wedge$  ifex-minimal t

```

```

definition bf-ifex-rel where
  bf-ifex-rel = {(a,b). ( $\forall$  ass. a ass  $\longleftrightarrow$  val-ifex b ass)  $\wedge$  ro-ifex b}

```

```

lemma ifex-var-noinfluence: x  $\notin$  ifex-var-set b  $\implies$  val-ifex b (ass(x:=val)) =
  val-ifex b ass
  <proof>

```

```

lemma roifex-var-not-in-subtree:
  assumes ro-ifex b and b = IF v t e
  shows v  $\notin$  ifex-var-set t and v  $\notin$  ifex-var-set e
  <proof>

```

**lemma** *roifex-set-var-subtree*:

**assumes** *ro-ifex b* **and**  $b = IF\ v\ t\ e$

**shows**  $val\text{-ifex}\ b\ (ass(v:=True)) = val\text{-ifex}\ t\ ass$

$val\text{-ifex}\ b\ (ass(v:=False)) = val\text{-ifex}\ e\ ass$

*<proof>*

**lemma** *roifex-Trueif-unique*:  $ro\text{-ifex}\ b \implies \forall\ ass.\ val\text{-ifex}\ b\ ass \implies b = Trueif$

*<proof>*

**lemma** *roifex-Falseif-unique*:  $ro\text{-ifex}\ b \implies \forall\ ass.\ \neg\ val\text{-ifex}\ b\ ass \implies b = Falseif$

*<proof>*

**lemma**  $(f, b) \in bf\text{-ifex}\text{-rel} \implies b = Trueif \longleftrightarrow f = (\lambda\ \cdot.\ True)$

*<proof>*

**lemma**  $(f, b) \in bf\text{-ifex}\text{-rel} \implies b = Falseif \longleftrightarrow f = (\lambda\ \cdot.\ False)$

*<proof>*

**lemma** *ifex-ordered-not-part*:  $ifex\text{-ordered}\ b \implies b = IF\ v\ b1\ b2 \implies w < v \implies$

$w \notin ifex\text{-var}\text{-set}\ b$

*<proof>*

**lemma** *ro-ifex-unique*:  $ro\text{-ifex}\ x \implies ro\text{-ifex}\ y \implies (\bigwedge\ ass.\ val\text{-ifex}\ x\ ass = val\text{-ifex}\ y\ ass) \implies x = y$

*<proof>*

**theorem** *bf-ifex-rel-single*: *single-valued bf-ifex-rel single-valued (bf-ifex-rel<sup>-1</sup>)*

*<proof>*

**lemma** *bf-ifex-eq*:  $(af, at) \in bf\text{-ifex}\text{-rel} \implies (bf, bt) \in bf\text{-ifex}\text{-rel} \implies (af = bf) \longleftrightarrow (at = bt)$

*<proof>*

**lemma** *nonempty-if-var-set*:  $ifex\text{-vars}\ (IF\ v\ t\ e) \neq []$  *<proof>*

**fun** *restrict where*

$restrict\ (IF\ v\ t\ e)\ var\ val = (let\ rt = restrict\ t\ var\ val;\ re = restrict\ e\ var\ val\ in$   
 $(if\ v = var\ then\ (if\ val\ then\ rt\ else\ re)\ else\ (IF\ v\ rt\ re)))\ |$

$restrict\ i\ -\ - = i$

**declare** *Let-def[simp]*

**lemma** *not-element-restrict*:  $var \notin ifex\text{-var}\text{-set}\ (restrict\ b\ var\ val)$

*<proof>*

**lemma** *restrict-assignment*:  $val\text{-ifex}\ b\ (ass(var := val)) \longleftrightarrow val\text{-ifex}\ (restrict\ b\ var\ val)\ ass$

*<proof>*

**lemma** *restrict-variables-subset*:  $\text{ifex-var-set } (\text{restrict } b \text{ var } val) \subseteq \text{ifex-var-set } b$   
 ⟨proof⟩

**lemma** *restrict-ifex-ordered-invar*:  $\text{ifex-ordered } b \implies \text{ifex-ordered } (\text{restrict } b \text{ var } val)$   
 ⟨proof⟩

**lemma** *restrict-val-invar*:  $\forall \text{ ass. } a \text{ ass} = \text{val-ifex } b \text{ ass} \implies$   
 $(\text{bf-restrict } \text{var } val \ a) \text{ ass} = \text{val-ifex } (\text{restrict } b \text{ var } val) \text{ ass}$   
 ⟨proof⟩

**lemma** *restrict-untouched-id*:  $x \notin \text{ifex-var-set } t \implies \text{restrict } t \ x \ val = t$   
 ⟨proof⟩

**fun** *ifex-top-var* ::  $'a \text{ ifex} \implies 'a \text{ option}$  **where**  
 $\text{ifex-top-var } (\text{IF } v \ t \ e) = \text{Some } v \mid$   
 $\text{ifex-top-var } - = \text{None}$

**fun** *restrict-top* ::  $('a :: \text{linorder}) \text{ ifex} \implies 'a \implies \text{bool} \implies 'a \text{ ifex}$  **where**  
 $\text{restrict-top } (\text{IF } v \ t \ e) \ \text{var } val = (\text{if } v = \text{var} \ \text{then } (\text{if } val \ \text{then } t \ \text{else } e) \ \text{else } (\text{IF } v$   
 $t \ e)) \mid$   
 $\text{restrict-top } i \ - \ - = i$

**lemma** *restrict-top-id*:  $\text{ifex-ordered } e \implies \text{ifex-top-var } e = \text{Some } v \implies v' < v \implies$   
 $\text{restrict-top } e \ v' \ val = e$   
 ⟨proof⟩

**lemma** *restrict-id*:  $\text{ifex-ordered } e \implies \text{ifex-top-var } e = \text{Some } v \implies v' < v \implies$   
 $\text{restrict } e \ v' \ val = e$   
 ⟨proof⟩

**lemma** *restrict-top-IF-id*:  $\text{ifex-ordered } (\text{IF } v \ t \ e) \implies v' < v \implies \text{restrict-top } (\text{IF}$   
 $v \ t \ e) \ v' \ val = (\text{IF } v \ t \ e)$   
 ⟨proof⟩

**lemma** *restrict-IF-id*: **assumes**  $o$ :  $\text{ifex-ordered } (\text{IF } v \ t \ e)$  **assumes**  $le$ :  $v' < v$   
**shows**  $\text{restrict } (\text{IF } v \ t \ e) \ v' \ val = (\text{IF } v \ t \ e)$   
 ⟨proof⟩

**lemma** *restrict-top-eq*:  $\text{ifex-ordered } (\text{IF } v \ t \ e) \implies \text{restrict } (\text{IF } v \ t \ e) \ v \ val =$   
 $\text{restrict-top } (\text{IF } v \ t \ e) \ v \ val$   
 ⟨proof⟩

**lemma** *restrict-top-ifex-ordered-invar*:  $\text{ifex-ordered } b \implies \text{ifex-ordered } (\text{restrict-top}$   
 $b \ \text{var } val)$   
 ⟨proof⟩

**fun** *lowest-tops* :: ('a :: linorder) ifex list  $\Rightarrow$  'a option **where**  
*lowest-tops* [] = None |  
*lowest-tops* ((IF v -)#r) = Some (case *lowest-tops* r of Some u  $\Rightarrow$  (min u v) |  
None  $\Rightarrow$  v) |  
*lowest-tops* (-#r) = *lowest-tops* r

**lemma** *lowest-tops-NoneD*: *lowest-tops* k = None  $\implies$  ( $\neg(\exists v t e. ((IF v t e) \in \text{set } k))$ )  
<proof>

**lemma** *lowest-tops-in*: *lowest-tops* k = Some l  $\implies$  l  $\in$  set (concat (map *ifex-vars* k))  
<proof>

**definition** *IFC* v t e  $\equiv$  (if t = e then t else IF v t e)

**function** *ifex-ite* :: 'a ifex  $\Rightarrow$  'a ifex  $\Rightarrow$  'a ifex  $\Rightarrow$  ('a :: linorder) ifex **where**  
*ifex-ite* i t e = (case *lowest-tops* [i, t, e] of Some x  $\Rightarrow$   
(IFC x (*ifex-ite* (restrict-top i x True) (restrict-top t x True)  
(restrict-top e x True))  
(*ifex-ite* (restrict-top i x False) (restrict-top t x False)  
(restrict-top e x False)))  
| None  $\Rightarrow$  (case i of Trueif  $\Rightarrow$  t | Falseif  $\Rightarrow$  e))  
<proof>

**lemma** *restrict-size-le*: size (restrict-top k var val)  $\leq$  size k  
<proof>

**lemma** *restrict-size-less*: *ifex-top-var* k = Some var  $\implies$  size (restrict-top k var val) < size k  
<proof>

**lemma** *lowest-tops-cases*:  
*lowest-tops* [i, t, e] = Some var  $\implies$  *ifex-top-var* i = Some var  $\vee$  *ifex-top-var* t  
= Some var  $\vee$  *ifex-top-var* e = Some var  
<proof>

**lemma** *lowest-tops-lowest*: *lowest-tops* es = Some a  $\implies$  e  $\in$  set es  $\implies$  *ifex-ordered*  
e  $\implies$  v  $\in$  *ifex-var-set* e  $\implies$  a  $\leq$  v  
<proof>

**lemma** *termlemma2*: *lowest-tops* [i, t, e] = Some xa  $\implies$   
(size (restrict-top i xa val) + size (restrict-top t xa val) + size (restrict-top e xa  
val)) <  
(size i + size t + size e)  
<proof>

**lemma** *termlemma*: *lowest-tops* [i, t, e] = Some xa  $\implies$   
(case (restrict-top i xa val, restrict-top t xa val, restrict-top e xa val) of

$(i, t, e) \Rightarrow \text{size } i + \text{size } t + \text{size } e <$   
 $(\text{case } (i, t, e) \text{ of } (i, t, e) \Rightarrow \text{size } i + \text{size } t + \text{size } e)$   
 $\langle \text{proof} \rangle$

**termination** *ifex-ite*  
 $\langle \text{proof} \rangle$

**definition** *const x - = x*

**declare** *const-def[simp]*

**lemma** *rel-true-false*:  $(a, \text{Trueif}) \in \text{bf-ifex-rel} \Rightarrow a = \text{const True}$   $(a, \text{Falseif}) \in$   
 $\text{bf-ifex-rel} \Rightarrow a = \text{const False}$   
 $\langle \text{proof} \rangle$

**lemma** *rel-if*:  $(a, \text{IF } v \text{ t } e) \in \text{bf-ifex-rel} \Rightarrow (ta, t) \in \text{bf-ifex-rel} \Rightarrow (ea, e) \in$   
 $\text{bf-ifex-rel} \Rightarrow a = (\lambda as. \text{if } as \text{ v then } ta \text{ as else } ea \text{ as})$   
 $\langle \text{proof} \rangle$

**lemma** *ifex-ordered-implied*:  $(a, b) \in \text{bf-ifex-rel} \Rightarrow \text{ifex-ordered } b$   $\langle \text{proof} \rangle$

**lemma** *ifex-minimal-implied*:  $(a, b) \in \text{bf-ifex-rel} \Rightarrow \text{ifex-minimal } b$   $\langle \text{proof} \rangle$

**lemma** *ifex-ite-induct2*[*case-names Trueif Falseif IF*]:

$(\bigwedge i \text{ t } e. \text{lowest-tops } [i, t, e] = \text{None} \Rightarrow i = \text{Trueif} \Rightarrow \text{sentence } i \text{ t } e) \Rightarrow$   
 $(\bigwedge i \text{ t } e. \text{lowest-tops } [i, t, e] = \text{None} \Rightarrow i = \text{Falseif} \Rightarrow \text{sentence } i \text{ t } e) \Rightarrow$   
 $(\bigwedge i \text{ t } e \text{ a. sentence } (\text{restrict-top } i \text{ a True}) (\text{restrict-top } t \text{ a True}) (\text{restrict-top } e \text{ a}$   
 $\text{True}) \Rightarrow$   
 $\text{sentence } (\text{restrict-top } i \text{ a False}) (\text{restrict-top } t \text{ a False}) (\text{restrict-top } e \text{ a}$   
 $\text{False}) \Rightarrow$   
 $\text{lowest-tops } [i, t, e] = \text{Some } a \Rightarrow \text{sentence } i \text{ t } e) \Rightarrow \text{sentence } i \text{ t } e$   
 $\langle \text{proof} \rangle$

**lemma** *ifex-ite-induct*[*case-names Trueif Falseif IF*]:

$(\bigwedge i \text{ t } e. \text{lowest-tops } [i, t, e] = \text{None} \Rightarrow i = \text{Trueif} \Rightarrow P \text{ i t } e) \Rightarrow$   
 $(\bigwedge i \text{ t } e. \text{lowest-tops } [i, t, e] = \text{None} \Rightarrow i = \text{Falseif} \Rightarrow P \text{ i t } e) \Rightarrow$   
 $(\bigwedge i \text{ t } e \text{ a. } (\bigwedge \text{val. } P (\text{restrict-top } i \text{ a val}) (\text{restrict-top } t \text{ a val}) (\text{restrict-top } e \text{ a}$   
 $\text{val})) \Rightarrow$   
 $\text{lowest-tops } [i, t, e] = \text{Some } a \Rightarrow P \text{ i t } e) \Rightarrow P \text{ i t } e$   
 $\langle \text{proof} \rangle$

**lemma** *restrict-top-subset*:  $x \in \text{ifex-var-set } (\text{restrict-top } i \text{ vr vl}) \Rightarrow x \in \text{ifex-var-set}$   
 $i$   
 $\langle \text{proof} \rangle$

**lemma** *ifex-vars-subset*:  $x \in \text{ifex-var-set } (\text{ifex-ite } i \text{ t } e) \Rightarrow (x \in \text{ifex-var-set } i) \vee$   
 $(x \in \text{ifex-var-set } t) \vee (x \in \text{ifex-var-set } e)$   
 $\langle \text{proof} \rangle$



**lemma** *three-ins*:  $i \in \text{set } [i, t, e] \ t \in \text{set } [i, t, e] \ e \in \text{set } [i, t, e] \langle \text{proof} \rangle$

**lemma** *hlp3*:  $\text{lowest-tops } (IF \ v \ uu \ uv \ \# \ r) \neq \text{lowest-tops } r \implies \text{lowest-tops } (IF \ v \ uu \ uv \ \# \ r) = \text{Some } v \langle \text{proof} \rangle$

**lemma** *hlp2*:  $IF \ vi \ vt \ ve \in \text{set } is \implies \text{lowest-tops } is = \text{Some } a \implies a \leq vi \langle \text{proof} \rangle$

**lemma** *hlp1*:  $i \in \text{set } is \implies \text{lowest-tops } is = \text{Some } a \implies \text{ifex-ordered } i \implies a \notin (\text{ifex-var-set } (\text{restrict-top } i \ a \ \text{val})) \langle \text{proof} \rangle$

**lemma** *order-ifex-ite-invar*:  $\text{ifex-ordered } i \implies \text{ifex-ordered } t \implies \text{ifex-ordered } e \implies \text{ifex-ordered } (\text{ifex-ite } i \ t \ e) \langle \text{proof} \rangle$

**lemma** *ifc-split*:  $P \ (IFC \ v \ t \ e) \longleftrightarrow ((t = e) \longrightarrow P \ t) \wedge (t \neq e \longrightarrow P \ (IF \ v \ t \ e)) \langle \text{proof} \rangle$

**lemma** *restrict-top-ifex-minimal-invar*:  $\text{ifex-minimal } i \implies \text{ifex-minimal } (\text{restrict-top } i \ a \ \text{val}) \langle \text{proof} \rangle$

**lemma** *minimal-ifex-ite-invar*:  $\text{ifex-minimal } i \implies \text{ifex-minimal } t \implies \text{ifex-minimal } e \implies \text{ifex-minimal } (\text{ifex-ite } i \ t \ e) \langle \text{proof} \rangle$

**lemma** *restrict-top-bf*:  $i \in \text{set } is \implies \text{lowest-tops } is = \text{Some } vr \implies \text{ifex-ordered } i \implies (\bigwedge \text{ass. } fi \ \text{ass} = \text{val-ifex } i \ \text{ass}) \implies \text{val-ifex } (\text{restrict-top } i \ vr \ vl) \ \text{ass} = \text{bf-restrict } vr \ vl \ fi \ \text{ass} \langle \text{proof} \rangle$

**lemma** *val-ifex-ite*:  
 $(\bigwedge \text{ass. } fi \ \text{ass} = \text{val-ifex } i \ \text{ass}) \implies$   
 $(\bigwedge \text{ass. } ft \ \text{ass} = \text{val-ifex } t \ \text{ass}) \implies$   
 $(\bigwedge \text{ass. } fe \ \text{ass} = \text{val-ifex } e \ \text{ass}) \implies$   
 $\text{ifex-ordered } i \implies \text{ifex-ordered } t \implies \text{ifex-ordered } e \implies$   
 $(\text{bf-ite } fi \ ft \ fe) \ \text{ass} = \text{val-ifex } (\text{ifex-ite } i \ t \ e) \ \text{ass} \langle \text{proof} \rangle$

**theorem** *ifex-ite-rel-bf*:  
 $(fi, i) \in \text{bf-ifex-rel} \implies$   
 $(ft, t) \in \text{bf-ifex-rel} \implies$   
 $(fe, e) \in \text{bf-ifex-rel} \implies$   
 $((\text{bf-ite } fi \ ft \ fe), (\text{ifex-ite } i \ t \ e)) \in \text{bf-ifex-rel} \langle \text{proof} \rangle$

**definition** *param-opt* **where**  $\text{param-opt } i \ t \ e =$

(if  $i = \text{Trueif}$  then  $\text{Some } t$  else  
 if  $i = \text{Falseif}$  then  $\text{Some } e$  else  
 if  $t = \text{Trueif} \wedge e = \text{Falseif}$  then  $\text{Some } i$  else  
 if  $t = e$  then  $\text{Some } t$  else  
 if  $e = \text{Trueif} \wedge i = t$  then  $\text{Some Trueif}$  else  
 if  $t = \text{Falseif} \wedge i = e$  then  $\text{Some Falseif}$  else  
 None)

**lemma** *param-opt-ifex-ite-eq*:  $\text{ro-ifex } i \implies \text{ro-ifex } t \implies \text{ro-ifex } e \implies$   
 $\text{param-opt } i \ t \ e = \text{Some } r \implies r = \text{ifex-ite } i \ t \ e$   
 ⟨proof⟩

**function** *ifex-ite-opt* ::  $'a \text{ ifex} \Rightarrow 'a \text{ ifex} \Rightarrow 'a \text{ ifex} \Rightarrow ('a :: \text{linorder}) \text{ ifex}$  **where**  
 $\text{ifex-ite-opt } i \ t \ e = (\text{case } \text{param-opt } i \ t \ e \text{ of } \text{Some } b \Rightarrow b \mid \text{None} \Rightarrow$   
 $\text{case } \text{lowest-tops } [i, t, e] \text{ of } \text{Some } x \Rightarrow$   
 $\text{IFC } x \ (\text{ifex-ite-opt } (\text{restrict-top } i \ x \ \text{True}) \ (\text{restrict-top } t \ x$   
 $\text{True}))$   
 $\text{restrict-top } e \ x \ \text{True}))$   
 $(\text{ifex-ite-opt } (\text{restrict-top } i \ x \ \text{False}) \ (\text{restrict-top } t \ x \ \text{False}))$   
 $(\text{restrict-top } e \ x \ \text{False}))$   
 $\mid \text{None} \Rightarrow (\text{case } i \text{ of } \text{Trueif} \Rightarrow t \mid \text{Falseif} \Rightarrow e))$   
 ⟨proof⟩

**termination** *ifex-ite-opt*  
 ⟨proof⟩

**lemma** *ifex-ite-opt-eq*:  
 $\text{ro-ifex } i \implies \text{ro-ifex } t \implies \text{ro-ifex } e \implies \text{ifex-ite-opt } i \ t \ e = \text{ifex-ite } i \ t \ e$   
 ⟨proof⟩

**lemma** *ro-ifexI*:  $(a, b) \in \text{bf-ifex-rel} \implies \text{ro-ifex } b$  ⟨proof⟩

**theorem** *ifex-ite-opt-rel-bf*:  
 $(fi, i) \in \text{bf-ifex-rel} \implies$   
 $(ft, t) \in \text{bf-ifex-rel} \implies$   
 $(fe, e) \in \text{bf-ifex-rel} \implies$   
 $((\text{bf-ite } fi \ ft \ fe), (\text{ifex-ite-opt } i \ t \ e)) \in \text{bf-ifex-rel}$   
 ⟨proof⟩

**lemma** *restrict-top-bf-ifex-rel*:  
 $(f, i) \in \text{bf-ifex-rel} \implies \exists f'. (f', \text{restrict-top } i \ \text{var } \text{val}) \in \text{bf-ifex-rel}$   
 ⟨proof⟩

**lemma** *param-opt-lowest-tops-lem*:  $\text{param-opt } i \ t \ e = \text{None} \implies \exists y. \text{lowest-tops}$   
 $[i, t, e] = \text{Some } y$   
 ⟨proof⟩

```

fun ifex-sat where
ifex-sat Trueif = Some (const False) |
ifex-sat Falseif = None |
ifex-sat (IF v t e) =
  (case ifex-sat e of
    Some a  $\Rightarrow$  Some (a(v:=False)) |
    None  $\Rightarrow$  (case ifex-sat t of
      Some a  $\Rightarrow$  Some (a(v:=True)) |
      None  $\Rightarrow$  None))

```

**lemma** *ifex-sat-untouched-False*:  $v \notin \text{ifex-var-set } i \implies \text{ifex-sat } i = \text{Some } a \implies a = \text{False}$   
 <proof>

**lemma** *ifex-upd-other*:  $v \notin \text{ifex-var-set } i \implies \text{val-ifex } i (a(v:=\text{any})) = \text{val-ifex } i a$   
 <proof>

```

fun ifex-no-twice where
ifex-no-twice (IF v t e) = (
  v  $\notin$  (ifex-var-set t  $\cup$  ifex-var-set e)  $\wedge$ 
  ifex-no-twice t  $\wedge$  ifex-no-twice e) |
ifex-no-twice - = True

```

**lemma** *ordered-ifex-no-twiceI*:  $\text{ifex-ordered } i \implies \text{ifex-no-twice } i$   
 <proof>

**lemma** *ifex-sat-NoneD*:  $\text{ifex-sat } i = \text{None} \implies \text{val-ifex } i \text{ ass} = \text{False}$   
 <proof>

**lemma** *ifex-sat-SomeD*:  $\text{ifex-no-twice } i \implies \text{ifex-sat } i = \text{Some } \text{ass} \implies \text{val-ifex } i \text{ ass} = \text{True}$   
 <proof>

**lemma** *ifex-sat-NoneI*:  $\text{ifex-no-twice } i \implies (\bigwedge \text{ass. val-ifex } i \text{ ass} = \text{False}) \implies \text{ifex-sat } i = \text{None}$

<proof>

```

fun ifex-sat-list where
ifex-sat-list Trueif = Some [] |
ifex-sat-list Falseif = None |
ifex-sat-list (IF v t e) =
  (case ifex-sat-list e of
    Some a  $\Rightarrow$  Some ((v,False)#a) |
    None  $\Rightarrow$  (case ifex-sat-list t of
      Some a  $\Rightarrow$  Some ((v,True)#a) |
      None  $\Rightarrow$  None))

```

**definition** *update-assignment-alt*  $u\ as = (\lambda v. \text{case map-of } u\ v\ \text{of } \text{None} \Rightarrow as\ v\ | \text{Some } n \Rightarrow n)$

**fun** *update-assignment* **where**

*update-assignment*  $((v,u)\#us)\ as = (\text{update-assignment } us\ as)(v:=u) |$

*update-assignment*  $[]\ as = as$

**lemma** *update-assignment-notin*:  $a \notin \text{fst } 'set\ us \Longrightarrow \text{update-assignment } us\ as\ a = as\ a$   
 $\langle proof \rangle$

**lemma** *update-assignment-alt*:  $\text{update-assignment } u\ as = \text{update-assignment-alt } u\ as$   
 $\langle proof \rangle$

**lemma** *update-assignment*:  $\text{distinct } (\text{map } \text{fst } ((v,u)\#us)) \Longrightarrow \text{update-assignment } ((v,u)\#us)\ as = \text{update-assignment } us\ (as(v:=u))$   
 $\langle proof \rangle$

**lemma** *ass-upd-same*:  $\text{update-assignment } ((v, u) \# a)\ ass\ v = u\ \langle proof \rangle$

**lemma** *ifex-sat-list-subset*:  $\text{ifex-sat-list } t = \text{Some } u \Longrightarrow \text{fst } 'set\ u \subseteq \text{ifex-var-set } t$   
 $\langle proof \rangle$

**lemma** *sat-list-distinct*:  $\text{ifex-no-twice } t \Longrightarrow \text{ifex-sat-list } t = \text{Some } u \Longrightarrow \text{distinct } (\text{map } \text{fst } u)$   
 $\langle proof \rangle$

**lemma** *ifex-sat-list-NoneD*:  $\text{ifex-sat-list } i = \text{None} \Longrightarrow \text{val-ifex } i\ ass = \text{False}$   
 $\langle proof \rangle$

**lemma** *ifex-sat-list-SomeD*:  $\text{ifex-no-twice } i \Longrightarrow \text{ifex-sat-list } i = \text{Some } u \Longrightarrow ass = \text{update-assignment } u\ ass' \Longrightarrow \text{val-ifex } i\ ass = \text{True}$   
 $\langle proof \rangle$

**fun** *sat-list-to-bdt* **where**

*sat-list-to-bdt*  $[] = \text{Trueif} |$

*sat-list-to-bdt*  $((v,u)\#us) = (\text{if } u\ \text{then } \text{IF } v\ (\text{sat-list-to-bdt } us)\ \text{Falseif } \text{else } \text{IF } v\ \text{Falseif } (\text{sat-list-to-bdt } us))$

**lemma** *ifex-sat-list*  $i = \text{Some } u \Longrightarrow \text{val-ifex } (\text{sat-list-to-bdt } u)\ as \Longrightarrow \text{val-ifex } i\ as$   
 $\langle proof \rangle$

**lemma** *bf-ifex-rel-consts*[*simp,intro!*]:

$(\text{bf-True}, \text{Trueif}) \in \text{bf-ifex-rel}$

$(\text{bf-False}, \text{Falseif}) \in \text{bf-ifex-rel}$

$\langle proof \rangle$

**lemma** *bf-ifex-rel-lit*[*simp,intro!*]:

$(\text{bf-lit } v, \text{IFC } v\ \text{Trueif } \text{Falseif}) \in \text{bf-ifex-rel}$

$\langle proof \rangle$

**lemma** *bf-ifex-rel-consts-ensured*[simp]:  
 $(bf-True, x) \in bf-ifex-rel \longleftrightarrow (x = Trueif)$   
 $(bf-False, x) \in bf-ifex-rel \longleftrightarrow (x = Falseif)$   
 $\langle proof \rangle$

**lemma** *bf-ifex-rel-consts-ensured-rev*[simp]:  
 $(x, Trueif) \in bf-ifex-rel \longleftrightarrow (x = bf-True)$   
 $(x, Falseif) \in bf-ifex-rel \longleftrightarrow (x = bf-False)$   
 $\langle proof \rangle$

**declare** *ifex-ite-opt.simps restrict-top.simps lowest-tops.simps*[simp del]

**end**

## 4 Option Helpers

These definitions were contributed by Peter Lammich.

**theory** *Option-Helpers*  
**imports** *Main HOL-Library.Monad-Syntax*  
**begin**

**primrec** *oassert* :: *bool*  $\Rightarrow$  *unit option* **where**  
 $oassert\ True = Some\ () \mid oassert\ False = None$

**lemma** *oassert-iff*[simp]:  
 $oassert\ \Phi = Some\ x \longleftrightarrow \Phi$   
 $oassert\ \Phi = None \longleftrightarrow \neg\Phi$   
 $\langle proof \rangle$

The idea is that we want the result of some computation to be *Some v* and the contents of *v* to satisfy some property *Q*.

**primrec** *ospec* :: (*'a option*)  $\Rightarrow$  (*'a*  $\Rightarrow$  *bool*)  $\Rightarrow$  *bool* **where**  
 $ospec\ None\ - = False$   
 $\mid ospec\ (Some\ v)\ Q = Q\ v$

**named-theorems** *ospec-rules*

**lemma** *oreturn-rule*[*ospec-rules*]:  $\llbracket P\ r \rrbracket \Longrightarrow ospec\ (Some\ r)\ P \langle proof \rangle$

**lemma** *obind-rule*[*ospec-rules*]:  $\llbracket ospec\ m\ Q; \bigwedge r. Q\ r \Longrightarrow ospec\ (f\ r)\ P \rrbracket \Longrightarrow ospec\ (m\ \gg\ f)\ P$   
 $\langle proof \rangle$

**lemma** *ospec-alt*:  $ospec\ m\ P = (case\ m\ of\ None\ \Rightarrow\ False \mid Some\ x\ \Rightarrow\ P\ x)$   
 $\langle proof \rangle$

**lemma** *ospec-bind-simp*:  $ospec (m \gg f) P \longleftrightarrow (ospec m (\lambda r. ospec (f r) P))$   
 ⟨proof⟩

**lemma** *ospec-cons*:  
**assumes** *ospec m Q*  
**assumes**  $\bigwedge r. Q r \implies P r$   
**shows** *ospec m P*  
 ⟨proof⟩

**lemma** *oreturn-synth*:  $ospec (Some x) (\lambda r. r=x)$  ⟨proof⟩

**lemma** *ospecD*:  $ospec x P \implies x = Some y \implies P y$  ⟨proof⟩

**lemma** *ospecD2*:  $ospec x P \implies \exists y. x = Some y \wedge P y$  ⟨proof⟩

**end**

## 5 Abstract ITE Implementation

**theory** *Abstract-Impl*

**imports** *BDT*

*Automatic-Refinement.Refine-Lib*

*Option-Helpers*

**begin**

**datatype**  $('a, 'ni)$  *IFEXD* = *TD* | *FD* | *IFD*  $'a$   $'ni$   $'ni$

**locale** *bdd-impl-pre* =

**fixes**  $R :: 's \Rightarrow ('ni \times ('a :: linorder) ifex)$  *set*

**fixes**  $I :: 's \Rightarrow bool$

**begin**

**definition** *les*::  $'s \Rightarrow 's \Rightarrow bool$  **where**

$les s s' == \forall ni n. (ni, n) \in R s \longrightarrow (ni, n) \in R s'$

**end**

**locale** *bdd-impl* = *bdd-impl-pre R* **for**  $R :: 's \Rightarrow ('ni \times ('a :: linorder) ifex)$  *set* +

**fixes**  $Timpl :: 's \rightarrow ('ni \times 's)$

**fixes**  $Fimpl :: 's \rightarrow ('ni \times 's)$

**fixes**  $IFimpl :: 'a \Rightarrow 'ni \Rightarrow 'ni \Rightarrow 's \rightarrow ('ni \times 's)$

**fixes**  $DESTRimpl :: 'ni \Rightarrow 's \rightarrow ('a, 'ni)$  *IFEXD*

**assumes** *Timpl-rule*:  $I s \implies ospec (Timpl s) (\lambda(ni, s'). (ni, Trueif) \in R s' \wedge I s' \wedge les s s')$

**assumes** *Fimpl-rule*:  $I s \implies ospec (Fimpl s) (\lambda(ni, s'). (ni, Falseif) \in R s' \wedge I s' \wedge les s s')$

**assumes** *IFimpl-rule*:  $\llbracket I s; (ni1, n1) \in R s; (ni2, n2) \in R s \rrbracket$   
 $\implies ospec (IFimpl v ni1 ni2 s) (\lambda(ni, s'). (ni, IFC v n1 n2) \in R s' \wedge I s' \wedge les s s')$

**assumes** *DESTRimpl-rule1*:  $I s \implies (ni, Trueif) \in R s \implies ospec (DESTRimpl$

$ni\ s$  ( $\lambda r. r = TD$ )  
**assumes** *DESTRIimpl-rule2*:  $I\ s \implies (ni, Falseif) \in R\ s \implies ospec\ (DESTRIimpl\ ni\ s)$   
 $ni\ s$  ( $\lambda r. r = FD$ )  
**assumes** *DESTRIimpl-rule3*:  $I\ s \implies (ni, IF\ v\ n1\ n2) \in R\ s \implies$   
 $ospec\ (DESTRIimpl\ ni\ s)$   
 $(\lambda r. \exists\ ni1\ ni2. r = (IFD\ v\ ni1\ ni2) \wedge (ni1, n1) \in R\ s$   
 $\wedge (ni2, n2) \in R\ s)$   
**begin**

**lemma** *les-refl[simp,intro!]*:  $les\ s\ s$  *<proof>*  
**lemma** *les-trans[trans]*:  $les\ s1\ s2 \implies les\ s2\ s3 \implies les\ s1\ s3$  *<proof>*  
**lemmas** *DESTRIimpl-rules* = *DESTRIimpl-rule1 DESTRIimpl-rule2 DESTRIimpl-rule3*

**lemma** *DESTRIimpl-rule-useless*:  
 $I\ s \implies (ni, n) \in R\ s \implies ospec\ (DESTRIimpl\ ni\ s)$  ( $\lambda r. (case\ r\ of$   
 $TD \Rightarrow (ni, Trueif) \in R\ s \mid$   
 $FD \Rightarrow (ni, Falseif) \in R\ s \mid$   
 $IFD\ v\ nt\ ne \Rightarrow (\exists\ t\ e. n = IF\ v\ t\ e \wedge (ni, IF\ v\ t\ e) \in R\ s))$ )  
*<proof>*

**lemma** *DESTRIimpl-rule*:  
 $I\ s \implies (ni, n) \in R\ s \implies ospec\ (DESTRIimpl\ ni\ s)$  ( $\lambda r. (case\ n\ of$   
 $Trueif \Rightarrow r = TD \mid$   
 $Falseif \Rightarrow r = FD \mid$   
 $IF\ v\ t\ e \Rightarrow (\exists\ tn\ en. r = IFD\ v\ tn\ en \wedge (tn, t) \in R\ s \wedge (en, e) \in R\ s))$ )  
*<proof>*

**definition** *case-ifexi fti ffi fii ni s*  $\equiv do\ \{$   
 $dest \leftarrow DESTRIimpl\ ni\ s;$   
 $case\ dest\ of$   
 $TD \Rightarrow fti\ s$   
 $\mid FD \Rightarrow ffi\ s$   
 $\mid IFD\ v\ ti\ ei \Rightarrow fii\ v\ ti\ ei\ s\}$

**lemma** *case-ifexi-rule*:  
**assumes** *INV*:  $I\ s$   
**assumes** *NI*:  $(ni, n) \in R\ s$   
**assumes** *FII*:  $\llbracket n = Trueif \rrbracket \implies ospec\ (fti\ s)$  ( $\lambda(r, s'). (r, ft) \in Q\ s \wedge I'\ s'$ )  
**assumes** *FFI*:  $\llbracket n = Falseif \rrbracket \implies ospec\ (ffi\ s)$  ( $\lambda(r, s'). (r, ff) \in Q\ s \wedge I'\ s'$ )  
**assumes** *FII*:  $\bigwedge ti\ ei\ v\ t\ e. \llbracket n = IF\ v\ t\ e; (ti, t) \in R\ s; (ei, e) \in R\ s \rrbracket \implies ospec$   
 $(fii\ v\ ti\ ei\ s)$  ( $\lambda(r, s'). (r, fi\ v\ t\ e) \in Q\ s \wedge I'\ s'$ )  
**shows**  $ospec\ (case-ifexi\ fti\ ffi\ fii\ ni\ s)$  ( $\lambda(r, s'). (r, case-ifex\ ft\ ff\ fi\ n) \in Q\ s \wedge I'$   
 $s'$ )  
*<proof>*

**abbreviation** *return x*  $\equiv \lambda s. Some\ (x, s)$

**primrec** *lowest-tops-impl where*  
 $lowest-tops-impl\ []\ s = Some\ (None, s) \mid$   
 $lowest-tops-impl\ (e\#\ es)\ s =$

```

case-ifexi
  (λs. lowest-tops-impl es s)
  (λs. lowest-tops-impl es s)
  (λv t e s. do {
    (rec,s) ← lowest-tops-impl es s;
    (case rec of
      Some u ⇒ Some ((Some (min u v)), s) |
      None ⇒ Some ((Some v), s))
    }) e s

```

**declare** *lowest-tops-impl.simps*[simp del]

```

fun lowest-tops-alt where
lowest-tops-alt [] = None |
lowest-tops-alt (e#es) = (
  let rec = lowest-tops-alt es in
  case-ifex
    rec
    rec
    (λv t e. (case rec of
      Some u ⇒ (Some (min u v)) |
      None ⇒ (Some v))
    ) e
)

```

**lemma** *lowest-tops-alt*:  $\text{lowest-tops } l = \text{lowest-tops-alt } l$   
 ⟨proof⟩

**lemma** *lowest-tops-impl-R*:  
**assumes** *list-all2* (*in-rel* (*R s*)) *li l I s*  
**shows** *ospec* (*lowest-tops-impl li s*) ( $\lambda(r,s'). r = \text{lowest-tops } l \wedge s'=s$ )  
 ⟨proof⟩

**definition** *restrict-top-impl where*  
*restrict-top-impl e vr vl s =*  
*case-ifexi*  
 (return e)  
 (return e)  
 (λv te ee. return (if v = vr then (if vl then te else ee) else e))  
 e s

**lemma** *restrict-top-alt*:  $\text{restrict-top } n \text{ var } val = (\text{case } n \text{ of}$   
 (*IF v t e*) ⇒ (if v = var then (if val then t else e) else (*IF v t e*))  
 | - ⇒ n)  
 ⟨proof⟩

**lemma** *restrict-top-impl-spec*:  $I s \implies (ni,n) \in R s \implies \text{ospec } (\text{restrict-top-impl } ni$



$vr\ vl\ s) (\lambda(res,s'). (res, restrict-top\ n\ vr\ vl) \in R\ s \wedge s'=s)$   
 $\langle proof \rangle$

**partial-function**(*option*) *ite-impl* **where**

*ite-impl* *i t e s* = do {  
 (*lt,-*)  $\leftarrow$  *lowest-tops-impl* [*i, t, e*] *s*;  
 (case *lt* of  
   Some *a*  $\Rightarrow$  do {  
     (*ti,-*)  $\leftarrow$  *restrict-top-impl* *i a True s*;  
     (*tt,-*)  $\leftarrow$  *restrict-top-impl* *t a True s*;  
     (*te,-*)  $\leftarrow$  *restrict-top-impl* *e a True s*;  
     (*fi,-*)  $\leftarrow$  *restrict-top-impl* *i a False s*;  
     (*ft,-*)  $\leftarrow$  *restrict-top-impl* *t a False s*;  
     (*fe,-*)  $\leftarrow$  *restrict-top-impl* *e a False s*;  
     (*tb,s*)  $\leftarrow$  *ite-impl* *ti tt te s*;  
     (*fb,s*)  $\leftarrow$  *ite-impl* *fi ft fe s*;  
     *IFimpl a tb fb s*  
   | None  $\Rightarrow$  *case-ifexi* ( $\lambda-.(Some\ (t,s))$ ) ( $\lambda-.(Some\ (e,s))$ ) ( $\lambda- - - . None$ ) *i s*  
 )}

**lemma** *ite-impl-R*: *I s*

$\Rightarrow in-rel\ (R\ s)\ ii\ i \Rightarrow in-rel\ (R\ s)\ ti\ t \Rightarrow in-rel\ (R\ s)\ ei\ e$   
 $\Rightarrow ospec\ (ite-impl\ ii\ ti\ ei\ s)\ (\lambda(r, s'). (r, ifex-ite\ i\ t\ e) \in R\ s' \wedge I\ s' \wedge les\ s)$   
 $s')$   
 $\langle proof \rangle$

**lemma** *case-ifexi-mono*[*partial-function-mono*]:

**assumes** [*partial-function-mono*]:

*mono-option* ( $\lambda F. fti\ F\ s$ )

*mono-option* ( $\lambda F. ffi\ F\ s$ )

$\wedge x31\ x32\ x33. mono-option\ (\lambda F. fii\ F\ x31\ x32\ x33\ s)$

**shows** *mono-option* ( $\lambda F. case-ifexi\ (fti\ F)\ (ffi\ F)\ (fii\ F)\ ni\ s$ )

$\langle proof \rangle$

**partial-function**(*option*) *val-impl* :: '*ni*  $\Rightarrow$  ('*a*  $\Rightarrow$  *bool*)  $\Rightarrow$  '*s*  $\Rightarrow$  (*bool*  $\times$  '*s*) *option*

**where**

*val-impl* *e ass s* = *case-ifexi*

( $\lambda s. Some\ (True,s)$ )

( $\lambda s. Some\ (False,s)$ )

( $\lambda v\ t\ e\ s. val-impl\ (if\ ass\ v\ then\ t\ else\ e)\ ass\ s$ )

*e s*

**lemma** *I s*  $\Rightarrow (ni,n) \in R\ s \Rightarrow ospec\ (val-impl\ ni\ ass\ s)\ (\lambda(r,s'). r = (val-ifex\ n\ ass) \wedge s'=s)$

$\langle proof \rangle$

**end**

**locale** *bdd-impl-cmp-pre* = *bdd-impl-pre*  
**begin**

**definition** *map-invar-impl* *m s* =

( $\forall ii\ ti\ ei\ ri. m\ (ii,ti,ei) = \text{Some}\ ri \longrightarrow$   
 $(\exists i\ t\ e. ((ri,ifex-ite-opt\ i\ t\ e) \in R\ s) \wedge (ii,i) \in R\ s \wedge (ti,t) \in R\ s \wedge (ei,e) \in R\ s))$ )

**lemma** *map-invar-impl-les*: *map-invar-impl* *m s*  $\implies$  *les* *s s'*  $\implies$  *map-invar-impl* *m s'*  
 ⟨*proof*⟩

**lemma** *map-invar-impl-update*: *map-invar-impl* *m s*  $\implies$

$(ii,i) \in R\ s \implies (ti,t) \in R\ s \implies (ei,e) \in R\ s \implies$   
 $(ri, ifex-ite-opt\ i\ t\ e) \in R\ s \implies \text{map-invar-impl}\ (m((ii,ti,ei) \mapsto ri))\ s$   
 ⟨*proof*⟩

**end**

**locale** *bdd-impl-cmp* = *bdd-impl* + *bdd-impl-cmp-pre* +

**fixes** *M* :: 'a  $\Rightarrow$  ('b  $\times$  'b  $\times$  'b)  $\Rightarrow$  'b *option*

**fixes** *U* :: 'a  $\Rightarrow$  ('b  $\times$  'b  $\times$  'b)  $\Rightarrow$  'b  $\Rightarrow$  'a

**fixes** *cmp* :: 'b  $\Rightarrow$  'b  $\Rightarrow$  *bool*

**assumes** *cmp-rule1*: *I s*  $\implies$   $(ni, i) \in R\ s \implies (ni', i) \in R\ s \implies \text{cmp}\ ni\ ni'$

**assumes** *cmp-rule2*: *I s*  $\implies \text{cmp}\ ni\ ni' \implies (ni, i) \in R\ s \implies (ni', i') \in R\ s$   
 $\implies i = i'$

**assumes** *map-invar-rule1*: *I s*  $\implies \text{map-invar-impl}\ (M\ s)\ s$

**assumes** *map-invar-rule2*: *I s*  $\implies (ii,it) \in R\ s \implies (ti,tt) \in R\ s \implies (ei,et) \in R\ s \implies$

$(ri, ifex-ite-opt\ it\ tt\ et) \in R\ s \implies U\ s\ (ii,ti,ei)\ ri = s' \implies I\ s'$

**assumes** *map-invar-rule3*: *I s*  $\implies R\ (U\ s\ (ii, ti, ei)\ ri) = R\ s$

**begin**

**lemma** *cmp-rule-eq*: *I s*  $\implies (ni, i) \in R\ s \implies (ni', i') \in R\ s \implies \text{cmp}\ ni\ ni' \longleftrightarrow i = i'$

⟨*proof*⟩

**lemma** *DESTRimpl-Some*: *I s*  $\implies (ni, i) \in R\ s \implies \text{ospec}\ (\text{DESTRimpl}\ ni\ s)$

( $\lambda r. \text{True}$ )

⟨*proof*⟩

**fun** *param-opt-impl* **where**

*param-opt-impl* *i t e s* = *do* {

*ii*  $\leftarrow$  *DESTRimpl* *i s*;

*ti*  $\leftarrow$  *DESTRimpl* *t s*;

*ei*  $\leftarrow$  *DESTRimpl* *e s*;

(*tn,s*)  $\leftarrow$  *Timpl* *s*;

(*fn,s*)  $\leftarrow$  *Fimpl* *s*;

```

Some ((if ii = TD then Some t else
if ii = FD then Some e else
if ti = TD ∧ ei = FD then Some i else
if cmp t e then Some t else
if ei = TD ∧ cmp i t then Some tn else
if ti = FD ∧ cmp i e then Some fn else
None), s)}

```

**declare** *param-opt-impl.simps*[*simp del*]

**lemma** *param-opt-impl-lesI*:

```

assumes I s (ii,i) ∈ R s (ti,t) ∈ R s (ei,e) ∈ R s
shows ospec (param-opt-impl ii ti ei s)
      (λ(r,s^). I s^ ∧ les s s^)
⟨proof⟩

```

**lemma** *param-opt-impl-R*:

```

assumes I s (ii,i) ∈ R s (ti,t) ∈ R s (ei,e) ∈ R s
shows ospec (param-opt-impl ii ti ei s)
      (λ(r,s^). case r of None ⇒ param-opt i t e = None
                | Some r ⇒ (∃ r'. param-opt i t e = Some r' ∧ (r, r')
                ∈ R s^))
⟨proof⟩

```

**partial-function**(*option*) *ite-impl-opt* **where**

```

ite-impl-opt i t e s = do {
  (ld, s) ← param-opt-impl i t e s;
  (case ld of Some b ⇒ Some (b, s) |
  None ⇒
  do {
    (lt,-) ← lowest-tops-impl [i, t, e] s;
    (case lt of
    Some a ⇒ do {
      (ti,-) ← restrict-top-impl i a True s;
      (tt,-) ← restrict-top-impl t a True s;
      (te,-) ← restrict-top-impl e a True s;
      (fi,-) ← restrict-top-impl i a False s;
      (ft,-) ← restrict-top-impl t a False s;
      (fe,-) ← restrict-top-impl e a False s;
      (tb,s) ← ite-impl-opt ti tt te s;
      (fb,s) ← ite-impl-opt fi ft fe s;
      IFimpl a tb fb s}
    | None ⇒ case-ifexi (λ-.(Some (t,s))) (λ-.(Some (e,s))) (λ- - - . None) i s
    )}})
}

```

**lemma** *ospec-and*: *ospec f P ⇒ ospec f Q ⇒ ospec f (λx. P x ∧ Q x)*  
 ⟨proof⟩

**lemma** *ite-impl-opt-R*:

```

  I s
  ⇒ in-rel (R s) ii i ⇒ in-rel (R s) ti t ⇒ in-rel (R s) ei e
  ⇒ ospec (ite-impl-opt ii ti ei s) (λ(r, s'). (r, ifex-ite-opt i t e) ∈ R s' ∧ I s' ∧
les s s')
⟨proof⟩

```

```

partial-function(option) ite-impl-lu where
ite-impl-lu i t e s = do {
  (case M s (i,t,e) of Some b ⇒ Some (b,s) | None ⇒ do {
    (ld, s) ← param-opt-impl i t e s;
    (case ld of Some b ⇒ Some (b, s) |
None ⇒
do {
  (lt,-) ← lowest-tops-impl [i, t, e] s;
  (case lt of
Some a ⇒ do {
    (ti,-) ← restrict-top-impl i a True s;
    (tt,-) ← restrict-top-impl t a True s;
    (te,-) ← restrict-top-impl e a True s;
    (fi,-) ← restrict-top-impl i a False s;
    (ft,-) ← restrict-top-impl t a False s;
    (fe,-) ← restrict-top-impl e a False s;
    (tb,s) ← ite-impl-lu ti tt te s;
    (fb,s) ← ite-impl-lu fi ft fe s;
    (r,s) ← IFimpl a tb fb s;
    let s = U s (i,t,e) r;
    Some (r,s)
  } |
None ⇒ None
  )}}})}

```

```

declare ifex-ite-opt.simps[simp del]

```

```

lemma ite-impl-lu-R: I s
  ⇒ (ii,i) ∈ R s ⇒ (ti,t) ∈ R s ⇒ (ei,e) ∈ R s
  ⇒ ospec (ite-impl-lu ii ti ei s)
    (λ(r, s'). (r, ifex-ite-opt i t e) ∈ R s' ∧ I s' ∧ les s s')
⟨proof⟩

```

```

end
end

```

## 6 Pointermap

```

theory Pointer-Map
imports Main
begin

```

We need a datastructure that supports the following two operations:

- Given an element, it can construct a pointer (i.e., a small representation) of that element. It will always construct the same pointer for equal elements.
- Given a pointer, we can retrieve the element

**record** *'a pointermap* =  
*entries* :: *'a list*  
*getentry* :: *'a* ⇒ *nat option*

**definition** *pointermap-sane* *m* ≡ (*distinct (entries m) ∧*  
 $(\forall n \in \{..<\text{length } (\text{entries } m)\}. \text{getentry } m (\text{entries } m ! n) = \text{Some } n) \wedge$   
 $(\forall p i. \text{getentry } m p = \text{Some } i \longrightarrow \text{entries } m ! i = p \wedge i < \text{length } (\text{entries } m)))$ )

**definition** *empty-pointermap* ≡ (*entries* = [], *getentry* = λ*p*. *None* )

**lemma** *pointermap-empty-sane*[*simp, intro!*]: *pointermap-sane empty-pointermap*  
 ⟨*proof*⟩

**definition** *pointermap-insert* *a m* ≡ (*entries* = (*entries m*)@[*a*], *getentry* = (*getentry* *m*)(*a* ↦ *length (entries m)*))

**definition** *pm-pth* *m p* ≡ *entries m ! p*

**definition** *pointermap-p-valid* *p m* ≡ *p < length (entries m)*

**definition** *pointermap-getmk* *a m* ≡ (*case getentry m a of Some p ⇒ (p,m) | None*  
 ⇒ *let u = pointermap-insert a m in (the (getentry u a), u)*)

**lemma** *pointermap-sane-appendD*: *pointermap-sane s* ⇒ *m ∉ set (entries s)* ⇒  
*pointermap-sane (pointermap-insert m s)*  
 ⟨*proof*⟩

**lemma** *lentries-noneD*: *getentry s a = None* ⇒ *pointermap-sane s* ⇒ *a ∉ set*  
*(entries s)*  
 ⟨*proof*⟩

**lemma** *pm-pth-append*: *pointermap-p-valid p m* ⇒ *pm-pth (pointermap-insert a*  
*m) p = pm-pth m p*  
 ⟨*proof*⟩

**lemma** *pointermap-insert-in*: *u = (pointermap-insert a m)* ⇒ *pm-pth u (the*  
*(getentry u a)) = a*  
 ⟨*proof*⟩

**lemma** *pointermap-insert-p-validI*: *pointermap-p-valid p m* ⇒ *pointermap-p-valid*  
*p (pointermap-insert a m)*  
 ⟨*proof*⟩

**thm** *nth-eq-iff-index-eq*

**lemma** *pth-eq-iff-index-eq*:  $\text{pointermap-sane } m \implies \text{pointermap-p-valid } p1 \ m \implies \text{pointermap-p-valid } p2 \ m \implies (\text{pm-pth } m \ p1 = \text{pm-pth } m \ p2) \longleftrightarrow (p1 = p2)$   
 ⟨proof⟩

**lemma** *pointermap-p-valid-updateI*:  $\text{pointermap-sane } m \implies \text{getentry } m \ a = \text{None} \implies u = \text{pointermap-insert } a \ m \implies p = \text{the } (\text{getentry } u \ a) \implies \text{pointermap-p-valid } p \ u$   
 ⟨proof⟩

**lemma** *pointermap-get-validI*:  $\text{pointermap-sane } m \implies \text{getentry } m \ a = \text{Some } p \implies \text{pointermap-p-valid } p \ m$   
 ⟨proof⟩

**lemma** *pointermap-sane-getmkD*:  
 assumes *sn*:  $\text{pointermap-sane } m$   
 assumes *res*:  $\text{pointermap-getmk } a \ m = (p, u)$   
 shows  $\text{pointermap-sane } u \wedge \text{pointermap-p-valid } p \ u$   
 ⟨proof⟩

**lemma** *pointermap-update-pthI*:  
 assumes *sn*:  $\text{pointermap-sane } m$   
 assumes *res*:  $\text{pointermap-getmk } a \ m = (p, u)$   
 shows  $\text{pm-pth } u \ p = a$   
 ⟨proof⟩

**lemma** *pointermap-p-valid-inv*:  
 assumes  $\text{pointermap-p-valid } p \ m$   
 assumes  $\text{pointermap-getmk } a \ m = (x, u)$   
 shows  $\text{pointermap-p-valid } p \ u$   
 ⟨proof⟩

**lemma** *pointermap-p-pth-inv*:  
 assumes *pv*:  $\text{pointermap-p-valid } p \ m$   
 assumes *u*:  $\text{pointermap-getmk } a \ m = (x, u)$   
 shows  $\text{pm-pth } u \ p = \text{pm-pth } m \ p$   
 ⟨proof⟩

**lemma** *pointermap-backward-valid*:  
 assumes *puv*:  $\text{pointermap-p-valid } p \ u$   
 assumes *u*:  $\text{pointermap-getmk } a \ m = (x, u)$   
 assumes *ne*:  $x \neq p$   
 shows  $\text{pointermap-p-valid } p \ m$

⟨proof⟩

end

## 7 Functional interpretation for the abstract implementation

```

theory Middle-Impl
imports Abstract-Impl Pointer-Map
begin

```

For the lack of a better name, the suffix *mi* stands for middle-implementation. This reflects that this “implementation” is neither entirely abstract, nor has it been made fully concrete: the data structures are decided, but not their implementations.

```

record bdd =
  dpm :: (nat × nat × nat) pointermap
  dcl :: ((nat × nat × nat), nat) map

```

**definition** *emptymi* ≡ ( $\lambda$  *dpm* = *empty-pointermap*, *dcl* = *Map.empty*)

```

fun destrmi :: nat ⇒ bdd ⇒ (nat, nat) IFEXD where
destrmi 0 bdd = FD |
destrmi (Suc 0) bdd = TD |
destrmi (Suc (Suc n)) bdd = (case pm-pth (dpm bdd) n of (v, t, e) ⇒ IFD v t e)
fun tmi where tmi bdd = (1, bdd)
fun fmi where fmi bdd = (0, bdd)
fun ifmi :: nat ⇒ nat ⇒ nat ⇒ bdd ⇒ (nat × bdd) where
ifmi v t e bdd = (if t = e
  then (t, bdd)
  else (let (r, pm) = pointermap-getmk (v, t, e) (dpm bdd) in
    (Suc (Suc r), dpm-update (const pm) bdd)))

```

```

fun Rmi-g :: nat ⇒ nat ifex ⇒ bdd ⇒ bool where
Rmi-g 0 Falseif bdd = True |
Rmi-g (Suc 0) Trueif bdd = True |
Rmi-g (Suc (Suc n)) (IF v t e) bdd = (pointermap-p-valid n (dpm bdd)
  ∧ (case pm-pth (dpm bdd) n of (nv, nt, ne) ⇒ nv = v ∧ Rmi-g nt t bdd ∧ Rmi-g ne e bdd)) |
Rmi-g - - - = False

```

**definition** *Rmi s* ≡ {(*a*,*b*)|*a b. Rmi-g a b s*}

**interpretation** *mi-pre*: *bdd-impl-cmp-pre Rmi* ⟨*proof*⟩

**definition** *bdd-node-valid bdd n* ≡ *n* ∈ *Domain* (*Rmi bdd*)

```

lemma [simp]:
  bdd-node-valid bdd 0
  bdd-node-valid bdd (Suc 0)
  ⟨proof⟩

```

**definition** *ifexd-valid bdd e* ≡ (case *e* of *IFD* - *t e* ⇒ *bdd-node-valid bdd t* ∧ *bdd-node-valid bdd e* | - ⇒ *True*)

**definition** *bdd-sane*  $bdd \equiv \text{pointermap-sane } (dpm \ bdd) \wedge \text{mi-pre.map-invar-impl } (dcl \ bdd) \ bdd$

**lemma** [*simp,intro!*]: *bdd-sane emptymi*  
 $\langle \text{proof} \rangle$

**lemma** *prod-split3*:  $P (\text{case } p \text{ of } (x, xa, xaa) \Rightarrow f \ x \ xa \ xaa) = (\forall x1 \ x2 \ x3. p = (x1, x2, x3) \longrightarrow P (f \ x1 \ x2 \ x3))$   
 $\langle \text{proof} \rangle$

**lemma** *IFI*:  $(c \Longrightarrow P \ x) \Longrightarrow (\neg c \Longrightarrow P \ y) \Longrightarrow P (\text{if } c \text{ then } x \ \text{else } y)$   $\langle \text{proof} \rangle$

**lemma** *fstsndI*:  $x = (a, b) \Longrightarrow \text{fst } x = a \wedge \text{snd } x = b$   $\langle \text{proof} \rangle$

**thm** *nat.split*

**lemma** *Rmi-g-2-split*:  $P (Rmi-g \ n \ x \ m) = ((x = \text{Falseif} \longrightarrow P (Rmi-g \ n \ x \ m)) \wedge (x = \text{Trueif} \longrightarrow P (Rmi-g \ n \ x \ m)) \wedge (\forall \ vs \ ts \ es. x = IF \ vs \ ts \ es \longrightarrow P (Rmi-g \ n \ x \ m)))$   
 $\langle \text{proof} \rangle$

**lemma** *rmigeq*:  $Rmi-g \ ni1 \ n1 \ s \Longrightarrow Rmi-g \ ni2 \ n2 \ s \Longrightarrow ni1 = ni2 \Longrightarrow n1 = n2$   
 $\langle \text{proof} \rangle$

**lemma** *rmigneq*:  $bdd\text{-sane } s \Longrightarrow Rmi-g \ ni1 \ n1 \ s \Longrightarrow Rmi-g \ ni2 \ n2 \ s \Longrightarrow ni1 \neq ni2 \Longrightarrow n1 \neq n2$   
 $\langle \text{proof} \rangle$

**lemma** *ifmi-les-hlp*:  $\text{pointermap-sane } (dpm \ s) \Longrightarrow \text{pointermap-getmk } (v, ni1, ni2) (dpm \ s) = (x1, dpm \ s') \Longrightarrow Rmi-g \ nia \ n \ s \Longrightarrow Rmi-g \ nia \ n \ s'$   
 $\langle \text{proof} \rangle$

**lemma** *ifmi-les*:

**assumes** *bdd-sane*  $s$   
**assumes** *ifmi*  $v \ ni1 \ ni2 \ s = (ni, s')$   
**shows** *mi-pre.les*  $s \ s'$

$\langle \text{proof} \rangle$

**lemma** *ifmi-notouch-dcl*:  $\text{ifmi } v \ ni1 \ ni2 \ s = (ni, s') \Longrightarrow dcl \ s' = dcl \ s$   
 $\langle \text{proof} \rangle$

**lemma** *ifmi-saneI*:  $bdd\text{-sane } s \Longrightarrow \text{ifmi } v \ ni1 \ ni2 \ s = (ni, s') \Longrightarrow bdd\text{-sane } s'$   
 $\langle \text{proof} \rangle$

**lemma** *rmigif*:  $Rmi-g \ ni \ (IF \ v \ n1 \ n2) \ s \Longrightarrow \exists n. ni = \text{Suc } (\text{Suc } n)$   
 $\langle \text{proof} \rangle$

**lemma** *in-lesI*:

**assumes** *mi-pre.les*  $s \ s'$   
**assumes**  $(ni1, n1) \in Rmi \ s$   
**assumes**  $(ni2, n2) \in Rmi \ s$



**shows**  $(ni1, n1) \in Rmi\ s' \ (ni2, n2) \in Rmi\ s'$   
 $\langle proof \rangle$

**lemma** *ifmi-modification-validI*:

**assumes** *sane*: *bdd-sane* *s*  
**assumes** *ifm*: *ifmi* *v* *ni1* *ni2* *s* =  $(ni, s')$   
**assumes** *vld*: *bdd-node-valid* *s* *n*  
**shows** *bdd-node-valid* *s'* *n*  
 $\langle proof \rangle$

**definition** *tmi'* *s*  $\equiv do \{oassert \ (bdd-sane \ s); \ Some \ (tmi \ s)\}$

**definition** *fmi'* *s*  $\equiv do \{oassert \ (bdd-sane \ s); \ Some \ (fmi \ s)\}$

**definition** *ifmi'* *v* *ni1* *ni2* *s*  $\equiv do \{oassert \ (bdd-sane \ s \wedge \ bdd-node-valid \ s \ ni1 \wedge \ bdd-node-valid \ s \ ni2); \ Some \ (ifmi \ v \ ni1 \ ni2 \ s)\}$

**lemma** *ifmi'-spec*:  $\llbracket bdd-sane \ s; \ bdd-node-valid \ s \ ni1; \ bdd-node-valid \ s \ ni2 \rrbracket \implies \ ospec \ (ifmi' \ v \ ni1 \ ni2 \ s) \ (\lambda r. \ r = ifmi \ v \ ni1 \ ni2 \ s)$   
 $\langle proof \rangle$

**lemma** *ifmi'-ifmi*:  $\llbracket bdd-sane \ s; \ bdd-node-valid \ s \ ni1; \ bdd-node-valid \ s \ ni2 \rrbracket \implies ifmi' \ v \ ni1 \ ni2 \ s = Some \ (ifmi \ v \ ni1 \ ni2 \ s)$   
 $\langle proof \rangle$

**definition** *destrmi'* *ni* *s*  $\equiv do \{oassert \ (bdd-sane \ s \wedge \ bdd-node-valid \ s \ ni); \ Some \ (destrmi \ ni \ s)\}$

**lemma** *destrmi-someD*: *destrmi'* *e* *bdd* = *Some* *x*  $\implies bdd-sane \ bdd \wedge \ bdd-node-valid \ bdd \ e$   
 $\langle proof \rangle$

**lemma** *Rmi-sv*:

**assumes** *bdd-sane* *s*  $(ni, n) \in Rmi \ s \ (ni', n') \in Rmi \ s$   
**shows**  $ni=ni' \implies n=n'$   
**and**  $ni \neq ni' \implies n \neq n'$   
 $\langle proof \rangle$

**lemma** *True-rep[simp]*: *bdd-sane* *s*  $\implies (ni, Trueif) \in Rmi \ s \longleftrightarrow ni = Suc \ 0$   
 $\langle proof \rangle$

**lemma** *False-rep[simp]*: *bdd-sane* *s*  $\implies (ni, Falseif) \in Rmi \ s \longleftrightarrow ni = 0$   
 $\langle proof \rangle$

**definition** *updS* *s* *x* *r* = *dcl-update*  $(\lambda m. \ m(x \mapsto r)) \ s$

**thm** *Rmi-g.induct*

**lemma** *updS-dpm*: *dpm* (*updS* *s* *x* *r*) = *dpm* *s*  
 $\langle proof \rangle$

**lemma** *updS-Rmi-g*: *Rmi-g* *n* *i* (*updS* *s* *x* *r*) = *Rmi-g* *n* *i* *s*

*<proof>*

**lemma** *updS-Rmi*:  $Rmi\ (updS\ s\ x\ r) = Rmi\ s$   
*<proof>*

**interpretation** *mi*: *bdd-impl-cmp bdd-sane Rmi tmi' fmi' ifmi' destrmi' dcl updS op =*  
*<proof>*

**lemma** *p-valid-RmiI*:  $(Suc\ (Suc\ na),\ b) \in Rmi\ bdd \implies pointermap-p-valid\ na$   
 $(dpm\ bdd)$   
*<proof>*

**lemma** *n-valid-RmiI*:  $(na,\ b) \in Rmi\ bdd \implies bdd-node-valid\ bdd\ na$   
*<proof>*

**lemma** *n-valid-Rmi-alt*:  $bdd-node-valid\ bdd\ na \longleftrightarrow (\exists\ b.\ (na,\ b) \in Rmi\ bdd)$   
*<proof>*

**lemma** *ifmi-result-validI*:

**assumes** *sane*: *bdd-sane s*

**assumes** *vld*: *bdd-node-valid s ni1 bdd-node-valid s ni2*

**assumes** *ifm*: *ifmi v ni1 ni2 s = (ni, s')*

**shows** *bdd-node-valid s' ni*

*<proof>*

**end**

## 8 Array List

Most of this has been contributed by Peter Lammich.

**theory** *Array-List*

**imports**

*Separation-Logic-Imperative-HOL.Array-Blit*

**begin**

This implements a datastructure that efficiently supports two operations: appending an element and looking up the *n*th element. The implementation is straightforward.

As underlying data structure an array is used. Since changing the length of an array requires copying, we double the size whenever the array needs to be expanded. We use a counter for the current length to track which elements are used and which are spares.

**type-synonym** *'a array-list* = *'a array*  $\times$  *nat*

**definition** *is-array-list*  $l \equiv \lambda(a,n).\ \exists_A l'.\ a \mapsto_a l' * \uparrow(n \leq \text{length } l' \wedge l = \text{take } n\ l' \wedge \text{length } l' > 0)$

**definition** *initial-capacity*  $\equiv 16::nat$

**definition** *arl-empty*  $\equiv do \{$   
   $a \leftarrow Array.new\ initial-capacity\ default;$   
   $return\ (a,0)$   
 $\}$

**lemma** [*sep-heap-rules*]:  $\langle emp \rangle arl-empty \langle is-array-list\ [] \rangle$   
 $\langle proof \rangle$

**definition** *arl-nth*  $\equiv \lambda(a,n)\ i.\ do \{$   
   $Array.nth\ a\ i$   
 $\}$

**lemma** [*sep-heap-rules*]:  $i < length\ l \implies \langle is-array-list\ l\ a \rangle arl-nth\ a\ i < \lambda x.$   
 $is-array-list\ l\ a * \uparrow(x = l!i) >$   
 $\langle proof \rangle$

**definition** *arl-append*  $\equiv \lambda(a,n)\ x.\ do \{$   
   $len \leftarrow Array.len\ a;$   
  
   $if\ n < len\ then\ do \{$   
     $a \leftarrow Array.upd\ n\ x\ a;$   
     $return\ (a, n+1)$   
   $\}$   $else\ do \{$   
     $let\ newcap = 2 * len;$   
     $a \leftarrow array-grow\ a\ newcap\ default;$   
     $a \leftarrow Array.upd\ n\ x\ a;$   
     $return\ (a, n+1)$   
   $\}$   
 $\}$

**lemma** [*sep-heap-rules*]:  
 $\langle is-array-list\ l\ a \rangle$   
   $arl-append\ a\ x$   
 $\langle \lambda a.\ is-array-list\ (l@[x])\ a \rangle_t$   
 $\langle proof \rangle$

**lemma** *is-array-list-prec*: *precise is-array-list*  
 $\langle proof \rangle$

**lemma** *is-array-list-lengthIA*:  $is-array-list\ l\ li \implies_A \uparrow(snd\ li = length\ l) * true$   
 $\langle proof \rangle$

**find-consts** *assn*  $\Rightarrow bool$

**lemma** *is-array-list-lengthI*:  $x \models is-array-list\ l\ li \implies snd\ li = length\ l$   
 $\langle proof \rangle$

**end**

## 9 Imperative implementation for Pointermap

```

theory Pointer-Map-Impl
imports Array-List
         Separation-Logic-Imperative-HOL.Sep-Main
         Separation-Logic-Imperative-HOL.Hash-Map-Impl
         Pointer-Map
begin

  record 'a pointermap-impl =
    entriesi :: 'a array-list
    getentryi :: ('a,nat) hashtable
  lemma pointermapieq-exhaust: entries a = entries b  $\implies$  getentry a = getentry b  $\implies a = (b :: 'a pointermap)$  <proof>

  definition is-pointermap-impl :: ('a::{hashable,heap}) pointermap  $\Rightarrow$  'a pointermap-impl
 $\Rightarrow$  assn where
    is-pointermap-impl b bi  $\equiv$ 
      is-array-list (entries b) (entriesi bi)
      * is-hashmap (getentry b) (getentryi bi)

  lemma is-pointermap-impl-prec: precise is-pointermap-impl
    <proof>

  definition pointermap-empty where
    pointermap-empty  $\equiv$  do {
      hm  $\leftarrow$  hm-new;
      arl  $\leftarrow$  arl-empty;
      return (entriesi = arl, getentryi = hm )
    }

  lemma [sep-heap-rules]: < emp > pointermap-empty <is-pointermap-impl empty-pointermap>t
    <proof>

  definition pm-pthi where
    pm-pthi m p  $\equiv$  arl-nth (entriesi m) p

  lemma [sep-heap-rules]: pointermap-sane m  $\implies$  pointermap-p-valid p m  $\implies$ 
    < is-pointermap-impl m mi > pm-pthi mi p < $\lambda$ ai. is-pointermap-impl m mi *
     $\uparrow$ (ai = pm-pth m p)>
    <proof>

  definition pointermap-getmki where
    pointermap-getmki a m  $\equiv$  do {
      lo  $\leftarrow$  ht-lookup a (getentryi m);
      (case lo of
        Some l  $\Rightarrow$  return (l,m) |
        None  $\Rightarrow$  do {
          p  $\leftarrow$  return (snd (entriesi m)));
    }

```

```

    ent ← arl-append (entriesi m) a;
    lut ← hm-update a p (getentryi m);
    u ← return (|entriesi = ent, getentryi = lut|);
    return (p,u)
  }
)
}

```

**lemmas** *pointermap-getmki-defs* = *pointermap-getmki-def* *pointermap-getmk-def*  
*pointermap-insert-def* *is-pointermap-impl-def*

**lemma** [*sep-heap-rules*]: *pointermap-sane*  $m \implies$  *pointermap-getmk*  $a\ m = (p,u)$   
 $\implies$   
 $\langle$  *is-pointermap-impl*  $m\ mi$   $\rangle$   
*pointermap-getmki*  $a\ mi$   
 $\langle \lambda(pi,ui). is\_pointermap\_impl\ u\ ui * \uparrow(pi = p) \rangle_t$   
 $\langle proof \rangle$

**end**

## 10 Imperative implementation

**theory** *Conc-Impl*

**imports** *Pointer-Map-Impl* *Middle-Impl*

**begin**

**record** *bddi* =

*dpmi* :: (*nat* × *nat* × *nat*) *pointermap-impl*

*dcli* :: ((*nat* × *nat* × *nat*), *nat*) *hashtable*

**lemma** *bdd-exhaust*: *dpm*  $a = dpm\ b \implies$  *dcl*  $a = dcl\ b \implies$   $a = (b :: bdd) \langle proof \rangle$

**instantiation** *prod* :: (*default*, *default*) *default*

**begin**

**definition** *default-prod* :: ( $'a \times 'b$ )  $\equiv$  (*default*, *default*)

**instance**  $\langle proof \rangle$

**end**

**instantiation** *nat* :: *default*

**begin**

**definition** *default-nat*  $\equiv 0 :: nat$

**instance**  $\langle proof \rangle$

**end**

**definition** *is-bdd-impl* (*bdd*::*bdd*) (*bddi*::*bddi*) = *is-pointermap-impl* (*dpm* *bdd*)  
(*dpmi* *bddi*) \* *is-hashmap* (*dcl* *bdd*) (*dcli* *bddi*)

**lemma** *is-bdd-impl-prec*: *precise is-bdd-impl*

$\langle proof \rangle$

**definition** *emptyci* :: *bddi* *Heap*  $\equiv$  *do* { *ep* ← *pointermap-empty*; *ehm* ← *hm-new*;

*return* ( $\langle dpmi=ep, dcli=ehm \rangle$ ) }

**definition** *tci* *bdd*  $\equiv$  *return* ( $1::nat, bdd::bddi$ )

**definition** *fci* *bdd*  $\equiv$  *return* ( $0::nat, bdd::bddi$ )

**definition** *ifci* *v t e bdd*  $\equiv$  (if  $t = e$  then *return* ( $t, bdd$ ) else do {  
 $(p, u) \leftarrow pointermap-getmki (v, t, e) (dpmi\ bdd)$ ;  
*return* (*Suc* (*Suc*  $p$ ), *dpmi-update* (*const*  $u$ ) *bdd*)  
 })

**definition** *destrci*  $:: nat \Rightarrow bddi \Rightarrow (nat, nat)$  *IFEXD Heap where*

*destrci*  $n\ bdd \equiv$  (case  $n$  of

$0 \Rightarrow$  *return* *FD* |

*Suc*  $0 \Rightarrow$  *return* *TD* |

*Suc* (*Suc*  $p$ )  $\Rightarrow pm-pthi (dpmi\ bdd)\ p \gg (\lambda(v,t,e). \textit{return} (IFD\ v\ t\ e))$ )

**term** *mi.les*

**lemma** *emptyci-rule*[*sep-heap-rules*]:  $\langle emp \rangle\ \textit{emptyci} \langle is-bdd-impl\ \textit{emptymi} \rangle_t$   
 $\langle proof \rangle$

**lemma** [*sep-heap-rules*]: *tmi'* *bdd* = *Some* ( $p, bdd'$ )  
 $\Rightarrow \langle is-bdd-impl\ bdd\ bddi \rangle$   
 $\quad tci\ bddi$   
 $\quad \langle \lambda(pi, bddi'). is-bdd-impl\ bdd'\ bddi' * \uparrow(pi = p) \rangle$   
 $\langle proof \rangle$

**lemma** [*sep-heap-rules*]: *fmi'* *bdd* = *Some* ( $p, bdd'$ )  
 $\Rightarrow \langle is-bdd-impl\ bdd\ bddi \rangle$   
 $\quad fci\ bddi$   
 $\quad \langle \lambda(pi, bddi'). is-bdd-impl\ bdd'\ bddi' * \uparrow(pi = p) \rangle$   
 $\langle proof \rangle$

**lemma** [*sep-heap-rules*]: *ifmi'* *v t e bdd* = *Some* ( $p, bdd'$ )  $\Rightarrow$   
 $\langle is-bdd-impl\ bdd\ bddi \rangle\ \textit{ifci}\ v\ t\ e\ bddi$   
 $\langle \lambda(pi, bddi'). is-bdd-impl\ bdd'\ bddi' * \uparrow(pi = p) \rangle_t$   
 $\langle proof \rangle$

**lemma** *destrci-rule*[*sep-heap-rules*]:  
 $\textit{destrmi}'\ n\ bdd = \textit{Some}\ r \Rightarrow$   
 $\langle is-bdd-impl\ bdd\ bddi \rangle\ \textit{destrci}\ n\ bddi$   
 $\langle \lambda r'. is-bdd-impl\ bdd\ bddi * \uparrow(r' = r) \rangle$   
 $\langle proof \rangle$

**term** *mi.restrict-top-impl*

**thm** *mi.case-ifexi-def*

**definition** *case-ifexici* *fti ffi fii ni bddi*  $\equiv$  do {  
 $\textit{dest} \leftarrow \textit{destrci}\ ni\ bddi$ ;  
 case  $\textit{dest}$  of *TD*  $\Rightarrow$  *fti* | *FD*  $\Rightarrow$  *ffi* | *IFD*  $v\ ti\ ei \Rightarrow$  *fii*  $v\ ti\ ei$   
 }

**lemma** [sep-decon-rules]:  
**assumes**  $S$ :  $mi.case\text{-}ifexi\ fti\ ffi\ fii\ ni\ bdd = Some\ r$   
**assumes** [sep-heap-rules]:  
 $destrmi'\ ni\ bdd = Some\ TD \implies fti\ bdd = Some\ r \implies \langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle$   
 $ftci\ \langle Q \rangle$   
 $destrmi'\ ni\ bdd = Some\ FD \implies ffi\ bdd = Some\ r \implies \langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle$   
 $ffci\ \langle Q \rangle$   
 $\bigwedge v\ t\ e.\ destrmi'\ ni\ bdd = Some\ (IFD\ v\ t\ e) \implies fii\ v\ t\ e\ bdd = Some\ r$   
 $\implies \langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle\ fici\ v\ t\ e\ \langle Q \rangle$   
**shows**  $\langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle\ case\text{-}ifexi\ fti\ ffi\ fii\ ni\ bddi\ \langle Q \rangle$   
 $\langle proof \rangle$

**definition**  $restrict\text{-}topci\ p\ vr\ vl\ bdd =$   
 $case\text{-}ifexi$   
 $(return\ p)$   
 $(return\ p)$   
 $(\lambda v\ te\ ee.\ return\ (if\ v = vr\ then\ (if\ vl\ then\ te\ else\ ee)\ else\ p))$   
 $p\ bdd$

**lemma** [sep-heap-rules]:  
**assumes**  $mi.restrict\text{-}top\text{-}impl\ p\ var\ val\ bdd = Some\ (r, bdd')$   
**shows**  $\langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle\ restrict\text{-}topci\ p\ var\ val\ bddi$   
 $\langle \lambda ri.\ is\text{-}bdd\text{-}impl\ bdd\ bddi * \uparrow(ri = r) \rangle$   
 $\langle proof \rangle$

**fun**  $lowest\text{-}topsci\ where$   
 $lowest\text{-}topsci\ []\ s = return\ None\ |$   
 $lowest\text{-}topsci\ (e\#es)\ s =$   
 $case\text{-}ifexi$   
 $(lowest\text{-}topsci\ es\ s)$   
 $(lowest\text{-}topsci\ es\ s)$   
 $(\lambda v\ t\ e.\ do\ \{$   
 $(rec) \leftarrow lowest\text{-}topsci\ es\ s;$   
 $(case\ rec\ of$   
 $Some\ u \Rightarrow return\ ((Some\ (min\ u\ v)))\ |$   
 $None \Rightarrow return\ ((Some\ v)))$   
 $\})\ e\ s$

**declare**  $lowest\text{-}topsci.simps[simp\ del]$

**lemma** [sep-heap-rules]:  
**assumes**  $mi.lowest\text{-}top\text{-}impl\ es\ bdd = Some\ (r, bdd')$   
**shows**  $\langle is\text{-}bdd\text{-}impl\ bdd\ bddi \rangle\ lowest\text{-}topsci\ es\ bddi$   
 $\langle \lambda(ri).\ is\text{-}bdd\text{-}impl\ bdd\ bddi * \uparrow(ri = r \wedge bdd' = bdd) \rangle$   
 $\langle proof \rangle$

**partial-function**( $heap$ )  $iteci\ where$

```

iteci i t e s = do {
  (lt) ← lowest-topsci [i, t, e] s;
  case lt of
    Some a ⇒ do {
      ti ← restrict-topci i a True s;
      tt ← restrict-topci t a True s;
      te ← restrict-topci e a True s;
      fi ← restrict-topci i a False s;
      ft ← restrict-topci t a False s;
      fe ← restrict-topci e a False s;
      (tb,s') ← iteci ti tt te s;
      (fb,s'') ← iteci fi ft fe s';
      (ifci a tb fb s'')
    }
  | None ⇒ do {
    case-ifexici (return (t,s)) (return (e,s)) (λ- - -. raise "Cannot happen") i s
  }
}
declare iteci.simps[code]

```

**lemma** *iteci-rule*:

```

( mi.ite-impl i t e bdd = Some (p,bdd') ) →
<is-bdd-impl bdd bddi>
  iteci i t e bddi
<λ(pi,bddi'). is-bdd-impl bdd' bddi' * ↑(pi=p )>_t
<proof>

```

**declare** *iteci-rule*[*THEN mp, sep-heap-rules*]

**definition** *param-optci where*

```

param-optci i t e bdd = do {
  (tr, bdd) ← tci bdd;
  (fl, bdd) ← fci bdd;
  id ← destrci i bdd;
  td ← destrci t bdd;
  ed ← destrci e bdd;
  return (
    if id = TD then Some t else
    if id = FD then Some e else
    if td = TD ∧ ed = FD then Some i else
    if t = e then Some t else
    if ed = TD ∧ i = t then Some tr else
    if td = FD ∧ i = e then Some fl else
    None, bdd)
}

```

**lemma** *param-optci-rule*:

```

( mi.param-opt-impl i t e bdd = Some (p,bdd') ) ⇒
<is-bdd-impl bdd bddi>

```



$\text{param-optci } i \ t \ e \ \text{bddi}$   
 $\langle \lambda(pi, \text{bddi}'). \text{is-bdd-impl } \text{bdd}' \ \text{bddi}' * \uparrow(pi=p) \rangle_t$   
 $\langle \text{proof} \rangle$

**lemma** *bdd-hm-lookup-rule*:

$(\text{dcl } \text{bdd } (i, t, e) = p) \implies$   
 $\langle \text{is-bdd-impl } \text{bdd } \text{bddi} \rangle$   
 $\text{hm-lookup } (i, t, e) \ (\text{dcli } \text{bddi})$   
 $\langle \lambda(pi). \text{is-bdd-impl } \text{bdd } \text{bddi} * \uparrow(pi = p) \rangle_t$   
 $\langle \text{proof} \rangle$

**lemma** *bdd-hm-update-rule*<sup>[sep-heap-rules]</sup>:

$\langle \text{is-bdd-impl } \text{bdd } \text{bddi} \rangle$   
 $\text{hm-update } k \ v \ (\text{dcli } \text{bddi})$   
 $\langle \lambda r. \text{is-bdd-impl } (\text{updS } \text{bdd } k \ v) \ (\text{dcli-update } (\text{const } r) \ \text{bddi}) * \text{true} \rangle$   
 $\langle \text{proof} \rangle$

**partial-function**(*heap*) *iteci-lu* **where**

$\text{iteci-lu } i \ t \ e \ s = \text{do } \{$   
 $\text{lu} \leftarrow \text{ht-lookup } (i, t, e) \ (\text{dcli } s);$   
 $(\text{case } \text{lu} \text{ of } \text{Some } b \Rightarrow \text{return } (b, s)$   
 $\mid \text{None} \Rightarrow \text{do } \{$   
 $\text{(po, s)} \leftarrow \text{param-optci } i \ t \ e \ s;$   
 $(\text{case } \text{po} \text{ of } \text{Some } b \Rightarrow \text{do } \{$   
 $\text{return } (b, s) \}$   
 $\mid \text{None} \Rightarrow \text{do } \{$   
 $(lt) \leftarrow \text{lowest-topsci } [i, t, e] \ s;$   
 $(\text{case } lt \text{ of } \text{Some } a \Rightarrow \text{do } \{$   
 $ti \leftarrow \text{restrict-topci } i \ a \ \text{True } s;$   
 $tt \leftarrow \text{restrict-topci } t \ a \ \text{True } s;$   
 $te \leftarrow \text{restrict-topci } e \ a \ \text{True } s;$   
 $fi \leftarrow \text{restrict-topci } i \ a \ \text{False } s;$   
 $ft \leftarrow \text{restrict-topci } t \ a \ \text{False } s;$   
 $fe \leftarrow \text{restrict-topci } e \ a \ \text{False } s;$   
 $(tb, s) \leftarrow \text{iteci-lu } ti \ tt \ te \ s;$   
 $(fb, s) \leftarrow \text{iteci-lu } fi \ ft \ fe \ s;$   
 $(r, s) \leftarrow \text{ifci } a \ tb \ fb \ s;$   
 $cl \leftarrow \text{hm-update } (i, t, e) \ r \ (\text{dcli } s);$   
 $\text{return } (r, \text{dcli-update } (\text{const } cl) \ s)$   
 $\}$   
 $\mid \text{None} \Rightarrow \text{raise } \text{"Cannot happen"} \}) \}$   
 $\}} \}$

**term** *ht-lookup*

**declare** *iteci-lu.simps*[code]

**thm** *iteci-lu.simps*[unfolded restrict-topci-def case-ifexici-def param-optci-def lowest-topsci.simps]

**partial-function**(*heap*) *iteci-lu-code* **where** *iteci-lu-code*  $i \ t \ e \ s = \text{do } \{$

$\text{lu} \leftarrow \text{hm-lookup } (i, t, e) \ (\text{dcli } s);$   
 $\text{case } \text{lu} \text{ of } \text{None} \Rightarrow \text{let } \text{po} = \text{if } i = 1 \text{ then } \text{Some } t$

```

else if i = 0 then Some e else if t = 1 ∧ e = 0 then Some
i else if t = e then Some t else if e = 1 ∧ i = t then Some 1 else if t = 0 ∧ i =
e then Some 0 else None
  in case po of None ⇒ do {
    id ← destrci i s;
    td ← destrci t s;
    ed ← destrci e s;
    let a = (case id of IFD v t e ⇒ v);
    let a = (case td of IFD v t e ⇒ min a v | - ⇒ a);
    let a = (case ed of IFD v t e ⇒ min a v | - ⇒ a);
    let ti = (case id of IFD v ti ei ⇒ if v = a then ti
else i | - ⇒ i);
    let tt = (case td of IFD v ti ei ⇒ if v = a then ti
else t | - ⇒ t);
    let te = (case ed of IFD v ti ei ⇒ if v = a then ti
else e | - ⇒ e);
    let fi = (case id of IFD v ti ei ⇒ if v = a then ei
else i | - ⇒ i);
    let ft = (case td of IFD v ti ei ⇒ if v = a then ei
else t | - ⇒ t);
    let fe = (case ed of IFD v ti ei ⇒ if v = a then ei
else e | - ⇒ e);
    (tb, s) ← iteci-lu-code ti tt te s;
    (fb, s) ← iteci-lu-code fi ft fe s;
    (r, s) ← ifci a tb fb s;
    cl ← hm-update (i, t, e) r (dcli s);
    return (r, dcli-update (const cl) s)
  }
| Some b ⇒ return (b, s)
}

```

**declare** *iteci-lu-code.simps*[code]

**lemma** *iteci-lu-code*[code-unfold]: *iteci-lu i t e s = iteci-lu-code i t e s*  
⟨proof⟩

**lemma** *iteci-lu-rule*:

```

( mi.ite-impl-lu i t e bdd = Some (p,bdd') ) →
<is-bdd-impl bdd bddi>
  iteci-lu i t e bddi
<λ(pi,bddi). is-bdd-impl bdd' bddi' * ↑(pi=p)>_t
⟨proof⟩

```

## 10.1 A standard library of functions

**declare** *iteci-rule*[THEN mp, sep-heap-rules]

```

definition notci e s ≡ do {
  (f,s) ← fci s;
  (t,s) ← tci s;
  iteci-lu e f t s
}
definition orci e1 e2 s ≡ do {
  (t,s) ← tci s;
  iteci-lu e1 t e2 s
}
definition andci e1 e2 s ≡ do {
  (f,s) ← fci s;
  iteci-lu e1 e2 f s
}
definition norci e1 e2 s ≡ do {
  (r,s) ← orci e1 e2 s;
  notci r s
}
definition nandci e1 e2 s ≡ do {
  (r,s) ← andci e1 e2 s;
  notci r s
}
definition biimpci a b s ≡ do {
  (nb,s) ← notci b s;
  iteci-lu a b nb s
}
definition xorci a b s ≡ do {
  (nb,s) ← notci b s;
  iteci-lu a nb b s
}
definition litci v bdd ≡ do {
  (t,bdd) ← tci bdd;
  (f,bdd) ← fci bdd;
  ifci v t f bdd
}
definition tautci v bdd ≡ do {
  d ← destrci v bdd;
  return (d = TD)
}

```

## 10.2 Printing

The following functions are exported unverified. They are intended for BDD debugging purposes.

**partial-function**(*heap*) *serializeci* :: *nat* ⇒ *bddi* ⇒ ((*nat* × *nat*) × *nat*) *list Heap*

**where**

```

serializeci p s = do {
  d ← destrci p s;
  (case d of

```

```

    IFD v t e ⇒ do {
      r ← serializeci t s;
      l ← serializeci e s;
      return (remdups (((p,t),1),((p,e),0)] @ r @ l)
    } |
  - ⇒ return []
)
}
declare serializeci.simps[code]

fun mapM where
  mapM f [] = return [] |
  mapM f (a#as) = do {
    r ← f a;
    rs ← mapM f as;
    return (r#rs)
  }
definition liftM f ma = do { a ← ma; return (f a) }
definition sequence = mapM id
term liftM (map f)
lemma liftM (map f) (sequence l) = sequence (map (liftM f) l)
  ⟨proof⟩

```

```

fun string-of-nat :: nat ⇒ string where
  string-of-nat n = (if n < 10 then [char-of-nat (48 + n)]
    else string-of-nat (n div 10) @ [char-of-nat (48 + (n mod
10))])

```

```

definition labelci :: bddi ⇒ nat ⇒ (string × string × string) Heap where
  labelci s n = do {
    d ← destrci n s;
    let son = string-of-nat n;
    let label = (case d of
      TD ⇒ "T" |
      FD ⇒ "F" |
      (IFD v -) ⇒ string-of-nat v);
    return (label, son, son @ "[label=" @ label @ "];
  )
}

```

```

definition graphifyci1 bdd a ≡ do {
  let ((f,t),y) = a;
  let c = (string-of-nat f @ " -> " @ string-of-nat t);
  return (c @ (case y of 0 ⇒ "[style=dotted]" | Suc - ⇒ "")) @ "
  )
}

```

```

definition trd = snd ∘ snd

```

**definition**  $fstp = apsnd\ fst$

**definition**  $the\text{-}thing\text{-}By\ f\ l = (let\ nub = remdups\ (map\ fst\ l)\ in\ map\ (\lambda e. (e, map\ snd\ (filter\ (\lambda g. (f\ e\ (fst\ g))))\ l)))\ nub)$

**definition**  $the\text{-}thing = the\text{-}thing\text{-}By\ (op =)$

**definition**  $graphifyci :: string \Rightarrow nat \Rightarrow bddi \Rightarrow string\ Heap$  **where**  
 $graphifyci\ name\ ep\ bdd \equiv do\ \{$   
 $\quad s \leftarrow serializeci\ ep\ bdd;$   
 $\quad let\ e = map\ fst\ s;$   
 $\quad l \leftarrow mapM\ (labelci\ bdd)\ (rev\ (remdups\ (map\ fst\ e\ @\ map\ snd\ e)));$   
 $\quad let\ grp = (map\ (\lambda l. foldr\ (\lambda a\ t. t\ @\ a\ @\ ";\"))\ (snd\ l)\ "\{rank=same;\}"\ @\ "'')$   
 $\quad \} (the\text{-}thing\ (map\ fstp\ l));$   
 $\quad e \leftarrow mapM\ (graphifyci1\ bdd)\ s;$   
 $\quad let\ emptyhlp = (case\ ep\ of\ 0 \Rightarrow "F;$   
 $\quad \quad \quad | Suc\ 0 \Rightarrow "T;$   
 $\quad \quad \quad | - \Rightarrow ""');$   
 $\quad return\ ("digraph\ " @\ name\ @\ " \{$   
 $\quad \quad \quad @\ concat\ (map\ trd\ l)\ @\ concat\ grp\ @\ concat\ e\ @\ emptyhlp\ @\ "'')$   
 $\quad \}$

**end**

## 11 Collapsing the levels

**theory**  $Level\text{-}Collapse$   
**imports**  $Conc\text{-}Impl$   
**begin**

The theory up to this point is implemented in a way that separated the different aspects into different levels. This is highly beneficial for us, since it allows us to tackle the difficulties arising in small chunks. However, exporting this to the user would be highly impractical. Thus, this theory collapses all the different levels (i.e. refinement steps) and relates the computations in the heap monad to  $boolfunc$ .

**definition**  $bddmi\text{-}rel\ cs \equiv \{(a,c) \mid a\ b\ c. (a,b) \in bf\text{-}ifex\text{-}rel \wedge (c,b) \in Rmi\ cs\}$

**definition**  $bdd\text{-}relator :: (nat\ boolfunc \times nat)\ set \Rightarrow bddi \Rightarrow assn$  **where**  
 $bdd\text{-}relator\ p\ s \equiv \exists_A\ cs. is\text{-}bdd\text{-}impl\ cs\ s * \uparrow(p \subseteq (bddmi\text{-}rel\ cs) \wedge bdd\text{-}sane\ cs) * true$

The  $assn$  predicate  $bdd\text{-}relator$  is the interface that is exposed to the user. (The contents of the definition are not exposed.)

**lemma**  $bdd\text{-}relator\text{-}mono[intro]: q \subseteq p \Longrightarrow bdd\text{-}relator\ p\ s \Longrightarrow_A\ bdd\text{-}relator\ q\ s$   
 $\langle proof \rangle$

**lemma** *bdd-relator-absorb-true*[*simp*]:  $\text{bdd-relator } p \ s * \text{true} = \text{bdd-relator } p \ s$   
 ⟨*proof*⟩

**thm** *bdd-relator-def*[*unfolded bddmi-rel-def, simplified*]

**lemma** *join-hlp1*:  $\text{is-bdd-impl } a \ s * \text{is-bdd-impl } b \ s \implies_A \text{is-bdd-impl } a \ s * \text{is-bdd-impl } b \ s * \uparrow(a = b)$   
 ⟨*proof*⟩

**lemma** *join-hlp*:  $\text{is-bdd-impl } a \ s * \text{is-bdd-impl } b \ s = \text{is-bdd-impl } b \ s * \text{is-bdd-impl } a \ s * \uparrow(a = b)$   
 ⟨*proof*⟩

**lemma** *add-true-asm*:

**assumes**  $\langle b * \text{true} \rangle \ p \ \langle a \rangle_t$

**shows**  $\langle b \rangle \ p \ \langle a \rangle_t$

⟨*proof*⟩

**lemma** *add-anything*:

**assumes**  $\langle b \rangle \ p \ \langle a \rangle$

**shows**  $\langle b * x \rangle \ p \ \langle \lambda r. a \ r * x \rangle_t$

⟨*proof*⟩

**lemma** *add-true*:

**assumes**  $\langle b \rangle \ p \ \langle a \rangle_t$

**shows**  $\langle b * \text{true} \rangle \ p \ \langle a \rangle_t$

⟨*proof*⟩

**definition** *node-relator where*  $\text{node-relator } x \ y \longleftrightarrow x \in y$

*sep-auto* behaves sub-optimal when having  $(bf, bdd) \in \text{computed-pointer-relation}$  as assumption in our cases. Using *node-relator* instead fixes this behavior with a custom solver for *simp*.

**lemma** *node-relatorI*:  $x \in y \implies \text{node-relator } x \ y$  ⟨*proof*⟩

**lemma** *node-relatorD*:  $\text{node-relator } x \ y \implies x \in y$  ⟨*proof*⟩

⟨*ML*⟩

This is the general form one wants to work with: if a function on the bdd is called with a set of already existing and valid pointers, the arguments to the function have to be in that set. The result is that one more pointer is the set of existing and valid pointers.

**thm** *iteci-rule*[*THEN mp*] *mi.ite-impl-R ifex-ite-rel-bf*

**lemma** *iteci-rule*[*sep-heap-rules*]:

$\llbracket \text{node-relator } (ib, ic) \ rp; \text{node-relator } (tb, tc) \ rp; \text{node-relator } (eb, ec) \ rp \rrbracket \implies$

$\langle \text{bdd-relator } rp \ s \rangle$

*iteci-lu ic tc ec s*

$\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-ite } ib \text{ } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *tci-rule*[*sep-heap-rules*]:

$\langle \text{bdd-relator } rp \text{ } s \rangle$   
*tci* *s*  
 $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-True},r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *fci-rule*[*sep-heap-rules*]:

$\langle \text{bdd-relator } rp \text{ } s \rangle$   
*fci* *s*  
 $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-False},r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{proof} \rangle$

IFC/ifmi/ifci require that the variable order is ensured by the user. Instead of using ifci, a combination of litci and iteci has to be used.

**lemma** [*sep-heap-rules*]:

$\llbracket (tb, tc) \in rp; (eb, ec) \in rp \rrbracket \implies$   
 $\langle \text{bdd-relator } rp \text{ } s \rangle$   
*ifci* *v* *tc* *ec* *s*  
 $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-if } v \text{ } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$

This probably doesn't hold.

$\langle \text{proof} \rangle$

**lemma** *notci-rule*[*sep-heap-rules*]:

**assumes** *node-relator* (*tb*, *tc*) *rp*  
**shows**  $\langle \text{bdd-relator } rp \text{ } s \rangle$  *notci* *tc* *s*  $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-not } tb,r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{proof} \rangle$

**lemma** *cirules1*[*sep-heap-rules*]:

**assumes** *node-relator* (*tb*, *tc*) *rp* *node-relator* (*eb*, *ec*) *rp*  
**shows**  
 $\langle \text{bdd-relator } rp \text{ } s \rangle$  *andci* *tc* *ec* *s*  $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-and } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{bdd-relator } rp \text{ } s \rangle$  *orci* *tc* *ec* *s*  $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-or } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{bdd-relator } rp \text{ } s \rangle$  *biimpci* *tc* *ec* *s*  $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-biimp } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$   
 $\langle \text{bdd-relator } rp \text{ } s \rangle$  *xorci* *tc* *ec* *s*  $\langle \lambda(r,s'). \text{bdd-relator } (\text{insert } (\text{bf-xor } tb \text{ } eb,r) \text{ } rp) \text{ } s' \rangle$

$\langle \text{proof} \rangle$

**lemma** *cirules2*[*sep-heap-rules*]:

**assumes** *node-relator* (*tb*, *tc*) *rp* *node-relator* (*eb*, *ec*) *rp*  
**shows**

```

    <bdd-relator rp s> nandci tc ec s < $\lambda(r,s')$ . bdd-relator (insert (bf-nand tb eb,r)
rp) s'>
    <bdd-relator rp s> norci tc ec s < $\lambda(r,s')$ . bdd-relator (insert (bf-nor tb eb,r)
rp) s'>
    <proof>

```

**lemma** *litci-rule*[*sep-heap-rules*]:

```

    <bdd-relator rp s> litci v s < $\lambda(r,s')$ . bdd-relator (insert (bf-lit v,r) rp) s'>
    <proof>

```

**lemma** *tautci-rule*[*sep-heap-rules*]:

```

    shows node-relator (tb, tc) rp  $\implies$  <bdd-relator rp s> tautci tc s < $\lambda r$ . bdd-relator
rp s *  $\uparrow(r \longleftrightarrow tb = \text{bf-True})$ >
    <proof>

```

**lemma** *emptyci-rule*[*sep-heap-rules*]:

```

    shows <emp> emptyci < $\lambda r$ . bdd-relator {} r>
    <proof>

```

**lemmas** [*simp*] = *bf-ite-def*

Efficient comparison of two nodes.

**definition** *eqci* *a b*  $\equiv$  *return (a = b)*

**lemma** *iteeq-rule*[*sep-heap-rules*]:

```

[[node-relator (xb, xc) rp; node-relator (yb, yc) rp]]  $\implies$ 
<bdd-relator rp s>
    eqci xc yc
    < $\lambda r$ .  $\uparrow(r \longleftrightarrow xb = yb)$ >t
    <proof>

```

**end**

## 12 Tests and examples

**theory** *BDD-Examples*

**imports** *Level-Collapse*

**begin**

Just two simple examples:

```

lemma <emp> do {
    s  $\leftarrow$  emptyci;
    (t,s)  $\leftarrow$  tci s;
    tautci t s
} < $\lambda r$ .  $\uparrow(r = \text{True})$ >t
<proof>

```



```

lemma <emp> do {
  s ← emptyci;
  (a,s) ← litci 0 s;
  (b,s) ← litci 1 s;
  (c,s) ← litci 2 s;
  (t1i,s) ← orci a b s;
  (t1,s) ← andci t1i c s;
  (t2i1,s) ← andci a c s;
  (t2i2,s) ← andci b c s;
  (t2,s) ← orci t2i1 t2i2 s;
  eqci t1 t2
} <↑>_t
<proof>

```

**end**

## References

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