

QR Decomposition

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Abstract

In this work we present a formalization of the QR decomposition, an algorithm which decomposes a real matrix A in the product of another two matrices Q and R , where Q is an orthogonal matrix and R is invertible and upper triangular. The algorithm is useful for the least squares problem, i.e. the computation of the best approximation of an unsolvable system of linear equations. As a side-product, the Gram-Schmidt process has also been formalized. A refinement using immutable arrays is presented as well. The development relies, among others, on the AFP entry *Implementing field extensions of the form $\mathbb{Q}[\sqrt{b}]$* by René Thiemann, which allows to execute the algorithm using symbolic computations. Verified code can be generated and executed using floats as well.

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1 Miscellaneous file for the QR algorithm

```

theory Miscellaneous-QR
imports
  Gauss-Jordan.Examples-Gauss-Jordan-Abstract
begin

```

These two lemmas maybe should be in the file *Code-Matrix.thy* of the Gauss-Jordan development.

```

lemma [code abstract]: vec-nth (a - b) = (%i. a$i - b$i) <proof>
lemma [code abstract]: vec-nth (c *_R x) = (%i. c *_R (x$i)) <proof>

```

This lemma maybe should be in the file *Mod-Type.thy* of the Gauss-Jordan development.

lemma *from-nat-le*:
fixes $i::'a::\{\text{mod-type}\}$
assumes i : *to-nat* $i < k$
and k : $k < \text{CARD}('a)$
shows $i < \text{from-nat } k$
 $\langle \text{proof} \rangle$

Some properties about orthogonal matrices.

lemma *orthogonal-mult*:
assumes *orthogonal* a b
shows *orthogonal* $(x *_R a)$ $(y *_R b)$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix-is-orthogonal*:
fixes $A::\text{real}^n{}^n$
assumes o : *orthogonal-matrix* A
shows $(\text{pairwise orthogonal (columns } A))$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix-norm*:
fixes $A::\text{real}^n{}^n$
assumes o : *orthogonal-matrix* A
shows $\text{norm (column } i \text{ } A) = 1$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix-card*:
fixes $A::\text{real}^n{}^n$
assumes o : *orthogonal-matrix* A
shows $\text{card (columns } A) = \text{ncols } A$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix-intro*:
fixes $A::\text{real}^n{}^n$
assumes p : $(\text{pairwise orthogonal (columns } A))$
and n : $\forall i. \text{norm (column } i \text{ } A) = 1$
and c : $\text{card (columns } A) = \text{ncols } A$
shows *orthogonal-matrix* A
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix2*:
fixes $A::\text{real}^n{}^n$
shows *orthogonal-matrix* $A = ((\text{pairwise orthogonal (columns } A)) \wedge (\forall i. \text{norm (column } i \text{ } A) = 1) \wedge (\text{card (columns } A) = \text{ncols } A))$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix'*: *orthogonal-matrix* ($Q :: \text{real } ^n ^n$) \longleftrightarrow $Q ** \text{transpose } Q = \text{mat } 1$
 ⟨*proof*⟩

lemma *orthogonal-matrix-intro2*:
fixes $A :: \text{real } ^n ^n$
assumes p : (*pairwise orthogonal* (rows A))
and n : $\forall i. \text{norm } (\text{row } i \ A) = 1$
and c : $\text{card } (\text{rows } A) = \text{nrows } A$
shows *orthogonal-matrix* A
 ⟨*proof*⟩

lemma *is-basis-imp-full-rank*:
fixes $A :: 'a :: \{\text{field}\} ^{\text{cols}} :: \{\text{mod-type}\} ^{\text{rows}} :: \{\text{mod-type}\}$
assumes b : *is-basis* (columns A)
and c : $\text{card } (\text{columns } A) = \text{ncols } A$
shows $\text{rank } A = \text{ncols } A$
 ⟨*proof*⟩

lemma *card-columns-le-ncols*:
 $\text{card } (\text{columns } A) \leq \text{ncols } A$
 ⟨*proof*⟩

lemma *full-rank-imp-is-basis*:
fixes $A :: 'a :: \{\text{field}\} ^n :: \{\text{mod-type}\} ^n :: \{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $\text{is-basis } (\text{columns } A) \wedge \text{card } (\text{columns } A) = \text{ncols } A$
 ⟨*proof*⟩

lemma *full-rank-imp-is-basis2*:
fixes $A :: 'a :: \{\text{field}\} ^n :: \{\text{mod-type}\} ^m :: \{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $\text{vec.independent } (\text{columns } A) \wedge \text{vec.span } (\text{columns } A) = \text{col-space } A$
 $\wedge \text{card } (\text{columns } A) = \text{ncols } A$
 ⟨*proof*⟩

corollary *full-rank-eq-is-basis*:
fixes $A :: 'a :: \{\text{field}\} ^n :: \{\text{mod-type}\} ^n :: \{\text{mod-type}\}$
shows $(\text{is-basis } (\text{columns } A) \wedge (\text{card } (\text{columns } A) = \text{ncols } A)) = (\text{rank } A = \text{ncols } A)$
 ⟨*proof*⟩

lemma *full-col-rank-imp-independent-columns*:
fixes $A :: 'a :: \{\text{field}\} ^n :: \{\text{mod-type}\} ^m :: \{\text{mod-type}\}$
assumes $\text{rank } A = \text{ncols } A$
shows $\text{vec.independent } (\text{columns } A)$
 ⟨*proof*⟩

lemma *matrix-vector-right-distrib-minus*:
fixes $A::'a::\{\text{ring-1}\}^{\wedge n \wedge m}$
shows $A * v (b - c) = (A * v b) - (A * v c)$
 $\langle \text{proof} \rangle$

lemma *inv-matrix-vector-mul-left*:
assumes $i: \text{invertible } A$
shows $(A * v x = A * v y) = (x=y)$
 $\langle \text{proof} \rangle$

lemma *norm-mult-vec*:
fixes $a::(\text{real}, 'b::\text{finite}) \text{ vec}$
shows $\text{norm } (x \cdot x) = \text{norm } x * \text{norm } x$
 $\langle \text{proof} \rangle$

lemma *norm-equivalence*:
fixes $A::\text{real}^{\wedge n \wedge m}$
shows $((\text{transpose } A) * v (A * v x) = 0) \longleftrightarrow (A * v x = 0)$
 $\langle \text{proof} \rangle$

lemma *invertible-transpose-mult*:
fixes $A::\text{real}^{\wedge \text{cols}::\{\text{mod-type}\} \wedge \text{rows}::\{\text{mod-type}\}}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $\text{invertible } (\text{transpose } A ** A)$
 $\langle \text{proof} \rangle$

lemma *matrix-inv-mult*:
fixes $A::'a::\{\text{semiring-1}\}^{\wedge n \wedge n}$
and $B::'a::\{\text{semiring-1}\}^{\wedge n \wedge n}$
assumes $\text{invertible } A$ **and** $\text{invertible } B$
shows $\text{matrix-inv } (A ** B) = \text{matrix-inv } B ** \text{matrix-inv } A$
 $\langle \text{proof} \rangle$

lemma *invertible-transpose*:
fixes $A::'a::\{\text{field}\}^{\wedge n \wedge n}$
assumes $\text{invertible } A$
shows $\text{invertible } (\text{transpose } A)$
 $\langle \text{proof} \rangle$

The following lemmas are generalizations of some parts of the library. They should be in the file *Generalizations.thy* of the Gauss-Jordan AFP entry.

context *vector-space*

begin

lemma *span-eq*: $(\text{span } S = \text{span } T) = (S \subseteq \text{span } T \wedge T \subseteq \text{span } S)$
 $\langle \text{proof} \rangle$

end

lemma *basis-orthogonal*:

fixes $B :: 'a::\text{real-inner set}$

assumes $fB: \text{finite } B$

shows $\exists C. \text{finite } C \wedge \text{card } C \leq \text{card } B \wedge \text{span } C$
 $= \text{span } B \wedge \text{pairwise orthogonal } C$

(is $\exists C. ?P B C)$

$\langle \text{proof} \rangle$

lemma *op-vec-scaleR*: $(*s) = (*_R)$

$\langle \text{proof} \rangle$

end

2 Projections

theory *Projections*

imports

Miscellaneous-QR

begin

2.1 Definitions of vector projection and projection of a vector onto a set.

definition $\text{proj } v \ u = (v \cdot u / (u \cdot u)) *_R \ u$

definition $\text{proj-onto } a \ S = (\text{sum } (\lambda x. \text{proj } a \ x) \ S)$

2.2 Properties

lemma *proj-onto-sum-rw*:

$\text{sum } (\lambda x. (x \cdot v / (x \cdot x)) *_R \ x) \ A = \text{sum } (\lambda x. (v \cdot x / (x \cdot x)) *_R \ x) \ A$
 $\langle \text{proof} \rangle$

lemma *vector-sub-project-orthogonal-proj*:

fixes $b \ x :: 'a::\text{euclidean-space}$

shows $\text{inner } b \ (x - \text{proj } x \ b) = 0$

$\langle \text{proof} \rangle$

lemma *orthogonal-proj-set*:

assumes $yC: y \in C$ **and** $C: \text{finite } C$ **and** $p: \text{pairwise orthogonal } C$

shows $\text{orthogonal } (a - \text{proj-onto } a \ C) \ y$

$\langle \text{proof} \rangle$

lemma *pairwise-orthogonal-proj-set*:

assumes $C: \text{finite } C$ **and** $p: \text{pairwise orthogonal } C$

shows $\text{pairwise orthogonal } (\text{insert } (a - \text{proj-onto } a \ C) \ C)$

<proof>

2.3 Orthogonal Complement

definition *orthogonal-complement* $W = \{x. \forall y \in W. \text{orthogonal } x \ y\}$

lemma *in-orthogonal-complement-imp-orthogonal*:

assumes $x: y \in S$

and $x \in \text{orthogonal-complement } S$

shows $\text{orthogonal } x \ y$

<proof>

lemma *subspace-orthogonal-complement*: *subspace (orthogonal-complement W)*

<proof>

lemma *orthogonal-complement-mono*:

assumes $A\text{-in-}B: A \subseteq B$

shows *orthogonal-complement B* \subseteq *orthogonal-complement A*

<proof>

lemma *B-in-orthogonal-complement-of-orthogonal-complement*:

shows $B \subseteq \text{orthogonal-complement (orthogonal-complement B)}$

<proof>

lemma *pythagorean-theorem-norm*:

assumes $o: \text{orthogonal } x \ y$

shows $\text{norm } (x+y)^2 = \text{norm } x^2 + \text{norm } y^2$

<proof>

lemma *in-orthogonal-complement-basis*:

fixes $B::'a::\{\text{euclidean-space}\}$ *set*

assumes S : *subspace S*

and $\text{ind-}B$: *independent B*

and B : $B \subseteq S$

and $\text{span-}B$: $S \subseteq \text{span } B$

shows $(v \in \text{orthogonal-complement } S) = (\forall a \in B. \text{orthogonal } a \ v)$

<proof>

See https://people.math.osu.edu/husen.1/teaching/571/least_squares.pdf

Part 1 of the Theorem 1.7 in the previous website, but the proof has been carried out in other way.

lemma *v-minus-p-orthogonal-complement*:

fixes $X::'a::\{\text{euclidean-space}\}$ *set*

assumes $\text{subspace-}S$: *subspace S*

and $\text{ind-}X$: *independent X*

and X : $X \subseteq S$

and $\text{span-}X$: $S \subseteq \text{span } X$

and o : pairwise orthogonal X
shows $(v - \text{proj-onto } v X) \in \text{orthogonal-complement } S$
 $\langle \text{proof} \rangle$

Part 2 of the Theorem 1.7 in the previous website.

lemma *UNIV-orthogonal-complement-decomposition*:

fixes $S::'a::\{\text{euclidean-space}\}$ set

assumes s : subspace S

shows $\text{UNIV} = S + (\text{orthogonal-complement } S)$

$\langle \text{proof} \rangle$

2.4 Normalization of vectors

definition *normalize*

where $\text{normalize } x = ((1/\text{norm } x) *_{\mathbb{R}} x)$

definition *normalize-set-of-vec*

where $\text{normalize-set-of-vec } X = \text{normalize}' X$

lemma *norm-normalize*:

assumes $x \neq 0$

shows $\text{norm } (\text{normalize } x) = 1$

$\langle \text{proof} \rangle$

lemma *normalize-0*: $(\text{normalize } x = 0) = (x = 0)$

$\langle \text{proof} \rangle$

lemma *norm-normalize-set-of-vec*:

assumes $x \neq 0$

and $x \in \text{normalize-set-of-vec } X$

shows $\text{norm } x = 1$

$\langle \text{proof} \rangle$

end

3 The Gram-Schmidt algorithm

theory *Gram-Schmidt*

imports

Miscellaneous-QR

Projections

begin

3.1 Gram-Schmidt algorithm

The algorithm is used to orthogonalise a set of vectors. The Gram-Schmidt process takes a set of vectors S and generates another orthogonal set that spans the same subspace as S .

We present three ways to compute the Gram-Schmidt algorithm.

1. The first one has been developed thinking about the simplicity of its formalisation. Given a list of vectors, the output is another list of orthogonal vectors with the same span. Such a list is constructed following the Gram-Schmidt process presented in any book, but in the reverse order (starting the process from the last element of the input list).
2. Based on previous formalization, another function has been defined to compute the process of the Gram-Schmidt algorithm in the natural order (starting from the first element of the input list).
3. The third way has as input and output a matrix. The algorithm is applied to the columns of a matrix, obtaining a matrix whose columns are orthogonal and where the column space is kept. This will be a previous step to compute the QR decomposition.

Every function can be executed with arbitrary precision (using rational numbers).

3.1.1 First way

definition *Gram-Schmidt-step* :: ('a::{real-inner} ^b) => ('a ^b) list => ('a ^b) list

where *Gram-Schmidt-step* a ys = ys @ [(a - proj-onto a (set ys))]

definition *Gram-Schmidt* xs = foldr *Gram-Schmidt-step* xs []

lemma *Gram-Schmidt-cons*:

Gram-Schmidt (a#xs) = *Gram-Schmidt-step* a (*Gram-Schmidt* xs)

<proof>

lemma *basis-orthogonal'*:

fixes xs::('a::{real-inner} ^b) list

shows length (*Gram-Schmidt* xs) = length (xs) ∧
span (set (*Gram-Schmidt* xs)) = span (set xs) ∧
pairwise orthogonal (set (*Gram-Schmidt* xs))

<proof>

lemma *card-Gram-Schmidt*:

fixes xs::('a::{real-inner} ^b) list

assumes distinct xs

shows card(set (*Gram-Schmidt* xs)) ≤ card (set (xs))

<proof>

lemma *orthogonal-basis-exists*:

fixes V :: (real ^b) list

assumes B: is-basis (set V)

and d : *distinct* V
shows $vec.independent$ (set ($Gram-Schmidt$ V)) \wedge (set V) \subseteq $vec.span$ (set ($Gram-Schmidt$ V))
 \wedge ($card$ (set ($Gram-Schmidt$ V)) = $vec.dim$ (set V)) \wedge *pairwise orthogonal* (set ($Gram-Schmidt$ V))
 $\langle proof \rangle$

corollary *orthogonal-basis-exists'*:

fixes V :: ($real$ ^{b}) *list*
assumes B : *is-basis* (set V)
and d : *distinct* V
shows *is-basis* (set ($Gram-Schmidt$ V))
 \wedge *distinct* ($Gram-Schmidt$ V) \wedge *pairwise orthogonal* (set ($Gram-Schmidt$ V))
 $\langle proof \rangle$

3.1.2 Second way

This definition applies the Gram Schmidt process starting from the first element of the list.

definition $Gram-Schmidt2$ $xs = Gram-Schmidt$ (rev xs)

lemma *basis-orthogonal2*:

fixes xs ::(a ::($real$ ^{b}) *list*)
shows $length$ ($Gram-Schmidt2$ xs) = $length$ (xs)
 \wedge $span$ (set ($Gram-Schmidt2$ xs)) = $span$ (set xs)
 \wedge *pairwise orthogonal* (set ($Gram-Schmidt2$ xs))
 $\langle proof \rangle$

lemma *card-Gram-Schmidt2*:

fixes xs ::(a ::($real$ ^{b}) *list*)
assumes *distinct* xs
shows $card$ (set ($Gram-Schmidt2$ xs)) \leq $card$ (set (xs))
 $\langle proof \rangle$

lemma *orthogonal-basis-exists2*:

fixes V :: ($real$ ^{b}) *list*
assumes B : *is-basis* (set V)
and d : *distinct* V
shows $vec.independent$ (set ($Gram-Schmidt2$ V)) \wedge (set V) \subseteq $vec.span$ (set ($Gram-Schmidt2$ V))
 \wedge ($card$ (set ($Gram-Schmidt2$ V)) = $vec.dim$ (set V)) \wedge *pairwise orthogonal* (set ($Gram-Schmidt2$ V))
 $\langle proof \rangle$

3.1.3 Third way

The following definitions applies the Gram Schmidt process in the columns of a given matrix. It is previous step to the computation of the QR decomposition.

definition *Gram-Schmidt-column-k* :: 'a::{real-inner} ^rows ^cols::mod-type => nat

=> 'a ^rows ^cols::mod-type

where *Gram-Schmidt-column-k* A k

= (χ a. (χ b. (if b = from-nat k

then (column b A - (proj-onto (column b A) {column i A | i. i < b}))

else (column b A)) \$ a))

definition *Gram-Schmidt-upt-k* A k = foldl *Gram-Schmidt-column-k* A [0..<(Suc k)]

definition *Gram-Schmidt-matrix* A = *Gram-Schmidt-upt-k* A (ncols A - 1)

Some definitions and lemmas in order to get execution.

definition *Gram-Schmidt-column-k-row* A k a =

vec-lambda(λb. (if b = from-nat k then

(column b A - (∑ x∈{column i A | i. i < b}. ((column b A) · x / (x · x)) *_R x))

else (column b A)) \$ a)

lemma *Gram-Schmidt-column-k-row-code*[code abstract]:

vec-nth (*Gram-Schmidt-column-k-row* A k a)

= (%b. (if b = from-nat k

then (column b A - (∑ x∈{column i A | i. i < b}. ((column b A) · x / (x · x)) *_R x))

else (column b A)) \$ a)

<proof>

lemma *Gram-Schmidt-column-k-code*[code abstract]:

vec-nth (*Gram-Schmidt-column-k* A k) = *Gram-Schmidt-column-k-row* A k

<proof>

Proofs

lemma *Gram-Schmidt-upt-k-suc*:

Gram-Schmidt-upt-k A (Suc k) = (*Gram-Schmidt-column-k* (*Gram-Schmidt-upt-k* A k) (Suc k))

<proof>

lemma *column-Gram-Schmidt-upt-k-preserves*:

fixes A::'a::{real-inner} ^rows ^cols::mod-type

assumes *i-less-suc*: to-nat i < (Suc k)

and *suc-less-card*: Suc k < CARD ('cols)

shows *column i* (*Gram-Schmidt-upt-k* A (Suc k)) = *column i* (*Gram-Schmidt-upt-k* A k)

<proof>

lemma *column-set-Gram-Schmidt-upt-k*:
fixes $A::'a::\{\text{real-inner}\}^{\wedge'}\text{cols}::\{\text{mod-type}\}^{\wedge'}\text{rows}$
assumes $k: \text{Suc } k < \text{CARD } ('cols)$
shows $\{\text{column } i \text{ (Gram-Schmidt-upt-k } A \text{ (Suc } k)) \mid i. \text{to-nat } i \leq (\text{Suc } k)\} =$
 $\{\text{column } i \text{ (Gram-Schmidt-upt-k } A \text{ } k) \mid i. \text{to-nat } i \leq k\} \cup \{\text{column (from-nat (Suc } k)) \text{ (Gram-Schmidt-upt-k } A \text{ } k)}\}$
 $- (\sum x \in \{\text{column } i \text{ (Gram-Schmidt-upt-k } A \text{ } k) \mid i. \text{to-nat } i \leq k\}. (x \cdot (\text{column (from-nat (Suc } k)) \text{ (Gram-Schmidt-upt-k } A \text{ } k)) / (x \cdot x)) *_R x)$
 $\langle \text{proof} \rangle$

lemma *orthogonal-Gram-Schmidt-upt-k*:
assumes $s: k < \text{ncols } A$
shows *pairwise orthogonal* $(\{\text{column } i \text{ (Gram-Schmidt-upt-k } A \text{ } k) \mid i. \text{to-nat } i \leq k\})$
 $\langle \text{proof} \rangle$

lemma *columns-Gram-Schmidt-matrix-rw*:
 $\{\text{column } i \text{ (Gram-Schmidt-matrix } A) \mid i. i \in \text{UNIV}\}$
 $= \{\text{column } i \text{ (Gram-Schmidt-upt-k } A \text{ (ncols } A - 1)) \mid i. \text{to-nat } i \leq (\text{ncols } A - 1)\}$
 $\langle \text{proof} \rangle$

corollary *orthogonal-Gram-Schmidt-matrix*:
shows *pairwise orthogonal* $(\{\text{column } i \text{ (Gram-Schmidt-matrix } A) \mid i. i \in \text{UNIV}\})$
 $\langle \text{proof} \rangle$

corollary *orthogonal-Gram-Schmidt-matrix2*:
shows *pairwise orthogonal* $(\text{columns (Gram-Schmidt-matrix } A))$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-column-k*:
fixes $A::'a::\{\text{real-inner}\}^{\wedge'}n::\{\text{mod-type}\}^{\wedge'}m::\{\text{mod-type}\}$
shows $\text{column } k \text{ (Gram-Schmidt-column-k } A \text{ (to-nat } k)) =$
 $(\text{column } k \text{ } A) - (\sum x \in \{\text{column } i \text{ } A \mid i. i < k\}. (x \cdot (\text{column } k \text{ } A) / (x \cdot x)) *_R x)$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-column-k'*:
fixes $A::'a::\{\text{real-inner}\}^{\wedge'}n::\{\text{mod-type}\}^{\wedge'}m::\{\text{mod-type}\}$
assumes $i\text{-not-}k: i \neq k$
shows $\text{column } i \text{ (Gram-Schmidt-column-k } A \text{ (to-nat } k)) = (\text{column } i \text{ } A)$
 $\langle \text{proof} \rangle$

definition $\text{cols-upt-k } A \text{ } k = \{\text{column } i \text{ } A \mid i. i \leq \text{from-nat } k\}$

lemma *cols-upt-k-insert*:

fixes $A::'a\ ^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k: (\text{Suc } k) < \text{ncols } A$
shows $\text{cols-upt-}k\ A\ (\text{Suc } k) = (\text{insert } (\text{column } (\text{from-nat } (\text{Suc } k))\ A)\ (\text{cols-upt-}k\ A\ k))$
<proof>

lemma *columns-eq-cols-upt-k*:
fixes $A::'a\ ^\text{cols}::\{\text{mod-type}\}^\text{rows}::\{\text{mod-type}\}$
shows $\text{cols-upt-}k\ A\ (\text{ncols } A - 1) = \text{columns } A$
<proof>

lemma *span-cols-upt-k-Gram-Schmidt-column-k*:
fixes $A::'a::\{\text{real-inner}\}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k < \text{ncols } A$
and $j < \text{ncols } A$
shows $\text{span } (\text{cols-upt-}k\ A\ k) = \text{span } (\text{cols-upt-}k\ (\text{Gram-Schmidt-column-}k\ A\ j)\ k)$
<proof>

corollary *span-Gram-Schmidt-column-k*:
fixes $A::'a::\{\text{real-inner}\}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k < \text{ncols } A$
shows $\text{span } (\text{columns } A) = \text{span } (\text{columns } (\text{Gram-Schmidt-column-}k\ A\ k))$
<proof>

corollary *span-Gram-Schmidt-upt-k*:
fixes $A::'a::\{\text{real-inner}\}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k < \text{ncols } A$
shows $\text{span } (\text{columns } A) = \text{span } (\text{columns } (\text{Gram-Schmidt-upt-}k\ A\ k))$
<proof>

corollary *span-Gram-Schmidt-matrix*:
fixes $A::'a::\{\text{real-inner}\}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $\text{span } (\text{columns } A) = \text{span } (\text{columns } (\text{Gram-Schmidt-matrix } A))$
<proof>

lemma *is-basis-columns-Gram-Schmidt-matrix*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $b: \text{is-basis } (\text{columns } A)$
and $c: \text{card } (\text{columns } A) = \text{ncols } A$
shows $\text{is-basis } (\text{columns } (\text{Gram-Schmidt-matrix } A))$
 $\wedge \text{card } (\text{columns } (\text{Gram-Schmidt-matrix } A)) = \text{ncols } A$
<proof>

From here on, we present some lemmas that will be useful for the formalisation of the QR decomposition.

lemma *column-gr-k-Gram-Schmidt-upt*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i > k$
and $i < \text{ncols } A$
shows $\text{column } (\text{from-nat } i) (\text{Gram-Schmidt-upt-k } A \ k) = \text{column } (\text{from-nat } i) A$
 $\langle \text{proof} \rangle$

lemma *columns-Gram-Schmidt-upt-k-rw*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k: \text{Suc } k < \text{ncols } A$
shows $\{\text{column } i (\text{Gram-Schmidt-upt-k } A (\text{Suc } k)) \mid i. i < \text{from-nat } (\text{Suc } k)\}$
 $= \{\text{column } i (\text{Gram-Schmidt-upt-k } A \ k) \mid i. i < \text{from-nat } (\text{Suc } k)\}$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-upt-k*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $k < \text{ncols } A$
shows $\text{column } (\text{from-nat } k) (\text{Gram-Schmidt-upt-k } A \ k) =$
 $(\text{column } (\text{from-nat } k) A) - (\sum x \in \{\text{column } i (\text{Gram-Schmidt-upt-k } A \ k) \mid i. i <$
 $(\text{from-nat } k)\}. (x \cdot (\text{column } (\text{from-nat } k) A) / (x \cdot x)) *_R x)$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-upt-k-preserves2*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $a \leq (\text{from-nat } i)$
and $i \leq j$
and $j < \text{ncols } A$
shows $\text{column } a (\text{Gram-Schmidt-upt-k } A \ i) = \text{column } a (\text{Gram-Schmidt-upt-k } A$
 $j)$
 $\langle \text{proof} \rangle$

lemma *set-columns-Gram-Schmidt-matrix*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $\{\text{column } i (\text{Gram-Schmidt-matrix } A) \mid i. i < k\} = \{\text{column } i (\text{Gram-Schmidt-upt-k}$
 $A (\text{to-nat } k)) \mid i. i < k\}$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-matrix*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $\text{column } k (\text{Gram-Schmidt-matrix } A)$
 $= (\text{column } k A) - (\sum x \in \{\text{column } i (\text{Gram-Schmidt-matrix } A) \mid i. i < k\}. (x \cdot$
 $(\text{column } k A) / (x \cdot x)) *_R x)$

<proof>

corollary *column-Gram-Schmidt-matrix2:*

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $(\text{column } k \ A) = \text{column } k \ (\text{Gram-Schmidt-matrix } A)$
 $+ (\sum_{x \in \{\text{column } i \ (\text{Gram-Schmidt-matrix } A) \mid i. i < k\}} (x \cdot (\text{column } k \ A) / (x \cdot x))) *_R x)$
<proof>

lemma *independent-columns-Gram-Schmidt-matrix:*

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $b: \text{vec.independent} \ (\text{columns } A)$
and $c: \text{card} \ (\text{columns } A) = \text{ncols } A$
shows $\text{vec.independent} \ (\text{columns} \ (\text{Gram-Schmidt-matrix } A)) \wedge \text{card} \ (\text{columns} \ (\text{Gram-Schmidt-matrix } A)) = \text{ncols } A$
<proof>

lemma *column-eq-Gram-Schmidt-matrix:*

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
and $c: \text{column } i \ (\text{Gram-Schmidt-matrix } A) = \text{column } ia \ (\text{Gram-Schmidt-matrix } A)$
shows $i = ia$
<proof>

lemma *scaleR-columns-Gram-Schmidt-matrix:*

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i \neq j$
and $\text{rank } A = \text{ncols } A$
shows $\text{column } j \ (\text{Gram-Schmidt-matrix } A) \cdot \text{column } i \ (\text{Gram-Schmidt-matrix } A) = 0$
<proof>

3.1.4 Examples of execution

Code lemma

lemmas *Gram-Schmidt-step-def*[*unfolded proj-onto-def proj-def*[*abs-def*],*code*]

value $let \ a = \text{map} \ (\text{list-to-vec}::\text{real list} \Rightarrow \text{real}^4) \ [[4, -2, -1, 2],$
 $[-6, 3, 4, -8], [5, -5, -3, -4]] \ in$
 $\text{map} \ \text{vec-to-list} \ (\text{Gram-Schmidt} \ a)$

value $let \ a = \text{map} \ (\text{list-to-vec}::\text{real list} \Rightarrow \text{real}^4) \ [[4, -2, -1, 2],$
 $[-6, 3, 4, -8], [5, -5, -3, -4]] \ in$
 $\text{map} \ \text{vec-to-list} \ (\text{Gram-Schmidt2} \ a)$


```

value let A = list-of-list-to-matrix [[4,-2,-1,2],
  [-6,3,4,-8], [5,-5,-3,-4]]::real^4^3 in
  matrix-to-list-of-list (Gram-Schmidt-matrix A)

```

```

end

```

4 QR Decomposition

```

theory QR-Decomposition
imports Gram-Schmidt
begin

```

4.1 The QR Decomposition of a matrix

First of all, it's worth noting what an orthogonal matrix is. In linear algebra, an orthogonal matrix is a square matrix with real entries whose columns and rows are orthogonal unit vectors.

Although in some texts the QR decomposition is presented over square matrices, it can be applied to any matrix. There are some variants of the algorithm, depending on the properties that the output matrices satisfy (see for instance, http://inst.eecs.berkeley.edu/~ee127a/book/login/1_mats_qr.html). We present two of them below.

Let A be a matrix with m rows and n columns (A is $m \times n$).

Case 1: Starting with a matrix whose column rank is maximum. We can define the QR decomposition to obtain:

- $A = Q ** R$.
- Q has m rows and n columns. Its columns are orthogonal unit vectors and *Finite-Cartesian-Product.transpose* $Q * Q = \text{mat}(1::'a)$. In addition, if A is a square matrix, then Q will be an orthonormal matrix.
- R is $n \times n$, invertible and upper triangular.

Case 2: The called full QR decomposition. We can obtain:

- $A = Q ** R$
- Q is an orthogonal matrix (Q is $m \times m$).
- R is $m \times n$ and upper triangular, but it isn't invertible.

We have decided to formalise the first one, because it's the only useful for solving the linear least squares problem (<http://math.mit.edu/linearalgebra/ila0403.pdf>).

If we have an unsolvable system $A *v x = b$, we can try to find an approximate solution. A plausible choice (not the only one) is to seek an x with the property that $\|A ** x - y\|$ (the magnitude of the error) is as small as possible. That x is the least squares approximation.

We will demonstrate that the best approximation (the solution for the linear least squares problem) is the x that satisfies:

$$(transpose A) ** A *v x = (transpose A) *v b$$

Now we want to compute that x .

If we are working with the first case, A can be substituted by $Q**R$ and then obtain the solution of the least squares approximation by means of the QR decomposition:

$$x = (inverse R)**(transpose Q) *v b$$

On the contrary, if we are working with the second case after substituting A by $Q**R$ we obtain:

$$(transpose R) ** R *v x = (transpose R) ** (transpose Q) *v b$$

But the R matrix is not invertible (so neither is $transpose R$). The left part of the equation $(transpose R) ** R$ is not going to be an upper triangular matrix, so it can't either be solved using backward-substitution.

4.1.1 Divide a vector by its norm

An orthogonal matrix is a matrix whose rows (and columns) are orthonormal vectors. So, in order to obtain the QR decomposition, we have to normalise (divide by the norm) the vectors obtained with the Gram-Schmidt algorithm.

definition *divide-by-norm* $A = (\chi a b. normalize (column b A) \$ a)$

Properties

lemma *norm-column-divide-by-norm*:

fixes $A::'a::\{real-inner\}^{\sim}cols^{\sim}rows$

assumes $a: column a A \neq 0$

shows $norm (column a (divide-by-norm A)) = 1$

<proof>

lemma *span-columns-divide-by-norm*:

shows $span (columns A) = span (columns (divide-by-norm A))$

<proof>

Code lemmas

definition *divide-by-norm-row* $A a = vec-lambda(\% b. ((1 / norm (column b A)) *_R column b A) \$ a)$

lemma *divide-by-norm-row-code**[code abstract]*:

$vec-nth (divide-by-norm-row A a) = (\% b. ((1 / norm (column b A)) *_R column b A) \$ a)$

<proof>

lemma *divide-by-norm-code* [*code abstract*]:
 $vec_nth (divide_by_norm A) = divide_by_norm_row A$
<proof>

4.1.2 The QR Decomposition

The QR decomposition. Given a real matrix A , the algorithm will return a pair (Q, R) where Q is a matrix whose columns are orthogonal unit vectors, R is upper triangular and $A = Q ** R$.

definition *QR-decomposition* $A = (let Q = divide_by_norm (Gram_Schmidt_matrix A) in (Q, (transpose Q) ** A))$

lemma *is-basis-columns-fst-QR-decomposition*:
fixes $A::real^{n::\{mod-type\}}^{m::\{mod-type\}}$
assumes $b: is_basis (columns A)$
and $c: card (columns A) = ncols A$
shows $is_basis (columns (fst (QR_decomposition A)))$
 $\wedge card (columns (fst (QR_decomposition A))) = ncols A$
<proof>

lemma *orthogonal-fst-QR-decomposition*:
shows $pairwise_orthogonal (columns (fst (QR_decomposition A)))$
<proof>

lemma *qk-uk-norm*:
 $(1 / (norm (column k ((Gram_Schmidt_matrix A)))))) *_R (column k ((Gram_Schmidt_matrix A)))$
 $= column k (fst(QR_decomposition A))$
<proof>

lemma *norm-columns-fst-QR-decomposition*:
fixes $A::real^{n::\{mod-type\}}^{m::\{mod-type\}}$
assumes $rank A = ncols A$
shows $norm (column i (fst (QR_decomposition A))) = 1$
<proof>

corollary *span-fst-QR-decomposition*:
fixes $A::real^{n::\{mod-type\}}^{m::\{mod-type\}}$
shows $vec.span (columns A) = vec.span (columns (fst (QR_decomposition A)))$
<proof>

corollary *col-space-QR-decomposition*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $\text{col-space } A = \text{col-space } (\text{fst } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *independent-columns-fst-QR-decomposition:*
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $b: \text{vec.independent } (\text{columns } A)$
and $c: \text{card } (\text{columns } A) = \text{ncols } A$
shows $\text{vec.independent } (\text{columns } (\text{fst } (\text{QR-decomposition } A)))$
 $\wedge \text{card } (\text{columns } (\text{fst } (\text{QR-decomposition } A))) = \text{ncols } A$
 $\langle \text{proof} \rangle$

lemma *orthogonal-matrix-fst-QR-decomposition:*
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $\text{transpose } (\text{fst } (\text{QR-decomposition } A)) ** (\text{fst } (\text{QR-decomposition } A)) =$
 $\text{mat } 1$
 $\langle \text{proof} \rangle$

corollary *orthogonal-matrix-fst-QR-decomposition':*
fixes $A::\text{real}^n::\{\text{mod-type}\}^n::\{\text{mod-type}\}$
assumes $\text{rank } A = \text{ncols } A$
shows $\text{orthogonal-matrix } (\text{fst } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *column-eq-fst-QR-decomposition:*
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
and $c: \text{column } i (\text{fst } (\text{QR-decomposition } A)) = \text{column } ia (\text{fst } (\text{QR-decomposition } A))$
shows $i = ia$
 $\langle \text{proof} \rangle$

corollary *column-QR-decomposition:*
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $\text{column } k ((\text{Gram-Schmidt-matrix } A))$
 $= (\text{column } k A) - (\sum_{x \in \{\text{column } i (\text{fst } (\text{QR-decomposition } A)) \mid i. i < k\}} (x \cdot (\text{column } k A) / (x \cdot x)) *_{\mathbb{R}} x)$
 $\langle \text{proof} \rangle$

lemma *column-QR-decomposition':*
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(\text{column } k A) = \text{column } k ((\text{Gram-Schmidt-matrix } A))$
 $+ (\sum_{x \in \{\text{column } i (\text{fst } (\text{QR-decomposition } A)) \mid i. i < k\}} (x \cdot (\text{column } k A) / (x$

• x) $*_R x$)
 ⟨proof⟩

lemma *norm-uk-eq*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $\text{norm } (\text{column } k \text{ } ((\text{Gram-Schmidt-matrix } A))) = ((\text{column } k \text{ } (\text{fst } (\text{QR-decomposition } A)))) \cdot (\text{column } k \text{ } A)$
 ⟨proof⟩

corollary *column-QR-decomposition2*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(\text{column } k \text{ } A)$
 $= (\sum_{x \in \{\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A)) \mid i. i \leq k\}} (x \cdot (\text{column } k \text{ } A))) *_R x$
 ⟨proof⟩

lemma *orthogonal-columns-fst-QR-decomposition*:

assumes $i\text{-not-ia}: (\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A))) \neq (\text{column } ia \text{ } (\text{fst } (\text{QR-decomposition } A)))$
shows $(\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A))) \cdot (\text{column } ia \text{ } (\text{fst } (\text{QR-decomposition } A))) = 0$
 ⟨proof⟩

lemma *scaler-column-fst-QR-decomposition*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i: i > j$
and $r: \text{rank } A = \text{ncols } A$
shows $\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A)) \cdot \text{column } j \text{ } A = 0$
 ⟨proof⟩

lemma *R-Qi-Aj*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
shows $(\text{snd } (\text{QR-decomposition } A)) \$ i \$ j = \text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A)) \cdot \text{column } j \text{ } A$
 ⟨proof⟩

lemma *sums-columns-Q-0*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(\sum_{x \in \{\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A)) \mid i. i > b\}} x \cdot \text{column } b \text{ } A * x) \$ a = 0$
 ⟨proof⟩

lemma *QR-decomposition-mult*:

fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$

assumes r : $\text{rank } A = \text{ncols } A$
shows $A = (\text{fst } (\text{QR-decomposition } A)) ** (\text{snd } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *upper-triangular-snd-QR-decomposition*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $\text{upper-triangular } (\text{snd } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *upper-triangular-invertible*:
fixes $A :: \text{real}^n::\{\text{finite,wellorder}\}^n::\{\text{finite,wellorder}\}$
assumes u : $\text{upper-triangular } A$
and d : $\forall i. A \$ i \$ i \neq 0$
shows $\text{invertible } A$
 $\langle \text{proof} \rangle$

lemma *invertible-snd-QR-decomposition*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $\text{invertible } (\text{snd } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *QR-decomposition*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $A = \text{fst } (\text{QR-decomposition } A) ** \text{snd } (\text{QR-decomposition } A) \wedge$
 $\text{pairwise orthogonal } (\text{columns } (\text{fst } (\text{QR-decomposition } A))) \wedge$
 $(\forall i. \text{norm } (\text{column } i \text{ } (\text{fst } (\text{QR-decomposition } A))) = 1) \wedge$
 $(\text{transpose } (\text{fst } (\text{QR-decomposition } A))) ** (\text{fst } (\text{QR-decomposition } A)) = \text{mat } 1$
 \wedge
 $\text{vec.independent } (\text{columns } (\text{fst } (\text{QR-decomposition } A))) \wedge$
 $\text{col-space } A = \text{col-space } (\text{fst } (\text{QR-decomposition } A)) \wedge$
 $\text{card } (\text{columns } A) = \text{card } (\text{columns } (\text{fst } (\text{QR-decomposition } A))) \wedge$
 $\text{invertible } (\text{snd } (\text{QR-decomposition } A)) \wedge$
 $\text{upper-triangular } (\text{snd } (\text{QR-decomposition } A))$
 $\langle \text{proof} \rangle$

lemma *QR-decomposition-square*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^n::\{\text{mod-type}\}$
assumes r : $\text{rank } A = \text{ncols } A$
shows $A = \text{fst } (\text{QR-decomposition } A) ** \text{snd } (\text{QR-decomposition } A) \wedge$
 $\text{orthogonal-matrix } (\text{fst } (\text{QR-decomposition } A)) \wedge$
 $\text{upper-triangular } (\text{snd } (\text{QR-decomposition } A)) \wedge$
 $\text{invertible } (\text{snd } (\text{QR-decomposition } A)) \wedge$

pairwise orthogonal (columns (fst (QR-decomposition A))) \wedge
($\forall i. \text{norm (column } i \text{ (fst (QR-decomposition A)))} = 1$) \wedge
vec.independent (columns (fst (QR-decomposition A))) \wedge
col-space A = col-space (fst (QR-decomposition A)) \wedge
card (columns A) = card (columns (fst (QR-decomposition A)))
 <proof>

QR for computing determinants

lemma *det-QR-decomposition:*

fixes $A::\text{real}^{\sim n}::\{\text{mod-type}\}^{\sim n}::\{\text{mod-type}\}$

assumes $r: \text{rank } A = \text{ncols } A$

shows $|\det A| = |(\text{prod } (\lambda i. \text{snd}(\text{QR-decomposition } A)\$i\$i) (\text{UNIV}::'n \text{ set}))|$

<proof>

end

5 Least Squares Approximation

theory *Least-Squares-Approximation*

imports

QR-Decomposition

begin

5.1 Second part of the Fundamental Theorem of Linear Algebra

See http://en.wikipedia.org/wiki/Fundamental_theorem_of_linear_algebra

lemma *null-space-orthogonal-complement-row-space:*

fixes $A::\text{real}^{\sim \text{cols}} \sim \text{rows}::\{\text{finite,wellorder}\}$

shows $\text{null-space } A = \text{orthogonal-complement (row-space } A)$

<proof>

lemma *left-null-space-orthogonal-complement-col-space:*

fixes $A::\text{real}^{\sim \text{cols}}::\{\text{finite,wellorder}\} \sim \text{rows}$

shows $\text{left-null-space } A = \text{orthogonal-complement (col-space } A)$

<proof>

5.2 Least Squares Approximation

See https://people.math.osu.edu/husen.1/teaching/571/least_squares.pdf

Part 3 of the Theorem 1.7 in the previous website.

lemma *least-squares-approximation:*

fixes $X::'a::\{\text{euclidean-space}\} \text{ set}$

assumes $\text{subspace-}S: \text{subspace } S$

and $\text{ind-}X: \text{independent } X$

and $X: X \subseteq S$

and $\text{span-}X: S \subseteq \text{span } X$
and $o: \text{pairwise orthogonal } X$
and $\text{not-eq}: \text{proj-onto } v \ X \neq y$
and $y: y \in S$
shows $\text{norm } (v - \text{proj-onto } v \ X) < \text{norm } (v - y)$
 $\langle \text{proof} \rangle$

lemma *least-squares-approximation2*:
fixes $S::'a::\{\text{euclidean-space}\}$ *set*
assumes $\text{subspace-}S: \text{subspace } S$
and $y: y \in S$
shows $\exists p \in S. \text{norm } (v - p) \leq \text{norm } (v - y) \wedge (v-p) \in \text{orthogonal-complement } S$
 $\langle \text{proof} \rangle$

corollary *least-squares-approximation3*:
fixes $S::'a::\{\text{euclidean-space}\}$ *set*
assumes $\text{subspace-}S: \text{subspace } S$
shows $\exists p \in S. \forall y \in S. \text{norm } (v - p) \leq \text{norm } (v - y) \wedge (v-p) \in \text{orthogonal-complement } S$
 $\langle \text{proof} \rangle$

lemma *norm-least-squares*:
fixes $A::\text{real}^{\wedge} \text{cols}::\{\text{finite,wellorder}\}^{\wedge} \text{rows}$
shows $\exists x. \forall x'. \text{norm } (b - A * v \ x) \leq \text{norm } (b - A * v \ x')$
 $\langle \text{proof} \rangle$

definition *set-least-squares-approximation* $A \ b = \{x. \forall y. \text{norm } (b - A * v \ x) \leq \text{norm } (b - A * v \ y)\}$

corollary *least-squares-approximation4*:
fixes $S::'a::\{\text{euclidean-space}\}$ *set*
assumes $\text{subspace-}S: \text{subspace } S$
shows $\exists ! p \in S. \forall y \in S - \{p\}. \text{norm } (v - p) < \text{norm } (v - y)$
 $\langle \text{proof} \rangle$

corollary *least-squares-approximation4'*:
fixes $S::'a::\{\text{euclidean-space}\}$ *set*
assumes $\text{subspace-}S: \text{subspace } S$
shows $\exists ! p \in S. \forall y \in S. \text{norm } (v - p) \leq \text{norm } (v - y)$
 $\langle \text{proof} \rangle$

corollary *least-squares-approximation5*:
fixes $S::'a::\{\text{euclidean-space}\}$ *set*
assumes $\text{subspace-}S: \text{subspace } S$
shows $\exists ! p \in S. \forall y \in S - \{p\}. \text{norm } (v - p) < \text{norm } (v - y) \wedge v-p \in \text{orthogonal-complement } S$

<proof>

corollary *least-squares-approximation5'*:

fixes $S::'a::\{\text{euclidean-space}\}$ *set*

assumes *subspace-S: subspace S*

shows $\exists!p \in S. \forall y \in S. \text{norm } (v - p) \leq \text{norm } (v - y) \wedge v - p \in \text{orthogonal-complement } S$

<proof>

corollary *least-squares-approximation6*:

fixes $S::'a::\{\text{euclidean-space}\}$ *set*

assumes *subspace-S: subspace S*

and $p \in S$

and $\forall y \in S. \text{norm } (v - p) \leq \text{norm } (v - y)$

shows $v - p \in \text{orthogonal-complement } S$

<proof>

corollary *least-squares-approximation7*:

fixes $S::'a::\{\text{euclidean-space}\}$ *set*

assumes *subspace-S: subspace S*

and $v - p \in \text{orthogonal-complement } S$

and $p \in S$

and $y \in S$

shows $\text{norm } (v - p) \leq \text{norm } (v - y)$

<proof>

lemma *in-set-least-squares-approximation*:

fixes $A::\text{real}^{\text{cols}}::\{\text{finite, wellorder}\}^{\text{rows}}$

assumes $o: A * v \ x - b \in \text{orthogonal-complement } (\text{col-space } A)$

shows $(x \in \text{set-least-squares-approximation } A \ b)$

<proof>

lemma *in-set-least-squares-approximation-eq*:

fixes $A::\text{real}^{\text{cols}}::\{\text{finite, wellorder}\}^{\text{rows}}$

shows $(x \in \text{set-least-squares-approximation } A \ b) = (\text{transpose } A ** A * v \ x = \text{transpose } A * v \ b)$

<proof>

lemma *in-set-least-squares-approximation-eq-full-rank*:

fixes $A::\text{real}^{\text{cols}}::\text{mod-type}^{\text{rows}}::\text{mod-type}$

assumes $r: \text{rank } A = \text{ncols } A$

shows $(x \in \text{set-least-squares-approximation } A \ b) = (x = \text{matrix-inv } (\text{transpose } A ** A) ** \text{transpose } A * v \ b)$

<proof>

lemma *in-set-least-squares-approximation-eq-full-rank-QR*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(x \in \text{set-least-squares-approximation } A \ b) = ((\text{snd } (\text{QR-decomposition } A))$
 $*v \ x = \text{transpose } (\text{fst } (\text{QR-decomposition } A)) *v \ b)$
 $\langle \text{proof} \rangle$

corollary *in-set-least-squares-approximation-eq-full-rank-QR2*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(x \in \text{set-least-squares-approximation } A \ b) = (x = \text{matrix-inv } (\text{snd } (\text{QR-decomposition } A))$
 $** \text{transpose } (\text{fst } (\text{QR-decomposition } A)) *v \ b)$
 $\langle \text{proof} \rangle$

lemma *set-least-squares-approximation-unique-solution*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(\text{set-least-squares-approximation } A \ b) = \{\text{matrix-inv } (\text{transpose } A **$
 $A)**\text{transpose } A *v \ b\}$
 $\langle \text{proof} \rangle$

lemma *set-least-squares-approximation-unique-solution-QR*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $r: \text{rank } A = \text{ncols } A$
shows $(\text{set-least-squares-approximation } A \ b) = \{\text{matrix-inv } (\text{snd } (\text{QR-decomposition } A))$
 $** \text{transpose } (\text{fst } (\text{QR-decomposition } A)) *v \ b\}$
 $\langle \text{proof} \rangle$

end

6 Examples of execution using floats

theory *Examples-QR-Abstract-Float*
imports
QR-Decomposition
Gauss-Jordan.Code-Real-Approx-By-Float-Haskell
begin

6.0.1 Examples

definition *example1* = $(\text{let } A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,0]]::\text{real}^{\wedge}3^{\wedge}3$
 in
 $\text{matrix-to-list-of-list } (\text{divide-by-norm } A))$

definition *example2* = $(\text{let } A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^{\wedge}3^{\wedge}3$
 in
 $\text{matrix-to-list-of-list } (\text{fst } (\text{QR-decomposition } A)))$

```

definition example3 = (let A = list-of-list-to-matrix [[1,2,4],[9,4,5],[0,0,4]]::real^3^3
in
  matrix-to-list-of-list (snd (QR-decomposition A)))

definition example4 = (let A = list-of-list-to-matrix [[1,2,4],[9,4,5],[0,0,4]]::real^3^3
in
  matrix-to-list-of-list (fst (QR-decomposition A) ** (snd (QR-decomposition A))))

definition example5 = (let A = list-of-list-to-matrix [[1,sqrt 2,4],[sqrt 5,4,5],[0,sqrt
7,4]]::real^3^3 in
  matrix-to-list-of-list (fst (QR-decomposition A)))

export-code example1 example2 example3 example4 example5 in SML mod-
ule-name QR

end

```

7 Examples of execution using symbolic computation

```

theory Examples-QR-Abstract-Symbolic
imports
  QR-Decomposition
  Real-Impl.Real-Unique-Impl
begin

```

7.1 Execution of the QR decomposition using symbolic computation

7.1.1 Some previous definitions and lemmas

The symbolic computation is based on the René Thiemann's work about implementing field extensions of the form $\mathbb{Q}[\sqrt{b}]$.

definition *show-vec-real* $v = (\chi \ i. \text{show-real } (v \ \$ \ i))$

lemma [*code abstract*]: $\text{vec-nth } (\text{show-vec-real } v) = (\% \ i. \text{show-real } (v \ \$ \ i))$
<proof>

definition *show-matrix-real* $A = (\chi \ i. \text{show-vec-real } (A \ \$ \ i))$

lemma[*code abstract*]: $\text{vec-nth } (\text{show-matrix-real } A) = (\% \ i. \text{show-vec-real } (A \ \$ \ i))$
<proof>

7.1.2 Examples

value *let* $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,0]]::\text{real}^3^3$ *in*
matrix-to-list-of-list (*show-matrix-real* (*divide-by-norm* A))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ *in*
matrix-to-list-of-list (*show-matrix-real* (*fst* (*QR-decomposition* A))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ *in*
matrix-to-list-of-list (*show-matrix-real* (*snd* (*QR-decomposition* A))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*fst* (*QR-decomposition* A)) ** (*snd* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4],[3,5,4]]::\text{real}^3^4$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*fst* (*QR-decomposition* A)) ** (*snd* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9],[2,0,2],[0,5,0]]::\text{real}^3^4$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*fst* (*QR-decomposition* A)) ** (*snd* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9],[2,0,2],[0,5,0]]::\text{real}^3^4$ *in*
matrix-to-list-of-list (*show-matrix-real* (*fst* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9],[2,0,2],[0,5,0]]::\text{real}^3^4$ *in*
vec-to-list (*show-vec-real* ((*column* 0 (*fst* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9],[2,0,2],[0,5,0]]::\text{real}^3^4$ *in*
vec-to-list (*show-vec-real* ((*column* 1 (*fst* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9],[2,0,2],[0,5,0]]::\text{real}^3^4$ *in*
matrix-to-list-of-list (*show-matrix-real* (*snd* (*QR-decomposition* A))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9]]::\text{real}^3^2$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*fst* (*QR-decomposition* A)) ** (*snd* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9]]::\text{real}^3^2$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*fst* (*QR-decomposition* A))))))

value *let* $A = \text{list-of-list-to-matrix } [[1,2,1],[9,4,9]]::\text{real}^3^2$ *in*
matrix-to-list-of-list (*show-matrix-real* ((*snd* (*QR-decomposition* A))))))

definition *example1* = (let *A* = list-of-list-to-matrix [[1,2,1],[9,4,9]]::real³² in
matrix-to-list-of-list (show-matrix-real ((snd (QR-decomposition *A*))))))

export-code *example1* in SML module-name *QR*

end

8 IArray Addenda QR

theory *IArray-Addenda-QR*

imports

HOL-Library.IArray

begin

The new file about Iarrays, with different instantiations from the presented ones in the Gauss-Jordan algorithm.

In order to make the formalisation of the QR algorithm easier, we have decided to present here some alternative instantiations for immutable arrays.

Let see an example. The following definition is the one presented in the Gauss-Jordan AFP entry to sum two vectors:

plus-iarray *A B* = *IArray.of-fun* ($\lambda n. A!!n + B !! n$) (*IArray.length* *A*)

While the following is the one we will present in this development:

plus-iarray *A B* =
 (let *length-A* = (*IArray.length* *A*);
length-B = (*IArray.length* *B*);
n = max *length-A* *length-B* ;
A' = *IArray.of-fun* ($\lambda a. \text{if } a < \text{length-A then } A!!a \text{ else } 0$) *n*;
B' = *IArray.of-fun* ($\lambda a. \text{if } a < \text{length-B then } B!!a \text{ else } 0$) *n*
 in *IArray.of-fun* ($\lambda a. A' !! a + B' !! a$) *n*)

Now the sum is done up to the length of the shortest vector and it is completed with zeros up to the length of the largest vector. This allows us to prove that *iarray* is an instance of *comm-monoid-add*, which is quite useful for the QR algorithm (we will be able to do sums involving immutable arrays).

These are just alternative definitions of the main operations over immutable arrays. They have the advantage of being an instance of *comm-monoid-add*; nevertheless, the performance is slower and proofs become more cumbersome. The user should decide what definitions to use (the presented here or the presented ones in the Gauss-Jordan AFP entry) depending on the algorithm to formalise.

lemma *iarray-exhaust2*:

(*xs* = *ys*) = (*IArray.list-of* *xs* = *IArray.list-of* *ys*)
 <proof>

lemma *of-fun-nth*:
assumes $i: i < n$
shows $(IArray.of-fun f n) !! i = f i$
 $\langle proof \rangle$

8.1 Some previous instances

instantiation *iarray* :: $(\{plus, zero\}) plus$
begin

definition *plus-iarray* :: $'a iarray \Rightarrow 'a iarray \Rightarrow 'a iarray$
where *plus-iarray* $A B =$
 $(let\ length-A = (IArray.length\ A);$
 $length-B = (IArray.length\ B);$
 $n = max\ length-A\ length-B ;$
 $A' = IArray.of-fun\ (\lambda a. if\ a < length-A\ then\ A!!a\ else\ 0)\ n;$
 $B' = IArray.of-fun\ (\lambda a. if\ a < length-B\ then\ B!!a\ else\ 0)\ n$
in
 $IArray.of-fun\ (\lambda a. A' !! a + B' !! a)\ n)$

instance $\langle proof \rangle$
end

instantiation *iarray* :: $(zero) zero$
begin
definition *zero-iarray* = $(IArray[] :: 'a iarray)$
instance $\langle proof \rangle$
end

instantiation *iarray* :: $(comm-monoid-add) comm-monoid-add$
begin

instance
 $\langle proof \rangle$
end

instantiation *iarray* :: $(uminus) uminus$
begin
definition *uminus-iarray* :: $'a iarray \Rightarrow 'a iarray$
where *uminus-iarray* $A = IArray.of-fun\ (\lambda n. - A!!n)\ (IArray.length\ A)$
instance $\langle proof \rangle$
end

instantiation *iarray* :: $(\{minus, zero\}) minus$
begin

definition *minus-iarray* :: $'a iarray \Rightarrow 'a iarray \Rightarrow 'a iarray$
where *minus-iarray* $A B =$

```

(let length-A= (IArray.length A);
 length-B= (IArray.length B);
 n=max length-A length-B ;
 A'= IArray.of-fun (λa. if a < length-A then A!!a else 0) n;
 B'=IArray.of-fun (λa. if a < length-B then B!!a else 0) n
 in
 IArray.of-fun (λa. A' !! a - B' !! a) n)

```

```

instance ⟨proof⟩
end

```

8.2 Some previous definitions and properties for IArrays

8.2.1 Lemmas

8.2.2 Definitions

```

fun all :: ('a ⇒ bool) ⇒ 'a iarray ⇒ bool
  where all p (IArray as) = (ALL a : set as. p a)
hide-const (open) all

```

```

fun exists :: ('a ⇒ bool) ⇒ 'a iarray ⇒ bool
  where exists p (IArray as) = (EX a : set as. p a)
hide-const (open) exists

```

8.3 Code generation

code-printing

```

constant IArray-Addenda-QR.exists → (SML) Vector.exists
| constant IArray-Addenda-QR.all → (SML) Vector.all

```

```

end

```

9 Matrices as nested IArrays

```

theory Matrix-To-IArray-QR

```

```

imports

```

```

  Rank-Nullity-Theorem.Mod-Type
  Gauss-Jordan.Elementary-Operations
  IArray-Addenda-QR

```

```

begin

```

The file is similar to the *Matrix-To-IArray.thy* one, presented in the Gauss-Jordan algorithm. But now, some proofs have changed slightly because of the new instantiations presented in the file *IArray-Addenda-QR.thy*.

9.1 Isomorphism between matrices implemented by vecs and matrices implemented by iarrays

9.1.1 Isomorphism between vec and iarray

definition *vec-to-iarray* :: 'aⁿ::{mod-type} ⇒ 'a iarray
where *vec-to-iarray* A = IArray.of-fun (λi. A \$ (from-nat i)) (CARD('n))

definition *iarray-to-vec* :: 'a iarray ⇒ 'aⁿ::{mod-type}
where *iarray-to-vec* A = (χ i. A !! (to-nat i))

lemma *vec-to-iarray-nth*:
fixes A::'aⁿ::{finite, mod-type}
assumes i: i < CARD('n)
shows (vec-to-iarray A) !! i = A \$ (from-nat i)
<proof>

lemma *vec-to-iarray-nth'*:
fixes A::'aⁿ::{mod-type}
shows (vec-to-iarray A) !! (to-nat i) = A \$ i
<proof>

lemma *iarray-to-vec-nth*:
shows (iarray-to-vec A) \$ i = A !! (to-nat i)
<proof>

lemma *vec-to-iarray-morph*:
fixes A::'aⁿ::{mod-type}
shows (A = B) = (vec-to-iarray A = vec-to-iarray B)
<proof>

lemma *inj-vec-to-iarray*:
shows inj vec-to-iarray
<proof>

lemma *iarray-to-vec-vec-to-iarray*:
fixes A::'aⁿ::{mod-type}
shows iarray-to-vec (vec-to-iarray A) = A
<proof>

lemma *vec-to-iarray-iarray-to-vec*:
assumes length-eq: IArray.length A = CARD('n::{mod-type})
shows vec-to-iarray (iarray-to-vec A::'aⁿ::{mod-type}) = A
<proof>

lemma *length-vec-to-iarray*:
fixes xa::'aⁿ::{mod-type}

shows $IArray.length (vec\text{-}to\text{-}iarray\ x) = CARD('n)$
 $\langle proof \rangle$

9.1.2 Isomorphism between matrix and nested iarrays

definition $matrix\text{-}to\text{-}iarray :: 'a \sim n :: \{mod\text{-}type\} \sim m :: \{mod\text{-}type\} \Rightarrow 'a\ iarray\ iarray$

where $matrix\text{-}to\text{-}iarray\ A = IArray (map (vec\text{-}to\text{-}iarray \circ ((\$) A) \circ (from\text{-}nat :: nat \Rightarrow 'm)) [0..<CARD('m)])$

definition $iarray\text{-}to\text{-}matrix :: 'a\ iarray\ iarray \Rightarrow 'a \sim n :: \{mod\text{-}type\} \sim m :: \{mod\text{-}type\}$
where $iarray\text{-}to\text{-}matrix\ A = (\chi\ i\ j. A\ !!\ (to\text{-}nat\ i)\ !!\ (to\text{-}nat\ j))$

lemma $matrix\text{-}to\text{-}iarray\text{-}morph:$

fixes $A :: 'a \sim n :: \{mod\text{-}type\} \sim m :: \{mod\text{-}type\}$

shows $(A = B) = (matrix\text{-}to\text{-}iarray\ A = matrix\text{-}to\text{-}iarray\ B)$

$\langle proof \rangle$

lemma $matrix\text{-}to\text{-}iarray\text{-}eq\text{-}of\text{-}fun:$

fixes $A :: 'a \sim columns :: \{mod\text{-}type\} \sim rows :: \{mod\text{-}type\}$

assumes $vec\text{-}eq\text{-}f: \forall i. vec\text{-}to\text{-}iarray (A \$ i) = f (to\text{-}nat\ i)$

and $n\text{-}eq\text{-}length: n = IArray.length (matrix\text{-}to\text{-}iarray\ A)$

shows $matrix\text{-}to\text{-}iarray\ A = IArray.of\text{-}fun\ f\ n$

$\langle proof \rangle$

lemma $map\text{-}vec\text{-}to\text{-}iarray\text{-}rw[simp]:$

fixes $A :: 'a \sim columns :: \{mod\text{-}type\} \sim rows :: \{mod\text{-}type\}$

shows $map (\lambda x. vec\text{-}to\text{-}iarray (A \$ from\text{-}nat\ x)) [0..<CARD('rows)] ! to\text{-}nat\ i = vec\text{-}to\text{-}iarray (A \$ i)$

$\langle proof \rangle$

lemma $matrix\text{-}to\text{-}iarray\text{-}nth:$

$matrix\text{-}to\text{-}iarray\ A\ !!\ to\text{-}nat\ i\ !!\ to\text{-}nat\ j = A\ \$\ i\ \$\ j$

$\langle proof \rangle$

lemma $vec\text{-}matrix: vec\text{-}to\text{-}iarray (A \$ i) = (matrix\text{-}to\text{-}iarray\ A) !! (to\text{-}nat\ i)$

$\langle proof \rangle$

lemma $iarray\text{-}to\text{-}matrix\text{-}matrix\text{-}to\text{-}iarray:$

fixes $A :: 'a \sim columns :: \{mod\text{-}type\} \sim rows :: \{mod\text{-}type\}$

shows $iarray\text{-}to\text{-}matrix (matrix\text{-}to\text{-}iarray\ A) = A$

$\langle proof \rangle$

9.2 Definition of operations over matrices implemented by iarrays

definition $mult\text{-}iarray :: 'a :: \{times\} iarray \Rightarrow 'a \Rightarrow 'a\ iarray$

where $mult\text{-}iarray\ A\ q = IArray.of\text{-}fun (\lambda n. q * A !! n) (IArray.length\ A)$

definition *row-iarray* :: nat => 'a iarray iarray => 'a iarray
 where *row-iarray* k A = A !! k

definition *column-iarray* :: nat => 'a iarray iarray => 'a iarray
 where *column-iarray* k A = IArray.of-fun (λm. A !! m !! k) (IArray.length A)

definition *nrows-iarray* :: 'a iarray iarray => nat
 where *nrows-iarray* A = IArray.length A

definition *ncols-iarray* :: 'a iarray iarray => nat
 where *ncols-iarray* A = IArray.length (A!!0)

definition *rows-iarray* A = {row-iarray i A | i. i ∈ {..*nrows-iarray* A}}

definition *columns-iarray* A = {column-iarray i A | i. i ∈ {..*ncols-iarray* A}}

definition *tabulate2* :: nat => nat => (nat => nat => 'a) => 'a iarray iarray
 where *tabulate2* m n f = IArray.of-fun (λi. IArray.of-fun (f i) n) m

definition *transpose-iarray* :: 'a iarray iarray => 'a iarray iarray
 where *transpose-iarray* A = *tabulate2* (*ncols-iarray* A) (*nrows-iarray* A) (λa b. A!!b!!a)

definition *matrix-matrix-mult-iarray* :: 'a::{times, comm-monoid-add} iarray iarray
 => 'a iarray iarray => 'a iarray iarray (**infixl** **i 70)
 where A **i B = *tabulate2* (*nrows-iarray* A) (*ncols-iarray* B) (λi j. sum (λk.
 ((A!!i)!!k) * ((B!!k)!!j)) {0..*ncols-iarray* A})

definition *matrix-vector-mult-iarray* :: 'a::{semiring-1} iarray iarray => 'a iarray
 => 'a iarray (**infixl** *iv 70)
 where A *iv x = IArray.of-fun (λi. sum (λj. ((A!!i)!!j) * (x!!j)) {0..*IArray.length*
 x}) (*nrows-iarray* A)

definition *vector-matrix-mult-iarray* :: 'a::{semiring-1} iarray => 'a iarray iarray
 => 'a iarray (**infixl** v*i 70)
 where x v*i A = IArray.of-fun (λj. sum (λi. ((A!!i)!!j) * (x!!i)) {0..*IArray.length*
 x}) (*ncols-iarray* A)

definition *mat-iarray* :: 'a::{zero} => nat => 'a iarray iarray
 where *mat-iarray* k n = *tabulate2* n n (λ i j. if i = j then k else 0)

definition *is-zero-iarray* :: 'a::{zero} iarray ⇒ bool
 where *is-zero-iarray* A = IArray-Addenda-QR.all (λi. A !! i = 0) (IArray[0..*IArray.length*
 A])

9.2.1 Properties of previous definitions

lemma *is-zero-iarray-eq-iff*:

fixes A::'a::{zero} ~ⁿ::{mod-type}

shows (A = 0) = (*is-zero-iarray* (*vec-to-iarray* A))

<proof>

lemma *mult-iarray-works*:

assumes $a < IArray.length\ A$ **shows** $mult\text{-}iarray\ A\ q\ !!\ a = q * A!!a$

<proof>

lemma *length-eq-card-rows*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $IArray.length\ (matrix\text{-}to\text{-}iarray\ A) = CARD('rows)$

<proof>

lemma *nrows-eq-card-rows*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $nrows\text{-}iarray\ (matrix\text{-}to\text{-}iarray\ A) = CARD('rows)$

<proof>

lemma *length-eq-card-columns*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $IArray.length\ (matrix\text{-}to\text{-}iarray\ A\ !!\ 0) = CARD('columns)$

<proof>

lemma *ncols-eq-card-columns*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $ncols\text{-}iarray\ (matrix\text{-}to\text{-}iarray\ A) = CARD('columns)$

<proof>

lemma *matrix-to-iarray-nrows*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $nrows\ A = nrows\text{-}iarray\ (matrix\text{-}to\text{-}iarray\ A)$

<proof>

lemma *matrix-to-iarray-ncols*:

fixes $A::'a\ ^\wedge\ columns::\{mod\text{-}type\}\ ^\wedge\ rows::\{mod\text{-}type\}$

shows $ncols\ A = ncols\text{-}iarray\ (matrix\text{-}to\text{-}iarray\ A)$

<proof>

lemma *vec-to-iarray-row[code-unfold]*: $vec\text{-}to\text{-}iarray\ (row\ i\ A) = row\text{-}iarray\ (to\text{-}nat\ i)\ (matrix\text{-}to\text{-}iarray\ A)$

<proof>

lemma *vec-to-iarray-row'*: $vec\text{-}to\text{-}iarray\ (row\ i\ A) = (matrix\text{-}to\text{-}iarray\ A)\ !!\ (to\text{-}nat\ i)$

<proof>

lemma *vec-to-iarray-column[code-unfold]*: $vec\text{-}to\text{-}iarray\ (column\ i\ A) = column\text{-}iarray\ (to\text{-}nat\ i)\ (matrix\text{-}to\text{-}iarray\ A)$

<proof>

lemma *vec-to-iarray-column'*:

assumes $k: k < n\text{cols } A$
shows $(\text{vec-to-iarray } (\text{column } (\text{from-nat } k) A)) = (\text{column-iarray } k (\text{matrix-to-iarray } A))$
 $\langle \text{proof} \rangle$

lemma *column-iarray-nth*:
assumes $i: i < n\text{rows-iarray } A$
shows $\text{column-iarray } j A !! i = A !! i !! j$
 $\langle \text{proof} \rangle$

lemma *vec-to-iarray-rows*: $\text{vec-to-iarray}' (\text{rows } A) = \text{rows-iarray } (\text{matrix-to-iarray } A)$
 $\langle \text{proof} \rangle$

lemma *vec-to-iarray-columns*: $\text{vec-to-iarray}' (\text{columns } A) = \text{columns-iarray } (\text{matrix-to-iarray } A)$
 $\langle \text{proof} \rangle$

9.3 Definition of elementary operations

definition *interchange-rows-iarray* :: $'a \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \text{ iarray iarray}$

where *interchange-rows-iarray* $A a b = \text{IArray.of-fun } (\lambda n. \text{if } n=a \text{ then } A!!b \text{ else if } n=b \text{ then } A!!a \text{ else } A!!n) (\text{IArray.length } A)$

definition *mult-row-iarray* :: $'a::\{\text{times}\} \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \text{ iarray iarray}$

where *mult-row-iarray* $A a q = \text{IArray.of-fun } (\lambda n. \text{if } n=a \text{ then } \text{mult-iarray } (A!!a) q \text{ else } A!!n) (\text{IArray.length } A)$

definition *row-add-iarray* :: $'a::\{\text{plus, times, zero}\} \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \text{ iarray iarray}$

where *row-add-iarray* $A a b q = \text{IArray.of-fun } (\lambda n. \text{if } n=a \text{ then } A!!a + \text{mult-iarray } (A!!b) q \text{ else } A!!n) (\text{IArray.length } A)$

definition *interchange-columns-iarray* :: $'a \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \text{ iarray iarray}$

where *interchange-columns-iarray* $A a b = \text{tabulate2 } (n\text{rows-iarray } A) (n\text{cols-iarray } A) (\lambda i j. \text{if } j = a \text{ then } A !! i !! b \text{ else if } j = b \text{ then } A !! i !! a \text{ else } A !! i !! j)$

definition *mult-column-iarray* :: $'a::\{\text{times}\} \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \text{ iarray iarray}$

where *mult-column-iarray* $A n q = \text{tabulate2 } (n\text{rows-iarray } A) (n\text{cols-iarray } A) (\lambda i j. \text{if } j = n \text{ then } A !! i !! j * q \text{ else } A !! i !! j)$

definition *column-add-iarray* :: $'a::\{\text{plus, times}\} \text{ iarray iarray} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \text{ iarray iarray}$

where *column-add-iarray* $A n m q = \text{tabulate2 } (n\text{rows-iarray } A) (n\text{cols-iarray } A) (\lambda i j. \text{if } j = n \text{ then } A !! i !! n + A !! i !! m * q \text{ else } A !! i !! j)$

9.3.1 Code generator

lemma *vec-to-iarray-plus*[code-unfold]: $\text{vec-to-iarray } (a + b) = (\text{vec-to-iarray } a) + (\text{vec-to-iarray } b)$
 ⟨proof⟩

lemma *matrix-to-iarray-plus*[code-unfold]: $\text{matrix-to-iarray } (A + B) = (\text{matrix-to-iarray } A) + (\text{matrix-to-iarray } B)$
 ⟨proof⟩

lemma *matrix-to-iarray-mat*[code-unfold]:
 $\text{matrix-to-iarray } (\text{mat } k :: 'a::\{\text{zero}\} \wedge 'n::\{\text{mod-type}\} \wedge 'm::\{\text{mod-type}\}) = \text{mat-iarray } k \text{ CARD}('n::\{\text{mod-type}\})$
 ⟨proof⟩

lemma *matrix-to-iarray-transpose*[code-unfold]:
shows $\text{matrix-to-iarray } (\text{transpose } A) = \text{transpose-iarray } (\text{matrix-to-iarray } A)$
 ⟨proof⟩

lemma *matrix-to-iarray-matrix-matrix-mult*[code-unfold]:
fixes $A::'a::\{\text{semiring-1}\} \wedge 'm::\{\text{mod-type}\} \wedge 'n::\{\text{mod-type}\}$ **and** $B::'a \wedge 'b::\{\text{mod-type}\} \wedge 'm::\{\text{mod-type}\}$
shows $\text{matrix-to-iarray } (A ** B) = (\text{matrix-to-iarray } A) **i (\text{matrix-to-iarray } B)$
 ⟨proof⟩

lemma *vec-to-iarray-matrix-matrix-mult*[code-unfold]:
fixes $A::'a::\{\text{semiring-1}\} \wedge 'm::\{\text{mod-type}\} \wedge 'n::\{\text{mod-type}\}$ **and** $x::'a \wedge 'm::\{\text{mod-type}\}$
shows $\text{vec-to-iarray } (A *v x) = (\text{matrix-to-iarray } A) *iv (\text{vec-to-iarray } x)$
 ⟨proof⟩

lemma *vec-to-iarray-vector-matrix-mult*[code-unfold]:
fixes $A::'a::\{\text{semiring-1}\} \wedge 'm::\{\text{mod-type}\} \wedge 'n::\{\text{mod-type}\}$ **and** $x::'a \wedge 'n::\{\text{mod-type}\}$
shows $\text{vec-to-iarray } (x v* A) = (\text{vec-to-iarray } x) v*i (\text{matrix-to-iarray } A)$
 ⟨proof⟩

lemma *matrix-to-iarray-interchange-rows*[code-unfold]:
fixes $A::'a::\{\text{semiring-1}\} \wedge 'columns::\{\text{mod-type}\} \wedge 'rows::\{\text{mod-type}\}$
shows $\text{matrix-to-iarray } (\text{interchange-rows } A \ i \ j) = \text{interchange-rows-iarray } (\text{matrix-to-iarray } A) \ (to\text{-nat } i) \ (to\text{-nat } j)$
 ⟨proof⟩

lemma *matrix-to-iarray-mult-row*[code-unfold]:
fixes $A::'a::\{\text{semiring-1}\} \wedge 'columns::\{\text{mod-type}\} \wedge 'rows::\{\text{mod-type}\}$
shows $\text{matrix-to-iarray } (\text{mult-row } A \ i \ q) = \text{mult-row-iarray } (\text{matrix-to-iarray } A) \ (to\text{-nat } i) \ q$
 ⟨proof⟩

```

lemma matrix-to-iarray-row-add[code-unfold]:
  fixes  $A::'a::\{\text{semiring-1}\}^{\wedge}\text{columns}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$ 
  shows  $\text{matrix-to-iarray} (\text{row-add } A \ i \ j \ q) = \text{row-add-iarray} (\text{matrix-to-iarray } A)$ 
   $(\text{to-nat } i) (\text{to-nat } j) \ q$ 
   $\langle \text{proof} \rangle$ 

lemma matrix-to-iarray-interchange-columns[code-unfold]:
  fixes  $A::'a::\{\text{semiring-1}\}^{\wedge}\text{columns}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$ 
  shows  $\text{matrix-to-iarray} (\text{interchange-columns } A \ i \ j) = \text{interchange-columns-iarray}$ 
   $(\text{matrix-to-iarray } A) (\text{to-nat } i) (\text{to-nat } j)$ 
   $\langle \text{proof} \rangle$ 

lemma matrix-to-iarray-mult-columns[code-unfold]:
  fixes  $A::'a::\{\text{semiring-1}\}^{\wedge}\text{columns}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$ 
  shows  $\text{matrix-to-iarray} (\text{mult-column } A \ i \ q) = \text{mult-column-iarray} (\text{matrix-to-iarray}$ 
   $A) (\text{to-nat } i) \ q$ 
   $\langle \text{proof} \rangle$ 

lemma matrix-to-iarray-column-add[code-unfold]:
  fixes  $A::'a::\{\text{semiring-1}\}^{\wedge}\text{columns}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$ 
  shows  $\text{matrix-to-iarray} (\text{column-add } A \ i \ j \ q) = \text{column-add-iarray} (\text{matrix-to-iarray}$ 
   $A) (\text{to-nat } i) (\text{to-nat } j) \ q$ 
   $\langle \text{proof} \rangle$ 

```

end

10 Gram Schmidt over IArrays

theory *Gram-Schmidt-IArrays*

imports

QR-Decomposition

Matrix-To-IArray-QR

begin

10.1 Some previous definitions, lemmas and instantiations about iarrays

definition *iarray-of-iarray-to-list-of-list* :: $'a \ \text{iarray} \ \text{iarray} \Rightarrow 'a \ \text{list} \ \text{list}$
where $\text{iarray-of-iarray-to-list-of-list } A = \text{map } \text{IArray.list-of} (\text{map } (!!)) \ A \ [0..<\text{IArray.length } A]$

instantiation $\text{iarray} :: (\text{scaleR}) \ \text{scaleR}$

begin

definition $\text{scaleR-iarray} \ k \ A = \text{IArray.of-fun} (\lambda i. \ k \ *_R \ (A \ !! \ i)) \ (\text{IArray.length } A)$

instance $\langle \text{proof} \rangle$

end

instantiation *iarray* :: (*times*) *times*
begin
definition *times-iarray* $A\ B = IArray.of\text{-}fun\ (\lambda i. A!!i * B!!i)\ (IArray.length\ A)$
instance $\langle proof \rangle$
end

lemma *plus-iarray-component*:
assumes $iA: i < IArray.length\ A$
and $iB: i < IArray.length\ B$
shows $(A+B)!!i = A!!i + B!!i$
 $\langle proof \rangle$

lemma *minus-iarray-component*:
assumes $iA: i < IArray.length\ A$
and $iB: i < IArray.length\ B$
shows $(A-B)!!i = A!!i - B!!i$
 $\langle proof \rangle$

lemma *length-plus-iarray*:
 $IArray.length\ (A+B) = \max\ (IArray.length\ A)\ (IArray.length\ B)$
 $\langle proof \rangle$

lemma *length-sum-iarray*:
assumes *finite* *S* **and** $S \neq \{\}$
shows $IArray.length\ (sum\ f\ S) = \text{Max}\ \{IArray.length\ (f\ x) \mid x. x \in S\}$
 $\langle proof \rangle$

lemma *sum-component-iarray*:
assumes $a: \forall x \in S. i < IArray.length\ (f\ x)$
and *f*: *finite* *S*
and $S: S \neq \{\}$ — If *S* is empty, then the sum will return the empty iarray and it makes no sense to access the component *i*
shows $sum\ f\ S!!i = (\sum x \in S. f\ x!!i)$
 $\langle proof \rangle$

lemma *length-zero-iarray*: $IArray.length\ 0 = 0$
 $\langle proof \rangle$

lemma *minus-zero-iarray*:
fixes $A::'a::\{group\text{-}add\}\ iarray$
shows $A - 0 = A$
 $\langle proof \rangle$

10.2 Inner mult over real iarrays

definition *inner-iarray* :: *real iarray => real iarray => real* (**infixl** \cdot *i* 70)
where *inner-iarray* $A B = \text{sum } (\lambda n. A !! n * B !! n) \{0..<IArray.length A\}$

lemma *vec-to-iarray-inner*:
 $a \cdot b = \text{vec-to-iarray } a \cdot i \text{vec-to-iarray } b$
<proof>

lemma *vec-to-iarray-scaleR*:
 $\text{vec-to-iarray } (a *_R x) = a *_R (\text{vec-to-iarray } x)$
<proof>

10.3 Gram Schmidt over IArrays

definition *Gram-Schmidt-column-k-iarrays* $A k$
 $= \text{tabulate2 } (\text{nrows-iarray } A) (\text{ncols-iarray } A) (\lambda a b. (\text{if } b = k$
 $\text{then } (\text{column-iarray } b A - \text{sum } (\lambda x. (((\text{column-iarray } b A) \cdot i x) / (x \cdot i x)) *_R$
 $x)$
 $(\text{set } (\text{List.map } (\lambda n. \text{column-iarray } n A) [0..<b])))$
 $\text{else } (\text{column-iarray } b A)) !! a)$

definition *Gram-Schmidt-upt-k-iarrays* $A k = \text{List.foldl } \text{Gram-Schmidt-column-k-iarrays} A [0..<(\text{Suc } k)]$

definition *Gram-Schmidt-matrix-iarrays* $A = \text{Gram-Schmidt-upt-k-iarrays } A (\text{ncols-iarray } A - 1)$

lemma *matrix-to-iarray-Gram-Schmidt-column-k*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $k: k < \text{ncols } A$
shows $\text{matrix-to-iarray } (\text{Gram-Schmidt-column-k } A k) = \text{Gram-Schmidt-column-k-iarrays} (\text{matrix-to-iarray } A) k$
<proof>

lemma *matrix-to-iarray-Gram-Schmidt-upt-k*:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
assumes $k: k < \text{ncols } A$
shows $\text{matrix-to-iarray } (\text{Gram-Schmidt-upt-k } A k) = \text{Gram-Schmidt-upt-k-iarrays} (\text{matrix-to-iarray } A) k$
<proof>

lemma *matrix-to-iarray-Gram-Schmidt-matrix*[*code-unfold*]:
fixes $A::\text{real}^{\wedge}\text{cols}::\{\text{mod-type}\}^{\wedge}\text{rows}::\{\text{mod-type}\}$
shows $\text{matrix-to-iarray } (\text{Gram-Schmidt-matrix } A) = \text{Gram-Schmidt-matrix-iarrays} (\text{matrix-to-iarray } A)$
<proof>

Examples:

```
value let A = list-of-list-to-matrix [[4,5],[8,1],[-1,5]]::real^2^3
  in iarray-of-iarray-to-list-of-list (matrix-to-iarray (Gram-Schmidt-matrix A))
```

```
value let A = IArray[IArray[4,5],IArray[8,1],IArray[-1,5]]
  in iarray-of-iarray-to-list-of-list (Gram-Schmidt-matrix-iarrays A)
```

end

11 QR Decomposition over iarrays

```
theory QR-Decomposition-IArrays
```

```
imports
```

```
  Gram-Schmidt-IArrays
```

```
begin
```

11.1 QR Decomposition refinement over iarrays

```
definition norm-iarray A = sqrt (A · i A)
```

```
definition divide-by-norm-iarray A = tabulate2 (nrows-iarray A) (ncols-iarray
A)
  (λa b. ((1/norm-iarray (column-iarray b A)) *R (column-iarray b A)) !! a)
```

```
definition QR-decomposition-iarrays A = (let Q = divide-by-norm-iarray (Gram-Schmidt-matrix-iarrays
A)
  in (Q, transpose-iarray Q **i A))
```

```
lemma vec-to-iarray-norm[code-unfold]:
  shows (norm A) = norm-iarray (vec-to-iarray A)
  ⟨proof⟩
```

```
lemma matrix-to-iarray-divide-by-norm[code-unfold]:
  fixes A::real^cols::{mod-type}^rows::{mod-type}
  shows matrix-to-iarray (divide-by-norm A) = divide-by-norm-iarray (matrix-to-iarray
A)
  ⟨proof⟩
```

```
lemma matrix-to-iarray-fst-QR-decomposition[code-unfold]:
  shows matrix-to-iarray (fst (QR-decomposition A)) = fst (QR-decomposition-iarrays
(matrix-to-iarray A))
  ⟨proof⟩
```

```
lemma matrix-to-iarray-snd-QR-decomposition[code-unfold]:
  shows matrix-to-iarray (snd (QR-decomposition A)) = snd (QR-decomposition-iarrays
(matrix-to-iarray A))
```

<proof>

definition *matrix-to-iarray-pair* $X = (\text{matrix-to-iarray } (\text{fst } X), \text{matrix-to-iarray } (\text{snd } X))$

lemma *matrix-to-iarray-QR-decomposition*[code-unfold]:

shows *matrix-to-iarray-pair* (*QR-decomposition* A) = *QR-decomposition-iarrays* (*matrix-to-iarray* A)

<proof>

end

12 Examples of execution using floats and IArrays

theory *Examples-QR-IArrays-Float*

imports

QR-Decomposition-IArrays

Gauss-Jordan.Code-Real-Approx-By-Float-Haskell

begin

12.1 Examples

definition *example1* = (let $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,0]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*divide-by-norm* A)))

definition *example2* = (let $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*fst* (*QR-decomposition* A))))

definition *example3* = (let $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*snd* (*QR-decomposition* A))))

definition *example4* = (let $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*fst* (*QR-decomposition* A) ** (*snd* (*QR-decomposition* A))))

definition *example5* = (let $A = \text{list-of-list-to-matrix } [[1,\text{sqrt } 2,4],[\text{sqrt } 5,4,5],[0,\text{sqrt } 7,4]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*fst* (*QR-decomposition* A))))

definition *example6* = (let $A = \text{list-of-list-to-matrix } [[1,\text{sqrt } 2,4],[\text{sqrt } 5,4,5],[0,\text{sqrt } 7,4]]::\text{real}^3^3$ in

iarray-of-iarray-to-list-of-list (*matrix-to-iarray* ((*fst* (*QR-decomposition* A))))

definition *example1b* = (let $A = \text{IArray}[\text{IArray}[1,2,4],\text{IArray}[9,4,5::\text{real}],\text{IArray}[0,0,0]]$

```

in
  iarray-of-iarray-to-list-of-list ((divide-by-norm-iarray A)))

definition example2b = (let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]] in
  iarray-of-iarray-to-list-of-list ((fst (QR-decomposition-iarrays A))))

definition example3b = (let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]]
in
  iarray-of-iarray-to-list-of-list ( (snd (QR-decomposition-iarrays A))))

definition example4b = (let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]]
in
  iarray-of-iarray-to-list-of-list (
    ((fst (QR-decomposition-iarrays A)) **i (snd (QR-decomposition-iarrays A))))))

definition example5b = (let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4],IArray[3,5,4]] in
  iarray-of-iarray-to-list-of-list (
    ((fst (QR-decomposition-iarrays A)) **i (snd (QR-decomposition-iarrays A))))))

definition example6b = (let A = IArray [IArray[1,sqrt 2,4],IArray[sqrt 5,4,5],IArray[0,sqrt
7,4]]
  in iarray-of-iarray-to-list-of-list (fst (QR-decomposition-iarrays A)))

The following example is presented in Chapter 1 of the book Numerical
Methods in Scientific Computing by Dahlquist and Bjorck

definition book-example = (let A = list-of-list-to-matrix
[[1,-0.6691],[1,-0.3907],[1,-0.1219],[1,0.3090],[1,0.5878]]::real^2^5;
b = list-to-vec [0.3704,0.5,0.6211,0.8333,0.9804]::real^5;
QR = (QR-decomposition A);
Q = fst QR;
R = snd QR
in IArray.list-of (vec-to-iarray (the (inverse-matrix R) ** transpose Q *v b)))

export-code example1 example2 example3 example4 example5 example6
example1b example2b example3b example4b example5b example6b
book-example
  in SML module-name QR

```

end

13 Examples of execution using symbolic computation and iarrays

```

theory Examples-QR-IArrays-Symbolic
imports
  Examples-QR-Abstract-Symbolic
  QR-Decomposition-IArrays
begin

```

13.1 Execution of the QR decomposition using symbolic computation and iarrays

definition *show-vec-real-iarrays* $v = IArray.of\text{-}fun (\lambda i. show\text{-}real (v !! i)) (IArray.length v)$

lemma *vec-to-iarray-show-vec-real*[code-unfold]: *vec-to-iarray* (*show-vec-real* v)
 $= show\text{-}vec\text{-}real\text{-}iarrays (vec\text{-}to\text{-}iarray v)$
<proof>

The following function is used to print elements of type `vec` as lists of characters; useful for printing vectors in the output panel.

definition *print-vec* $= IArray.list\text{-}of \circ show\text{-}vec\text{-}real\text{-}iarrays \circ vec\text{-}to\text{-}iarray$

definition *show-matrix-real-iarrays* $A = IArray.of\text{-}fun (\lambda i. show\text{-}vec\text{-}real\text{-}iarrays (A !! i)) (IArray.length A)$

lemma *matrix-to-iarray-show-matrix-real*[code-unfold]: *matrix-to-iarray* (*show-matrix-real* v)
 $= show\text{-}matrix\text{-}real\text{-}iarrays (matrix\text{-}to\text{-}iarray v)$
<proof>

The following functions are useful to print matrices as lists of lists of characters; useful for printing in the output panel.

definition *print-vec-mat* $= IArray.list\text{-}of \circ show\text{-}vec\text{-}real\text{-}iarrays$

definition *print-mat-aux* $A = IArray.of\text{-}fun (\lambda i. print\text{-}vec\text{-}mat (A !! i)) (IArray.length A)$

definition *print-mat* $= IArray.list\text{-}of \circ print\text{-}mat\text{-}aux \circ matrix\text{-}to\text{-}iarray$

13.1.1 Examples

value *let* $A = list\text{-}of\text{-}list\text{-}to\text{-}matrix [[1,2,4],[9,4,5],[0,0,0]]::real^3^3$ *in*
iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*show-matrix-real* (*divide-by-norm* A))))

value *let* $A = list\text{-}of\text{-}list\text{-}to\text{-}matrix [[1,2,4],[9,4,5],[0,0,4]]::real^3^3$ *in*
iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*show-matrix-real* (*fst* (*QR-decomposition* A))))

value *let* $A = list\text{-}of\text{-}list\text{-}to\text{-}matrix [[1,2,4],[9,4,5],[0,0,4]]::real^3^3$ *in*
iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*show-matrix-real* (*snd* (*QR-decomposition* A))))

value *let* $A = list\text{-}of\text{-}list\text{-}to\text{-}matrix [[1,2,4],[9,4,5],[0,0,4]]::real^3^3$ *in*
iarray-of-iarray-to-list-of-list (*matrix-to-iarray* (*show-matrix-real* ((*fst* (*QR-decomposition* A)) ** (*snd* (*QR-decomposition* A))))))

value let $A = \text{list-of-list-to-matrix } [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::\text{real}^3^4$ in rank
 $A = \text{ncols } A$

value let $A = \text{list-of-list-to-matrix } [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::\text{real}^3^4$;
 $b = \text{list-to-vec } [1,2,3,4]::\text{real}^4$ in
 $\text{print-result-solve } (\text{solve } A \ b)$

value let $A = \text{list-of-list-to-matrix } [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::\text{real}^3^4$;
 $b = \text{list-to-vec } [1,2,3,4]::\text{real}^4$
in
 $\text{vec-to-list } (\text{show-vec-real } (\text{the } (\text{inverse-matrix } (\text{snd } (\text{QR-decomposition } A)))) **$
 $\text{transpose } (\text{fst } (\text{QR-decomposition } A)) *v \ b))$

value let $A = \text{list-of-list-to-matrix } [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::\text{real}^3^4$;
 $b = \text{list-to-vec } [1,2,3,4]::\text{real}^4$
in $\text{matrix-to-list-of-list } (\text{show-matrix-real } ((\text{snd } (\text{QR-decomposition } A))))$

least squares solution

definition $A \equiv \text{list-of-list-to-matrix } [[1,3/5,3],[9,4,5/3],[0,0,4],[1,2,3]]::\text{real}^3^4$
definition $b \equiv \text{list-to-vec } [1,2,3,4]::\text{real}^4$

value let $Q = \text{fst } (\text{QR-decomposition } A)$; $R = \text{snd } (\text{QR-decomposition } A)$
in $\text{print-vec } ((\text{the } (\text{inverse-matrix } R)) ** \text{transpose } Q *v \ b))$

A times least squares solution

value let $Q = \text{fst } (\text{QR-decomposition } A)$; $R = \text{snd } (\text{QR-decomposition } A)$
in $\text{print-vec } (A *v \ (\text{the } (\text{inverse-matrix } R)) ** \text{transpose } Q *v \ b))$

The matrix Q

value $\text{print-mat } (\text{fst } (\text{QR-decomposition } A))$

The matrix R

value $\text{print-mat } (\text{snd } (\text{QR-decomposition } A))$

The inverse of matrix R

value let $R = \text{snd } (\text{QR-decomposition } A)$ in $\text{print-mat } (\text{the } (\text{inverse-matrix } R))$

The least squares solution is in the left null space of A

value let $Q = \text{fst } (\text{QR-decomposition } A)$; $R = \text{snd } (\text{QR-decomposition } A)$;
 $b2 = (A *v \ (\text{the } (\text{inverse-matrix } R)) ** \text{transpose } Q *v \ b)$
in $\text{print-vec } ((b - b2)v * A)$

value let $A = \text{list-of-list-to-matrix } [[1,2,4],[9,4,5],[0,0,4],[3,5,4]]::\text{real}^3^4$ in
 $\text{iarray-of-iarray-to-list-of-list } (\text{matrix-to-iarray}$
 $(\text{show-matrix-real } ((\text{fst } (\text{QR-decomposition } A)) ** (\text{snd } (\text{QR-decomposition}$
 $A))))))$

```

value let A = IArray[IArray[1,2,4],IArray[9,4,5::real],IArray[0,0,0]] in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays (divide-by-norm-iarray
A))

value let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]] in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays (fst (QR-decomposition-iarrays
A))))

value let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]] in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays (snd (QR-decomposition-iarrays
A))))

value let A = list-of-list-to-matrix [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::real^3^4 in rank
A = ncols A

value let A = list-of-list-to-matrix [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::real^3^4;
  b = list-to-vec [1,2,3,4]::real^4 in
  print-result-solve (solve A b)

value let A = list-of-list-to-matrix [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::real^3^4;
  b = list-to-vec [1,2,3,4]::real^4
  in
  vec-to-list (show-vec-real (the (inverse-matrix (snd (QR-decomposition A))) **
transpose (fst (QR-decomposition A)) *v b))

value let A = list-of-list-to-matrix [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::real^3^4;
  b = list-to-vec [1,2,3,4]::real^4
  in matrix-to-list-of-list (show-matrix-real ((snd (QR-decomposition A))))

value let A = list-of-list-to-matrix [[1,2,3],[9,4,5],[0,0,4],[1,2,3]]::real^3^4;
  b = list-to-vec [1,2,3,4]::real^4;
  b2 = (A *v (the (inverse-matrix (snd (QR-decomposition A))) ** transpose (fst
(QR-decomposition A)) *v b))
  in
  vec-to-list (show-vec-real ((b - b2)v* A))

value let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4]] in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays
((fst (QR-decomposition-iarrays A)) **i (snd (QR-decomposition-iarrays A))))

value let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4],IArray[3,5,4]] in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays
((fst (QR-decomposition-iarrays A)) **i (snd (QR-decomposition-iarrays A))))

```

The following example is presented in Chapter 1 of the book *Numerical Methods in Scientific Computing* by Dahlquist and Bjorck

```

value let A = list-of-list-to-matrix
[[1,-0.6691],[1,-0.3907],[1,-0.1219],[1,0.3090],[1,0.5878]]::real^2^5;
  b = list-to-vec [0.3704,0.5,0.6211,0.8333,0.9804]::real^5;

```

```

QR = (QR-decomposition A);
Q = fst QR;
R = snd QR
in print-vec (the (inverse-matrix R) ** transpose Q *v b)

```

```

definition example = (let A = IArray[IArray[1,2,4],IArray[9,4,5],IArray[0,0,4],IArray[3,5,4]]in
  iarray-of-iarray-to-list-of-list (show-matrix-real-iarrays
    ((fst (QR-decomposition-iarrays A)) **i (snd (QR-decomposition-iarrays A))))))

```

```

export-code example in SML module-name QR

```

```

end

```

14 Generalization of the Second Part of the Fundamental Theorem of Linear Algebra

```

theory Generalizations2

```

```

imports

```

```

  Rank-Nullity-Theorem.Fundamental-Subspaces

```

```

begin

```

14.1 Conjugate class

```

class cnj = field +
  fixes cnj :: 'a⇒'a
  assumes cnj-idem[simp]: cnj (cnj a) = a
  and cnj-add: cnj (a+b) = cnj a + cnj b
  and cnj-mult: cnj (a * b) = cnj a * cnj b
begin

```

```

lemma two-not-one: 2 ≠ (1::'a)
  ⟨proof⟩

```

```

lemma cnj-0[simp]: cnj 0 = 0
  ⟨proof⟩

```

```

lemma cnj-0-eq[simp]: (cnj a = 0) = (a = 0)
  ⟨proof⟩

```

```

lemma a-cnj-a-0: (a*cnj a = 0) = (a = 0)
  ⟨proof⟩

```

```

end

```

lemma *cnj-sum*: $cnj (\sum xa \in A. ((f xa))) = (\sum xa \in A. cnj (f xa))$
⟨*proof*⟩

instantiation *real* :: *cnj*
begin

definition (*cnj-real* :: *real* ⇒ *real*) = *id*

instance
⟨*proof*⟩
end

instantiation *complex* :: *cnj*
begin

definition (*cnj-complex* :: *complex* ⇒ *complex*) = *Complex.cnj*

instance
⟨*proof*⟩
end

14.2 Real_of_extended class

class *real-of-extended* = *real-vector* + *cnj* +
fixes *real-of* :: 'a ⇒ *real*
assumes *real-add*: $real-of ((a::'a) + b) = real-of a + real-of b$
and *real-uminus*: $real-of (-a) = - real-of a$
and *real-scalar-mult*: $real-of (c *_R a) = c * (real-of a)$
and *real-a-cnj-ge-0*: $real-of (a * cnj a) \geq 0$
begin

lemma *real-minus*: $real-of (a - b) = real-of a - real-of b$
⟨*proof*⟩

lemma *real-0[simp]*: $real-of 0 = 0$
⟨*proof*⟩

lemma *real-sum*:
 $real-of (sum (\lambda i. f i) A) = sum (\lambda i. real-of (f i)) A$
⟨*proof*⟩

end

instantiation *real* :: *real-of-extended*
begin

definition *real-of-real* :: *real* \Rightarrow *real* **where** *real-of-real* = *id*

instance
 \langle *proof* \rangle
end

instantiation *complex* :: *real-of-extended*
begin

definition *real-of-complex* :: *complex* \Rightarrow *real* **where** *real-of-complex* = *Re*

instance
 \langle *proof* \rangle
end

14.3 Generalizing HMA

14.3.1 Inner product spaces

We generalize the *real-inner class* to more general inner product spaces.

locale *inner-product-space* = *vector-space scale*
for *scale* :: ('a::*{field, cnj, real-of-extended}*) \Rightarrow 'b::*ab-group-add* \Rightarrow 'b) +
fixes *inner* :: 'b \Rightarrow 'b \Rightarrow 'a
assumes *inner-commute*: *inner* *x* *y* = *cnj* (*inner* *y* *x*)
and *inner-add-left*: *inner* (*x*+*y*) *z* = *inner* *x* *z* + *inner* *y* *z*
and *inner-scaleR-left* [*simp*]: *inner* (*scale* *r* *x*) *y* = *r* * *inner* *x* *y*
and *inner-ge-zero* [*simp*]: $0 \leq$ *real-of* (*inner* *x* *x*)
and *inner-eq-zero-iff* [*simp*]: *inner* *x* *x* = 0 \longleftrightarrow *x*=0

and *real-scalar-mult2*: *real-of* (*inner* *x* *x*) *_R *A* = *inner* *x* *x* * *A*
and *inner-gt-zero-iff*: $0 <$ *real-of* (*inner* *x* *x*) \longleftrightarrow *x* \neq 0

interpretation *RV-inner*: *inner-product-space scaleR inner*
 \langle *proof* \rangle

interpretation *RR-inner*: *inner-product-space scaleR* (*)
 \langle *proof* \rangle

interpretation *CC-inner*: *inner-product-space ((*)::complex \Rightarrow complex \Rightarrow complex)*
 λ *x y. x***cnj* *y*
 \langle *proof* \rangle

context *inner-product-space*
begin

lemma *inner-zero-left* [*simp*]: *inner* 0 *x* = 0
 \langle *proof* \rangle

lemma *inner-minus-left* [simp]: $\text{inner } (- x) y = - \text{inner } x y$
<proof>

lemma *inner-diff-left*: $\text{inner } (x - y) z = \text{inner } x z - \text{inner } y z$
<proof>

lemma *inner-sum-left*: $\text{inner } (\sum x \in A. f x) y = (\sum x \in A. \text{inner } (f x) y)$
<proof>

Transfer distributivity rules to right argument.

lemma *inner-add-right*: $\text{inner } x (y + z) = \text{inner } x y + \text{inner } x z$
<proof>

lemma *inner-scaleR-right* [simp]: $\text{inner } x (\text{scale } r y) = (\text{cnj } r) * (\text{inner } x y)$
<proof>

lemma *inner-zero-right* [simp]: $\text{inner } x 0 = 0$
<proof>

lemma *inner-minus-right* [simp]: $\text{inner } x (- y) = - \text{inner } x y$
<proof>

lemma *inner-diff-right*: $\text{inner } x (y - z) = \text{inner } x y - \text{inner } x z$
<proof>

lemma *inner-sum-right*: $\text{inner } x (\sum y \in A. f y) = (\sum y \in A. \text{inner } x (f y))$
<proof>

lemmas *inner-add* [algebra-simps] = *inner-add-left inner-add-right*

lemmas *inner-diff* [algebra-simps] = *inner-diff-left inner-diff-right*

lemmas *inner-scaleR* = *inner-scaleR-left inner-scaleR-right*

Legacy theorem names

lemmas *inner-left-distrib* = *inner-add-left*

lemmas *inner-right-distrib* = *inner-add-right*

lemmas *inner-distrib* = *inner-left-distrib inner-right-distrib*

lemma *aux-Cauchy*:

shows $0 \leq \text{real-of } (\text{inner } x x + (\text{cnj } a) * (\text{inner } x y) + a * ((\text{cnj } (\text{inner } x y)) + (\text{cnj } a) * (\text{inner } y y)))$

<proof>

lemma *real-inner-inner*: $\text{real-of } (\text{inner } x x * \text{inner } y y) = \text{real-of } (\text{inner } x x) *$

real-of (inner y y)
<proof>

lemma *Cauchy-Schwarz-ineq:*

*real-of (cnj (inner x y) * inner x y) ≤ real-of (inner x x) * real-of (inner y y)*
<proof>

end

hide-const (open) *norm*

context *inner-product-space*
begin

definition *norm x = (sqrt (real-of (inner x x)))*

lemmas *norm-eq-sqrt-inner = norm-def*

lemma *inner-cnj-ge-zero[simp]: real-of ((inner x y) * cnj (inner x y)) ≥ 0*
<proof>

lemma *power2-norm-eq-inner: (norm x)² = real-of (inner x x)*
<proof>

lemma *Cauchy-Schwarz-ineq2:*

*sqrt (real-of (cnj (inner x y) * inner x y)) ≤ norm x * norm y*
<proof>

end

14.3.2 Orthogonality

hide-const (open) *orthogonal*

context *inner-product-space*
begin

definition *orthogonal x y ↔ inner x y = 0*

lemma *orthogonal-clauses:*

orthogonal a 0

orthogonal a x ⇒ orthogonal a (scale c x)

orthogonal a x ⇒ orthogonal a (- x)

orthogonal a x ⇒ orthogonal a y ⇒ orthogonal a (x + y)

orthogonal a x ⇒ orthogonal a y ⇒ orthogonal a (x - y)

orthogonal 0 a

orthogonal x a ⇒ orthogonal (scale c x) a

orthogonal x a ⇒ orthogonal (- x) a

orthogonal x a ⇒ orthogonal y a ⇒ orthogonal (x + y) a

orthogonal x a ⇒ orthogonal y a ⇒ orthogonal (x - y) a

<proof>

lemma *inner-commute-zero*: $(\text{inner } xa \ x = 0) = (\text{inner } x \ xa = 0)$
<proof>

lemma *vector-sub-project-orthogonal*:
 $\text{inner } b \ (x - \text{scale } (\text{inner } x \ b / (\text{inner } b \ b)) \ b) = 0$
<proof>

lemma *orthogonal-commute*: $\text{orthogonal } x \ y \longleftrightarrow \text{orthogonal } y \ x$
<proof>

lemma *pairwise-orthogonal-insert*:
assumes *pairwise orthogonal* S
and $\bigwedge y. y \in S \implies \text{orthogonal } x \ y$
shows *pairwise orthogonal* $(\text{insert } x \ S)$
<proof>

end

lemma *sum-0-all*:
assumes $a: \forall a \in A. f \ a \geq (0 :: \text{real})$
and $s0: \text{sum } f \ A = 0$ **and** $f: \text{finite } A$
shows $\forall a \in A. f \ a = 0$
<proof>

14.4 Vecs as inner product spaces

locale *vec-real-inner* = $F?$: *inner-product-space* $((*) :: 'a \Rightarrow 'a \Rightarrow 'a)$ *inner-field*
for *inner-field* $:: 'a \Rightarrow 'a \Rightarrow 'a :: \{\text{field}, \text{cnj}, \text{real-of-extended}\}$
+ fixes *inner* $:: 'a \wedge^n \Rightarrow 'a \wedge^n \Rightarrow 'a$
assumes *inner-vec-def*: $\text{inner } x \ y = \text{sum } (\lambda i. \text{inner-field } (x \$ i) \ (y \$ i)) \ \text{UNIV}$
begin

lemma *inner-ge-zero [simp]*: $0 \leq \text{real-of } (\text{inner } x \ x)$
<proof>

lemma *real-scalar-mult2*: $\text{real-of } (\text{inner } x \ x) \ *_{\mathbb{R}} \ A = \text{inner } x \ x \ * \ A$
<proof>

lemma *i1*: $\text{inner } x \ y = \text{cnj } (\text{inner } y \ x)$
<proof>

lemma *i2*: $\text{inner } (x + y) \ z = \text{inner } x \ z + \text{inner } y \ z$
<proof>

lemma *i3*: $\text{inner } (r \ *s \ x) \ y = r \ * \ \text{inner } x \ y$
<proof>

lemma *i4*: **assumes** *inner x x = 0*
shows *x = 0*
 \langle *proof* \rangle

lemma *inner-0-0[simp]*: *inner 0 0 = 0*
 \langle *proof* \rangle

sublocale *v?*: *inner-product-space* ((*s*) :: '*a* \Rightarrow '*a*^{*n*} \Rightarrow '*a*^{*n*}) *inner*
 \langle *proof* \rangle
end

14.5 Matrices and inner product

locale *matrix* =
COLS?: *vec-real-inner* $\lambda x y. x * cnj y$ *inner-cols*
+ *ROWS?*: *vec-real-inner* $\lambda x y. x * cnj y$ *inner-rows*
for *inner-cols* :: '*a*^{*cols*}::{*finite*, *wellorder*} \Rightarrow '*a*^{*cols*}::{*finite*, *wellorder*} \Rightarrow
'*a*::{*field*, *cnj*, *real-of-extended*}
and *inner-rows* :: '*a*^{*rows*}::{*finite*, *wellorder*} \Rightarrow '*a*^{*rows*}::{*finite*, *wellorder*} \Rightarrow
'*a*
begin

lemma *dot-lmul-matrix*: *inner-rows* (*x v* A*) *y* = *inner-cols* *x* ((χ *i j. cnj* (*A* \$ *i* \$ *j*)) **v y*)
 \langle *proof* \rangle

end

14.6 Orthogonal complement generalized

context *inner-product-space*
begin

definition *orthogonal-complement* $W = \{x. \forall y \in W. \text{orthogonal } y x\}$

lemma *subspace-orthogonal-complement*: *subspace* (*orthogonal-complement* *W*)
 \langle *proof* \rangle

lemma *orthogonal-complement-mono*:
assumes *A-in-B*: $A \subseteq B$
shows *orthogonal-complement* $B \subseteq$ *orthogonal-complement* *A*
 \langle *proof* \rangle

lemma *B-in-orthogonal-complement-of-orthogonal-complement*:
shows $B \subseteq$ *orthogonal-complement* (*orthogonal-complement* *B*)
 \langle *proof* \rangle

end

14.7 Generalizing projections

context *inner-product-space*

begin

Projection of two vectors: v onto u

definition $\text{proj } v \ u = \text{scale } (\text{inner } v \ u / \text{inner } u \ u) \ u$

Projection of a onto S

definition $\text{proj-onto } a \ S = (\text{sum } (\lambda x. \text{proj } a \ x) \ S)$

lemma *vector-sub-project-orthogonal-proj*:

shows $\text{inner } b \ (x - \text{proj } x \ b) = 0$

<proof>

lemma *orthogonal-proj-set*:

assumes $y \in C$ **and** C : *finite* C **and** p : *pairwise orthogonal* C

shows *orthogonal* $(a - \text{proj-onto } a \ C) \ y$

<proof>

lemma *pairwise-orthogonal-proj-set*:

assumes C : *finite* C **and** p : *pairwise orthogonal* C

shows *pairwise orthogonal* $(\text{insert } (a - \text{proj-onto } a \ C) \ C)$

<proof>

end

lemma *orthogonal-real-eq*: $\text{RV-inner.orthogonal} = \text{real-inner-class.orthogonal}$

<proof>

14.8 Second Part of the Fundamental Theorem of Linear Algebra generalized

context *matrix*

begin

lemma *cnj-cnj-matrix[simp]*: $(\chi \ i \ j. \text{cnj } ((\chi \ i \ j. \text{cnj } (A \ \$ \ i \ \$ \ j)) \ \$ \ i \ \$ \ j)) = A$

<proof>

lemma *cnj-transpose[simp]*: $(\chi \ i \ j. \text{cnj } (\text{transpose } A \ \$ \ i \ \$ \ j)) = \text{transpose } (\chi \ i \ j. \text{cnj } (A \ \$ \ i \ \$ \ j))$

<proof>

lemma *null-space-orthogonal-complement-row-space*:

fixes $A::'a \ ^\wedge \text{cols}::\{\text{finite}, \text{wellorder}\} \ ^\wedge \text{rows}::\{\text{finite}, \text{wellorder}\}$

shows $\text{null-space } A = \text{COLS.v.orthogonal-complement } (\text{row-space } (\chi \ i \ j. \text{cnj } (A \ \$ \ i \ \$ \ j)))$

<proof>

lemma *left-null-space-orthogonal-complement-col-space:*
fixes $A::'a\ ^\wedge cols::\{finite, wellorder\}^\wedge rows::\{finite, wellorder\}$
shows $left\text{-null-space } A = ROWS.v.orthogonal\text{-complement } (col\text{-space } (\chi\ i\ j. cnj\ (A\ \$\ i\ \$\ j)))$
 $\langle proof \rangle$

end

We can get the explicit results for complex and real matrices

interpretation *real-matrix: matrix $\lambda x\ y::real\ ^\wedge cols::\{finite, wellorder\}$.*
 $sum\ (\lambda i. (x\ \$\ i) * (y\ \$\ i))\ UNIV\ \lambda x\ y. sum\ (\lambda i. (x\ \$\ i) * (y\ \$\ i))\ UNIV$
 $\langle proof \rangle$

interpretation *complex-matrix: matrix $\lambda x\ y::complex\ ^\wedge cols::\{finite, wellorder\}$.*
 $sum\ (\lambda i. (x\ \$\ i) * cnj\ (y\ \$\ i))\ UNIV\ \lambda x\ y. sum\ (\lambda i. (x\ \$\ i) * cnj\ (y\ \$\ i))\ UNIV$
 $\langle proof \rangle$

lemma *null-space-orthogonal-complement-row-space-complex:*
fixes $A::complex\ ^\wedge cols::\{finite, wellorder\}^\wedge rows::\{finite, wellorder\}$
shows $null\text{-space } A = complex\text{-matrix.orthogonal-complement } (row\text{-space } (\chi\ i\ j. cnj\ (A\ \$\ i\ \$\ j)))$
 $\langle proof \rangle$

lemma *left-null-space-orthogonal-complement-col-space-complex:*
fixes $A::complex\ ^\wedge cols::\{finite, wellorder\}^\wedge rows::\{finite, wellorder\}$
shows $left\text{-null-space } A = complex\text{-matrix.orthogonal-complement } (col\text{-space } (\chi\ i\ j. cnj\ (A\ \$\ i\ \$\ j)))$
 $\langle proof \rangle$

lemma *null-space-orthogonal-complement-row-space-reals:*
fixes $A::real\ ^\wedge cols::\{finite, wellorder\}^\wedge rows::\{finite, wellorder\}$
shows $null\text{-space } A = real\text{-matrix.orthogonal-complement } (row\text{-space } A)$
 $\langle proof \rangle$

lemma *left-null-space-orthogonal-complement-col-space-real:*
fixes $A::real\ ^\wedge cols::\{finite, wellorder\}^\wedge rows::\{finite, wellorder\}$
shows $left\text{-null-space } A = real\text{-matrix.orthogonal-complement } (col\text{-space } A)$
 $\langle proof \rangle$

end

15 Improvements to get better performance of the algorithm

theory *QR-Efficient*

```
imports QR-Decomposition-IArrays
begin
```

15.1 Improvements for computing the Gram Schmidt algorithm and QR decomposition using vecs

Essentially, we try to avoid removing duplicates in each iteration. They will not affect the *sum-list* since the duplicates will be the vector zero.

15.1.1 New definitions

definition *Gram-Schmidt-column-k-efficient* $A\ k$
 $= (\chi\ a\ b.\ (if\ b = from\ nat\ k$
then $column\ b\ A - sum\ list\ (map\ (\lambda x.\ ((column\ b\ A \cdot x) / (x \cdot x)) *_{\mathbb{R}}\ x)$
 $((map\ (\lambda n.\ column\ (from\ nat\ n)\ A)\ [0..<to\ nat\ b])))\ else\ column\ b\ A)\ \$\ a)$

15.1.2 General properties about *sum-list*

lemma *sum-list-remdups*:
assumes $!!i\ j.\ i < length\ xs \wedge j < length\ xs \wedge i \neq j$
 $\wedge xs\ !\ i = xs\ !\ j \longrightarrow xs\ !\ i = 0 \wedge xs\ !\ j = 0$
shows $sum\ list\ (remdups\ xs) = sum\ list\ xs$
 $\langle proof \rangle$

lemma *sum-list-remdups-2*:
fixes $f :: 'a :: \{zero, monoid\text{-}add\} \Rightarrow 'a$
assumes $!!i\ j.\ i < length\ xs \wedge j < length\ xs \wedge i \neq j \wedge (xs\ !\ i) = (xs\ !\ j)$
 $\longrightarrow f\ (xs\ !\ i) = 0 \wedge f\ (xs\ !\ j) = 0$
shows $sum\ list\ (map\ f\ (remdups\ xs)) = sum\ list\ (map\ f\ xs)$
 $\langle proof \rangle$

15.1.3 Proving a code equation to improve the performance

lemma *set-map-column*:
 $set\ (map\ (\lambda n.\ column\ (from\ nat\ n)\ G)\ [0..<to\ nat\ b]) = \{column\ i\ G \mid i.\ i < b\}$
 $\langle proof \rangle$

lemma *column-Gram-Schmidt-column-k-repeated-0*:
fixes $A :: 'a :: \{real\text{-}inner\} \wedge 'n :: \{mod\text{-}type\} \wedge 'm :: \{mod\text{-}type\}$
assumes $i\text{-not-}k:\ i \neq k$ **and** $ik:\ i < k$
and $c\text{-eq}:\ column\ k\ (Gram\ Schmidt\ column\ k\ A\ (to\ nat\ k))$
 $= column\ i\ (Gram\ Schmidt\ column\ k\ A\ (to\ nat\ k))$
and $o:$ *pairwise orthogonal* $\{column\ i\ A \mid i.\ i < k\}$
shows $column\ k\ (Gram\ Schmidt\ column\ k\ A\ (to\ nat\ k)) = 0$
and $column\ i\ (Gram\ Schmidt\ column\ k\ A\ (to\ nat\ k)) = 0$
 $\langle proof \rangle$

lemma *column-Gram-Schmidt-upt-k-repeated-0'*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i\text{-not-}k: i \neq j$ **and** $ij: i < j$ **and** $j: j \leq \text{from-nat } k$
and $c\text{-eq}: \text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ } k)$
 $= \text{column } i \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ } k)$
and $k: k < \text{ncols } A$
shows $\text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ } k) = 0$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-upt-k-repeated-0*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i\text{-not-}k: i \neq j$ **and** $ij: i < j$ **and** $j: j \leq k$
and $c\text{-eq}: \text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k))$
 $= \text{column } i \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k))$
shows $\text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k)) = 0$
 $\langle \text{proof} \rangle$

corollary *column-Gram-Schmidt-upt-k-repeated*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $i\text{-not-}k: i \neq j$ **and** $ij: i \leq k$ **and** $j \leq k$
and $c\text{-eq}: \text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k))$
 $= \text{column } i \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k))$
shows $\text{column } j \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k)) = 0$
and $\text{column } i \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ (to-nat } k)) = 0$
 $\langle \text{proof} \rangle$

lemma *column-Gram-Schmidt-column-k-eq-efficient*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $\text{Gram-Schmidt-upt-}k \text{ } A \text{ } k = \text{foldl Gram-Schmidt-column-k-efficient } A$
 $[0..<\text{Suc } k]$
and $\text{suc-}k: \text{Suc } k < \text{ncols } A$
shows $\text{column } b \text{ (Gram-Schmidt-column-}k \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ } k) \text{ (Suc } k))$
 $= \text{column } b \text{ (Gram-Schmidt-column-}k\text{-efficient (Gram-Schmidt-upt-}k \text{ } A \text{ } k) \text{ (Suc } k))$
 $\langle \text{proof} \rangle$

lemma *Gram-Schmidt-upt-k-efficient-induction*:
fixes $A::\text{real}^n::\{\text{mod-type}\}^m::\{\text{mod-type}\}$
assumes $\text{Gram-Schmidt-upt-}k \text{ } A \text{ } k = \text{foldl Gram-Schmidt-column-k-efficient } A$
 $[0..<\text{Suc } k]$
and $\text{suc-}k: \text{Suc } k < \text{ncols } A$
shows $\text{Gram-Schmidt-column-}k \text{ (Gram-Schmidt-upt-}k \text{ } A \text{ } k) \text{ (Suc } k)$

= *Gram-Schmidt-column-k-efficient* (*Gram-Schmidt-upt-k* A k) (*Suc* k)
 ⟨*proof*⟩

lemma *Gram-Schmidt-upt-k-efficient*:
fixes A::*real*ⁿ::{*mod-type*}^m::{*mod-type*}
assumes k: k < n*cols* A
shows *Gram-Schmidt-upt-k* A k = *foldl Gram-Schmidt-column-k-efficient* A [0..*Suc* k]
 ⟨*proof*⟩

This equation is now more efficient than the original definition of the algorithm, since it is not removing duplicates in each iteration, which is more expensive in time than adding zeros (if there appear duplicates while applying the algorithm, they are zeros and then the *sum-list* is the same in each step).

lemma *Gram-Schmidt-matrix-efficient*[*code-unfold*]:
fixes A::*real*ⁿ::{*mod-type*}^m::{*mod-type*}
shows *Gram-Schmidt-matrix* A = *foldl Gram-Schmidt-column-k-efficient* A [0..*ncols* A]
 ⟨*proof*⟩

15.2 Improvements for computing the Gram Schmidt algorithm and QR decomposition using immutable arrays

15.2.1 New definitions

definition *Gram-Schmidt-column-k-iarrays-efficient* A k =
tabulate2 (*nrows-iarray* A) (*ncols-iarray* A) (λa b. let *column-b-A* = *column-iarray* b A in
 (if b = k then (*column-b-A* - *sum-list* (*map* (λx . ((*column-b-A* · i x) / (x · i x))
 *_R x)
 ((*List.map* (λn . *column-iarray* n A) [0..*b*]))))
 else *column-b-A*) !! a)

definition *Gram-Schmidt-matrix-iarrays-efficient* A
 = *foldl Gram-Schmidt-column-k-iarrays-efficient* A [0..*ncols-iarray* A]

definition *QR-decomposition-iarrays-efficient* A =
 (let Q = *divide-by-norm-iarray* (*Gram-Schmidt-matrix-iarrays-efficient* A)
 in (Q, *transpose-iarray* Q **i A))

15.2.2 General properties

lemma *tabulate2-nth*:
assumes i: i < nr and j: j < nc
shows (*tabulate2* nr nc f) !! i !! j = f i j
 ⟨*proof*⟩

lemma *vec-to-iarray-minus*[code-unfold]:
 $vec\text{-to-iarray } (a - b) = (vec\text{-to-iarray } a) - (vec\text{-to-iarray } b)$
 ⟨proof⟩

lemma *vec-to-iarray-minus-nth*:
assumes $A: i < IArray.length (vec\text{-to-iarray } A)$
and $B: i < IArray.length (vec\text{-to-iarray } B)$
shows $(vec\text{-to-iarray } A - vec\text{-to-iarray } B) !! i$
 $= vec\text{-to-iarray } A !! i - vec\text{-to-iarray } B !! i$
 ⟨proof⟩

lemma *sum-list-map-vec-to-iarray*:
assumes $xs \neq []$
shows $sum\text{-list } (map (vec\text{-to-iarray } \circ f) xs) = vec\text{-to-iarray } (sum\text{-list } (map f xs))$
 ⟨proof⟩

15.2.3 Proving the equivalence

lemma *matrix-to-iarray-Gram-Schmidt-column-k-efficient*:
fixes $A::real^{n::\{mod\text{-type}\}}^{m::\{mod\text{-type}\}}$
assumes $k: k < ncols A$
shows $matrix\text{-to-iarray } (Gram\text{-Schmidt-column-k-efficient } A k)$
 $= Gram\text{-Schmidt-column-k-iarrays-efficient } (matrix\text{-to-iarray } A) k$
 ⟨proof⟩

lemma *matrix-to-iarray-Gram-Schmidt-upt-k-efficient*:
fixes $A::real^{n::\{mod\text{-type}\}}^{m::\{mod\text{-type}\}}$
assumes $k: k < ncols A$
shows $matrix\text{-to-iarray } (Gram\text{-Schmidt-upt-k } A k)$
 $= foldl Gram\text{-Schmidt-column-k-iarrays-efficient } (matrix\text{-to-iarray } A) [0..<Suc k]$
 ⟨proof⟩

lemma *matrix-to-iarray-Gram-Schmidt-matrix-efficient*[code-unfold]:
fixes $A::real^{n::\{mod\text{-type}\}}^{m::\{mod\text{-type}\}}$
shows $matrix\text{-to-iarray } (Gram\text{-Schmidt-matrix } A)$
 $= Gram\text{-Schmidt-matrix-iarrays-efficient } (matrix\text{-to-iarray } A)$
 ⟨proof⟩

lemma *QR-decomposition-iarrays-efficient*[code]:
 $QR\text{-decomposition-iarrays } (matrix\text{-to-iarray } A)$
 $= QR\text{-decomposition-iarrays-efficient } (matrix\text{-to-iarray } A)$
 ⟨proof⟩

15.3 Other code equations that improve the performance

lemma *inner-iarray-code*[code]:

inner-iarray A B = *sum-list* (map ($\lambda n. A !! n * B !! n$) [0..*IArray.length* A])
<proof>

definition *Gram-Schmidt-column-k-iarrays-efficient2* A k =

tabulate2 (*nrows-iarray* A) (*ncols-iarray* A)

(let col-k = *column-iarray* k A;

col = (col-k - *sum-list* (map ($\lambda x. ((col-k \cdot i x) / (x \cdot i x)) *_R x$)

((*List.map* ($\lambda n. column-iarray\ n\ A\ [0..<k]$))))

in ($\lambda a\ b. (if\ b = k\ then\ col\ else\ column-iarray\ b\ A) !! a$))

lemma *Gram-Schmidt-column-k-iarrays-efficient-eq*[code]: *Gram-Schmidt-column-k-iarrays-efficient*
A k

= *Gram-Schmidt-column-k-iarrays-efficient2* A k

<proof>

end