

# Quasi-Borel Spaces

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## Abstract

The notion of quasi-Borel spaces was introduced by Heunen et al. [1]. The theory provides a suitable denotational model for higher-order probabilistic programming languages with continuous distributions.

This entry is a formalization of the theory of quasi-Borel spaces, including construction of quasi-Borel spaces (product, coproduct, function spaces), the adjunction between the category of measurable spaces and the category of quasi-Borel spaces, and the probability monad on quasi-Borel spaces. This entry also contains the formalization of the Bayesian regression presented in the work of Heunen et al.

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## 1 Standard Borel Spaces

```

theory StandardBorel
  imports HOL-Probability.Probability
begin

```

A standard Borel space is the Borel space associated with a Polish space. Here, we define standard Borel spaces in another, but equivalent, way. See [1] Proposition 5.

**abbreviation**  $real\text{-}borel \equiv borel :: real\ measure$

**abbreviation**  $nat\text{-}borel \equiv borel :: nat\ measure$

**abbreviation**  $ennreal\text{-}borel \equiv borel :: ennreal\ measure$

**abbreviation**  $bool\text{-}borel \equiv borel :: bool\ measure$

## 1.1 Definition

**locale**  $standard\text{-}borel =$

**fixes**  $M :: 'a\ measure$

**assumes**  $exist\text{-}fg: \exists f \in M \rightarrow_M real\text{-}borel. \exists g \in real\text{-}borel \rightarrow_M M.$   
 $\forall x \in space\ M. (g \circ f)\ x = x$

**begin**

**abbreviation**  $fg \equiv (SOME\ k. (fst\ k) \in M \rightarrow_M real\text{-}borel \wedge$   
 $(snd\ k) \in real\text{-}borel \rightarrow_M M \wedge$   
 $(\forall x \in space\ M. ((snd\ k) \circ (fst\ k))\ x = x))$

**definition**  $f \equiv (fst\ fg)$

**definition**  $g \equiv (snd\ fg)$

**lemma**

**shows**  $f\text{-}meas[simp,measurable] : f \in M \rightarrow_M real\text{-}borel$   
**and**  $g\text{-}meas[simp,measurable] : g \in real\text{-}borel \rightarrow_M M$   
**and**  $gf\text{-}comp\text{-}id[simp]: \bigwedge x. x \in space\ M \implies (g \circ f)\ x = x$   
 $\bigwedge x. x \in space\ M \implies g\ (f\ x) = x$

$\langle proof \rangle$

**lemma**  $standard\text{-}borel\text{-}sets[simp]:$

**assumes**  $sets\ M = sets\ Y$

**shows**  $standard\text{-}borel\ Y$

$\langle proof \rangle$

**lemma**  $f\text{-}inj:$

$inj\text{-}on\ f\ (space\ M)$

$\langle proof \rangle$

**lemma**  $singleton\text{-}sets:$

**assumes**  $x \in space\ M$

**shows**  $\{x\} \in sets\ M$

$\langle proof \rangle$

**lemma**  $countable\text{-}space\text{-}discrete:$

**assumes**  $countable\ (space\ M)$

**shows**  $sets\ M = sets\ (count\text{-}space\ (space\ M))$

$\langle proof \rangle$

**end**

**lemma** *standard-borelI*:

**assumes**  $f \in Y \rightarrow_M \text{real-borel}$   
 $g \in \text{real-borel} \rightarrow_M Y$   
**and**  $\bigwedge y. y \in \text{space } Y \implies (g \circ f) y = y$   
**shows** *standard-borel*  $Y$   
*<proof>*

**locale** *standard-borel-space-UNIV* = *standard-borel* +

**assumes** *space-UNIV*:  $\text{space } M = \text{UNIV}$

**begin**

**lemma** *gf-comp-id'[simp]*:

$g \circ f = \text{id}$   $g (f x) = x$   
*<proof>*

**lemma** *f-inj'*:

*inj*  $f$   
*<proof>*

**lemma** *g-surj'*:

*surj*  $g$   
*<proof>*

**end**

**lemma** *standard-borel-space-UNIV*:

**assumes**  $f \in Y \rightarrow_M \text{real-borel}$   
 $g \in \text{real-borel} \rightarrow_M Y$   
 $(g \circ f) = \text{id}$   
**and**  $\text{space } Y = \text{UNIV}$   
**shows** *standard-borel-space-UNIV*  $Y$   
*<proof>*

**lemma** *standard-borel-space-UNIV'*:

**assumes** *standard-borel*  $Y$   
**and**  $\text{space } Y = \text{UNIV}$   
**shows** *standard-borel-space-UNIV*  $Y$   
*<proof>*

## 1.2 $\mathbb{R}$ , $\mathbb{N}$ , Boolean, $[0, \infty]$

$\mathbb{R}$  is a standard Borel space.

**interpretation** *real* : *standard-borel-space-UNIV* *real-borel*

*<proof>*

A non-empty Borel subspace of  $\mathbb{R}$  is also a standard Borel space.

**lemma** *real-standard-borel-subset*:  
**assumes**  $U \in \text{sets real-borel}$   
**and**  $U \neq \{\}$   
**shows** *standard-borel (restrict-space real-borel U)*  
*<proof>*

A non-empty measurable subset of a standard Borel space is also a standard Borel space.

**lemma**(*in standard-borel*) *standard-borel-subset*:  
**assumes**  $U \in \text{sets } M$   
 $U \neq \{\}$   
**shows** *standard-borel (restrict-space M U)*  
*<proof>*

$\mathbb{N}$  is a standard Borel space.

**interpretation** *nat : standard-borel-space-UNIV nat-borel*  
*<proof>*

For a countable space  $X$ ,  $X$  is a standard Borel space iff  $X$  is a discrete space.

**lemma** *countable-standard-iff*:  
**assumes**  $\text{space } X \neq \{\}$   
**and** *countable (space X)*  
**shows** *standard-borel X  $\longleftrightarrow$  sets X = sets (count-space (space X))*  
*<proof>*

$\mathbb{B}$  is a standard Borel space.

**lemma** *to-bool-measurable*:  
**assumes**  $f - \{ \text{True} \} \cap \text{space } M \in \text{sets } M$   
**shows**  $f \in M \rightarrow_M \text{bool-borel}$   
*<proof>*

**interpretation** *bool : standard-borel-space-UNIV bool-borel*  
*<proof>*

$[0, \infty]$  (the set of extended non-negative real numbers) is a standard Borel space.

**interpretation** *ennreal : standard-borel-space-UNIV ennreal-borel*  
*<proof>*

### 1.3 $\mathbb{R} \times \mathbb{R}$

**definition** *real-to-01open* :: *real  $\Rightarrow$  real* **where**  
*real-to-01open r  $\equiv$  arctan r / pi + 1 / 2*

**definition** *real-to-01open-inverse* :: *real  $\Rightarrow$  real* **where**  
*real-to-01open-inverse r  $\equiv$  tan (pi \* r - (pi / 2))*

**lemma** *real-to-01open-inverse-correct*:  
*real-to-01open-inverse*  $\circ$  *real-to-01open* = *id*  
 ⟨*proof*⟩

**lemma** *real-to-01open-inverse-correct'*:  
**assumes**  $0 < r < 1$   
**shows** *real-to-01open* (*real-to-01open-inverse*  $r$ ) =  $r$   
 ⟨*proof*⟩

**lemma** *real-to-01open-01* :  
 $0 < \text{real-to-01open } r \wedge \text{real-to-01open } r < 1$   
 ⟨*proof*⟩

**lemma** *real-to-01open-continuous*:  
*continuous-on UNIV real-to-01open*  
 ⟨*proof*⟩

**lemma** *real-to-01open-inverse-continuous*:  
*continuous-on*  $\{0 < .. < 1\}$  *real-to-01open-inverse*  
 ⟨*proof*⟩

**lemma** *real-to-01open-inverse-measurable*:  
*real-to-01open-inverse*  $\in$  *restrict-space real-borel*  $\{0 < .. < 1\} \rightarrow_M$  *real-borel*  
 ⟨*proof*⟩

**fun** *r01-binary-expansion''* :: *real*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat*  $\times$  *real*  $\times$  *real* **where**  
*r01-binary-expansion''*  $r$  0 = (if  $1/2 \leq r$  then (1,1,1/2)  
   else (0,1/2,0)) |  
*r01-binary-expansion''*  $r$  (Suc  $n$ ) = (let (-,ur,lr) = *r01-binary-expansion''*  $r$   $n$ ;  
    $k = (ur + lr)/2$  in  
   (if  $k \leq r$  then (1,ur,k)  
   else (0,k,lr)))

$a_n$  where  $r = 0.a_0a_1a_2\dots$  for  $0 < r < 1$ .

**definition** *r01-binary-expansion'* :: *real*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat* **where**  
*r01-binary-expansion'*  $r$   $n \equiv$  *fst* (*r01-binary-expansion''*  $r$   $n$ )

$a_n = 0$  or  $1$ .

**lemma** *real01-binary-expansion'-0or1*:  
*r01-binary-expansion'*  $r$   $n \in \{0,1\}$   
 ⟨*proof*⟩

**definition** *r01-binary-sum* :: (*nat*  $\Rightarrow$  *nat*)  $\Rightarrow$  *nat*  $\Rightarrow$  *real* **where**  
*r01-binary-sum*  $a$   $n \equiv$  ( $\sum_{i=0..n}$  *real* ( $a$   $i$ ) \*  $((1/2) \wedge$  (Suc  $i$ )))

**definition** *r01-binary-sum-lim* :: (*nat*  $\Rightarrow$  *nat*)  $\Rightarrow$  *real* **where**  
*r01-binary-sum-lim*  $\equiv$  *lim*  $\circ$  *r01-binary-sum*

**definition**  $r01\text{-binary-expression} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$  **where**  
 $r01\text{-binary-expression} \equiv r01\text{-binary-sum} \circ r01\text{-binary-expansion}'$

**lemma**  $r01\text{-binary-expansion-lr-r-ur}$ :

**assumes**  $0 < r \ r < 1$

**shows**  $(\text{snd} (\text{snd} (r01\text{-binary-expansion}'' r n))) \leq r \wedge$   
 $r < (\text{fst} (\text{snd} (r01\text{-binary-expansion}'' r n)))$

$\langle \text{proof} \rangle$

$0 \leq lr \wedge lr < ur \wedge ur \leq 1.$

**lemma**  $r01\text{-binary-expansion-lr-ur-nn}$ :

**shows**  $0 \leq \text{snd} (\text{snd} (r01\text{-binary-expansion}'' r n)) \wedge$

$\text{snd} (\text{snd} (r01\text{-binary-expansion}'' r n)) < \text{fst} (\text{snd} (r01\text{-binary-expansion}'' r$   
 $n)) \wedge$

$\text{fst} (\text{snd} (r01\text{-binary-expansion}'' r n)) \leq 1$

$\langle \text{proof} \rangle$

**lemma**  $r01\text{-binary-expansion-diff}$ :

**shows**  $(\text{fst} (\text{snd} (r01\text{-binary-expansion}'' r n))) - (\text{snd} (\text{snd} (r01\text{-binary-expansion}''$   
 $r n))) = (1/2)^\wedge(\text{Suc } n)$

$\langle \text{proof} \rangle$

$lrn = Sn.$

**lemma**  $r01\text{-binary-expression-eq-lr}$ :

$\text{snd} (\text{snd} (r01\text{-binary-expansion}'' r n)) = r01\text{-binary-expression } r \ n$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-binary-expansion}'\text{-sum-range}$ :

$\exists k::\text{nat}. (\text{snd} (\text{snd} (r01\text{-binary-expansion}'' r n))) = \text{real } k/2^\wedge(\text{Suc } n) \wedge$   
 $k < 2^\wedge(\text{Suc } n) \wedge$   
 $((r01\text{-binary-expansion}' r n) = 0 \longrightarrow \text{even } k) \wedge$   
 $((r01\text{-binary-expansion}' r n) = 1 \longrightarrow \text{odd } k)$

$\langle \text{proof} \rangle$

$an = bn \leftrightarrow Sn = S'n.$

**lemma**  $r01\text{-binary-expansion}'\text{-expression-eq}$ :

$r01\text{-binary-expansion}' r1 = r01\text{-binary-expansion}' r2 \longleftrightarrow$   
 $r01\text{-binary-expression } r1 = r01\text{-binary-expression } r2$

$\langle \text{proof} \rangle$

**lemma**  $\text{power2-e}$ :

$\bigwedge e::\text{real}. 0 < e \implies \exists n::\text{nat}. \text{real-of-rat } (1/2)^\wedge n < e$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-binary-expression-converges-to-r}$ :

**assumes**  $0 < r$

**and**  $r < 1$

**shows**  $\text{LIMSEQ } (r01\text{-binary-expression } r) \ r$

$\langle proof \rangle$

**lemma** *r01-binary-expression-correct*:

**assumes**  $0 < r$

**and**  $r < 1$

**shows**  $r = (\sum n. \text{real } (r01\text{-binary-expansion}' r n) * (1/2)^\wedge(Suc n))$

$\langle proof \rangle$

$S0 \leq S1 \leq S2 \leq \dots$

**lemma** *binary-sum-incseq*:

*incseq* (*r01-binary-sum* *a*)

$\langle proof \rangle$

**lemma** *r01-eq-iff*:

**assumes**  $0 < r1$   $r1 < 1$

$0 < r2$   $r2 < 1$

**shows**  $r1 = r2 \iff r01\text{-binary-expansion}' r1 = r01\text{-binary-expansion}' r2$

$\langle proof \rangle$

**lemma** *power-half-summable*:

*summable*  $(\lambda n. ((1::\text{real}) / 2)^\wedge Suc n)$

$\langle proof \rangle$

**lemma** *binary-expression-summable*:

**assumes**  $\bigwedge n. a n \in \{0,1 :: \text{nat}\}$

**shows** *summable*  $(\lambda n. \text{real } (a n) * (1/2)^\wedge(Suc n))$

$\langle proof \rangle$

**lemma** *binary-expression-gteq0*:

**assumes**  $\bigwedge n. a n \in \{0,1 :: \text{nat}\}$

**shows**  $0 \leq (\sum n. \text{real } (a (n + k)) * (1 / 2)^\wedge Suc (n + k))$

$\langle proof \rangle$

**lemma** *binary-expression-leeq1*:

**assumes**  $\bigwedge n. a n \in \{0,1 :: \text{nat}\}$

**shows**  $(\sum n. \text{real } (a (n + k)) * (1 / 2)^\wedge Suc (n + k)) \leq 1$

$\langle proof \rangle$

**lemma** *binary-expression-less-than*:

**assumes**  $\bigwedge n. a n \in \{0,1 :: \text{nat}\}$

**shows**  $(\sum n. \text{real } (a (n + k)) * (1 / 2)^\wedge Suc (n + k)) \leq (\sum n. (1 / 2)^\wedge Suc (n + k))$

$\langle proof \rangle$

**lemma** *lim-sum-ai*:

**assumes**  $\bigwedge n. a n \in \{0,1 :: \text{nat}\}$

**shows**  $\text{lim } (\lambda n. (\sum i=0..n. \text{real } (a i) * (1/2)^\wedge(Suc i))) = (\sum n::\text{nat}. \text{real } (a n) * (1/2)^\wedge(Suc n))$



*<proof>*

**lemma** *half-1-minus-sum:*

$$1 - (\sum_{i < k}. ((1::real) / 2) ^ Suc i) = (1/2) ^ k$$

*<proof>*

**lemma** *half-sum:*

$$(\sum n. ((1::real) / 2) ^ (Suc (n + k))) = (1/2) ^ k$$

*<proof>*

**lemma** *ai-exists0-less-than-sum:*

**assumes**  $\bigwedge n. a n \in \{0,1\}$

$$i \geq m$$

**and**  $a i = 0$

**shows**  $(\sum n::nat. real (a (n + m)) * (1/2) ^ (Suc (n + m))) < (1 / 2) ^ m$

*<proof>*

**lemma** *ai-exists0-less-than1:*

**assumes**  $\bigwedge n. a n \in \{0,1\}$

**and**  $\exists i. a i = 0$

**shows**  $(\sum n::nat. real (a n) * (1/2) ^ (Suc n)) < 1$

*<proof>*

**lemma** *ai-1-gt:*

**assumes**  $\bigwedge n. a n \in \{0,1\}$

**and**  $a i = 1$

**shows**  $(1/2) ^ (Suc i) \leq (\sum n::nat. real (a (n+i)) * (1/2) ^ (Suc (n+i)))$

*<proof>*

**lemma** *ai-exists1-gt0:*

**assumes**  $\bigwedge n. a n \in \{0,1\}$

**and**  $\exists i. a i = 1$

**shows**  $0 < (\sum n::nat. real (a n) * (1/2) ^ (Suc n))$

*<proof>*

**lemma** *r01-binary-expression-ex0:*

**assumes**  $0 < r < 1$

**shows**  $\exists i. r01-binary-expansion' r i = 0$

*<proof>*

**lemma** *r01-binary-expression-ex1:*

**assumes**  $0 < r < 1$

**shows**  $\exists i. r01-binary-expansion' r i = 1$

*<proof>*

**lemma** *r01-binary-expansion'-gt1:*

$1 \leq r \iff (\forall n. r01-binary-expansion' r n = 1)$

*<proof>*

**lemma** *r01-binary-expansion'-lt0*:  
 $r \leq 0 \iff (\forall n. r01\text{-binary-expansion}' r n = 0)$   
 ⟨proof⟩

The sequence 111111... does not appear in  $r = 0.a_1a_2\dots$

**lemma** *r01-binary-expression-ex0-strong*:  
**assumes**  $0 < r \ r < 1$   
**shows**  $\exists i \geq n. r01\text{-binary-expansion}' r i = 0$   
 ⟨proof⟩

A binary expression is well-formed when 111... does not appear in the tail of the sequence

**definition** *biexp01-well-formed* ::  $(nat \Rightarrow nat) \Rightarrow bool$  **where**  
 $biexp01\text{-well-formed } a \equiv (\forall n. a n \in \{0,1\}) \wedge (\forall n. \exists m \geq n. a m = 0)$

**lemma** *biexp01-well-formedE*:  
**assumes** *biexp01-well-formed*  $a$   
**shows**  $(\forall n. a n \in \{0,1\}) \wedge (\forall n. \exists m \geq n. a m = 0)$   
 ⟨proof⟩

**lemma** *biexp01-well-formedI*:  
**assumes**  $\bigwedge n. a n \in \{0,1\}$   
**and**  $\bigwedge n. \exists m \geq n. a m = 0$   
**shows** *biexp01-well-formed*  $a$   
 ⟨proof⟩

**lemma** *r01-binary-expansion-well-formed*:  
**assumes**  $0 < r \ r < 1$   
**shows** *biexp01-well-formed*  $(r01\text{-binary-expansion}' r)$   
 ⟨proof⟩

**lemma** *biexp01-well-formed-comb*:  
**assumes** *biexp01-well-formed*  $a$   
**and** *biexp01-well-formed*  $b$   
**shows** *biexp01-well-formed*  $(\lambda n. \text{if even } n \text{ then } a (n \text{ div } 2) \text{ else } b ((n-1) \text{ div } 2))$   
 ⟨proof⟩

**lemma** *nat-complete-induction*:  
**assumes**  $P (0 :: nat)$   
**and**  $\bigwedge m. m \leq n \implies P m \implies P (Suc n)$   
**shows**  $P n$   
 ⟨proof⟩

$(\sum m. \text{real } (a m) * (1 / 2) ^ Suc m) n = a n.$

**lemma** *biexp01-well-formed-an*:

**assumes** *biexp01-well-formed a*  
**shows**  $r01\text{-binary-expansion}' (\sum m. \text{real } (a \ m) * (1 / 2) ^ \wedge \text{Suc } m) \ n = a \ n$   
*<proof>*

**lemma** *f01-borel-measurable:*  
**assumes**  $f - \{0::\text{real}\} \in \text{sets } \text{real-borel}$   
 $f - \{1\} \in \text{sets } \text{borel}$   
**and**  $\bigwedge r::\text{real}. f \ r \in \{0,1\}$   
**shows**  $f \in \text{borel-measurable } \text{real-borel}$   
*<proof>*

**lemma** *r01-binary-expansion'-measurable:*  
 $(\lambda r. \text{real } (r01\text{-binary-expansion}' \ r \ n)) \in \text{borel-measurable } (\text{borel} :: \text{real measure})$   
*<proof>*

**definition**  $r01\text{-to-r01-r01-fst}' :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat}$  **where**  
 $r01\text{-to-r01-r01-fst}' \ r \ n \equiv r01\text{-binary-expansion}' \ r \ (2*n)$

**lemma** *r01-to-r01-r01-fst'in01:*  
 $\bigwedge n. r01\text{-to-r01-r01-fst}' \ r \ n \in \{0,1\}$   
*<proof>*

**definition**  $r01\text{-to-r01-r01-fst-sum} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$  **where**  
 $r01\text{-to-r01-r01-fst-sum} \equiv r01\text{-binary-sum} \circ r01\text{-to-r01-r01-fst}'$

**definition**  $r01\text{-to-r01-r01-fst} :: \text{real} \Rightarrow \text{real}$  **where**  
 $r01\text{-to-r01-r01-fst} = \text{lim} \circ r01\text{-to-r01-r01-fst-sum}$

**lemma** *r01-to-r01-r01-fst-def':*  
 $r01\text{-to-r01-r01-fst} \ r = (\sum n. \text{real } (r01\text{-binary-expansion}' \ r \ (2*n)) * (1/2) ^ \wedge (n+1))$   
*<proof>*

**lemma** *r01-to-r01-r01-fst-measurable:*  
 $r01\text{-to-r01-r01-fst} \in \text{borel-measurable } \text{borel}$   
*<proof>*

**definition**  $r01\text{-to-r01-r01-snd}' :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat}$  **where**  
 $r01\text{-to-r01-r01-snd}' \ r \ n = r01\text{-binary-expansion}' \ r \ (2*n + 1)$

**lemma** *r01-to-r01-r01-snd'in01:*  
 $\bigwedge n. r01\text{-to-r01-r01-snd}' \ r \ n \in \{0,1\}$   
*<proof>*

**definition**  $r01\text{-to-}r01\text{-}r01\text{-snd-sum} :: \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$  **where**  
 $r01\text{-to-}r01\text{-}r01\text{-snd-sum} \equiv r01\text{-binary-sum} \circ r01\text{-to-}r01\text{-}r01\text{-snd}'$

**definition**  $r01\text{-to-}r01\text{-}r01\text{-snd} :: \text{real} \Rightarrow \text{real}$  **where**  
 $r01\text{-to-}r01\text{-}r01\text{-snd} = \text{lim} \circ r01\text{-to-}r01\text{-}r01\text{-snd-sum}$

**lemma**  $r01\text{-to-}r01\text{-}r01\text{-snd-def}'$ :  
 $r01\text{-to-}r01\text{-}r01\text{-snd} r = (\sum n. \text{real} (r01\text{-binary-expansion}' r (2*n + 1)) * (1/2) \frown (n+1))$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-to-}r01\text{-}r01\text{-snd-measurable}$ :  
 $r01\text{-to-}r01\text{-}r01\text{-snd} \in \text{borel-measurable borel}$   
 $\langle \text{proof} \rangle$

**definition**  $r01\text{-to-}r01\text{-}r01 :: \text{real} \Rightarrow \text{real} \times \text{real}$  **where**  
 $r01\text{-to-}r01\text{-}r01 r = (r01\text{-to-}r01\text{-}r01\text{-fst} r, r01\text{-to-}r01\text{-}r01\text{-snd} r)$

**lemma**  $r01\text{-to-}r01\text{-}r01\text{-image}$ :  
 $r01\text{-to-}r01\text{-}r01 r \in \{0..1\} \times \{0..1\}$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-to-}r01\text{-}r01\text{-measurable}$ :  
 $r01\text{-to-}r01\text{-}r01 \in \text{real-borel} \rightarrow_M \text{real-borel} \otimes_M \text{real-borel}$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-to-}r01\text{-}r01\text{-3over4}$ :  
 $r01\text{-to-}r01\text{-}r01 (3/4) = (1/2, 1/2)$   
 $\langle \text{proof} \rangle$

**definition**  $r01\text{-}r01\text{-to-}r01' :: \text{real} \times \text{real} \Rightarrow \text{nat} \Rightarrow \text{nat}$  **where**  
 $r01\text{-}r01\text{-to-}r01' rs \equiv (\lambda n. \text{if even } n \text{ then } r01\text{-binary-expansion}' (\text{fst } rs) (n \text{ div } 2)$   
 $\text{else } r01\text{-binary-expansion}' (\text{snd } rs) ((n-1) \text{ div } 2))$

**lemma**  $r01\text{-}r01\text{-to-}r01'\text{in}01$ :  
 $\bigwedge n. r01\text{-}r01\text{-to-}r01' rs n \in \{0,1\}$   
 $\langle \text{proof} \rangle$

**lemma**  $r01\text{-}r01\text{-to-}r01'\text{-well-formed}$ :  
**assumes**  $0 < r1$   $r1 < 1$   
**and**  $0 < r2$   $r2 < 1$   
**shows**  $\text{biexp}01\text{-well-formed} (r01\text{-}r01\text{-to-}r01' (r1, r2))$   
 $\langle \text{proof} \rangle$

**definition**  $r01\text{-}r01\text{-to-}r01\text{-sum} :: \text{real} \times \text{real} \Rightarrow \text{nat} \Rightarrow \text{real}$  **where**  
 $r01\text{-}r01\text{-to-}r01\text{-sum} \equiv r01\text{-binary-sum} \circ r01\text{-}r01\text{-to-}r01'$

**definition**  $r01-r01-to-r01 :: real \times real \Rightarrow real$  **where**

$r01-r01-to-r01 \equiv lim \circ r01-r01-to-r01-sum$

**lemma**  $r01-r01-to-r01-def'$ :

$r01-r01-to-r01 (r1,r2) = (\sum n. real (r01-r01-to-r01' (r1,r2) n) * (1/2) \frown(n+1))$   
 $\langle proof \rangle$

**lemma**  $r01-r01-to-r01-measurable$ :

$r01-r01-to-r01 \in real-borel \otimes_M real-borel \rightarrow_M real-borel$   
 $\langle proof \rangle$

**lemma**  $r01-r01-to-r01-image$ :

**assumes**  $0 < r1$   $r1 < 1$

**shows**  $r01-r01-to-r01 (r1,r2) \in \{0 < .. < 1\}$

$\langle proof \rangle$

**lemma**  $r01-r01-to-r01-image'$ :

**assumes**  $0 < r2$   $r2 < 1$

**shows**  $r01-r01-to-r01 (r1,r2) \in \{0 < .. < 1\}$

$\langle proof \rangle$

**lemma**  $r01-r01-to-r01-binary-nth$ :

**assumes**  $0 < r1$   $r1 < 1$

**and**  $0 < r2$   $r2 < 1$

**shows**  $r01-binary-expansion' r1 n = r01-binary-expansion' (r01-r01-to-r01 (r1,r2)) (2*n) \wedge$

$r01-binary-expansion' r2 n = r01-binary-expansion' (r01-r01-to-r01 (r1,r2)) (2*n + 1)$

$\langle proof \rangle$

**lemma**  $r01-r01--r01--r01-r01-id$ :

**assumes**  $0 < r1$   $r1 < 1$

$0 < r2$   $r2 < 1$

**shows**  $(r01-to-r01-r01 \circ r01-r01-to-r01) (r1,r2) = (r1,r2)$

$\langle proof \rangle$

We first show that  $M \otimes_M N$  is a standard Borel space for standard Borel spaces  $M$  and  $N$ .

**lemma**  $pair-measurable[measurable]$ :

**assumes**  $f \in X \rightarrow_M Y$

**and**  $g \in X' \rightarrow_M Y'$

**shows**  $map-prod f g \in X \otimes_M X' \rightarrow_M Y \otimes_M Y'$

$\langle proof \rangle$

**lemma**  $pair-standard-borel-standard$ :

**assumes**  $standard-borel M$

**and**  $standard-borel N$

**shows** *standard-borel* ( $M \otimes_M N$ )  
*<proof>*

**lemma** *pair-standard-borel-space-UNIV*:  
**assumes** *standard-borel-space-UNIV M*  
**and** *standard-borel-space-UNIV N*  
**shows** *standard-borel-space-UNIV* ( $M \otimes_M N$ )  
*<proof>*

**locale** *pair-standard-borel = s1: standard-borel M + s2: standard-borel N*  
**for**  $M :: 'a \text{ measure}$  **and**  $N :: 'b \text{ measure}$   
**begin**

**sublocale** *standard-borel*  $M \otimes_M N$   
*<proof>*

**end**

**locale** *pair-standard-borel-space-UNIV = s1: standard-borel-space-UNIV M + s2:*  
*standard-borel-space-UNIV N*  
**for**  $M :: 'a \text{ measure}$  **and**  $N :: 'b \text{ measure}$   
**begin**

**sublocale** *pair-standard-borel*  $M N$   
*<proof>*

**sublocale** *standard-borel-space-UNIV*  $M \otimes_M N$   
*<proof>*

**end**

$\mathbb{R} \times \mathbb{R}$  is a standard Borel space.

**interpretation** *real-real : pair-standard-borel-space-UNIV real-borel real-borel*  
*<proof>*

#### 1.4 $\mathbb{N} \times \mathbb{R}$

$\mathbb{N} \times \mathbb{R}$  is a standard Borel space.

**interpretation** *nat-real: pair-standard-borel-space-UNIV nat-borel real-borel*  
*<proof>*

**end**

## 2 Quasi-Borel Spaces

**theory** *QuasiBorel*  
**imports** *StandardBorel*

begin

## 2.1 Definitions

We formalize quasi-Borel spaces introduced by Heunen et al. [1].

### 2.1.1 Quasi-Borel Spaces

**definition** *qbs-closed1* :: (real  $\Rightarrow$  'a) set  $\Rightarrow$  bool

where *qbs-closed1* *Mx*  $\equiv$  ( $\forall a \in Mx. \forall f \in \text{real-borel} \rightarrow_M \text{real-borel}. a \circ f \in Mx$ )

**definition** *qbs-closed2* :: ['a set, (real  $\Rightarrow$  'a) set]  $\Rightarrow$  bool

where *qbs-closed2* *X Mx*  $\equiv$  ( $\forall x \in X. (\lambda r. x) \in Mx$ )

**definition** *qbs-closed3* :: (real  $\Rightarrow$  'a) set  $\Rightarrow$  bool

where *qbs-closed3* *Mx*  $\equiv$  ( $\forall P::\text{real} \Rightarrow \text{nat}. \forall Fi::\text{nat} \Rightarrow \text{real} \Rightarrow 'a.$   
 $(\forall i. P - \{i\} \in \text{sets real-borel})$   
 $\rightarrow (\forall i. Fi i \in Mx)$   
 $\rightarrow (\lambda r. Fi (P r) r) \in Mx$ )

**lemma** *separate-measurable*:

fixes *P* :: real  $\Rightarrow$  nat

assumes  $\bigwedge i. P - \{i\} \in \text{sets real-borel}$

shows  $P \in \text{real-borel} \rightarrow_M \text{nat-borel}$

*<proof>*

**lemma** *measurable-separate*:

fixes *P* :: real  $\Rightarrow$  nat

assumes  $P \in \text{real-borel} \rightarrow_M \text{nat-borel}$

shows  $P - \{i\} \in \text{sets real-borel}$

*<proof>*

**definition** *is-quasi-borel* *X Mx*  $\longleftrightarrow Mx \subseteq UNIV \rightarrow X \wedge \text{qbs-closed1 } Mx \wedge$   
*qbs-closed2* *X Mx*  $\wedge \text{qbs-closed3 } Mx$

**lemma** *is-quasi-borel-intro[simp]*:

assumes  $Mx \subseteq UNIV \rightarrow X$

and *qbs-closed1* *Mx* *qbs-closed2* *X Mx* *qbs-closed3* *Mx*

shows *is-quasi-borel* *X Mx*

*<proof>*

**typedef** 'a *quasi-borel* = {(*X*::'a set, *Mx*). *is-quasi-borel* *X Mx*}

*<proof>*

**definition** *qbs-space* :: 'a *quasi-borel*  $\Rightarrow$  'a set **where**

*qbs-space* *X*  $\equiv \text{fst } (\text{Rep-quasi-borel } X)$

**definition** *qbs-Mx* :: 'a *quasi-borel*  $\Rightarrow$  (real  $\Rightarrow$  'a) set **where**

*qbs-Mx* *X*  $\equiv \text{snd } (\text{Rep-quasi-borel } X)$

**lemma** *qbs-decomp* :  
 $(qbs\text{-space } X, qbs\text{-Mx } X) \in \{(X :: 'a \text{ set}, Mx). \text{is-quasi-borel } X \text{ Mx}\}$   
 ⟨proof⟩

**lemma** *qbs-Mx-to-X[dest]*:  
**assumes**  $\alpha \in qbs\text{-Mx } X$   
**shows**  $\alpha \in UNIV \rightarrow qbs\text{-space } X$   
 $\alpha \ r \in qbs\text{-space } X$   
 ⟨proof⟩

**lemma** *qbs-closed1I*:  
**assumes**  $\bigwedge \alpha \ f. \alpha \in Mx \implies f \in \text{real-borel} \rightarrow_M \text{real-borel} \implies \alpha \circ f \in Mx$   
**shows** *qbs-closed1*  $Mx$   
 ⟨proof⟩

**lemma** *qbs-closed1-dest[simp]*:  
**assumes**  $\alpha \in qbs\text{-Mx } X$   
**and**  $f \in \text{real-borel} \rightarrow_M \text{real-borel}$   
**shows**  $\alpha \circ f \in qbs\text{-Mx } X$   
 ⟨proof⟩

**lemma** *qbs-closed2I*:  
**assumes**  $\bigwedge x. x \in X \implies (\lambda r. x) \in Mx$   
**shows** *qbs-closed2*  $X \ Mx$   
 ⟨proof⟩

**lemma** *qbs-closed2-dest[simp]*:  
**assumes**  $x \in qbs\text{-space } X$   
**shows**  $(\lambda r. x) \in qbs\text{-Mx } X$   
 ⟨proof⟩

**lemma** *qbs-closed3I*:  
**assumes**  $\bigwedge (P :: \text{real} \Rightarrow \text{nat}) \ Fi. (\bigwedge i. P \text{ - ' } \{i\} \in \text{sets real-borel}) \implies (\bigwedge i. Fi \ i \in Mx)$   
 $\implies (\lambda r. Fi \ (P \ r) \ r) \in Mx$   
**shows** *qbs-closed3*  $Mx$   
 ⟨proof⟩

**lemma** *qbs-closed3I'*:  
**assumes**  $\bigwedge (P :: \text{real} \Rightarrow \text{nat}) \ Fi. P \in \text{real-borel} \rightarrow_M \text{nat-borel} \implies (\bigwedge i. Fi \ i \in Mx)$   
 $\implies (\lambda r. Fi \ (P \ r) \ r) \in Mx$   
**shows** *qbs-closed3*  $Mx$   
 ⟨proof⟩

**lemma** *qbs-closed3-dest[simp]*:  
**fixes**  $P :: \text{real} \Rightarrow \text{nat}$  **and**  $Fi :: \text{nat} \Rightarrow \text{real} \Rightarrow -$



**assumes**  $\bigwedge i. P - \{i\} \in \text{sets real-borel}$   
**and**  $\bigwedge i. Fi\ i \in \text{qbs-Mx } X$   
**shows**  $(\lambda r. Fi\ (P\ r)\ r) \in \text{qbs-Mx } X$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-closed3-dest'*:  
**fixes**  $P :: \text{real} \Rightarrow \text{nat}$  **and**  $Fi :: \text{nat} \Rightarrow \text{real} \Rightarrow -$   
**assumes**  $P \in \text{real-borel} \rightarrow_M \text{nat-borel}$   
**and**  $\bigwedge i. Fi\ i \in \text{qbs-Mx } X$   
**shows**  $(\lambda r. Fi\ (P\ r)\ r) \in \text{qbs-Mx } X$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-closed3-dest2*:  
**assumes** *countable*  $I$   
**and** [*measurable*]:  $P \in \text{real-borel} \rightarrow_M \text{count-space } I$   
**and**  $\bigwedge i. i \in I \implies Fi\ i \in \text{qbs-Mx } X$   
**shows**  $(\lambda r. Fi\ (P\ r)\ r) \in \text{qbs-Mx } X$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-closed3-dest2'*:  
**assumes** *countable*  $I$   
**and** [*measurable*]:  $P \in \text{real-borel} \rightarrow_M \text{count-space } I$   
**and**  $\bigwedge i. i \in \text{range } P \implies Fi\ i \in \text{qbs-Mx } X$   
**shows**  $(\lambda r. Fi\ (P\ r)\ r) \in \text{qbs-Mx } X$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-space-Mx*:  
 $\text{qbs-space } X = \{\alpha\ x \mid x\ \alpha. \alpha \in \text{qbs-Mx } X\}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-space-eq-Mx*:  
**assumes**  $\text{qbs-Mx } X = \text{qbs-Mx } Y$   
**shows**  $\text{qbs-space } X = \text{qbs-space } Y$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-eqI*:  
**assumes**  $\text{qbs-Mx } X = \text{qbs-Mx } Y$   
**shows**  $X = Y$   
 $\langle \text{proof} \rangle$

## 2.1.2 Morphism of Quasi-Borel Spaces

**definition** *qbs-morphism* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*'a*  $\Rightarrow$  *'b*) *set* (**infix**  $\rightarrow_Q$  60) **where**  
 $X \rightarrow_Q Y \equiv \{f \in \text{qbs-space } X \rightarrow \text{qbs-space } Y. \forall \alpha \in \text{qbs-Mx } X. f \circ \alpha \in \text{qbs-Mx } Y\}$

**lemma** *qbs-morphismI*:  
**assumes**  $\bigwedge \alpha. \alpha \in \text{qbs-Mx } X \implies f \circ \alpha \in \text{qbs-Mx } Y$

**shows**  $f \in X \rightarrow_Q Y$   
*<proof>*

**lemma** *qbs-morphismE[dest]*:  
**assumes**  $f \in X \rightarrow_Q Y$   
**shows**  $f \in \text{qbs-space } X \rightarrow \text{qbs-space } Y$   
 $\bigwedge x. x \in \text{qbs-space } X \implies f x \in \text{qbs-space } Y$   
 $\bigwedge \alpha. \alpha \in \text{qbs-Mx } X \implies f \circ \alpha \in \text{qbs-Mx } Y$   
*<proof>*

**lemma** *qbs-morphism-ident[simp]*:  
 $id \in X \rightarrow_Q X$   
*<proof>*

**lemma** *qbs-morphism-ident'[simp]*:  
 $(\lambda x. x) \in X \rightarrow_Q X$   
*<proof>*

**lemma** *qbs-morphism-comp*:  
**assumes**  $f \in X \rightarrow_Q Y$   $g \in Y \rightarrow_Q Z$   
**shows**  $g \circ f \in X \rightarrow_Q Z$   
*<proof>*

**lemma** *qbs-morphism-cong*:  
**assumes**  $\bigwedge x. x \in \text{qbs-space } X \implies f x = g x$   
**and**  $f \in X \rightarrow_Q Y$   
**shows**  $g \in X \rightarrow_Q Y$   
*<proof>*

**lemma** *qbs-morphism-const*:  
**assumes**  $y \in \text{qbs-space } Y$   
**shows**  $(\lambda \cdot. y) \in X \rightarrow_Q Y$   
*<proof>*

### 2.1.3 Empty Space

**definition** *empty-quasi-borel* :: 'a quasi-borel **where**  
 $\text{empty-quasi-borel} \equiv \text{Abs-quasi-borel } (\{\}, \{\})$

**lemma** *eqb-correct*:  $\text{Rep-quasi-borel empty-quasi-borel} = (\{\}, \{\})$   
*<proof>*

**lemma** *eqb-space[simp]*:  $\text{qbs-space empty-quasi-borel} = \{\}$   
*<proof>*

**lemma** *eqb-Mx[simp]*:  $\text{qbs-Mx empty-quasi-borel} = \{\}$   
*<proof>*

**lemma** *qbs-empty-equiv* :  $qbs\text{-space } X = \{\} \longleftrightarrow qbs\text{-Mx } X = \{\}$   
 ⟨proof⟩

**lemma** *empty-quasi-borel-iff*:  
 $qbs\text{-space } X = \{\} \longleftrightarrow X = \text{empty-quasi-borel}$   
 ⟨proof⟩

#### 2.1.4 Unit Space

**definition** *unit-quasi-borel* :: *unit quasi-borel* ( $!Q$ ) **where**  
*unit-quasi-borel*  $\equiv$  *Abs-quasi-borel* ( $UNIV, UNIV$ )

**lemma** *uqb-correct*: *Rep-quasi-borel unit-quasi-borel* = ( $UNIV, UNIV$ )  
 ⟨proof⟩

**lemma** *uqb-space[simp]*: *qbs-space unit-quasi-borel* =  $\{\}$   
 ⟨proof⟩

**lemma** *uqb-Mx[simp]*: *qbs-Mx unit-quasi-borel* =  $\{\lambda r. ()\}$   
 ⟨proof⟩

**lemma** *unit-quasi-borel-terminal*:  
 $\exists! f. f \in X \rightarrow_Q \text{unit-quasi-borel}$   
 ⟨proof⟩

**definition** *to-unit-quasi-borel* ::  $'a \Rightarrow \text{unit } (!Q)$  **where**  
*to-unit-quasi-borel*  $\equiv$   $(\lambda \cdot. ())$

**lemma** *to-unit-quasi-borel-morphism* :  
 $!Q \in X \rightarrow_Q \text{unit-quasi-borel}$   
 ⟨proof⟩

#### 2.1.5 Subspaces

**definition** *sub-qbs* :: [*'a quasi-borel, 'a set*]  $\Rightarrow$  *'a quasi-borel* **where**  
*sub-qbs*  $X U \equiv \text{Abs-quasi-borel } (qbs\text{-space } X \cap U, \{f \in UNIV \rightarrow qbs\text{-space } X \cap U. f \in qbs\text{-Mx } X\})$

**lemma** *sub-qbs-closed*:  
 $qbs\text{-closed1 } \{f \in UNIV \rightarrow qbs\text{-space } X \cap U. f \in qbs\text{-Mx } X\}$   
 $qbs\text{-closed2 } (qbs\text{-space } X \cap U) \{f \in UNIV \rightarrow qbs\text{-space } X \cap U. f \in qbs\text{-Mx } X\}$   
 $qbs\text{-closed3 } \{f \in UNIV \rightarrow qbs\text{-space } X \cap U. f \in qbs\text{-Mx } X\}$   
 ⟨proof⟩

**lemma** *sub-qbs-correct[simp]*: *Rep-quasi-borel (sub-qbs X U)* =  $(qbs\text{-space } X \cap U, \{f \in UNIV \rightarrow qbs\text{-space } X \cap U. f \in qbs\text{-Mx } X\})$   
 ⟨proof⟩

**lemma** *sub-qbs-space[simp]*: *qbs-space (sub-qbs X U)* =  $qbs\text{-space } X \cap U$   
 ⟨proof⟩

**lemma** *sub-qbs-Mx[simp]*:  $qbs\text{-}Mx (sub\text{-}qbs\ X\ U) = \{f \in UNIV \rightarrow qbs\text{-}space\ X \cap U. f \in qbs\text{-}Mx\ X\}$   
 ⟨proof⟩

**lemma** *sub-qbs*:

**assumes**  $U \subseteq qbs\text{-}space\ X$

**shows**  $(qbs\text{-}space (sub\text{-}qbs\ X\ U), qbs\text{-}Mx (sub\text{-}qbs\ X\ U)) = (U, \{f \in UNIV \rightarrow U. f \in qbs\text{-}Mx\ X\})$

⟨proof⟩

### 2.1.6 Image Spaces

**definition** *map-qbs* ::  $[ 'a \Rightarrow 'b ] \Rightarrow 'a\ quasi\text{-}borel \Rightarrow 'b\ quasi\text{-}borel$  **where**  
 $map\text{-}qbs\ f\ X = Abs\text{-}quasi\text{-}borel (f\ ' (qbs\text{-}space\ X), \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\})$

**lemma** *map-qbs-f*:

$\{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\} \subseteq UNIV \rightarrow f\ ' (qbs\text{-}space\ X)$   
 ⟨proof⟩

**lemma** *map-qbs-closed1*:

$qbs\text{-}closed1\ \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\}$   
 ⟨proof⟩

**lemma** *map-qbs-closed2*:

$qbs\text{-}closed2\ (f\ ' (qbs\text{-}space\ X))\ \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\}$   
 ⟨proof⟩

**lemma** *map-qbs-closed3*:

$qbs\text{-}closed3\ \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\}$   
 ⟨proof⟩

**lemma** *map-qbs-correct[simp]*:

$Rep\text{-}quasi\text{-}borel (map\text{-}qbs\ f\ X) = (f\ ' (qbs\text{-}space\ X), \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\})$   
 ⟨proof⟩

**lemma** *map-qbs-space[simp]*:

$qbs\text{-}space (map\text{-}qbs\ f\ X) = f\ ' (qbs\text{-}space\ X)$   
 ⟨proof⟩

**lemma** *map-qbs-Mx[simp]*:

$qbs\text{-}Mx (map\text{-}qbs\ f\ X) = \{\beta. \exists \alpha \in qbs\text{-}Mx\ X. \beta = f \circ \alpha\}$   
 ⟨proof⟩

**inductive-set** *generating-Mx* ::  $'a\ set \Rightarrow (real \Rightarrow 'a)\ set \Rightarrow (real \Rightarrow 'a)\ set$   
**for**  $X :: 'a\ set$  **and**  $Mx :: (real \Rightarrow 'a)\ set$   
**where**

*Basic*:  $\alpha \in Mx \implies \alpha \in \text{generating-Mx } X \ Mx$   
 | *Const*:  $x \in X \implies (\lambda r. x) \in \text{generating-Mx } X \ Mx$   
 | *Comp*:  $f \in \text{real-borel} \rightarrow_M \text{real-borel} \implies \alpha \in \text{generating-Mx } X \ Mx \implies \alpha \circ f \in \text{generating-Mx } X \ Mx$   
 | *Part*:  $(\bigwedge i. Fi \ i \in \text{generating-Mx } X \ Mx) \implies P \in \text{real-borel} \rightarrow_M \text{nat-borel} \implies (\lambda r. Fi \ (P \ r) \ r) \in \text{generating-Mx } X \ Mx$

**lemma** *generating-Mx-to-space*:  
**assumes**  $Mx \subseteq UNIV \rightarrow X$   
**shows**  $\text{generating-Mx } X \ Mx \subseteq UNIV \rightarrow X$   
*<proof>*

**lemma** *generating-Mx-closed1*:  
*qbs-closed1* ( $\text{generating-Mx } X \ Mx$ )  
*<proof>*

**lemma** *generating-Mx-closed2*:  
*qbs-closed2*  $X$  ( $\text{generating-Mx } X \ Mx$ )  
*<proof>*

**lemma** *generating-Mx-closed3*:  
*qbs-closed3* ( $\text{generating-Mx } X \ Mx$ )  
*<proof>*

**lemma** *generating-Mx-Mx*:  
 $\text{generating-Mx} \ (\text{qbs-space } X) \ (\text{qbs-Mx } X) = \text{qbs-Mx } X$   
*<proof>*

## 2.1.7 Ordering of Quasi-Borel Spaces

**instantiation** *quasi-borel* :: (type) *order-bot*  
**begin**

**inductive** *less-eq-quasi-borel* :: 'a *quasi-borel*  $\Rightarrow$  'a *quasi-borel*  $\Rightarrow$  bool **where**  
 $\text{qbs-space } X \subset \text{qbs-space } Y \implies \text{less-eq-quasi-borel } X \ Y$   
 |  $\text{qbs-space } X = \text{qbs-space } Y \implies \text{qbs-Mx } Y \subseteq \text{qbs-Mx } X \implies \text{less-eq-quasi-borel } X \ Y$

**lemma** *le-quasi-borel-iff*:  
 $X \leq Y \iff (\text{if } \text{qbs-space } X = \text{qbs-space } Y \text{ then } \text{qbs-Mx } Y \subseteq \text{qbs-Mx } X \text{ else } \text{qbs-space } X \subset \text{qbs-space } Y)$   
*<proof>*

**definition** *less-quasi-borel* :: 'a *quasi-borel*  $\Rightarrow$  'a *quasi-borel*  $\Rightarrow$  bool **where**  
 $\text{less-quasi-borel } X \ Y \iff (X \leq Y \wedge \neg Y \leq X)$

**definition** *bot-quasi-borel* :: 'a *quasi-borel* **where**  
 $\text{bot-quasi-borel} = \text{empty-quasi-borel}$

**instance**  
 ⟨proof⟩  
**end**

**definition** *inf-quasi-borel* :: [*'a quasi-borel, 'a quasi-borel*] ⇒ *'a quasi-borel* **where**  
*inf-quasi-borel X X' = Abs-quasi-borel (qbs-space X ∩ qbs-space X', qbs-Mx X ∩ qbs-Mx X')*

**lemma** *inf-quasi-borel-correct*: *Rep-quasi-borel (inf-quasi-borel X X') = (qbs-space X ∩ qbs-space X', qbs-Mx X ∩ qbs-Mx X')*  
 ⟨proof⟩

**lemma** *inf-qbs-space[simp]*: *qbs-space (inf-quasi-borel X X') = qbs-space X ∩ qbs-space X'*  
 ⟨proof⟩

**lemma** *inf-qbs-Mx[simp]*: *qbs-Mx (inf-quasi-borel X X') = qbs-Mx X ∩ qbs-Mx X'*  
 ⟨proof⟩

**definition** *max-quasi-borel* :: *'a set* ⇒ *'a quasi-borel* **where**  
*max-quasi-borel X = Abs-quasi-borel (X, UNIV → X)*

**lemma** *max-quasi-borel-correct*: *Rep-quasi-borel (max-quasi-borel X) = (X, UNIV → X)*  
 ⟨proof⟩

**lemma** *max-qbs-space[simp]*: *qbs-space (max-quasi-borel X) = X*  
 ⟨proof⟩

**lemma** *max-qbs-Mx[simp]*: *qbs-Mx (max-quasi-borel X) = UNIV → X*  
 ⟨proof⟩

**instantiation** *quasi-borel* :: (*type*) *semilattice-sup*  
**begin**

**definition** *sup-quasi-borel* :: *'a quasi-borel* ⇒ *'a quasi-borel* ⇒ *'a quasi-borel* **where**  
*sup-quasi-borel X Y ≡ (if qbs-space X = qbs-space Y then inf-quasi-borel X Y*  
   *else if qbs-space X ⊂ qbs-space Y then Y*  
   *else if qbs-space Y ⊂ qbs-space X then X*  
   *else max-quasi-borel (qbs-space X ∪ qbs-space Y))*

**instance**  
 ⟨proof⟩  
**end**

**end**

## 2.2 Relation to Measurable Spaces

**theory** *Measure-QuasiBorel-Adjunction*  
**imports** *QuasiBorel*  
**begin**

We construct the adjunction between **Meas** and **QBS**, where **Meas** is the category of measurable spaces and measurable functions and **QBS** is the category of quasi-Borel spaces and morphisms.

### 2.2.1 The Functor $R$

**definition** *measure-to-qbs* :: 'a measure  $\Rightarrow$  'a quasi-borel **where**  
*measure-to-qbs*  $X \equiv \text{Abs-quasi-borel } (\text{space } X, \text{real-borel } \rightarrow_M X)$

**lemma** *R-Mx-correct*:  $\text{real-borel } \rightarrow_M X \subseteq \text{UNIV} \rightarrow \text{space } X$   
*<proof>*

**lemma** *R-qbs-closed1*:  $\text{qbs-closed1 } (\text{real-borel } \rightarrow_M X)$   
*<proof>*

**lemma** *R-qbs-closed2*:  $\text{qbs-closed2 } (\text{space } X) (\text{real-borel } \rightarrow_M X)$   
*<proof>*

**lemma** *R-qbs-closed3*:  $\text{qbs-closed3 } (\text{real-borel } \rightarrow_M (X :: \text{'a measure}))$   
*<proof>*

**lemma** *R-correct[simp]*:  $\text{Rep-quasi-borel } (\text{measure-to-qbs } X) = (\text{space } X, \text{real-borel } \rightarrow_M X)$   
*<proof>*

**lemma** *qbs-space-R[simp]*:  $\text{qbs-space } (\text{measure-to-qbs } X) = \text{space } X$   
*<proof>*

**lemma** *qbs-Mx-R[simp]*:  $\text{qbs-Mx } (\text{measure-to-qbs } X) = \text{real-borel } \rightarrow_M X$   
*<proof>*

The following lemma says that *measure-to-qbs* is a functor from **Meas** to **QBS**.

**lemma** *r-preserves-morphisms*:  
 $X \rightarrow_M Y \subseteq (\text{measure-to-qbs } X) \rightarrow_Q (\text{measure-to-qbs } Y)$   
*<proof>*

### 2.2.2 The Functor $L$

**definition** *sigma-Mx* :: 'a quasi-borel  $\Rightarrow$  'a set set **where**  
*sigma-Mx*  $X \equiv \{U \cap \text{qbs-space } X \mid U. \forall \alpha \in \text{qbs-Mx } X. \alpha - ' U \in \text{sets real-borel}\}$

**definition** *qbs-to-measure* :: 'a quasi-borel  $\Rightarrow$  'a measure **where**

*qbs-to-measure*  $X \equiv \text{Abs-measure } (qbs\text{-space } X, \text{sigma-Mx } X, \lambda A. (\text{if } A = \{\} \text{ then } 0 \text{ else if } A \in - \text{sigma-Mx } X \text{ then } 0 \text{ else } \infty))$

**lemma** *measure-space-L*: *measure-space* (*qbs-space*  $X$ ) (*sigma-Mx*  $X$ ) ( $\lambda A. (\text{if } A = \{\} \text{ then } 0 \text{ else if } A \in - \text{sigma-Mx } X \text{ then } 0 \text{ else } \infty)$ )  
 ⟨*proof*⟩

**lemma** *L-correct[simp]*: *Rep-measure* (*qbs-to-measure*  $X$ ) = (*qbs-space*  $X$ , *sigma-Mx*  $X$ ,  $\lambda A. (\text{if } A = \{\} \text{ then } 0 \text{ else if } A \in - \text{sigma-Mx } X \text{ then } 0 \text{ else } \infty)$ )  
 ⟨*proof*⟩

**lemma** *space-L[simp]*: *space* (*qbs-to-measure*  $X$ ) = *qbs-space*  $X$   
 ⟨*proof*⟩

**lemma** *sets-L[simp]*: *sets* (*qbs-to-measure*  $X$ ) = *sigma-Mx*  $X$   
 ⟨*proof*⟩

**lemma** *emeasure-L[simp]*: *emeasure* (*qbs-to-measure*  $X$ ) = ( $\lambda A. \text{if } A = \{\} \vee A \notin \text{sigma-Mx } X \text{ then } 0 \text{ else } \infty$ )  
 ⟨*proof*⟩

**lemma** *qbs-Mx-sigma-Mx-contr*:  
**assumes** *qbs-space*  $X = \text{qbs-space } Y$   
**and** *qbs-Mx*  $X \subseteq \text{qbs-Mx } Y$   
**shows** *sigma-Mx*  $Y \subseteq \text{sigma-Mx } X$   
 ⟨*proof*⟩

The following lemma says that *qbs-to-measure* is a functor from **QBS** to **Meas**.

**lemma** *l-preserves-morphisms*:  
 $X \rightarrow_Q Y \subseteq (\text{qbs-to-measure } X) \rightarrow_M (\text{qbs-to-measure } Y)$   
 ⟨*proof*⟩

**abbreviation** *qbs-borel*  $\equiv \text{measure-to-qbs borel}$

**declare** [[*coercion measure-to-qbs*]]

**abbreviation** *real-quasi-borel*  $:: \text{real quasi-borel } (\mathbb{R}_Q)$  **where**  
*real-quasi-borel*  $\equiv \text{qbs-borel}$

**abbreviation** *nat-quasi-borel*  $:: \text{nat quasi-borel } (\mathbb{N}_Q)$  **where**  
*nat-quasi-borel*  $\equiv \text{qbs-borel}$

**abbreviation** *ennreal-quasi-borel*  $:: \text{ennreal quasi-borel } (\mathbb{R}_{Q \geq 0})$  **where**  
*ennreal-quasi-borel*  $\equiv \text{qbs-borel}$

**abbreviation** *bool-quasi-borel*  $:: \text{bool quasi-borel } (\mathbb{B}_Q)$  **where**  
*bool-quasi-borel*  $\equiv \text{qbs-borel}$



**lemma** *qbs-Mx-is-morphisms*:  
 $qbs-Mx X = real-quasi-borel \rightarrow_Q X$   
 ⟨proof⟩

**lemma** *qbs-Mx-subset-of-measurable*:  
 $qbs-Mx X \subseteq real-borel \rightarrow_M qbs-to-measure X$   
 ⟨proof⟩

**lemma** *L-max-of-measurables*:  
**assumes** *space*  $M = qbs-space X$   
**and**  $qbs-Mx X \subseteq real-borel \rightarrow_M M$   
**shows**  $sets M \subseteq sets (qbs-to-measure X)$   
 ⟨proof⟩

**lemma** *qbs-Mx-are-measurable[simp,measurable]*:  
**assumes**  $\alpha \in qbs-Mx X$   
**shows**  $\alpha \in real-borel \rightarrow_M qbs-to-measure X$   
 ⟨proof⟩

**lemma** *measure-to-qbs-cong-sets*:  
**assumes**  $sets M = sets N$   
**shows**  $measure-to-qbs M = measure-to-qbs N$   
 ⟨proof⟩

**lemma** *lr-sets[simp,measurable-cong]*:  
 $sets X \subseteq sets (qbs-to-measure (measure-to-qbs X))$   
 ⟨proof⟩

**lemma**(in *standard-borel*) *standard-borel-lr-sets-ident[simp, measurable-cong]*:  
 $sets (qbs-to-measure (measure-to-qbs M)) = sets M$   
 ⟨proof⟩

### 2.2.3 The Adjunction

**lemma** *lr-adjunction-correspondence* :  
 $X \rightarrow_Q (measure-to-qbs Y) = (qbs-to-measure X) \rightarrow_M Y$   
 ⟨proof⟩

**lemma**(in *standard-borel*) *standard-borel-r-full-faithful*:  
 $M \rightarrow_M Y = measure-to-qbs M \rightarrow_Q measure-to-qbs Y$   
 ⟨proof⟩

**lemma** *qbs-morphism-dest[dest]*:  
**assumes**  $f \in X \rightarrow_Q measure-to-qbs Y$   
**shows**  $f \in qbs-to-measure X \rightarrow_M Y$   
 ⟨proof⟩

**lemma**(in *standard-borel*) *qbs-morphism-dest[dest]*:

**assumes**  $k \in \text{measure-to-qbs } M \rightarrow_Q \text{ measure-to-qbs } Y$   
**shows**  $k \in M \rightarrow_M Y$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-morphism-measurable-intro*[intro!]:

**assumes**  $f \in \text{qbs-to-measure } X \rightarrow_M Y$   
**shows**  $f \in X \rightarrow_Q \text{ measure-to-qbs } Y$   
 $\langle \text{proof} \rangle$

**lemma**(in *standard-borel*) *qbs-morphism-measurable-intro*[intro!]:

**assumes**  $k \in M \rightarrow_M Y$   
**shows**  $k \in \text{measure-to-qbs } M \rightarrow_Q \text{ measure-to-qbs } Y$   
 $\langle \text{proof} \rangle$

We can use the measurability prover when we reason about morphisms.

**lemma**

**assumes**  $f \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$   
**shows**  $(\lambda x. 2 * f x + (f x) \hat{~} 2) \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$   
 $\langle \text{proof} \rangle$

**lemma**

**assumes**  $f \in X \rightarrow_Q \mathbb{R}_Q$   
**and**  $\alpha \in \text{qbs-Mx } X$   
**shows**  $(\lambda x. 2 * f (\alpha x) + (f (\alpha x)) \hat{~} 2) \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-morphisn-from-countable*:

**fixes**  $X :: 'a \text{ quasi-borel}$   
**assumes** *countable* (*qbs-space*  $X$ )  
 $\text{qbs-Mx } X \subseteq \text{real-borel} \rightarrow_M \text{count-space } (\text{qbs-space } X)$   
**and**  $\bigwedge i. i \in \text{qbs-space } X \implies f i \in \text{qbs-space } Y$   
**shows**  $f \in X \rightarrow_Q Y$   
 $\langle \text{proof} \rangle$

**corollary** *nat-qbs-morphism*:

**assumes**  $\bigwedge n. f n \in \text{qbs-space } Y$   
**shows**  $f \in \mathbb{N}_Q \rightarrow_Q Y$   
 $\langle \text{proof} \rangle$

**corollary** *bool-qbs-morphism*:

**assumes**  $\bigwedge b. f b \in \text{qbs-space } Y$   
**shows**  $f \in \mathbb{B}_Q \rightarrow_Q Y$   
 $\langle \text{proof} \rangle$

## 2.2.4 The Adjunction w.r.t. Ordering

**lemma** *l-mono*:

*mono qbs-to-measure*

*<proof>*

**lemma** *r-mono*:

*mono measure-to-qbs*

*<proof>*

**lemma** *rl-order-adjunction*:

$X \leq \text{qbs-to-measure } Y \iff \text{measure-to-qbs } X \leq Y$   
*<proof>*

**end**

## 2.3 Product Spaces

**theory** *Binary-Product-QuasiBorel*

**imports** *Measure-QuasiBorel-Adjunction*

**begin**

### 2.3.1 Binary Product Spaces

**definition** *pair-qbs-Mx* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*real*  $\Rightarrow$  *'a*  $\times$  *'b*) *set*  
**where**

*pair-qbs-Mx X Y*  $\equiv$  {*f. fst*  $\circ$  *f*  $\in$  *qbs-Mx X*  $\wedge$  *snd*  $\circ$  *f*  $\in$  *qbs-Mx Y*}

**definition** *pair-qbs* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*'a*  $\times$  *'b*) *quasi-borel* (**infix**  
 $\otimes_Q$  *80*) **where**

*pair-qbs X Y* = *Abs-quasi-borel (qbs-space X*  $\times$  *qbs-space Y, pair-qbs-Mx X Y)*

**lemma** *pair-qbs-f[simp]*: *pair-qbs-Mx X Y*  $\subseteq$  *UNIV*  $\rightarrow$  *qbs-space X*  $\times$  *qbs-space Y*  
*<proof>*

**lemma** *pair-qbs-closed1*: *qbs-closed1 (pair-qbs-Mx (X::'a quasi-borel) (Y::'b quasi-borel))*  
*<proof>*

**lemma** *pair-qbs-closed2*: *qbs-closed2 (qbs-space X*  $\times$  *qbs-space Y) (pair-qbs-Mx X*  
*Y)*  
*<proof>*

**lemma** *pair-qbs-closed3*: *qbs-closed3 (pair-qbs-Mx (X::'a quasi-borel) (Y::'b quasi-borel))*  
*<proof>*

**lemma** *pair-qbs-correct*: *Rep-quasi-borel (X*  $\otimes_Q$  *Y) = (qbs-space X*  $\times$  *qbs-space*  
*Y, pair-qbs-Mx X Y)*  
*<proof>*

**lemma** *pair-qbs-space[simp]*: *qbs-space (X*  $\otimes_Q$  *Y) = qbs-space X*  $\times$  *qbs-space Y*  
*<proof>*

**lemma** *pair-qbs-Mx[simp]*: *qbs-Mx (X*  $\otimes_Q$  *Y) = pair-qbs-Mx X Y*

*<proof>*

**lemma** *pair-qbs-morphismI*:

**assumes**  $\bigwedge \alpha \beta. \alpha \in \text{qbs-Mx } X \implies \beta \in \text{qbs-Mx } Y$   
 $\implies f \circ (\lambda r. (\alpha \ r, \beta \ r)) \in \text{qbs-Mx } Z$

**shows**  $f \in (X \otimes_Q Y) \rightarrow_Q Z$

*<proof>*

**lemma** *fst-qbs-morphism*:

$\text{fst} \in X \otimes_Q Y \rightarrow_Q X$

*<proof>*

**lemma** *snd-qbs-morphism*:

$\text{snd} \in X \otimes_Q Y \rightarrow_Q Y$

*<proof>*

**lemma** *qbs-morphism-pair-iff*:

$f \in X \rightarrow_Q Y \otimes_Q Z \iff \text{fst} \circ f \in X \rightarrow_Q Y \wedge \text{snd} \circ f \in X \rightarrow_Q Z$

*<proof>*

**lemma** *qbs-morphism-Pair1*:

**assumes**  $x \in \text{qbs-space } X$

**shows**  $\text{Pair } x \in Y \rightarrow_Q X \otimes_Q Y$

*<proof>*

**lemma** *qbs-morphism-Pair1'*:

**assumes**  $x \in \text{qbs-space } X$

**and**  $f \in X \otimes_Q Y \rightarrow_Q Z$

**shows**  $(\lambda y. f \ (x,y)) \in Y \rightarrow_Q Z$

*<proof>*

**lemma** *qbs-morphism-Pair2*:

**assumes**  $y \in \text{qbs-space } Y$

**shows**  $(\lambda x. (x,y)) \in X \rightarrow_Q X \otimes_Q Y$

*<proof>*

**lemma** *qbs-morphism-Pair2'*:

**assumes**  $y \in \text{qbs-space } Y$

**and**  $f \in X \otimes_Q Y \rightarrow_Q Z$

**shows**  $(\lambda x. f \ (x,y)) \in X \rightarrow_Q Z$

*<proof>*

**lemma** *qbs-morphism-fst''*:

**assumes**  $f \in X \rightarrow_Q Y$

**shows**  $(\lambda k. f \ (\text{fst } k)) \in X \otimes_Q Z \rightarrow_Q Y$

*<proof>*

**lemma** *qbs-morphism-snd''*:

**assumes**  $f \in X \rightarrow_Q Y$

**shows**  $(\lambda k. f (snd k)) \in Z \otimes_Q X \rightarrow_Q Y$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-tuple*:

**assumes**  $f \in Z \rightarrow_Q X$

**and**  $g \in Z \rightarrow_Q Y$

**shows**  $(\lambda z. (f z, g z)) \in Z \rightarrow_Q X \otimes_Q Y$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-map-prod*:

**assumes**  $f \in X \rightarrow_Q Y$

**and**  $g \in X' \rightarrow_Q Y'$

**shows**  $map-prod f g \in X \otimes_Q X' \rightarrow_Q Y \otimes_Q Y'$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-pair-swap'*:

$(\lambda(x,y). (y,x)) \in (X::'a \text{ quasi-borel}) \otimes_Q (Y::'b \text{ quasi-borel}) \rightarrow_Q Y \otimes_Q X$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-pair-swap*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q Z$

**shows**  $(\lambda(x,y). f (y,x)) \in Y \otimes_Q X \rightarrow_Q Z$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-pair-assoc1*:

$(\lambda((x,y),z). (x,(y,z))) \in (X \otimes_Q Y) \otimes_Q Z \rightarrow_Q X \otimes_Q (Y \otimes_Q Z)$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-pair-assoc2*:

$(\lambda(x,(y,z)). ((x,y),z)) \in X \otimes_Q (Y \otimes_Q Z) \rightarrow_Q (X \otimes_Q Y) \otimes_Q Z$   
 $\langle proof \rangle$

**lemma** *pair-qbs-fst*:

**assumes**  $qbs-space Y \neq \{\}$

**shows**  $map-qbs fst (X \otimes_Q Y) = X$   
 $\langle proof \rangle$

**lemma** *pair-qbs-snd*:

**assumes**  $qbs-space X \neq \{\}$

**shows**  $map-qbs snd (X \otimes_Q Y) = Y$   
 $\langle proof \rangle$

The following lemma corresponds to [1] Proposition 19(1).

**lemma** *r-preserves-product* :

$measure-to-qbs (X \otimes_M Y) = measure-to-qbs X \otimes_Q measure-to-qbs Y$   
 $\langle proof \rangle$

**lemma** *l-product-sets*[*simp,measurable-cong*]:  
 $sets (qbs\text{-to-measure } X \otimes_M qbs\text{-to-measure } Y) \subseteq sets (qbs\text{-to-measure } (X \otimes_Q Y))$   
 ⟨*proof*⟩

**lemma**(**in** *pair-standard-borel*) *l-r-r-sets*[*simp,measurable-cong*]:  
 $sets (qbs\text{-to-measure } (measure\text{-to-qbs } M \otimes_Q measure\text{-to-qbs } N)) = sets (M \otimes_M N)$   
 ⟨*proof*⟩

**end**

### 2.3.2 Product Spaces

**theory** *Product-QuasiBorel*

**imports** *Binary-Product-QuasiBorel*

**begin**

**definition** *prod-qbs-Mx* :: [*'a set, 'a ⇒ 'b quasi-borel*] ⇒ (*real ⇒ 'a ⇒ 'b*) *set*  
**where**  
 $prod\text{-qbs-Mx } I M \equiv \{ \alpha \mid \alpha. \forall i. (i \in I \longrightarrow (\lambda r. \alpha r i) \in qbs\text{-Mx } (M i)) \wedge (i \notin I \longrightarrow (\lambda r. \alpha r i) = (\lambda r. undefined)) \}$

**lemma** *prod-qbs-MxI*:  
**assumes**  $\bigwedge i. i \in I \implies (\lambda r. \alpha r i) \in qbs\text{-Mx } (M i)$   
**and**  $\bigwedge i. i \notin I \implies (\lambda r. \alpha r i) = (\lambda r. undefined)$   
**shows**  $\alpha \in prod\text{-qbs-Mx } I M$   
 ⟨*proof*⟩

**lemma** *prod-qbs-MxE*:  
**assumes**  $\alpha \in prod\text{-qbs-Mx } I M$   
**shows**  $\bigwedge i. i \in I \implies (\lambda r. \alpha r i) \in qbs\text{-Mx } (M i)$   
**and**  $\bigwedge i. i \notin I \implies (\lambda r. \alpha r i) = (\lambda r. undefined)$   
**and**  $\bigwedge i r. i \notin I \implies \alpha r i = undefined$   
 ⟨*proof*⟩

**definition** *PiQ* :: [*'a set ⇒ ('a ⇒ 'b quasi-borel) ⇒ ('a ⇒ 'b) quasi-borel*] **where**  
 $PiQ I M \equiv Abs\text{-quasi-borel } (\Pi_E i \in I. qbs\text{-space } (M i), prod\text{-qbs-Mx } I M)$

**syntax**

*-PiQ* :: *pttrn ⇒ 'i set ⇒ 'a quasi-borel ⇒ ('i => 'a) quasi-borel* (( $\exists \Pi_Q -\in - / -$ )  
 10)

**translations**

$\Pi_Q x \in I. M == CONST PiQ I (\lambda x. M)$

**lemma** *PiQ-f*:  $prod\text{-qbs-Mx } I M \subseteq UNIV \rightarrow (\Pi_E i \in I. qbs\text{-space } (M i))$

*<proof>*

**lemma** *PiQ-closed1: qbs-closed1 (prod-qbs-Mx I M)*  
*<proof>*

**lemma** *PiQ-closed2: qbs-closed2 ( $\prod_E i \in I. \text{qbs-space } (M \ i)$ ) (prod-qbs-Mx I M)*  
*<proof>*

**lemma** *PiQ-closed3: qbs-closed3 (prod-qbs-Mx I M)*  
*<proof>*

**lemma** *PiQ-correct: Rep-quasi-borel (PiQ I M) = ( $\prod_E i \in I. \text{qbs-space } (M \ i)$ ), prod-qbs-Mx I M)*  
*<proof>*

**lemma** *PiQ-space[simp]: qbs-space (PiQ I M) = ( $\prod_E i \in I. \text{qbs-space } (M \ i)$ )*  
*<proof>*

**lemma** *PiQ-Mx[simp]: qbs-Mx (PiQ I M) = prod-qbs-Mx I M*  
*<proof>*

**lemma** *qbs-morphism-component-singleton:*  
**assumes**  $i \in I$   
**shows**  $(\lambda x. x \ i) \in (\prod_Q i \in I. (M \ i)) \rightarrow_Q M \ i$   
*<proof>*

**lemma** *product-qbs-canonical1:*  
**assumes**  $\bigwedge i. i \in I \implies f \ i \in Y \rightarrow_Q X \ i$   
**and**  $\bigwedge i. i \notin I \implies f \ i = (\lambda y. \text{undefined})$   
**shows**  $(\lambda y \ i. f \ i \ y) \in Y \rightarrow_Q (\prod_Q i \in I. X \ i)$   
*<proof>*

**lemma** *product-qbs-canonical2:*  
**assumes**  $\bigwedge i. i \in I \implies f \ i \in Y \rightarrow_Q X \ i$   
 $\bigwedge i. i \notin I \implies f \ i = (\lambda y. \text{undefined})$   
 $g \in Y \rightarrow_Q (\prod_Q i \in I. X \ i)$   
 $\bigwedge i. i \in I \implies f \ i = (\lambda x. x \ i) \circ g$   
**and**  $y \in \text{qbs-space } Y$   
**shows**  $g \ y = (\lambda i. f \ i \ y)$   
*<proof>*

**lemma** *merge-qbs-morphism:*  
 $\text{merge } I \ J \in (\prod_Q i \in I. (M \ i)) \otimes_Q (\prod_Q j \in J. (M \ j)) \rightarrow_Q (\prod_Q i \in I \cup J. (M \ i))$   
*<proof>*

The following lemma corresponds to [1] Proposition 19(1).

**lemma** *r-preserves-product':*  
 $\text{measure-to-qbs } (\prod_M i \in I. M \ i) = (\prod_Q i \in I. \text{measure-to-qbs } (M \ i))$

*<proof>*

$$\prod_{i=0,1} X_i \cong X_1 \times X_2.$$

**lemma** *product-binary-product:*

$$\exists f g. f \in (\prod_Q i \in UNIV. \text{if } i \text{ then } X \text{ else } Y) \rightarrow_Q X \otimes_Q Y \wedge g \in X \otimes_Q Y \rightarrow_Q (\prod_Q i \in UNIV. \text{if } i \text{ then } X \text{ else } Y) \wedge$$

$$g \circ f = id \wedge f \circ g = id$$

*<proof>*

**end**

## 2.4 Coproduct Spaces

**theory** *Binary-CoProduct-QuasiBorel*

**imports** *Measure-QuasiBorel-Adjunction*

**begin**

### 2.4.1 Binary Coproduct Spaces

**definition** *copair-qbs-Mx* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*real*  $\Rightarrow$  *'a + 'b*) *set*  
**where**

*copair-qbs-Mx X Y*  $\equiv$

{*g.  $\exists S \in$  sets real-borel.*

*(S = {}  $\longrightarrow$  ( $\exists \alpha 1 \in$  qbs-Mx X. *g* = ( $\lambda r. Inl (\alpha 1 r)$ )))  $\wedge$*

*(S = UNIV  $\longrightarrow$  ( $\exists \alpha 2 \in$  qbs-Mx Y. *g* = ( $\lambda r. Inr (\alpha 2 r)$ )))  $\wedge$*

*((S  $\neq$  {}  $\wedge$  S  $\neq$  UNIV)  $\longrightarrow$*

*( $\exists \alpha 1 \in$  qbs-Mx X.*

*$\exists \alpha 2 \in$  qbs-Mx Y.*

**g* = ( $\lambda r::real. (\text{if } (r \in S) \text{ then } Inl (\alpha 1 r) \text{ else } Inr (\alpha 2 r))$ ))*}

**definition** *copair-qbs* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*'a + 'b*) *quasi-borel*  
(**infixr** *<+><sub>Q</sub>* 65) **where**

*copair-qbs X Y*  $\equiv$  *Abs-quasi-borel (qbs-space X <+> qbs-space Y, copair-qbs-Mx X Y)*

The followin is an equivalent definition of *copair-qbs-Mx*.

**definition** *copair-qbs-Mx2* :: [*'a quasi-borel, 'b quasi-borel*]  $\Rightarrow$  (*real*  $\Rightarrow$  *'a + 'b*) *set* **where**

*copair-qbs-Mx2 X Y*  $\equiv$

{*g. (if qbs-space X = {}  $\wedge$  qbs-space Y = {} then False*

*else if qbs-space X  $\neq$  {}  $\wedge$  qbs-space Y = {} then*

*( $\exists \alpha 1 \in$  qbs-Mx X. *g* = ( $\lambda r. Inl (\alpha 1 r)$ ))*

*else if qbs-space X = {}  $\wedge$  qbs-space Y  $\neq$  {} then*

*( $\exists \alpha 2 \in$  qbs-Mx Y. *g* = ( $\lambda r. Inr (\alpha 2 r)$ ))*

*else*

*( $\exists S \in$  sets real-borel.  $\exists \alpha 1 \in$  qbs-Mx X.  $\exists \alpha 2 \in$  qbs-Mx Y.*

**g* = ( $\lambda r::real. (\text{if } (r \in S) \text{ then } Inl (\alpha 1 r) \text{ else } Inr (\alpha 2 r))$ ))*}



**lemma** *copair-qbs-Mx-equiv*:  $\text{copair-qbs-Mx } (X :: 'a \text{ quasi-borel}) (Y :: 'b \text{ quasi-borel})$   
 $= \text{copair-qbs-Mx2 } X Y$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-f[simp]*:  $\text{copair-qbs-Mx } X Y \subseteq \text{UNIV} \rightarrow \text{qbs-space } X \langle + \rangle$   
 $\text{qbs-space } Y$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-closed1*:  $\text{qbs-closed1 } (\text{copair-qbs-Mx } X Y)$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-closed2*:  $\text{qbs-closed2 } (\text{qbs-space } X \langle + \rangle \text{qbs-space } Y) (\text{copair-qbs-Mx } X Y)$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-closed3*:  $\text{qbs-closed3 } (\text{copair-qbs-Mx } X Y)$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-correct*:  $\text{Rep-quasi-borel } (\text{copair-qbs } X Y) = (\text{qbs-space } X \langle + \rangle$   
 $\text{qbs-space } Y, \text{copair-qbs-Mx } X Y)$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-space[simp]*:  $\text{qbs-space } (\text{copair-qbs } X Y) = \text{qbs-space } X \langle + \rangle$   
 $\text{qbs-space } Y$   
 $\langle \text{proof} \rangle$

**lemma** *copair-qbs-Mx[simp]*:  $\text{qbs-Mx } (\text{copair-qbs } X Y) = \text{copair-qbs-Mx } X Y$   
 $\langle \text{proof} \rangle$

**lemma** *Inl-qbs-morphism*:  
 $\text{Inl} \in X \rightarrow_Q X \langle + \rangle_Q Y$   
 $\langle \text{proof} \rangle$

**lemma** *Inr-qbs-morphism*:  
 $\text{Inr} \in Y \rightarrow_Q X \langle + \rangle_Q Y$   
 $\langle \text{proof} \rangle$

**lemma** *case-sum-preserves-morphisms*:  
**assumes**  $f \in X \rightarrow_Q Z$   
**and**  $g \in Y \rightarrow_Q Z$   
**shows**  $\text{case-sum } f g \in X \langle + \rangle_Q Y \rightarrow_Q Z$   
 $\langle \text{proof} \rangle$

**lemma** *map-sum-preserves-morphisms*:  
**assumes**  $f \in X \rightarrow_Q Y$   
**and**  $g \in X' \rightarrow_Q Y'$

**shows**  $\text{map-sum } f g \in X \langle + \rangle_Q X' \rightarrow_Q Y \langle + \rangle_Q Y'$   
 $\langle \text{proof} \rangle$

**end**

## 2.4.2 Countable Coproduct Spaces

**theory** *CoProduct-QuasiBorel*

**imports**

*Product-QuasiBorel*

*Binary-CoProduct-QuasiBorel*

**begin**

**definition** *coprod-qbs-Mx* :: [*'a set, 'a  $\Rightarrow$  'b quasi-borel*]  $\Rightarrow$  (*real  $\Rightarrow$  'a  $\times$  'b*) *set*  
**where**

*coprod-qbs-Mx I X*  $\equiv$  {  $\lambda r. (f r, \alpha (f r) r) \mid f \alpha. f \in \text{real-borel} \rightarrow_M \text{count-space } I$   
 $\wedge (\forall i \in \text{range } f. \alpha i \in \text{qbs-Mx } (X i))$  }

**lemma** *coprod-qbs-MxI*:

**assumes**  $f \in \text{real-borel} \rightarrow_M \text{count-space } I$

**and**  $\bigwedge i. i \in \text{range } f \implies \alpha i \in \text{qbs-Mx } (X i)$

**shows**  $(\lambda r. (f r, \alpha (f r) r)) \in \text{coprod-qbs-Mx } I X$

$\langle \text{proof} \rangle$

**definition** *coprod-qbs-Mx'* :: [*'a set, 'a  $\Rightarrow$  'b quasi-borel*]  $\Rightarrow$  (*real  $\Rightarrow$  'a  $\times$  'b*) *set*  
**where**

*coprod-qbs-Mx' I X*  $\equiv$  {  $\lambda r. (f r, \alpha (f r) r) \mid f \alpha. f \in \text{real-borel} \rightarrow_M \text{count-space } I$   
 $\wedge (\forall i. (i \in \text{range } f \vee \text{qbs-space } (X i) \neq \{\}) \longrightarrow \alpha i \in \text{qbs-Mx } (X i))$  }

**lemma** *coproduct-qbs-Mx-eq*:

$\text{coprod-qbs-Mx } I X = \text{coprod-qbs-Mx}' I X$

$\langle \text{proof} \rangle$

**definition** *coprod-qbs* :: [*'a set, 'a  $\Rightarrow$  'b quasi-borel*]  $\Rightarrow$  (*'a  $\times$  'b*) *quasi-borel* **where**  
*coprod-qbs I X*  $\equiv \text{Abs-quasi-borel } (\text{SIGMA } i:I. \text{qbs-space } (X i), \text{coprod-qbs-Mx } I X)$

**syntax**

*-coprod-qbs* :: *pttrn  $\Rightarrow$  'i set  $\Rightarrow$  'a quasi-borel  $\Rightarrow$  ('i  $\times$  'a) quasi-borel* (( $\exists \Pi_Q \text{-}\in\text{-}/$   
 $\text{-})$  10)

**translations**

$\Pi_Q x \in I. M \equiv \text{CONST } \text{coprod-qbs } I (\lambda x. M)$

**lemma** *coprod-qbs-f[simp]*:  $\text{coprod-qbs-Mx } I X \subseteq \text{UNIV} \rightarrow (\text{SIGMA } i:I. \text{qbs-space } (X i))$

$\langle \text{proof} \rangle$

**lemma** *coprod-qbs-closed1*: *qbs-closed1 (coprod-qbs-Mx I X)*

$\langle proof \rangle$

**lemma** *coprod-qbs-closed2*: *qbs-closed2* ( $SIGMA\ i:I. qbs-space\ (X\ i)$ ) (*coprod-qbs-Mx*  $I\ X$ )

$\langle proof \rangle$

**lemma** *coprod-qbs-closed3*:  
*qbs-closed3* (*coprod-qbs-Mx*  $I\ X$ )

$\langle proof \rangle$

**lemma** *coprod-qbs-correct*: *Rep-quasi-borel* (*coprod-qbs*  $I\ X$ ) = ( $SIGMA\ i:I. qbs-space\ (X\ i)$ , *coprod-qbs-Mx*  $I\ X$ )

$\langle proof \rangle$

**lemma** *coproduct-qbs-space[simp]*: *qbs-space* (*coprod-qbs*  $I\ X$ ) = ( $SIGMA\ i:I. qbs-space\ (X\ i)$ )

$\langle proof \rangle$

**lemma** *coproduct-qbs-Mx[simp]*: *qbs-Mx* (*coprod-qbs*  $I\ X$ ) = *coprod-qbs-Mx*  $I\ X$

$\langle proof \rangle$

**lemma** *ini-morphism*:

**assumes**  $j \in I$

**shows**  $(\lambda x. (j, x)) \in X\ j \rightarrow_Q (\coprod_Q\ i \in I. X\ i)$

$\langle proof \rangle$

**lemma** *coprod-qbs-canonical1*:

**assumes** *countable*  $I$

**and**  $\bigwedge i. i \in I \implies f\ i \in X\ i \rightarrow_Q Y$

**shows**  $(\lambda(i, x). f\ i\ x) \in (\coprod_Q\ i \in I. X\ i) \rightarrow_Q Y$

$\langle proof \rangle$

**lemma** *coprod-qbs-canonical1'*:

**assumes** *countable*  $I$

**and**  $\bigwedge i. i \in I \implies (\lambda x. f\ (i, x)) \in X\ i \rightarrow_Q Y$

**shows**  $f \in (\coprod_Q\ i \in I. X\ i) \rightarrow_Q Y$

$\langle proof \rangle$

$\coprod_{i=0,1} X_i \cong X_1 + X_2.$

**lemma** *coproduct-binary-coproduct*:

$\exists f\ g. f \in (\coprod_Q\ i \in UNIV. \text{if } i \text{ then } X \text{ else } Y) \rightarrow_Q X <+>_Q Y \wedge g \in X <+>_Q Y \rightarrow_Q (\coprod_Q\ i \in UNIV. \text{if } i \text{ then } X \text{ else } Y) \wedge$

$g \circ f = id \wedge f \circ g = id$

$\langle proof \rangle$

### 2.4.3 Lists

**abbreviation** *list-of*  $X \equiv \coprod_Q\ n \in (UNIV :: nat\ set). (\coprod_Q\ i \in \{..<n\}. X)$

**abbreviation**  $list-nil :: nat \times (nat \Rightarrow 'a)$  **where**

$list-nil \equiv (0, \lambda n. undefined)$

**abbreviation**  $list-cons :: ['a, nat \times (nat \Rightarrow 'a)] \Rightarrow nat \times (nat \Rightarrow 'a)$  **where**

$list-cons\ x\ l \equiv (Suc\ (fst\ l), (\lambda n. if\ n = 0\ then\ x\ else\ (snd\ l)\ (n - 1)))$

**definition**  $list-head :: nat \times (nat \Rightarrow 'a) \Rightarrow 'a$  **where**

$list-head\ l = snd\ l\ 0$

**definition**  $list-tail :: nat \times (nat \Rightarrow 'a) \Rightarrow nat \times (nat \Rightarrow 'a)$  **where**

$list-tail\ l = (fst\ l - 1, \lambda m. (snd\ l)\ (Suc\ m))$

**lemma**  $list-simp1$ :

$list-nil \neq list-cons\ x\ l$

$\langle proof \rangle$

**lemma**  $list-simp2$ :

**assumes**  $list-cons\ a\ al = list-cons\ b\ bl$

**shows**  $a = b\ al = bl$

$\langle proof \rangle$

**lemma**  $list-simp3$ :

**shows**  $list-head\ (list-cons\ a\ l) = a$

$\langle proof \rangle$

**lemma**  $list-simp4$ :

**assumes**  $l \in qbs-space\ (list-of\ X)$

**shows**  $list-tail\ (list-cons\ a\ l) = l$

$\langle proof \rangle$

**lemma**  $list-decomp1$ :

**assumes**  $l \in qbs-space\ (list-of\ X)$

**shows**  $l = list-nil \vee$

$(\exists a\ l'. a \in qbs-space\ X \wedge l' \in qbs-space\ (list-of\ X) \wedge l = list-cons\ a\ l')$

$\langle proof \rangle$

**lemma**  $list-simp5$ :

**assumes**  $l \in qbs-space\ (list-of\ X)$

**and**  $l \neq list-nil$

**shows**  $l = list-cons\ (list-head\ l)\ (list-tail\ l)$

$\langle proof \rangle$

**lemma**  $list-simp6$ :

$list-nil \in qbs-space\ (list-of\ X)$

$\langle proof \rangle$

**lemma**  $list-simp7$ :

**assumes**  $a \in qbs-space\ X$

**and**  $l \in qbs-space\ (list-of\ X)$

**shows**  $list-cons\ a\ l \in qbs-space\ (list-of\ X)$

$\langle \text{proof} \rangle$

**lemma** *list-destruct-rule*:

**assumes**  $l \in \text{qbs-space } (\text{list-of } X)$   
           $P \text{ list-nil}$   
**and**  $\bigwedge a l'. a \in \text{qbs-space } X \implies l' \in \text{qbs-space } (\text{list-of } X) \implies P (\text{list-cons } a$   
 $l')$   
**shows**  $P l$   
 $\langle \text{proof} \rangle$

**lemma** *list-induct-rule*:

**assumes**  $l \in \text{qbs-space } (\text{list-of } X)$   
           $P \text{ list-nil}$   
**and**  $\bigwedge a l'. a \in \text{qbs-space } X \implies l' \in \text{qbs-space } (\text{list-of } X) \implies P l' \implies P$   
 $(\text{list-cons } a l')$   
**shows**  $P l$   
 $\langle \text{proof} \rangle$

**fun** *from-list* ::  $'a \text{ list} \Rightarrow \text{nat} \times (\text{nat} \Rightarrow 'a)$  **where**

*from-list* [] = *list-nil* |  
*from-list* (a#l) = *list-cons* a (*from-list* l)

**fun** *to-list'* ::  $\text{nat} \Rightarrow (\text{nat} \Rightarrow 'a) \Rightarrow 'a \text{ list}$  **where**

*to-list'* 0 = [] |  
*to-list'* (Suc n) f = f 0 # *to-list'* n ( $\lambda n. f (\text{Suc } n)$ )

**definition** *to-list* ::  $\text{nat} \times (\text{nat} \Rightarrow 'a) \Rightarrow 'a \text{ list}$  **where**

*to-list*  $\equiv \text{case-prod } \text{to-list}'$

**lemma** *to-list-simp1*:

**shows** *to-list* *list-nil* = []  
 $\langle \text{proof} \rangle$

**lemma** *to-list-simp2*:

**assumes**  $l \in \text{qbs-space } (\text{list-of } X)$   
**shows** *to-list* (*list-cons* a l) = a # *to-list* l  
 $\langle \text{proof} \rangle$

**lemma** *from-list-length*:

*fst* (*from-list* l) = *length* l  
 $\langle \text{proof} \rangle$

**lemma** *from-list-in-list-of*:

**assumes**  $set l \subseteq \text{qbs-space } X$   
**shows** *from-list* l  $\in \text{qbs-space } (\text{list-of } X)$   
 $\langle \text{proof} \rangle$

**lemma** *from-list-in-list-of'*:

**shows**  $from-list\ l \in qbs-space\ (list-of\ (Abs-quasi-borel\ (UNIV, UNIV)))$   
 ⟨proof⟩

**lemma** *list-cons-in-list-of*:  
**assumes**  $set\ (a\#\ l) \subseteq qbs-space\ X$   
**shows**  $list-cons\ a\ (from-list\ l) \in qbs-space\ (list-of\ X)$   
 ⟨proof⟩

**lemma** *from-list-to-list-ident*:  
 $(to-list\ \circ\ from-list)\ l = l$   
 ⟨proof⟩

**lemma** *to-list-from-list-ident*:  
**assumes**  $l \in qbs-space\ (list-of\ X)$   
**shows**  $(from-list\ \circ\ to-list)\ l = l$   
 ⟨proof⟩

**definition** *rec-list'* ::  $'b \Rightarrow ('a \Rightarrow (nat \times (nat \Rightarrow 'a))) \Rightarrow 'b \Rightarrow 'b \Rightarrow (nat \times (nat \Rightarrow 'a)) \Rightarrow 'b$  **where**  
 $rec-list'\ t0\ f\ l \equiv (rec-list\ t0\ (\lambda x\ l'.\ f\ x\ (from-list\ l'))\ (to-list\ l))$

**lemma** *rec-list'-simp1*:  
 $rec-list'\ t\ f\ list-nil = t$   
 ⟨proof⟩

**lemma** *rec-list'-simp2*:  
**assumes**  $l \in qbs-space\ (list-of\ X)$   
**shows**  $rec-list'\ t\ f\ (list-cons\ x\ l) = f\ x\ l\ (rec-list'\ t\ f\ l)$   
 ⟨proof⟩

**end**

## 2.5 Function Spaces

**theory** *Exponent-QuasiBorel*  
**imports** *CoProduct-QuasiBorel*  
**begin**

### 2.5.1 Function Spaces

**definition** *exp-qbs-Mx* ::  $['a\ quasi-borel,\ 'b\ quasi-borel] \Rightarrow (real \Rightarrow 'a \Rightarrow 'b)$  **set**  
**where**  
 $exp-qbs-Mx\ X\ Y \equiv \{g :: real \Rightarrow 'a \Rightarrow 'b.\ case-prod\ g \in \mathbf{R}_Q \otimes_Q X \rightarrow_Q Y\}$

**definition** *exp-qbs* ::  $['a\ quasi-borel,\ 'b\ quasi-borel] \Rightarrow ('a \Rightarrow 'b)$  **quasi-borel** (**infixr**  
 $\Rightarrow_Q$  61) **where**  
 $X \Rightarrow_Q Y \equiv Abs-quasi-borel\ (X \rightarrow_Q Y,\ exp-qbs-Mx\ X\ Y)$

**lemma** *exp-qbs-f[simp]*:  $\text{exp-qbs-Mx } X \ Y \subseteq \text{UNIV} \rightarrow (X :: 'a \text{ quasi-borel}) \rightarrow_Q (Y :: 'b \text{ quasi-borel})$

*<proof>*

**lemma** *exp-qbs-closed1*:  $\text{qbs-closed1 } (\text{exp-qbs-Mx } X \ Y)$

*<proof>*

**lemma** *exp-qbs-closed2*:  $\text{qbs-closed2 } (X \rightarrow_Q Y) (\text{exp-qbs-Mx } X \ Y)$

*<proof>*

**lemma** *exp-qbs-closed3*:  $\text{qbs-closed3 } (\text{exp-qbs-Mx } X \ Y)$

*<proof>*

**lemma** *exp-qbs-correct*:  $\text{Rep-quasi-borel } (\text{exp-qbs } X \ Y) = (X \rightarrow_Q Y, \text{exp-qbs-Mx } X \ Y)$

*<proof>*

**lemma** *exp-qbs-space[simp]*:  $\text{qbs-space } (\text{exp-qbs } X \ Y) = X \rightarrow_Q Y$

*<proof>*

**lemma** *exp-qbs-Mx[simp]*:  $\text{qbs-Mx } (\text{exp-qbs } X \ Y) = \text{exp-qbs-Mx } X \ Y$

*<proof>*

**lemma** *qbs-exp-morphismI*:

**assumes**  $\bigwedge \alpha \ \beta. \ \alpha \in \text{qbs-Mx } X \implies$

$\beta \in \text{pair-qbs-Mx real-quasi-borel } Y \implies$

$(\lambda(r,x). (f \circ \alpha) \ r \ x) \circ \beta \in \text{qbs-Mx } Z$

**shows**  $f \in X \rightarrow_Q \text{exp-qbs } Y \ Z$

*<proof>*

**definition** *qbs-eval* ::  $(('a \Rightarrow 'b) \times 'a) \Rightarrow 'b$  **where**

$\text{qbs-eval } a \equiv (\text{fst } a) (\text{snd } a)$

**lemma** *qbs-eval-morphism*:

$\text{qbs-eval} \in (\text{exp-qbs } X \ Y) \otimes_Q X \rightarrow_Q Y$

*<proof>*

**lemma** *curry-morphism*:

$\text{curry} \in \text{exp-qbs } (X \otimes_Q Y) \ Z \rightarrow_Q \text{exp-qbs } X \ (\text{exp-qbs } Y \ Z)$

*<proof>*

**lemma** *curry-preserves-morphisms*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q Z$

**shows**  $\text{curry } f \in X \rightarrow_Q \text{exp-qbs } Y \ Z$

*<proof>*

**lemma** *uncurry-morphism*:

*case-prod*  $\in \text{exp-qbs } X (\text{exp-qbs } Y Z) \rightarrow_Q \text{exp-qbs } (X \otimes_Q Y) Z$   
 ⟨proof⟩

**lemma** *uncurry-preserves-morphisms*:

**assumes**  $f \in X \rightarrow_Q \text{exp-qbs } Y Z$   
**shows** *case-prod*  $f \in X \otimes_Q Y \rightarrow_Q Z$   
 ⟨proof⟩

**lemma** *arg-swap-morphism*:

**assumes**  $f \in X \rightarrow_Q \text{exp-qbs } Y Z$   
**shows**  $(\lambda y x. f x y) \in Y \rightarrow_Q \text{exp-qbs } X Z$   
 ⟨proof⟩

**lemma** *exp-qbs-comp-morphism*:

**assumes**  $f \in W \rightarrow_Q \text{exp-qbs } X Y$   
**and**  $g \in W \rightarrow_Q \text{exp-qbs } Y Z$   
**shows**  $(\lambda w. g w \circ f w) \in W \rightarrow_Q \text{exp-qbs } X Z$   
 ⟨proof⟩

**lemma** *case-sum-morphism*:

*case-prod case-sum*  $\in \text{exp-qbs } X Z \otimes_Q \text{exp-qbs } Y Z \rightarrow_Q \text{exp-qbs } (X <+>_Q Y) Z$   
 ⟨proof⟩

**lemma** *not-qbs-morphism*:

*Not*  $\in \mathbb{B}_Q \rightarrow_Q \mathbb{B}_Q$   
 ⟨proof⟩

**lemma** *or-qbs-morphism*:

$(\vee) \in \mathbb{B}_Q \rightarrow_Q \text{exp-qbs } \mathbb{B}_Q \mathbb{B}_Q$   
 ⟨proof⟩

**lemma** *and-qbs-morphism*:

$(\wedge) \in \mathbb{B}_Q \rightarrow_Q \text{exp-qbs } \mathbb{B}_Q \mathbb{B}_Q$   
 ⟨proof⟩

**lemma** *implies-qbs-morphism*:

$(\longrightarrow) \in \mathbb{B}_Q \rightarrow_Q \mathbb{B}_Q \Rightarrow_Q \mathbb{B}_Q$   
 ⟨proof⟩

**lemma** *less-nat-qbs-morphism*:

$(<) \in \mathbb{N}_Q \rightarrow_Q \text{exp-qbs } \mathbb{N}_Q \mathbb{B}_Q$   
 ⟨proof⟩

**lemma** *less-real-qbs-morphism*:

$(<) \in \mathbb{R}_Q \rightarrow_Q \text{exp-qbs } \mathbb{R}_Q \mathbb{B}_Q$   
 ⟨proof⟩



```

lemma rec-list-morphism':
  rec-list' ∈ qbs-space (exp-qbs Y (exp-qbs (exp-qbs X (exp-qbs (list-of X) (exp-qbs
Y Y))) (exp-qbs (list-of X) Y)))
  ⟨proof⟩

```

**end**

### 3 Probability Spaces

#### 3.1 Probability Measures

```

theory Probability-Space-QuasiBorel
  imports Exponent-QuasiBorel
begin

```

##### 3.1.1 Probability Measures

```

type-synonym 'a qbs-prob-t = 'a quasi-borel * (real ⇒ 'a) * real measure

```

```

locale in-Mx =
  fixes X :: 'a quasi-borel
  and α :: real ⇒ 'a
  assumes in-Mx[simp]: α ∈ qbs-Mx X

```

```

locale qbs-prob = in-Mx X α + real-distribution μ
  for X :: 'a quasi-borel and α and μ
begin
declare prob-space-axioms[simp]

```

```

lemma m-in-space-prob-algebra[simp]:
  μ ∈ space (prob-algebra real-borel)
  ⟨proof⟩
end

```

```

locale pair-qbs-probs = qp1:qbs-prob X α μ + qp2:qbs-prob Y β ν
  for X :: 'a quasi-borel and α μ and Y :: 'b quasi-borel and β ν
begin

```

```

sublocale pair-prob-space μ ν
  ⟨proof⟩

```

```

lemma ab-measurable[measurable]:
  map-prod α β ∈ real-borel ⊗M real-borel →M qbs-to-measure (X ⊗Q Y)
  ⟨proof⟩

```

```

lemma ab-g-in-Mx[simp]:
  map-prod α β ∘ real-real.g ∈ pair-qbs-Mx X Y

```

*<proof>*

**sublocale** *qbs-prob*  $X \otimes_Q Y$  *map-prod*  $\alpha \beta \circ \text{real-real.g distr } (\mu \otimes_M \nu)$  *real-borel*  
*real-real.f*  
*<proof>*

**end**

**locale** *pair-qbs-prob* = *qp1:qbs-prob*  $X \alpha \mu$  + *qp2:qbs-prob*  $Y \beta \nu$   
**for**  $X :: 'a \text{ quasi-borel}$  **and**  $\alpha \mu$  **and**  $Y :: 'a \text{ quasi-borel}$  **and**  $\beta \nu$   
**begin**

**sublocale** *pair-qbs-probs*  
*<proof>*

**lemma** *same-spaces[simp]*:  
**assumes**  $Y = X$   
**shows**  $\beta \in \text{qbs-Mx } X$   
*<proof>*

**end**

**lemma** *prob-algebra-real-prob-measure*:  
 $p \in \text{space } (\text{prob-algebra } (\text{real-borel})) = \text{real-distribution } p$   
*<proof>*

**lemma** *qbs-probI*:  
**assumes**  $\alpha \in \text{qbs-Mx } X$   
**and** *sets*  $\mu = \text{sets borel}$   
**and** *prob-space*  $\mu$   
**shows** *qbs-prob*  $X \alpha \mu$   
*<proof>*

**lemma** *qbs-empty-not-qbs-prob* :  $\neg \text{qbs-prob } (\text{empty-quasi-borel}) f M$   
*<proof>*

**definition** *qbs-prob-eq* :: [ $'a \text{ qbs-prob-t}$ ,  $'a \text{ qbs-prob-t}$ ]  $\Rightarrow \text{bool}$  **where**  
*qbs-prob-eq*  $p1 p2 \equiv$   
*(let*  $(qbs1, a1, m1) = p1;$   
 $(qbs2, a2, m2) = p2$  *in*  
 $\text{qbs-prob } qbs1 a1 m1 \wedge \text{qbs-prob } qbs2 a2 m2 \wedge qbs1 = qbs2 \wedge$   
 $\text{distr } m1 (\text{qbs-to-measure } qbs1) a1 = \text{distr } m2 (\text{qbs-to-measure } qbs2) a2)$

**definition** *qbs-prob-eq2* :: [ $'a \text{ qbs-prob-t}$ ,  $'a \text{ qbs-prob-t}$ ]  $\Rightarrow \text{bool}$  **where**  
*qbs-prob-eq2*  $p1 p2 \equiv$   
*(let*  $(qbs1, a1, m1) = p1;$   
 $(qbs2, a2, m2) = p2$  *in*  
 $\text{qbs-prob } qbs1 a1 m1 \wedge \text{qbs-prob } qbs2 a2 m2 \wedge qbs1 = qbs2 \wedge$   
 $(\forall f \in qbs1 \rightarrow_Q \text{real-quasi-borel}.$

$$(\int x. f (a1 x) \partial m1) = (\int x. f (a2 x) \partial m2))$$

**definition** *qbs-prob-eq3* :: [*'a qbs-prob-t, 'a qbs-prob-t*]  $\Rightarrow$  *bool* **where**  
*qbs-prob-eq3* *p1 p2*  $\equiv$   
 (let (*qbs1, a1, m1*) = *p1*;  
       (*qbs2, a2, m2*) = *p2* in  
 (*qbs-prob qbs1 a1 m1*  $\wedge$  *qbs-prob qbs2 a2 m2*  $\wedge$  *qbs1* = *qbs2*  $\wedge$   
 ( $\forall f \in$  *qbs1*  $\rightarrow_Q$  *real-quasi-borel*.  
 ( $\forall k \in$  *qbs-space qbs1*.  $0 \leq f k$ )  $\longrightarrow$   
 ( $\int x. f (a1 x) \partial m1$ ) = ( $\int x. f (a2 x) \partial m2$ ))))

**definition** *qbs-prob-eq4* :: [*'a qbs-prob-t, 'a qbs-prob-t*]  $\Rightarrow$  *bool* **where**  
*qbs-prob-eq4* *p1 p2*  $\equiv$   
 (let (*qbs1, a1, m1*) = *p1*;  
       (*qbs2, a2, m2*) = *p2* in  
 (*qbs-prob qbs1 a1 m1*  $\wedge$  *qbs-prob qbs2 a2 m2*  $\wedge$  *qbs1* = *qbs2*  $\wedge$   
 ( $\forall f \in$  *qbs1*  $\rightarrow_Q$   $\mathbb{R}_{\geq 0}$ .  
 ( $\int ^+ x. f (a1 x) \partial m1$ ) = ( $\int ^+ x. f (a2 x) \partial m2$ ))))

**lemma**(in *qbs-prob*) *qbs-prob-eq-refl[simp]*:  
*qbs-prob-eq* (*X,  $\alpha, \mu$* ) (*X,  $\alpha, \mu$* )  
*<proof>*

**lemma**(in *qbs-prob*) *qbs-prob-eq2-refl[simp]*:  
*qbs-prob-eq2* (*X,  $\alpha, \mu$* ) (*X,  $\alpha, \mu$* )  
*<proof>*

**lemma**(in *qbs-prob*) *qbs-prob-eq3-refl[simp]*:  
*qbs-prob-eq3* (*X,  $\alpha, \mu$* ) (*X,  $\alpha, \mu$* )  
*<proof>*

**lemma**(in *qbs-prob*) *qbs-prob-eq4-refl[simp]*:  
*qbs-prob-eq4* (*X,  $\alpha, \mu$* ) (*X,  $\alpha, \mu$* )  
*<proof>*

**lemma**(in *pair-qbs-prob*) *qbs-prob-eq-intro*:  
**assumes** *X = Y*  
**and** *distr  $\mu$  (qbs-to-measure X)  $\alpha$  = distr  $\nu$  (qbs-to-measure X)  $\beta$*   
**shows** *qbs-prob-eq (X,  $\alpha, \mu$ ) (Y,  $\beta, \nu$ )*  
*<proof>*

**lemma**(in *pair-qbs-prob*) *qbs-prob-eq2-intro*:  
**assumes** *X = Y*  
**and**  $\bigwedge f. f \in$  *qbs-to-measure X*  $\rightarrow_M$  *real-borel*  
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
**shows** *qbs-prob-eq2 (X,  $\alpha, \mu$ ) (Y,  $\beta, \nu$ )*  
*<proof>*

**lemma**(in *pair-qbs-prob*) *qbs-prob-eq3-intro*:

**assumes**  $X = Y$   
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel} \implies (\forall k \in \text{qbs-space } X. 0 \leq f k)$   
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
**shows**  $\text{qbs-prob-eq3 } (X, \alpha, \mu) (Y, \beta, \nu)$   
 $\langle \text{proof} \rangle$

**lemma**(in *pair-qbs-prob*) *qbs-prob-eq4-intro*:  
**assumes**  $X = Y$   
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{ennreal-borel}$   
 $\implies (\int^+ x. f (\alpha x) \partial \mu) = (\int^+ x. f (\beta x) \partial \nu)$   
**shows**  $\text{qbs-prob-eq4 } (X, \alpha, \mu) (Y, \beta, \nu)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-dest*:  
**assumes**  $\text{qbs-prob-eq } (X, \alpha, \mu) (Y, \beta, \nu)$   
**shows**  $\text{qbs-prob } X \alpha \mu$   
 $\text{qbs-prob } Y \beta \nu$   
 $Y = X$   
**and**  $\text{distr } \mu (\text{qbs-to-measure } X) \alpha = \text{distr } \nu (\text{qbs-to-measure } X) \beta$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq2-dest*:  
**assumes**  $\text{qbs-prob-eq2 } (X, \alpha, \mu) (Y, \beta, \nu)$   
**shows**  $\text{qbs-prob } X \alpha \mu$   
 $\text{qbs-prob } Y \beta \nu$   
 $Y = X$   
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel}$   
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq3-dest*:  
**assumes**  $\text{qbs-prob-eq3 } (X, \alpha, \mu) (Y, \beta, \nu)$   
**shows**  $\text{qbs-prob } X \alpha \mu$   
 $\text{qbs-prob } Y \beta \nu$   
 $Y = X$   
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel} \implies (\forall k \in \text{qbs-space } X. 0 \leq f k)$   
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq4-dest*:  
**assumes**  $\text{qbs-prob-eq4 } (X, \alpha, \mu) (Y, \beta, \nu)$   
**shows**  $\text{qbs-prob } X \alpha \mu$   
 $\text{qbs-prob } Y \beta \nu$   
 $Y = X$   
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{ennreal-borel}$   
 $\implies (\int^+ x. f (\alpha x) \partial \mu) = (\int^+ x. f (\beta x) \partial \nu)$   
 $\langle \text{proof} \rangle$

**definition** *qbs-prob-t-ennintegral* :: [*'a qbs-prob-t, 'a  $\Rightarrow$  ennreal*]  $\Rightarrow$  *ennreal* **where**  
*qbs-prob-t-ennintegral p f*  $\equiv$   
 (if  $f \in (\text{fst } p) \rightarrow_Q \text{ennreal-quasi-borel}$   
 then  $(\int^+ x. f (\text{fst } (\text{snd } p) x) \partial (\text{snd } (\text{snd } p)))$  else 0)

**definition** *qbs-prob-t-integral* :: [*'a qbs-prob-t, 'a  $\Rightarrow$  real*]  $\Rightarrow$  *real* **where**  
*qbs-prob-t-integral p f*  $\equiv$   
 (if  $f \in (\text{fst } p) \rightarrow_Q \mathbb{R}_Q$   
 then  $(\int x. f (\text{fst } (\text{snd } p) x) \partial (\text{snd } (\text{snd } p)))$   
 else 0)

**definition** *qbs-prob-t-integrable* :: [*'a qbs-prob-t, 'a  $\Rightarrow$  real*]  $\Rightarrow$  *bool* **where**  
*qbs-prob-t-integrable p f*  $\equiv f \in \text{fst } p \rightarrow_Q \text{real-quasi-borel} \wedge \text{integrable } (\text{snd } (\text{snd } p))$   
 $(f \circ (\text{fst } (\text{snd } p)))$

**definition** *qbs-prob-t-measure* :: *'a qbs-prob-t  $\Rightarrow$  'a measure* **where**  
*qbs-prob-t-measure p*  $\equiv \text{distr } (\text{snd } (\text{snd } p)) (\text{qbs-to-measure } (\text{fst } p)) (\text{fst } (\text{snd } p))$

**lemma** *qbs-prob-eq-symp*:  
*symp qbs-prob-eq*  
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-transp*:  
*transp qbs-prob-eq*  
 $\langle \text{proof} \rangle$

**quotient-type** *'a qbs-prob-space* = *'a qbs-prob-t / partial: qbs-prob-eq*  
**morphisms** *rep-qbs-prob-space qbs-prob-space*  
 $\langle \text{proof} \rangle$

**interpretation** *qbs-prob-space* : *quot-type qbs-prob-eq Abs-qbs-prob-space Rep-qbs-prob-space*  
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-space-induct*:  
**assumes**  $\bigwedge X \alpha \mu. \text{qbs-prob } X \alpha \mu \Longrightarrow P (\text{qbs-prob-space } (X, \alpha, \mu))$   
**shows**  $P s$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-space-induct'*:  
**assumes**  $\bigwedge X \alpha \mu. \text{qbs-prob } X \alpha \mu \Longrightarrow s = \text{qbs-prob-space } (X, \alpha, \mu) \Longrightarrow P$   
 $(\text{qbs-prob-space } (X, \alpha, \mu))$   
**shows**  $P s$   
 $\langle \text{proof} \rangle$

**lemma** *rep-qbs-prob-space*:  
 $\exists X \alpha \mu. p = \text{qbs-prob-space } (X, \alpha, \mu) \wedge \text{qbs-prob } X \alpha \mu$   
 $\langle \text{proof} \rangle$

**lemma**(in *qbs-prob*) *in-Rep*:  
 $(X, \alpha, \mu) \in \text{Rep-qbs-prob-space } (\text{qbs-prob-space } (X, \alpha, \mu))$   
 $\langle \text{proof} \rangle$

**lemma**(in *qbs-prob*) *if-in-Rep*:  
**assumes**  $(X', \alpha', \mu') \in \text{Rep-qbs-prob-space } (\text{qbs-prob-space } (X, \alpha, \mu))$   
**shows**  $X' = X$   
 $\text{qbs-prob } X' \alpha' \mu'$   
 $\text{qbs-prob-eq } (X, \alpha, \mu) (X', \alpha', \mu')$   
 $\langle \text{proof} \rangle$

**lemma**(in *qbs-prob*) *in-Rep-induct*:  
**assumes**  $\bigwedge Y \beta \nu. (Y, \beta, \nu) \in \text{Rep-qbs-prob-space } (\text{qbs-prob-space } (X, \alpha, \mu)) \implies P$   
 $(Y, \beta, \nu)$   
**shows**  $P (\text{rep-qbs-prob-space } (\text{qbs-prob-space } (X, \alpha, \mu)))$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-2-implies-3* :  
**assumes** *qbs-prob-eq2*  $p1 \ p2$   
**shows** *qbs-prob-eq3*  $p1 \ p2$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-3-implies-1* :  
**assumes** *qbs-prob-eq3*  $(p1 :: 'a \text{ qbs-prob-t}) \ p2$   
**shows** *qbs-prob-eq*  $p1 \ p2$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-1-implies-2* :  
**assumes** *qbs-prob-eq*  $p1 \ (p2 :: 'a \text{ qbs-prob-t})$   
**shows** *qbs-prob-eq2*  $p1 \ p2$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-1-implies-4* :  
**assumes** *qbs-prob-eq*  $p1 \ p2$   
**shows** *qbs-prob-eq4*  $p1 \ p2$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-4-implies-3* :  
**assumes** *qbs-prob-eq4*  $p1 \ p2$   
**shows** *qbs-prob-eq3*  $p1 \ p2$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-equiv12* :  
 $\text{qbs-prob-eq} = \text{qbs-prob-eq2}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-eq-equiv13* :  
 $\text{qbs-prob-eq} = \text{qbs-prob-eq3}$

*<proof>*

**lemma** *qbs-prob-eq-equiv14* :  
*qbs-prob-eq* = *qbs-prob-eq4*  
*<proof>*

**lemma** *qbs-prob-eq-equiv23* :  
*qbs-prob-eq2* = *qbs-prob-eq3*  
*<proof>*

**lemma** *qbs-prob-eq-equiv24* :  
*qbs-prob-eq2* = *qbs-prob-eq4*  
*<proof>*

**lemma** *qbs-prob-eq-equiv34* :  
*qbs-prob-eq3* = *qbs-prob-eq4*  
*<proof>*

**lemma** *qbs-prob-eq-equiv31* :  
*qbs-prob-eq* = *qbs-prob-eq3*  
*<proof>*

**lemma** *qbs-prob-space-eq* :  
**assumes** *qbs-prob-eq* (*X*, $\alpha$ , $\mu$ ) (*Y*, $\beta$ , $\nu$ )  
**shows** *qbs-prob-space* (*X*, $\alpha$ , $\mu$ ) = *qbs-prob-space* (*Y*, $\beta$ , $\nu$ )  
*<proof>*

**lemma**(**in** *pair-qbs-prob*) *qbs-prob-space-eq* :  
**assumes** *Y* = *X*  
**and** *distr*  $\mu$  (*qbs-to-measure* *X*)  $\alpha$  = *distr*  $\nu$  (*qbs-to-measure* *X*)  $\beta$   
**shows** *qbs-prob-space* (*X*, $\alpha$ , $\mu$ ) = *qbs-prob-space* (*Y*, $\beta$ , $\nu$ )  
*<proof>*

**lemma**(**in** *pair-qbs-prob*) *qbs-prob-space-eq2* :  
**assumes** *Y* = *X*  
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel}$   
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
**shows** *qbs-prob-space* (*X*, $\alpha$ , $\mu$ ) = *qbs-prob-space* (*Y*, $\beta$ , $\nu$ )  
*<proof>*

**lemma**(**in** *pair-qbs-prob*) *qbs-prob-space-eq3* :  
**assumes** *Y* = *X*  
**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel} \implies (\forall k \in \text{qbs-space } X. 0 \leq f k)$   
 $\implies (\int x. f (\alpha x) \partial \mu) = (\int x. f (\beta x) \partial \nu)$   
**shows** *qbs-prob-space* (*X*, $\alpha$ , $\mu$ ) = *qbs-prob-space* (*Y*, $\beta$ , $\nu$ )  
*<proof>*

**lemma**(**in** *pair-qbs-prob*) *qbs-prob-space-eq4* :  
**assumes** *Y* = *X*

**and**  $\bigwedge f. f \in \text{qbs-to-measure } X \rightarrow_M \text{ennreal-borel}$   
 $\implies (\int^{+x}. f (\alpha x) \partial \mu) = (\int^{+x}. f (\beta x) \partial \nu)$   
**shows**  $\text{qbs-prob-space } (X, \alpha, \mu) = \text{qbs-prob-space } (Y, \beta, \nu)$   
 $\langle \text{proof} \rangle$

**lemma**(**in** *pair-qbs-prob*) *qbs-prob-space-eq-inverse*:  
**assumes**  $\text{qbs-prob-space } (X, \alpha, \mu) = \text{qbs-prob-space } (Y, \beta, \nu)$   
**shows**  $\text{qbs-prob-eq } (X, \alpha, \mu) (Y, \beta, \nu)$   
**and**  $\text{qbs-prob-eq2 } (X, \alpha, \mu) (Y, \beta, \nu)$   
**and**  $\text{qbs-prob-eq3 } (X, \alpha, \mu) (Y, \beta, \nu)$   
**and**  $\text{qbs-prob-eq4 } (X, \alpha, \mu) (Y, \beta, \nu)$   
 $\langle \text{proof} \rangle$

**lift-definition** *qbs-prob-space-qbs* ::  $'a \text{ qbs-prob-space} \Rightarrow 'a \text{ quasi-borel}$   
**is** *fst*  $\langle \text{proof} \rangle$

**lemma**(**in** *qbs-prob*) *qbs-prob-space-qbs-computation[simp]*:  
 $\text{qbs-prob-space-qbs } (\text{qbs-prob-space } (X, \alpha, \mu)) = X$   
 $\langle \text{proof} \rangle$

**lemma** *rep-qbs-prob-space'*:  
**assumes**  $\text{qbs-prob-space-qbs } s = X$   
**shows**  $\exists \alpha \mu. s = \text{qbs-prob-space } (X, \alpha, \mu) \wedge \text{qbs-prob } X \alpha \mu$   
 $\langle \text{proof} \rangle$

**lift-definition** *qbs-prob-ennintegral* ::  $['a \text{ qbs-prob-space}, 'a \Rightarrow \text{ennreal}] \Rightarrow \text{ennreal}$   
**is** *qbs-prob-t-ennintegral*  
 $\langle \text{proof} \rangle$

**lift-definition** *qbs-prob-integral* ::  $['a \text{ qbs-prob-space}, 'a \Rightarrow \text{real}] \Rightarrow \text{real}$   
**is** *qbs-prob-t-integral*  
 $\langle \text{proof} \rangle$

**syntax**  
 $\text{-qbs-prob-ennintegral} :: \text{pttrn} \Rightarrow \text{ennreal} \Rightarrow 'a \text{ qbs-prob-space} \Rightarrow \text{ennreal} (\int^+_Q ((2 \text{-./ -}) / \partial \text{-}) [60,61] 110)$

**translations**  
 $\int^+_Q x. f \partial p \equiv \text{CONST } \text{qbs-prob-ennintegral } p (\lambda x. f)$

**syntax**  
 $\text{-qbs-prob-integral} :: \text{pttrn} \Rightarrow \text{real} \Rightarrow 'a \text{ qbs-prob-space} \Rightarrow \text{real} (\int_Q ((2 \text{-./ -}) / \partial \text{-}) [60,61] 110)$

**translations**  
 $\int_Q x. f \partial p \equiv \text{CONST } \text{qbs-prob-integral } p (\lambda x. f)$

We define the function  $l_X \in L(P(X)) \rightarrow_M G(X)$ .



**lift-definition**  $qbs\text{-}prob\text{-}measure :: 'a\ qbs\text{-}prob\text{-}space \Rightarrow 'a\ measure$   
**is**  $qbs\text{-}prob\text{-}t\text{-}measure$   
 $\langle proof \rangle$

**declare**  $[[coercion\ qbs\text{-}prob\text{-}measure]]$

**lemma**(**in**  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}measure\text{-}computation[simp]$ :  
 $qbs\text{-}prob\text{-}measure\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu)) = distr\ \mu\ (qbs\text{-}to\text{-}measure\ X)\ \alpha$   
 $\langle proof \rangle$

**definition**  $qbs\text{-}emeasure :: 'a\ qbs\text{-}prob\text{-}space \Rightarrow 'a\ set \Rightarrow ennreal$  **where**  
 $qbs\text{-}emeasure\ s \equiv emeasure\ (qbs\text{-}prob\text{-}measure\ s)$

**lemma**(**in**  $qbs\text{-}prob$ )  $qbs\text{-}emeasure\text{-}computation[simp]$ :  
**assumes**  $U \in sets\ (qbs\text{-}to\text{-}measure\ X)$   
**shows**  $qbs\text{-}emeasure\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu))\ U = emeasure\ \mu\ (\alpha - ' U)$   
 $\langle proof \rangle$

**definition**  $qbs\text{-}measure :: 'a\ qbs\text{-}prob\text{-}space \Rightarrow 'a\ set \Rightarrow real$  **where**  
 $qbs\text{-}measure\ s \equiv measure\ (qbs\text{-}prob\text{-}measure\ s)$

**interpretation**  $qbs\text{-}prob\text{-}measure\text{-}prob\text{-}space : prob\text{-}space\ qbs\text{-}prob\text{-}measure\ (s::'a\ qbs\text{-}prob\text{-}space)$  **for**  $s$   
 $\langle proof \rangle$

**lemma**  $qbs\text{-}prob\text{-}measure\text{-}space$ :  
 $qbs\text{-}space\ (qbs\text{-}prob\text{-}space\text{-}qbs\ s) = space\ (qbs\text{-}prob\text{-}measure\ s)$   
 $\langle proof \rangle$

**lemma**  $qbs\text{-}prob\text{-}measure\text{-}sets[measurable\text{-}cong]$ :  
 $sets\ (qbs\text{-}to\text{-}measure\ (qbs\text{-}prob\text{-}space\text{-}qbs\ s)) = sets\ (qbs\text{-}prob\text{-}measure\ s)$   
 $\langle proof \rangle$

**lemma**(**in**  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}ennintegral\text{-}def$ :  
**assumes**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows**  $qbs\text{-}prob\text{-}ennintegral\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu))\ f = (\int^+ x. f\ (\alpha\ x)\ \partial\ \mu)$   
 $\langle proof \rangle$

**lemma**(**in**  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}ennintegral\text{-}def2$ :  
**assumes**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows**  $qbs\text{-}prob\text{-}ennintegral\ (qbs\text{-}prob\text{-}space\ (X,\alpha,\mu))\ f = integral^N\ (distr\ \mu\ (qbs\text{-}to\text{-}measure\ X)\ \alpha)\ f$   
 $\langle proof \rangle$

**lemma** (**in**  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}ennintegral\text{-}not\text{-}morphism$ :  
**assumes**  $f \notin X \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $qbs\text{-}prob\text{-}ennintegral (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f = 0$   
 ⟨proof⟩

**lemma**  $qbs\text{-}prob\text{-}ennintegral\text{-}def2$ :

**assumes**  $qbs\text{-}prob\text{-}space\text{-}qbs s = (X :: 'a\ quasi\text{-}borel)$   
**and**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows**  $qbs\text{-}prob\text{-}ennintegral s f = integral^N (qbs\text{-}prob\text{-}measure s) f$   
 ⟨proof⟩

**lemma**(in  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}integral\text{-}def$ :

**assumes**  $f \in X \rightarrow_Q real\text{-}quasi\text{-}borel$   
**shows**  $qbs\text{-}prob\text{-}integral (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f = (\int x. f (\alpha x) \partial \mu)$   
 ⟨proof⟩

**lemma**(in  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}integral\text{-}def2$ :

$qbs\text{-}prob\text{-}integral (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f = integral^L (distr \mu (qbs\text{-}to\text{-}measure X) \alpha) f$   
 ⟨proof⟩

**lemma**  $qbs\text{-}prob\text{-}integral\text{-}def2$ :

$qbs\text{-}prob\text{-}integral (s :: 'a\ qbs\text{-}prob\text{-}space) f = integral^L (qbs\text{-}prob\text{-}measure s) f$   
 ⟨proof⟩

**definition**  $qbs\text{-}prob\text{-}var :: 'a\ qbs\text{-}prob\text{-}space \Rightarrow ('a \Rightarrow real) \Rightarrow real$  **where**  
 $qbs\text{-}prob\text{-}var s f \equiv qbs\text{-}prob\text{-}integral s (\lambda x. (f x - qbs\text{-}prob\text{-}integral s f)^2)$

**lemma**(in  $qbs\text{-}prob$ )  $qbs\text{-}prob\text{-}var\text{-}computation$ :

**assumes**  $f \in X \rightarrow_Q real\text{-}quasi\text{-}borel$   
**shows**  $qbs\text{-}prob\text{-}var (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f = (\int x. (f (\alpha x) - (\int x. f (\alpha x) \partial \mu))^2 \partial \mu)$   
 ⟨proof⟩

**lift-definition**  $qbs\text{-}integrable :: ['a\ qbs\text{-}prob\text{-}space, 'a \Rightarrow real] \Rightarrow bool$

**is**  $qbs\text{-}prob\text{-}t\text{-}integrable$

⟨proof⟩

**lemma**(in  $qbs\text{-}prob$ )  $qbs\text{-}integrable\text{-}def$ :

$qbs\text{-}integrable (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f = (f \in X \rightarrow_Q \mathbb{R}_Q \wedge integrable \mu (f \circ \alpha))$   
 ⟨proof⟩

**lemma**  $qbs\text{-}integrable\text{-}morphism$ :

**assumes**  $qbs\text{-}prob\text{-}space\text{-}qbs s = X$

**and**  $qbs\text{-}integrable s f$

**shows**  $f \in X \rightarrow_Q \mathbb{R}_Q$

⟨proof⟩

**lemma**(in  $qbs\text{-}prob$ )  $qbs\text{-}integrable\text{-}measurable[simp, measurable]$ :

**assumes**  $qbs\text{-}integrable (qbs\text{-}prob\text{-}space (X, \alpha, \mu)) f$

**shows**  $f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel}$   
*<proof>*

**lemma** *qbs-integrable-iff-integrable:*

$(\text{qbs-integrable } (s::'a \text{ qbs-prob-space}) f) = (\text{integrable } (\text{qbs-prob-measure } s) f)$   
*<proof>*

**lemma**(**in** *qbs-prob*) *qbs-integrable-iff-integrable-distr:*

$\text{qbs-integrable } (\text{qbs-prob-space } (X, \alpha, \mu)) f = \text{integrable } (\text{distr } \mu (\text{qbs-to-measure } X) \alpha) f$   
*<proof>*

**lemma**(**in** *qbs-prob*) *qbs-integrable-iff-integrable:*

**assumes**  $f \in \text{qbs-to-measure } X \rightarrow_M \text{real-borel}$   
**shows**  $\text{qbs-integrable } (\text{qbs-prob-space } (X, \alpha, \mu)) f = \text{integrable } \mu (\lambda x. f (\alpha x))$   
*<proof>*

**lemma** *qbs-integrable-if-integrable:*

**assumes**  $\text{integrable } (\text{qbs-prob-measure } s) f$   
**shows**  $\text{qbs-integrable } (s::'a \text{ qbs-prob-space}) f$   
*<proof>*

**lemma** *integrable-if-qbs-integrable:*

**assumes**  $\text{qbs-integrable } (s::'a \text{ qbs-prob-space}) f$   
**shows**  $\text{integrable } (\text{qbs-prob-measure } s) f$   
*<proof>*

**lemma** *qbs-integrable-iff-bounded:*

**assumes**  $\text{qbs-prob-space-qbs } s = X$   
**shows**  $\text{qbs-integrable } s f \longleftrightarrow f \in X \rightarrow_Q \mathbb{R}_Q \wedge \text{qbs-prob-ennintegral } s (\lambda x. \text{ennreal } |f x|) < \infty$   
(**is** ?lhs = ?rhs)  
*<proof>*

**lemma** *qbs-integrable-cong:*

**assumes**  $\text{qbs-prob-space-qbs } s = X$   
 $\bigwedge x. x \in \text{qbs-space } X \implies f x = g x$   
**and**  $\text{qbs-integrable } s f$   
**shows**  $\text{qbs-integrable } s g$   
*<proof>*

**lemma** *qbs-integrable-const[simp]:*

$\text{qbs-integrable } s (\lambda x. c)$   
*<proof>*

**lemma** *qbs-integrable-add[simp]:*

**assumes**  $\text{qbs-integrable } s f$   
**and**  $\text{qbs-integrable } s g$   
**shows**  $\text{qbs-integrable } s (\lambda x. f x + g x)$

*<proof>*

**lemma** *qbs-integrable-diff[simp]*:  
 **assumes** *qbs-integrable s f*  
 **and** *qbs-integrable s g*  
 **shows** *qbs-integrable s ( $\lambda x. f x - g x$ )*  
*<proof>*

**lemma** *qbs-integrable-mult-iff[simp]*:  
*(qbs-integrable s ( $\lambda x. c * f x$ )) = (c = 0  $\vee$  qbs-integrable s f)*  
*<proof>*

**lemma** *qbs-integrable-mult[simp]*:  
 **assumes** *qbs-integrable s f*  
 **shows** *qbs-integrable s ( $\lambda x. c * f x$ )*  
*<proof>*

**lemma** *qbs-integrable-abs[simp]*:  
 **assumes** *qbs-integrable s f*  
 **shows** *qbs-integrable s ( $\lambda x. |f x|$ )*  
*<proof>*

**lemma** *qbs-integrable-sq[simp]*:  
 **assumes** *qbs-integrable s f*  
 **and** *qbs-integrable s ( $\lambda x. (f x)^2$ )*  
 **shows** *qbs-integrable s ( $\lambda x. (f x - c)^2$ )*  
*<proof>*

**lemma** *qbs-ennintegral-eq-qbs-integral*:  
 **assumes** *qbs-prob-space-qbs s = X*  
 *qbs-integrable s f*  
 **and**  $\bigwedge x. x \in \text{qbs-space } X \implies 0 \leq f x$   
 **shows** *qbs-prob-ennintegral s ( $\lambda x. \text{ennreal } (f x)$ ) = ennreal (qbs-prob-integral s f)*  
*<proof>*

**lemma** *qbs-prob-ennintegral-cong*:  
 **assumes** *qbs-prob-space-qbs s = X*  
 **and**  $\bigwedge x. x \in \text{qbs-space } X \implies f x = g x$   
 **shows** *qbs-prob-ennintegral s f = qbs-prob-ennintegral s g*  
*<proof>*

**lemma** *qbs-prob-ennintegral-const*:  
*qbs-prob-ennintegral (s::'a qbs-prob-space) ( $\lambda x. c$ ) = c*  
*<proof>*

**lemma** *qbs-prob-ennintegral-add*:  
 **assumes** *qbs-prob-space-qbs s = X*

$f \in (X :: 'a \text{ quasi-borel}) \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**and**  $g \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows**  $\text{qbs-prob-ennintegral } s (\lambda x. f x + g x) = \text{qbs-prob-ennintegral } s f + \text{qbs-prob-ennintegral } s g$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-cmult*:  
**assumes**  $\text{qbs-prob-space-qbs } s = X$   
**and**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows**  $\text{qbs-prob-ennintegral } s (\lambda x. c * f x) = c * \text{qbs-prob-ennintegral } s f$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-cmult-noninfty*:  
**assumes**  $c \neq \infty$   
**shows**  $\text{qbs-prob-ennintegral } s (\lambda x. c * f x) = c * \text{qbs-prob-ennintegral } s f$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-cong*:  
**assumes**  $\text{qbs-prob-space-qbs } s = X$   
**and**  $\bigwedge x. x \in \text{qbs-space } X \implies f x = g x$   
**shows**  $\text{qbs-prob-integral } s f = \text{qbs-prob-integral } s g$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-nonneg*:  
**assumes**  $\text{qbs-prob-space-qbs } s = X$   
**and**  $\bigwedge x. x \in \text{qbs-space } X \implies 0 \leq f x$   
**shows**  $0 \leq \text{qbs-prob-integral } s f$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-mono*:  
**assumes**  $\text{qbs-prob-space-qbs } s = X$   
 $\text{qbs-integrable } (s :: 'a \text{ qbs-prob-space}) f$   
 $\text{qbs-integrable } s g$   
**and**  $\bigwedge x. x \in \text{qbs-space } X \implies f x \leq g x$   
**shows**  $\text{qbs-prob-integral } s f \leq \text{qbs-prob-integral } s g$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-const*:  
 $\text{qbs-prob-integral } (s :: 'a \text{ qbs-prob-space}) (\lambda x. c) = c$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-add*:  
**assumes**  $\text{qbs-integrable } (s :: 'a \text{ qbs-prob-space}) f$   
**and**  $\text{qbs-integrable } s g$   
**shows**  $\text{qbs-prob-integral } s (\lambda x. f x + g x) = \text{qbs-prob-integral } s f + \text{qbs-prob-integral } s g$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-diff*:

**assumes** *qbs-integrable* (*s*::'a *qbs-prob-space*) *f*  
**and** *qbs-integrable* *s g*  
**shows**  $qbs\text{-prob-integral } s (\lambda x. f x - g x) = qbs\text{-prob-integral } s f - qbs\text{-prob-integral } s g$   
*<proof>*

**lemma** *qbs-prob-integral-cmult*:  
 $qbs\text{-prob-integral } s (\lambda x. c * f x) = c * qbs\text{-prob-integral } s f$   
*<proof>*

**lemma** *real-qbs-prob-integral-def*:  
**assumes** *qbs-integrable* (*s*::'a *qbs-prob-space*) *f*  
**shows**  $qbs\text{-prob-integral } s f = enn2real (qbs\text{-prob-ennintegral } s (\lambda x. ennreal (f x))) - enn2real (qbs\text{-prob-ennintegral } s (\lambda x. ennreal (- f x)))$   
*<proof>*

**lemma** *qbs-prob-var-eq*:  
**assumes** *qbs-integrable* (*s*::'a *qbs-prob-space*) *f*  
**and** *qbs-integrable* *s* ( $\lambda x. (f x)^2$ )  
**shows**  $qbs\text{-prob-var } s f = qbs\text{-prob-integral } s (\lambda x. (f x)^2) - (qbs\text{-prob-integral } s f)^2$   
*<proof>*

**lemma** *qbs-prob-var-affine*:  
**assumes** *qbs-integrable* *s f*  
**shows**  $qbs\text{-prob-var } s (\lambda x. a * f x + b) = a^2 * qbs\text{-prob-var } s f$   
(is ?lhs = ?rhs)  
*<proof>*

**lemma** *qbs-prob-integral-Markov-inequality*:  
**assumes** *qbs-prob-space-qbs* *s = X*  
**and** *qbs-integrable* *s f*  
 $\bigwedge x. x \in qbs\text{-space } X \implies 0 \leq f x$   
**and**  $0 < c$   
**shows**  $qbs\text{-emeasure } s \{x \in qbs\text{-space } X. c \leq f x\} \leq ennreal (1/c * qbs\text{-prob-integral } s f)$   
*<proof>*

**lemma** *qbs-prob-integral-Markov-inequality'*:  
**assumes** *qbs-prob-space-qbs* *s = X*  
*qbs-integrable* *s f*  
 $\bigwedge x. x \in qbs\text{-space } (qbs\text{-prob-space-qbs } s) \implies 0 \leq f x$   
**and**  $0 < c$   
**shows**  $qbs\text{-measure } s \{x \in qbs\text{-space } (qbs\text{-prob-space-qbs } s). c \leq f x\} \leq (1/c * qbs\text{-prob-integral } s f)$   
*<proof>*

**lemma** *qbs-prob-integral-Markov-inequality-abs*:  
**assumes** *qbs-prob-space-qbs* *s = X*

$qbs\text{-integrable } s f$   
**and**  $0 < c$   
**shows**  $qbs\text{-emeasure } s \{x \in qbs\text{-space } X. c \leq |f x|\} \leq \text{ennreal } (1/c * qbs\text{-prob-integral } s (\lambda x. |f x|))$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-Markov-inequality-abs'*:  
**assumes**  $qbs\text{-prob-space-qbs } s = X$   
 $qbs\text{-integrable } s f$   
**and**  $0 < c$   
**shows**  $qbs\text{-measure } s \{x \in qbs\text{-space } X. c \leq |f x|\} \leq (1/c * qbs\text{-prob-integral } s (\lambda x. |f x|))$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-real-Markov-inequality*:  
**assumes**  $qbs\text{-prob-space-qbs } s = \mathbb{R}_Q$   
 $qbs\text{-integrable } s f$   
**and**  $0 < c$   
**shows**  $qbs\text{-emeasure } s \{r. c \leq |f r|\} \leq \text{ennreal } (1/c * qbs\text{-prob-integral } s (\lambda x. |f x|))$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-real-Markov-inequality'*:  
**assumes**  $qbs\text{-prob-space-qbs } s = \mathbb{R}_Q$   
 $qbs\text{-integrable } s f$   
**and**  $0 < c$   
**shows**  $qbs\text{-measure } s \{r. c \leq |f r|\} \leq 1/c * qbs\text{-prob-integral } s (\lambda x. |f x|)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-integral-Chebyshev-inequality*:  
**assumes**  $qbs\text{-prob-space-qbs } s = X$   
 $qbs\text{-integrable } s f$   
 $qbs\text{-integrable } s (\lambda x. (f x)^2)$   
**and**  $0 < b$   
**shows**  $qbs\text{-measure } s \{x \in qbs\text{-space } X. b \leq |f x - qbs\text{-prob-integral } s f|\} \leq 1 / b^2 * qbs\text{-prob-var } s f$   
 $\langle \text{proof} \rangle$

**end**

## 3.2 The Probability Monad

**theory** *Monad-QuasiBorel*  
**imports** *Probability-Space-QuasiBorel*  
**begin**

### 3.2.1 The Probability Monad $P$

**definition** *monadP-qbs-Px* :: 'a quasi-borel  $\Rightarrow$  'a qbs-prob-space set **where**  
 $monadP\text{-qbs-Px } X \equiv \{s. qbs\text{-prob-space-qbs } s = X\}$

**locale** *in-Px* =  
**fixes**  $X :: 'a \text{ quasi-borel}$  **and**  $s :: 'a \text{ qbs-prob-space}$   
**assumes**  $\text{in-Px}: s \in \text{monadP-qbs-Px } X$   
**begin**

**lemma** *qbs-prob-space-X[simp]*:  
 $\text{qbs-prob-space-qbs } s = X$   
 $\langle \text{proof} \rangle$

**end**

**locale** *in-MPx* =  
**fixes**  $X :: 'a \text{ quasi-borel}$  **and**  $\beta :: \text{real} \Rightarrow 'a \text{ qbs-prob-space}$   
**assumes**  $\text{ex}: \exists \alpha \in \text{qbs-Mx } X. \exists g \in \text{real-borel} \rightarrow_M \text{prob-algebra real-borel}.$   
 $\forall r. \beta \ r = \text{qbs-prob-space } (X, \alpha, g \ r)$   
**begin**

**lemma** *rep-inMPx*:  
 $\exists \alpha \ g. \alpha \in \text{qbs-Mx } X \wedge g \in \text{real-borel} \rightarrow_M \text{prob-algebra real-borel} \wedge$   
 $\beta = (\lambda r. \text{qbs-prob-space } (X, \alpha, g \ r))$   
 $\langle \text{proof} \rangle$

**end**

**definition** *monadP-qbs-MPx* ::  $'a \text{ quasi-borel} \Rightarrow (\text{real} \Rightarrow 'a \text{ qbs-prob-space}) \text{ set}$   
**where**  
 $\text{monadP-qbs-MPx } X \equiv \{\beta. \text{in-MPx } X \ \beta\}$

**definition** *monadP-qbs* ::  $'a \text{ quasi-borel} \Rightarrow 'a \text{ qbs-prob-space quasi-borel}$  **where**  
 $\text{monadP-qbs } X \equiv \text{Abs-quasi-borel } (\text{monadP-qbs-Px } X, \text{monadP-qbs-MPx } X)$

**lemma**(**in** *qbs-prob*) *qbs-prob-space-in-Px*:  
 $\text{qbs-prob-space } (X, \alpha, \mu) \in \text{monadP-qbs-Px } X$   
 $\langle \text{proof} \rangle$

**lemma** *rep-monadP-qbs-Px*:  
**assumes**  $s \in \text{monadP-qbs-Px } X$   
**shows**  $\exists \alpha \ \mu. s = \text{qbs-prob-space } (X, \alpha, \mu) \wedge \text{qbs-prob } X \ \alpha \ \mu$   
 $\langle \text{proof} \rangle$

**lemma** *rep-monadP-qbs-MPx*:  
**assumes**  $\beta \in \text{monadP-qbs-MPx } X$   
**shows**  $\exists \alpha \ g. \alpha \in \text{qbs-Mx } X \wedge g \in \text{real-borel} \rightarrow_M \text{prob-algebra real-borel} \wedge$   
 $\beta = (\lambda r. \text{qbs-prob-space } (X, \alpha, g \ r))$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-MPx*:  
**assumes**  $\alpha \in \text{qbs-Mx } X$



**and**  $g \in \text{real-borel} \rightarrow_M \text{prob-algebra real-borel}$   
**shows**  $\text{qbs-prob } X \alpha (g \ r)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-f[simp]}$ :  $\text{monadP-qbs-MPx } X \subseteq \text{UNIV} \rightarrow \text{monadP-qbs-Px } X$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-closed1}$ :  $\text{qbs-closed1 } (\text{monadP-qbs-MPx } X)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-closed2}$ :  $\text{qbs-closed2 } (\text{monadP-qbs-Px } X) (\text{monadP-qbs-MPx } X)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-closed3}$ :  $\text{qbs-closed3 } (\text{monadP-qbs-MPx } (X :: 'a \text{ quasi-borel}))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-correct}$ :  $\text{Rep-quasi-borel } (\text{monadP-qbs } X) = (\text{monadP-qbs-Px } X, \text{monadP-qbs-MPx } X)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-space[simp]}$ :  $\text{qbs-space } (\text{monadP-qbs } X) = \text{monadP-qbs-Px } X$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-Mx[simp]}$ :  $\text{qbs-Mx } (\text{monadP-qbs } X) = \text{monadP-qbs-MPx } X$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{monadP-qbs-empty-iff}$ :  
 $\text{qbs-space } X = \{\} \longleftrightarrow \text{qbs-space } (\text{monadP-qbs } X) = \{\}$   
 $\langle \text{proof} \rangle$

If  $\beta \in \text{MPx}$ , there exists  $X \alpha g$  s.t.  $\beta \ r = [X, \alpha, g \ r]$ . We define a function which picks  $X \alpha g$  from  $\beta \in \text{MPx}$ .

**definition**  $\text{rep-monadP-qbs-MPx} :: (\text{real} \Rightarrow 'a \text{ qbs-prob-space}) \Rightarrow 'a \text{ quasi-borel} \times (\text{real} \Rightarrow 'a) \times (\text{real} \Rightarrow \text{real measure})$  **where**  
 $\text{rep-monadP-qbs-MPx } \beta \equiv \text{let } X = \text{qbs-prob-space-qbs } (\beta \ \text{undefined});$   
 $\alpha g = (\text{SOME } k. (\text{fst } k) \in \text{qbs-Mx } X \wedge (\text{snd } k) \in \text{real-borel}$   
 $\rightarrow_M \text{prob-algebra real-borel}$   
 $\wedge \beta = (\lambda r. \text{qbs-prob-space } (X, \text{fst } k, \text{snd } k \ r)))$   
 $\text{in } (X, \alpha g)$

**lemma**  $\text{qbs-prob-measure-measurable[measurable]}$ :  
 $\text{qbs-prob-measure} \in \text{qbs-to-measure } (\text{monadP-qbs } (X :: 'a \text{ quasi-borel})) \rightarrow_M \text{prob-algebra}$   
 $(\text{qbs-to-measure } X)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{qbs-l-inj}$ :  
 $\text{inj-on qbs-prob-measure } (\text{monadP-qbs-Px } X)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-measure-measurable*'[*measurable*]:  
*qbs-prob-measure*  $\in$  *qbs-to-measure* (*monadP-qbs* ( $X :: 'a$  *quasi-borel*))  $\rightarrow_M$  *sub-prob-algebra* (*qbs-to-measure*  $X$ )  
 ⟨*proof*⟩

### 3.2.2 Return

**definition** *qbs-return* :: [ $'a$  *quasi-borel*,  $'a$ ]  $\Rightarrow$   $'a$  *qbs-prob-space* **where**  
*qbs-return*  $X$   $x \equiv$  *qbs-prob-space* ( $X, \lambda r. x, Eps$  *real-distribution*)

**lemma**(*in real-distribution*) *qbs-return-qbs-prob*:  
**assumes**  $x \in$  *qbs-space*  $X$   
**shows** *qbs-prob*  $X$  ( $\lambda r. x$ )  $M$   
 ⟨*proof*⟩

**lemma**(*in real-distribution*) *qbs-return-computation* :  
**assumes**  $x \in$  *qbs-space*  $X$   
**shows** *qbs-return*  $X$   $x =$  *qbs-prob-space* ( $X, \lambda r. x, M$ )  
 ⟨*proof*⟩

**lemma** *qbs-return-morphism*:  
*qbs-return*  $X \in X \rightarrow_Q$  *monadP-qbs*  $X$   
 ⟨*proof*⟩

**lemma** *qbs-return-morphism'*:  
**assumes**  $f \in X \rightarrow_Q Y$   
**shows** ( $\lambda x. \text{qbs-return } Y (f x) \in X \rightarrow_Q \text{monadP-qbs } Y$ )  
 ⟨*proof*⟩

### 3.2.3 Bind

**definition** *qbs-bind* ::  $'a$  *qbs-prob-space*  $\Rightarrow$  ( $'a \Rightarrow 'b$  *qbs-prob-space*)  $\Rightarrow$   $'b$  *qbs-prob-space*  
**where**  
*qbs-bind*  $s f \equiv$  (*let* (*qbsx*,  $\alpha, \mu$ ) = *rep-qbs-prob-space*  $s$ ;  
                   (*qbsy*,  $\beta, g$ ) = *rep-monadP-qbs-MPx* ( $f \circ \alpha$ )  
                   *in* *qbs-prob-space* (*qbsy*,  $\beta, \mu \ggg g$ ))

**adhoc-overloading** *Monad-Syntax.bind* *qbs-bind*

**lemma**(*in qbs-prob*) *qbs-bind-computation*:  
**assumes**  $s =$  *qbs-prob-space* ( $X, \alpha, \mu$ )  
            $f \in X \rightarrow_Q$  *monadP-qbs*  $Y$   
            $\beta \in$  *qbs-Mx*  $Y$   
**and** [*measurable*]:  $g \in$  *real-borel*  $\rightarrow_M$  *prob-algebra* *real-borel*  
           **and** ( $f \circ \alpha$ ) = ( $\lambda r. \text{qbs-prob-space } (Y, \beta, g r)$ )  
**shows** *qbs-prob*  $Y$   $\beta$  ( $\mu \ggg g$ )  
            $s \ggg f =$  *qbs-prob-space* ( $Y, \beta, \mu \ggg g$ )  
 ⟨*proof*⟩

**lemma** *qbs-bind-morphism'*:

**assumes**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$

**shows**  $(\lambda x. x \ggg f) \in \text{monadP-qbs } X \rightarrow_Q \text{monadP-qbs } Y$

*<proof>*

**lemma** *qbs-return-comp*:

**assumes**  $\alpha \in \text{qbs-Mx } X$

**shows**  $(\text{qbs-return } X \circ \alpha) = (\lambda r. \text{qbs-prob-space } (X, \alpha, \text{return real-borel } r))$

*<proof>*

**lemma** *qbs-bind-return'*:

**assumes**  $x \in \text{monadP-qbs-Px } X$

**shows**  $x \ggg \text{qbs-return } X = x$

*<proof>*

**lemma** *qbs-bind-return*:

**assumes**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$

**and**  $x \in \text{qbs-space } X$

**shows**  $\text{qbs-return } X x \ggg f = f x$

*<proof>*

**lemma** *qbs-bind-assoc*:

**assumes**  $s \in \text{monadP-qbs-Px } X$

$f \in X \rightarrow_Q \text{monadP-qbs } Y$

**and**  $g \in Y \rightarrow_Q \text{monadP-qbs } Z$

**shows**  $s \ggg (\lambda x. f x \ggg g) = (s \ggg f) \ggg g$

*<proof>*

**lemma** *qbs-bind-cong*:

**assumes**  $s \in \text{monadP-qbs-Px } X$

$\bigwedge x. x \in \text{qbs-space } X \implies f x = g x$

**and**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$

**shows**  $s \ggg f = s \ggg g$

*<proof>*

### 3.2.4 The Functorial Action $P(f)$

**definition** *monadP-qbs-Pf* :: [*'a quasi-borel, 'b quasi-borel, 'a  $\Rightarrow$  'b, 'a qbs-prob-space*]

$\Rightarrow$  *'b qbs-prob-space* **where**

$\text{monadP-qbs-Pf } - Y f s x \equiv s x \ggg \text{qbs-return } Y \circ f$

**lemma** *monadP-qbs-Pf-morphism*:

**assumes**  $f \in X \rightarrow_Q Y$

**shows**  $\text{monadP-qbs-Pf } X Y f \in \text{monadP-qbs } X \rightarrow_Q \text{monadP-qbs } Y$

*<proof>*

**lemma**(in *qbs-prob*) *monadP-qbs-Pf-computation*:

**assumes**  $s = \text{qbs-prob-space } (X, \alpha, \mu)$

**and**  $f \in X \rightarrow_Q Y$

**shows**  $qbs\text{-}prob\ Y\ (f \circ \alpha)\ \mu$   
**and**  $monadP\text{-}qbs\text{-}Pf\ X\ Y\ f\ s = qbs\text{-}prob\text{-}space\ (Y, f \circ \alpha, \mu)$   
 $\langle proof \rangle$

We show that  $P$  is a functor i.e.  $P$  preserves identity and composition.

**lemma**  $monadP\text{-}qbs\text{-}Pf\text{-}id$ :  
**assumes**  $s \in monadP\text{-}qbs\text{-}Px\ X$   
**shows**  $monadP\text{-}qbs\text{-}Pf\ X\ X\ id\ s = s$   
 $\langle proof \rangle$

**lemma**  $monadP\text{-}qbs\text{-}Pf\text{-}comp$ :  
**assumes**  $s \in monadP\text{-}qbs\text{-}Px\ X$   
 $f \in X \rightarrow_Q Y$   
**and**  $g \in Y \rightarrow_Q Z$   
**shows**  $((monadP\text{-}qbs\text{-}Pf\ Y\ Z\ g) \circ (monadP\text{-}qbs\text{-}Pf\ X\ Y\ f))\ s = monadP\text{-}qbs\text{-}Pf\ X\ Z\ (g \circ f)\ s$   
 $\langle proof \rangle$

### 3.2.5 Join

**definition**  $qbs\text{-}join :: 'a\ qbs\text{-}prob\text{-}space\ qbs\text{-}prob\text{-}space \Rightarrow 'a\ qbs\text{-}prob\text{-}space$  **where**  
 $qbs\text{-}join \equiv (\lambda sst.\ sst \ggg id)$

**lemma**  $qbs\text{-}join\text{-}morphism$ :  
 $qbs\text{-}join \in monadP\text{-}qbs\ (monadP\text{-}qbs\ X) \rightarrow_Q monadP\text{-}qbs\ X$   
 $\langle proof \rangle$

**lemma**  $qbs\text{-}join\text{-}computation$ :  
**assumes**  $qbs\text{-}prob\ (monadP\text{-}qbs\ X)\ \beta\ \mu$   
 $ssx = qbs\text{-}prob\text{-}space\ (monadP\text{-}qbs\ X, \beta, \mu)$   
 $\alpha \in qbs\text{-}Mx\ X$   
 $g \in real\text{-}borel \rightarrow_M\ prob\text{-}algebra\ real\text{-}borel$   
**and**  $\beta = (\lambda r.\ qbs\text{-}prob\text{-}space\ (X, \alpha, g\ r))$   
**shows**  $qbs\text{-}prob\ X\ \alpha\ (\mu \ggg g)\ qbs\text{-}join\ ssx = qbs\text{-}prob\text{-}space\ (X, \alpha, \mu \ggg g)$   
 $\langle proof \rangle$

### 3.2.6 Strength

**definition**  $qbs\text{-}strength :: ['a\ quasi\text{-}borel, 'b\ quasi\text{-}borel, 'a \times 'b\ qbs\text{-}prob\text{-}space] \Rightarrow ('a \times 'b)\ qbs\text{-}prob\text{-}space$  **where**  
 $qbs\text{-}strength\ W\ X = (\lambda (w, sx).\ let\ (-, \alpha, \mu) = rep\text{-}qbs\text{-}prob\text{-}space\ sx$   
 $in\ qbs\text{-}prob\text{-}space\ (W \otimes_Q X, \lambda r.\ (w, \alpha\ r), \mu))$

**lemma**(**in**  $qbs\text{-}prob$ )  $qbs\text{-}strength\text{-}computation$ :  
**assumes**  $w \in qbs\text{-}space\ W$   
**and**  $sx = qbs\text{-}prob\text{-}space\ (X, \alpha, \mu)$   
**shows**  $qbs\text{-}prob\ (W \otimes_Q X)\ (\lambda r.\ (w, \alpha\ r))\ \mu$   
 $qbs\text{-}strength\ W\ X\ (w, sx) = qbs\text{-}prob\text{-}space\ (W \otimes_Q X, \lambda r.\ (w, \alpha\ r), \mu)$   
 $\langle proof \rangle$

**lemma** *qbs-strength-natural*:

**assumes**  $f \in X \rightarrow_Q X'$   
 $g \in Y \rightarrow_Q Y'$   
 $x \in \text{qbs-space } X$   
**and**  $sy \in \text{monadP-qbs-Px } Y$   
**shows**  $(\text{monadP-qbs-Pf } (X \otimes_Q Y) (X' \otimes_Q Y') (\text{map-prod } f \ g) \circ \text{qbs-strength } X \ Y) (x, sy) = (\text{qbs-strength } X' \ Y' \circ \text{map-prod } f \ (\text{monadP-qbs-Pf } Y \ Y' \ g)) (x, sy)$   
**(is ?lhs = ?rhs)**  
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-ab-r*:

**assumes**  $\alpha \in \text{qbs-Mx } X$   
 $\beta \in \text{monadP-qbs-MPx } Y$   
 $\gamma \in \text{qbs-Mx } Y$   
**and**  $[\text{measurable}]: g \in \text{real-borel} \rightarrow_M \text{prob-algebra real-borel}$   
**and**  $\beta = (\lambda r. \text{qbs-prob-space } (Y, \gamma, g \ r))$   
**shows**  $\text{qbs-prob } (X \otimes_Q Y) (\text{map-prod } \alpha \ \gamma \circ \text{real-real.g}) (\text{distr } (\text{return real-borel } r \otimes_M g \ r) \text{ real-borel real-real.f})$   
 $\text{qbs-strength } X \ Y (\alpha \ r, \beta \ r) = \text{qbs-prob-space } (X \otimes_Q Y, \text{map-prod } \alpha \ \gamma \circ \text{real-real.g}, \text{distr } (\text{return real-borel } r \otimes_M g \ r) \text{ real-borel real-real.f})$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-morphism*:

$\text{qbs-strength } X \ Y \in X \otimes_Q \text{monadP-qbs } Y \rightarrow_Q \text{monadP-qbs } (X \otimes_Q Y)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-bind-morphism''*:

$(\lambda(f, x). x \ggg f) \in \text{exp-qbs } X (\text{monadP-qbs } Y) \otimes_Q (\text{monadP-qbs } X) \rightarrow_Q (\text{monadP-qbs } Y)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-bind-morphism'''*:

$(\lambda f x. x \ggg f) \in \text{exp-qbs } X (\text{monadP-qbs } Y) \rightarrow_Q \text{exp-qbs } (\text{monadP-qbs } X)$   
 $(\text{monadP-qbs } Y)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-bind-morphism*:

**assumes**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$   
**and**  $g \in X \rightarrow_Q \text{exp-qbs } Y (\text{monadP-qbs } Z)$   
**shows**  $(\lambda x. f \ x \ggg g \ x) \in X \rightarrow_Q \text{monadP-qbs } Z$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-bind-morphism''''*:

**assumes**  $x \in \text{monadP-qbs-Px } X$   
**shows**  $(\lambda f. x \ggg f) \in \text{exp-qbs } X (\text{monadP-qbs } Y) \rightarrow_Q \text{monadP-qbs } Y$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-law1*:

**assumes**  $x \in \text{qbs-space } (\text{unit-quasi-borel } \otimes_Q \text{ monadP-qbs } X)$   
**shows**  $\text{snd } x = (\text{monadP-qbs-Pf } (\text{unit-quasi-borel } \otimes_Q X) X \text{ snd } \circ \text{qbs-strength } \text{unit-quasi-borel } X) x$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-law2*:

**assumes**  $x \in \text{qbs-space } ((X \otimes_Q Y) \otimes_Q \text{ monadP-qbs } Z)$   
**shows**  $(\text{qbs-strength } X (Y \otimes_Q Z) \circ (\text{map-prod id } (\text{qbs-strength } Y Z)) \circ (\lambda((x,y),z). (x,(y,z)))) x =$   
 $(\text{monadP-qbs-Pf } ((X \otimes_Q Y) \otimes_Q Z) (X \otimes_Q (Y \otimes_Q Z)) (\lambda((x,y),z). (x,(y,z)))) \circ \text{qbs-strength } (X \otimes_Q Y) Z) x$   
**(is ?lhs = ?rhs)**  
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-law3*:

**assumes**  $x \in \text{qbs-space } (X \otimes_Q Y)$   
**shows**  $\text{qbs-return } (X \otimes_Q Y) x = (\text{qbs-strength } X Y \circ (\text{map-prod id } (\text{qbs-return } Y))) x$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-strength-law4*:

**assumes**  $x \in \text{qbs-space } (X \otimes_Q \text{ monadP-qbs } (\text{monadP-qbs } Y))$   
**shows**  $(\text{qbs-strength } X Y \circ \text{map-prod id } \text{qbs-join}) x = (\text{qbs-join } \circ \text{monadP-qbs-Pf } (X \otimes_Q \text{ monadP-qbs } Y) (\text{monadP-qbs } (X \otimes_Q Y)) (\text{qbs-strength } X Y) \circ \text{qbs-strength } X (\text{monadP-qbs } Y)) x$   
**(is ?lhs = ?rhs)**  
 $\langle \text{proof} \rangle$

**lemma** *qbs-return-Mxpair*:

**assumes**  $\alpha \in \text{qbs-Mx } X$   
**and**  $\beta \in \text{qbs-Mx } Y$   
**shows**  $\text{qbs-return } (X \otimes_Q Y) (\alpha r, \beta k) = \text{qbs-prob-space } (X \otimes_Q Y, \text{map-prod } \alpha \beta \circ \text{real-real.g}, \text{distr } (\text{return real-borel } r \otimes_M \text{return real-borel } k) \text{ real-borel real-real.f})$   
 $\text{qbs-prob } (X \otimes_Q Y) (\text{map-prod } \alpha \beta \circ \text{real-real.g}) (\text{distr } (\text{return real-borel } r \otimes_M \text{return real-borel } k) \text{ real-borel real-real.f})$   
 $\langle \text{proof} \rangle$

**lemma** *pair-return-return*:

**assumes**  $l \in \text{space } M$   
**and**  $r \in \text{space } N$   
**shows**  $\text{return } M l \otimes_M \text{return } N r = \text{return } (M \otimes_M N) (l,r)$   
 $\langle \text{proof} \rangle$

**lemma** *bind-bind-return-distr*:

**assumes** *real-distribution*  $\mu$   
**and** *real-distribution*  $\nu$

**shows**  $\mu \gg (\lambda r. \nu \gg (\lambda l. \text{distr} (\text{return real-borel } r \otimes_M \text{return real-borel } l) \text{ real-borel real-real.f}))$   
 $= \text{distr} (\mu \otimes_M \nu) \text{ real-borel real-real.f}$   
**(is ?lhs = ?rhs)**  
 <proof>

**lemma**(in *pair-qbs-probs*) *qbs-bind-return-qp*:

**shows**  $\text{qbs-prob-space } (Y, \beta, \nu) \gg (\lambda y. \text{qbs-prob-space } (X, \alpha, \mu) \gg (\lambda x. \text{qbs-return } (X \otimes_Q Y) (x,y))) = \text{qbs-prob-space } (X \otimes_Q Y, \text{map-prod } \alpha \beta \circ \text{real-real.g}, \text{distr} (\mu \otimes_M \nu) \text{ real-borel real-real.f})$   
 $\text{qbs-prob } (X \otimes_Q Y) (\text{map-prod } \alpha \beta \circ \text{real-real.g}) (\text{distr} (\mu \otimes_M \nu) \text{ real-borel real-real.f})$   
 <proof>

**lemma**(in *pair-qbs-probs*) *qbs-bind-return-pq*:

**shows**  $\text{qbs-prob-space } (X, \alpha, \mu) \gg (\lambda x. \text{qbs-prob-space } (Y, \beta, \nu) \gg (\lambda y. \text{qbs-return } (X \otimes_Q Y) (x,y))) = \text{qbs-prob-space } (X \otimes_Q Y, \text{map-prod } \alpha \beta \circ \text{real-real.g}, \text{distr} (\mu \otimes_M \nu) \text{ real-borel real-real.f})$   
 $\text{qbs-prob } (X \otimes_Q Y) (\text{map-prod } \alpha \beta \circ \text{real-real.g}) (\text{distr} (\mu \otimes_M \nu) \text{ real-borel real-real.f})$   
 <proof>

**lemma** *qbs-bind-return-rotate*:

**assumes**  $p \in \text{monadP-qbs-Px } X$   
**and**  $q \in \text{monadP-qbs-Px } Y$   
**shows**  $q \gg (\lambda y. p \gg (\lambda x. \text{qbs-return } (X \otimes_Q Y) (x,y))) = p \gg (\lambda x. q \gg (\lambda y. \text{qbs-return } (X \otimes_Q Y) (x,y)))$   
 <proof>

**lemma** *qbs-pair-bind-return1*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \text{monadP-qbs } Z$   
 $p \in \text{monadP-qbs-Px } X$   
**and**  $q \in \text{monadP-qbs-Px } Y$   
**shows**  $q \gg (\lambda y. p \gg (\lambda x. f (x,y))) = (q \gg (\lambda y. p \gg (\lambda x. \text{qbs-return } (X \otimes_Q Y) (x,y)))) \gg f$   
**(is ?lhs = ?rhs)**  
 <proof>

**lemma** *qbs-pair-bind-return2*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \text{monadP-qbs } Z$   
 $p \in \text{monadP-qbs-Px } X$   
**and**  $q \in \text{monadP-qbs-Px } Y$   
**shows**  $p \gg (\lambda x. q \gg (\lambda y. f (x,y))) = (p \gg (\lambda x. q \gg (\lambda y. \text{qbs-return } (X \otimes_Q Y) (x,y)))) \gg f$   
**(is ?lhs = ?rhs)**  
 <proof>

**lemma** *qbs-bind-rotate*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \text{monadP-qbs } Z$

$p \in \text{monadP-qbs-Px } X$   
**and**  $q \in \text{monadP-qbs-Px } Y$   
**shows**  $q \gg (\lambda y. p \gg (\lambda x. f(x,y))) = p \gg (\lambda x. q \gg (\lambda y. f(x,y)))$   
 ⟨proof⟩

**lemma**(in *pair-qbs-probs*) *qbs-bind-bind-return*:  
**assumes**  $f \in X \otimes_Q Y \rightarrow_Q Z$   
**shows**  $\text{qbs-prob } Z (f \circ (\text{map-prod } \alpha \beta \circ \text{real-real.g})) (\text{distr } (\mu \otimes_M \nu) \text{ real-borel } \text{real-real.f})$   
**and**  $\text{qbs-prob-space } (X, \alpha, \mu) \gg (\lambda x. \text{qbs-prob-space } (Y, \beta, \nu) \gg (\lambda y. \text{qbs-return } Z (f(x,y)))) = \text{qbs-prob-space } (Z, f \circ (\text{map-prod } \alpha \beta \circ \text{real-real.g}), \text{distr } (\mu \otimes_M \nu) \text{ real-borel } \text{real-real.f})$   
 (is ?lhs = ?rhs)  
 ⟨proof⟩

### 3.2.7 Properties of Return and Bind

**lemma** *qbs-prob-measure-return*:  
**assumes**  $x \in \text{qbs-space } X$   
**shows**  $\text{qbs-prob-measure } (\text{qbs-return } X x) = \text{return } (\text{qbs-to-measure } X) x$   
 ⟨proof⟩

**lemma** *qbs-prob-measure-bind*:  
**assumes**  $s \in \text{monadP-qbs-Px } X$   
**and**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$   
**shows**  $\text{qbs-prob-measure } (s \gg f) = \text{qbs-prob-measure } s \gg \text{qbs-prob-measure } \circ f$   
 (is ?lhs = ?rhs)  
 ⟨proof⟩

**lemma** *qbs-of-return*:  
**assumes**  $x \in \text{qbs-space } X$   
**shows**  $\text{qbs-prob-space-qbs } (\text{qbs-return } X x) = X$   
 ⟨proof⟩

**lemma** *qbs-of-bind*:  
**assumes**  $s \in \text{monadP-qbs-Px } X$   
**and**  $f \in X \rightarrow_Q \text{monadP-qbs } Y$   
**shows**  $\text{qbs-prob-space-qbs } (s \gg f) = Y$   
 ⟨proof⟩

### 3.2.8 Properties of Integrals

**lemma** *qbs-integrable-return*:  
**assumes**  $x \in \text{qbs-space } X$   
**and**  $f \in X \rightarrow_Q \mathbb{R}_Q$   
**shows**  $\text{qbs-integrable } (\text{qbs-return } X x) f$   
 ⟨proof⟩



**lemma** *qbs-integrable-bind-return:*

**assumes**  $s \in \text{monadP-qbs-Px } Y$

$f \in Z \rightarrow_Q \mathbb{R}_Q$

**and**  $g \in Y \rightarrow_Q Z$

**shows**  $\text{qbs-integrable } (s \gg (\lambda y. \text{qbs-return } Z (g y))) f = \text{qbs-integrable } s (f \circ g)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-morphism:*

**assumes**  $L \in X \rightarrow_Q \text{monadP-qbs } Y$

**and**  $f \in X \rightarrow_Q \text{exp-qbs } Y \mathbb{R}_{Q \geq 0}$

**shows**  $(\lambda x. \text{qbs-prob-ennintegral } (L x) (f x)) \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-morphism-ennintegral-fst:*

**assumes**  $q \in \text{monadP-qbs-Px } Y$

**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\lambda x. \int^+_Q y. f (x, y) \partial q) \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-morphism-ennintegral-snd:*

**assumes**  $p \in \text{monadP-qbs-Px } X$

**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\lambda y. \int^+_Q x. f (x, y) \partial p) \in Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-morphism':*

**assumes**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\lambda s. \text{qbs-prob-ennintegral } s f) \in \text{monadP-qbs } X \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-return:*

**assumes**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**and**  $x \in \text{qbs-space } X$

**shows**  $\text{qbs-prob-ennintegral } (\text{qbs-return } X x) f = f x$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-bind:*

**assumes**  $s \in \text{monadP-qbs-Px } X$

$f \in X \rightarrow_Q \text{monadP-qbs } Y$

**and**  $g \in Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $\text{qbs-prob-ennintegral } (s \gg f) g = \text{qbs-prob-ennintegral } s (\lambda y. (\text{qbs-prob-ennintegral } (f y) g))$   
 $(\text{is } ?lhs = ?rhs)$   
 $\langle \text{proof} \rangle$

**lemma** *qbs-prob-ennintegral-bind-return:*

**assumes**  $s \in \text{monadP-qbs-Px } Y$

$f \in Z \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**and**  $g \in Y \rightarrow_Q Z$   
**shows**  $qbs\text{-prob-ennintegral } (s \gg (\lambda y. qbs\text{-return } Z (g y))) f = qbs\text{-prob-ennintegral } s (f \circ g)$   
 $\langle proof \rangle$

**lemma** *qbs-prob-integral-morphism'*:  
**assumes**  $f \in X \rightarrow_Q \mathbb{R}_Q$   
**shows**  $(\lambda s. qbs\text{-prob-integral } s f) \in monadP\text{-qbs } X \rightarrow_Q \mathbb{R}_Q$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-integral-fst*:  
**assumes**  $q \in monadP\text{-qbs-}Px Y$   
**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
**shows**  $(\lambda x. \int_Q y. f (x, y) \partial q) \in X \rightarrow_Q \mathbb{R}_Q$   
 $\langle proof \rangle$

**lemma** *qbs-morphism-integral-snd*:  
**assumes**  $p \in monadP\text{-qbs-}Px X$   
**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
**shows**  $(\lambda y. \int_Q x. f (x, y) \partial p) \in Y \rightarrow_Q \mathbb{R}_Q$   
 $\langle proof \rangle$

**lemma** *qbs-prob-integral-morphism*:  
**assumes**  $L \in X \rightarrow_Q monadP\text{-qbs } Y$   
 $f \in X \rightarrow_Q exp\text{-qbs } Y \mathbb{R}_Q$   
**and**  $\bigwedge x. x \in qbs\text{-space } X \implies qbs\text{-integrable } (L x) (f x)$   
**shows**  $(\lambda x. qbs\text{-prob-integral } (L x) (f x)) \in X \rightarrow_Q \mathbb{R}_Q$   
 $\langle proof \rangle$

**lemma** *qbs-prob-integral-morphism''*:  
**assumes**  $f \in X \rightarrow_Q \mathbb{R}_Q$   
**and**  $L \in Y \rightarrow_Q monadP\text{-qbs } X$   
**shows**  $(\lambda y. qbs\text{-prob-integral } (L y) f) \in Y \rightarrow_Q \mathbb{R}_Q$   
 $\langle proof \rangle$

**lemma** *qbs-prob-integral-return*:  
**assumes**  $f \in X \rightarrow_Q \mathbb{R}_Q$   
**and**  $x \in qbs\text{-space } X$   
**shows**  $qbs\text{-prob-integral } (qbs\text{-return } X x) f = f x$   
 $\langle proof \rangle$

**lemma** *qbs-prob-integral-bind*:  
**assumes**  $s \in monadP\text{-qbs-}Px X$   
 $f \in X \rightarrow_Q monadP\text{-qbs } Y$   
 $g \in Y \rightarrow_Q \mathbb{R}_Q$   
**and**  $\exists K. \forall y \in qbs\text{-space } Y. |g y| \leq K$   
**shows**  $qbs\text{-prob-integral } (s \gg f) g = qbs\text{-prob-integral } s (\lambda y. (qbs\text{-prob-integral } (f y) g))$

(is ?lhs = ?rhs)  
 ⟨proof⟩

**lemma** *qbs-prob-integral-bind-return*:

assumes  $s \in \text{monadP-qbs-Px } Y$   
 $f \in Z \rightarrow_Q \mathbb{R}_Q$   
 and  $g \in Y \rightarrow_Q Z$   
 shows  $\text{qbs-prob-integral } (s \gg (\lambda y. \text{qbs-return } Z (g y))) f = \text{qbs-prob-integral } s$   
 $(f \circ g)$   
 ⟨proof⟩

**lemma** *qbs-prob-var-bind-return*:

assumes  $s \in \text{monadP-qbs-Px } Y$   
 $f \in Z \rightarrow_Q \mathbb{R}_Q$   
 and  $g \in Y \rightarrow_Q Z$   
 shows  $\text{qbs-prob-var } (s \gg (\lambda y. \text{qbs-return } Z (g y))) f = \text{qbs-prob-var } s (f \circ g)$   
 ⟨proof⟩

end

### 3.3 Binary Product Measure

**theory** *Pair-QuasiBorel-Measure*

imports *Monad-QuasiBorel*

begin

#### 3.3.1 Binary Product Measure

Special case of [1] Proposition 23 where  $\Omega = \mathbb{R} \times \mathbb{R}$  and  $X = X \times Y$ . Let  $[\alpha, \mu] \in P(X)$  and  $[\beta, \nu] \in P(Y)$ .  $\alpha \times \beta$  is the  $\alpha$  in Proposition 23.

**definition** *qbs-prob-pair-measure-t* :: [*'a qbs-prob-t, 'b qbs-prob-t*]  $\Rightarrow$  (*'a*  $\times$  *'b*)  
*qbs-prob-t* **where**

*qbs-prob-pair-measure-t*  $p \ q \equiv$  (let  $(X, \alpha, \mu) = p$ ;  
 $(Y, \beta, \nu) = q$  in  
 $(X \otimes_Q Y, \text{map-prod } \alpha \ \beta \circ \text{real-real.g}, \text{distr } (\mu \otimes_M \nu)$   
*real-borel real-real.f*))

**lift-definition** *qbs-prob-pair-measure* :: [*'a qbs-prob-space, 'b qbs-prob-space*]  $\Rightarrow$  (*'a*  
 $\times$  *'b*) *qbs-prob-space* (**infix**  $\otimes_{Qmes}$  80)

**is** *qbs-prob-pair-measure-t*

⟨proof⟩

**lemma**(in *pair-qbs-probs*) *qbs-prob-pair-measure-computation*:

$(\text{qbs-prob-space } (X, \alpha, \mu)) \otimes_{Qmes} (\text{qbs-prob-space } (Y, \beta, \nu)) = \text{qbs-prob-space } (X$   
 $\otimes_Q Y, \text{map-prod } \alpha \ \beta \circ \text{real-real.g}, \text{distr } (\mu \otimes_M \nu) \text{ real-borel real-real.f})$   
 $\text{qbs-prob } (X \otimes_Q Y) (\text{map-prod } \alpha \ \beta \circ \text{real-real.g}) (\text{distr } (\mu \otimes_M \nu) \text{ real-borel}$   
*real-real.f*)  
 ⟨proof⟩

**lemma** *qbs-prob-pair-measure-qbs*:

*qbs-prob-space-qbs* ( $p \otimes_{Qmes} q$ ) = *qbs-prob-space-qbs*  $p \otimes_Q$  *qbs-prob-space-qbs*  $q$   
 <proof>

**lemma**(in *pair-qbs-probs*) *qbs-prob-pair-measure-measure*:

**shows** *qbs-prob-measure* (*qbs-prob-space* ( $X, \alpha, \mu$ )  $\otimes_{Qmes}$  *qbs-prob-space* ( $Y, \beta, \nu$ ))  
 = *distr* ( $\mu \otimes_M \nu$ ) (*qbs-to-measure* ( $X \otimes_Q Y$ )) (*map-prod*  $\alpha \beta$ )  
 <proof>

**lemma** *qbs-prob-pair-measure-morphism*:

*case-prod* *qbs-prob-pair-measure*  $\in$  *monadP-qbs*  $X \otimes_Q$  *monadP-qbs*  $Y \rightarrow_Q$  *monadP-qbs* ( $X \otimes_Q Y$ )  
 <proof>

**lemma**(in *pair-qbs-probs*) *qbs-prob-pair-measure-nnintegral*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows** ( $\int^+_Q z. f z \partial(\text{qbs-prob-space } (X, \alpha, \mu) \otimes_{Qmes} \text{qbs-prob-space } (Y, \beta, \nu))$ )  
 = ( $\int^+_Q z. (f \circ \text{map-prod } \alpha \beta) z \partial(\mu \otimes_M \nu)$ )  
 (is ?lhs = ?rhs)  
 <proof>

**lemma**(in *pair-qbs-probs*) *qbs-prob-pair-measure-integral*:

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
**shows** ( $\int_Q z. f z \partial(\text{qbs-prob-space } (X, \alpha, \mu) \otimes_{Qmes} \text{qbs-prob-space } (Y, \beta, \nu))$ )  
 = ( $\int z. (f \circ \text{map-prod } \alpha \beta) z \partial(\mu \otimes_M \nu)$ )  
 (is ?lhs = ?rhs)  
 <proof>

**lemma** *qbs-prob-pair-measure-eq-bind*:

**assumes**  $p \in \text{monadP-qbs-Px } X$   
**and**  $q \in \text{monadP-qbs-Px } Y$   
**shows**  $p \otimes_{Qmes} q = p \gg (\lambda x. q \gg (\lambda y. \text{qbs-return } (X \otimes_Q Y) (x, y)))$   
 <proof>

### 3.3.2 Fubini Theorem

**lemma** *qbs-prob-ennintegral-Fubini-fst*:

**assumes**  $p \in \text{monadP-qbs-Px } X$   
 $q \in \text{monadP-qbs-Px } Y$   
**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
**shows** ( $\int^+_Q x. \int^+_Q y. f (x, y) \partial q \partial p$ ) = ( $\int^+_Q z. f z \partial(p \otimes_{Qmes} q)$ )  
 (is ?lhs = ?rhs)  
 <proof>

**lemma** *qbs-prob-ennintegral-Fubini-snd*:

**assumes**  $p \in \text{monadP-qbs-Px } X$   
 $q \in \text{monadP-qbs-Px } Y$   
**and**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\int^{+Q} y. \int^{+Q} x. f(x,y) \partial p \partial q) = (\int^{+Q} x. f x \partial(p \otimes_{Qmes} q))$   
**(is ?lhs = ?rhs)**  
 <proof>

**lemma** *qbs-prob-ennintegral-indep1:*

**assumes**  $p \in \text{monadP-qbs-Px } X$

**and**  $f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\int^{+Q} z. f(\text{fst } z) \partial(p \otimes_{Qmes} q)) = (\int^{+Q} x. f x \partial p)$   
**(is ?lhs = -)**

<proof>

**lemma** *qbs-prob-ennintegral-indep2:*

**assumes**  $q \in \text{monadP-qbs-Px } Y$

**and**  $f \in Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\int^{+Q} z. f(\text{snd } z) \partial(p \otimes_{Qmes} q)) = (\int^{+Q} y. f y \partial q)$   
**(is ?lhs = -)**

<proof>

**lemma** *qbs-ennintegral-indep-mult:*

**assumes**  $p \in \text{monadP-qbs-Px } X$

$q \in \text{monadP-qbs-Px } Y$

$f \in X \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**and**  $g \in Y \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\int^{+Q} z. f(\text{fst } z) * g(\text{snd } z) \partial(p \otimes_{Qmes} q)) = (\int^{+Q} x. f x \partial p) * (\int^{+Q} y. g y \partial q)$   
**(is ?lhs = ?rhs)**

<proof>

**lemma**(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-integrable:*

**assumes** *qbs-integrable*  $(\text{qbs-prob-space } (X, \alpha, \mu) \otimes_{Qmes} \text{qbs-prob-space } (Y, \beta, \nu))$   
 $f$

**shows**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
*integrable*  $(\mu \otimes_M \nu) (f \circ (\text{map-prod } \alpha \beta))$

<proof>

**lemma**(**in** *pair-qbs-probs*) *qbs-prob-pair-measure-integrable':*

**assumes**  $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$

**and** *integrable*  $(\mu \otimes_M \nu) (f \circ (\text{map-prod } \alpha \beta))$

**shows** *qbs-integrable*  $(\text{qbs-prob-space } (X, \alpha, \mu) \otimes_{Qmes} \text{qbs-prob-space } (Y, \beta, \nu))$

$f$

<proof>

**lemma** *qbs-integrable-pair-swap:*

**assumes** *qbs-integrable*  $(p \otimes_{Qmes} q) f$

**shows** *qbs-integrable*  $(q \otimes_{Qmes} p) (\lambda(x,y). f(y,x))$

<proof>

**lemma** *qbs-integrable-pair1:*

**assumes**  $p \in \text{monadP-qbs-Px } X$   
 $q \in \text{monadP-qbs-Px } Y$   
 $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
 $\text{qbs-integrable } p (\lambda x. \int_Q y. |f(x,y)| \partial q)$   
**and**  $\bigwedge x. x \in \text{qbs-space } X \implies \text{qbs-integrable } q (\lambda y. f(x,y))$   
**shows**  $\text{qbs-integrable } (p \otimes_{Qmes} q) f$   
 <proof>

**lemma** *qbs-integrable-pair2*:  
**assumes**  $p \in \text{monadP-qbs-Px } X$   
 $q \in \text{monadP-qbs-Px } Y$   
 $f \in X \otimes_Q Y \rightarrow_Q \mathbb{R}_Q$   
 $\text{qbs-integrable } q (\lambda y. \int_Q x. |f(x,y)| \partial p)$   
**and**  $\bigwedge y. y \in \text{qbs-space } Y \implies \text{qbs-integrable } p (\lambda x. f(x,y))$   
**shows**  $\text{qbs-integrable } (p \otimes_{Qmes} q) f$   
 <proof>

**lemma** *qbs-integrable-fst*:  
**assumes**  $\text{qbs-integrable } (p \otimes_{Qmes} q) f$   
**shows**  $\text{qbs-integrable } p (\lambda x. \int_Q y. f(x,y) \partial q)$   
 <proof>

**lemma** *qbs-integrable-snd*:  
**assumes**  $\text{qbs-integrable } (p \otimes_{Qmes} q) f$   
**shows**  $\text{qbs-integrable } q (\lambda y. \int_Q x. f(x,y) \partial p)$   
 <proof>

**lemma** *qbs-integrable-indep-mult*:  
**assumes**  $\text{qbs-integrable } p f$   
**and**  $\text{qbs-integrable } q g$   
**shows**  $\text{qbs-integrable } (p \otimes_{Qmes} q) (\lambda x. f(\text{fst } x) * g(\text{snd } x))$   
 <proof>

**lemma** *qbs-integrable-indep1*:  
**assumes**  $\text{qbs-integrable } p f$   
**shows**  $\text{qbs-integrable } (p \otimes_{Qmes} q) (\lambda x. f(\text{fst } x))$   
 <proof>

**lemma** *qbs-integrable-indep2*:  
**assumes**  $\text{qbs-integrable } q g$   
**shows**  $\text{qbs-integrable } (p \otimes_{Qmes} q) (\lambda x. g(\text{snd } x))$   
 <proof>

**lemma** *qbs-prob-integral-Fubini-fst*:  
**assumes**  $\text{qbs-integrable } (p \otimes_{Qmes} q) f$   
**shows**  $(\int_Q x. \int_Q y. f(x,y) \partial q \partial p) = (\int_Q z. f z \partial(p \otimes_{Qmes} q))$   
 (is ?lhs = ?rhs)  
 <proof>

**lemma** *qbs-prob-integral-Fubini-snd:*

**assumes** *qbs-integrable*  $(p \otimes_{Qmes} q) f$   
**shows**  $(\int_Q y. \int_Q x. f(x,y) \partial p \partial q) = (\int_Q z. f z \partial(p \otimes_{Qmes} q))$   
**(is ?lhs = ?rhs)**

*<proof>*

**lemma** *qbs-prob-integral-indep1:*

**assumes** *qbs-integrable*  $p f$   
**shows**  $(\int_Q z. f(fst z) \partial(p \otimes_{Qmes} q)) = (\int_Q x. f x \partial p)$   
*<proof>*

**lemma** *qbs-prob-integral-indep2:*

**assumes** *qbs-integrable*  $q g$   
**shows**  $(\int_Q z. g(snd z) \partial(p \otimes_{Qmes} q)) = (\int_Q y. g y \partial q)$   
*<proof>*

**lemma** *qbs-prob-integral-indep-mult:*

**assumes** *qbs-integrable*  $p f$   
**and** *qbs-integrable*  $q g$   
**shows**  $(\int_Q z. f(fst z) * g(snd z) \partial(p \otimes_{Qmes} q)) = (\int_Q x. f x \partial p) * (\int_Q y. g y \partial q)$   
**(is ?lhs = ?rhs)**  
*<proof>*

**lemma** *qbs-prob-var-indep-plus:*

**assumes** *qbs-integrable*  $(p \otimes_{Qmes} q) f$   
*qbs-integrable*  $(p \otimes_{Qmes} q) (\lambda z. (f z)^2)$   
*qbs-integrable*  $(p \otimes_{Qmes} q) g$   
*qbs-integrable*  $(p \otimes_{Qmes} q) (\lambda z. (g z)^2)$   
*qbs-integrable*  $(p \otimes_{Qmes} q) (\lambda z. (f z) * (g z))$   
**and**  $(\int_Q z. f z * g z \partial(p \otimes_{Qmes} q)) = (\int_Q z. f z \partial(p \otimes_{Qmes} q)) * (\int_Q z. g z \partial(p \otimes_{Qmes} q))$   
**shows** *qbs-prob-var*  $(p \otimes_{Qmes} q) (\lambda z. f z + g z) = \text{qbs-prob-var } (p \otimes_{Qmes} q) f + \text{qbs-prob-var } (p \otimes_{Qmes} q) g$   
*<proof>*

**lemma** *qbs-prob-var-indep-plus':*

**assumes** *qbs-integrable*  $p f$   
*qbs-integrable*  $p (\lambda x. (f x)^2)$   
*qbs-integrable*  $q g$   
**and** *qbs-integrable*  $q (\lambda x. (g x)^2)$   
**shows** *qbs-prob-var*  $(p \otimes_{Qmes} q) (\lambda z. f(fst z) + g(snd z)) = \text{qbs-prob-var } p f + \text{qbs-prob-var } q g$   
*<proof>*

**end**

### 3.4 Measure as QBS Measure

**theory** *Measure-as-QuasiBorel-Measure*  
**imports** *Pair-QuasiBorel-Measure*

**begin**

**lemma** *distr-id'*:

**assumes** *sets N = sets M*

$f \in N \rightarrow_M N$

**and**  $\bigwedge x. x \in \text{space } N \implies f x = x$

**shows**  $\text{distr } N M f = N$

*<proof>*

Every probability measure on a standard Borel space can be represented as a measure on a quasi-Borel space [1], Proposition 23.

**locale** *standard-borel-prob-space = standard-borel P + p:prob-space P*

**for**  $P :: 'a \text{ measure}$

**begin**

**sublocale** *qbs-prob measure-to-qbs P g distr P real-borel f*

*<proof>*

**lift-definition** *as-qbs-measure :: 'a qbs-prob-space is*

*(measure-to-qbs P, g, distr P real-borel f)*

*<proof>*

**lemma** *as-qbs-measure-retract:*

**assumes** *[measurable]: a ∈ P →<sub>M</sub> real-borel*

**and** *[measurable]: b ∈ real-borel →<sub>M</sub> P*

**and** *[simp]: ⋀ x. x ∈ space P ⟹ (b ∘ a) x = x*

**shows** *qbs-prob (measure-to-qbs P) b (distr P real-borel a)*

*as-qbs-measure = qbs-prob-space (measure-to-qbs P, b, distr P real-borel a)*

*<proof>*

**lemma** *measure-as-qbs-measure-qbs:*

*qbs-prob-space-qbs as-qbs-measure = measure-to-qbs P*

*<proof>*

**lemma** *measure-as-qbs-measure-image:*

*as-qbs-measure ∈ monadP-qbs-Px (measure-to-qbs P)*

*<proof>*

**lemma** *as-qbs-measure-as-measure[simp]:*

*distr (distr P real-borel f) (qbs-to-measure (measure-to-qbs P)) g = P*

*<proof>*

**lemma** *measure-as-qbs-measure-recover:*

*qbs-prob-measure as-qbs-measure = P*



*<proof>*

**end**

**lemma**(in *standard-borel*) *qbs-prob-measure-recover*:

**assumes**  $q \in \text{monadP-qbs-Px}$  (*measure-to-qbs M*)

**shows** *standard-borel-prob-space.as-qbs-measure* (*qbs-prob-measure q*) =  $q$

*<proof>*

**lemma**(in *standard-borel-prob-space*) *ennintegral-as-qbs-ennintegral*:

**assumes**  $k \in \text{borel-measurable P}$

**shows**  $(\int^+_{\mathcal{Q}} x. k x \partial \text{as-qbs-measure}) = (\int^+ x. k x \partial P)$

*<proof>*

**lemma**(in *standard-borel-prob-space*) *integral-as-qbs-integral*:

$(\int_{\mathcal{Q}} x. k x \partial \text{as-qbs-measure}) = (\int x. k x \partial P)$

*<proof>*

**lemma**(in *standard-borel*) *measure-with-args-morphism*:

**assumes** [*measurable*]:  $\mu \in X \rightarrow_M \text{prob-algebra M}$

**shows** *standard-borel-prob-space.as-qbs-measure*  $\circ \mu \in \text{measure-to-qbs } X \rightarrow_{\mathcal{Q}}$   
*monadP-qbs* (*measure-to-qbs M*)

*<proof>*

**lemma**(in *standard-borel*) *measure-with-args-recover*:

**assumes**  $\mu \in \text{space } X \rightarrow \text{space} (\text{prob-algebra M})$

**and**  $x \in \text{space } X$

**shows** *qbs-prob-measure* (*standard-borel-prob-space.as-qbs-measure* ( $\mu x$ )) =  $\mu$

$x$

*<proof>*

### 3.5 Example of Probability Measures

Probability measures on  $\mathbb{R}$  can be represented as probability measures on the quasi-Borel space  $\mathbb{R}$ .

#### 3.5.1 Normal Distribution

**definition** *normal-distribution* :: *real*  $\times$  *real*  $\Rightarrow$  *real measure* **where**

*normal-distribution*  $\mu\sigma = (\text{if } 0 < (\text{snd } \mu\sigma) \text{ then density lborel } (\lambda x. \text{ennreal } (\text{normal-density}$

$(\text{fst } \mu\sigma) (\text{snd } \mu\sigma) x)$

$\text{else return lborel } 0)$

**lemma** *normal-distribution-measurable*:

*normal-distribution*  $\in \text{real-borel} \otimes_M \text{real-borel} \rightarrow_M \text{prob-algebra real-borel}$

*<proof>*

**definition** *qbs-normal-distribution* :: *real*  $\Rightarrow$  *real*  $\Rightarrow$  *real qbs-prob-space* **where**

*qbs-normal-distribution*  $\equiv$  *curry* (*standard-borel-prob-space.as-qbs-measure*  $\circ$  *normal-distribution*)

**lemma** *qbs-normal-distribution-morphism*:

*qbs-normal-distribution*  $\in \mathbf{R}_Q \rightarrow_Q \text{exp-qbs } \mathbf{R}_Q$  (*monadP-qbs*  $\mathbf{R}_Q$ )  
 ⟨*proof*⟩

**context**

**fixes**  $\mu \sigma :: \text{real}$   
**assumes** *sigma*: $\sigma > 0$

**begin**

**interpretation** *n-dist:standard-borel-prob-space normal-distribution*  $(\mu, \sigma)$

⟨*proof*⟩

**lemma** *qbs-normal-distribution-def2*:

*qbs-normal-distribution*  $\mu \sigma = \text{n-dist.as-qbs-measure}$   
 ⟨*proof*⟩

**lemma** *qbs-normal-distribution-integral*:

$(\int_Q x. f x \partial (\text{qbs-normal-distribution } \mu \sigma)) = (\int x. f x \partial (\text{density lborel } (\lambda x. \text{ennreal } (\text{normal-density } \mu \sigma x))))$   
 ⟨*proof*⟩

**lemma** *qbs-normal-distribution-expectation*:

**assumes**  $f \in \text{real-borel} \rightarrow_M \text{real-borel}$   
**shows**  $(\int_Q x. f x \partial (\text{qbs-normal-distribution } \mu \sigma)) = (\int x. \text{normal-density } \mu \sigma x * f x \partial \text{lborel})$   
 ⟨*proof*⟩

**end**

### 3.5.2 Uniform Distribution

**definition** *interval-uniform-distribution*  $:: \text{real} \Rightarrow \text{real} \Rightarrow \text{real measure}$  **where**

*interval-uniform-distribution*  $a b \equiv$  (if  $a < b$  then *uniform-measure lborel*  $\{a <..<b\}$   
 else *return lborel*  $0$ )

**lemma** *sets-interval-uniform-distribution[measurable-cong]*:

*sets* (*interval-uniform-distribution*  $a b$ ) = *borel*  
 ⟨*proof*⟩

**lemma** *interval-uniform-distribution-measurable*:

$(\lambda r. \text{interval-uniform-distribution } (\text{fst } r) (\text{snd } r)) \in \text{real-borel} \otimes_M \text{real-borel} \rightarrow_M \text{prob-algebra real-borel}$   
 ⟨*proof*⟩

**definition** *qbs-interval-uniform-distribution*  $:: \text{real} \Rightarrow \text{real} \Rightarrow \text{real qbs-prob-space}$

**where**

$qbs\text{-interval-uniform-distribution} \equiv \text{curry } (\text{standard-borel-prob-space.as-qbs-measure} \circ (\lambda r. \text{interval-uniform-distribution } (\text{fst } r) (\text{snd } r)))$

**lemma**  $qbs\text{-interval-uniform-distribution-morphism}$ :

$qbs\text{-interval-uniform-distribution} \in \mathbb{R}_Q \rightarrow_Q \text{exp-qbs } \mathbb{R}_Q \text{ (monadP-qbs } \mathbb{R}_Q)$   
 $\langle \text{proof} \rangle$

**context**

**fixes**  $a b :: \text{real}$

**assumes**  $a\text{-less-than-}b:a < b$

**begin**

**definition**  $ab\text{-qbs-uniform-distribution} \equiv qbs\text{-interval-uniform-distribution } a b$

**interpretation**  $ab\text{-u-dist}$ :  $\text{standard-borel-prob-space interval-uniform-distribution } a b$

$\langle \text{proof} \rangle$

**lemma**  $qbs\text{-interval-uniform-distribution-def2}$ :

$ab\text{-qbs-uniform-distribution} = ab\text{-u-dist.as-qbs-measure}$   
 $\langle \text{proof} \rangle$

**lemma**  $qbs\text{-uniform-distribution-expectation}$ :

**assumes**  $f \in \mathbb{R}_Q \rightarrow_Q \mathbb{R}_{Q \geq 0}$

**shows**  $(\int^+ x. f x \partial ab\text{-qbs-uniform-distribution}) = (\int^+ x \in \{a < .. < b\}. f x \partial \text{lborel})$   
 $/ (b - a)$

(**is**  $?lhs = ?rhs$ )

$\langle \text{proof} \rangle$

**end**

### 3.5.3 Bernoulli Distribution

**definition**  $qbs\text{-bernoulli} :: \text{real} \Rightarrow \text{bool qbs-prob-space where}$

$qbs\text{-bernoulli} \equiv \text{standard-borel-prob-space.as-qbs-measure} \circ (\lambda x. \text{measure-pmf } (\text{bernoulli-pmf } x))$

**lemma**  $bernoulli\text{-measurable}$ :

$(\lambda x. \text{measure-pmf } (\text{bernoulli-pmf } x)) \in \text{real-borel} \rightarrow_M \text{prob-algebra bool-borel}$   
 $\langle \text{proof} \rangle$

**lemma**  $qbs\text{-bernoulli-morphism}$ :

$qbs\text{-bernoulli} \in \mathbb{R}_Q \rightarrow_Q \text{monadP-qbs } \mathbb{B}_Q$   
 $\langle \text{proof} \rangle$

**lemma**  $qbs\text{-bernoulli-measure}$ :

$qbs\text{-prob-measure } (qbs\text{-bernoulli } p) = \text{measure-pmf } (\text{bernoulli-pmf } p)$

*<proof>*

**context**

**fixes**  $p :: \text{real}$

**assumes**  $p \geq 0$  and  $p \leq 1$

**begin**

**lemma** *qbs-bernoulli-expectation*:

$(\int_Q x. f x \text{ } \partial \text{qbs-bernoulli } p) = f \text{ True} * p + f \text{ False} * (1 - p)$

*<proof>*

**end**

**end**

### 3.6 Bayesian Linear Regression

**theory** *Bayesian-Linear-Regression*

**imports** *Measure-as-QuasiBorel-Measure*

**begin**

We formalize the Bayesian linear regression presented in [1] section VI.

#### 3.6.1 Prior

**abbreviation**  $\nu \equiv \text{density lborel } (\lambda x. \text{ennreal } (\text{normal-density } 0 \ 3 \ x))$

**interpretation**  $\nu$ : *standard-borel-prob-space*  $\nu$

*<proof>*

**term**  $\nu.\text{as-qbs-measure} :: \text{real qbs-prob-space}$

**definition** *prior* ::  $(\text{real} \Rightarrow \text{real}) \text{ qbs-prob-space}$  **where**

$\text{prior} \equiv \text{do } \{ s \leftarrow \nu.\text{as-qbs-measure} ;$

$b \leftarrow \nu.\text{as-qbs-measure} ;$

$\text{qbs-return } (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) (\lambda r. s * r + b) \}$

**lemma**  *$\nu$ -as-qbs-measure-eq*:

$\nu.\text{as-qbs-measure} = \text{qbs-prob-space } (\mathbb{R}_Q, \text{id}, \nu)$

*<proof>*

**interpretation**  $\nu$ -qp: *pair-qbs-prob*  $\mathbb{R}_Q \text{ id } \nu \ \mathbb{R}_Q \text{ id } \nu$

*<proof>*

**lemma**  *$\nu$ -as-qbs-measure-in-Pr*:

$\nu.\text{as-qbs-measure} \in \text{monadP-qbs-Px } \mathbb{R}_Q$

*<proof>*

**lemma** *sets-real-real-real[measurable-cong]*:

$\text{sets } (\text{qbs-to-measure } ((\mathbb{R}_Q \otimes_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_Q)) = \text{sets } ((\text{borel} \otimes_M \text{borel}) \otimes_M \text{borel})$

*<proof>*

**lemma** *lin-morphism*:

$(\lambda(s, b) \ r. \ s * r + b) \in \mathbb{R}_Q \otimes_Q \mathbb{R}_Q \rightarrow_Q \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$   
*<proof>*

**lemma** *lin-measurable[measurable]*:

$(\lambda(s, b) \ r. \ s * r + b) \in \text{real-borel} \otimes_M \text{real-borel} \rightarrow_M \text{qbs-to-measure} (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)$   
*<proof>*

**lemma** *prior-computation*:

$\text{qbs-prob} (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) ((\lambda(s, b) \ r. \ s * r + b) \circ \text{real-real.g}) (\text{distr} (\nu \otimes_M \nu) \text{real-borel real-real.f})$   
 $\text{prior} = \text{qbs-prob-space} (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q, (\lambda(s, b) \ r. \ s * r + b) \circ \text{real-real.g}, \text{distr} (\nu \otimes_M \nu) \text{real-borel real-real.f})$   
*<proof>*

The following lemma corresponds to the equation (5).

**lemma** *prior-measure*:

$\text{qbs-prob-measure prior} = \text{distr} (\nu \otimes_M \nu) (\text{qbs-to-measure} (\text{exp-qbs } \mathbb{R}_Q \ \mathbb{R}_Q))$   
 $(\lambda(s, b) \ r. \ s * r + b)$   
*<proof>*

**lemma** *prior-in-space*:

$\text{prior} \in \text{qbs-space} (\text{monadP-qbs} (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q))$   
*<proof>*

### 3.6.2 Likelihood

**abbreviation**  $d \ \mu \ x \equiv \text{normal-density } \mu \ (1/2) \ x$

**lemma** *d-positive* :  $0 < d \ \mu \ x$

*<proof>*

**definition** *obs* ::  $(\text{real} \Rightarrow \text{real}) \Rightarrow \text{ennreal}$  **where**

$\text{obs } f \equiv d \ (f \ 1) \ 2.5 * d \ (f \ 2) \ 3.8 * d \ (f \ 3) \ 4.5 * d \ (f \ 4) \ 6.2 * d \ (f \ 5) \ 8$

**lemma** *obs-morphism*:

$\text{obs} \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \rightarrow_Q \mathbb{R}_{Q \geq 0}$   
*<proof>*

**lemma** *obs-measurable[measurable]*:

$\text{obs} \in \text{qbs-to-measure} (\text{exp-qbs } \mathbb{R}_Q \ \mathbb{R}_Q) \rightarrow_M \text{ennreal-borel}$   
*<proof>*

### 3.6.3 Posterior

**lemma** *id-obs-morphism*:

$(\lambda f. \ (f, \text{obs } f)) \in \mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q \rightarrow_Q (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$

*<proof>*

**lemma** *push-forward-measure-in-space:*

*monadP-qbs-Pf* ( $\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ ) ( $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$ ) ( $\lambda f. (f, \text{obs } f)$ ) *prior*  $\in$   
*qbs-space* (*monadP-qbs* ( $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$ ))

*<proof>*

**lemma** *push-forward-measure-computation:*

*qbs-prob* ( $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$ ) ( $\lambda l. (((\lambda(s, b) r. s * r + b) \circ \text{real-real.g})$   
 $l, ((\text{obs} \circ (\lambda(s, b) r. s * r + b)) \circ \text{real-real.g}) l)$ ) (*distr* ( $\nu \otimes_M \nu$ ) *real-borel*  
*real-real.f*)

*monadP-qbs-Pf* ( $\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ ) ( $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$ ) ( $\lambda f. (f, \text{obs } f)$ ) *prior* =  
*qbs-prob-space* ( $(\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) \otimes_Q \mathbb{R}_{Q \geq 0}$ , ( $\lambda l. (((\lambda(s, b) r. s * r + b) \circ \text{real-real.g})$   
 $l, ((\text{obs} \circ (\lambda(s, b) r. s * r + b)) \circ \text{real-real.g}) l)$ ), *distr* ( $\nu \otimes_M \nu$ ) *real-borel*  
*real-real.f*)

*<proof>*

### 3.6.4 Normalizer

We use the unit space for an error.

**definition** *norm-qbs-measure* :: ( $'a \times \text{ennreal}$ ) *qbs-prob-space*  $\Rightarrow$   $'a$  *qbs-prob-space*  
+ *unit* **where**

*norm-qbs-measure*  $p \equiv$  (*let* ( $XR, \alpha\beta, \nu$ ) = *rep-qbs-prob-space*  $p$  *in*  
    *if* *emeasure* (*density*  $\nu$  (*snd*  $\circ \alpha\beta$ )) *UNIV* = 0 *then* *Inr* ()  
    *else if* *emeasure* (*density*  $\nu$  (*snd*  $\circ \alpha\beta$ )) *UNIV* =  $\infty$  *then* *Inr* ()  
    *else* *Inl* (*qbs-prob-space* (*map-qbs* *fst*  $XR$ , *fst*  $\circ \alpha\beta$ , *density*  $\nu$   
( $\lambda r. \text{snd} (\alpha\beta r) / \text{emeasure} (\text{density } \nu (\text{snd} \circ \alpha\beta)) \text{ UNIV}$ ))))

**lemma** *norm-qbs-measure-qbs-prob:*

**assumes** *qbs-prob* ( $X \otimes_Q \mathbb{R}_{Q \geq 0}$ ) ( $\lambda r. (\alpha r, \beta r)$ )  $\mu$   
*emeasure* (*density*  $\mu$   $\beta$ ) *UNIV*  $\neq 0$

**and** *emeasure* (*density*  $\mu$   $\beta$ ) *UNIV*  $\neq \infty$

**shows** *qbs-prob*  $X \alpha$  (*density*  $\mu$  ( $\lambda r. (\beta r) / \text{emeasure} (\text{density } \mu \beta) \text{ UNIV}$ ))

*<proof>*

**lemma** *norm-qbs-measure-computation:*

**assumes** *qbs-prob* ( $X \otimes_Q \mathbb{R}_{Q \geq 0}$ ) ( $\lambda r. (\alpha r, \beta r)$ )  $\mu$

**shows** *norm-qbs-measure* (*qbs-prob-space* ( $X \otimes_Q \mathbb{R}_{Q \geq 0}$ , ( $\lambda r. (\alpha r, \beta r)$ ),  $\mu$ )) =  
(*if* *emeasure* (*density*  $\mu$   $\beta$ ) *UNIV* = 0 *then* *Inr* ()

*else if* *emeasure*  
(*density*  $\mu$   $\beta$ ) *UNIV* =  $\infty$  *then* *Inr* ()

*else* *Inl* (*qbs-prob-space*  
( $X, \alpha$ , *density*  $\mu$  ( $\lambda r. (\beta r) / \text{emeasure} (\text{density } \mu \beta) \text{ UNIV}$ ))))

*<proof>*

**lemma** *norm-qbs-measure-morphism:*

*norm-qbs-measure*  $\in$  *monadP-qbs* ( $X \otimes_Q \mathbb{R}_{Q \geq 0}$ )  $\rightarrow_Q$  *monadP-qbs*  $X <+>_Q 1_Q$   
*<proof>*

The following is the semantics of the entire program.

**definition** *program* :: (real  $\Rightarrow$  real) qbs-prob-space + unit **where**  
*program*  $\equiv$  norm-qbs-measure (monadP-qbs-Pf ( $\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ )) (( $\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ )  $\otimes_Q$   $\mathbb{R}_{Q \geq 0}$ ) ( $\lambda f. (f, \text{obs } f)$ ) prior

**lemma** *program-in-space*:

*program*  $\in$  qbs-space (monadP-qbs ( $\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q$ ))  $\langle + \rangle_Q 1_Q$   
 $\langle$ proof $\rangle$

We calculate the normalizing constant.

**lemma** *complete-the-square*:

**fixes** a b c x :: real

**assumes** a  $\neq$  0

**shows**  $a*x^2 + b*x + c = a * (x + (b / (2*a)))^2 - ((b^2 - 4*a*c)/(4*a))$

$\langle$ proof $\rangle$

**lemma** *complete-the-square2'*:

**fixes** a b c x :: real

**assumes** a  $\neq$  0

**shows**  $a*x^2 - 2*b*x + c = a * (x - (b / a))^2 - ((b^2 - a*c)/a)$

$\langle$ proof $\rangle$

**lemma** *normal-density-mu-x-swap*:

normal-density  $\mu \sigma x =$  normal-density  $x \sigma \mu$

$\langle$ proof $\rangle$

**lemma** *normal-density-plus-shift*:

normal-density  $\mu \sigma (x + y) =$  normal-density  $(\mu - x) \sigma y$

$\langle$ proof $\rangle$

**lemma** *normal-density-times*:

**assumes**  $\sigma > 0 \sigma' > 0$

**shows** normal-density  $\mu \sigma x * normal-density \mu' \sigma' x = (1 / \text{sqrt } (2 * \text{pi} * (\sigma^2 + \sigma'^2))) * \text{exp } (- (\mu - \mu')^2 / (2 * (\sigma^2 + \sigma'^2))) * normal-density ((\mu*\sigma'^2 + \mu'*\sigma^2)/(\sigma^2 + \sigma'^2)) (\sigma * \sigma' / \text{sqrt } (\sigma^2 + \sigma'^2)) x$

(is ?lhs = ?rhs)

$\langle$ proof $\rangle$

**lemma** *normal-density-times'*:

**assumes**  $\sigma > 0 \sigma' > 0$

**shows** a \* normal-density  $\mu \sigma x * normal-density \mu' \sigma' x = a * (1 / \text{sqrt } (2 * \text{pi} * (\sigma^2 + \sigma'^2))) * \text{exp } (- (\mu - \mu')^2 / (2 * (\sigma^2 + \sigma'^2))) * normal-density ((\mu*\sigma'^2 + \mu'*\sigma^2)/(\sigma^2 + \sigma'^2)) (\sigma * \sigma' / \text{sqrt } (\sigma^2 + \sigma'^2)) x$

$\langle$ proof $\rangle$

**lemma** *normal-density-times-minusx*:

**assumes**  $\sigma > 0 \sigma' > 0 a \neq a'$

**shows** normal-density  $(\mu - a*x) \sigma y * normal-density (\mu' - a'*x) \sigma' y = (1$

$/ |a' - a| * \text{normal-density } ((\mu' - \mu)/(a' - a)) (\text{sqrt } ((\sigma^2 + \sigma'^2)/(a' - a)^2)) x * \text{normal-density } (((\mu - a*x)*\sigma'^2 + (\mu' - a'*x)*\sigma^2)/(\sigma^2 + \sigma'^2)) (\sigma * \sigma' / \text{sqrt } (\sigma^2 + \sigma'^2)) y$   
 <proof>

The following is the normalizing constant of the program.

**abbreviation**  $C \equiv \text{ennreal } ((4 * \text{sqrt } 2 / (\text{pi}^2 * \text{sqrt } (66961 * \text{pi}))) * (\text{exp } (- (1674761 / 1674025))))$

**lemma** *program-normalizing-constant:*

$\text{emeasure } (\text{density } (\text{distr } (\nu \otimes_M \nu) \text{ real-borel real-real.f}) (\text{obs } \circ (\lambda(s, b) r. s * r + b) \circ \text{real-real.g})) \text{ UNIV} = C$   
 (is ?lhs = ?rhs)  
 <proof>

The program returns a probability measure, rather than error.

**lemma** *program-result:*

$\text{qbs-prob } (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q) ((\lambda(s, b) r. s * r + b) \circ \text{real-real.g}) (\text{density } (\text{distr } (\nu \otimes_M \nu) \text{ real-borel real-real.f}) (\lambda r. (\text{obs } \circ (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}) r / C))$   
 $\text{program} = \text{Inl } (\text{qbs-prob-space } (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q, (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}, \text{density } (\text{distr } (\nu \otimes_M \nu) \text{ real-borel real-real.f}) (\lambda r. (\text{obs } \circ (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}) r / C)))$   
 <proof>

**lemma** *program-inl:*

$\text{program} \in \text{Inl } ' (\text{qbs-space } (\text{monadP-qbs } (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q)))$   
 <proof>

**lemma** *program-result-measure:*

$\text{qbs-prob-measure } (\text{qbs-prob-space } (\mathbb{R}_Q \Rightarrow_Q \mathbb{R}_Q, (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}, \text{density } (\text{distr } (\nu \otimes_M \nu) \text{ real-borel real-real.f}) (\lambda r. (\text{obs } \circ (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}) r / C)))$   
 $= \text{density } (\text{qbs-prob-measure prior}) (\lambda k. \text{obs } k / C)$   
 (is ?lhs = ?rhs)  
 <proof>

**lemma** *program-result-measure':*

$\text{qbs-prob-measure } (\text{qbs-prob-space } (\text{exp-qbs } \mathbb{R}_Q \mathbb{R}_Q, (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}, \text{density } (\text{distr } (\nu \otimes_M \nu) \text{ real-borel real-real.f}) (\lambda r. (\text{obs } \circ (\lambda(s, b) r. s * r + b) \circ \text{real-real.g}) r / C)))$   
 $= \text{distr } (\text{density } (\nu \otimes_M \nu) (\lambda(s, b). \text{obs } (\lambda r. s * r + b) / C)) (\text{qbs-to-measure } (\text{exp-qbs } \mathbb{R}_Q \mathbb{R}_Q)) (\lambda(s, b) r. s * r + b)$   
 <proof>

**end**



## References

- [1] C. Heunen, O. Kammar, S. Staton, and H. Yang. A convenient category for higher-order probability theory. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '17*. IEEE Press, 2017.