

Ptolemy's Theorem

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Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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1 Ptolemy's Theorem

```
theory Ptolemy's-Theorem
imports
  HOL-Analysis.Multivariate-Analysis
begin
```

1.1 Preliminaries

1.1.1 Additions to Rat theory

```
hide-const (open) normalize
```

1.1.2 Additions to Transcendental theory

Lemmas about \arcsin and \arccos commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument y is between -1 and 1 . As the argumentation for $-1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term $\cos (\arccos y)$ is already provided in the Isabelle distribution with name $\cos\text{-arccos-abs}$. Here, we further provide the analogue on \arcsin for rewriting $\sin (\arcsin y)$.

lemma $\sin\text{-arcsin-abs}$: $|y| \leq 1 \implies \sin (\arcsin y) = y$
by ($\text{simp add: abs-le-iff}$)

The further lemmas are the required variants from existing lemmas $\arccos\text{-lbound}$ and $\arccos\text{-ubound}$.

lemma $\arccos\text{-lbound-abs}$ [simp]:
 $|y| \leq 1 \implies 0 \leq \arccos y$
by ($\text{simp add: arccos-lbound}$)

lemma $\arccos\text{-ubound-abs}$ [simp]:
 $|y| \leq 1 \implies \arccos y \leq \pi$
by ($\text{simp add: arccos-ubound}$)

As we choose angles to be between 0 between $2 * \pi$, we need some lemmas to reason about the sign of $\sin x$ for angles x .

lemma $\sin\text{-ge-zero-iff}$:
assumes $0 \leq x < 2 * \pi$
shows $0 \leq \sin x \iff x \leq \pi$
proof
assume $0 \leq \sin x$
show $x \leq \pi$
proof (rule ccontr)
assume $\neg x \leq \pi$
from this $\langle x < 2 * \pi \rangle$ **have** $\sin x < 0$
using sin-lt-zero **by auto**
from this $\langle 0 \leq \sin x \rangle$ **show** False **by auto**
qed
next
assume $x \leq \pi$
from this $\langle 0 \leq x \rangle$ **show** $0 \leq \sin x$ **by** ($\text{simp add: sin-ge-zero}$)
qed

lemma $\sin\text{-less-zero-iff}$:
assumes $0 \leq x < 2 * \pi$
shows $\sin x < 0 \iff \pi < x$
using $\text{assms sin-ge-zero-iff}$ **by fastforce**

1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma *axis-nth-eq-0* [*simp*]:
 assumes $i \neq j$
 shows $\text{axis } i \ x \ \$ \ j = 0$
using *assms unfolding axis-def* **by** *simp*

lemma *norm-axis*:
 fixes $x :: \text{real}$
 shows $\text{norm } (\text{axis } i \ x) = \text{abs } x$
by (*simp add: norm-eq-sqrt-inner inner-axis-axis*)

lemma *norm-eq-on-real-2-vec*:
 fixes $x :: \text{real}^2$
 shows $\text{norm } x = \text{sqrt } ((x \ \$ \ 1)^2 + (x \ \$ \ 2)^2)$
by (*simp add: norm-eq-sqrt-inner inner-vec-def UNIV-2 power2-eq-square*)

lemma *dist-eq-on-real-2-vec*:
 fixes $a \ b :: \text{real}^2$
 shows $\text{dist } a \ b = \text{sqrt } ((a \ \$ \ 1 - b \ \$ \ 1)^2 + (a \ \$ \ 2 - b \ \$ \ 2)^2)$
unfolding *dist-norm norm-eq-on-real-2-vec* **by** *simp*

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition *of-radiant* $:: \text{real}^2 \Rightarrow \text{real}^2$
where
 $\text{of-radiant } \omega = \text{axis } 1 \ (\cos \ \omega) + \text{axis } 2 \ (\sin \ \omega)$

definition *normalize* $:: \text{real}^2 \Rightarrow \text{real}^2$
where
 $\text{normalize } p = (\text{if } p = 0 \text{ then } \text{axis } 1 \ 1 \ \text{else } (1 / \text{norm } p) *_{\mathbb{R}} p)$

definition *radiant-of* $:: \text{real}^2 \Rightarrow \text{real}$
where
 $\text{radiant-of } p = (\text{THE } \omega. \ 0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{of-radiant } \omega = \text{normalize } p)$

The vector *of-radiant* ω is the vector with length 1 and angle ω to the first axis. We normalize vectors to length 1 keeping their orientation with the *normalize* function. Conversely, *radiant-of* p is the angle of vector p to the first axis, where we choose *radiant-of* to return angles between 0 and $2 * \pi$, following the usual high-school convention. With these definitions, we can express the main result $\text{norm } p *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } p) = p$. Note that the main result holds for any definition of *radiant-of* 0. So, we choose to define *normalize* 0 and *radiant-of* 0, such that *radiant-of* 0 = 0.

1.2.2 Lemmas on *of-radiant*

lemma *nth-of-radiant-1* [*simp*]:

of-radiant ω \$ 1 = $\cos \omega$

unfolding *of-radiant-def* **by** *simp*

lemma *nth-of-radiant-2* [*simp*]:

of-radiant ω \$ 2 = $\sin \omega$

unfolding *of-radiant-def* **by** *simp*

lemma *norm-of-radiant*:

norm (*of-radiant* ω) = 1

unfolding *of-radiant-def* *norm-eq-on-real-2-vec* **by** *simp*

lemma *of-radiant-plus-2pi*:

of-radiant ($\omega + 2 * \pi$) = *of-radiant* ω

unfolding *of-radiant-def* **by** *simp*

lemma *of-radiant-minus-2pi*:

of-radiant ($\omega - 2 * \pi$) = *of-radiant* ω

proof –

have *of-radiant* ($\omega - 2 * \pi$) = *of-radiant* ($\omega - 2 * \pi + 2 * \pi$)

by (*simp only*: *of-radiant-plus-2pi*)

also have ... = *of-radiant* ω **by** *simp*

finally show *?thesis* .

qed

1.2.3 Lemmas on *normalize*

lemma *normalize-eq*:

norm $p *_R$ *normalize* p = p

unfolding *normalize-def* **by** *simp*

lemma *norm-normalize*:

norm (*normalize* p) = 1

unfolding *normalize-def* **by** (*auto simp add*: *norm-axis*)

lemma *nth-normalize* [*simp*]:

$|$ *normalize* p \$ i $|$ ≤ 1

using *norm-normalize* *component-le-norm-cart* **by** *metis*

lemma *normalize-square*:

$($ *normalize* p \$ 1 $)^2 + ($ *normalize* p \$ 2 $)^2 = 1$

using *dot-square-norm*[*of normalize p*]

by (*simp add*: *inner-vec-def UNIV-2 power2-eq-square norm-normalize*)

lemma *nth-normalize-ge-zero-iff*:

$0 \leq$ *normalize* p \$ i $\iff 0 \leq p$ \$ i

proof

assume $0 \leq$ *normalize* p \$ i

from this show $0 \leq p \ \$ \ i$
unfolding *normalize-def* **by** (*auto split: if-split-asm simp add: zero-le-divide-iff*)
next
assume $0 \leq p \ \$ \ i$
have $0 \leq \text{axis } 1 \ (1 :: \text{real}) \ \$ \ i$
using *exhaust-2[of i]* **by** *auto*
from this $\langle 0 \leq p \ \$ \ i \rangle$ **show** $0 \leq \text{normalize } p \ \$ \ i$
unfolding *normalize-def* **by** *auto*
qed

lemma *nth-normalize-less-zero-iff*:
 $\text{normalize } p \ \$ \ i < 0 \longleftrightarrow p \ \$ \ i < 0$
using *nth-normalize-ge-zero-iff leD leI* **by** *metis*

lemma *normalize-boundary-iff*:
 $|\text{normalize } p \ \$ \ 1| = 1 \longleftrightarrow p \ \$ \ 2 = 0$
proof
assume $|\text{normalize } p \ \$ \ 1| = 1$
from this have $1: (p \ \$ \ 1) \ ^2 = \text{norm } p \ ^2$
unfolding *normalize-def* **by** (*auto split: if-split-asm simp add: power2-eq-iff*)
moreover have $(p \ \$ \ 1) \ ^2 + (p \ \$ \ 2) \ ^2 = \text{norm } p \ ^2$
using *norm-eq-on-real-2-vec* **by** *auto*
ultimately show $p \ \$ \ 2 = 0$ **by** *simp*
next
assume $p \ \$ \ 2 = 0$
from this have $|p \ \$ \ 1| = \text{norm } p$
by (*auto simp add: norm-eq-on-real-2-vec*)
from this show $|\text{normalize } p \ \$ \ 1| = 1$
unfolding *normalize-def* **by** *simp*
qed

lemma *between-normalize-if-distant-from-0*:
assumes $\text{norm } p \geq 1$
shows *between* $(0, p)$ $(\text{normalize } p)$
using *assms* **by** (*auto simp add: between-mem-segment closed-segment-def normalize-def*)

lemma *between-normalize-if-near-0*:
assumes $\text{norm } p \leq 1$
shows *between* $(0, \text{normalize } p)$ p
proof –
have $0 \leq \text{norm } p$ **by** *simp*
from *assms* **have** $p = (\text{norm } p / \text{norm } p) *_{\mathbb{R}} p \wedge 0 \leq \text{norm } p \wedge \text{norm } p \leq 1$
by *auto*
from this have $\exists u. p = (u / \text{norm } p) *_{\mathbb{R}} p \wedge 0 \leq u \wedge u \leq 1$ **by** *blast*
from this show *?thesis*
by (*auto simp add: between-mem-segment closed-segment-def normalize-def*)
qed

1.2.4 Lemmas on *radiant-of*

lemma *radiant-of*:

$0 \leq \text{radiant-of } p \wedge \text{radiant-of } p < 2 * \pi \wedge \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p$

proof –

let $?a = \text{if } 0 \leq p \text{ \$ } 2 \text{ then } \arccos (\text{normalize } p \text{ \$ } 1) \text{ else } \pi + \arccos (- (\text{normalize } p \text{ \$ } 1))$

have $0 \leq ?a \wedge ?a < 2 * \pi \wedge \text{of-radiant } ?a = \text{normalize } p$

proof –

have $0 \leq ?a$ **by** *auto*

moreover have $?a < 2 * \pi$

proof cases

assume $0 \leq p \text{ \$ } 2$

from this have $?a \leq \pi$ **by** *simp*

from this show *?thesis*

using *pi-gt-zero* **by** *linarith*

next

assume $\neg 0 \leq p \text{ \$ } 2$

have $\arccos (- \text{normalize } p \text{ \$ } 1) < \pi$

proof –

have $|\text{normalize } p \text{ \$ } 1| \neq 1$

using $\langle \neg 0 \leq p \text{ \$ } 2 \rangle$ **by** (*simp only: normalize-boundary-iff*)

from this have $\arccos (- \text{normalize } p \text{ \$ } 1) \neq \pi$

unfolding *arccos-minus-1[symmetric]* **by** (*subst arccos-eq-iff*) *auto*

moreover have $\arccos (- \text{normalize } p \text{ \$ } 1) \leq \pi$ **by** *simp*

ultimately show $\arccos (- \text{normalize } p \text{ \$ } 1) < \pi$ **by** *linarith*

qed

from this $\langle \neg 0 \leq p \text{ \$ } 2 \rangle$ **show** *?thesis* **by** *simp*

qed

moreover have *of-radiant* $?a = \text{normalize } p$

proof –

have *of-radiant* $?a \text{ \$ } i = \text{normalize } p \text{ \$ } i$ **for** i

proof –

have *of-radiant* $?a \text{ \$ } 1 = \text{normalize } p \text{ \$ } 1$

unfolding *of-radiant-def* **by** (*simp add: cos-arccos-abs*)

moreover have *of-radiant* $?a \text{ \$ } 2 = \text{normalize } p \text{ \$ } 2$

proof cases

assume $0 \leq p \text{ \$ } 2$

have $\sin (\arccos (\text{normalize } p \text{ \$ } 1)) = \text{sqrt } (1 - (\text{normalize } p \text{ \$ } 1) ^ 2)$

by (*simp add: sin-arccos-abs*)

also have $\dots = \text{normalize } p \text{ \$ } 2$

proof –

have $1 - (\text{normalize } p \text{ \$ } 1) ^ 2 = (\text{normalize } p \text{ \$ } 2) ^ 2$

using *normalize-square[of p]* **by** *auto*

from this $\langle 0 \leq p \text{ \$ } 2 \rangle$ **show** *?thesis* **by** (*simp add: nth-normalize-ge-zero-iff*)

qed

finally show *?thesis*

using $\langle 0 \leq p \text{ \$ } 2 \rangle$ **unfolding** *of-radiant-def* **by** *auto*

next

```

    assume  $\neg 0 \leq p \ \$ 2$ 
    have  $-\sin(\arccos(-\text{normalize } p \ \$ 1)) = -\sqrt{1 - (\text{normalize } p \ \$ 1)^2}$ 
    by (simp add: sin-arccos-abs)
    also have  $\dots = \text{normalize } p \ \$ 2$ 
    proof -
      have  $1 - (\text{normalize } p \ \$ 1)^2 = (\text{normalize } p \ \$ 2)^2$ 
      using normalize-square[of p] by auto
      from this  $\langle \neg 0 \leq p \ \$ 2 \rangle$  show ?thesis
      using nth-normalize-ge-zero-iff by fastforce
    qed
    finally show ?thesis
      using  $\langle \neg 0 \leq p \ \$ 2 \rangle$  unfolding of-radiant-def by auto
    qed
    ultimately show ?thesis by (metis exhaust-2[of i])
  qed
  from this show ?thesis by (simp add: vec-eq-iff)
  qed
  ultimately show ?thesis by blast
  qed
  moreover {
    fix  $\omega$ 
    assume  $0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{of-radiant } \omega = \text{normalize } p$ 
    from this have  $0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{normalize } p = \text{of-radiant } \omega$  by auto
    from this have  $\cos \omega = \text{normalize } p \ \$ 1 \wedge \sin \omega = \text{normalize } p \ \$ 2$  by auto
    have  $\omega = ?a$ 
    proof cases
      assume  $0 \leq p \ \$ 2$ 
      from this have  $\omega \leq \pi$ 
      using  $\langle 0 \leq \omega \rangle \langle \omega < 2 * \pi \rangle \langle \sin \omega = \text{normalize } p \ \$ 2 \rangle$ 
      by (simp add: sin-ge-zero-iff[symmetric] nth-normalize-ge-zero-iff)
      from  $\langle 0 \leq \omega \rangle$  this have  $\omega = \arccos(\cos \omega)$  by (simp add: arccos-cos)
      from  $\langle \cos \omega = \text{normalize } p \ \$ 1 \rangle$  this have  $\omega = \arccos(\text{normalize } p \ \$ 1)$ 
      by (simp add: arccos-eq-iff)
      from this show  $\omega = ?a$  using  $\langle 0 \leq p \ \$ 2 \rangle$  by auto
    next
      assume  $\neg 0 \leq p \ \$ 2$ 
      from this have  $\omega > \pi$ 
      using  $\langle 0 \leq \omega \rangle \langle \omega < 2 * \pi \rangle \langle \sin \omega = \text{normalize } p \ \$ 2 \rangle$ 
      by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)
      from this  $\langle \omega < 2 * \pi \rangle$  have  $\omega - \pi = \arccos(\cos(\omega - \pi))$ 
      by (auto simp only: arccos-cos)
      from this  $\langle \cos \omega = \text{normalize } p \ \$ 1 \rangle$  have  $\omega - \pi = \arccos(-\text{normalize } p \ \$ 1)$ 
      by simp
      from this have  $\omega = \pi + \arccos(-\text{normalize } p \ \$ 1)$  by simp
      from this show  $\omega = ?a$  using  $\langle \neg 0 \leq p \ \$ 2 \rangle$  by auto
    qed
  }
  ultimately show ?thesis

```

unfolding *radiant-of-def* **by** (rule *theI*)
qed

lemma *radiant-of-bounds* [*simp*]:
 $0 \leq \text{radiant-of } p \text{ radiant-of } p < 2 * \pi$
using *radiant-of* **by** *auto*

lemma *radiant-of-weak-ubound* [*simp*]:
 $\text{radiant-of } p \leq 2 * \pi$
using *radiant-of-bounds(2)[of p]* **by** *linarith*

1.2.5 Main Equations for Transforming to Polar Form

lemma *polar-form-eq*:
 $\text{norm } p *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } p) = p$
using *radiant-of normalize-eq* **by** *simp*

lemma *relative-polar-form-eq*:
 $Q + \text{dist } P \ Q *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } (P - Q)) = P$
proof –
have $\text{norm } (P - Q) *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } (P - Q)) = P - Q$
unfolding *polar-form-eq* ..
moreover **have** $\text{dist } P \ Q = \text{norm } (P - Q)$ **by** (*simp add: dist-norm*)
ultimately show *?thesis* **by** (*metis add commute diff-add-cancel*)
qed

1.3 Ptolemy's Theorem

lemma *dist-circle-segment*:
assumes $0 \leq \text{radius}$ $0 \leq \alpha$ $\alpha \leq \beta$ $\beta \leq 2 * \pi$
shows $\text{dist } (\text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \alpha) (\text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \beta)$
 $= 2 * \text{radius} * \sin ((\beta - \alpha) / 2)$
(is *?lhs = ?rhs*)
proof –
have *trigonometry*: $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = (2 * \sin ((\beta - \alpha) / 2))^2$
proof –
have *sin-diff-minus*: $\sin ((\alpha - \beta) / 2) = - \sin ((\beta - \alpha) / 2)$
by (*simp only: sin-minus[symmetric] minus-divide-left minus-diff-eq*)
have $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 =$
 $(2 * \sin ((\alpha + \beta) / 2) * \sin ((\beta - \alpha) / 2))^2 + (2 * \sin ((\alpha - \beta) / 2) * \cos$
 $((\alpha + \beta) / 2))^2$
by (*simp only: cos-diff-cos sin-diff-sin*)
also have $\dots = (2 * \sin ((\beta - \alpha) / 2))^2 * ((\sin ((\alpha + \beta) / 2))^2 + (\cos ((\alpha$
 $+ \beta) / 2))^2)$
unfolding *sin-diff-minus* **by** *algebra*
also have $\dots = (2 * \sin ((\beta - \alpha) / 2))^2$ **by** *simp*
finally show *?thesis* .
qed
from *assms* **have** $0 \leq \sin ((\beta - \alpha) / 2)$ **by** (*simp add: sin-ge-zero*)

have $?lhs = \text{sqrt} (\text{radius}^2 * ((\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2))$
unfolding *dist-eq-on-real-2-vec* **by** *simp algebra*
also have $\dots = \text{sqrt} (\text{radius}^2 * (2 * \sin ((\beta - \alpha) / 2))^2)$ **by** (*simp add: trigonometry*)
also have $\dots = ?rhs$
using $\langle 0 \leq \text{radius} \rangle \langle 0 \leq \sin ((\beta - \alpha) / 2) \rangle$ **by** (*simp add: real-sqrt-mult*)
finally show *?thesis* .
qed

theorem *ptolemy-trigonometric:*

fixes $\omega_1 \omega_2 \omega_3 :: \text{real}$
shows $\sin (\omega_1 + \omega_2) * \sin (\omega_2 + \omega_3) = \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin (\omega_1 + \omega_3)$
proof –
have $\sin (\omega_1 + \omega_2) * \sin (\omega_2 + \omega_3) = ((\sin \omega_2)^2 + (\cos \omega_2)^2) * \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin (\omega_1 + \omega_2 + \omega_3)$
by (*simp only: sin-add cos-add*) *algebra*
also have $\dots = \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin (\omega_1 + \omega_2 + \omega_3)$ **by** *simp*
finally show *?thesis* .
qed

theorem *ptolemy:*

fixes $A B C D \text{ center} :: \text{real} \wedge 2$
assumes $\text{dist center } A = \text{radius}$ **and** $\text{dist center } B = \text{radius}$
assumes $\text{dist center } C = \text{radius}$ **and** $\text{dist center } D = \text{radius}$
assumes *ordering-of-points:*
 $\text{radiant-of } (A - \text{center}) \leq \text{radiant-of } (B - \text{center})$
 $\text{radiant-of } (B - \text{center}) \leq \text{radiant-of } (C - \text{center})$
 $\text{radiant-of } (C - \text{center}) \leq \text{radiant-of } (D - \text{center})$
shows $\text{dist } A C * \text{dist } B D = \text{dist } A B * \text{dist } C D + \text{dist } A D * \text{dist } B C$
proof –
from $\langle \text{dist center } A = \text{radius} \rangle$ **have** $0 \leq \text{radius}$ **by** *auto*
define $\alpha \beta \gamma \delta$
where $\alpha = \text{radiant-of } (A - \text{center})$ **and** $\beta = \text{radiant-of } (B - \text{center})$
and $\gamma = \text{radiant-of } (C - \text{center})$ **and** $\delta = \text{radiant-of } (D - \text{center})$
from *ordering-of-points* **have** *angle-basics:*
 $\alpha \leq \beta \beta \leq \gamma \gamma \leq \delta$
 $0 \leq \alpha \alpha \leq 2 * \text{pi} \ 0 \leq \beta \beta \leq 2 * \text{pi}$
 $0 \leq \gamma \gamma \leq 2 * \text{pi} \ 0 \leq \delta \delta \leq 2 * \text{pi}$
unfolding $\alpha\text{-def } \beta\text{-def } \gamma\text{-def } \delta\text{-def}$ **by** *auto*
from *assms(1-4)* **have**
 $A = \text{center} + \text{radius} *_{\text{R}} \text{of-radiant } \alpha \ B = \text{center} + \text{radius} *_{\text{R}} \text{of-radiant } \beta$
 $C = \text{center} + \text{radius} *_{\text{R}} \text{of-radiant } \gamma \ D = \text{center} + \text{radius} *_{\text{R}} \text{of-radiant } \delta$
unfolding $\alpha\text{-def } \beta\text{-def } \gamma\text{-def } \delta\text{-def}$
using *relative-polar-form-eq dist-commute* **by** *metis+*

from this have *dist-egs:*

$$\text{dist } A C = 2 * \text{radius} * \sin ((\gamma - \alpha) / 2)$$

$$\text{dist } B D = 2 * \text{radius} * \sin ((\delta - \beta) / 2)$$

```

dist A B = 2 * radius * sin ((β - α) / 2)
dist C D = 2 * radius * sin ((δ - γ) / 2)
dist A D = 2 * radius * sin ((δ - α) / 2)
dist B C = 2 * radius * sin ((γ - β) / 2)
using angle-basics ⟨radius ≥ 0⟩ dist-circle-segment by (auto)

have dist A C * dist B D = 4 * radius ^ 2 * sin ((γ - α) / 2) * sin ((δ - β) / 2)
unfolding dist-eqs by (simp add: power2-eq-square)
also have ... = 4 * radius ^ 2 * (sin ((β - α) / 2) * sin ((δ - γ) / 2) + sin ((γ - β) / 2) * sin ((δ - α) / 2))
proof -
  define ω1 ω2 ω3 where ω1 = (β - α) / 2 and ω2 = (γ - β) / 2 and ω3 = (δ - γ) / 2
  have (γ - α) / 2 = ω1 + ω2 and (δ - β) / 2 = ω2 + ω3 and (δ - α) / 2 = ω1 + ω2 + ω3
  unfolding ω1-def ω2-def ω3-def by (auto simp add: field-simps)
  have sin ((γ - α) / 2) * sin ((δ - β) / 2) = sin (ω1 + ω2) * sin (ω2 + ω3)
  using ⟨(γ - α) / 2 = ω1 + ω2⟩ ⟨(δ - β) / 2 = ω2 + ω3⟩ by (simp only:)
  also have ... = sin ω1 * sin ω3 + sin ω2 * sin (ω1 + ω2 + ω3)
  by (rule ptolemy-trigonometric)
  also have ... = (sin ((β - α) / 2) * sin ((δ - γ) / 2) + sin ((γ - β) / 2) * sin ((δ - α) / 2))
  using ω1-def ω2-def ω3-def ⟨(δ - α) / 2 = ω1 + ω2 + ω3⟩ by (simp only:)
  finally show ?thesis by simp
qed
also have ... = dist A B * dist C D + dist A D * dist B C
unfolding dist-eqs by (simp add: distrib-left power2-eq-square)
finally show ?thesis .
qed

end

```

References

- [1] J. Harrison. Ptolemy's theorem. <https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml>.
- [2] F. Wiedijk. Formalizing 100 theorems. <http://www.cs.ru.nl/~freek/100/>.
- [3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817 [Online; accessed 6-August-2016].