Ptolemy's Theorem

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December 14, 2021

Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

Contents

1 Ptolemy's Theorem

theory *Ptolemys-Theorem*
imports
HOL−Analysis.Multivariate-Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory

hide-const (**open**) *normalize*

1.1.2 Additions to Transcendental theory

Lemmas about *arcsin* and *arccos* commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between $-\ 1$ and 1. As the argumentation for $-\ 1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term $cos\ (arccos\ y)$ is already provided in the Isabelle distribution with name *cos-arccos-abs*. Here, we further provide the analogue on *arcsin* for rewriting $sin\ (arcsin\ y)$.

lemma *sin-arcsin-abs*: $|y| \leq 1 \Longrightarrow sin\ (arcsin\ y) = y$
  ⟨*proof*⟩

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

lemma *arccos-lbound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow 0 \leq arccos\ y$
⟨*proof*⟩

lemma *arccos-ubound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow arccos\ y \leq pi$
⟨*proof*⟩

As we choose angles to be between 0 between $2 * pi$, we need some lemmas to reason about the sign of $sin\ x$ for angles x.

lemma *sin-ge-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $0 \leq sin\ x \longleftrightarrow x \leq pi$
⟨*proof*⟩

lemma *sin-less-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $sin\ x < 0 \longleftrightarrow pi < x$
⟨*proof*⟩

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [*simp*]:
  assumes $i \neq j$
  shows $axis\ i\ x\ \$\ j = 0$
⟨*proof*⟩

lemma *norm-axis*:
  fixes $x :: real$
  shows $norm\ (axis\ i\ x) = abs\ x$
⟨*proof*⟩

lemma *norm-eq-on-real-2-vec*:
  fixes x :: *real* ^ *2*
  shows *norm* $x = sqrt\ ((x\ \$\ 1)\ \hat{}\ 2 + (x\ \$\ 2)\ \hat{}\ 2)$
⟨*proof*⟩

lemma *dist-eq-on-real-2-vec*:
  fixes $a\ b$ :: *real* ^ *2*
  shows *dist* $a\ b = sqrt\ ((a\ \$\ 1 - b\ \$\ 1)\ \hat{}\ 2 + (a\ \$\ 2 - b\ \$\ 2)\ \hat{}\ 2)$
⟨*proof*⟩

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* :: *real* ⇒ *real* ^ *2*
where
  of-radiant ω = *axis 1* (*cos* ω) + *axis 2* (*sin* ω)

definition *normalize* :: *real* ^ *2* ⇒ *real* ^ *2*
where
  normalize p = (*if* $p = 0$ *then axis 1 1 else* (*1* / *norm* p) $*_R$ p)

definition *radiant-of* :: *real* ^ *2* ⇒ *real*
where
  radiant-of p = (*THE* ω. $0 \leq \omega \wedge \omega < 2 * pi \wedge$ *of-radiant* ω = *normalize* p)

The vector *of-radiant* ω is the vector with length *1* and angle ω to the first axis. We normalize vectors to length *1* keeping their orientation with the normalize function. Conversely, *radiant-of* p is the angle of vector p to the first axis, where we choose *radiant-of* to return angles between *0* and $2 * pi$, following the usual high-school convention. With these definitions, we can express the main result *norm* $p *_R$ *of-radiant* (*radiant-of* p) = p. Note that the main result holds for any definition of *radiant-of 0*. So, we choose to define *normalize 0* and *radiant-of 0*, such that *radiant-of 0* = *0*.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [*simp*]:
  of-radiant ω \$ *1* = *cos* ω
⟨*proof*⟩

lemma *nth-of-radiant-2* [*simp*]:
  of-radiant ω \$ *2* = *sin* ω
⟨*proof*⟩

lemma *norm-of-radiant*:
  norm (*of-radiant* ω) = *1*
⟨*proof*⟩

lemma *of-radiant-plus-2pi*:
$\mathit{of\text{-}radiant}\ (\omega + 2 * pi) = \mathit{of\text{-}radiant}\ \omega$
⟨*proof*⟩

lemma *of-radiant-minus-2pi*:
$\mathit{of\text{-}radiant}\ (\omega - 2 * pi) = \mathit{of\text{-}radiant}\ \omega$
⟨*proof*⟩

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:
$norm\ p *_R normalize\ p = p$
⟨*proof*⟩

lemma *norm-normalize*:
$norm\ (normalize\ p) = 1$
⟨*proof*⟩

lemma *nth-normalize* [*simp*]:
$|normalize\ p\ \$\ i| \leq 1$
⟨*proof*⟩

lemma *normalize-square*:
$(normalize\ p\ \$\ 1)^2 + (normalize\ p\ \$\ 2)^2 = 1$
⟨*proof*⟩

lemma *nth-normalize-ge-zero-iff*:
$0 \leq normalize\ p\ \$\ i \longleftrightarrow 0 \leq p\ \$\ i$
⟨*proof*⟩

lemma *nth-normalize-less-zero-iff*:
$normalize\ p\ \$\ i < 0 \longleftrightarrow p\ \$\ i < 0$
⟨*proof*⟩

lemma *normalize-boundary-iff*:
$|normalize\ p\ \$\ 1| = 1 \longleftrightarrow p\ \$\ 2 = 0$
⟨*proof*⟩

lemma *between-normalize-if-distant-from-0*:
assumes $norm\ p \geq 1$
shows $between\ (0, p)\ (normalize\ p)$
⟨*proof*⟩

lemma *between-normalize-if-near-0*:
assumes $norm\ p \leq 1$
shows $between\ (0, normalize\ p)\ p$
⟨*proof*⟩

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:
$0 \leq$ *radiant-of* $p \wedge$ *radiant-of* $p < 2 * pi \wedge$ *of-radiant* (*radiant-of* p) = *normalize* p
⟨*proof*⟩

lemma *radiant-of-bounds* [*simp*]:
$0 \leq$ *radiant-of* p *radiant-of* $p < 2 * pi$
⟨*proof*⟩

lemma *radiant-of-weak-ubound* [*simp*]:
radiant-of $p \leq 2 * pi$
⟨*proof*⟩

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:
norm $p *_R$ *of-radiant* (*radiant-of* p) = p
⟨*proof*⟩

lemma *relative-polar-form-eq*:
Q + *dist* P Q $*_R$ *of-radiant* (*radiant-of* ($P - Q$)) = P
⟨*proof*⟩

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:
assumes $0 \leq$ *radius* $0 \leq \alpha$ $\alpha \leq \beta$ $\beta \leq 2 * pi$
shows *dist* (*center* + *radius* $*_R$ *of-radiant* α) (*center* + *radius* $*_R$ *of-radiant* β) = $2 *$ *radius* $*$ *sin* (($\beta - \alpha$) / 2)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

theorem *ptolemy-trigonometric*:
fixes ω_1 ω_2 ω_3 :: *real*
shows *sin* ($\omega_1 + \omega_2$) $*$ *sin* ($\omega_2 + \omega_3$) = *sin* ω_1 $*$ *sin* ω_3 + *sin* ω_2 $*$ *sin* ($\omega_1 + \omega_2 + \omega_3$)
⟨*proof*⟩

theorem *ptolemy*:
fixes A B C D *center* :: *real* ^ *2*
assumes *dist center* A = *radius* **and** *dist center* B = *radius*
assumes *dist center* C = *radius* **and** *dist center* D = *radius*
assumes *ordering-of-points*:
radiant-of (A − *center*) ≤ *radiant-of* (B − *center*)
radiant-of (B − *center*) ≤ *radiant-of* (C − *center*)
radiant-of (C − *center*) ≤ *radiant-of* (D − *center*)
shows *dist* A C $*$ *dist* B D = *dist* A B $*$ *dist* C D + *dist* A D $*$ *dist* B C
⟨*proof*⟩

end

References

[1] J. Harrison. Ptolemy's theorem. https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml.

[2] F. Wiedijk. Formalizing 100 theorems. http://www.cs.ru.nl/~freek/100/.

[3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].