Public Announcement Logic

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Abstract

This work is a formalization of public announcement logic with countably many agents. It includes proofs of soundness and completeness for variants of the axiom system PA + DIST! + NEC! [1]. The completeness proofs build on the Epistemic Logic theory.

Contents

theory *PAL* **imports** *Epistemic-Logic.Epistemic-Logic* **begin**

1 Syntax

```
datatype 'i pfm
  = FF (‹⊥!›)
  | Pro' id (‹Pro!›)
  | Dis ‹'i pfm› ‹'i pfm› (infixr ‹∨!› 60)
  | Con ‹'i pfm› ‹'i pfm› (infixr ‹∧!› 65)
  | Imp ‹'i pfm› ‹'i pfm› (infixr ‹⟶!› 55)
  | K' 'i ‹'i pfm› (‹K!›)
  | Ann ‹'i pfm› ‹'i pfm› (‹[-]! -› [80, 80] 80)

abbreviation PIff :: ‹'i pfm ⇒ 'i pfm ⇒ 'i pfm› (infixr ‹⟷!› 55) where
  ‹p ⟷! q ≡ (p ⟶! q) ∧! (q ⟶! p)›

abbreviation PNeg (‹¬! -› [70] 70) where
  ‹¬! p ≡ p ⟶! ⊥!›

abbreviation PL (‹L!›) where
  ‹L! i p ≡ (¬! (K! i (¬! p)))›

primrec anns :: ‹'i pfm ⇒ 'i pfm set› where
  ‹anns ⊥! = {}›
| ‹anns (Pro! -) = {}›
| ‹anns (p ∨! q) = (anns p ∪ anns q)›
| ‹anns (p ∧! q) = (anns p ∪ anns q)›
| ‹anns (p ⟶! q) = (anns p ∪ anns q)›
| ‹anns (K! i p) = anns p›
| ‹anns ([r]! p) = {r} ∪ anns r ∪ anns p›
```

2 Semantics

```
fun
  psemantics :: ‹('i, 'w) kripke ⇒ 'w ⇒ 'i pfm ⇒ bool› (‹-, - ⊨! -› [50, 50, 50]
50) and
  restrict :: ‹('i, 'w) kripke ⇒ 'i pfm ⇒ ('i, 'w) kripke› (‹-[-!]› [50, 50] 50) where
  ‹M, w ⊨! ⊥! ⟷ False›
| ‹M, w ⊨! Pro! x ⟷ π M w x›
| ‹M, w ⊨! p ∨! q ⟷ M, w ⊨! p ∨ M, w ⊨! q›
| ‹M, w ⊨! p ∧! q ⟷ M, w ⊨! p ∧ M, w ⊨! q›
| ‹M, w ⊨! p ⟶! q ⟷ M, w ⊨! p ⟶ M, w ⊨! q›
| ‹M, w ⊨! K! i p ⟷ (∀ v ∈ 𝒲 M ∩ 𝒦 M i w. M, v ⊨! p)›
| ‹M, w ⊨! [r]! p ⟷ M, w ⊨! r ⟶ M[r!], w ⊨! p›
| ‹M[r!] = M (|𝒲 := {w. w ∈ 𝒲 M ∧ M, w ⊨! r}|)›
```

abbreviation *validPStar* :: ‹(('i, 'w) kripke ⇒ bool) ⇒ 'i pfm set ⇒ 'i pfm ⇒ bool›
(‹-; - $\Vdash_!\star$ -› [50, 50, 50] 50) **where**
‹P; G $\Vdash_!\star$ $p \equiv \forall M.\ P\ M \longrightarrow (\forall w \in \mathcal{W}\ M.\ (\forall q \in G.\ M, w \models_! q) \longrightarrow M, w \models_! p)$›

primrec *static* :: ‹'i pfm ⇒ bool› **where**
‹static $\bot_!$ = True›
| ‹static ($Pro_!$ -) = True›
| ‹static ($p \lor_! q$) = (static p ∧ static q)›
| ‹static ($p \land_! q$) = (static p ∧ static q)›
| ‹static ($p \longrightarrow_! q$) = (static p ∧ static q)›
| ‹static ($K_!$ i p) = static p›
| ‹static ($[r]_!$ p) = False›

primrec *lower* :: ‹'i pfm ⇒ 'i fm› **where**
‹lower $\bot_!$ = ⊥›
| ‹lower ($Pro_!$ x) = Pro x›
| ‹lower ($p \lor_! q$) = (lower p ∨ lower q)›
| ‹lower ($p \land_! q$) = (lower p ∧ lower q)›
| ‹lower ($p \longrightarrow_! q$) = (lower p ⟶ lower q)›
| ‹lower ($K_!$ i p) = K i (lower p)›
| ‹lower ($[r]_!$ p) = undefined›

primrec *lift* :: ‹'i fm ⇒ 'i pfm› **where**
‹lift ⊥ = $\bot_!$›
| ‹lift (Pro x) = $Pro_!$ x›
| ‹lift (p ∨ q) = (lift p $\lor_!$ lift q)›
| ‹lift (p ∧ q) = (lift p $\land_!$ lift q)›
| ‹lift (p ⟶ q) = (lift p $\longrightarrow_!$ lift q)›
| ‹lift (K i p) = $K_!$ i (lift p)›

lemma *lower-semantics*:
assumes ‹static p›
shows ‹(M, w ⊨ lower p) ⟷ (M, w $\models_!$ p)›
⟨proof⟩

lemma *lift-semantics*: ‹(M, w ⊨ p) ⟷ (M, w $\models_!$ lift p)›
⟨proof⟩

lemma *lower-lift*: ‹lower (lift p) = p›
⟨proof⟩

lemma *lift-lower*: ‹static p ⟹ lift (lower p) = p›
⟨proof⟩

3 Soundness of Reduction

primrec *reduce'* :: ‹'i pfm ⇒ 'i pfm ⇒ 'i pfm› **where**

‹reduce′ r ⊥! = (r ⟶! ⊥!)›
| ‹reduce′ r (Pro! x) = (r ⟶! Pro! x)›
| ‹reduce′ r (p ∨! q) = (reduce′ r p ∨! reduce′ r q)›
| ‹reduce′ r (p ∧! q) = (reduce′ r p ∧! reduce′ r q)›
| ‹reduce′ r (p ⟶! q) = (reduce′ r p ⟶! reduce′ r q)›
| ‹reduce′ r (K! i p) = (r ⟶! K! i (reduce′ r p))›
| ‹reduce′ r ([p]! q) = undefined›

primrec *reduce* :: ‹′i pfm ⇒ ′i pfm› **where**
‹reduce ⊥! = ⊥!›
| ‹reduce (Pro! x) = Pro! x›
| ‹reduce (p ∨! q) = (reduce p ∨! reduce q)›
| ‹reduce (p ∧! q) = (reduce p ∧! reduce q)›
| ‹reduce (p ⟶! q) = (reduce p ⟶! reduce q)›
| ‹reduce (K! i p) = K! i (reduce p)›
| ‹reduce ([r]! p) = reduce′ (reduce r) (reduce p)›

lemma *static-reduce′*: ‹static p ⟹ static r ⟹ static (reduce′ r p)›
⟨proof⟩

lemma *static-reduce*: ‹static (reduce p)›
⟨proof⟩

lemma *reduce′-semantics*:
assumes ‹static q›
shows ‹(M, w ⊨! [p]! q) = (M, w ⊨! reduce′ p q)›
⟨proof⟩

lemma *reduce-semantics*: ‹M, w ⊨! p ⟷ M, w ⊨! reduce p›
⟨proof⟩

4 Chains of Implications

primrec *implyP* :: ‹′i pfm list ⇒ ′i pfm ⇒ ′i pfm› (**infixr** ‹⇝!› *56*) **where**
‹([] ⇝! q) = q›
| ‹(p # ps ⇝! q) = (p ⟶! ps ⇝! q)›

lemma *lift-implyP*: ‹lift (ps ⇝ q) = (map lift ps ⇝! lift q)›
⟨proof⟩

lemma *reduce-implyP*: ‹reduce (ps ⇝! q) = (map reduce ps ⇝! reduce q)›
⟨proof⟩

5 Proof System

primrec *peval* :: ‹(id ⇒ bool) ⇒ (′i pfm ⇒ bool) ⇒ ′i pfm ⇒ bool› **where**
‹peval - - ⊥! = False›
| ‹peval g - (Pro! x) = g x›

| ‹$peval\ g\ h\ (p \vee_! q) = (peval\ g\ h\ p \vee peval\ g\ h\ q)$›
| ‹$peval\ g\ h\ (p \wedge_! q) = (peval\ g\ h\ p \wedge peval\ g\ h\ q)$›
| ‹$peval\ g\ h\ (p \longrightarrow_! q) = (peval\ g\ h\ p \longrightarrow peval\ g\ h\ q)$›
| ‹$peval$ - $h\ (K_!\ i\ p) = h\ (K_!\ i\ p)$›
| ‹$peval$ - $h\ ([r]_!\ p) = h\ ([r]_!\ p)$›

abbreviation ‹$ptautology\ p \equiv \forall g\ h.\ peval\ g\ h\ p$›

inductive PAK :: ‹$('i\ pfm \Rightarrow bool) \Rightarrow ('i\ pfm \Rightarrow bool) \Rightarrow 'i\ pfm \Rightarrow bool$›
(‹-; - $\vdash_!$ -› [$50, 50, 50$] 50)
for $A\ B$ :: ‹$'i\ pfm \Rightarrow bool$› **where**
$PA1$: ‹$ptautology\ p \Longrightarrow A;\ B \vdash_!\ p$›
| $PA2$: ‹$A;\ B \vdash_!\ K_!\ i\ p \wedge_! K_!\ i\ (p \longrightarrow_! q) \longrightarrow_! K_!\ i\ q$›
| PAx: ‹$A\ p \Longrightarrow A;\ B \vdash_!\ p$›
| $PR1$: ‹$A;\ B \vdash_!\ p \Longrightarrow A;\ B \vdash_!\ p \longrightarrow_! q \Longrightarrow A;\ B \vdash_!\ q$›
| $PR2$: ‹$A;\ B \vdash_!\ p \Longrightarrow A;\ B \vdash_!\ K_!\ i\ p$›
| $PAnn$: ‹$A;\ B \vdash_!\ p \Longrightarrow B\ r \Longrightarrow A;\ B \vdash_!\ [r]_!\ p$›
| PFF: ‹$A;\ B \vdash_!\ [r]_!\ \bot_! \longleftrightarrow_! (r \longrightarrow_! \bot_!)$›
| $PPro$: ‹$A;\ B \vdash_!\ [r]_!\ Pro_!\ x \longleftrightarrow_! (r \longrightarrow_! Pro_!\ x)$›
| $PDis$: ‹$A;\ B \vdash_!\ [r]_!\ (p \vee_! q) \longleftrightarrow_! [r]_!\ p \vee_! [r]_!\ q$›
| $PCon$: ‹$A;\ B \vdash_!\ [r]_!\ (p \wedge_! q) \longleftrightarrow_! [r]_!\ p \wedge_! [r]_!\ q$›
| $PImp$: ‹$A;\ B \vdash_!\ [r]_!\ (p \longrightarrow_! q) \longleftrightarrow_! ([r]_!\ p \longrightarrow_! [r]_!\ q)$›
| PK: ‹$A;\ B \vdash_!\ [r]_!\ K_!\ i\ p \longleftrightarrow_! (r \longrightarrow_! K_!\ i\ ([r]_!\ p))$›

abbreviation PAK-$assms$ (‹-; -; - $\vdash_!$ -› [$50, 50, 50, 50$] 50) **where**
‹$A;\ B;\ G \vdash_!\ p \equiv \exists qs.\ set\ qs \subseteq G \wedge (A;\ B \vdash_!\ qs \leadsto_! p)$›

lemma *eval-peval*: ‹$eval\ h\ (g\ o\ lift)\ p = peval\ h\ g\ (lift\ p)$›
⟨*proof*⟩

lemma *tautology-ptautology*: ‹$tautology\ p \Longrightarrow ptautology\ (lift\ p)$›
⟨*proof*⟩

theorem *AK-PAK*: ‹$A\ o\ lift \vdash p \Longrightarrow A;\ B \vdash_!\ lift\ p$›
⟨*proof*⟩

abbreviation *validP*
:: ‹$(('i, 'i\ fm\ set)\ kripke \Rightarrow bool) \Rightarrow 'i\ pfm\ set \Rightarrow 'i\ pfm \Rightarrow bool$›
(‹-; - $\models_!$ -› [$50, 50, 50$] 50)
where ‹$P;\ G \models_!\ p \equiv P;\ G \models_{!\star}\ p$›

lemma *set-map-inv*:
assumes ‹$set\ xs \subseteq f\ `\ X$›
shows ‹$\exists ys.\ set\ ys \subseteq X \wedge map\ f\ ys = xs$›
⟨*proof*⟩

lemma *strong-static-completeness'*:
assumes ‹$static\ p$› **and** ‹$\forall q \in G.\ static\ q$› **and** ‹$P;\ G \models_!\ p$›
and ‹$P;\ lower\ `\ G \models\star\ lower\ p \Longrightarrow A\ o\ lift;\ lower\ `\ G \vdash lower\ p$›

shows ‹A; B; $G \vdash_! p$›
⟨*proof*⟩

theorem *strong-static-completeness*:
assumes ‹*static* p› **and** ‹$\forall\, q \in G$. *static* q› **and** ‹P; $G \Vdash_! p$›
and ‹$\bigwedge G\ p$. P; $G \Vdash p \Longrightarrow A$ *o lift*; $G \vdash p$›
shows ‹A; B; $G \vdash_! p$›
⟨*proof*⟩

corollary *static-completeness'*:
assumes ‹*static* p› **and** ‹P; $\{\}$ $\Vdash_{!\star} p$›
and ‹P; $\{\}$ $\Vdash$ *lower* $p \Longrightarrow A$ *o lift* $\vdash$ *lower* p›
shows ‹A; $B \vdash_! p$›
⟨*proof*⟩

corollary *static-completeness*:
assumes ‹*static* p› **and** ‹P; $\{\}$ $\Vdash_{!\star} p$› **and** ‹$\bigwedge p$. P; $\{\}$ $\Vdash p \Longrightarrow A$ *o lift* $\vdash p$›
shows ‹A; $B \vdash_! p$›
⟨*proof*⟩

corollary
assumes ‹*static* p› ‹(λ-. *True*); $\{\}$ $\Vdash_! p$›
shows ‹A; $B \vdash_! p$›
⟨*proof*⟩

6 Soundness

lemma *peval-semantics*:
‹*peval* (*val* w) (λq. (|$\mathcal{W} = W$, $\mathcal{K} = r$, $\pi =$ *val*|), $w \models_! q$) p = ((|$\mathcal{W} = W$, $\mathcal{K} = r$, $\pi =$ *val*|), $w \models_! p$)›
⟨*proof*⟩

lemma *ptautology*:
assumes ‹*ptautology* p›
shows ‹M, $w \models_! p$›
⟨*proof*⟩

theorem *soundness$_P$*:
assumes
‹$\bigwedge M\ p\ w$. $A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_! p$›
‹$\bigwedge M\ r$. $P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
shows ‹A; $B \vdash_! p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_! p$›
⟨*proof*⟩

corollary ‹(λ-. *False*); $B \vdash_! p \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_! p$›
⟨*proof*⟩

7 Strong Soundness

lemma *ptautology-imply-superset*:
  assumes ‹$set\ ps \subseteq set\ qs$›
  shows ‹$ptautology\ (ps \leadsto_! r \longrightarrow_! qs \leadsto_! r)$›
⟨*proof*⟩

lemma *PK-imply-weaken*:
  assumes ‹$A;\ B \vdash_! ps \leadsto_! q$› ‹$set\ ps \subseteq set\ ps'$›
  shows ‹$A;\ B \vdash_! ps' \leadsto_! q$›
⟨*proof*⟩

lemma *implyP-append*: ‹$(ps\ @\ ps' \leadsto_! q) = (ps \leadsto_! ps' \leadsto_! q)$›
  ⟨*proof*⟩

lemma *PK-ImpI*:
  assumes ‹$A;\ B \vdash_! p\ \#\ G \leadsto_! q$›
  shows ‹$A;\ B \vdash_! G \leadsto_! (p \longrightarrow_! q)$›
⟨*proof*⟩

corollary *soundness-imply$_P$*:
  assumes
    ‹$\bigwedge M\ p\ w.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p$›
    ‹$\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
  shows ‹$A;\ B \vdash_! qs \leadsto_! p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow \forall q \in set\ qs.\ M,\ w \models_! q$
$\Longrightarrow M,\ w \models_! p$›
⟨*proof*⟩

theorem *strong-soundness$_P$*:
  assumes
    ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models_! p$›
    ‹$\bigwedge M\ r.\ P\ M \Longrightarrow B\ r \Longrightarrow P\ (M[r!])$›
  shows ‹$A;\ B;\ G \vdash_! p \Longrightarrow P;\ G \models_{!\star} p$›
⟨*proof*⟩

8 Completeness

lemma *ConE*:
  assumes ‹$A;\ B \vdash_! p \wedge_! q$›
  shows ‹$A;\ B \vdash_! p$› ‹$A;\ B \vdash_! q$›
  ⟨*proof*⟩

lemma *Iff-Dis*:
  assumes ‹$A;\ B \vdash_! p \longleftrightarrow_! p'$› ‹$A;\ B \vdash_! q \longleftrightarrow_! q'$›
  shows ‹$A;\ B \vdash_! ((p \vee_! q) \longleftrightarrow_! (p' \vee_! q'))$›
⟨*proof*⟩

lemma *Iff-Con*:
  assumes ‹$A;\ B \vdash_! p \longleftrightarrow_! p'$› ‹$A;\ B \vdash_! q \longleftrightarrow_! q'$›

shows ‹A; $B \vdash_! (p \wedge_! q) \longleftrightarrow_! (p' \wedge_! q')$›
⟨*proof*⟩

lemma *Iff-Imp*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› ‹$A$; $B \vdash_! q \longleftrightarrow_! q'$›
shows ‹A; $B \vdash_! ((p \longrightarrow_! q) \longleftrightarrow_! (p' \longrightarrow_! q'))$›
⟨*proof*⟩

lemma *Iff-sym*: ‹(A; $B \vdash_! p \longleftrightarrow_! q$) = ($A$; $B \vdash_! q \longleftrightarrow_! p$)›
⟨*proof*⟩

lemma *Iff-Iff*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› ‹$A$; $B \vdash_! p \longleftrightarrow_! q$›
shows ‹A; $B \vdash_! p' \longleftrightarrow_! q$›
⟨*proof*⟩

lemma *K'-A2'*: ‹A; $B \vdash_! K_!\ i\ (p \longrightarrow_! q) \longrightarrow_! K_!\ i\ p \longrightarrow_! K_!\ i\ q$›
⟨*proof*⟩

lemma *K'-map*:
assumes ‹A; $B \vdash_! p \longrightarrow_! q$›
shows ‹A; $B \vdash_! K_!\ i\ p \longrightarrow_! K_!\ i\ q$›
⟨*proof*⟩

lemma *ConI*:
assumes ‹A; $B \vdash_! p$› ‹A; $B \vdash_! q$›
shows ‹A; $B \vdash_! p \wedge_! q$›
⟨*proof*⟩

lemma *Iff-wk*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! q$›
shows ‹A; $B \vdash_! (r \longrightarrow_! p) \longleftrightarrow_! (r \longrightarrow_! q)$›
⟨*proof*⟩

lemma *Iff-reduce'*:
assumes ‹*static* p›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_!$ *reduce'* r p›
⟨*proof*⟩

lemma *Iff-Ann1*:
assumes r: ‹A; $B \vdash_! r \longleftrightarrow_! r'$› **and** ‹*static* p›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_! [r']_!\ p$›
⟨*proof*⟩

lemma *Iff-Ann2*:
assumes ‹A; $B \vdash_! p \longleftrightarrow_! p'$› **and** ‹$B$ r›
shows ‹A; $B \vdash_! [r]_!\ p \longleftrightarrow_! [r]_!\ p'$›
⟨*proof*⟩

lemma *Iff-reduce*:
  assumes ‹$\forall r \in$ *anns* p. B r›
  shows ‹A; $B \vdash_! p \longleftrightarrow_!$ *reduce* p›
  ⟨*proof*⟩

lemma *anns-implyP* [*simp*]:
  ‹*anns* ($ps \leadsto_! q$) = *anns* $q \cup (\bigcup p \in$ *set* ps. *anns* p)›
  ⟨*proof*⟩

lemma *strong-completeness*$_P{}'$:
  assumes ‹P; $G \models\!\!\models_! p$›
    and ‹$\forall r \in$ *anns* p. B r› ‹$\forall q \in G$. $\forall r \in$ *anns* q. B r›
    and ‹P; *lower* ‘ *reduce* ‘ $G \models\!\!\models_\star$ *lower* (*reduce* p) $\Longrightarrow$
      A *o lift*; *lower* ‘ *reduce* ‘ $G \vdash$ *lower* (*reduce* p)›
  shows ‹A; B; $G \vdash_! p$›
⟨*proof*⟩

theorem *strong-completeness*$_P$:
  assumes ‹P; $G \models\!\!\models_! p$›
    and ‹$\forall r \in$ *anns* p. B r› ‹$\forall q \in G$. $\forall r \in$ *anns* q. B r›
    and ‹$\bigwedge G$ p. P; $G \models\!\!\models_\star p \Longrightarrow A$ *o lift*; $G \vdash p$›
  shows ‹A; B; $G \vdash_! p$›
  ⟨*proof*⟩

theorem *main*$_P$:
  assumes ‹$\bigwedge M$ w p. A $p \Longrightarrow P$ $M \Longrightarrow w \in \mathcal{W}$ $M \Longrightarrow M$, $w \models_! p$›
    and ‹$\bigwedge M$ r. P $M \Longrightarrow B$ $r \Longrightarrow P$ ($M[r!]$)›
    and ‹$\forall r \in$ *anns* p. B r› ‹$\forall q \in G$. $\forall r \in$ *anns* q. B r›
    and ‹$\bigwedge G$ p. P; $G \models\!\!\models_\star p \Longrightarrow A$ *o lift*; $G \vdash p$›
  shows ‹P; $G \models\!\!\models_! p \longleftrightarrow A$; B; $G \vdash_! p$›
  ⟨*proof*⟩

corollary *strong-completeness*$_{PB}$:
  assumes ‹P; $G \models\!\!\models_! p$›
    and ‹$\bigwedge G$ p. P; $G \models\!\!\models_\star p \Longrightarrow A$ *o lift*; $G \vdash p$›
  shows ‹A; (λ-. *True*); $G \vdash_! p$›
  ⟨*proof*⟩

corollary *completeness*$_P{}'$:
  assumes ‹P; {} $\models\!\!\models_! p$›
    and ‹$\forall r \in$ *anns* p. B r›
    and ‹$\bigwedge p$. P; {} $\models\!\!\models$ *lower* $p \Longrightarrow A$ *o lift* $\vdash$ *lower* p›
  shows ‹A; $B \vdash_! p$›
  ⟨*proof*⟩

corollary *completeness*$_P$:
  assumes ‹P; {} $\models\!\!\models_! p$›
    and ‹$\forall r \in$ *anns* p. B r›
    and ‹$\bigwedge p$. P; {} $\models\!\!\models p \Longrightarrow A$ *o lift* $\vdash p$›

shows ‹A; $B \vdash_! p$›
⟨*proof*⟩

corollary *completeness*$_{PA}$:
assumes ‹(λ-. *True*); {} $\models_! p$›
shows ‹A; (λ-. *True*) $\vdash_! p$›
⟨*proof*⟩

9 System PAL + K

abbreviation *SystemPK* (‹- $\vdash_{!K}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!K} p \equiv$ (λ-. *False*); (λ-. *True*); $G \vdash_! p$›

lemma *strong-soundness*$_{PK}$: ‹$G \vdash_{!K} p \Longrightarrow$ (λ-. *True*); $G \models_{!\star} p$›
⟨*proof*⟩

abbreviation *validPK* (‹- $\models_{!K}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!K} p \equiv$ (λ-. *True*); $G \models_! p$›

lemma *strong-completeness*$_{PK}$:
assumes ‹$G \models_{!K} p$›
shows ‹$G \vdash_{!K} p$›
⟨*proof*⟩

theorem *main*$_{PK}$: ‹$G \models_{!K} p \longleftrightarrow G \vdash_{!K} p$›
⟨*proof*⟩

corollary ‹$G \models_{!K} p \Longrightarrow$ (λ-. *True*); $G \models_{!\star} p$›
⟨*proof*⟩

10 System PAL + T

Also known as System PAL + M

inductive *AxPT* :: ‹*'i pfm* ⇒ *bool*› **where**
‹*AxPT* ($K_!$ i $p \longrightarrow_! p$)›

abbreviation *SystemPT* (‹- $\vdash_{!T}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!T} p \equiv$ *AxPT*; (λ-. *True*); $G \vdash_! p$›

lemma *soundness-AxPT*: ‹*AxPT* $p \Longrightarrow$ *reflexive* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_!$ p›
⟨*proof*⟩

lemma *reflexive-restrict*: ‹*reflexive* $M \Longrightarrow$ *reflexive* ($M[r!]$)›
⟨*proof*⟩

lemma *strong-soundness*$_{PT}$: ‹$G \vdash_{!T} p \Longrightarrow$ *reflexive*; $G \models_{!\star} p$›

⟨*proof*⟩

lemma *AxT-AxPT*: ‹*AxT* = *AxPT o lift*›
⟨*proof*⟩

abbreviation *validPT* (‹- $\models_{!T}$ -› [*50*, *50*] *50*) **where**
‹*G* $\models_{!T}$ *p* ≡ *reflexive*; *G* $\models_{!}$ *p*›

lemma *strong-completeness*$_{PT}$:
assumes ‹*G* $\models_{!T}$ *p*›
shows ‹*G* $\vdash_{!T}$ *p*›
⟨*proof*⟩

theorem *main*$_{PT}$: ‹*G* $\models_{!T}$ *p* ⟷ *G* $\vdash_{!T}$ *p*›
⟨*proof*⟩

corollary ‹*G* $\models_{!T}$ *p* ⟹ *reflexive*; *G* $\models_{!\star}$ *p*›
⟨*proof*⟩

11 System PAL + KB

inductive *AxPB* :: ‹*'i pfm* ⇒ *bool*› **where**
‹*AxPB* (*p* $\longrightarrow_{!}$ $K_{!}$ *i* ($L_{!}$ *i p*))›

abbreviation *SystemPKB* (‹- $\vdash_{!KB}$ -› [*50*, *50*] *50*) **where**
‹*G* $\vdash_{!KB}$ *p* ≡ *AxPB*; (λ-. *True*); *G* $\vdash_{!}$ *p*›

lemma *soundness-AxPB*: ‹*AxPB p* ⟹ *symmetric M* ⟹ *w* ∈ $\mathcal{W}$ *M* ⟹ *M*, *w* $\models_{!}$ *p*›
⟨*proof*⟩

lemma *symmetric-restrict*: ‹*symmetric M* ⟹ *symmetric* (*M*[*r*!])›
⟨*proof*⟩

lemma *strong-soundness*$_{PKB}$: ‹*G* $\vdash_{!KB}$ *p* ⟹ *symmetric*; *G* $\models_{!\star}$ *p*›
⟨*proof*⟩

lemma *AxB-AxPB*: ‹*AxB* = *AxPB o lift*›
⟨*proof*⟩

abbreviation *validPKB* (‹- $\models_{!KB}$ -› [*50*, *50*] *50*) **where**
‹*G* $\models_{!KB}$ *p* ≡ *symmetric*; *G* $\models_{!}$ *p*›

lemma *strong-completeness*$_{PKB}$:
assumes ‹*G* $\models_{!KB}$ *p*›
shows ‹*G* $\vdash_{!KB}$ *p*›
⟨*proof*⟩

theorem *main*$_{PKB}$: ‹*G* $\models_{!KB}$ *p* ⟷ *G* $\vdash_{!KB}$ *p*›

⟨*proof*⟩

corollary ‹$G \models_{!KB} p \Longrightarrow$ *symmetric*; $G \models_{!\star} p$›
⟨*proof*⟩

12 System PAL + K4

inductive *AxP4* :: ‹'*i pfm* ⇒ *bool*› **where**
‹*AxP4* ($K_!$ *i p* $\longrightarrow_!$ $K_!$ *i* ($K_!$ *i p*))›

abbreviation *SystemPK4* (‹- $\vdash_{!K4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!K4} p \equiv$ *AxP4*; (λ-. *True*); $G \vdash_! p$›

lemma *pos-introspection*:
assumes ‹*transitive M*› ‹$w \in \mathcal{W}$ *M*›
shows ‹*M*, $w \models_!$ ($K_!$ *i p* $\longrightarrow_!$ $K_!$ *i* ($K_!$ *i p*))›
⟨*proof*⟩

lemma *soundness-AxP4*: ‹*AxP4 p* $\Longrightarrow$ *transitive M* $\Longrightarrow$ $w \in \mathcal{W}$ *M* $\Longrightarrow$ *M*, $w \models_!$ *p*›
⟨*proof*⟩

lemma *transitive-restrict*: ‹*transitive M* $\Longrightarrow$ *transitive* (*M*[*r*!])›
⟨*proof*⟩

lemma *strong-soundness*$_{PK4}$: ‹$G \vdash_{!K4} p \Longrightarrow$ *transitive*; $G \models_{!\star} p$›
⟨*proof*⟩

lemma *Ax4-AxP4*: ‹*Ax4* = *AxP4 o lift*›
⟨*proof*⟩

abbreviation *validPK4* (‹- $\models_{!K4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{!K4} p \equiv$ *transitive*; $G \models_! p$›

lemma *strong-completeness*$_{PK4}$:
assumes ‹$G \models_{!K4} p$›
shows ‹$G \vdash_{!K4} p$›
⟨*proof*⟩

theorem *main*$_{PK4}$: ‹$G \models_{!K4} p \longleftrightarrow G \vdash_{!K4} p$›
⟨*proof*⟩

corollary ‹$G \models_{!K4} p \Longrightarrow$ *transitive*; $G \models_{!\star} p$›
⟨*proof*⟩

13 System PAL + K5

inductive *AxP5* :: ‹'*i pfm* ⇒ *bool*› **where**

‹AxP5 ($L_!$ i p ⟶$_!$ $K_!$ i ($L_!$ i p))›

abbreviation *SystemPK5* (‹- $\vdash_{!K5}$ -› [50, 50] 50) **where**
‹G $\vdash_{!K5}$ p ≡ AxP5; (λ-. True); G $\vdash_!$ p›

lemma *soundness-AxP5*: ‹AxP5 p ⟹ Euclidean M ⟹ w ∈ 𝒲 M ⟹ M, w $\models_!$ p›
⟨proof⟩

lemma *Euclidean-restrict*: ‹Euclidean M ⟹ Euclidean (M[r!])›
⟨proof⟩

lemma *strong-soundness$_{PK5}$*: ‹G $\vdash_{!K5}$ p ⟹ Euclidean; G $\Vdash_{!\star}$ p›
⟨proof⟩

lemma *Ax5-AxP5*: ‹Ax5 = AxP5 o lift›
⟨proof⟩

abbreviation *validPK5* (‹- $\Vdash_{!K5}$ -› [50, 50] 50) **where**
‹G $\Vdash_{!K5}$ p ≡ Euclidean; G $\Vdash_!$ p›

lemma *strong-completeness$_{PK5}$*:
assumes ‹G $\Vdash_{!K5}$ p›
shows ‹G $\vdash_{!K5}$ p›
⟨proof⟩

theorem *main$_{PK5}$*: ‹G $\Vdash_{!K5}$ p ⟷ G $\vdash_{!K5}$ p›
⟨proof⟩

corollary ‹G $\Vdash_{!K5}$ p ⟹ Euclidean; G $\Vdash_{!\star}$ p›
⟨proof⟩

14 System PAL + S4

abbreviation *SystemPS4* (‹- $\vdash_{!S4}$ -› [50, 50] 50) **where**
‹G $\vdash_{!S4}$ p ≡ AxPT ⊕ AxP4; (λ-. True); G $\vdash_!$ p›

lemma *soundness-AxPT4*: ‹(AxPT ⊕ AxP4) p ⟹ refltrans M ⟹ w ∈ 𝒲 M ⟹ M, w $\models_!$ p›
⟨proof⟩

lemma *refltrans-restrict*: ‹refltrans M ⟹ refltrans (M[r!])›
⟨proof⟩

lemma *strong-soundness$_{PS4}$*: ‹G $\vdash_{!S4}$ p ⟹ refltrans; G $\Vdash_{!\star}$ p›
⟨proof⟩

lemma *AxT4-AxPT4*: ‹(AxT ⊕ Ax4) = (AxPT ⊕ AxP4) o lift›
⟨proof⟩

abbreviation *validPS4* (‹- $\models_{!S4}$ -› [50, 50] 50) **where**
‹$G \models_{!S4} p \equiv$ *refltrans*; $G \models_{!} p$›

theorem *strong-completeness*$_{PS4}$:
assumes ‹$G \models_{!S4} p$›
shows ‹$G \vdash_{!S4} p$›
⟨*proof*⟩

theorem *main*$_{PS4}$: ‹$G \models_{!S4} p \longleftrightarrow G \vdash_{!S4} p$›
⟨*proof*⟩

corollary ‹$G \models_{!S4} p \Longrightarrow$ *refltrans*; $G \models_{!\star} p$›
⟨*proof*⟩

15 System PAL + S5

abbreviation *SystemPS5* (‹- $\vdash_{!S5}$ -› [50, 50] 50) **where**
‹$G \vdash_{!S5} p \equiv AxPT \oplus AxPB \oplus AxP4$; (λ-. *True*); $G \vdash_{!} p$›

abbreviation *AxPTB4* :: ‹'i pfm ⇒ bool› **where**
‹$AxPTB4 \equiv AxPT \oplus AxPB \oplus AxP4$›

lemma *soundness-AxPTB4*: ‹$AxPTB4\ p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_{!} p$›
⟨*proof*⟩

lemma *equivalence-restrict*: ‹*equivalence* $M \Longrightarrow$ *equivalence* $(M[r!])$›
⟨*proof*⟩

lemma *strong-soundness*$_{PS5}$: ‹$G \vdash_{!S5} p \Longrightarrow$ *equivalence*; $G \models_{!\star} p$›
⟨*proof*⟩

lemma *AxTB4-AxPTB4*: ‹$AxTB4 = AxPTB4\ o\ lift$›
⟨*proof*⟩

abbreviation *validPS5* (‹- $\models_{!S5}$ -› [50, 50] 50) **where**
‹$G \models_{!S5} p \equiv$ *equivalence*; $G \models_{!} p$›

theorem *strong-completeness*$_{PS5}$:
assumes ‹$G \models_{!S5} p$›
shows ‹$G \vdash_{!S5} p$›
⟨*proof*⟩

theorem *main*$_{PS5}$: ‹$G \models_{!S5} p \longleftrightarrow G \vdash_{!S5} p$›
⟨*proof*⟩

corollary ‹$G \models_{!S5} p \Longrightarrow$ *equivalence*; $G \models_{!\star} p$›
⟨*proof*⟩

16 System PAL + S5'

abbreviation *SystemPS5'* (‹- $\vdash_{!S5}{}''$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{!S5}{}' p \equiv AxPT \oplus AxP5$; ($\lambda$-. *True*); $G \vdash_{!} p$›

abbreviation *AxPT5* :: ‹$'i$ *pfm* ⇒ *bool*› **where**
‹$AxPT5 \equiv AxPT \oplus AxP5$›

lemma *soundness-AxPT5*: ‹$AxPT5\ p \Longrightarrow equivalence\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models_{!} p$›
⟨*proof*⟩

lemma *strong-soundness*$_{PS5}$*'*: ‹$G \vdash_{!S5}{}' p \Longrightarrow equivalence; G \models_{!\star} p$›
⟨*proof*⟩

lemma *AxT5-AxPT5*: ‹$AxT5 = AxPT5\ o\ lift$›
⟨*proof*⟩

theorem *strong-completeness*$_{PS5}$*'*:
assumes ‹$G \models_{!S5} p$›
shows ‹$G \vdash_{!S5}{}' p$›
⟨*proof*⟩

theorem *main*$_{PS5}$*'*: ‹$G \models_{!S5} p \longleftrightarrow G \vdash_{!S5}{}' p$›
⟨*proof*⟩

end

References

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