Ptolemy's Theorem

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Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

Contents

1 Ptolemy's Theorem

theory *Ptolemys-Theorem*
imports
  HOL−Analysis.Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory

hide-const (**open**) *normalize*

1.1.2 Additions to Transcendental theory

Lemmas about *arcsin* and *arccos* commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between -1 and 1. As the argumentation for $-1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term *cos* (*arccos* y) is already provided in the Isabelle distribution with name *cos-arccos-abs*. Here, we further provide the analogue on *arcsin* for rewriting *sin* (*arcsin* y).

lemma *sin-arcsin-abs*: $|y| \leq 1 \Longrightarrow sin\ (arcsin\ y) = y$
  by (*simp add*: *abs-le-iff*)

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

lemma *arccos-lbound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow 0 \leq arccos\ y$
by (*simp add*: *arccos-lbound*)

lemma *arccos-ubound-abs* [*simp*]:
  $|y| \leq 1 \Longrightarrow arccos\ y \leq pi$
by (*simp add*: *arccos-ubound*)

As we choose angles to be between *0* between $2 * pi$, we need some lemmas to reason about the sign of *sin* x for angles x.

lemma *sin-ge-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $0 \leq sin\ x \longleftrightarrow x \leq pi$
proof
  assume $0 \leq sin\ x$
  show $x \leq pi$
  proof (*rule ccontr*)
    assume $\neg\ x \leq pi$
    from *this* ‹$x < 2 * pi$› **have** $sin\ x < 0$
      using *sin-lt-zero* **by** *auto*
    from *this* ‹$0 \leq sin\ x$› **show** *False* **by** *auto*
  qed
next
  assume $x \leq pi$
  from *this* ‹$0 \leq x$› **show** $0 \leq sin\ x$ **by** (*simp add*: *sin-ge-zero*)
qed

lemma *sin-less-zero-iff*:
  assumes $0 \leq x\ x < 2 * pi$
  shows $sin\ x < 0 \longleftrightarrow pi < x$
using *assms sin-ge-zero-iff* **by** *fastforce*

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [*simp*]:
  assumes $i \neq j$
  shows *axis i x* \$ *j* = *0*
using *assms* **unfolding** *axis-def* **by** *simp*

lemma *norm-axis*:
  fixes *x* :: *real*
  shows *norm* (*axis i x*) = *abs x*
by (*simp add*: *norm-eq-sqrt-inner inner-axis-axis*)

lemma *norm-eq-on-real-2-vec*:
  fixes *x* :: *real* ˆ *2*
  shows *norm x* = *sqrt* ((*x* \$ *1*) ˆ *2* + (*x* \$ *2*) ˆ *2*)
by (*simp add*: *norm-eq-sqrt-inner inner-vec-def UNIV-2 power2-eq-square*)

lemma *dist-eq-on-real-2-vec*:
  fixes *a b* :: *real* ˆ *2*
  shows *dist a b* = *sqrt* ((*a* \$ *1* − *b* \$ *1*) ˆ *2* + (*a* \$ *2* − *b* \$ *2*) ˆ *2*)
unfolding *dist-norm norm-eq-on-real-2-vec* **by** *simp*

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* :: *real* ⇒ *real* ˆ *2*
where
  of-radiant ω = *axis 1* (*cos* ω) + *axis 2* (*sin* ω)

definition *normalize* :: *real* ˆ *2* ⇒ *real* ˆ *2*
where
  normalize p = (*if p* = *0 then axis 1 1 else* (*1* / *norm p*) $*_R$ *p*)

definition *radiant-of* :: *real* ˆ *2* ⇒ *real*
where
  radiant-of p = (*THE* ω. *0* ≤ ω ∧ ω < *2* ∗ *pi* ∧ *of-radiant* ω = *normalize p*)

The vector *of-radiant* ω is the vector with length *1* and angle ω to the first axis. We normalize vectors to length *1* keeping their orientation with the normalize function. Conversely, *radiant-of p* is the angle of vector *p* to the first axis, where we choose *radiant-of* to return angles between *0* and *2* ∗ *pi*, following the usual high-school convention. With these definitions, we can express the main result *norm p* $*_R$ *of-radiant* (*radiant-of p*) = *p*. Note that the main result holds for any definition of *radiant-of 0*. So, we choose to define *normalize 0* and *radiant-of 0*, such that *radiant-of 0* = *0*.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [*simp*]:
of-radiant ω \$ *1* = *cos* ω
unfolding *of-radiant-def* **by** *simp*

lemma *nth-of-radiant-2* [*simp*]:
of-radiant ω \$ *2* = *sin* ω
unfolding *of-radiant-def* **by** *simp*

lemma *norm-of-radiant*:
norm (*of-radiant* ω) = *1*
unfolding *of-radiant-def norm-eq-on-real-2-vec* **by** *simp*

lemma *of-radiant-plus-2pi*:
of-radiant (ω + *2* * *pi*) = *of-radiant* ω
unfolding *of-radiant-def* **by** *simp*

lemma *of-radiant-minus-2pi*:
of-radiant (ω − *2* * *pi*) = *of-radiant* ω
proof −
have *of-radiant* (ω − *2* * *pi*) = *of-radiant* (ω − *2* * *pi* + *2* * *pi*)
by (*simp only*: *of-radiant-plus-2pi*)
also have ... = *of-radiant* ω **by** *simp*
finally show *?thesis* **.**
qed

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:
norm p $*_R$ *normalize p* = *p*
unfolding *normalize-def* **by** *simp*

lemma *norm-normalize*:
norm (*normalize p*) = *1*
unfolding *normalize-def* **by** (*auto simp add*: *norm-axis*)

lemma *nth-normalize* [*simp*]:
|*normalize p* \$ *i*| ≤ *1*
using *norm-normalize component-le-norm-cart* **by** *metis*

lemma *normalize-square*:
(*normalize p* \$ *1*)^2 + (*normalize p* \$ *2*)^2 = *1*
using *dot-square-norm*[*of normalize p*]
by (*simp add*: *inner-vec-def UNIV-2 power2-eq-square norm-normalize*)

lemma *nth-normalize-ge-zero-iff*:
0 ≤ *normalize p* \$ *i* ⟷ *0* ≤ *p* \$ *i*
proof
assume *0* ≤ *normalize p* \$ *i*

from *this* **show** $0 \leq p \ \$ \ i$
unfolding *normalize-def* **by** (*auto split*: *if-split-asm simp add*: *zero-le-divide-iff*)
next
assume $0 \leq p \ \$ \ i$
have $0 \leq axis\ 1\ (1 :: real)\ \$ \ i$
using *exhaust-2*[*of i*] **by** *auto*
from *this* ‹$0 \leq p \ \$ \ i$› **show** $0 \leq normalize\ p \ \$ \ i$
unfolding *normalize-def* **by** *auto*
qed

lemma *nth-normalize-less-zero-iff*:
$normalize\ p \ \$ \ i < 0 \longleftrightarrow p \ \$ \ i < 0$
using *nth-normalize-ge-zero-iff leD leI* **by** *blast*

lemma *normalize-boundary-iff*:
$|normalize\ p \ \$ \ 1| = 1 \longleftrightarrow p \ \$ \ 2 = 0$
proof
assume $|normalize\ p \ \$ \ 1| = 1$
from *this* **have** *1*: $(p \ \$ \ 1) \ \hat{}\ 2 = norm\ p \ \hat{}\ 2$
unfolding *normalize-def* **by** (*auto split*: *if-split-asm simp add*: *power2-eq-iff*)
moreover have $(p \ \$ \ 1) \ \hat{}\ 2 + (p \ \$ \ 2) \ \hat{}\ 2 = norm\ p \ \hat{}\ 2$
using *norm-eq-on-real-2-vec* **by** *auto*
ultimately show $p \ \$ \ 2 = 0$ **by** *simp*
next
assume $p \ \$ \ 2 = 0$
from *this* **have** $|p \ \$ \ 1| = norm\ p$
by (*auto simp add*: *norm-eq-on-real-2-vec*)
from *this* **show** $|normalize\ p \ \$ \ 1| = 1$
unfolding *normalize-def* **by** *simp*
qed

lemma *between-normalize-if-distant-from-0*:
assumes $norm\ p \geq 1$
shows $between\ (0,\ p)\ (normalize\ p)$
using *assms* **by** (*auto simp add*: *between-mem-segment closed-segment-def normalize-def*)

lemma *between-normalize-if-near-0*:
assumes $norm\ p \leq 1$
shows $between\ (0,\ normalize\ p)\ p$
proof −
have $0 \leq norm\ p$ **by** *simp*
from *assms* **have** $p = (norm\ p\ /\ norm\ p) *_R p \wedge 0 \leq norm\ p \wedge norm\ p \leq 1$
by *auto*
from *this* **have** $\exists u.\ p = (u\ /\ norm\ p) *_R p \wedge 0 \leq u \wedge u \leq 1$ **by** *blast*
from *this* **show** *?thesis*
by (*auto simp add*: *between-mem-segment closed-segment-def normalize-def*)
qed

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:
  $0 \le$ *radiant-of p* $\wedge$ *radiant-of p* $< 2 * pi \wedge$ *of-radiant* (*radiant-of p*) = *normalize p*
proof −
 let *?a = if 0 ≤ p $ 2 then arccos* (*normalize p $ 1*) *else pi + arccos* (− (*normalize p $ 1*))
  have *0 ≤ ?a ∧ ?a < 2 ∗ pi ∧ of-radiant ?a = normalize p*
  proof −
    have *0 ≤ ?a* **by** *auto*
    moreover have *?a < 2 ∗ pi*
    proof *cases*
      assume *0 ≤ p $ 2*
      from *this* **have** *?a ≤ pi* **by** *simp*
      from *this* **show** *?thesis*
        using *pi-gt-zero* **by** *linarith*
    next
      assume ¬ *0 ≤ p $ 2*
      have *arccos* (− *normalize p $ 1*) < *pi*
      proof −
        have |*normalize p $ 1*| ≠ *1*
          using ‹¬ *0 ≤ p $ 2*› **by** (*simp only*: *normalize-boundary-iff*)
        from *this* **have** *arccos* (− *normalize p $ 1*) ≠ *pi*
          unfolding *arccos-minus-1*[*symmetric*] **by** (*subst arccos-eq-iff*) *auto*
        moreover have *arccos* (− *normalize p $ 1*) ≤ *pi* **by** *simp*
        ultimately show *arccos* (− *normalize p $ 1*) < *pi* **by** *linarith*
      qed
      from *this* ‹¬ *0 ≤ p $ 2*› **show** *?thesis* **by** *simp*
    qed
    moreover have *of-radiant ?a = normalize p*
    proof −
      have *of-radiant ?a $ i = normalize p $ i* **for** *i*
      proof −
        have *of-radiant ?a $ 1 = normalize p $ 1*
          unfolding *of-radiant-def* **by** (*simp add*: *cos-arccos-abs*)
        moreover have *of-radiant ?a $ 2 = normalize p $ 2*
        proof *cases*
          assume *0 ≤ p $ 2*
          have *sin* (*arccos* (*normalize p $ 1*)) = *sqrt* (*1* − (*normalize p $ 1*) ^ *2*)
            by (*simp add*: *sin-arccos-abs*)
          also have ... = *normalize p $ 2*
          proof −
            have *1* − (*normalize p $ 1*)^2 = (*normalize p $ 2*)^2
              using *normalize-square*[*of p*] **by** *auto*
          from *this* ‹*0 ≤ p $ 2*› **show** *?thesis* **by** (*simp add*: *nth-normalize-ge-zero-iff*)
          qed
          finally show *?thesis*
            using ‹*0 ≤ p $ 2*› **unfolding** *of-radiant-def* **by** *auto*
        next

```
        assume ¬ 0 ≤ p $ 2
        have − sin (arccos (− normalize p $ 1)) = − sqrt (1 − (normalize p $ 1)^2)
          by (simp add: sin-arccos-abs)
        also have . . . = normalize p $ 2
        proof −
          have 1 − (normalize p $ 1)^2 = (normalize p $ 2)^2
            using normalize-square[of p] by auto
          from this ⟨¬ 0 ≤ p $ 2⟩ show ?thesis
            using nth-normalize-ge-zero-iff by fastforce
        qed
        finally show ?thesis
          using ⟨¬ 0 ≤ p $ 2⟩ unfolding of-radiant-def by auto
      qed
      ultimately show ?thesis by (metis exhaust-2[of i])
    qed
    from this show ?thesis by (simp add: vec-eq-iff)
  qed
  ultimately show ?thesis by blast
qed
moreover {
  fix ω
  assume 0 ≤ ω ∧ ω < 2 ∗ pi ∧ of-radiant ω = normalize p
  from this have 0 ≤ ω ω < 2 ∗ pi normalize p = of-radiant ω by auto
  from this have cos ω = normalize p $ 1 sin ω = normalize p $ 2 by auto
  have ω = ?a
  proof cases
    assume 0 ≤ p $ 2
    from this have ω ≤ pi
      using ⟨0 ≤ ω⟩ ⟨ω < 2 ∗ pi⟩ ⟨sin ω = normalize p $ 2⟩
      by (simp add: sin-ge-zero-iff[symmetric] nth-normalize-ge-zero-iff)
    from ⟨0 ≤ ω⟩ this have ω = arccos (cos ω) by (simp add: arccos-cos)
    from ⟨cos ω = normalize p $ 1⟩ this have ω = arccos (normalize p $ 1)
      by (simp add: arccos-eq-iff)
    from this show ω = ?a using ⟨0 ≤ p $ 2⟩ by auto
  next
    assume ¬ 0 ≤ p $ 2
    from this have ω > pi
      using ⟨0 ≤ ω⟩ ⟨ω < 2 ∗ pi⟩ ⟨sin ω = normalize p $ 2⟩
      by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)
    from this ⟨ω < 2 ∗ pi⟩ have ω − pi = arccos (cos (ω − pi))
      by (auto simp only: arccos-cos)
    from this ⟨cos ω = normalize p $ 1⟩ have ω − pi = arccos (− normalize p $ 1) by simp
    from this have ω = pi + arccos (− normalize p $ 1) by simp
    from this show ω = ?a using ⟨¬ 0 ≤ p $ 2⟩ by auto
  qed
}
ultimately show ?thesis
```

  unfolding *radiant-of-def* **by** (*rule theI*)
qed

lemma *radiant-of-bounds* [*simp*]:
  $0 \le$ *radiant-of p radiant-of p* $< 2 * pi$
using *radiant-of* **by** *auto*

lemma *radiant-of-weak-ubound* [*simp*]:
  radiant-of p $\le 2 * pi$
using *radiant-of-bounds*(*2*)[*of p*] **by** *linarith*

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:
  norm p $*_R$ *of-radiant* (*radiant-of p*) = *p*
using *radiant-of normalize-eq* **by** *simp*

lemma *relative-polar-form-eq*:
  Q + *dist P Q* $*_R$ *of-radiant* (*radiant-of* (*P* − *Q*)) = *P*
proof −
  have *norm* (*P* − *Q*) $*_R$ *of-radiant* (*radiant-of* (*P* − *Q*)) = *P* − *Q*
    unfolding *polar-form-eq* **..**
  moreover have *dist P Q* = *norm* (*P* − *Q*) **by** (*simp add*: *dist-norm*)
  ultimately show *?thesis* **by** (*metis add.commute diff-add-cancel*)
qed

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:
  assumes $0 \le$ *radius* $0 \le \alpha$ $\alpha \le \beta$ $\beta \le 2 * pi$
  shows *dist* (*center* + *radius* $*_R$ *of-radiant* α) (*center* + *radius* $*_R$ *of-radiant* β) $= 2 *$ *radius* $*$ *sin* $((\beta - \alpha) / 2)$
    (**is** *?lhs* = *?rhs*)
proof −
  have *trigonometry*: $(cos\ \alpha - cos\ \beta)^2 + (sin\ \alpha - sin\ \beta)^2 = (2 * sin\ ((\beta - \alpha) / 2))^2$
  proof −
    have *sin-diff-minus*: $sin\ ((\alpha - \beta) / 2) = - sin\ ((\beta - \alpha) / 2)$
      by (*simp only*: *sin-minus*[*symmetric*] *minus-divide-left minus-diff-eq*)
    have $(cos\ \alpha - cos\ \beta)^2 + (sin\ \alpha - sin\ \beta)^2 =$
      $(2 * sin\ ((\alpha + \beta) / 2) * sin\ ((\beta - \alpha) / 2))^2 + (2 * sin\ ((\alpha - \beta) / 2) * cos\ ((\alpha + \beta) / 2))^2$
      by (*simp only*: *cos-diff-cos sin-diff-sin*)
    also have $\ldots = (2 * sin\ ((\beta - \alpha) / 2))^2 * ((sin\ ((\alpha + \beta) / 2))^2 + (cos\ ((\alpha + \beta) / 2))^2)$
      unfolding *sin-diff-minus* **by** *algebra*
    also have $\ldots = (2 * sin\ ((\beta - \alpha) / 2))^2$ **by** *simp*
    finally show *?thesis* **.**
  qed
  from *assms* **have** $0 \le sin\ ((\beta - \alpha) / 2)$ **by** (*simp add*: *sin-ge-zero*)

have *?lhs* = *sqrt* (*radius*^{2} ∗ ((*cos* α − *cos* β)^2 + (*sin* α − *sin* β)^2))
  unfolding *dist-eq-on-real-2-vec* **by** *simp algebra*
 also have ... = *sqrt* (*radius*^{2} ∗ (*2* ∗ *sin* ((β − α) / *2*))^2) **by** (*simp add*: *trigonometry*)
also have ... = *?rhs*
  using ‹*0* ≤ *radius*› ‹*0* ≤ *sin* ((β − α) / *2*)› **by** (*simp add*: *real-sqrt-mult*)
finally show *?thesis* **.**
qed

theorem *ptolemy-trigonometric*:
fixes ω_1 ω_2 ω_3 :: *real*
shows *sin* (ω_1 + ω_2) ∗ *sin* (ω_2 + ω_3) = *sin* ω_1 ∗ *sin* ω_3 + *sin* ω_2 ∗ *sin* (ω_1 + ω_2 + ω_3)
proof −
have *sin* (ω_1 + ω_2) ∗ *sin* (ω_2 + ω_3) = ((*sin* ω_2)^2 + (*cos* ω_2)^2) ∗ *sin* ω_1 ∗ *sin* ω_3 + *sin* ω_2 ∗ *sin* (ω_1 + ω_2 + ω_3)
  by (*simp only*: *sin-add cos-add*) *algebra*
also have ... = *sin* ω_1 ∗ *sin* ω_3 + *sin* ω_2 ∗ *sin* (ω_1 + ω_2 + ω_3) **by** *simp*
finally show *?thesis* **.**
qed

theorem *ptolemy*:
fixes *A B C D center* :: *real* ^ *2*
assumes *dist center A* = *radius* **and** *dist center B* = *radius*
assumes *dist center C* = *radius* **and** *dist center D* = *radius*
assumes *ordering-of-points*:
  radiant-of (*A* − *center*) ≤ *radiant-of* (*B* − *center*)
  radiant-of (*B* − *center*) ≤ *radiant-of* (*C* − *center*)
  radiant-of (*C* − *center*) ≤ *radiant-of* (*D* − *center*)
shows *dist A C* ∗ *dist B D* = *dist A B* ∗ *dist C D* + *dist A D* ∗ *dist B C*
proof −
from ‹*dist center A* = *radius*› **have** *0* ≤ *radius* **by** *auto*
define α β γ δ
  where α = *radiant-of* (*A* − *center*) **and** β = *radiant-of* (*B* − *center*)
  and γ = *radiant-of* (*C* − *center*) **and** δ = *radiant-of* (*D* − *center*)
from *ordering-of-points* **have** *angle-basics*:
  α ≤ β β ≤ γ γ ≤ δ
  0 ≤ α α ≤ *2* ∗ *pi* *0* ≤ β β ≤ *2* ∗ *pi*
  0 ≤ γ γ ≤ *2* ∗ *pi* *0* ≤ δ δ ≤ *2* ∗ *pi*
  unfolding *α-def β-def γ-def δ-def* **by** *auto*
from *assms*(*1*−*4*) **have**
  A = *center* + *radius* $*_R$ *of-radiant* α *B* = *center* + *radius* $*_R$ *of-radiant* β
  C = *center* + *radius* $*_R$ *of-radiant* γ *D* = *center* + *radius* $*_R$ *of-radiant* δ
  unfolding *α-def β-def γ-def δ-def*
  using *relative-polar-form-eq dist-commute* **by** *metis+*

from *this* **have** *dist-eqs*:
  dist A C = *2* ∗ *radius* ∗ *sin* ((γ − α) / *2*)
  dist B D = *2* ∗ *radius* ∗ *sin* ((δ − β) / *2*)

$dist\ A\ B = 2 * radius * sin\ ((\beta - \alpha) / 2)$
$dist\ C\ D = 2 * radius * sin\ ((\delta - \gamma) / 2)$
$dist\ A\ D = 2 * radius * sin\ ((\delta - \alpha) / 2)$
$dist\ B\ C = 2 * radius * sin\ ((\gamma - \beta) / 2)$
using *angle-basics* ‹$radius \geq 0$› *dist-circle-segment* **by** (*auto*)

have $dist\ A\ C * dist\ B\ D = 4 * radius \ \hat{}\ 2 * sin\ ((\gamma - \alpha) / 2) * sin\ ((\delta - \beta) / 2)$
unfolding *dist-eqs* **by** (*simp add*: *power2-eq-square*)
also have $\ldots = 4 * radius \ \hat{}\ 2 * (sin\ ((\beta - \alpha) / 2) * sin\ ((\delta - \gamma) / 2) + sin\ ((\gamma - \beta) / 2) * sin\ ((\delta - \alpha) / 2))$
proof −
define ω_1 ω_2 ω_3 **where** $\omega_1 = (\beta - \alpha) / 2$ **and** $\omega_2 = (\gamma - \beta) / 2$ **and** $\omega_3 = (\delta - \gamma) / 2$
have $(\gamma - \alpha) / 2 = \omega_1 + \omega_2$ **and** $(\delta - \beta) / 2 = \omega_2 + \omega_3$ **and** $(\delta - \alpha) / 2 = \omega_1 + \omega_2 + \omega_3$
unfolding ω_1*-def* ω_2*-def* ω_3*-def* **by** (*auto simp add*: *field-simps*)
have $sin\ ((\gamma - \alpha) / 2) * sin\ ((\delta - \beta) / 2) = sin\ (\omega_1 + \omega_2) * sin\ (\omega_2 + \omega_3)$
using ‹$(\gamma - \alpha) / 2 = \omega_1 + \omega_2$› ‹$(\delta - \beta) / 2 = \omega_2 + \omega_3$› **by** (*simp only*:)
also have $\ldots = sin\ \omega_1 * sin\ \omega_3 + sin\ \omega_2 * sin\ (\omega_1 + \omega_2 + \omega_3)$
by (*rule ptolemy-trigonometric*)
also have $\ldots = (sin\ ((\beta - \alpha) / 2) * sin\ ((\delta - \gamma) / 2) + sin\ ((\gamma - \beta) / 2) * sin\ ((\delta - \alpha) / 2))$
using ω_1*-def* ω_2*-def* ω_3*-def* ‹$(\delta - \alpha) / 2 = \omega_1 + \omega_2 + \omega_3$› **by** (*simp only*:)
finally show *?thesis* **by** *simp*
qed
also have $\ldots = dist\ A\ B * dist\ C\ D + dist\ A\ D * dist\ B\ C$
unfolding *dist-eqs* **by** (*simp add*: *distrib-left power2-eq-square*)
finally show *?thesis* **.**
qed

end

References

[1] J. Harrison. Ptolemy's theorem. https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml.

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[3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].