Ptolemy's Theorem

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Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

Contents

1 Ptolemy's Theorem

theory *Ptolemys-Theorem*
imports
  HOL−Analysis.Multivariate-Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory

hide-const (**open**) *normalize*

1.1.2 Additions to Transcendental theory

Lemmas about *arcsin* and *arccos* commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between $-\ 1$ and 1. As the argumentation for $-\ 1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term *cos* (*arccos* y) is already provided in the Isabelle distribution with name *cos-arccos-abs*. Here, we further provide the analogue on *arcsin* for rewriting *sin* (*arcsin* y).

lemma *sin-arcsin-abs*: $|y| \leq 1 \Longrightarrow sin\ (arcsin\ y) = y$
  ⟨*proof*⟩

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

lemma *arccos-lbound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow 0 \leq arccos\ y$
⟨*proof*⟩

lemma *arccos-ubound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow arccos\ y \leq pi$
⟨*proof*⟩

As we choose angles to be between 0 between $2 * pi$, we need some lemmas to reason about the sign of *sin* x for angles x.

lemma *sin-ge-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $0 \leq sin\ x \longleftrightarrow x \leq pi$
⟨*proof*⟩

lemma *sin-less-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $sin\ x < 0 \longleftrightarrow pi < x$
⟨*proof*⟩

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [*simp*]:
  assumes $i \neq j$
  shows *axis i x* \$ $j = 0$
⟨*proof*⟩

lemma *norm-axis*:
  fixes x :: *real*
  shows $norm\ (axis\ i\ x) = abs\ x$
⟨*proof*⟩

lemma *norm-eq-on-real-2-vec*:
  fixes x :: *real* ^ *2*
  shows *norm* x = *sqrt* ((x $ *1*) ^ *2* + (x $ *2*) ^ *2*)
⟨*proof*⟩

lemma *dist-eq-on-real-2-vec*:
  fixes a b :: *real* ^ *2*
  shows *dist* a b = *sqrt* ((a $ *1* − b $ *1*) ^ *2* + (a $ *2* − b $ *2*) ^ *2*)
⟨*proof*⟩

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* :: *real* ⇒ *real* ^ *2*
where
  of-radiant ω = *axis 1* (*cos* ω) + *axis 2* (*sin* ω)

definition *normalize* :: *real* ^ *2* ⇒ *real* ^ *2*
where
  normalize p = (*if* p = *0* *then axis 1 1 else* (*1* / *norm* p) $*_R$ p)

definition *radiant-of* :: *real* ^ *2* ⇒ *real*
where
  radiant-of p = (*THE* ω. *0* $\leq \omega \wedge \omega <$ *2* $*$ *pi* $\wedge$ *of-radiant* ω = *normalize* p)

The vector *of-radiant* ω is the vector with length *1* and angle ω to the first axis. We normalize vectors to length *1* keeping their orientation with the normalize function. Conversely, *radiant-of* p is the angle of vector p to the first axis, where we choose *radiant-of* to return angles between *0* and *2* $*$ *pi*, following the usual high-school convention. With these definitions, we can express the main result *norm* p $*_R$ *of-radiant* (*radiant-of* p) = p. Note that the main result holds for any definition of *radiant-of 0*. So, we choose to define *normalize 0* and *radiant-of 0*, such that *radiant-of 0* = *0*.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [*simp*]:
  of-radiant ω $ *1* = *cos* ω
⟨*proof*⟩

lemma *nth-of-radiant-2* [*simp*]:
  of-radiant ω $ *2* = *sin* ω
⟨*proof*⟩

lemma *norm-of-radiant*:
  norm (*of-radiant* ω) = *1*
⟨*proof*⟩

lemma *of-radiant-plus-2pi*:
  $of\text{-}radiant\ (\omega + 2 * pi) = of\text{-}radiant\ \omega$
⟨*proof*⟩

lemma *of-radiant-minus-2pi*:
  $of\text{-}radiant\ (\omega - 2 * pi) = of\text{-}radiant\ \omega$
⟨*proof*⟩

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:
  $norm\ p *_R normalize\ p = p$
⟨*proof*⟩

lemma *norm-normalize*:
  $norm\ (normalize\ p) = 1$
⟨*proof*⟩

lemma *nth-normalize* [*simp*]:
  $|normalize\ p\ \$\ i| \leq 1$
⟨*proof*⟩

lemma *normalize-square*:
  $(normalize\ p\ \$\ 1)^2 + (normalize\ p\ \$\ 2)^2 = 1$
⟨*proof*⟩

lemma *nth-normalize-ge-zero-iff*:
  $0 \leq normalize\ p\ \$\ i \longleftrightarrow 0 \leq p\ \$\ i$
⟨*proof*⟩

lemma *nth-normalize-less-zero-iff*:
  $normalize\ p\ \$\ i < 0 \longleftrightarrow p\ \$\ i < 0$
⟨*proof*⟩

lemma *normalize-boundary-iff*:
  $|normalize\ p\ \$\ 1| = 1 \longleftrightarrow p\ \$\ 2 = 0$
⟨*proof*⟩

lemma *between-normalize-if-distant-from-0*:
  assumes $norm\ p \geq 1$
  shows $between\ (0, p)\ (normalize\ p)$
⟨*proof*⟩

lemma *between-normalize-if-near-0*:
  assumes $norm\ p \leq 1$
  shows $between\ (0, normalize\ p)\ p$
⟨*proof*⟩

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:
$0 \le \textit{radiant-of } p \wedge \textit{radiant-of } p < 2 * \textit{pi} \wedge \textit{of-radiant } (\textit{radiant-of } p) = \textit{normalize } p$
⟨*proof*⟩

lemma *radiant-of-bounds* [*simp*]:
$0 \le \textit{radiant-of } p$ $\textit{radiant-of } p < 2 * \textit{pi}$
⟨*proof*⟩

lemma *radiant-of-weak-ubound* [*simp*]:
$\textit{radiant-of } p \le 2 * \textit{pi}$
⟨*proof*⟩

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:
$\textit{norm } p *_R \textit{of-radiant } (\textit{radiant-of } p) = p$
⟨*proof*⟩

lemma *relative-polar-form-eq*:
$Q + \textit{dist } P\ Q *_R \textit{of-radiant } (\textit{radiant-of } (P - Q)) = P$
⟨*proof*⟩

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:
assumes $0 \le \textit{radius}$ $0 \le \alpha$ $\alpha \le \beta$ $\beta \le 2 * \textit{pi}$
shows $\textit{dist } (\textit{center} + \textit{radius} *_R \textit{of-radiant } \alpha)\ (\textit{center} + \textit{radius} *_R \textit{of-radiant } \beta) = 2 * \textit{radius} * \textit{sin } ((\beta - \alpha) / 2)$
(**is** $?lhs = ?rhs$)
⟨*proof*⟩

theorem *ptolemy-trigonometric*:
fixes $\omega_1\ \omega_2\ \omega_3 :: \textit{real}$
shows $\textit{sin } (\omega_1 + \omega_2) * \textit{sin } (\omega_2 + \omega_3) = \textit{sin } \omega_1 * \textit{sin } \omega_3 + \textit{sin } \omega_2 * \textit{sin } (\omega_1 + \omega_2 + \omega_3)$
⟨*proof*⟩

theorem *ptolemy*:
fixes $A\ B\ C\ D\ \textit{center} :: \textit{real} \hat{\ } 2$
assumes $\textit{dist center } A = \textit{radius}$ **and** $\textit{dist center } B = \textit{radius}$
assumes $\textit{dist center } C = \textit{radius}$ **and** $\textit{dist center } D = \textit{radius}$
assumes *ordering-of-points*:
$\textit{radiant-of } (A - \textit{center}) \le \textit{radiant-of } (B - \textit{center})$
$\textit{radiant-of } (B - \textit{center}) \le \textit{radiant-of } (C - \textit{center})$
$\textit{radiant-of } (C - \textit{center}) \le \textit{radiant-of } (D - \textit{center})$
shows $\textit{dist } A\ C * \textit{dist } B\ D = \textit{dist } A\ B * \textit{dist } C\ D + \textit{dist } A\ D * \textit{dist } B\ C$
⟨*proof*⟩

end

References

[1] J. Harrison. Ptolemy's theorem. https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml.

[2] F. Wiedijk. Formalizing 100 theorems. http://www.cs.ru.nl/~freek/100/.

[3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].