Pushdown Systems

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Abstract

We formalize pushdown systems and the correctness of the pushdown reachability algorithms post* (forward search), pre* (backward search) and dual* (bi-directional search). For pre* we refine the algorithm to an executable version for which one can generate code using Isabelle's code generator. For pre* and post* we follow Stefan Schwoon's PhD thesis [Sch02a]. The dual* algorithm is from a paper by Jensen et. al presented at ATVA2021 [JSS+21]. The formalization is described in our FMCAD2022 paper [SSST22] in which we also document how we have used it to do differential testing against a C++ implementation of pushdown reachability called PDAAAL. Lammich et al. [Lam09, LMW09] formalized the pre* algorithm for dynamic pushdown networks (DPN) which is a generalization of pushdown systems. Our work is independent from that because the post* of DPNs is not regular and additionally the DPN formalization does not support epsilon transitions which we use for post* and dual*.

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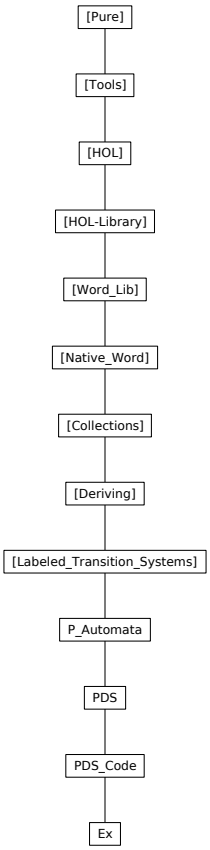

Figure 1: Theory dependency graph

1 Introduction

Pushdown reachability was studied by Büchi in 1964 [Büc64] and has been used for, among other things, interprocedural control-flow analysis of recursive programs [EK99, CNDE05], model checking [ES01, Sch02b, SSE05, BEM97] and communication network analysis [JKM+18, JKS+20, vDJJ+21]. In this formalization we formalize the pre* and post* algorithms [Sch02a] and the dual* algorithm [JSS+21]. For pre* we have also an executable version. In our FMCAD2022 paper [SSST22] we describe the formalization and use it to do differential testing against a C++ implementation of pushdown reachability called PDAAAL [JSS+21]. The differential testing revealed a number of bugs in PDAAAL that we were then able to fix.

theory *P_Automata* **imports** *Labeled_Transition_Systems.LTS* **begin**

2 Automata

2.1 P-Automaton locale

locale *P_Automaton* = *LTS transition_relation*
  for *transition_relation* :: "('state::finite, 'label) transition set" +
  fixes *Init* :: "'ctr_loc::enum ⇒ 'state"
    and *finals* :: "'state set"
begin

definition *initials* :: "'state set" **where**
  "*initials* ≡ *Init* ‘ *UNIV*"

lemma *initials_list*:
  "*initials* = *set* (*map Init Enum.enum*)"
  using *enum_UNIV* **unfolding** *initials_def* **by** *force*

definition *accepts_aut* :: "'ctr_loc ⇒ 'label list ⇒ *bool*" **where**
  "*accepts_aut* ≡ λ*p w*. (∃ *q* ∈ *finals*. (*Init p*, *w*, *q*) ∈ *trans_star*)"

definition *lang_aut* :: "('ctr_loc ∗ 'label list) *set*" **where**
  "*lang_aut* = {(*p*,*w*). *accepts_aut p w*}"

definition *nonempty* **where**
  "*nonempty* ⟷ *lang_aut* ≠ {}"

lemma *nonempty_alt*:
  "*nonempty* ⟷ (∃ *p*. ∃ *q* ∈ *finals*. ∃ *w*. (*Init p*, *w*, *q*) ∈ *trans_star*)"
  unfolding *lang_aut_def nonempty_def accepts_aut_def* **by** *auto*

typedef 'a *mark_state* = "{(*Q* :: 'a *set*, *I*). *I* ⊆ *Q*}"
  by *auto*
setup-lifting *type_definition_mark_state*
lift-definition *get_visited* :: "'a *mark_state* ⇒ 'a *set*" **is** *fst* .
lift-definition *get_next* :: "'a *mark_state* ⇒ 'a *set*" **is** *snd* .
lift-definition *make_mark_state* :: "'a *set* ⇒ 'a *set* ⇒ 'a *mark_state*" **is** "λ*Q J*. (*Q* ∪ *J*, *J*)" **by** *auto*
lemma *get_next_get_visited*: "*get_next ms* ⊆ *get_visited ms*"
  by *transfer auto*
lemma *get_next_set_next*[*simp*]: "*get_next* (*make_mark_state Q J*) = *J*"
  by *transfer auto*
lemma *get_visited_set_next*[*simp*]: "*get_visited* (*make_mark_state Q J*) = *Q* ∪ *J*"
  by *transfer auto*

function *mark* **where**
  "*mark ms* ⟷
    (*let Q* = *get_visited ms*; *I* = *get_next ms in*
    if I ∩ *finals* ≠ {} *then True*
    else let J = (⋃(*q*,*w*,*q'*)∈*transition_relation*. *if q* ∈ *I* ∧ *q'* ∉ *Q then* {*q'*} *else* {}) *in*
      if J = {} *then False else mark* (*make_mark_state Q J*))"
  by *auto*
termination by (*relation* "*measure* (λ*ms*. *card* (*UNIV* :: 'a *set*) − *card* (*get_visited ms* :: 'a *set*))")
    (*fastforce intro*!: *diff_less_mono2 psubset_card_mono split*: *if_splits*)+

declare *mark.simps*[*simp del*]

lemma *trapped_transitions*: "(*p*, *w*, *q*) ∈ *trans_star* ⟹
  ∀ *p* ∈ *Q*. (∀ γ *q*. (*p*, γ, *q*) ∈ *transition_relation* ⟶ *q* ∈ *Q*) ⟹
  p ∈ *Q* ⟹ *q* ∈ *Q*"
  by (*induct p w q rule*: *trans_star.induct*) *auto*

lemma *mark_complete*: "(*p*, *w*, *q*) ∈ *trans_star* ⟹ (*get_visited ms* − *get_next ms*) ∩ *finals* = {} ⟹

```
  ∀ p ∈ get_visited ms − get_next ms. ∀ q γ. (p, γ, q) ∈ transition_relation ⟶ q ∈ get_visited ms ⟹
  p ∈ get_visited ms ⟹ q ∈ finals ⟹ mark ms"
proof (induct p w q arbitrary: ms rule: trans_star.induct)
  case (trans_star_refl p)
  then show ?case by (subst mark.simps) (auto simp: Let_def)
next
  case step: (trans_star_step p γ q' w q)
  define J where "J ≡ ⋃(q, w, q')∈transition_relation. if q ∈ get_next ms ∧ q' ∉ get_visited ms then {q'} else
{}"
  show ?case
  proof (cases "J = {}")
    case True
    then have "q' ∈ get_visited ms"
      by (smt (z3) DiffI Diff_disjoint Int_iff J_def SUP_bot_conv(2) case_prod_conv insertI1
          step.hyps(1) step.prems(2) step.prems(3))
    with True show ?thesis
      using step(1,2,4,5,7)
      by (subst mark.simps)
        (auto 10 0 intro!: step(3) elim!: set_mp[of _ "get_next ms"] simp: split_beta J_def
          dest: trapped_transitions[of q' w q "get_visited ms"])
  next
    case False
    then have [simp]: "get_visited ms ∪ J − J = get_visited ms"
      by (auto simp: J_def split: if_splits)
    then have "p ∈ get_visited ms ⟹ (p, γ, q) ∈ transition_relation ⟹ q ∉ get_visited ms ⟹ q ∈ J" for p γ q
      using step(5)
      by (cases "p ∈ get_next ms")
        (auto simp only: J_def simp_thms if_True if_False intro!: UN_I[of "(p, γ, q)"])
    with False show ?thesis
      using step(1,4,5,6,7)
      by (subst mark.simps)
        (auto 0 2 simp add: Let_def J_def[symmetric] disj_commute
        intro!: step(3)[of "make_mark_state (get_visited ms) J"])
  qed
qed


lemma mark_sound: "mark ms ⟹ (∃ p ∈ get_next ms. ∃ q ∈ finals. ∃ w. (p, w, q) ∈ trans_star)"
  by (induct ms rule: mark.induct)
    (subst (asm) (2) mark.simps, fastforce dest: trans_star_step simp: Let_def split: if_splits)

lemma nonempty_code[code]: "nonempty = mark (make_mark_state {} (set (map Init Enum.enum)))"
  using mark_complete[of _ _ _ "make_mark_state {} initials"]
      mark_sound[of "make_mark_state {} initials"] nonempty_alt
  unfolding initials_def initials_list[symmetric] by auto


end
```

2.2 Intersection P-Automaton locale

```
locale Intersection_P_Automaton =
  A1: P_Automaton ts1 Init finals1 +
  A2: P_Automaton ts2 Init finals2
  for ts1 :: "('state :: finite, 'label) transition set"
    and Init :: "'ctr_loc :: enum ⇒ 'state"
    and finals1 :: "'state set"
    and ts2 :: "('state, 'label) transition set"
    and finals2 :: "'state set"
begin

sublocale pa: P_Automaton "inters ts1 ts2" "(λp. (Init p, Init p))" "inters_finals finals1 finals2"
  .
```

```
definition accepts_aut_inters where
  "accepts_aut_inters p w = pa.accepts_aut p w"

definition lang_aut_inters :: "('ctr_loc * 'label list) set" where
  "lang_aut_inters = {(p,w). accepts_aut_inters p w}"

lemma trans_star_inter:
  assumes "(p1, w, p2) ∈ A1.trans_star"
  assumes "(q1, w, q2) ∈ A2.trans_star"
  shows "((p1,q1), w :: 'label list, (p2,q2)) ∈ pa.trans_star"
  using assms
proof (induction w arbitrary: p1 q1)
  case (Cons α w1')
  obtain p' where p'_p: "(p1, α, p') ∈ ts1 ∧ (p', w1', p2) ∈ A1.trans_star"
    using Cons by (metis LTS.trans_star_cons)
  obtain q' where q'_p: "(q1, α, q') ∈ ts2 ∧(q', w1', q2) ∈ A2.trans_star"
    using Cons by (metis LTS.trans_star_cons)
  have ind: "((p', q'), w1', p2, q2) ∈ pa.trans_star"
  proof −
    have "Suc (length w1') = length (α#w1')"
      by auto
    moreover
    have "(p', w1', p2) ∈ A1.trans_star"
      using p'_p by simp
    moreover
    have "(q', w1', q2) ∈ A2.trans_star"
      using q'_p by simp
    ultimately
    show "((p', q'), w1', p2, q2) ∈ pa.trans_star"
      using Cons(1) by auto
  qed
  moreover
  have "((p1, q1), α, (p', q')) ∈ (inters ts1 ts2)"
    by (simp add: inters_def p'_p q'_p)
  ultimately
  have "((p1, q1), α#w1', p2, q2) ∈ pa.trans_star"
    by (meson LTS.trans_star.trans_star_step)
  moreover
  have "length ((α#w1')) > 0"
    by auto
  moreover
  have "hd ((α#w1')) = α"
    by auto
  ultimately
  show ?case
    by force
next
  case Nil
  then show ?case
    by (metis LTS.trans_star.trans_star_refl LTS.trans_star_empty)
qed

lemma inters_trans_star1:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star"
  shows "(fst p1q2, w, fst p2q2) ∈ A1.trans_star"
  using assms
proof (induction rule: LTS.trans_star.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS.trans_star.trans_star_refl)
next
  case (2 p γ q' w q)
  then have ind: "(fst q', w, fst q) ∈ A1.trans_star"
```

```
    by auto
  from 2(1) have "(p, γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    unfolding inters_def by auto
  then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, γ, p2) ∈ ts1 ∧ (q1, γ, q2) ∈ ts2)"
    by simp
  then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, γ, p2) ∈ ts1 ∧ (q1, γ, q2) ∈ ts2)"
    by auto
  then show ?case
    using LTS.trans_star.trans_star_step ind by fastforce
qed

lemma inters_trans_star:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star"
  shows "(snd p1q2, w, snd p2q2) ∈ A2.trans_star"
  using assms
proof (induction rule: LTS.trans_star.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS.trans_star.trans_star_refl)
next
  case (2 p γ q' w q)
  then have ind: "(snd q', w, snd q) ∈ A2.trans_star"
    by auto
  from 2(1) have "(p, γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    unfolding inters_def by auto
  then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, γ, p2) ∈ ts1 ∧ (q1, γ, q2) ∈ ts2)"
    by simp
  then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, γ, p2) ∈ ts1 ∧ (q1, γ, q2) ∈ ts2)"
    by auto
  then show ?case
    using LTS.trans_star.trans_star_step ind by fastforce
qed

lemma inters_trans_star_iff:
  "((p1,q2), w :: 'label list, (p2,q2)) ∈ pa.trans_star ⟷ (p1, w, p2) ∈ A1.trans_star ∧ (q2, w, q2) ∈ A2.trans_star"
  by (metis fst_conv inters_trans_star inters_trans_star1 snd_conv trans_star_inter)

lemma inters_accept_iff: "accepts_aut_inters p w ⟷ A1.accepts_aut p w ∧ A2.accepts_aut p w"
proof
  assume "accepts_aut_inters p w"
  then show "A1.accepts_aut p w ∧ A2.accepts_aut p w"
    unfolding accepts_aut_inters_def A1.accepts_aut_def A2.accepts_aut_def pa.accepts_aut_def
    unfolding inters_finals_def
    using inters_trans_star_iff[of _ _ w _ ]
    using SigmaE fst_conv inters_trans_star inters_trans_star1 snd_conv
    by (metis (no_types, lifting))
next
  assume a: "A1.accepts_aut p w ∧ A2.accepts_aut p w"
  then have "(∃ q∈finals1. (Init p, w, q) ∈ A1.trans_star) ∧
             (∃ q∈finals2. (Init p, w, q) ∈ A2.trans_star)"
    unfolding A1.accepts_aut_def A2.accepts_aut_def by auto
  then show "accepts_aut_inters p w"
    unfolding accepts_aut_inters_def pa.accepts_aut_def inters_finals_def
    by (auto simp: P_Automaton.accepts_aut_def intro: trans_star_inter)
qed

lemma lang_aut_alt:
  "pa.lang_aut = {(p, w). (p, w) ∈ lang_aut_inters}"
  unfolding pa.lang_aut_def lang_aut_inters_def accepts_aut_inters_def pa.accepts_aut_def
  by auto
```

lemma *inters_lang*: *"lang_aut_inters = A1.lang_aut ∩ A2.lang_aut"*
unfolding *lang_aut_inters_def A1.lang_aut_def A2.lang_aut_def* **using** *inters_accept_iff* **by** *auto*

end

3 Automata with epsilon

3.1 P-Automaton with epsilon locale

locale *P_Automaton_ε = LTS_ε transition_relation* **for** *transition_relation* :: *"('state::finite, 'label option) transition set" +*
fixes *finals* :: *"'state set"* **and** *Init* :: *"'ctr_loc :: enum ⇒ 'state"*
begin

definition *accepts_aut_ε* :: *"'ctr_loc ⇒ 'label list ⇒ bool"* **where**
"accepts_aut_ε ≡ λp w. (∃ q ∈ finals. (Init p, w, q) ∈ trans_star_ε)"

definition *lang_aut_ε* :: *"('ctr_loc * 'label list) set"* **where**
"lang_aut_ε = {(p,w). accepts_aut_ε p w}"

definition *nonempty_ε* **where**
"nonempty_ε ⟷ lang_aut_ε ≠ {}"

end

3.2 Intersection P-Automaton with epsilon locale

locale *Intersection_P_Automaton_ε =*
A1: *P_Automaton_ε ts1 finals1 Init +*
A2: *P_Automaton_ε ts2 finals2 Init*
for *ts1* :: *"('state :: finite, 'label option) transition set"*
and *finals1* :: *"'state set"*
and *Init* :: *"'ctr_loc :: enum ⇒ 'state"*
and *ts2* :: *"('state, 'label option) transition set"*
and *finals2* :: *"'state set"*
begin

abbreviation *ε* :: *"'label option"* **where**
"ε == None"

sublocale *pa*: *P_Automaton_ε "inters_ε ts1 ts2" "inters_finals finals1 finals2" "(λp. (Init p, Init p))"*
.

definition *accepts_aut_inters_ε* **where**
"accepts_aut_inters_ε p w = pa.accepts_aut_ε p w"

definition *lang_aut_inters_ε* :: *"('ctr_loc * 'label list) set"* **where**
"lang_aut_inters_ε = {(p,w). accepts_aut_inters_ε p w}"

lemma *trans_star_trans_star_ε_inter*:
assumes *"LTS_ε.ε_exp w1 w"*
assumes *"LTS_ε.ε_exp w2 w"*
assumes *"(p1, w1, p2) ∈ A1.trans_star"*
assumes *"(q1, w2, q2) ∈ A2.trans_star"*
shows *"((p1,q1), w :: 'label list, (p2,q2)) ∈ pa.trans_star_ε"*
using *assms*
proof (*induction "length w1 + length w2" arbitrary*: *w1 w2 w p1 q1 rule*: *less_induct*)
case *less*
then show *?case*
proof (*cases "∃α w1' w2' β. w1=Some α#w1' ∧ w2=Some β#w2'"*)
case *True*
from *True* **obtain** *α β w1' w2'* **where** *True''*:

```
    "w1=Some α#w1'"
    "w2=Some β#w2'"
    by auto
  have "α = β"
    by (metis True''(1) True''(2) LTS_ε.ε_exp_Some_hd less.prems(1) less.prems(2))
  then have True':
    "w1=Some α#w1'"
    "w2=Some α#w2'"
    using True'' by auto
  define w' where "w' = tl w"
  obtain p' where p'_p: "(p1, Some α, p') ∈ ts1 ∧ (p', w1', p2) ∈ A1.trans_star"
    using less True'(1) by (metis LTS_ε.trans_star_cons_ε)
  obtain q' where q'_p: "(q1, Some α, q') ∈ ts2 ∧(q', w2', q2) ∈ A2.trans_star"
    using less True'(2) by (metis LTS_ε.trans_star_cons_ε)
  have ind: "((p', q'), w', p2, q2) ∈ pa.trans_star_ε"
  proof −
    have "length w1' + length w2' < length w1 + length w2"
      using True'(1) True'(2) by simp
    moreover
    have "LTS_ε.ε_exp w1' w'"
      by (metis (no_types) LTS_ε.ε_exp_def less(2) True'(1) list.map(2) list.sel(3)
          option.simps(3) removeAll.simps(2) w'_def)
    moreover
    have "LTS_ε.ε_exp w2' w'"
      by (metis (no_types) LTS_ε.ε_exp_def less(3) True'(2) list.map(2) list.sel(3)
          option.simps(3) removeAll.simps(2) w'_def)
    moreover
    have "(p', w1', p2) ∈ A1.trans_star"
      using p'_p by simp
    moreover
    have "(q', w2', q2) ∈ A2.trans_star"
      using q'_p by simp
    ultimately
    show "((p', q'), w', p2, q2) ∈ pa.trans_star_ε"
      using less(1)[of w1' w2' w' p' q'] by auto
  qed
  moreover
  have "((p1, q1), Some α, (p', q')) ∈ (inters_ε ts1 ts2)"
    by (simp add: inters_ε_def p'_p q'_p)
  ultimately
  have "((p1, q1), α#w', p2, q2) ∈ pa.trans_star_ε"
    by (meson LTS_ε.trans_star_ε.trans_star_ε_step_γ)
  moreover
  have "length w > 0"
    using less(3) True' LTS_ε.ε_exp_Some_length by metis
  moreover
  have "hd w = α"
    using less(3) True' LTS_ε.ε_exp_Some_hd by metis
  ultimately
  show ?thesis
    using w'_def by force
next
  case False
  note False_outer_outer_outer_outer = False
  show ?thesis
  proof (cases "w1 = [] ∧ w2 = []")
    case True
    then have same: "p1 = p2 ∧ q1 = q2"
      by (metis LTS.trans_star_empty less.prems(3) less.prems(4))
    have "w = []"
      using True less(2) LTS_ε.exp_empty_empty by auto
    then show ?thesis
      using less True
```

```
    by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl same)
next
  case False
  note False_outer_outer_outer = False
  show ?thesis
  proof (cases "∃ w1'. w1=ε#w1'")
    case True
    then obtain w1' where True':
      "w1=ε#w1'"
      by auto
    obtain p' where p'_p: "(p1, ε, p') ∈ ts1 ∧ (p', w1', p2) ∈ A1.trans_star"
      using less True'(1) by (metis LTS_ε.trans_star_cons_ε)
    have q'_p: "(q1, w2, q2) ∈ A2.trans_star"
      using less by metis
    have ind: "((p', q1), w, p2, q2) ∈ pa.trans_star_ε"
    proof −
      have "length w1' + length w2 < length w1 + length w2"
        using True'(1) by simp
      moreover
      have "LTS_ε.ε_exp w1' w"
        by (metis (no_types) LTS_ε.ε_exp_def less(2) True'(1) removeAll.simps(2))
      moreover
      have "LTS_ε.ε_exp w2 w"
        by (metis (no_types) less(3))
      moreover
      have "(p', w1', p2) ∈ A1.trans_star"
        using p'_p by simp
      moreover
      have "(q1, w2, q2) ∈ A2.trans_star"
        using q'_p by simp
      ultimately
      show "((p', q1), w, p2, q2) ∈ pa.trans_star_ε"
        using less(1)[of w1' w2 w p' q1] by auto
    qed
    moreover
    have "((p1, q1), ε, (p', q1)) ∈ (inters_ε ts1 ts2)"
      by (simp add: inters_ε_def p'_p q'_p)
    ultimately
    have "((p1, q1), w, p2, q2) ∈ pa.trans_star_ε"
      using LTS_ε.trans_star_ε.simps by fastforce
    then
    show ?thesis
      by force
  next
    case False
    note False_outer_outer = False
    then show ?thesis
    proof (cases "∃ w2'. w2 = ε # w2'")
      case True
      then obtain w2' where True':
        "w2=ε#w2'"
        by auto
      have p'_p: "(p1, w1, p2) ∈ A1.trans_star"
        using less by (metis)
      obtain q' where q'_p: "(q1, ε, q') ∈ ts2 ∧(q', w2', q2) ∈ A2.trans_star"
        using less True'(1) by (metis LTS_ε.trans_star_cons_ε)
      have ind: "((p1, q'), w, p2, q2) ∈ pa.trans_star_ε"
      proof −
        have "length w1 + length w2' < length w1 + length w2"
          using True'(1) True'(1) by simp
        moreover
        have "LTS_ε.ε_exp w1 w"
          by (metis (no_types) less(2))
```

```
          moreover
          have "LTS_ε.ε_exp w2' w"
            by (metis (no_types) LTS_ε.ε_exp_def less(3) True'(1) removeAll.simps(2))
          moreover
          have "(p1, w1, p2) ∈ A1.trans_star"
            using p'_p by simp
          moreover
          have "(q', w2', q2) ∈ A2.trans_star"
            using q'_p by simp
          ultimately
          show "((p1, q'), w, p2, q2) ∈ pa.trans_star_ε"
            using less(1)[of w1 w2' w p1 q'] by auto
        qed
        moreover
        have "((p1, q1), ε, (p1, q')) ∈ inters_ε ts1 ts2"
          by (simp add: inters_ε_def p'_p q'_p)
        ultimately
        have "((p1, q1), w, p2, q2) ∈ pa.trans_star_ε"
          using LTS_ε.trans_star_ε.simps by fastforce
        then
        show ?thesis
          by force
      next
        case False
        then have "(w1 = [] ∧ (∃α w2'. w2 = Some α # w2')) ∨ ((∃α w1'. w1 = Some α # w1') ∧ w2 = [])"
          using False_outer_outer False_outer_outer_outer False_outer_outer_outer_outer
          by (metis neq_Nil_conv option.exhaust_sel)
        then show ?thesis
          by (metis LTS_ε.ε_exp_def LTS_ε.ε_exp_Some_length less.prems(1) less.prems(2)
              less_numeral_extra(3) list.simps(8) list.size(3) removeAll.simps(1))
      qed
    qed
  qed
qed

lemma trans_star_ε_inter:
  assumes "(p1, w :: 'label list, p2) ∈ A1.trans_star_ε"
  assumes "(q1, w, q2) ∈ A2.trans_star_ε"
  shows "((p1, q1), w, (p2, q2)) ∈ pa.trans_star_ε"
proof −
  have "∃w1'. LTS_ε.ε_exp w1' w ∧ (p1, w1', p2) ∈ A1.trans_star"
    using assms by (simp add: LTS_ε.trans_star_ε_ε_exp_trans_star)
  then obtain w1' where "LTS_ε.ε_exp w1' w ∧ (p1, w1', p2) ∈ A1.trans_star"
    by auto
  moreover
  have "∃w2'. LTS_ε.ε_exp w2' w ∧ (q1, w2', q2) ∈ A2.trans_star"
    using assms by (simp add: LTS_ε.trans_star_ε_ε_exp_trans_star)
  then obtain w2' where "LTS_ε.ε_exp w2' w ∧ (q1, w2', q2) ∈ A2.trans_star"
    by auto
  ultimately
  show ?thesis
    using trans_star_trans_star_ε_inter by metis
qed

lemma inters_trans_star_ε1:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star_ε"
  shows "(fst p1q2, w, fst p2q2) ∈ A1.trans_star_ε"
  using assms
proof (induction rule: LTS_ε.trans_star_ε.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl)
```

```
next
  case (2 p γ q' w q)
  then have ind: "(fst q', w, fst q) ∈ A1.trans_star_ε"
    by auto
  from 2(1) have "(p, Some γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts1}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ, q2) ∈
ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ,
q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_γ ind by fastforce
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have ?case
      by auto
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts1}"
    then have ?case
      by auto
  }
  ultimately
  show ?case
    by auto
next
  case (3 p q' w q)
  then have ind: "(fst q', w, fst q) ∈ A1.trans_star_ε"
    by auto
  from 3(1) have "(p, ε, q') ∈
                  {((p1, q1), α, (p2, q2)) | p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, (p2, q1)) | p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, (p1, q2)) | p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have "∃ p1 p2 q1. p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then obtain p1 p2 q1 where "p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then have ?case
```

```
    using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have "∃ p1 q1 q2. p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then obtain p1 q1 q2 where "p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  ultimately
  show ?case
    by auto
qed

lemma inters_trans_star_ε:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ pa.trans_star_ε"
  shows "(snd p1q2, w, snd p2q2) ∈ A2.trans_star_ε"
  using assms
proof (induction rule: LTS_ε.trans_star_ε.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl)
next
  case (2 p γ q' w q)
  then have ind: "(snd q', w, snd q) ∈ A2.trans_star_ε"
    by auto
  from 2(1) have "(p, Some γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ, q2) ∈
ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ,
q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_γ ind by fastforce
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have ?case
      by auto
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have ?case
      by auto
  }
  ultimately
  show ?case
    by auto
next
  case (3 p q' w q)
```

```
  then have ind: "(snd q', w, snd q) ∈ A2.trans_star_ε"
    by auto
  from 3(1) have "(p, ε, q') ∈
                  {((p1, q1), α, (p2, q2)) | p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, (p2, q1)) | p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, (p1, q2)) | p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have "∃ p1 p2 q1. p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then obtain p1 p2 q1 where "p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have "∃ p1 q1 q2. p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then obtain p1 q1 q2 where "p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  ultimately
  show ?case
    by auto
qed

lemma inters_trans_star_ε_iff:
  "((p1,q2), w :: 'label list, (p2,q2)) ∈ pa.trans_star_ε ⟷
   (p1, w, p2) ∈ A1.trans_star_ε ∧ (q2, w, q2) ∈ A2.trans_star_ε"
  by (metis fst_conv inters_trans_star_ε inters_trans_star_ε1 snd_conv trans_star_ε_inter)

lemma inters_ε_accept_ε_iff:
  "accepts_aut_inters_ε p w ⟷ A1.accepts_aut_ε p w ∧ A2.accepts_aut_ε p w"
proof
  assume "accepts_aut_inters_ε p w"
  then show "A1.accepts_aut_ε p w ∧ A2.accepts_aut_ε p w"
    unfolding accepts_aut_inters_ε_def A1.accepts_aut_ε_def A2.accepts_aut_ε_def pa.accepts_aut_ε_def
    unfolding inters_finals_def
    using inters_trans_star_ε_iff[of _ _ w _ ]
    using SigmaE fst_conv inters_trans_star_ε inters_trans_star_ε1 snd_conv
    by (metis (no_types, lifting))
next
  assume a: "A1.accepts_aut_ε p w ∧ A2.accepts_aut_ε p w"
  then have "(∃ q∈finals1. (Init p, w, q) ∈ A1.trans_star_ε) ∧
             (∃ q∈finals2. (Init p, w, q) ∈ LTS_ε.trans_star_ε ts2)"
    unfolding A1.accepts_aut_ε_def A2.accepts_aut_ε_def by auto
  then show "accepts_aut_inters_ε p w"
```

unfolding *accepts_aut_inters_ε_def pa.accepts_aut_ε_def inters_finals_def*
by (*auto simp*: *P_Automaton_ε.accepts_aut_ε_def intro*: *trans_star_ε_inter*)
qed

lemma *inters_ε_lang_ε*: *"lang_aut_inters_ε = A1.lang_aut_ε ∩ A2.lang_aut_ε"*
unfolding *lang_aut_inters_ε_def P_Automaton_ε.lang_aut_ε_def* **using** *inters_ε_accept_ε_iff* **by** *auto*

end

end
theory *PDS* **imports** *"P_Automata" "HOL−Library.While_Combinator"* **begin**

4 PDS

datatype *'label operation = pop | swap 'label | push 'label 'label*
type-synonym (*'ctr_loc*, *'label*) *rule* = *"('ctr_loc × 'label) × ('ctr_loc × 'label operation)"*
type-synonym (*'ctr_loc*, *'label*) *conf* = *"'ctr_loc × 'label list"*

We define push down systems.

locale *PDS* =
fixes Δ :: *"('ctr_loc*, *'label*::*finite) rule set"*

begin

primrec *lbl* :: *"'label operation ⇒ 'label list"* **where**
"lbl pop = []"
| *"lbl (swap γ) = [γ]"*
| *"lbl (push γ γ') = [γ, γ']"*

definition *is_rule* :: *"'ctr_loc × 'label ⇒ 'ctr_loc × 'label operation ⇒ bool"* (**infix** ‹↪› *80*) **where**
"pγ ↪ p'w ≡ (pγ,p'w) ∈ Δ"

inductive-set *transition_rel* :: *"(('ctr_loc*, *'label) conf × unit × ('ctr_loc*, *'label) conf) set"* **where**
"(p, *γ) ↪ (p'*, *w) ⟹*
((p, *γ#w')*, *()*, *(p'*, *(lbl w)@w'*)) ∈ *transition_rel"*

interpretation *LTS transition_rel* **.**

notation *step_relp* (**infix** ‹⇒› *80*)
notation *step_starp* (**infix** ‹⇒*› *80*)

lemma *step_relp_def2*:
"(p, *γw') ⇒ (p'*,*ww') ⟷ (∃γ w' w. γw' = γ#w' ∧ ww' = (lbl w)@w' ∧ (p*, *γ) ↪ (p'*, *w))"*
by (*metis* (*no_types*, *lifting*) *PDS.transition_rel.intros step_relp_def transition_rel.cases*)

end

5 PDS with P automata

type-synonym (*'ctr_loc*, *'label*) *sat_rule* = *"('ctr_loc*, *'label) transition set ⇒ ('ctr_loc*, *'label) transition set ⇒ bool"*

datatype (*'ctr_loc*, *'noninit*, *'label*) *state* =
is_Init: *Init* (*the_Ctr_Loc*: *'ctr_loc*)
| *is_Noninit*: *Noninit* (*the_St*: *'noninit*)
| *is_Isolated*: *Isolated* (*the_Ctr_Loc*: *'ctr_loc*) (*the_Label*: *'label*)

lemma *finitely_many_states*:
assumes *"finite (UNIV :: 'ctr_loc set)"*
assumes *"finite (UNIV :: 'noninit set)"*
assumes *"finite (UNIV :: 'label set)"*

```
  shows "finite (UNIV :: ('ctr_loc, 'noninit, 'label) state set)"
proof −
  define Isolated' :: "('ctr_loc * 'label) ⇒ ('ctr_loc, 'noninit, 'label) state" where
    "Isolated' == λ(c :: 'ctr_loc, l:: 'label). Isolated c l"
  define Init' :: "'ctr_loc ⇒ ('ctr_loc, 'noninit, 'label) state" where
    "Init' = Init"
  define Noninit' :: "'noninit ⇒ ('ctr_loc, 'noninit, 'label) state" where
    "Noninit' = Noninit"

  have split: "UNIV = (Init' ` UNIV) ∪ (Noninit' ` UNIV) ∪ (Isolated' ` (UNIV :: (('ctr_loc * 'label) set)))"
    unfolding Init'_def Noninit'_def
  proof (rule; rule; rule; rule)
    fix x :: "('ctr_loc, 'noninit, 'label) state"
    assume "x ∈ UNIV"
    moreover
    assume "x ∉ range Isolated'"
    moreover
    assume "x ∉ range Noninit"
    ultimately
    show "x ∈ range Init"
      by (metis Isolated'_def prod.simps(2) range_eqI state.exhaust)
  qed

  have "finite (Init' ` (UNIV:: 'ctr_loc set))"
    using assms by auto
  moreover
  have "finite (Noninit' ` (UNIV:: 'noninit set))"
    using assms by auto
  moreover
  have "finite (UNIV :: (('ctr_loc * 'label) set))"
    using assms by (simp add: finite_Prod_UNIV)
  then have "finite (Isolated' ` (UNIV :: (('ctr_loc * 'label) set)))"
    by auto
  ultimately
  show "finite (UNIV :: ('ctr_loc, 'noninit, 'label) state set)"
    unfolding split by auto
qed

instantiation state :: (finite, finite, finite) finite begin

instance by standard (simp add: finitely_many_states)

end


locale PDS_with_P_automata = PDS Δ
  for Δ :: "('ctr_loc::enum, 'label::finite) rule set"
    +
  fixes final_inits :: "('ctr_loc::enum) set"
  fixes final_noninits :: "('noninit::finite) set"
begin


definition finals :: "('ctr_loc, 'noninit::finite, 'label) state set" where
  "finals = Init ` final_inits ∪ Noninit ` final_noninits"

lemma F_not_Ext: "¬(∃f∈finals. is_Isolated f)"
  using finals_def by fastforce


definition inits :: "('ctr_loc, 'noninit, 'label) state set" where
  "inits = {q. is_Init q}"
```

lemma *inits_code*[*code*]: *"inits = set (map Init Enum.enum)"*
by (*auto simp*: *inits_def is_Init_def simp flip*: *UNIV_enum*)

definition *noninits* :: *"('ctr_loc, 'noninit, 'label) state set"* **where**
"noninits = {q. is_Noninit q}"

definition *isols* :: *"('ctr_loc, 'noninit, 'label) state set"* **where**
"isols = {q. is_Isolated q}"

sublocale *LTS transition_rel* **.**
notation *step_relp* (**infix** ‹⇒› *80*)
notation *step_starp* (**infix** ‹⇒*› *80*)

definition *accepts* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ ('ctr_loc, 'label) conf ⇒ bool"* **where**
"accepts ts ≡ λ(p,w). (∃ q ∈ finals. (Init p,w,q) ∈ LTS.trans_star ts)"

lemma *accepts_accepts_aut*: *"accepts ts (p, w) ⟷ P_Automaton.accepts_aut ts Init finals p w"*
unfolding *accepts_def P_Automaton.accepts_aut_def inits_def* **by** *auto*

definition *accepts_ε* :: *"(('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ ('ctr_loc, 'label) conf ⇒ bool"* **where**
"accepts_ε ts ≡ λ(p,w). (∃ q ∈ finals. (Init p,w,q) ∈ LTS_ε.trans_star_ε ts)"

abbreviation *ε* :: *"'label option"* **where**
"ε == None"

lemma *accepts_mono*[*mono*]: *"mono accepts"*
proof (*rule*, *rule*)
fix *c* :: *"('ctr_loc, 'label) conf"*
fix *ts ts'* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set"*
assume *accepts_ts*: *"accepts ts c"*
assume *tsts'*: *"ts ⊆ ts'"*
obtain *p l* **where** *pl_p*: *"c = (p,l)"*
by (*cases c*)
obtain *q* **where** *q_p*: *"q ∈ finals ∧ (Init p, l, q) ∈ LTS.trans_star ts"*
using *accepts_ts* **unfolding** *pl_p accepts_def* **by** *auto*
then have *"(Init p, l, q) ∈ LTS.trans_star ts'"*
using *tsts' LTS_trans_star_mono monoD* **by** *blast*
then have *"accepts ts' (p,l)"*
unfolding *accepts_def* **using** *q_p* **by** *auto*
then show *"accepts ts' c"*
unfolding *pl_p* **.**
qed

lemma *accepts_cons*: *"(Init p, γ, Init p') ∈ ts ⟹ accepts ts (p', w) ⟹ accepts ts (p, γ # w)"*
using *LTS.trans_star.trans_star_step accepts_def* **by** *fastforce*

definition *lang* :: *"(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ ('ctr_loc, 'label) conf set"* **where**
"lang ts = {c. accepts ts c}"

lemma *lang_lang_aut*: *"lang ts = (λ(s,w). (s, w)) ` (P_Automaton.lang_aut ts Init finals)"*
unfolding *lang_def P_Automaton.lang_aut_def*
by (*auto simp*: *inits_def accepts_def P_Automaton.accepts_aut_def image_iff intro*!: *exI*[*of* _ *"Init _"*])

lemma *lang_aut_lang*: *"P_Automaton.lang_aut ts Init finals = lang ts"*
unfolding *lang_lang_aut*
by (*auto 0 3 simp*: *P_Automaton.lang_aut_def P_Automaton.accepts_aut_def inits_def image_iff*)

definition *lang_ε* :: *"(('ctr_loc, 'noninit, 'label) state, 'label option) transition set ⇒ ('ctr_loc, 'label) conf set"*
where
"lang_ε ts = {c. accepts_ε ts c}"

5.1 Saturations

definition *saturated* :: "('c, 'l) sat_rule ⇒ ('c, 'l) transition set ⇒ bool" **where**
"saturated rule ts ⟷ (∄ ts'. rule ts ts')"

definition *saturation* :: "('c, 'l) sat_rule ⇒ ('c, 'l) transition set ⇒ ('c, 'l) transition set ⇒ bool" **where**
"saturation rule ts ts' ⟷ rule** ts ts' ∧ saturated rule ts'"

lemma *no_infinite*:
assumes "⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"
assumes "∀ i :: nat. rule (tts i) (tts (Suc i))"
shows "False"
proof −
define f **where** "f i = card (tts i)" **for** i
have f_Suc: "∀ i. f i < f (Suc i)"
using assms f_def lessI **by** metis
have "∀ i. ∃ j. f j > i"
proof
fix i
show "∃ j. i < f j"
proof(induction i)
case 0
then show ?case
by (metis f_Suc neq0_conv)
next
case (Suc i)
then show ?case
by (metis Suc_lessI f_Suc)
qed
qed
then have "∃ j. f j > card (UNIV :: ('c, 'l) transition set)"
by auto
then show False
by (metis card_seteq f_def finite_UNIV le_eq_less_or_eq nat_neq_iff subset_UNIV)
qed

lemma *saturation_termination*:
assumes "⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"
shows "¬(∃ tts. (∀ i :: nat. rule (tts i) (tts (Suc i))))"
using assms no_infinite **by** blast

lemma *saturation_exi*:
assumes "⋀ts ts' :: ('c ::finite, 'l::finite) transition set. rule ts ts' ⟹ card ts' = Suc (card ts)"
shows "∃ ts'. saturation rule ts ts'"
proof (rule ccontr)
assume a: "∄ ts'. saturation rule ts ts'"
define g **where** "g ts = (SOME ts'. rule ts ts')" **for** ts
define tts **where** "tts i = (g ^^ i) ts" **for** i
have "∀ i :: nat. rule** ts (tts i) ∧ rule (tts i) (tts (Suc i))"
proof
fix i
show "rule** ts (tts i) ∧ rule (tts i) (tts (Suc i))"
proof (induction i)
case 0
have "rule ts (g ts)"
by (metis g_def a rtranclp.rtrancl_refl saturation_def saturated_def someI)
then show ?case
using tts_def a saturation_def **by** auto
next
case (Suc i)
then have sat_Suc: "rule** ts (tts (Suc i))"
by fastforce
then have "rule (g ((g ^^ i) ts)) (g (g ((g ^^ i) ts)))"
by (metis Suc.IH tts_def g_def a r_into_rtranclp rtranclp_trans saturation_def saturated_def

```
      someI)
    then have "rule (tts (Suc i)) (tts (Suc (Suc i)))"
      unfolding tts_def by simp
    then show ?case
      using sat_Suc by auto
  qed
 qed
 then have "∀ i. rule (tts i) (tts (Suc i))"
  by auto
 then show False
  using no_infinite assms by auto
qed
```

5.2 Saturation rules

```
inductive pre_star_rule :: "(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label)
state, 'label) transition set ⇒ bool" where
 add_trans: "(p, γ) ↪ (p', w) ⟹ (Init p', lbl w, q) ∈ LTS.trans_star ts ⟹
  (Init p, γ, q) ∉ ts ⟹ pre_star_rule ts (ts ∪ {(Init p, γ, q)})"

definition pre_star1 :: "(('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state,
'label) transition set" where
 "pre_star1 ts =
  (⋃((p, γ), (p', w)) ∈ Δ. ⋃q ∈ LTS.reach ts (Init p') (lbl w). {(Init p, γ, q)})"

lemma pre_star1_mono: "mono pre_star1"
 unfolding pre_star1_def
 by (auto simp: mono_def LTS.trans_star_code[symmetric] elim!: bexI[rotated]
  LTS_trans_star_mono[THEN monoD, THEN subsetD])

lemma pre_star_rule_pre_star1:
 assumes "X ⊆ pre_star1 ts"
 shows "pre_star_rule** ts (ts ∪ X)"
proof −
 have "finite X"
  by simp
 from this assms show ?thesis
 proof (induct X arbitrary: ts rule: finite_induct)
  case (insert x F)
  then obtain p γ p' w q where ∗: "(p, γ) ↪ (p', w)"
   "(Init p', lbl w, q) ∈ LTS.trans_star ts" and x:
   "x = (Init p, γ, q)"
   by (auto simp: pre_star1_def is_rule_def LTS.trans_star_code)
  with insert show ?case
  proof (cases "(Init p, γ, q) ∈ ts")
   case False
   with insert(1,2,4) x show ?thesis
    by (intro converse_rtranclp_into_rtranclp[of pre_star_rule, OF add_trans[OF ∗ False]])
     (auto intro!: insert(3)[of "insert x ts", simplified x Un_insert_left]
      intro: pre_star1_mono[THEN monoD, THEN set_mp, of ts])
  qed (simp add: insert_absorb)
 qed simp
qed

lemma pre_star_rule_pre_star1s: "pre_star_rule** ts (((λs. s ∪ pre_star1 s) ^^ k) ts)"
 by (induct k) (auto elim!: rtranclp_trans intro: pre_star_rule_pre_star1)

definition "pre_star_loop = while_option (λs. s ∪ pre_star1 s ≠ s) (λs. s ∪ pre_star1 s)"
definition "pre_star_exec = the o pre_star_loop"
definition "pre_star_exec_check A = (if inits ⊆ LTS.srcs A then pre_star_loop A else None)"

definition "accept_pre_star_exec_check A c = (if inits ⊆ LTS.srcs A then Some (accepts (pre_star_exec A) c)
else None)"
```

lemma *while_option_finite_subset_Some*: **fixes** $C :: "'a\ set"$
assumes *"mono f"* **and** $"!!X.\ X \subseteq C \Longrightarrow f\ X \subseteq C"$ **and** *"finite C"* **and** X: $"X \subseteq C"$ $"X \subseteq f\ X"$
shows $"\exists P.\ while\_option\ (\lambda A.\ f\ A \neq A)\ f\ X = Some\ P"$
proof(*rule measure_while_option_Some*[**where**
f= $"\%A::'a\ set.\ card\ C - card\ A"$ **and** P= $"\%A.\ A \subseteq C \wedge A \subseteq f\ A"$ **and** s= X])
fix A **assume** A: $"A \subseteq C \wedge A \subseteq f\ A"$ $"f\ A \neq A"$
show $"(f\ A \subseteq C \wedge f\ A \subseteq f\ (f\ A)) \wedge card\ C - card\ (f\ A) < card\ C - card\ A"$
(**is** $"?L \wedge ?R"$)
proof
show *?L* **by**(*metis A*(*1*) *assms*(*2*) *monoD*[*OF* ‹*mono f*›])
show *?R*
by (*metis A assms*(*2,3*) *card_seteq diff_less_mono2 equalityI linorder_le_less_linear rev_finite_subset*)
qed
qed (*simp add*: *X*)

lemma *pre_star_exec_terminates*: $"\exists t.\ pre\_star\_loop\ s = Some\ t"$
unfolding *pre_star_loop_def*
by (*rule while_option_finite_subset_Some*[**where** *C=UNIV*])
(*auto simp*: *mono_def dest*: *pre_star1_mono*[*THEN monoD*])

lemma *pre_star_exec_code*[*code*]:
$"pre\_star\_exec\ s = (let\ s' = pre\_star1\ s\ in\ if\ s' \subseteq s\ then\ s\ else\ pre\_star\_exec\ (s \cup s'))"$
unfolding *pre_star_exec_def pre_star_loop_def o_apply*
by (*subst while_option_unfold*)(*auto simp*: *Let_def*)

lemma *saturation_pre_star_exec*: $"saturation\ pre\_star\_rule\ ts\ (pre\_star\_exec\ ts)"$
proof −
from *pre_star_exec_terminates* **obtain** t **where** t: $"pre\_star\_loop\ ts = Some\ t"$
by *blast*
obtain k **where** k: $"t = ((\lambda s.\ s \cup pre\_star1\ s)\ ^^\ k)\ ts"$ **and** le: $"pre\_star1\ t \subseteq t"$
using *while_option_stop2*[*OF t*[*unfolded pre_star_loop_def*]] **by** *auto*
have $"(\bigcup\{us.\ pre\_star\_rule\ t\ us\}) - t \subseteq pre\_star1\ t"$
by (*auto simp*: *pre_star1_def LTS.trans_star_code*[*symmetric*] *prod.splits is_rule_def pre_star_rule.simps*)
from *subset_trans*[*OF this le*] **show** *?thesis*
unfolding *saturation_def saturated_def pre_star_exec_def o_apply k t*
by (*auto 9 0 simp*: *pre_star_rule_pre_star1s subset_eq pre_star_rule.simps*)
qed

inductive *post_star_rules* :: $"(('ctr\_loc, 'noninit, 'label)\ state, 'label\ option)\ transition\ set \Rightarrow (('ctr\_loc, 'noninit, 'label)\ state, 'label\ option)\ transition\ set \Rightarrow bool"$ **where**
add_trans_pop:
$"(p, \gamma) \hookrightarrow (p', pop) \Longrightarrow$
$(Init\ p, [\gamma], q) \in LTS\_\varepsilon.trans\_star\_\varepsilon\ ts \Longrightarrow$
$(Init\ p', \varepsilon, q) \notin ts \Longrightarrow$
$post\_star\_rules\ ts\ (ts \cup \{(Init\ p', \varepsilon, q)\})"$
| *add_trans_swap*:
$"(p, \gamma) \hookrightarrow (p', swap\ \gamma') \Longrightarrow$
$(Init\ p, [\gamma], q) \in LTS\_\varepsilon.trans\_star\_\varepsilon\ ts \Longrightarrow$
$(Init\ p', Some\ \gamma', q) \notin ts \Longrightarrow$
$post\_star\_rules\ ts\ (ts \cup \{(Init\ p', Some\ \gamma', q)\})"$
| *add_trans_push_1*:
$"(p, \gamma) \hookrightarrow (p', push\ \gamma'\ \gamma'') \Longrightarrow$
$(Init\ p, [\gamma], q) \in LTS\_\varepsilon.trans\_star\_\varepsilon\ ts \Longrightarrow$
$(Init\ p', Some\ \gamma', Isolated\ p'\ \gamma') \notin ts \Longrightarrow$
$post\_star\_rules\ ts\ (ts \cup \{(Init\ p', Some\ \gamma', Isolated\ p'\ \gamma')\})"$
| *add_trans_push_2*:
$"(p, \gamma) \hookrightarrow (p', push\ \gamma'\ \gamma'') \Longrightarrow$
$(Init\ p, [\gamma], q) \in LTS\_\varepsilon.trans\_star\_\varepsilon\ ts \Longrightarrow$
$(Isolated\ p'\ \gamma', Some\ \gamma'', q) \notin ts \Longrightarrow$
$(Init\ p', Some\ \gamma', Isolated\ p'\ \gamma') \in ts \Longrightarrow$
$post\_star\_rules\ ts\ (ts \cup \{(Isolated\ p'\ \gamma', Some\ \gamma'', q)\})"$

lemma *pre_star_rule_mono*:
"*pre_star_rule ts ts'* $\Longrightarrow$ *ts* $\subset$ *ts'*"
unfolding *pre_star_rule.simps* **by** *auto*

lemma *post_star_rules_mono*:
"*post_star_rules ts ts'* $\Longrightarrow$ *ts* $\subset$ *ts'*"
proof(*induction rule*: *post_star_rules.induct*)
case (*add_trans_pop p* γ *p'* *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_swap p* γ *p'* γ' *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_push_1 p* γ *p'* γ' γ'' *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_push_2 p* γ *p'* γ' γ'' *q ts*)
then show *?case* **by** *auto*
qed

lemma *pre_star_rule_card_Suc*: "*pre_star_rule ts ts'* $\Longrightarrow$ *card ts'* = *Suc* (*card ts*)"
unfolding *pre_star_rule.simps* **by** *auto*

lemma *post_star_rules_card_Suc*: "*post_star_rules ts ts'* $\Longrightarrow$ *card ts'* = *Suc* (*card ts*)"
proof(*induction rule*: *post_star_rules.induct*)
case (*add_trans_pop p* γ *p'* *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_swap p* γ *p'* γ' *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_push_1 p* γ *p'* γ' γ'' *q ts*)
then show *?case* **by** *auto*
next
case (*add_trans_push_2 p* γ *p'* γ' γ'' *q ts*)
then show *?case* **by** *auto*
qed

lemma *pre_star_saturation_termination*:
"$\neg$($\exists$ *tts*. ($\forall$ *i* :: *nat*. *pre_star_rule* (*tts i*) (*tts* (*Suc i*))))"
using *no_infinite pre_star_rule_card_Suc* **by** *blast*

lemma *post_star_saturation_termination*:
"$\neg$($\exists$ *tts*. ($\forall$ *i* :: *nat*. *post_star_rules* (*tts i*) (*tts* (*Suc i*))))"
using *no_infinite post_star_rules_card_Suc* **by** *blast*

lemma *pre_star_saturation_exi*:
shows "$\exists$ *ts'*. *saturation pre_star_rule ts ts'*"
using *pre_star_rule_card_Suc saturation_exi* **by** *blast*

lemma *post_star_saturation_exi*:
shows "$\exists$ *ts'*. *saturation post_star_rules ts ts'*"
using *post_star_rules_card_Suc saturation_exi* **by** *blast*

lemma *pre_star_rule_incr*: "*pre_star_rule A B* $\Longrightarrow$ *A* $\subseteq$ *B*"
proof(*induction rule*: *pre_star_rule.inducts*)
case (*add_trans p* γ *p'* *w q rel*)
then show *?case*
by *auto*
qed

lemma *post_star_rules_incr*: *"post_star_rules A B ⟹ A ⊆ B"*
proof(*induction rule*: *post_star_rules.inducts*)
  case (*add_trans_pop p γ p' q ts*)
  then show *?case*
    by *auto*
next
  case (*add_trans_swap p γ p' γ' q ts*)
  then show *?case*
    by *auto*
next
  case (*add_trans_push_1 p γ p' γ' γ'' q ts*)
  then show *?case*
    by *auto*
next
  case (*add_trans_push_2 p γ p' γ' γ'' q ts*)
  then show *?case*
    by *auto*
qed

lemma *saturation_rtranclp_pre_star_rule_incr*: *"pre_star_rule** A B ⟹ A ⊆ B"*
proof (*induction rule*: *rtranclp_induct*)
  case *base*
  then show *?case* **by** *auto*
next
  case (*step y z*)
  then show *?case*
    using *pre_star_rule_incr* **by** *auto*
qed

lemma *saturation_rtranclp_post_star_rule_incr*: *"post_star_rules** A B ⟹ A ⊆ B"*
proof (*induction rule*: *rtranclp_induct*)
  case *base*
  then show *?case* **by** *auto*
next
  case (*step y z*)
  then show *?case*
    using *post_star_rules_incr* **by** *auto*
qed

lemma *pre_star'_incr_trans_star*:
  *"pre_star_rule** A A' ⟹ LTS.trans_star A ⊆ LTS.trans_star A'"*
  using *mono_def LTS_trans_star_mono saturation_rtranclp_pre_star_rule_incr* **by** *metis*

lemma *post_star'_incr_trans_star*:
  *"post_star_rules** A A' ⟹ LTS.trans_star A ⊆ LTS.trans_star A'"*
  using *mono_def LTS_trans_star_mono saturation_rtranclp_post_star_rule_incr* **by** *metis*

lemma *post_star'_incr_trans_star_ε*:
  *"post_star_rules** A A' ⟹ LTS_ε.trans_star_ε A ⊆ LTS_ε.trans_star_ε A'"*
  using *mono_def LTS_ε_trans_star_ε_mono saturation_rtranclp_post_star_rule_incr* **by** *metis*

lemma *pre_star_lim'_incr_trans_star*:
  "saturation pre_star_rule A A' ⟹ LTS.trans_star A ⊆ LTS.trans_star A'"
  by (*simp add*: *pre_star'_incr_trans_star saturation_def*)

lemma *post_star_lim'_incr_trans_star*:
  "saturation post_star_rules A A' ⟹ LTS.trans_star A ⊆ LTS.trans_star A'"
  by (*simp add*: *post_star'_incr_trans_star saturation_def*)

lemma *post_star_lim'_incr_trans_star_ε*:
  "saturation post_star_rules A A' ⟹ LTS_ε.trans_star_ε A ⊆ LTS_ε.trans_star_ε A'"
  by (*simp add*: *post_star'_incr_trans_star_ε saturation_def*)

5.3 Pre* lemmas

lemma *inits_srcs_iff_Ctr_Loc_srcs*:
"*inits* ⊆ *LTS.srcs A* ⟷ (∄ *q γ q'*. (*q*, *γ*, *Init q'*) ∈ *A*)"
proof
assume "*inits* ⊆ *LTS.srcs A*"
then show "∄ *q γ q'*. (*q*, *γ*, *Init q'*) ∈ *A*"
by (*simp add*: *Collect_mono_iff LTS.srcs_def inits_def*)
next
assume "∄ *q γ q'*. (*q*, *γ*, *Init q'*) ∈ *A*"
show "*inits* ⊆ *LTS.srcs A*"
by (*metis LTS.srcs_def2 inits_def* ‹∄ *q γ q'*. (*q*, *γ*, *Init q'*) ∈ *A*› *mem_Collect_eq*
state.collapse(*1*) *subsetI*)
qed

lemma *lemma_3_1*:
assumes "*p'w* ⇒* *pv*"
assumes "*pv* ∈ *lang A*"
assumes "*saturation pre_star_rule A A'*"
shows "*accepts A' p'w*"
using *assms*
proof (*induct rule*: *converse_rtranclp_induct*)
case *base*
define *p* **where** "*p* = *fst pv*"
define *v* **where** "*v* = *snd pv*"
from *base* **have** "∃ *q* ∈ *finals*. (*Init p*, *v*, *q*) ∈ *LTS.trans_star A'*"
unfolding *lang_def p_def v_def* **using** *pre_star_lim'_incr_trans_star accepts_def* **by** *fastforce*
then show *?case*
unfolding *accepts_def p_def v_def* **by** *auto*
next
case (*step p'w p''u*)
define *p'* **where** "*p'* = *fst p'w*"
define *w* **where** "*w* = *snd p'w*"
define *p''* **where** "*p''* = *fst p''u*"
define *u* **where** "*u* = *snd p''u*"
have *p'w_def*: "*p'w* = (*p'*, *w*)"
using *p'_def w_def* **by** *auto*
have *p''u_def*: "*p''u* = (*p''*, *u*)"
using *p''_def u_def* **by** *auto*

then have "*accepts A'* (*p''*, *u*)"
using *step* **by** *auto*
then obtain *q* **where** *q_p*: "*q* ∈ *finals* ∧ (*Init p''*, *u*, *q*) ∈ *LTS.trans_star A'*"
unfolding *accepts_def* **by** *auto*
have "∃ *γ w1 u1*. *w*=*γ*#*w1* ∧ *u*=*lbl u1*@*w1* ∧ (*p'*, *γ*) ↪ (*p''*, *u1*)"
using *p''u_def p'w_def step.hyps*(*1*) *step_relp_def2* **by** *auto*
then obtain *γ w1 u1* **where** *γ_w1_u1_p*: "*w*=*γ*#*w1* ∧ *u*=*lbl u1*@*w1* ∧ (*p'*, *γ*) ↪ (*p''*, *u1*)"
by *blast*

then have "∃ *q1*. (*Init p''*, *lbl u1*, *q1*) ∈ *LTS.trans_star A'* ∧ (*q1*, *w1*, *q*) ∈ *LTS.trans_star A'*"
using *q_p LTS.trans_star_split* **by** *auto*

then obtain *q1* **where** *q1_p*: "(*Init p''*, *lbl u1*, *q1*) ∈ *LTS.trans_star A'* ∧ (*q1*, *w1*, *q*) ∈ *LTS.trans_star A'*"
by *auto*

then have *in_A'*: "(*Init p'*, *γ*, *q1*) ∈ *A'*"
using *γ_w1_u1_p add_trans*[*of p' γ p'' u1 q1 A'*] *saturated_def saturation_def step.prems* **by** *metis*

then have "(*Init p'*, *γ*#*w1*, *q*) ∈ *LTS.trans_star A'*"
using *LTS.trans_star.trans_star_step q1_p* **by** *meson*
then have *t_in_A'*: "(*Init p'*, *w*, *q*) ∈ *LTS.trans_star A'*"
using *γ_w1_u1_p* **by** *blast*

from *q_p t_in_A'* **have** "*q* ∈ *finals* ∧ (*Init p'*, *w*, *q*) ∈ *LTS.trans_star A'*"

```
    by auto
  then show ?case
    unfolding accepts_def p'w_def by auto
qed

lemma word_into_init_empty_states:
  fixes A :: "(('ctr_loc, 'noninit, 'label) state, 'label) transition set"
  assumes "(p, w, ss, Init q) ∈ LTS.trans_star_states A"
  assumes "inits ⊆ LTS.srcs A"
  shows "w = [] ∧ p = Init q ∧ ss=[p]"
proof −
  define q1 :: "('ctr_loc, 'noninit, 'label) state" where
    "q1 = Init q"
  have q1_path: "(p, w, ss, q1) ∈ LTS.trans_star_states A"
    by (simp add: assms(1) q1_def)
  moreover
  have "q1 ∈ inits"
    by (simp add: inits_def q1_def)
  ultimately
  have "w = [] ∧ p = q1 ∧ ss=[p]"
  proof(induction rule: LTS.trans_star_states.induct[OF q1_path])
    case (1 p)
    then show ?case by auto
  next
    case (2 p γ q' w ss q)
    have "∄ q γ q'. (q, γ, Init q') ∈ A"
      using assms(2) unfolding inits_def LTS.srcs_def by (simp add: Collect_mono_iff)
    then show ?case
      using 2 assms(2) by (metis inits_def is_Init_def mem_Collect_eq)
  qed
  then show ?thesis
    using q1_def by fastforce
qed


lemma word_into_init_empty:
  fixes A :: "(('ctr_loc, 'noninit, 'label) state, 'label) transition set"
  assumes "(p, w, Init q) ∈ LTS.trans_star A"
  assumes "inits ⊆ LTS.srcs A"
  shows "w = [] ∧ p = Init q"
  using assms word_into_init_empty_states LTS.trans_star_trans_star_states by metis

lemma step_relp_append_aux:
  assumes "pu ⇒* p1y"
  shows "(fst pu, snd pu @ v) ⇒* (fst p1y, snd p1y @ v)"
  using assms
proof (induction rule: rtranclp_induct)
  case base
  then show ?case by auto
next
  case (step p'w p1y)
  define p where "p = fst pu"
  define u where "u = snd pu"
  define p' where "p' = fst p'w"
  define w where "w = snd p'w"
  define p1 where "p1 = fst p1y"
  define y where "y = snd p1y"
  have step_1: "(p,u) ⇒* (p',w)"
    by (simp add: p'_def p_def step.hyps(1) u_def w_def)
  have step_2: "(p',w) ⇒ (p1,y)"
    by (simp add: p'_def p1_def step.hyps(2) w_def y_def)
  have step_3: "(p, u @ v) ⇒* (p', w @ v)"
    by (simp add: p'_def p_def step.IH u_def w_def)
```

```
  note step' = step_1 step_2 step_3

  from step'(2) have "∃γ w' wa. w = γ # w' ∧ y = lbl wa @ w' ∧ (p', γ) ↪ (p1, wa)"
    using step_relp_def2[of p' w p1 y] by auto
  then obtain γ w' wa where γ_w'_wa_p: " w = γ # w' ∧ y = lbl wa @ w' ∧ (p', γ) ↪ (p1, wa)"
    by metis
  then have "(p, u @ v) ⇒* (p1, y @ v)"
    by (metis (no_types, lifting) PDS.step_relp_def2 append.assoc append_Cons rtranclp.simps step_3)
  then show ?case
    by (simp add: p1_def p_def u_def y_def)
qed

lemma step_relp_append:
  assumes "(p, u) ⇒* (p1, y)"
  shows "(p, u @ v) ⇒* (p1, y @ v)"
  using assms step_relp_append_aux by auto

lemma step_relp_append_empty:
  assumes "(p, u) ⇒* (p1, [])"
  shows "(p, u @ v) ⇒* (p1, v)"
  using step_relp_append[OF assms] by auto

lemma lemma_3_2_a':
  assumes "inits ⊆ LTS.srcs A"
  assumes "pre_star_rule** A A'"
  assumes "(Init p, w, q) ∈ LTS.trans_star A'"
  shows "∃p' w'. (Init p', w', q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p', w')"
  using assms(2) assms(3)
proof (induction arbitrary: p q w rule: rtranclp_induct)
  case base
  then have "(Init p, w, q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p, w)"
    by auto
  then show ?case
    by auto
next
  case (step Aiminus1 Ai)

  from step(2) obtain p1 γ p2 w2 q' where p1_γ_p2_w2_q'_p:
    "Ai = Aiminus1 ∪ {(Init p1, γ, q')}"
    "(p1, γ) ↪ (p2, w2)"
    "(Init p2, lbl w2, q') ∈ LTS.trans_star Aiminus1"
    "(Init p1, γ, q') ∉ Aiminus1"
    by (meson pre_star_rule.cases)

  define t :: "(('ctr_loc, 'noninit, 'label) state, 'label) transition"
    where "t = (Init p1, γ, q')"

  obtain ss where ss_p: "(Init p, w, ss, q) ∈ LTS.trans_star_states Ai"
    using step(4) LTS.trans_star_trans_star_states by metis

  define j where "j = count (transitions_of' (Init p, w, ss, q)) t"

  from j_def ss_p show ?case
  proof (induction j arbitrary: p q w ss)
    case 0
    then have "(Init p, w, q) ∈ LTS.trans_star Aiminus1"
      using count_zero_remove_trans_star_states_trans_star p1_γ_p2_w2_q'_p(1) t_def by metis
    then show ?case
      using step.IH by metis
  next
    case (Suc j')
    have "∃u v u_ss v_ss.
```

ss = u_ss@v_ss ∧ w = u@[γ]@v ∧
(Init p,u,u_ss, Init p1) ∈ LTS.trans_star_states Aiminus1 ∧
(Init p1,[γ],q′) ∈ LTS.trans_star Ai ∧
(q′,v,v_ss,q) ∈ LTS.trans_star_states Ai ∧
(Init p, w, ss, q) = ((Init p, u, u_ss, Init p1), γ) @@^γ (q′, v, v_ss, q)"
using *split_at_first_t[of "Init p" w ss q Ai j′ "Init p1" γ q′ Aiminus1]*
using *Suc(2,3) t_def p1_γ_p2_w2_q′_p(1,4) t_def* **by** *auto*
then obtain *u v u_ss v_ss* **where** *u_v_u_ss_v_ss_p*:
"ss = u_ss@v_ss ∧ w = u@[γ]@v"
"(Init p,u,u_ss, Init p1) ∈ LTS.trans_star_states Aiminus1"
"(Init p1,[γ],q′) ∈ LTS.trans_star Ai"
"(q′,v,v_ss,q) ∈ LTS.trans_star_states Ai"
"(Init p, w, ss, q) = ((Init p, u, u_ss, Init p1), γ) @@^γ (q′, v, v_ss, q)"
by *blast*
from *this(2)* **have** *"∃ p′′ w′′. (Init p′′, w′′, Init p1) ∈ LTS.trans_star A ∧ (p, u) ⇒* (p′′, w′′)"*
using *Suc(1)[of p u _ "Init p1"] step.IH step.prems(1)*
by (*meson LTS.trans_star_states_trans_star LTS.trans_star_trans_star_states*)
from *this this(1)* **have** *VIII*: *"(p, u) ⇒* (p1, [])"*
using *word_into_init_empty assms(1)* **by** *blast*

note *IX = p1_γ_p2_w2_q′_p(2)*
note *III = p1_γ_p2_w2_q′_p(3)*
from *III* **have** *III_2*: *"∃ w2_ss. (Init p2, lbl w2, w2_ss, q′) ∈ LTS.trans_star_states Aiminus1"*
using *LTS.trans_star_trans_star_states[of "Init p2" "lbl w2" q′ Aiminus1]* **by** *auto*
then obtain *w2_ss* **where** *III_2*: *"(Init p2, lbl w2, w2_ss, q′) ∈ LTS.trans_star_states Aiminus1"*
by *blast*

from *III* **have** *V*:
"(Init p2, lbl w2, w2_ss, q′) ∈ LTS.trans_star_states Aiminus1"
"(q′, v, v_ss, q) ∈ LTS.trans_star_states Ai"
using *III_2* ‹*(q′, v, v_ss, q) ∈ LTS.trans_star_states Ai*› **by** *auto*

define *w2v* **where** *"w2v = lbl w2 @ v"*
define *w2v_ss* **where** *"w2v_ss = w2_ss @ tl v_ss"*

from *V(1)* **have** *"(Init p2, lbl w2, w2_ss, q′) ∈ LTS.trans_star_states Ai"*
using *trans_star_states_mono p1_γ_p2_w2_q′_p(1)* **using** *Un_iff subsetI* **by** (*metis* (*no_types*))
then have *V_merged*: *"(Init p2, w2v, w2v_ss, q) ∈ LTS.trans_star_states Ai"*
using *V(2)* **unfolding** *w2v_def w2v_ss_def* **by** (*meson LTS.trans_star_states_append*)

have *j′_count*: *"j′ = count (transitions_of′ (Init p2, w2v, w2v_ss, q)) t"*
proof −
define *countts* **where**
"countts == λx. count (transitions_of′ x) t"

have *"countts (Init p, w, ss, q) = Suc j′ "*
using *Suc.prems(1) countts_def* **by** *force*
moreover
have *"countts (Init p, u, u_ss, Init p1) = 0"*
using *LTS.avoid_count_zero countts_def p1_γ_p2_w2_q′_p(4) t_def u_v_u_ss_v_ss_p(2)*
by *fastforce*
moreover
from *u_v_u_ss_v_ss_p(5)* **have** *"countts (Init p, w, ss, q) = countts (Init p, u, u_ss, Init p1) + 1 + countts (q′, v, v_ss, q)"*
using *count_combine_trans_star_states countts_def t_def u_v_u_ss_v_ss_p(2)*
u_v_u_ss_v_ss_p(4) **by** *fastforce*
ultimately
have *"Suc j′ = 0 + 1 + countts (q′, v, v_ss, q)"*
by *auto*
then have *"j′ = countts (q′, v, v_ss, q)"*
by *auto*
moreover
have *"countts (Init p2, lbl w2, w2_ss, q′) = 0"*

```
      using III_2 LTS.avoid_count_zero countts_def p1_γ_p2_w2_q'_p(4) t_def by fastforce
    moreover
    have "(Init p2, w2v, w2v_ss, q) = (Init p2, lbl w2, w2_ss, q') @@' (q', v, v_ss, q)"
      using w2v_def w2v_ss_def by auto
   then have "countts (Init p2, w2v, w2v_ss, q) = countts (Init p2, lbl w2, w2_ss, q') + countts (q', v, v_ss, q)"
      using ‹(Init p2, lbl w2, w2_ss, q') ∈ LTS.trans_star_states Ai›
        count_append_trans_star_states countts_def t_def u_v_u_ss_v_ss_p(4) by fastforce
    ultimately
    show ?thesis
      by (simp add: countts_def)
  qed

  have "∃ p' w'. (Init p', w', q) ∈ LTS.trans_star A ∧ (p2, w2v) ⇒* (p', w')"
    using Suc(1) using j'_count V_merged by auto
  then obtain p' w' where p'_w'_p: "(Init p', w', q) ∈ LTS.trans_star A" "(p2, w2v) ⇒* (p', w')"
    by blast

  note X = p'_w'_p(2)

  have "(p,w) = (p,u@[γ]@v)"
    using ‹ss = u_ss @ v_ss ∧ w = u @ [γ] @ v› by blast

  have "(p,u@[γ]@v) ⇒* (p1,γ#v)"
    using VIII step_relp_append_empty by auto

  from X have "(p1,γ#v) ⇒ (p2, w2v)"
    by (metis IX LTS.step_relp_def transition_rel.intros w2v_def)

  from X have
    "(p2, w2v) ⇒* (p', w')"
    by simp

  have "(p, w) ⇒* (p', w')"
    using X ‹(p, u @ [γ] @ v) ⇒* (p1, γ # v)› ‹(p, w) = (p, u @ [γ] @ v)› ‹(p1, γ # v) ⇒ (p2, w2v)› by auto

  then have "(Init p', w', q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p', w')"
    using p'_w'_p(1) by auto
  then show ?case
    by metis
 qed
qed

lemma lemma_3_2_a:
 assumes "inits ⊆ LTS.srcs A"
 assumes "saturation pre_star_rule A A'"
 assumes "(Init p, w, q) ∈ LTS.trans_star A'"
 shows "∃ p' w'. (Init p', w', q) ∈ LTS.trans_star A ∧ (p, w) ⇒* (p', w')"
 using assms lemma_3_2_a' saturation_def by metis

— Corresponds to one direction of Schwoon's theorem 3.2
theorem pre_star_rule_subset_pre_star_lang:
 assumes "inits ⊆ LTS.srcs A"
 assumes "pre_star_rule** A A'"
 shows "{c. accepts A' c} ⊆ pre_star (lang A)"
proof
 fix c :: "'ctr_loc × 'label list"
 assume c_a: "c ∈ {w. accepts A' w}"
 define p where "p = fst c"
 define w where "w = snd c"
 from p_def w_def c_a have "accepts A' (p,w)"
   by auto
 then have "∃ q ∈ finals. (Init p, w, q) ∈ LTS.trans_star A'"
   unfolding accepts_def by auto
```

```
  then obtain q where q_p: "q ∈ finals" "(Init p, w, q) ∈ LTS.trans_star A'"
    by auto
  then have "∃ p' w'. (p,w) ⇒* (p',w') ∧ (Init p', w', q) ∈ LTS.trans_star A"
    using lemma_3_2_a' assms(1) assms(2) by metis
  then obtain p' w' where p'_w'_p: "(p,w) ⇒* (p',w')" "(Init p', w', q) ∈ LTS.trans_star A"
    by auto
  then have "(p', w') ∈ lang A"
    unfolding lang_def unfolding accepts_def using q_p(1) by auto
  then have "(p,w) ∈ pre_star (lang A)"
    unfolding pre_star_def using p'_w'_p(1) by auto
  then show "c ∈ pre_star (lang A)"
    unfolding p_def w_def by auto
qed

— Corresponds to Schwoon's theorem 3.2
theorem pre_star_rule_accepts_correct:
  assumes "inits ⊆ LTS.srcs A"
  assumes "saturation pre_star_rule A A'"
  shows "{c. accepts A' c} = pre_star (lang A)"
proof (rule; rule)
  fix c :: "'ctr_loc × 'label list"
  define p where "p = fst c"
  define w where "w = snd c"
  assume "c ∈ pre_star (lang A)"
  then have "(p,w) ∈ pre_star (lang A)"
    unfolding p_def w_def by auto
  then have "∃ p' w'. (p',w') ∈ lang A ∧ (p,w) ⇒* (p',w')"
    unfolding pre_star_def by force
  then obtain p' w' where "(p',w') ∈ lang A ∧ (p,w) ⇒* (p',w')"
    by auto
  then have "∃ q ∈ finals. (Init p, w, q) ∈ LTS.trans_star A'"
    using lemma_3_1 assms(2) unfolding accepts_def by force
  then have "accepts A' (p,w)"
    unfolding accepts_def by auto
  then show "c ∈ {c. accepts A' c}"
    using p_def w_def by auto
next
  fix c :: "'ctr_loc × 'label list"
  assume "c ∈ {w. accepts A' w}"
  then show "c ∈ pre_star (lang A)"
    using pre_star_rule_subset_pre_star_lang assms unfolding saturation_def by auto
qed

— Corresponds to Schwoon's theorem 3.2
theorem pre_star_rule_correct:
  assumes "inits ⊆ LTS.srcs A"
  assumes "saturation pre_star_rule A A'"
  shows "lang A' = pre_star (lang A)"
  using assms(1) assms(2) lang_def pre_star_rule_accepts_correct by auto

theorem pre_star_exec_accepts_correct:
  assumes "inits ⊆ LTS.srcs A"
  shows "{c. accepts (pre_star_exec A) c} = pre_star (lang A)"
  using pre_star_rule_accepts_correct[of A "pre_star_exec A"] saturation_pre_star_exec[of A]
    assms by auto

theorem pre_star_exec_lang_correct:
  assumes "inits ⊆ LTS.srcs A"
  shows "lang (pre_star_exec A) = pre_star (lang A)"
  using pre_star_rule_correct[of A "pre_star_exec A"] saturation_pre_star_exec[of A] assms by auto

theorem pre_star_exec_check_accepts_correct:
  assumes "pre_star_exec_check A ≠ None"
```

shows *"{c. accepts (the (pre_star_exec_check A)) c} = pre_star (lang A)"*
using *pre_star_exec_accepts_correct assms* **unfolding** *pre_star_exec_check_def pre_star_exec_def*
by (*auto split*: *if_splits*)

theorem *pre_star_exec_check_correct*:
assumes *"pre_star_exec_check A ≠ None"*
shows *"lang (the (pre_star_exec_check A)) = pre_star (lang A)"*
using *pre_star_exec_check_accepts_correct assms* **unfolding** *lang_def* **by** *auto*

theorem *accept_pre_star_exec_correct_True*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"accepts (pre_star_exec A) c"*
shows *"c ∈ pre_star (lang A)"*
using *pre_star_exec_accepts_correct assms*(*1*) *assms*(*2*) **by** *blast*

theorem *accept_pre_star_exec_correct_False*:
assumes *"inits ⊆ LTS.srcs A"*
assumes *"¬accepts (pre_star_exec A) c"*
shows *"c ∉ pre_star (lang A)"*
using *pre_star_exec_accepts_correct assms*(*1*) *assms*(*2*) **by** *blast*

theorem *accept_pre_star_exec_correct_Some_True*:
assumes *"accept_pre_star_exec_check A c = Some True"*
shows *"c ∈ pre_star (lang A)"*
proof −
have *"inits ⊆ LTS.srcs A"*
using *assms* **unfolding** *accept_pre_star_exec_check_def*
by (*auto split*: *if_splits*)
moreover
have *"accepts (pre_star_exec A) c"*
using *assms*
using *accept_pre_star_exec_check_def calculation* **by** *auto*
ultimately
show *"c ∈ pre_star (lang A)"*
using *accept_pre_star_exec_correct_True* **by** *auto*
qed

theorem *accept_pre_star_exec_correct_Some_False*:
assumes *"accept_pre_star_exec_check A c = Some False"*
shows *"c ∉ pre_star (lang A)"*
proof −
have *"inits ⊆ LTS.srcs A"*
using *assms* **unfolding** *accept_pre_star_exec_check_def*
by (*auto split*: *if_splits*)
moreover
have *"¬accepts (pre_star_exec A) c"*
using *assms*
using *accept_pre_star_exec_check_def calculation* **by** *auto*
ultimately
show *"c ∉ pre_star (lang A)"*
using *accept_pre_star_exec_correct_False* **by** *auto*
qed

theorem *accept_pre_star_exec_correct_None*:
assumes *"accept_pre_star_exec_check A c = None"*
shows *"¬inits ⊆ LTS.srcs A"*
using *assms* **unfolding** *accept_pre_star_exec_check_def* **by** *auto*

5.4 Post* lemmas

lemma *lemma_3_3′*:
assumes *"pv ⇒* p′w"*
and *"(fst pv, snd pv) ∈ lang_ε A"*
and *"saturation post_star_rules A A′"*

```
  shows "accepts_ε A' (fst p'w, snd p'w)"
  using assms
proof (induct arbitrary: pv rule: rtranclp_induct)
  case base
  show ?case
    using assms post_star_lim'_incr_trans_star_ε
    by (auto simp: lang_ε_def accepts_ε_def)
next
  case (step p''u p'w)
  define p' where "p' = fst p'w"
  define w  where "w = snd p'w"
  define p'' where "p'' = fst p''u"
  define u  where "u = snd p''u"
  have p'w_def: "p'w = (p', w)"
    using p'_def w_def by auto
  have p''u_def: "p''u = (p'', u)"
    using p''_def u_def by auto

  then have "accepts_ε A' (p'', u)"
    using assms(2) p''_def step.hyps(3) step.prems(2) u_def by metis
  then have "∃ q. q ∈ finals ∧ (Init p'', u, q) ∈ LTS_ε.trans_star_ε A'"
    by (auto simp: accepts_ε_def)
  then obtain q where q_p: "q ∈ finals ∧ (Init p'', u, q) ∈ LTS_ε.trans_star_ε A'"
    by metis
  then have "∃ u_ε. q ∈ finals ∧ LTS_ε.ε_exp u_ε u ∧ (Init p'', u_ε, q) ∈ LTS.trans_star A'"
    using LTS_ε.trans_star_ε_iff_ε_exp_trans_star[of "Init p''" u q A'] by auto
  then obtain u_ε where II: "q ∈ finals" "LTS_ε.ε_exp u_ε u" "(Init p'', u_ε, q) ∈ LTS.trans_star A'"
    by blast
  have "∃ γ u1 w1. u=γ#u1 ∧ w=lbl w1@u1 ∧ (p'', γ) ↪ (p', w1)"
    using p''u_def p'w_def step.hyps(2) step_relp_def2 by auto
  then obtain γ u1 w1 where III: "u=γ#u1" "w=lbl w1@u1" "(p'', γ) ↪ (p', w1)"
    by blast

  have p'_inits: "Init p' ∈ inits"
    unfolding inits_def by auto
  have p''_inits: "Init p'' ∈ inits"
    unfolding inits_def by auto

  have "∃ γ_ε u1_ε. LTS_ε.ε_exp γ_ε [γ] ∧ LTS_ε.ε_exp u1_ε u1 ∧ (Init p'', γ_ε@u1_ε, q) ∈ LTS.trans_star
A'"
  proof −
    have "∃ γ_ε u1_ε. LTS_ε.ε_exp γ_ε [γ] ∧ LTS_ε.ε_exp u1_ε u1 ∧ u_ε = γ_ε @ u1_ε"
      using LTS_ε.ε_exp_split'[of u_ε γ u1] II(2) III(1) by auto
    then obtain γ_ε u1_ε where "LTS_ε.ε_exp γ_ε [γ] ∧ LTS_ε.ε_exp u1_ε u1 ∧ u_ε = γ_ε @ u1_ε"
      by auto
    then have "(Init p'', γ_ε@u1_ε , q) ∈ LTS.trans_star A'"
      using II(3) by auto
    then show ?thesis
      using ‹LTS_ε.ε_exp γ_ε [γ] ∧ LTS_ε.ε_exp u1_ε u1 ∧ u_ε = γ_ε @ u1_ε› by blast
  qed
  then obtain γ_ε u1_ε where
    iii: "LTS_ε.ε_exp γ_ε [γ]" and
    iv: "LTS_ε.ε_exp u1_ε u1" "(Init p'', γ_ε@u1_ε, q) ∈ LTS.trans_star A'"
    by blast
  then have VI: "∃ q1. (Init p'', γ_ε, q1) ∈ LTS.trans_star A' ∧ (q1, u1_ε, q) ∈ LTS.trans_star A'"
    by (simp add: LTS.trans_star_split)
  then obtain q1 where VI: "(Init p'', γ_ε, q1) ∈ LTS.trans_star A'" "(q1, u1_ε, q) ∈ LTS.trans_star A'"
    by blast

  then have VI_2: "(Init p'', [γ], q1) ∈ LTS_ε.trans_star_ε A'" "(q1, u1, q) ∈ LTS_ε.trans_star_ε A'"
    by (meson LTS_ε.trans_star_ε_iff_ε_exp_trans_star iii VI(2) iv(1))+

  show ?case
```

```
  proof (cases w1)
    case pop
    then have r: "(p'', γ) ↪ (p', pop)"
      using III(3) by blast
    then have "(Init p', ε, q1) ∈ A'"
      using VI_2(1) add_trans_pop assms saturated_def saturation_def p'_inits by metis
    then have "(Init p', w, q) ∈ LTS_ε.trans_star_ε A'"
      using III(2) VI_2(2) pop LTS_ε.trans_star_ε.trans_star_ε_step_ε by fastforce
    then have "accepts_ε A' (p', w)"
      unfolding accepts_ε_def using II(1) by blast
    then show ?thesis
      using p'_def w_def by force
  next
    case (swap γ')
    then have r: "(p'', γ) ↪ (p', swap γ')"
      using III(3) by blast
    then have "(Init p', Some γ', q1) ∈ A'"
      by (metis VI_2(1) add_trans_swap assms(3) saturated_def saturation_def)
    have "(Init p', w, q) ∈ LTS_ε.trans_star_ε A'"
      using III(2) LTS_ε.trans_star_ε.trans_star_ε_step_γ VI_2(2) append_Cons append_self_conv2
        lbl.simps(3) swap ‹(Init p', Some γ', q1) ∈ A'› by fastforce
    then have "accepts_ε A' (p', w)"
      unfolding accepts_ε_def
      using II(1) by blast
    then show ?thesis
      using p'_def w_def by force
  next
    case (push γ' γ'')
    then have r: "(p'', γ) ↪ (p', push γ' γ'')"
      using III(3) by blast
    from this VI_2 iii post_star_rules.intros(3)[OF this, of q1 A', OF VI_2(1)]
    have "(Init p', Some γ', Isolated p' γ') ∈ A'"
      using assms(3) by (meson saturated_def saturation_def)
    from this r VI_2 iii post_star_rules.intros(4)[OF r, of q1 A', OF VI_2(1)]
    have "(Isolated p' γ', Some γ'', q1) ∈ A'"
      using assms(3) using saturated_def saturation_def by metis
    have "(Init p', [γ'], Isolated p' γ') ∈ LTS_ε.trans_star_ε A' ∧
          (Isolated p' γ', [γ''], q1) ∈ LTS_ε.trans_star_ε A' ∧
          (q1, u1, q) ∈ LTS_ε.trans_star_ε A'"
      by (metis LTS_ε.trans_star_ε.simps VI_2(2) ‹(Init p', Some γ', Isolated p' γ') ∈ A'›
          ‹(Isolated p' γ', Some γ'', q1) ∈ A'›)
    have "(Init p', w, q) ∈ LTS_ε.trans_star_ε A'"
      using III(2) VI_2(2) ‹(Init p', Some γ', Isolated p' γ') ∈ A'›
        ‹(Isolated p' γ', Some γ'', q1) ∈ A'› push LTS_ε.append_edge_edge_trans_star_ε by auto
    then have "accepts_ε A' (p', w)"
      unfolding accepts_ε_def
      using II(1) by blast
    then show ?thesis
      using p'_def w_def by force

  qed
qed

lemma lemma_3_3:
  assumes "(p,v) ⇒* (p',w)"
  and "(p, v) ∈ lang_ε A"
  and "saturation post_star_rules A A'"
  shows "accepts_ε A' (p', w)"
  using assms lemma_3_3' by force

lemma init_only_hd:
  assumes "(ss, w) ∈ LTS.path_with_word A"
  assumes "inits ⊆ LTS.srcs A"
```

assumes *"count (transitions_of (ss, w)) t > 0"*
assumes *"t = (Init p1, γ, q1)"*
shows *"hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"*
using *assms LTS.source_only_hd* **by** (*metis LTS.srcs_def2 inits_srcs_iff_Ctr_Loc_srcs*)

lemma *no_edge_to_Ctr_Loc_avoid_Ctr_Loc*:
assumes *"(p, w, qq) ∈ LTS.trans_star Aiminus1"*
assumes *"w ≠ []"*
assumes *"inits ⊆ LTS.srcs Aiminus1"*
shows *"qq ∉ inits"*
using *assms LTS.no_end_in_source* **by** (*metis subset_iff*)

lemma *no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε*:
assumes *"(p, [γ], qq) ∈ LTS_ε.trans_star_ε Aiminus1"*
assumes *"inits ⊆ LTS.srcs Aiminus1"*
shows *"qq ∉ inits"*
using *assms LTS_ε.no_edge_to_source_ε* **by** (*metis subset_iff*)

lemma *no_edge_to_Ctr_Loc_post_star_rules'*:
assumes *"post_star_rules** A Ai"*
assumes *"∄ q γ q'. (q, γ, Init q') ∈ A"*
shows *"∄ q γ q'. (q, γ, Init q') ∈ Ai"*
using *assms*
proof (*induction rule*: *rtranclp_induct*)
case *base*
then show *?case* **by** *auto*
next
case (*step Aiminus1 Ai*)
then have *ind*: *"∄ q γ q'. (q, γ, Init q') ∈ Aiminus1"*
by *auto*
from *step(2)* **show** *?case*
proof (*cases rule*: *post_star_rules.cases*)
case (*add_trans_pop p γ p' q*)
have *"q ∉ inits"*
using *ind no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε inits_srcs_iff_Ctr_Loc_srcs*
by (*metis local.add_trans_pop(3)*)
then have *"∄ qq. q = Init qq"*
by (*simp add*: *inits_def is_Init_def*)
then show *?thesis*
using *ind local.add_trans_pop(1)* **by** *auto*
next
case (*add_trans_swap p γ p' γ' q*)
have *"q ∉ inits"*
using *add_trans_swap ind no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε inits_srcs_iff_Ctr_Loc_srcs*
by *metis*
then have *"∄ qq. q = Init qq"*
by (*simp add*: *inits_def is_Init_def*)
then show *?thesis*
using *ind local.add_trans_swap(1)* **by** *auto*
next
case (*add_trans_push_1 p γ p' γ' γ'' q*)
have *"q ∉ inits"*
using *add_trans_push_1 ind no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε inits_srcs_iff_Ctr_Loc_srcs*
by *metis*
then have *"∄ qq. q = Init qq"*
by (*simp add*: *inits_def is_Init_def*)
then show *?thesis*
using *ind local.add_trans_push_1(1)* **by** *auto*
next
case (*add_trans_push_2 p γ p' γ' γ'' q*)
have *"q ∉ inits"*
using *add_trans_push_2 ind no_edge_to_Ctr_Loc_avoid_Ctr_Loc_ε inits_srcs_iff_Ctr_Loc_srcs*
by *metis*

```
    then have "∄ qq. q = Init qq"
      by (simp add: inits_def is_Init_def)
    then show ?thesis
      using ind local.add_trans_push_2(1) by auto
  qed
qed

lemma no_edge_to_Ctr_Loc_post_star_rules:
  assumes "post_star_rules** A Ai"
  assumes "inits ⊆ LTS.srcs A"
  shows "inits ⊆ LTS.srcs Ai"
  using assms no_edge_to_Ctr_Loc_post_star_rules' inits_srcs_iff_Ctr_Loc_srcs by metis

lemma source_and_sink_isolated:
  assumes "N ⊆ LTS.srcs A"
  assumes "N ⊆ LTS.sinks A"
  shows "∀ p γ q. (p, γ, q) ∈ A ⟶ p ∉ N ∧ q ∉ N"
  by (metis LTS.srcs_def2 LTS.sinks_def2 assms(1) assms(2) in_mono)

lemma post_star_rules_Isolated_source_invariant':
  assumes "post_star_rules** A A'"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p', Some γ', Isolated p' γ') ∉ A'"
  shows "∄ p γ. (p, γ, Isolated p' γ') ∈ A'"
  using assms
proof (induction rule: rtranclp_induct)
  case base
  then show ?case
    unfolding isols_def is_Isolated_def using LTS.isolated_no_edges by fastforce
next
  case (step Aiminus1 Ai)
  from step(2) show ?case
  proof (cases rule: post_star_rules.cases)
    case (add_trans_pop p''' γ'' p'' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then have nin: "∄ p γ. (p, γ, Isolated p' γ') ∈ Aiminus1"
      using local.add_trans_pop(1) step.IH step.prems(1,2) by fastforce
    then have "Isolated p' γ' ≠ q"
      using add_trans_pop(4) LTS_ε.trans_star_not_to_source_ε LTS.srcs_def2
      by (metis local.add_trans_pop(3) state.distinct(3))
    then have "∄ p γ. (p, γ, Isolated p' γ') = (Init p'', ε, q)"
      by auto
    then show ?thesis
      using nin add_trans_pop(1) by auto
  next
    case (add_trans_swap p'''' γ'' p'' γ''' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then have nin: "∄ p γ. (p, γ, Isolated p' γ') ∈ Aiminus1"
      using local.add_trans_swap(1) step.IH step.prems(1,2) by fastforce
    then have "Isolated p' γ' ≠ q"
      using LTS.srcs_def2
      by (metis state.distinct(4) LTS_ε.trans_star_not_to_source_ε local.add_trans_swap(3))
    then have "∄ p γ. (p, γ, Isolated p' γ') = (Init p'', Some γ''', q)"
      by auto
    then show ?thesis
      using nin add_trans_swap(1) by auto
  next
    case (add_trans_push_1 p'''' γ'' p'' γ''' γ''''' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then show ?thesis
```

```
    using add_trans_push_1(1) Un_iff state.inject(2) prod.inject singleton_iff step.IH
      step.prems(1,2) by blast
  next
    case (add_trans_push_2 p'''' γ'' p'' γ''' γ'''' q)
    have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) .
    then have nin: "∄p γ. (p, γ, Isolated p' γ') ∈ Aiminus1"
      using local.add_trans_push_2(1) step.IH step.prems(1) by fastforce
    then have "Isolated p' γ' ≠ q"
      using LTS.srcs_def2 local.add_trans_push_2(3)
      by (metis state.disc(1,3) LTS_ε.trans_star_not_to_source_ε)
    then have "∄p γ. (p, γ, Isolated p' γ') = (Init p'', ε, q)"
      by auto
    then show ?thesis
      using nin add_trans_push_2(1) by auto
  qed
qed

lemma post_star_rules_Isolated_source_invariant:
  assumes "post_star_rules** A A'"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p', Some γ', Isolated p' γ') ∉ A'"
  shows "Isolated p' γ' ∈ LTS.srcs A'"
  by (meson LTS.srcs_def2 assms(1) assms(2) assms(3) post_star_rules_Isolated_source_invariant')

lemma post_star_rules_Isolated_sink_invariant':
  assumes "post_star_rules** A A'"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p', Some γ', Isolated p' γ') ∉ A'"
  shows "∄p γ. (Isolated p' γ', γ, p) ∈ A'"
  using assms
proof (induction rule: rtranclp_induct)
  case base
  then show ?case
    unfolding isols_def is_Isolated_def
    using LTS.isolated_no_edges by fastforce
next
  case (step Aiminus1 Ai)
  from step(2) show ?case
  proof (cases rule: post_star_rules.cases)
    case (add_trans_pop p''' γ'' p'' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then have nin: "∄p γ. (Isolated p' γ', γ, p) ∈ Aiminus1"
      using local.add_trans_pop(1) step.IH step.prems(1,2) by fastforce
    then have "Isolated p' γ' ≠ q"
      using add_trans_pop(4)
        LTS_ε.trans_star_not_to_source_ε[of "Init p'''" "[γ'']" q Aiminus1 "Isolated p' γ'"]
        post_star_rules_Isolated_source_invariant local.add_trans_pop(1) step.hyps(1) step.prems(1,2)
        UnI1 local.add_trans_pop(3) by (metis (full_types) state.distinct(3))
    then have "∄p γ. (p, γ, Isolated p' γ') = (Init p'', ε, q)"
      by auto
    then show ?thesis
      using nin add_trans_pop(1) by auto
  next
    case (add_trans_swap p'''' γ'' p'' γ''' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then have nin: "∄p γ. (Isolated p' γ', γ, p) ∈ Aiminus1"
      using local.add_trans_swap(1) step.IH step.prems(1,2) by fastforce
    then have "Isolated p' γ' ≠ q"
      using LTS_ε.trans_star_not_to_source_ε[of "Init p''''" "[γ'']" q Aiminus1]
        local.add_trans_swap(3) post_star_rules_Isolated_source_invariant[of _ Aiminus1 p' γ'] UnCI
```

```
      local.add_trans_swap(1) step.hyps(1) step.prems(1,2) state.simps(7) by metis
    then have "∄p γ. (p, γ, Isolated p' γ') = (Init p'', Some γ''', q)"
      by auto
    then show ?thesis
      using nin add_trans_swap(1) by auto
  next
    case (add_trans_push_1 p'''' γ'' p'' γ''' γ''''' q)
    then have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then show ?thesis
      using add_trans_push_1(1) Un_iff state.inject prod.inject singleton_iff step.IH
        step.prems(1,2) by blast
  next
    case (add_trans_push_2 p'''' γ'' p'' γ''' γ'''' q)
    have "(Init p', Some γ', Isolated p' γ') ∉ Ai"
      using step.prems(2) by blast
    then have nin: "∄p γ. (Isolated p' γ', γ, p) ∈ Aiminus1"
      using local.add_trans_push_2(1) step.IH step.prems(1,2) by fastforce
    then have "Isolated p' γ' ≠ q"
      using state.disc(3)
        LTS_ε.trans_star_not_to_source_ε[of "Init p''''" "[γ'']" q Aiminus1 "Isolated p' γ'"]
        local.add_trans_push_2(3)
      using post_star_rules_Isolated_source_invariant[of _ Aiminus1 p' γ'] UnCI
        local.add_trans_push_2(1) step.hyps(1) step.prems(1,2) state.disc(1) by metis
    then have "∄p γ. (Isolated p' γ', γ, p) = (Init p'', ε, q)"
      by auto
    then show ?thesis
      using nin add_trans_push_2(1)
      using local.add_trans_push_2 step.prems(2) by auto
  qed
qed

lemma post_star_rules_Isolated_sink_invariant:
  assumes "post_star_rules** A A'"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p', Some γ', Isolated p' γ') ∉ A'"
  shows "Isolated p' γ' ∈ LTS.sinks A'"
  by (meson LTS.sinks_def2 assms(1,2,3) post_star_rules_Isolated_sink_invariant')


— Corresponds to Schwoon's lemma 3.4
lemma rtranclp_post_star_rules_constains_successors_states:
  assumes "post_star_rules** A A'"
  assumes "inits ⊆ LTS.srcs A"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p, w, ss, q) ∈ LTS.trans_star_states A'"
  shows "(¬is_Isolated q ⟶ (∃p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε
w))) ∧
        (is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w))"
  using assms
proof (induction arbitrary: p q w ss rule: rtranclp_induct)
  case base
  {
    assume ctr_loc: "is_Init q ∨ is_Noninit q"
    then have "(Init p, LTS_ε.remove_ε w, q) ∈ LTS_ε.trans_star_ε A"
      using base LTS_ε.trans_star_states_trans_star_ε by metis
    then have "∃p' w'. (p', w', q) ∈ LTS_ε.trans_star_ε A"
      by auto
    then have ?case
      using ctr_loc ‹(Init p, LTS_ε.remove_ε w, q) ∈ LTS_ε.trans_star_ε A› by blast
  }
  moreover
  {
```

```
    assume "is_Isolated q"
    then have ?case
    proof (cases w)
      case Nil
      then have False using base
        using LTS.trans_star_empty LTS.trans_star_states_trans_star ‹is_Isolated q›
        by (metis state.disc(7))
      then show ?thesis
        by metis
    next
      case (Cons γ w_rest)
      then have "(Init p, γ#w_rest, ss, q) ∈ LTS.trans_star_states A"
        using base Cons by blast
      then have "∃ s γ'. (s, γ', q) ∈ A"
        using LTS.trans_star_states_transition_relation by metis
      then have False
        using ‹is_Isolated q› isols_def base.prems(2) LTS.isolated_no_edges
        by (metis mem_Collect_eq subset_eq)
      then show ?thesis
        by auto
    qed
  }
  ultimately
  show ?case
   by (meson state.exhaust_disc)
next
  case (step Aiminus1 Ai)
  from step(2) have "∃ p1 γ p2 w2 q1. Ai = Aiminus1 ∪ {(p1, γ, q1)} ∧ (p1, γ, q1) ∉ Aiminus1"
    by (cases rule: post_star_rules.cases) auto
  then obtain p1 γ q1 where p1_γ_p2_w2_q'_p:
    "Ai = Aiminus1 ∪ {(p1, γ, q1)}"
    "(p1, γ, q1) ∉ Aiminus1"
    by auto

  define t where "t = (p1, γ, q1)"
  define j where "j = count (transitions_of' (Init p, w, ss, q)) t"

  note ss_p = step(6)

  from j_def ss_p show ?case
  proof (induction j arbitrary: p q w ss)
    case 0
    then have "(Init p, w, ss, q) ∈ LTS.trans_star_states Aiminus1"
      using count_zero_remove_path_with_word_trans_star_states p1_γ_p2_w2_q'_p(1) t_def
      by metis
    then show ?case
      using step by auto
  next
    case (Suc j)
    from step(2) show ?case
    proof (cases rule: post_star_rules.cases)
      case (add_trans_pop p2 γ2 p1 q1)
      note III = add_trans_pop(3)
      note VI = add_trans_pop(2)
      have t_def: "t = (Init p1, ε, q1)"
        using local.add_trans_pop(1) local.add_trans_pop p1_γ_p2_w2_q'_p(1) t_def by blast
      have init_Ai: "inits ⊆ LTS.srcs Ai"
        using step(1,2) step(4)
        using no_edge_to_Ctr_Loc_post_star_rules
        using r_into_rtranclp by (metis)
      have t_hd_once: "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
      proof −
        have "(ss, w) ∈ LTS.path_with_word Ai"
```

```
    using Suc(3) LTS.trans_star_states_path_with_word by metis
  moreover
  have "inits ⊆ LTS.srcs Ai"
    using init_Ai by auto
  moreover
  have "0 < count (transitions_of (ss, w)) t"
    by (metis Suc.prems(1) transitions_of'.simps zero_less_Suc)
  moreover
  have "t = (Init p1, ε, q1)"
    using t_def by auto
  moreover
  have "Init p1 ∈ inits"
    by (simp add: inits_def)
  ultimately
  show "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
    using init_only_hd[of ss w Ai t p1 ε q1] by auto
qed

have "transition_list (ss, w) ≠ []"
  by (metis LTS.trans_star_states_path_with_word LTS.path_with_word.simps Suc.prems(1)
      Suc.prems(2) count_empty less_not_refl2 list.distinct(1) transition_list.simps(1)
      transitions_of'.simps transitions_of.simps(2) zero_less_Suc)
then have ss_w_split: "([Init p1,q1], [ε]) @´ (tl ss,  tl w) = (ss, w)"
  using t_hd_once t_def hd_transition_list_append_path_with_word by metis
then have ss_w_split': "(Init p1, [ε], [Init p1,q1], q1) @@´ (q1, tl w, tl ss, q) = (Init p1, w, ss, q)"
  by auto
have VII: "p = p1"
proof −
  have "(Init p, w, ss, q) ∈ LTS.trans_star_states Ai"
    using Suc.prems(2) by blast
  moreover
  have "t = hd (transition_list' (Init p, w, ss, q))"
    using ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1›
    by fastforce
  moreover
  have "transition_list' (Init p, w, ss, q) ≠ []"
    by (simp add: ‹transition_list (ss, w) ≠ []›)
  moreover
  have "t = (Init p1, ε, q1)"
    using t_def by auto
  ultimately
  show "p = p1"
    using LTS.hd_is_hd by fastforce
qed
have "j=0"
  using Suc(2) ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1›
  by force
have "(Init p1, [ε], [Init p1, q1], q1) ∈ LTS.trans_star_states Ai"
proof −
  have "(Init p1, ε, q1) ∈ Ai"
    using local.add_trans_pop(1) by auto
  moreover
  have "(Init p1, ε, q1) ∉ Aiminus1"
    by (simp add: local.add_trans_pop)
  ultimately
  show "(Init p1, [ε], [Init p1, q1], q1) ∈ LTS.trans_star_states Ai"
    by (meson LTS.trans_star_states.trans_star_states_refl
        LTS.trans_star_states.trans_star_states_step)
qed
have "(q1, tl w, tl ss, q) ∈ LTS.trans_star_states Aiminus1"
proof −
  from Suc(3) have "(ss, w) ∈ LTS.path_with_word Ai"
    by (meson LTS.trans_star_states_path_with_word)
```

then have *tl_ss_w_Ai*: "(*tl ss*, *tl w*) ∈ *LTS.path_with_word Ai*"
by (*metis LTS.path_with_word.simps* ‹*transition_list* (*ss*, *w*) ≠ []› *list.sel*(*3*)
transition_list.simps(*2*))
from *t_hd_once* **have** *zero_p1_ε_q1*: "*0* = *count* (*transitions_of* (*tl ss*, *tl w*)) (*Init p1*, *ε*, *q1*)"
using *count_append_path_with_word_γ*[*of* "[*hd ss*]" "[]" "*tl ss*" "*hd w*" "*tl w*" "*Init p1*" *ε q1*, *simplified*]
‹(*Init p1*, [*ε*], [*Init p1*, *q1*], *q1*) ∈ *LTS.trans_star_states Ai*›
‹*transition_list* (*ss*, *w*) ≠ []› *Suc.prems*(*2*) *VII*
LTS.transition_list_Cons[*of* "*Init p*" *w ss q Ai ε q1*] **by** (*auto simp*: *t_def*)
have *Ai_Aiminus1*: "*Ai* = *Aiminus1* ∪ {(*Init p1*, *ε*, *q1*)}"
using *local.add_trans_pop*(*1*) **by** *auto*
from *t_hd_once tl_ss_w_Ai zero_p1_ε_q1 Ai_Aiminus1*
count_zero_remove_path_with_word[*OF tl_ss_w_Ai*, *of* "*Init p1*" *ε q1 Aiminus1*]
have "(*tl ss*, *tl w*) ∈ *LTS.path_with_word Aiminus1*"
by *auto*
moreover
have "*hd* (*tl ss*) = *q1*"
using *Suc.prems*(*2*) *VII* ‹*transition_list* (*ss*, *w*) ≠ []› *t_def*
LTS.transition_list_Cons t_hd_once **by** *fastforce*
moreover
have "*last ss* = *q*"
by (*metis LTS.trans_star_states_last Suc.prems*(*2*))
ultimately
show "(*q1*, *tl w*, *tl ss*, *q*) ∈ *LTS.trans_star_states Aiminus1*"
by (*metis* (*no_types*, *lifting*) *LTS.trans_star_states_path_with_word*
LTS.path_with_word_trans_star_states LTS.path_with_word_not_empty Suc.prems(*2*)
last_ConsR list.collapse)
qed
have "*w* = *ε* # (*tl w*)"
by (*metis Suc*(*3*) *VII* ‹*transition_list* (*ss*, *w*) ≠ []› *list.distinct*(*1*) *list.exhaust_sel*
list.sel(*1*) *t_def LTS.transition_list_Cons t_hd_once*)
then have *w_tl_ε*: "*LTS_ε.remove_ε w* = *LTS_ε.remove_ε* (*tl w*)"
by (*metis LTS_ε.remove_ε_def removeAll.simps*(*2*))

have "∃ *γ2'*. *LTS_ε.ε_exp γ2'* [*γ2*] ∧ (*Init p2*, *γ2'*, *q1*) ∈ *LTS.trans_star Aiminus1*"
using *add_trans_pop*
by (*simp add*: *LTS_ε.trans_star_ε_ε_exp_trans_star*)
then obtain *γ2'* **where** "*LTS_ε.ε_exp γ2'* [*γ2*] ∧ (*Init p2*, *γ2'*, *q1*) ∈ *LTS.trans_star Aiminus1*"
by *blast*
then have "∃ *ss2*. (*Init p2*, *γ2'*, *ss2*, *q1*) ∈ *LTS.trans_star_states Aiminus1*"
by (*simp add*: *LTS.trans_star_trans_star_states*)
then obtain *ss2* **where** *IIII_1*: "(*Init p2*, *γ2'*, *ss2*, *q1*) ∈ *LTS.trans_star_states Aiminus1*"
by *blast*
have *IIII_2*: "(*q1*, *tl w*, *tl ss*, *q*) ∈ *LTS.trans_star_states Aiminus1*"
using *ss_w_split'* *Suc*(*3*) *Suc*(*2*) ‹*j*=*0*›
using ‹(*q1*, *tl w*, *tl ss*, *q*) ∈ *LTS.trans_star_states Aiminus1*› **by** *blast*
have *IIII*: "(*Init p2*, *γ2'* @ *tl w*, *ss2* @ (*tl* (*tl ss*)), *q*) ∈ *LTS.trans_star_states Aiminus1*"
using *IIII_1 IIII_2* **by** (*meson LTS.trans_star_states_append*)

from *Suc*(*1*)[*of p2* "*γ2'* @ *tl w*" "*ss2* @ (*tl* (*tl ss*))" *q*]
have *V*: "(¬*is_Isolated q* ⟶
(∃ *p' w'*. (*Init p'*, *w'*, *q*) ∈ *LTS_ε.trans_star_ε A* ∧ (*p'*, *w'*) ⇒* (*p2*, *LTS_ε.remove_ε* (*γ2'* @ *tl w*)))) ∧
(*is_Isolated q* ⟶ (*the_Ctr_Loc q*, [*the_Label q*]) ⇒* (*p2*, *LTS_ε.remove_ε* (*γ2'* @ *tl w*)))"
using *IIII*
using *step.IH step.prems*(*1*,*2*,*3*) **by** *blast*

have "¬*is_Isolated q* ∨ *is_Isolated q*"
using *state.exhaust_disc* **by** *blast*
then show *?thesis*
proof
assume *ctr_q*: "¬*is_Isolated q*"
then have "∃ *p' w'*. (*Init p'*, *w'*, *q*) ∈ *LTS_ε.trans_star_ε A* ∧ (*p'*, *w'*) ⇒* (*p2*, *LTS_ε.remove_ε* (*γ2'* @ *tl w*))"
using *V* **by** *auto*

```
      then obtain p' w' where
        VIII:"(Init p', w', q) ∈ LTS_ε.trans_star_ε A" and steps: "(p', w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl
w))"
        by blast
      then have "(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w)) ∧
            (p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, LTS_ε.remove_ε (tl w))"
      proof −
        have γ2'_γ2: "LTS_ε.remove_ε γ2' = [γ2]"
            by (metis LTS_ε.ε_exp_def LTS_ε.remove_ε_def ‹LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈
LTS.trans_star Aiminus1›)
        have "(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w))"
          using steps by auto
        moreover
        have rule: "(p2, γ2) ↪ (p, pop)"
          using VI VII by auto
        from steps have steps': "(p', w') ⇒* (p2, γ2 # (LTS_ε.remove_ε (tl w)))"
          using γ2'_γ2
          by (metis Cons_eq_appendI LTS_ε.remove_ε_append_dist self_append_conv2)
        from rule steps' have "(p2, γ2 # (LTS_ε.remove_ε (tl w))) ⇒* (p, LTS_ε.remove_ε (tl w))"
          using VIII
          by (metis PDS.transition_rel.intros append_self_conv2 lbl.simps(1) r_into_rtranclp
              step_relp_def)
        then have "(p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, LTS_ε.remove_ε (tl w))"
          by (simp add: LTS_ε.remove_ε_append_dist γ2'_γ2)
        ultimately
        show "(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w)) ∧
            (p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, LTS_ε.remove_ε (tl w))"
          by auto
      qed
      then have "(p',w') ⇒* (p, LTS_ε.remove_ε (tl w)) ∧ (Init p', w', q) ∈ LTS_ε.trans_star_ε A"
        using VIII by force
      then have "∃p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p, LTS_ε.remove_ε w)"
        using w_tl_ε by auto
      then show ?thesis
        using ctr_q ‹p = p1› by blast
    next
      assume "is_Isolated q"
      from V have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p2, γ2#(LTS_ε.remove_ε w))"
        by (metis LTS_ε.ε_exp_def LTS_ε.remove_ε_append_dist LTS_ε.remove_ε_def
            ‹LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈ LTS.trans_star Aiminus1› ‹is_Isolated q›
            append_Cons append_self_conv2 w_tl_ε)

      then have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p1, LTS_ε.remove_ε w)"
        using VI by (metis append_Nil lbl.simps(1) rtranclp.simps step_relp_def2)
      then have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w)"
        using VII by auto
      then show ?thesis
        using ‹is_Isolated q› by blast
    qed
  next
    case (add_trans_swap p2 γ2 p1 γ' q1)
    note III = add_trans_swap(3)
    note VI = add_trans_swap(2)
    have t_def: "t = (Init p1, Some γ', q1)"
      using local.add_trans_swap(1) local.add_trans_swap p1_γ_p2_w2_q'_p(1) t_def by blast
    have init_Ai: "inits ⊆ LTS.srcs Ai"
      using step(1,2,4) no_edge_to_Ctr_Loc_post_star_rules by (meson r_into_rtranclp)
    have t_hd_once: "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
    proof −
      have "(ss, w) ∈ LTS.path_with_word Ai"
        using Suc(3) LTS.trans_star_states_path_with_word by metis
      moreover
      have "inits ⊆ LTS.srcs Ai"
```

```
    using init_Ai by auto
  moreover
  have "0 < count (transitions_of (ss, w)) t"
    by (metis Suc.prems(1) transitions_of'.simps zero_less_Suc)
  moreover
  have "t = (Init p1, Some γ', q1)"
    using t_def
    by auto
  moreover
  have "Init p1 ∈ inits"
    using inits_def by force
  ultimately
  show "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
    using init_only_hd[of ss w Ai t p1 _ q1] by auto
qed

have "transition_list (ss, w) ≠ []"
  by (metis LTS.trans_star_states_path_with_word LTS.path_with_word.simps Suc.prems(1,2)
      count_empty less_not_refl2 list.distinct(1) transition_list.simps(1) transitions_of'.simps
      transitions_of.simps(2) zero_less_Suc)
then have ss_w_split: "([Init p1,q1], [Some γ']) @´ (tl ss,  tl w) = (ss, w)"
  using  t_hd_once t_def hd_transition_list_append_path_with_word by metis
then have ss_w_split': "(Init p1, [Some γ'], [Init p1,q1], q1) @@´ (q1, tl w, tl ss, q) = (Init p1, w, ss, q)"
  by auto
have VII: "p = p1"
proof −
  have "(Init p, w, ss, q) ∈ LTS.trans_star_states Ai"
    using Suc.prems(2) by blast
  moreover
  have "t = hd (transition_list' (Init p, w, ss, q))"
    using ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1› by fastforce
  moreover
  have "transition_list' (Init p, w, ss, q) ≠ []"
    by (simp add: ‹transition_list (ss, w) ≠ []›)
  moreover
  have "t = (Init p1, Some γ', q1)"
    using t_def by auto
  ultimately
  show "p = p1"
    using LTS.hd_is_hd by fastforce
qed
have "j=0"
  using Suc(2) ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1› by force
have "(Init p1, [Some γ'], [Init p1, q1], q1) ∈ LTS.trans_star_states Ai"
proof −
  have "(Init p1, Some γ', q1) ∈ Ai"
    using local.add_trans_swap(1) by auto
  moreover
  have "(Init p1, Some γ', q1) ∉ Aiminus1"
    using local.add_trans_swap(4) by blast
  ultimately
  show "(Init p1, [Some γ'], [Init p1, q1], q1) ∈ LTS.trans_star_states Ai"
    by (meson LTS.trans_star_states.trans_star_states_refl LTS.trans_star_states.trans_star_states_step)
qed
have "(q1, tl w, tl ss, q) ∈ LTS.trans_star_states Aiminus1"
proof −
  from Suc(3) have "(ss, w) ∈ LTS.path_with_word Ai"
    by (meson LTS.trans_star_states_path_with_word)
  then have tl_ss_w_Ai: "(tl ss, tl w) ∈ LTS.path_with_word Ai"
    by (metis LTS.path_with_word.simps ‹transition_list (ss, w) ≠ []› list.sel(3)
        transition_list.simps(2))
  from t_hd_once have zero_p1_ε_q1: "0 = count (transitions_of (tl ss, tl w)) (Init p1, Some γ', q1)"
    using count_append_path_with_word_γ[of "[hd ss]" "[]" "tl ss" "hd w" "tl w" "Init p1" "Some γ'" q1,
```

simplified]
⟨(*Init p1*, [*Some γ'*], [*Init p1*, *q1*], *q1*) ∈ *LTS.trans_star_states Ai*⟩ ⟨*transition_list* (*ss*, *w*) ≠ []⟩
Suc.prems(*2*) *VII LTS.transition_list_Cons*[*of "Init p" w ss q Ai "Some γ'" q1*]
by (*auto simp*: *t_def*)
have *Ai_Aiminus1*: *"Ai = Aiminus1 ∪ {(Init p1, Some γ', q1)}"*
using *local.add_trans_swap*(*1*) **by** *auto*
from *t_hd_once tl_ss_w_Ai zero_p1_ε_q1 Ai_Aiminus1*
count_zero_remove_path_with_word[*OF tl_ss_w_Ai*, *of "Init p1" _ q1 Aiminus1*]
have *"(tl ss, tl w) ∈ LTS.path_with_word Aiminus1"*
by *auto*
moreover
have *"hd* (*tl ss*) = *q1"*
using *Suc.prems*(*2*) *VII* ⟨*transition_list* (*ss*, *w*) ≠ []⟩ *t_def LTS.transition_list_Cons t_hd_once* **by** *fastforce*
moreover
have *"last ss = q"*
by (*metis LTS.trans_star_states_last Suc.prems*(*2*))
ultimately
show *"(q1, tl w, tl ss, q) ∈ LTS.trans_star_states Aiminus1"*
by (*metis* (*no_types*, *lifting*) *LTS.trans_star_states_path_with_word*
LTS.path_with_word_trans_star_states LTS.path_with_word_not_empty Suc.prems(*2*)
last_ConsR list.collapse)
qed
have *"w = Some γ' # (tl w)"*
by (*metis Suc*(*3*) *VII* ⟨*transition_list* (*ss*, *w*) ≠ []⟩ *list.distinct*(*1*) *list.exhaust_sel*
list.sel(*1*) *t_def LTS.transition_list_Cons t_hd_once*)
then have *w_tl_ε*: *"LTS_ε.remove_ε w = LTS_ε.remove_ε (Some γ'#tl w)"*
using *LTS_ε.remove_ε_def removeAll.simps*(*2*)
by *presburger*
have *"∃γ2'. LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈ LTS.trans_star Aiminus1"*
using *add_trans_swap* **by** (*simp add*: *LTS_ε.trans_star_ε_ε_exp_trans_star*)
then obtain *γ2'* **where** *"LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈ LTS.trans_star Aiminus1"*
by *blast*
then have *"∃ss2. (Init p2, γ2', ss2, q1) ∈ LTS.trans_star_states Aiminus1"*
by (*simp add*: *LTS.trans_star_trans_star_states*)
then obtain *ss2* **where** *IIII_1*: *"(Init p2, γ2', ss2, q1) ∈ LTS.trans_star_states Aiminus1"*
by *blast*
have *IIII_2*: *"(q1, tl w, tl ss, q) ∈ LTS.trans_star_states Aiminus1"*
using *ss_w_split' Suc*(*3*) *Suc*(*2*) ⟨*j=0*⟩
using ⟨(*q1*, *tl w*, *tl ss*, *q*) ∈ *LTS.trans_star_states Aiminus1*⟩ **by** *blast*
have *IIII*: *"(Init p2, γ2' @ tl w, ss2 @ (tl* (*tl ss*)), *q*) ∈ *LTS.trans_star_states Aiminus1"*
using *IIII_1 IIII_2* **by** (*meson LTS.trans_star_states_append*)

from *Suc*(*1*)[*of p2 "γ2' @ tl w" "ss2 @ (tl (tl ss))" q*]
have *V*: *"(¬is_Isolated q ⟶*
(∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒ (p2, LTS_ε.remove_ε (γ2' @ tl w)))) ∧*
(is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒ (p2, LTS_ε.remove_ε (γ2' @ tl w)))"*
using *IIII*
using *step.IH step.prems*(*1*,*2*,*3*) **by** *blast*

have *"¬is_Isolated q ∨ is_Isolated q"*
using *state.exhaust_disc* **by** *blast*
then show *?thesis*
proof
assume *ctr_q*: *"¬is_Isolated q"*
then have *"∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w))"*
using *V* **by** *auto*
then obtain *p' w'* **where**
VIII:*"(Init p', w', q) ∈ LTS_ε.trans_star_ε A"* **and** *steps*: *"(p', w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w))"*
by *blast*
then have *"(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w)) ∧*

```
            (p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, γ' # LTS_ε.remove_ε (tl w))"
      proof −
        have γ2'_γ2: "LTS_ε.remove_ε γ2' = [γ2]"
            by (metis LTS_ε.ε_exp_def LTS_ε.remove_ε_def ‹LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈
LTS.trans_star Aiminus1›)
        have "(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w))"
          using steps by auto
        moreover
        have rule: "(p2, γ2) ↪ (p, swap γ')"
          using VI VII by auto
        from steps have steps': "(p', w') ⇒* (p2, γ2 # (LTS_ε.remove_ε (tl w)))"
          using γ2'_γ2
          by (metis Cons_eq_appendI LTS_ε.remove_ε_append_dist self_append_conv2)
        from rule steps' have "(p2,  γ2 # (LTS_ε.remove_ε (tl w))) ⇒* (p, γ' # LTS_ε.remove_ε (tl w))"
          using  VIII
          using PDS.transition_rel.intros append_self_conv2 lbl.simps(1) r_into_rtranclp step_relp_def
          by fastforce
        then have "(p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, γ' # LTS_ε.remove_ε (tl w))"
          by (simp add: LTS_ε.remove_ε_append_dist γ2'_γ2)
        ultimately
        show "(p',w') ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w)) ∧
              (p2, LTS_ε.remove_ε (γ2' @ tl w)) ⇒* (p, γ' # LTS_ε.remove_ε (tl w))"
          by auto
      qed
      then have "(p',w') ⇒* (p, γ' # LTS_ε.remove_ε (tl w)) ∧ (Init p', w', q) ∈ LTS_ε.trans_star_ε A"
        using VIII by force
      then have "∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p, LTS_ε.remove_ε w)"
        using LTS_ε.remove_ε_Cons_tl by (metis ‹w = Some γ' # tl w›)
      then show ?thesis
        using ctr_q ‹p = p1› by blast
    next
      assume "is_Isolated q"
      from V this have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p2, LTS_ε.remove_ε (γ2' @ tl w))"
        by auto
      then have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p2, γ2#(LTS_ε.remove_ε (tl w)))"
        by (metis LTS_ε.ε_exp_def LTS_ε.remove_ε_append_dist LTS_ε.remove_ε_def
            ‹LTS_ε.ε_exp γ2' [γ2] ∧ (Init p2, γ2', q1) ∈ LTS.trans_star Aiminus1› append_Cons
            append_self_conv2)
      then have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p1, γ' # LTS_ε.remove_ε (tl w))"
        using VI
        by (metis (no_types) append_Cons append_Nil lbl.simps(2) rtranclp.rtrancl_into_rtrancl
            step_relp_def2)
      then have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p, γ' # LTS_ε.remove_ε (tl w))"
        using VII by auto
      then show ?thesis
        using ‹is_Isolated q›
        by (metis LTS_ε.remove_ε_Cons_tl w_tl_ε)
    qed
  next
    case (add_trans_push_1 p2 γ2 p1 γ1 γ'' q1')
    then have t_def: "t = (Init p1, Some γ1, Isolated p1 γ1)"
      using local.add_trans_pop(1) local.add_trans_pop p1_γ_p2_w2_q'_p(1) t_def by blast
    have init_Ai: "inits ⊆ LTS.srcs Ai"
      using step(1,2) step(4)
      using no_edge_to_Ctr_Loc_post_star_rules
      by (meson r_into_rtranclp)
    have t_hd_once: "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
    proof −
      have "(ss, w) ∈ LTS.path_with_word Ai"
        using Suc(3) LTS.trans_star_states_path_with_word by metis
      moreover
      have  "inits ⊆ LTS.srcs Ai"
        using init_Ai by auto
```

```
      moreover
      have "0 < count (transitions_of (ss, w)) t"
        by (metis Suc.prems(1) transitions_of'.simps zero_less_Suc)
      moreover
      have "t = (Init p1, Some γ1, Isolated p1 γ1)"
        using t_def by auto
      moreover
      have "Init p1 ∈ inits"
        using inits_def by fastforce
      ultimately
      show "hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1"
        using init_only_hd[of ss w Ai t] by auto
    qed
    have "transition_list (ss, w) ≠ []"
      by (metis LTS.trans_star_states_path_with_word LTS.path_with_word.simps Suc.prems(1,2)
          count_empty less_not_refl2 list.distinct(1) transition_list.simps(1)
          transitions_of'.simps transitions_of.simps(2) zero_less_Suc)

    have VII: "p = p1"
    proof −
      have "(Init p, w, ss, q) ∈ LTS.trans_star_states Ai"
        using Suc.prems(2) by blast
      moreover
      have "t = hd (transition_list' (Init p, w, ss, q))"
        using ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1› by fastforce
      moreover
      have "transition_list' (Init p, w, ss, q) ≠ []"
        by (simp add: ‹transition_list (ss, w) ≠ []›)
      moreover
      have "t = (Init p1, Some γ1, Isolated p1 γ1)"
        using t_def by auto
      ultimately
      show "p = p1"
        using LTS.hd_is_hd by fastforce
    qed
    from add_trans_push_1(4) have "∄p γ. (Isolated p1 γ1, γ, p) ∈ Aiminus1"
      using post_star_rules_Isolated_sink_invariant[of A Aiminus1 p1 γ1] step.hyps(1)
        step.prems(1,2,3) unfolding LTS.sinks_def by blast
    then have "∄p γ. (Isolated p1 γ1, γ, p) ∈ Ai"
      using local.add_trans_push_1(1) by blast
    then have ss_w_short: "ss = [Init p1, Isolated p1 γ1] ∧ w = [Some γ1]"
      using Suc.prems(2) VII ‹hd (transition_list (ss, w)) = t ∧ count (transitions_of (ss, w)) t = 1› t_def
       LTS.nothing_after_sink[of "Init p1" "Isolated p1 γ1" "tl (tl ss)" "Some γ1" "tl w" Ai] ‹transition_list (ss,
w) ≠ []›
       LTS.trans_star_states_path_with_word[of "Init p" w ss q Ai]
       LTS.transition_list_Cons[of "Init p" w ss q Ai]
       by (auto simp: LTS.sinks_def2)
    then have q_ext: "q = Isolated p1 γ1"
      using LTS.trans_star_states_last Suc.prems(2) by fastforce
    have "(p1, [γ1]) ⇒* (p, LTS_ε.remove_ε w)"
      using ss_w_short unfolding LTS_ε.remove_ε_def
      using VII by force
    have "(the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w)"
      by (simp add: ‹(p1, [γ1]) ⇒* (p, LTS_ε.remove_ε w)› q_ext)
    then show ?thesis
      using q_ext by auto
  next
    case (add_trans_push_2 p2 γ2 p1 γ1 γ'' q')
    note IX = add_trans_push_2(3)
    note XIII = add_trans_push_2(2)
    have t_def: "t = (Isolated p1 γ1, Some γ'', q')"
      using local.add_trans_push_2(1,4) p1_γ_p2_w2_q'_p(1) t_def by blast
    have init_Ai: "inits ⊆ LTS.srcs Ai"
```

using *step(1,2) step(4)*
using *no_edge_to_Ctr_Loc_post_star_rules*
by (*meson r_into_rtranclp*)

from *Suc(2,3) split_at_first_t*[*of* "*Init p*" *w ss q Ai j* "*Isolated p1 γ1*" "*Some γ''*" *q' Aiminus1*] *t_def*
have "∃ *u v u_ss v_ss*.
ss = *u_ss* @ *v_ss* ∧
w = *u* @ [*Some γ''*] @ *v* ∧
(*Init p*, *u*, *u_ss*, *Isolated p1 γ1*) ∈ *LTS.trans_star_states Aiminus1* ∧
(*Isolated p1 γ1*, [*Some γ''*], *q'*) ∈ *LTS.trans_star Ai* ∧ (*q'*, *v*, *v_ss*, *q*) ∈ *LTS.trans_star_states Ai*"
using *local.add_trans_push_2(1,4)* **by** *blast*
then obtain *u v u_ss v_ss* **where**
ss_split: "*ss* = *u_ss* @ *v_ss*" **and**
w_split: "*w* = *u* @ [*Some γ''*] @ *v*" **and**
X_1: "(*Init p*, *u*, *u_ss*, *Isolated p1 γ1*) ∈ *LTS.trans_star_states Aiminus1*" **and**
out_trans: "(*Isolated p1 γ1*, [*Some γ''*], *q'*) ∈ *LTS.trans_star Ai*" **and**
path: "(*q'*, *v*, *v_ss*, *q*) ∈ *LTS.trans_star_states Ai*"
by *auto*
from *step(3)*[*of p u u_ss* "*Isolated p1 γ1*"] *X_1* **have**
"(¬*is_Isolated* (*Isolated p1 γ1*) ⟶
(∃ *p' w'*. (*Init p'*, *w'*, *Isolated p1 γ1*) ∈ *LTS_ε.trans_star_ε A* ∧ (*p'*, *w'*) ⇒* (*p*, *LTS_ε.remove_ε u*))) ∧
(*is_Isolated* (*Isolated p1 γ1*) ⟶
(*the_Ctr_Loc* (*Isolated p1 γ1*), [*the_Label* (*Isolated p1 γ1*)]) ⇒* (*p*, *LTS_ε.remove_ε u*))"
using *step.prems(1,2,3)* **by** *auto*
then have "(*the_Ctr_Loc* (*Isolated p1 γ1*), [*the_Label* (*Isolated p1 γ1*)]) ⇒* (*p*, *LTS_ε.remove_ε u*)"
by *auto*
then have *p1_γ1_p_u*: "(*p1*, [*γ1*]) ⇒* (*p*, *LTS_ε.remove_ε u*)"
by *auto*
from *IX* **have** "∃*γ2ε γ2ss*. *LTS_ε.ε_exp γ2ε* [*γ2*] ∧ (*Init p2*, *γ2ε*, *γ2ss*, *q'*) ∈ *LTS.trans_star_states Aiminus1*"
by (*meson LTS.trans_star_trans_star_states LTS_ε.trans_star_ε_ε_exp_trans_star*)
then obtain *γ2ε γ2ss* **where** *XI_1*: "*LTS_ε.ε_exp γ2ε* [*γ2*] ∧ (*Init p2*, *γ2ε*, *γ2ss*, *q'*) ∈ *LTS.trans_star_states Aiminus1*"
by *blast*
have "(*q'*, *v*, *v_ss*, *q*) ∈ *LTS.trans_star_states Ai*"
using *path* **.**
have *ind*:
"(¬*is_Isolated q* ⟶ (∃ *p' w'*. (*Init p'*, *w'*, *q*) ∈ *LTS_ε.trans_star_ε A* ∧ (*p'*, *w'*) ⇒* (*p2*, *LTS_ε.remove_ε* (*γ2ε* @ *v*)))) ∧
(*is_Isolated q* ⟶ (*the_Ctr_Loc q*, [*the_Label q*]) ⇒* (*p2*, *LTS_ε.remove_ε* (*γ2ε* @ *v*)))"
proof −
have *γ2ss_len*: "*length γ2ss* = *Suc* (*length γ2ε*)"
by (*meson LTS.trans_star_states_length XI_1*)

have *v_ss_empty*: "*v_ss* ≠ []"
by (*metis LTS.trans_star_states.simps path list.distinct(1)*)

have *γ2ss_last*: "*last γ2ss* = *hd v_ss*"
by (*metis LTS.trans_star_states_hd LTS.trans_star_states_last XI_1 path*)

have *cv*: "*j* = *count* (*transitions_of* ((*v_ss*, *v*))) *t*"
proof −
have *last_u_ss*: "*Isolated p1 γ1* = *last u_ss*"
by (*meson LTS.trans_star_states_last X_1*)
have *q'_hd_v_ss*: "*q'* = *hd v_ss*"
by (*meson LTS.trans_star_states_hd path*)

have "*count* (*transitions_of'* (((*Init p*, *u*, *u_ss*, *Isolated p1 γ1*), *Some γ''*) @@^γ (*q'*, *v*, *v_ss*, *q*)))
(*Isolated p1 γ1*, *Some γ''*, *q'*) =
count (*transitions_of'* (*Init p*, *u*, *u_ss*, *Isolated p1 γ1*)) (*Isolated p1 γ1*, *Some γ''*, *q'*) +
(*if Isolated p1 γ1* = *last u_ss* ∧ *q'* = *hd v_ss* ∧ *Some γ''* = *Some γ''* *then 1 else 0*) +
count (*transitions_of'* (*q'*, *v*, *v_ss*, *q*)) (*Isolated p1 γ1*, *Some γ''*, *q'*)"

```
        using count_append_trans_star_states_γ_length[of u_ss u v_ss "Init p" "Isolated p1 γ1" "Some γ''" q'
v q "Isolated p1 γ1" "Some γ''" q'] t_def ss_split w_split X_1
        by (meson LTS.trans_star_states_length v_ss_empty)
       then have "count (transitions_of (u_ss @ v_ss, u @ Some γ'' # v)) (last u_ss, Some γ'', hd v_ss) =
Suc (count (transitions_of (u_ss, u)) (last u_ss, Some γ'', hd v_ss) + count (transitions_of (v_ss, v)) (last u_ss,
Some γ'', hd v_ss))"
        using last_u_ss q'_hd_v_ss by auto
      then have "j = count (transitions_of' ((q',v, v_ss, q))) t"
        using last_u_ss q'_hd_v_ss X_1 ss_split w_split add_trans_push_2(4) Suc(2)
          LTS.avoid_count_zero[of "Init p" u u_ss "Isolated p1 γ1" Aiminus1 "Isolated p1 γ1" "Some γ''" q']
        by (auto simp: t_def)
      then show "j = count (transitions_of ((v_ss, v))) t"
        by force
    qed
    have p2_q'_states_Aiminus1: "(Init p2, γ2ε, γ2ss, q') ∈ LTS.trans_star_states Aiminus1"
      using XI_1 by blast
    then have cγ2: "count (transitions_of (γ2ss, γ2ε)) t = 0"
      using LTS.avoid_count_zero local.add_trans_push_2(4) t_def by fastforce
    have "j = count (transitions_of ((γ2ss, γ2ε) @´ (v_ss, v))) t"
      using LTS.count_append_path_with_word[of γ2ss γ2ε v_ss v "Isolated p1 γ1" "Some γ''" q'] t_def
        cγ2 cv γ2ss_len v_ss_empty γ2ss_last
      by force
    then have j_count: "j = count (transitions_of' (Init p2, γ2ε @ v, γ2ss @ tl v_ss, q)) t"
      by simp

    have "(γ2ss, γ2ε) ∈ LTS.path_with_word Aiminus1"
      by (meson LTS.trans_star_states_path_with_word p2_q'_states_Aiminus1)
    then have γ2ss_path: "(γ2ss, γ2ε) ∈ LTS.path_with_word Ai"
      using add_trans_push_2(1)
      path_with_word_mono'[of γ2ss γ2ε Aiminus1 Ai] by auto

    have path': "(v_ss, v) ∈ LTS.path_with_word Ai"
      by (meson LTS.trans_star_states_path_with_word path)
    have "(γ2ss, γ2ε) @´ (v_ss, v) ∈ LTS.path_with_word Ai"
      using γ2ss_path path' LTS.append_path_with_word_path_with_word γ2ss_last
      by auto
    then have "(γ2ss @ tl v_ss, γ2ε @ v) ∈ LTS.path_with_word Ai"
      by auto


    have "(Init p2, γ2ε @ v, γ2ss @ tl v_ss, q) ∈ LTS.trans_star_states Ai"
      by (metis (no_types, lifting) LTS.path_with_word_trans_star_states
          LTS.trans_star_states_append LTS.trans_star_states_hd XI_1 path γ2ss_path γ2ss_last)

    from this Suc(1)[of p2 "γ2ε @ v" "γ2ss @ tl v_ss" q]
    show
      "(¬is_Isolated q ⟶ (∃p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p2, LTS_ε.remove_ε
(γ2ε @ v)))) ∧
       (is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p2, LTS_ε.remove_ε (γ2ε @ v)))"
      using j_count by fastforce
  qed

  show ?thesis
  proof (cases "is_Init q ∨ is_Noninit q")
    case True
    have "(∃p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p2, LTS_ε.remove_ε (γ2ε @ v)))"
      using True ind by fastforce
       then obtain p' w' where p'_w'_p: "(Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p', w') ⇒* (p2,
LTS_ε.remove_ε (γ2ε @ v))"
      by auto
    then have "(p', w') ⇒* (p2, LTS_ε.remove_ε (γ2ε @ v))"
      by auto
    have p2_γ2εv_p1_γ1_γ''_v: "(p2, LTS_ε.remove_ε (γ2ε @ v)) ⇒* (p1, γ1#γ''#LTS_ε.remove_ε v)"
```

```
      proof −
        have "γ2 #(LTS_ε.remove_ε v) = LTS_ε.remove_ε (γ2ε @ v)"
          using XI_1
          by (metis LTS_ε.ε_exp_def LTS_ε.remove_ε_append_dist LTS_ε.remove_ε_def append_Cons
              self_append_conv2)
        moreover
        from XIII have "(p2, γ2 #(LTS_ε.remove_ε v)) ⇒* (p1, γ1#γ''#LTS_ε.remove_ε v)"
          by (metis PDS.transition_rel.intros append_Cons append_Nil lbl.simps(3) r_into_rtranclp
              step_relp_def)
        ultimately
        show "(p2, LTS_ε.remove_ε (γ2ε @ v)) ⇒* (p1, γ1#γ''#LTS_ε.remove_ε v)"
          by auto
      qed
    have p1_γ1γ''v_p_uv: "(p1, γ1#γ''#LTS_ε.remove_ε v) ⇒* (p, (LTS_ε.remove_ε u) @ (γ''#LTS_ε.remove_ε
v))"
        by (metis p1_γ1_p_u append_Cons append_Nil step_relp_append)
      have "(p, (LTS_ε.remove_ε u) @ (γ''#LTS_ε.remove_ε v)) = (p, LTS_ε.remove_ε w)"
        by (metis (no_types, lifting) Cons_eq_append_conv LTS_ε.remove_ε_Cons_tl
            LTS_ε.remove_ε_append_dist w_split hd_Cons_tl list.inject list.sel(1) list.simps(3)
            self_append_conv2)
      then show ?thesis
        using True p1_γ1γ''v_p_uv p2_γ2εv_p1_γ1_γ''_v p'_w'_p by fastforce
    next
      case False
      then have q_nlq_p2_γ2εv: "(the_Ctr_Loc q, [the_Label q]) ⇒* (p2, LTS_ε.remove_ε (γ2ε @ v))"
        using ind state.exhaust_disc
        by blast
      have p2_γ2εv_p1_γ1γ''v: "(p2, LTS_ε.remove_ε (γ2ε @ v)) ⇒* (p1, γ1 # γ'' # LTS_ε.remove_ε v)"
        by (metis (mono_tags) LTS_ε.ε_exp_def LTS_ε.remove_ε_append_dist LTS_ε.remove_ε_def XIII
            XI_1 append_Cons append_Nil lbl.simps(3) r_into_rtranclp step_relp_def2)

      have p1_γ1γ''_v_p_uγ''v: "(p1, γ1 # γ'' # LTS_ε.remove_ε v) ⇒* (p, LTS_ε.remove_ε u @ γ'' #
LTS_ε.remove_ε v)"
        by (metis p1_γ1_p_u append_Cons append_Nil step_relp_append)

      have "(p, LTS_ε.remove_ε u @ γ'' # LTS_ε.remove_ε v) = (p, LTS_ε.remove_ε w)"
        by (metis LTS_ε.remove_ε_Cons_tl LTS_ε.remove_ε_append_dist append_Cons append_Nil w_split
            hd_Cons_tl list.distinct(1) list.inject)

      then show ?thesis
        using False p1_γ1γ''_v_p_uγ''v p2_γ2εv_p1_γ1γ''v q_nlq_p2_γ2εv
        by (metis (no_types, lifting) ind rtranclp_trans)
    qed
  qed
 qed
qed

— Corresponds to Schwoon's lemma 3.4
lemma rtranclp_post_star_rules_constains_successors:
  assumes "post_star_rules** A A'"
  assumes "inits ⊆ LTS.srcs A"
  assumes "isols ⊆ LTS.isolated A"
  assumes "(Init p, w, q) ∈ LTS.trans_star A'"
  shows "(¬is_Isolated q ⟶ (∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε
w))) ∧
        (is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒* (p, LTS_ε.remove_ε w))"
  using rtranclp_post_star_rules_constains_successors_states assms
  by (metis LTS.trans_star_trans_star_states)


— Corresponds to Schwoon's lemma 3.4
lemma post_star_rules_saturation_constains_successors:
  assumes "saturation post_star_rules A A'"
```

assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
assumes *"(Init p, w, q) ∈ LTS.trans_star A'"*
shows *"(¬is_Isolated q ⟶ (∃ p' w'. (Init p', w', q) ∈ LTS_ε.trans_star_ε A ∧ (p',w') ⇒* (p, LTS_ε.remove_ε w))) ∧*
(is_Isolated q ⟶ (the_Ctr_Loc q, [the_Label q]) ⇒ (p, LTS_ε.remove_ε w))"*
using *rtranclp_post_star_rules_constains_successors assms saturation_def* **by** *metis*

— Corresponds to one direction of Schwoon's theorem 3.3
theorem *post_star_rules_subset_post_star_lang*:
assumes *"post_star_rules** A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"{c. accepts_ε A' c} ⊆ post_star (lang_ε A)"*
proof
fix *c :: "('ctr_loc, 'label) conf"*
define *p* **where** *"p = fst c"*
define *w* **where** *"w = snd c"*
assume *"c ∈ {c. accepts_ε A' c}"*
then have *"accepts_ε A' (p,w)"*
unfolding *p_def w_def* **by** *auto*
then obtain *q* **where** *q_p: "q ∈ finals" "(Init p, w, q) ∈ LTS_ε.trans_star_ε A'"*
unfolding *accepts_ε_def* **by** *auto*
then obtain *w'* **where** *w'_def: "LTS_ε.ε_exp w' w ∧ (Init p, w', q) ∈ LTS.trans_star A'"*
by (*meson LTS_ε.trans_star_ε_iff_ε_exp_trans_star*)
then have *path: "(Init p, w', q) ∈ LTS.trans_star A'"*
by *auto*
have *"¬ is_Isolated q"*
using *F_not_Ext q_p(1)* **by** *blast*
then obtain *p' w'a* **where** *"(Init p', w'a, q) ∈ LTS_ε.trans_star_ε A ∧ (p', w'a) ⇒* (p, LTS_ε.remove_ε w')"*
using *rtranclp_post_star_rules_constains_successors*[*OF assms(1) assms(2) assms(3) path*] **by** *auto*
then have *"(Init p', w'a, q) ∈ LTS_ε.trans_star_ε A ∧ (p', w'a) ⇒* (p, w)"*
using *w'_def*
by (*metis LTS_ε.ε_exp_def LTS_ε.remove_ε_def*
‹*LTS_ε.ε_exp w' w ∧ (Init p, w', q) ∈ LTS.trans_star A'*›)
then have *"(p,w) ∈ post_star (lang_ε A)"*
using ‹*q ∈ finals*› **unfolding** *LTS.post_star_def accepts_ε_def lang_ε_def* **by** *fastforce*
then show *"c ∈ post_star (lang_ε A)"*
unfolding *p_def w_def* **by** *auto*
qed

— Corresponds to Schwoon's theorem 3.3
theorem *post_star_rules_accepts_ε_correct*:
assumes *"saturation post_star_rules A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"{c. accepts_ε A' c} = post_star (lang_ε A)"*
proof (*rule*; *rule*)
fix *c :: "('ctr_loc, 'label) conf"*
define *p* **where** *"p = fst c"*
define *w* **where** *"w = snd c"*
assume *"c ∈ post_star (lang_ε A)"*
then obtain *p' w'* **where** *"(p', w') ⇒* (p, w) ∧ (p', w') ∈ lang_ε A"*
by (*auto simp*: *post_star_def p_def w_def*)
then have *"accepts_ε A' (p, w)"*
using *lemma_3_3*[*of p' w' p w A A'*] *assms(1)* **by** *auto*
then have *"accepts_ε A' c"*
unfolding *p_def w_def* **by** *auto*
then show *"c ∈ {c. accepts_ε A' c}"*
by *auto*
next
fix *c :: "('ctr_loc, 'label) conf"*
assume *"c ∈ {c. accepts_ε A' c}"*

then show *"c ∈ post_star (lang_ε A)"*
using *assms post_star_rules_subset_post_star_lang* **unfolding** *saturation_def* **by** *blast*
qed

— Corresponds to Schwoon's theorem 3.3
theorem *post_star_rules_correct*:
assumes *"saturation post_star_rules A A'"*
assumes *"inits ⊆ LTS.srcs A"*
assumes *"isols ⊆ LTS.isolated A"*
shows *"lang_ε A' = post_star (lang_ε A)"*
using *assms lang_ε_def post_star_rules_accepts_ε_correct* **by** *presburger*

end

5.5 Intersection Automata

definition *accepts_inters* :: *"(('ctr_loc, 'noninit, 'label) state ∗ ('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state ∗ ('ctr_loc, 'noninit, 'label) state) set ⇒ ('ctr_loc, 'label) conf ⇒ bool"* **where**
"accepts_inters ts finals ≡ λ(p,w). (∃ qq ∈ finals. ((Init p, Init p),w,qq) ∈ LTS.trans_star ts)"

lemma *accepts_inters_accepts_aut_inters*:
assumes *"ts12 = inters ts1 ts2"*
assumes *"finals12 = inters_finals finals1 finals2"*
shows *"accepts_inters ts12 finals12 (p,w) ⟷*
Intersection_P_Automaton.accepts_aut_inters ts1 Init finals1 ts2
finals2 p w"
by (*simp add*: *Intersection_P_Automaton.accepts_aut_inters_def PDS_with_P_automata.inits_def P_Automaton.accepts_aut_def accepts_inters_def assms*)

definition *lang_inters* :: *"(('ctr_loc, 'noninit, 'label) state ∗ ('ctr_loc, 'noninit, 'label) state, 'label) transition set ⇒ (('ctr_loc, 'noninit, 'label) state ∗ ('ctr_loc, 'noninit, 'label) state) set ⇒ ('ctr_loc, 'label) conf set"* **where**
"lang_inters ts finals = {c. accepts_inters ts finals c}"

lemma *lang_inters_lang_aut_inters*:
assumes *"ts12 = inters ts1 ts2"*
assumes *"finals12 = inters_finals finals1 finals2"*
shows *"(λ(p,w). (p, w)) ‘ lang_inters ts12 finals12 =*
Intersection_P_Automaton.lang_aut_inters ts1 Init finals1 ts2 finals2"
using *assms*
by (*auto simp*: *Intersection_P_Automaton.lang_aut_inters_def*
Intersection_P_Automaton.inters_accept_iff
accepts_inters_accepts_aut_inters lang_inters_def is_Init_def
PDS_with_P_automata.inits_def P_Automaton.accepts_aut_def image_iff)

lemma *inters_accept_iff*:
assumes *"ts12 = inters ts1 ts2"*
assumes *"finals12 = inters_finals (PDS_with_P_automata.finals final_initss1 final_noninits1)*
(PDS_with_P_automata.finals final_initss2 final_noninits2)"
shows
"accepts_inters ts12 finals12 (p,w) ⟷
PDS_with_P_automata.accepts final_initss1 final_noninits1 ts1 (p,w) ∧
PDS_with_P_automata.accepts final_initss2 final_noninits2 ts2 (p,w)"
using *accepts_inters_accepts_aut_inters Intersection_P_Automaton.inters_accept_iff assms*
by (*simp add*: *Intersection_P_Automaton.inters_accept_iff accepts_inters_accepts_aut_inters*
PDS_with_P_automata.accepts_accepts_aut)

lemma *inters_lang*:
assumes *"ts12 = inters ts1 ts2"*
assumes *"finals12 =*
inters_finals (PDS_with_P_automata.finals final_initss1 final_noninits1)
(PDS_with_P_automata.finals final_initss2 final_noninits2)"
shows *"lang_inters ts12 finals12 =*
PDS_with_P_automata.lang final_initss1 final_noninits1 ts1 ∩
PDS_with_P_automata.lang final_initss2 final_noninits2 ts2"

using *assms* **by** (*auto simp add*: *PDS_with_P_automata.lang_def inters_accept_iff lang_inters_def*)

5.6 Intersection epsilon-Automata

context *PDS_with_P_automata* **begin**

interpretation *LTS transition_rel* **.**
notation *step_relp* (**infix** ‹⇒› *80*)
notation *step_starp* (**infix** ‹⇒*› *80*)

definition *accepts_ε_inters* :: "(('*ctr_loc*, '*noninit*, '*label*) *state* ∗ ('*ctr_loc*, '*noninit*, '*label*) *state*, '*label option*) *transition set* ⇒ ('*ctr_loc*, '*label*) *conf* ⇒ *bool*" **where**
"*accepts_ε_inters ts* ≡ λ(*p*,*w*). (∃ *q1* ∈ *finals*. ∃ *q2* ∈ *finals*. ((*Init p*, *Init p*),*w*,(*q1*,*q2*)) ∈ *LTS_ε.trans_star_ε ts*)"

definition *lang_ε_inters* :: "(('*ctr_loc*, '*noninit*, '*label*) *state* ∗ ('*ctr_loc*, '*noninit*, '*label*) *state*, '*label option*) *transition set* ⇒ ('*ctr_loc*, '*label*) *conf set*" **where**
"*lang_ε_inters ts* = {*c*. *accepts_ε_inters ts c*}"

lemma *trans_star_trans_star_ε_inter*:
assumes "*LTS_ε.ε_exp w1 w*"
assumes "*LTS_ε.ε_exp w2 w*"
assumes "(*p1*, *w1*, *p2*) ∈ *LTS.trans_star ts1*"
assumes "(*q1*, *w2*, *q2*) ∈ *LTS.trans_star ts2*"
shows "((*p1*,*q1*), *w* :: '*label list*, (*p2*,*q2*)) ∈ *LTS_ε.trans_star_ε* (*inters_ε ts1 ts2*)"
using *assms*
proof (*induction* "*length w1* + *length w2*" *arbitrary*: *w1 w2 w p1 q1 rule*: *less_induct*)
case *less*
then show *?case*
proof (*cases* "∃ α *w1'* *w2'* β. *w1*=*Some* α#*w1'* ∧ *w2*=*Some* β#*w2'*")
case *True*
from *True* **obtain** α β *w1'* *w2'* **where** *True''*:
"*w1*=*Some* α#*w1'*"
"*w2*=*Some* β#*w2'*"
by *auto*
have "α = β"
by (*metis True''*(*1*) *True''*(*2*) *LTS_ε.ε_exp_Some_hd less.prems*(*1*) *less.prems*(*2*))
then have *True'*:
"*w1*=*Some* α#*w1'*"
"*w2*=*Some* α#*w2'*"
using *True''* **by** *auto*
define *w'* **where** "*w'* = *tl w*"
obtain *p'* **where** *p'_p*: "(*p1*, *Some* α, *p'*) ∈ *ts1* ∧ (*p'*, *w1'*, *p2*) ∈ *LTS.trans_star ts1*"
using *less True'*(*1*) **by** (*metis LTS_ε.trans_star_cons_ε*)
obtain *q'* **where** *q'_p*: "(*q1*, *Some* α, *q'*) ∈ *ts2* ∧(*q'*, *w2'*, *q2*) ∈ *LTS.trans_star ts2*"
using *less True'*(*2*) **by** (*metis LTS_ε.trans_star_cons_ε*)
have *ind*: "((*p'*, *q'*), *w'*, *p2*, *q2*) ∈ *LTS_ε.trans_star_ε* (*inters_ε ts1 ts2*)"
proof −
have "*length w1'* + *length w2'* < *length w1* + *length w2*"
using *True'*(*1*) *True'*(*2*) **by** *simp*
moreover
have "*LTS_ε.ε_exp w1' w'*"
by (*metis* (*no_types*) *LTS_ε.ε_exp_def less*(*2*) *True'*(*1*) *list.map*(*2*) *list.sel*(*3*)
option.simps(*3*) *removeAll.simps*(*2*) *w'_def*)
moreover
have "*LTS_ε.ε_exp w2' w'*"
by (*metis* (*no_types*) *LTS_ε.ε_exp_def less*(*3*) *True'*(*2*) *list.map*(*2*) *list.sel*(*3*)
option.simps(*3*) *removeAll.simps*(*2*) *w'_def*)
moreover
have "(*p'*, *w1'*, *p2*) ∈ *LTS.trans_star ts1*"
using *p'_p* **by** *simp*
moreover
have "(*q'*, *w2'*, *q2*) ∈ *LTS.trans_star ts2*"
using *q'_p* **by** *simp*

```
    ultimately
    show "((p', q'), w', p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
      using less(1)[of w1' w2' w' p' q'] by auto
  qed
  moreover
  have "((p1, q1), Some α, (p', q')) ∈ (inters_ε ts1 ts2)"
    by (simp add: inters_ε_def p'_p q'_p)
  ultimately
  have "((p1, q1), α#w', p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
    by (meson LTS_ε.trans_star_ε.trans_star_ε_step_γ)
  moreover
  have "length w > 0"
    using less(3) True' LTS_ε.ε_exp_Some_length by metis
  moreover
  have "hd w = α"
    using less(3) True' LTS_ε.ε_exp_Some_hd by metis
  ultimately
  show ?thesis
    using w'_def by force
next
  case False
  note False_outer_outer_outer_outer = False
  show ?thesis
  proof (cases "w1 = [] ∧ w2 = []")
    case True
    then have same: "p1 = p2 ∧ q1 = q2"
      by (metis LTS.trans_star_empty less.prems(3) less.prems(4))
    have "w = []"
      using True less(2) LTS_ε.exp_empty_empty by auto
    then show ?thesis
      using less True
      by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl same)
  next
    case False
    note False_outer_outer_outer = False
    show ?thesis
    proof (cases "∃ w1'. w1=ε#w1'")
      case True
      then obtain w1' where True':
        "w1=ε#w1'"
        by auto
      obtain p' where p'_p: "(p1, ε, p') ∈ ts1 ∧ (p', w1', p2) ∈ LTS.trans_star ts1"
        using less True'(1) by (metis LTS_ε.trans_star_cons_ε)
      have q'_p: " (q1, w2, q2) ∈ LTS.trans_star ts2"
        using less by (metis)
      have ind: "((p', q1), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
      proof −
        have "length w1' + length w2 < length w1 + length w2"
          using True'(1) by simp
        moreover
        have "LTS_ε.ε_exp w1' w"
          by (metis (no_types) LTS_ε.ε_exp_def less(2) True'(1) removeAll.simps(2))
        moreover
        have "LTS_ε.ε_exp w2 w"
          by (metis (no_types) less(3))
        moreover
        have "(p', w1', p2) ∈ LTS.trans_star ts1"
          using p'_p by simp
        moreover
        have "(q1, w2, q2) ∈ LTS.trans_star ts2"
          using q'_p by simp
        ultimately
        show "((p', q1), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
```

```
      using less(1)[of w1' w2 w p' q1] by auto
  qed
  moreover
  have "((p1, q1), ε, (p', q1)) ∈ (inters_ε ts1 ts2)"
    by (simp add: inters_ε_def p'_p q'_p)
  ultimately
  have "((p1, q1), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
    using LTS_ε.trans_star_ε.simps by fastforce
  then
  show ?thesis
    by force
next
  case False
  note False_outer_outer = False
  then show ?thesis
  proof (cases "∃ w2'. w2 = ε # w2'")
    case True
    then obtain w2' where True': "w2=ε#w2'"
      by auto
    have p'_p: "(p1, w1, p2) ∈ LTS.trans_star ts1"
      using less by (metis)
    obtain q' where q'_p: "(q1, ε, q') ∈ ts2 ∧(q', w2', q2) ∈ LTS.trans_star ts2"
      using less True'(1) by (metis LTS_ε.trans_star_cons_ε)
    have ind: "((p1, q'), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
    proof −
      have "length w1 + length w2' < length w1 + length w2"
        using True'(1) True'(1) by simp
      moreover
      have "LTS_ε.ε_exp w1 w"
        by (metis (no_types) less(2))
      moreover
      have "LTS_ε.ε_exp w2' w"
        by (metis (no_types) LTS_ε.ε_exp_def less(3) True'(1) removeAll.simps(2))
      moreover
      have "(p1, w1, p2) ∈ LTS.trans_star ts1"
        using p'_p by simp
      moreover
      have "(q', w2', q2) ∈ LTS.trans_star ts2"
        using q'_p by simp
      ultimately
      show "((p1, q'), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
        using less(1)[of w1 w2' w p1 q'] by auto
    qed
    moreover
    have "((p1, q1), ε, (p1, q')) ∈ (inters_ε ts1 ts2)"
      by (simp add: inters_ε_def p'_p q'_p)
    ultimately
    have "((p1, q1), w, p2, q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
      using LTS_ε.trans_star_ε.simps by fastforce
    then
    show ?thesis
      by force
  next
    case False
    then have "(w1 = [] ∧ (∃ α w2'. w2 = Some α # w2')) ∨ ((∃ α w1'. w1 = Some α # w1') ∧ w2 = [])"
      using False_outer_outer False_outer_outer_outer False_outer_outer_outer_outer
      by (metis neq_Nil_conv option.exhaust_sel)
    then show ?thesis
      by (metis LTS_ε.ε_exp_def LTS_ε.ε_exp_Some_length less.prems(1,2) less_numeral_extra(3)
          list.simps(8) list.size(3) removeAll.simps(1))
  qed
qed
```

```
    qed
  qed
qed

lemma trans_star_ε_inter:
  assumes "(p1, w :: 'label list, p2) ∈ LTS_ε.trans_star_ε ts1"
  assumes "(q1, w, q2) ∈ LTS_ε.trans_star_ε ts2"
  shows "((p1, q1), w, (p2, q2)) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
proof −
  have "∃ w1'. LTS_ε.ε_exp w1' w ∧ (p1, w1', p2) ∈ LTS.trans_star ts1"
    using assms by (simp add: LTS_ε.trans_star_ε_ε_exp_trans_star)
  then obtain w1' where "LTS_ε.ε_exp w1' w ∧ (p1, w1', p2) ∈ LTS.trans_star ts1"
    by auto
  moreover
  have "∃ w2'. LTS_ε.ε_exp w2' w ∧ (q1, w2', q2) ∈ LTS.trans_star ts2"
    using assms by (simp add: LTS_ε.trans_star_ε_ε_exp_trans_star)
  then obtain w2' where "LTS_ε.ε_exp w2' w ∧ (q1, w2', q2) ∈ LTS.trans_star ts2"
    by auto
  ultimately
  show ?thesis
    using trans_star_trans_star_ε_inter by metis
qed

lemma inters_trans_star_ε1:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
  shows "(fst p1q2, w, fst p2q2) ∈ LTS_ε.trans_star_ε ts1"
  using assms
proof (induction rule: LTS_ε.trans_star_ε.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl)
next
  case (2 p γ q' w q)
  then have ind: "(fst q', w, fst q) ∈ LTS_ε.trans_star_ε ts1"
    by auto
  from 2(1) have "(p, Some γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts1}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ, q2) ∈
ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ,
q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_γ ind by fastforce
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have ?case
      by auto
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts1}"
    then have ?case
      by auto
```

```
  }
  ultimately
  show ?case
    by auto
next
  case (3 p q' w q)
  then have ind: "(fst q', w, fst q) ∈ LTS_ε.trans_star_ε ts1"
    by auto
  from 3(1) have "(p, ε, q') ∈
                  {((p1, q1), α, (p2, q2)) | p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, (p2, q1)) | p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                  {((p1, q1), ε, (p1, q2)) | p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have "∃ p1 p2 q1. p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then obtain p1 p2 q1 where "p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have "∃ p1 q1 q2. p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then obtain p1 q1 q2 where "p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  ultimately
  show ?case
    by auto
qed

lemma inters_trans_star_ε:
  assumes "(p1q2, w :: 'label list, p2q2) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2)"
  shows "(snd p1q2, w, snd p2q2) ∈ LTS_ε.trans_star_ε ts2"
  using assms
proof (induction rule: LTS_ε.trans_star_ε.induct[OF assms(1)])
  case (1 p)
  then show ?case
    by (simp add: LTS_ε.trans_star_ε.trans_star_ε_refl)
next
  case (2 p γ q' w q)
  then have ind: "(snd q', w, snd q) ∈ LTS_ε.trans_star_ε ts2"
    by auto
  from 2(1) have "(p, Some γ, q') ∈
                  {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                  {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
```

```
                {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ, q2) ∈
ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, Some γ, p2) ∈ ts1 ∧ (q1, Some γ,
q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_γ ind by fastforce
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have ?case
      by auto
  }
  moreover
  {
    assume "(p, Some γ, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have ?case
      by auto
  }
  ultimately
  show ?case
    by auto
next
  case (3 p q' w q)
  then have ind: "(snd q', w, snd q) ∈ LTS_ε.trans_star_ε ts2"
    by auto
  from 3(1) have "(p, ε, q') ∈
                {((p1, q1), α, (p2, q2)) | p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2} ∪
                {((p1, q1), ε, (p2, q1)) | p1 p2 q1. (p1, ε, p2) ∈ ts1} ∪
                {((p1, q1), ε, (p1, q2)) | p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    unfolding inters_ε_def by auto
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), α, p2, q2) |p1 q1 α p2 q2. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"
    then have "∃ p1 q1. p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by simp
    then obtain p1 q1 where "p = (p1, q1) ∧ (∃ p2 q2. q' = (p2, q2) ∧ (p1, ε, p2) ∈ ts1 ∧ (q1, ε, q2) ∈ ts2)"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p2, q1) |p1 p2 q1. (p1, ε, p2) ∈ ts1}"
    then have "∃ p1 p2 q1. p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then obtain p1 p2 q1 where "p = (p1, q1) ∧ q' = (p2, q1) ∧ (p1, ε, p2) ∈ ts1"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  moreover
  {
    assume "(p, ε, q') ∈ {((p1, q1), ε, p1, q2) |p1 q1 q2. (q1, ε, q2) ∈ ts2}"
    then have "∃ p1 q1 q2. p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
```

```
    then obtain p1 q1 q2 where "p = (p1, q1) ∧ q' = (p1, q2) ∧ (q1, ε, q2) ∈ ts2"
      by auto
    then have ?case
      using LTS_ε.trans_star_ε.trans_star_ε_step_ε ind by fastforce
  }
  ultimately
  show ?case
    by auto
qed

lemma inters_trans_star_ε_iff:
  "((p1,q2), w :: 'label list, (p2,q2)) ∈ LTS_ε.trans_star_ε (inters_ε ts1 ts2) ⟷
   (p1, w, p2) ∈ LTS_ε.trans_star_ε ts1 ∧ (q2, w, q2) ∈ LTS_ε.trans_star_ε ts2"
  by (metis fst_conv inters_trans_star_ε inters_trans_star_ε1 snd_conv trans_star_ε_inter)

lemma inters_ε_accept_ε_iff:
  "accepts_ε_inters (inters_ε ts1 ts2) c ⟷ accepts_ε ts1 c ∧ accepts_ε ts2 c"
proof
  assume "accepts_ε_inters (inters_ε ts1 ts2) c"
  then show "accepts_ε ts1 c ∧ accepts_ε ts2 c"
    using accepts_ε_def accepts_ε_inters_def inters_trans_star_ε inters_trans_star_ε1 by fastforce
next
  assume asm: "accepts_ε ts1 c ∧ accepts_ε ts2 c"
  define p where "p = fst c"
  define w where "w = snd c"

  from asm have "accepts_ε ts1 (p,w) ∧ accepts_ε ts2 (p,w)"
    using p_def w_def by auto
  then have "(∃ q∈finals. (Init p, w, q) ∈ LTS_ε.trans_star_ε ts1) ∧
             (∃ q∈finals. (Init p, w, q) ∈ LTS_ε.trans_star_ε ts2)"
    unfolding accepts_ε_def by auto
  then show "accepts_ε_inters (inters_ε ts1 ts2) c"
    using accepts_ε_inters_def p_def trans_star_ε_inter w_def by fastforce
qed

lemma inters_ε_lang_ε: "lang_ε_inters (inters_ε ts1 ts2) = lang_ε ts1 ∩ lang_ε ts2"
  unfolding lang_ε_inters_def lang_ε_def using inters_ε_accept_ε_iff by auto
```

5.7 Dual search

```
lemma dual1:
  "post_star (lang_ε A1) ∩ pre_star (lang A2) = {c. ∃ c1 ∈ lang_ε A1. ∃ c2 ∈ lang A2. c1 ⇒* c ∧ c ⇒* c2}"
proof −
  have "post_star (lang_ε A1) ∩ pre_star (lang A2) = {c. c ∈ post_star (lang_ε A1) ∧ c ∈ pre_star (lang A2)}"
    by auto
  moreover
  have "... = {c. (∃ c1 ∈ lang_ε A1. c1 ⇒* c) ∧ (∃ c2 ∈ lang A2. c ⇒* c2)}"
    unfolding post_star_def pre_star_def by auto
  moreover
  have "... = {c. ∃ c1 ∈ lang_ε A1. ∃ c2 ∈ lang A2. c1 ⇒* c ∧ c ⇒* c2}"
    by auto
  ultimately
  show ?thesis by metis
qed

lemma dual2:
  "post_star (lang_ε A1) ∩ pre_star (lang A2) ≠ {} ⟷ (∃ c1 ∈ lang_ε A1. ∃ c2 ∈ lang A2. c1 ⇒* c2)"
proof (rule)
  assume "post_star (lang_ε A1) ∩ pre_star (lang A2) ≠ {}"
  then show "∃ c1∈lang_ε A1. ∃ c2∈lang A2. c1 ⇒* c2"
    by (auto simp: pre_star_def post_star_def intro: rtranclp_trans)
next
  assume "∃ c1∈lang_ε A1. ∃ c2∈lang A2. c1 ⇒* c2"
  then show "post_star (lang_ε A1) ∩ pre_star (lang A2) ≠ {}"
```

using *dual1* **by** *auto*
qed

lemma *LTS_ε_of_LTS_Some*: "(p, *Some* γ, q′) ∈ *LTS_ε_of_LTS A2′* ⟷ (p, γ, q′) ∈ *A2′*"
unfolding *LTS_ε_of_LTS_def ε_edge_of_edge_def* **by** (*auto simp add*: *rev_image_eqI*)

lemma *LTS_ε_of_LTS_None*: "(p, *None*, q′) ∉ *LTS_ε_of_LTS A2′*"
unfolding *LTS_ε_of_LTS_def ε_edge_of_edge_def* **by** (*auto*)

lemma *trans_star_ε_LTS_ε_of_LTS_trans_star*:
assumes "(p,w,q) ∈ *LTS_ε.trans_star_ε* (*LTS_ε_of_LTS A2′*)"
shows "(p,w,q) ∈ *LTS.trans_star A2′*"
using *assms*
proof (*induction rule*: *LTS_ε.trans_star_ε.induct*[*OF assms*(*1*)])
case (*1 p*)
then show *?case*
by (*simp add*: *LTS.trans_star.trans_star_refl*)
next
case (*2 p γ q′ w q*)
moreover
have "(p, γ, q′) ∈ *A2′*"
using *2*(*1*) **using** *LTS_ε_of_LTS_Some* **by** *metis*
moreover
have "(q′, w, q) ∈ *LTS.trans_star A2′*"
using "*2.IH*" *2*(*2*) **by** *auto*
ultimately show *?case*
by (*meson LTS.trans_star.trans_star_step*)
next
case (*3 p q′ w q*)
then show *?case*
using *LTS_ε_of_LTS_None* **by** *fastforce*
qed

lemma *trans_star_trans_star_ε_LTS_ε_of_LTS*:
assumes "(p,w,q) ∈ *LTS.trans_star A2′*"
shows "(p,w,q) ∈ *LTS_ε.trans_star_ε* (*LTS_ε_of_LTS A2′*)"
using *assms*
proof (*induction rule*: *LTS.trans_star.induct*[*OF assms*(*1*)])
case (*1 p*)
then show *?case*
by (*simp add*: *LTS_ε.trans_star_ε.trans_star_ε_refl*)
next
case (*2 p γ q′ w q*)
then show *?case*
by (*meson LTS_ε.trans_star_ε.trans_star_ε_step_γ LTS_ε_of_LTS_Some*)
qed

lemma *accepts_ε_LTS_ε_of_LTS_iff_accepts*: "*accepts_ε* (*LTS_ε_of_LTS A2′*) (p, w) ⟷ *accepts A2′* (p, w)"
using *accepts_ε_def accepts_def trans_star_ε_LTS_ε_of_LTS_trans_star*
trans_star_trans_star_ε_LTS_ε_of_LTS **by** *fastforce*

lemma *lang_ε_LTS_ε_of_LTS_is_lang*: "*lang_ε* (*LTS_ε_of_LTS A2′*) = *lang A2′*"
unfolding *lang_ε_def lang_def* **using** *accepts_ε_LTS_ε_of_LTS_iff_accepts* **by** *auto*

theorem *dual_star_correct_early_termination*:
assumes "*inits* ⊆ *LTS.srcs A1*"
assumes "*inits* ⊆ *LTS.srcs A2*"
assumes "*isols* ⊆ *LTS.isolated A1*"
assumes "*isols* ⊆ *LTS.isolated A2*"
assumes "*post_star_rules*** *A1 A1′*"
assumes "*pre_star_rule*** *A2 A2′*"
assumes "*lang_ε_inters* (*inters_ε A1′* (*LTS_ε_of_LTS A2′*)) ≠ {}"

shows "$\exists\, c1 \in lang\_\varepsilon\ A1.\ \exists\, c2 \in lang\ A2.\ c1 \Rightarrow^* c2$"
proof −
have "$\{c.\ accepts\_\varepsilon\ A1'\ c\} \subseteq post\_star\ (lang\_\varepsilon\ A1)$"
using *assms* **using** *post_star_rules_subset_post_star_lang* **by** *auto*
then have *A1'_correct*: "$lang\_\varepsilon\ A1' \subseteq post\_star\ (lang\_\varepsilon\ A1)$"
unfolding *lang_ε_def* **by** *auto*

have "$\{c.\ accepts\ A2'\ c\} \subseteq pre\_star\ (lang\ A2)$"
using *pre_star_rule_subset_pre_star_lang*[*of A2 A2'*] *assms* **by** *auto*
then have *A2'_correct*: "$lang\ A2' \subseteq pre\_star\ (lang\ A2)$"
unfolding *lang_def* **by** *auto*

have "$lang\_\varepsilon\_inters\ (inters\_\varepsilon\ A1'\ (LTS\_\varepsilon\_of\_LTS\ A2')) = lang\_\varepsilon\ A1' \cap lang\_\varepsilon\ (LTS\_\varepsilon\_of\_LTS\ A2')$"
using *inters_ε_lang_ε*[*of A1'* "$(LTS\_\varepsilon\_of\_LTS\ A2')$"] **by** *auto*
moreover
have "$... = lang\_\varepsilon\ A1' \cap lang\ A2'$"
using *lang_ε_LTS_ε_of_LTS_is_lang* **by** *auto*
moreover
have "$... \subseteq post\_star\ (lang\_\varepsilon\ A1) \cap pre\_star\ (lang\ A2)$"
using *A1'_correct A2'_correct* **by** *auto*
ultimately
have *inters_correct*: "$lang\_\varepsilon\_inters\ (inters\_\varepsilon\ A1'\ (LTS\_\varepsilon\_of\_LTS\ A2')) \subseteq post\_star\ (lang\_\varepsilon\ A1) \cap pre\_star\ (lang\ A2)$"
by *metis*

from *assms* **have** "$post\_star\ (lang\_\varepsilon\ A1) \cap pre\_star\ (lang\ A2) \neq \{\}$"
using *inters_correct* **by** *auto*
then show "$\exists\, c1 \in lang\_\varepsilon\ A1.\ \exists\, c2 \in lang\ A2.\ c1 \Rightarrow^* c2$"
using *dual2* **by** *auto*

qed

theorem *dual_star_correct_saturation*:
assumes "$inits \subseteq LTS.srcs\ A1$"
assumes "$inits \subseteq LTS.srcs\ A2$"
assumes "$isols \subseteq LTS.isolated\ A1$"
assumes "$isols \subseteq LTS.isolated\ A2$"
assumes "$saturation\ post\_star\_rules\ A1\ A1'$"
assumes "$saturation\ pre\_star\_rule\ A2\ A2'$"
shows "$lang\_\varepsilon\_inters\ (inters\_\varepsilon\ A1'\ (LTS\_\varepsilon\_of\_LTS\ A2')) \neq \{\} \longleftrightarrow (\exists\, c1 \in lang\_\varepsilon\ A1.\ \exists\, c2 \in lang\ A2.\ c1 \Rightarrow^* c2)$"
proof −
have "$\{c.\ accepts\_\varepsilon\ A1'\ c\} = post\_star\ (lang\_\varepsilon\ A1)$"
using *post_star_rules_accepts_ε_correct*[*of A1 A1'*] *assms* **by** *auto*
then have *A1'_correct*: "$lang\_\varepsilon\ A1' = post\_star\ (lang\_\varepsilon\ A1)$"
unfolding *lang_ε_def* **by** *auto*

have "$\{c.\ accepts\ A2'\ c\} = pre\_star\ (lang\ A2)$"
using *pre_star_rule_accepts_correct*[*of A2 A2'*] *assms* **by** *auto*
then have *A2'_correct*: "$lang\ A2' = pre\_star\ (lang\ A2)$"
unfolding *lang_def* **by** *auto*

have "$lang\_\varepsilon\_inters\ (inters\_\varepsilon\ A1'\ (LTS\_\varepsilon\_of\_LTS\ A2')) = lang\_\varepsilon\ A1' \cap lang\_\varepsilon\ (LTS\_\varepsilon\_of\_LTS\ A2')$"
using *inters_ε_lang_ε*[*of A1'* "$(LTS\_\varepsilon\_of\_LTS\ A2')$"] **by** *auto*
moreover
have "$... = lang\_\varepsilon\ A1' \cap lang\ A2'$"
using *lang_ε_LTS_ε_of_LTS_is_lang* **by** *auto*
moreover
have "$... = post\_star\ (lang\_\varepsilon\ A1) \cap pre\_star\ (lang\ A2)$"
using *A1'_correct A2'_correct* **by** *auto*
ultimately
have *inters_correct*: "$lang\_\varepsilon\_inters\ (inters\_\varepsilon\ A1'\ (LTS\_\varepsilon\_of\_LTS\ A2')) = post\_star\ (lang\_\varepsilon\ A1) \cap pre\_star\ (lang\ A2)$"

```
    by metis

  show ?thesis
  proof
    assume "lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {}"
    then have "post_star (lang_ε A1) ∩ pre_star (lang A2) ≠ {}"
      using inters_correct by auto
    then show "∃ c1∈lang_ε A1. ∃ c2∈lang A2. c1 ⇒* c2"
      using dual2 by auto
  next
    assume "∃ c1∈lang_ε A1. ∃ c2∈lang A2. c1 ⇒* c2"
    then have "post_star (lang_ε A1) ∩ pre_star (lang A2) ≠ {}"
      using dual2 by auto
    then show "lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {}"
      using inters_correct by auto
  qed
qed

theorem dual_star_correct_early_termination_configs:
  assumes "inits ⊆ LTS.srcs A1"
  assumes "inits ⊆ LTS.srcs A2"
  assumes "isols ⊆ LTS.isolated A1"
  assumes "isols ⊆ LTS.isolated A2"
  assumes "lang_ε A1 = {c1}"
  assumes "lang A2 = {c2}"
  assumes "post_star_rules** A1 A1'"
  assumes "pre_star_rule** A2 A2'"
  assumes "lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {}"
  shows "c1 ⇒* c2"
  using dual_star_correct_early_termination assms by (metis singletonD)

theorem dual_star_correct_saturation_configs:
  assumes "inits ⊆ LTS.srcs A1"
  assumes "inits ⊆ LTS.srcs A2"
  assumes "isols ⊆ LTS.isolated A1"
  assumes "isols ⊆ LTS.isolated A2"
  assumes "lang_ε A1 = {c1}"
  assumes "lang A2 = {c2}"
  assumes "saturation post_star_rules A1 A1'"
  assumes "saturation pre_star_rule A2 A2'"
  shows "lang_ε_inters (inters_ε A1' (LTS_ε_of_LTS A2')) ≠ {} ⟷ c1 ⇒* c2"
  using assms dual_star_correct_saturation by auto

end

end
theory PDS_Code
  imports PDS "Deriving.Derive"
begin

global-interpretation pds: PDS_with_P_automata Δ F_ctr_loc F_ctr_loc_st
  for Δ :: "('ctr_loc::{enum, linorder}, 'label::{finite, linorder}) rule set"
  and F_ctr_loc :: "('ctr_loc) set"
  and F_ctr_loc_st :: "('state::finite) set"
  defines pre_star = "PDS_with_P_automata.pre_star_exec Δ"
  and pre_star_check = "PDS_with_P_automata.pre_star_exec_check Δ"
  and inits = "PDS_with_P_automata.inits"
  and finals = "PDS_with_P_automata.finals F_ctr_loc F_ctr_loc_st"
  and accepts = "PDS_with_P_automata.accepts F_ctr_loc F_ctr_loc_st"
  and language = "PDS_with_P_automata.lang F_ctr_loc F_ctr_loc_st"
  and step_starp = "rtranclp (LTS.step_relp (PDS.transition_rel Δ))"
 and accepts_pre_star_check = "PDS_with_P_automata.accept_pre_star_exec_check Δ F_ctr_loc F_ctr_loc_st"
  .
```

global-interpretation *inter*: *Intersection_P_Automaton*
initial_automaton Init "finals initial_F_ctr_loc initial_F_ctr_loc_st"
"pre_star Δ final_automaton" "finals final_F_ctr_loc final_F_ctr_loc_st"
for Δ :: *"('ctr_loc::{enum, linorder}, 'label::{finite, linorder}) rule set"*
and *initial_automaton* :: *"(('ctr_loc, 'state::finite, 'label) state, 'label) transition set"*
and *initial_F_ctr_loc* :: *"'ctr_loc set"*
and *initial_F_ctr_loc_st* :: *"'state set"*
and *final_automaton* :: *"(('ctr_loc, 'state, 'label) state, 'label) transition set"*
and *final_F_ctr_loc* :: *"'ctr_loc set"*
and *final_F_ctr_loc_st* :: *"'state set"*
defines *nonempty_inter* = *"P_Automaton.nonempty*
(inters initial_automaton (pre_star Δ final_automaton))
((λp. (Init p, Init p)))
(inters_finals (finals initial_F_ctr_loc initial_F_ctr_loc_st)
(finals final_F_ctr_loc final_F_ctr_loc_st))"
.

definition *"check Δ I IF IF_st F FF FF_st =*
(if pds.inits ⊆ LTS.srcs F then Some (nonempty_inter Δ I IF IF_st F FF FF_st) else None)"

lemma *check_None*: *"check Δ I IF IF_st F FF FF_st = None ⟷ ¬ (inits ⊆ LTS.srcs F)"*
unfolding *check_def* **by** *auto*

lemma *check_Some*: *"check Δ I IF IF_st F FF FF_st = Some b ⟷*
(inits ⊆ LTS.srcs F ∧ b = (∃ p w p' w'.
(p, w) ∈ language IF IF_st I ∧
(p', w') ∈ language FF FF_st F ∧
step_starp Δ (p, w) (p', w')))"
unfolding *check_def nonempty_inter_def P_Automaton.nonempty_def*
inter.lang_aut_alt inter.inters_lang
pds.lang_aut_lang
by (*auto 0 5 simp*: *pds.pre_star_exec_lang_correct pds.pre_star_def image_iff*
elim!: *bexI*[*rotated*])

declare *P_Automaton.mark.simps*[*code*]

export-code *check* **checking** *SML*

end

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