Ptolemy's Theorem

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Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

Contents

1 Ptolemy's Theorem

theory *Ptolemys-Theorem*
imports
  HOL−Analysis.Multivariate-Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory

hide-const (**open**) *normalize*

1.1.2 Additions to Transcendental theory

Lemmas about *arcsin* and *arccos* commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between $-\ 1$ and 1. As the argumentation for $-\ 1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term *cos* (*arccos y*) is already provided in the Isabelle distribution with name *cos-arccos-abs*. Here, we further provide the analogue on *arcsin* for rewriting *sin* (*arcsin y*).

lemma *sin-arcsin-abs*: $|y| \leq 1 \Longrightarrow sin\ (arcsin\ y) = y$
  ⟨*proof*⟩

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

lemma *arccos-lbound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow 0 \leq arccos\ y$
⟨*proof*⟩

lemma *arccos-ubound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow arccos\ y \leq pi$
⟨*proof*⟩

As we choose angles to be between 0 between $2 * pi$, we need some lemmas to reason about the sign of *sin x* for angles x.

lemma *sin-ge-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $0 \leq sin\ x \longleftrightarrow x \leq pi$
⟨*proof*⟩

lemma *sin-less-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $sin\ x < 0 \longleftrightarrow pi < x$
⟨*proof*⟩

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [*simp*]:
  assumes $i \neq j$
  shows *axis i x* \$ $j = 0$
⟨*proof*⟩

lemma *norm-axis*:
  fixes x :: *real*
  shows $norm\ (axis\ i\ x) = abs\ x$
⟨*proof*⟩

lemma *norm-eq-on-real-2-vec*:
  fixes x :: *real* ^ *2*
  shows $norm\ x = sqrt\ ((x\ \$\ 1)\ \hat{}\ 2 + (x\ \$\ 2)\ \hat{}\ 2)$
⟨*proof*⟩

lemma *dist-eq-on-real-2-vec*:
  fixes $a\ b$ :: *real* ^ *2*
  shows $dist\ a\ b = sqrt\ ((a\ \$\ 1 - b\ \$\ 1)\ \hat{}\ 2 + (a\ \$\ 2 - b\ \$\ 2)\ \hat{}\ 2)$
⟨*proof*⟩

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* :: *real* ⇒ *real* ^ *2*
where
  $of\text{-}radiant\ \omega = axis\ 1\ (cos\ \omega) + axis\ 2\ (sin\ \omega)$

definition *normalize* :: *real* ^ *2* ⇒ *real* ^ *2*
where
  $normalize\ p = (if\ p = 0\ then\ axis\ 1\ 1\ else\ (1\ /\ norm\ p) *_R p)$

definition *radiant-of* :: *real* ^ *2* ⇒ *real*
where
  $radiant\text{-}of\ p = (THE\ \omega.\ 0 \le \omega \wedge \omega < 2 * pi \wedge of\text{-}radiant\ \omega = normalize\ p)$

The vector *of-radiant* ω is the vector with length *1* and angle ω to the first axis. We normalize vectors to length *1* keeping their orientation with the normalize function. Conversely, *radiant-of* p is the angle of vector p to the first axis, where we choose *radiant-of* to return angles between *0* and $2 * pi$, following the usual high-school convention. With these definitions, we can express the main result $norm\ p *_R of\text{-}radiant\ (radiant\text{-}of\ p) = p$. Note that the main result holds for any definition of *radiant-of 0*. So, we choose to define *normalize 0* and *radiant-of 0*, such that $radiant\text{-}of\ 0 = 0$.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [*simp*]:
  $of\text{-}radiant\ \omega\ \$\ 1 = cos\ \omega$
⟨*proof*⟩

lemma *nth-of-radiant-2* [*simp*]:
  $of\text{-}radiant\ \omega\ \$\ 2 = sin\ \omega$
⟨*proof*⟩

lemma *norm-of-radiant*:
  $norm\ (of\text{-}radiant\ \omega) = 1$
⟨*proof*⟩

lemma *of-radiant-plus-2pi*:
of-radiant $(\omega + 2 * pi) = $ *of-radiant* ω
⟨*proof*⟩

lemma *of-radiant-minus-2pi*:
of-radiant $(\omega - 2 * pi) = $ *of-radiant* ω
⟨*proof*⟩

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:
$norm\ p *_R normalize\ p = p$
⟨*proof*⟩

lemma *norm-normalize*:
$norm\ (normalize\ p) = 1$
⟨*proof*⟩

lemma *nth-normalize* [*simp*]:
$|normalize\ p\ \$\ i| \leq 1$
⟨*proof*⟩

lemma *normalize-square*:
$(normalize\ p\ \$\ 1)^2 + (normalize\ p\ \$\ 2)^2 = 1$
⟨*proof*⟩

lemma *nth-normalize-ge-zero-iff*:
$0 \leq normalize\ p\ \$\ i \longleftrightarrow 0 \leq p\ \$\ i$
⟨*proof*⟩

lemma *nth-normalize-less-zero-iff*:
$normalize\ p\ \$\ i < 0 \longleftrightarrow p\ \$\ i < 0$
⟨*proof*⟩

lemma *normalize-boundary-iff*:
$|normalize\ p\ \$\ 1| = 1 \longleftrightarrow p\ \$\ 2 = 0$
⟨*proof*⟩

lemma *between-normalize-if-distant-from-0*:
assumes $norm\ p \geq 1$
shows $between\ (0,\ p)\ (normalize\ p)$
⟨*proof*⟩

lemma *between-normalize-if-near-0*:
assumes $norm\ p \leq 1$
shows $between\ (0,\ normalize\ p)\ p$
⟨*proof*⟩

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:
$0 \leq radiant\text{-}of\ p \wedge radiant\text{-}of\ p < 2 * pi \wedge of\text{-}radiant\ (radiant\text{-}of\ p) = normalize\ p$
⟨*proof*⟩

lemma *radiant-of-bounds* [*simp*]:
$0 \leq radiant\text{-}of\ p\ \ radiant\text{-}of\ p < 2 * pi$
⟨*proof*⟩

lemma *radiant-of-weak-ubound* [*simp*]:
$radiant\text{-}of\ p \leq 2 * pi$
⟨*proof*⟩

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:
$norm\ p *_R of\text{-}radiant\ (radiant\text{-}of\ p) = p$
⟨*proof*⟩

lemma *relative-polar-form-eq*:
$Q + dist\ P\ Q *_R of\text{-}radiant\ (radiant\text{-}of\ (P - Q)) = P$
⟨*proof*⟩

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:
assumes $0 \leq radius\ \ 0 \leq \alpha\ \ \alpha \leq \beta\ \ \beta \leq 2 * pi$
shows $dist\ (center + radius *_R of\text{-}radiant\ \alpha)\ (center + radius *_R of\text{-}radiant\ \beta) = 2 * radius * sin\ ((\beta - \alpha) / 2)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

theorem *ptolemy-trigonometric*:
fixes $\omega_1\ \omega_2\ \omega_3$ :: *real*
shows $sin\ (\omega_1 + \omega_2) * sin\ (\omega_2 + \omega_3) = sin\ \omega_1 * sin\ \omega_3 + sin\ \omega_2 * sin\ (\omega_1 + \omega_2 + \omega_3)$
⟨*proof*⟩

theorem *ptolemy*:
fixes $A\ B\ C\ D\ center$ :: *real* ^ *2*
assumes $dist\ center\ A = radius$ **and** $dist\ center\ B = radius$
assumes $dist\ center\ C = radius$ **and** $dist\ center\ D = radius$
assumes *ordering-of-points*:
$radiant\text{-}of\ (A - center) \leq radiant\text{-}of\ (B - center)$
$radiant\text{-}of\ (B - center) \leq radiant\text{-}of\ (C - center)$
$radiant\text{-}of\ (C - center) \leq radiant\text{-}of\ (D - center)$
shows $dist\ A\ C * dist\ B\ D = dist\ A\ B * dist\ C\ D + dist\ A\ D * dist\ B\ C$
⟨*proof*⟩

end

References

[1] J. Harrison. Ptolemy's theorem. https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml.

[2] F. Wiedijk. Formalizing 100 theorems. http://www.cs.ru.nl/~freek/100/.

[3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].