Abstract

This entry provides an analytic proof to Ptolemy’s Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison’s HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy’s Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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1.3 Ptolemy’s Theorem
1.1.2 Additions to Transcendental theory

Lemmas about \( \arcsin \) and \( \arccos \) commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument \( y \) is between \(-1\) and \(1\). As the argumentation for \(-1 \leq y \) and \(y \leq 1\) is often very similar, we prefer to prove \(|y| \leq 1\) to the two goals above.

The lemma for rewriting the term \( \cos (\arccos y) \) is already provided in the Isabelle distribution with name \(\text{cos-arccos-abs}\). Here, we further provide the analogue on \( \arcsin \) for rewriting \( \sin (\arcsin y) \).

\textbf{lemma sin-arcsin-abs}: \[|y| \leq 1 \implies \sin (\arcsin y) = y\]

by (\text{simp add: abs-le-iff})

The further lemmas are the required variants from existing lemmas \(\text{arccos-lbound}\) and \(\text{arccos-ubound}\).

\textbf{lemma arccos-lbound-abs}: \[|y| \leq 1 \implies 0 \leq \arccos y\]

by (\text{simp add: arccos-lbound})

\textbf{lemma arccos-ubound-abs}: \[|y| \leq 1 \implies \arccos y \leq \pi\]

by (\text{simp add: arccos-ubound})

As we choose angles to be between 0 and \(2 \pi\), we need some lemmas to reason about the sign of \(\sin x\) for angles \(x\).

\textbf{lemma sin-ge-zero-iff}: \[\text{assumes } 0 \leq x \text{ x } < 2 \ast \pi \text{ shows } 0 \leq \sin x \longleftrightarrow x \leq \pi\]

proof
  assume \(0 \leq \sin x\)
  show \(x \leq \pi\)
  proof (rule ccontr)
    assume \(\neg x \leq \pi\)
    from this \((x < 2 \ast \pi)\) have \(\sin x < 0\)
    using \(\sin-lt-zero\) by auto
    from this \((0 \leq \sin x)\) show \(False\) by auto
  qed
next
  assume \(x \leq \pi\)
  from this \((0 \leq x)\) show \(0 \leq \sin x\) by (\text{simp add: sin-ge-zero})
qed

\textbf{lemma sin-less-zero-iff}: \[\text{assumes } 0 \leq x \text{ x } < 2 \ast \pi \text{ shows } \sin x < 0 \longleftrightarrow \pi < x\]

using \(\text{assms sin-ge-zero-iff}\) by \text{fastforce}
1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

lemma axis-nth-eq-0 [simp]:
assumes i ≠ j
shows axis i x $ j = 0
using assms unfolding axis-def by simp

lemma norm-axis:
fixes x :: real
shows norm (axis i x) = abs x
by (simp add: norm-eq-sqrt-inner inner-axis-axis)

lemma norm-eq-on-real-2-vec:
fixes x :: real ^ 2
shows norm x = sqrt (x $ 1 ^ 2 + x $ 2 ^ 2)
by (simp add: norm-eq-sqrt-inner inner-vec-def UNIV-2 power2-eq-square)

lemma dist-eq-on-real-2-vec:
fixes a b :: real ^ 2
shows dist a b = sqrt ((a $ 1 - b $ 1) ^ 2 + (a $ 2 - b $ 2) ^ 2)
unfolding dist-norm norm-eq-on-real-2-vec by simp

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

definition of-radiant :: real ⇒ real ^ 2
where
of-radiant ω = axis 1 (cos ω) + axis 2 (sin ω)
definition normalize :: real ^ 2 ⇒ real ^ 2
where
normalize p = (if p = 0 then axis 1 1 else 1 / norm p) *R p
definition radiant-of :: real ^ 2 ⇒ real
where
radiant-of p = (THE ω. 0 ≤ ω ∧ ω < 2 * pi ∧ of-radiant ω = normalize p)

The vector of-radiant ω is the vector with length 1 and angle ω to the first axis. We normalize vectors to length 1 keeping their orientation with the normalize function. Conversely, radiant-of p is the angle of vector p to the first axis, where we choose radiant-of to return angles between 0 and 2 * pi, following the usual high-school convention. With these definitions, we can express the main result norm p *R of-radiant (radiant-of p) = p. Note that the main result holds for any definition of radiant-of 0. So, we choose to define normalize 0 and radiant-of 0, such that radiant-of 0 = 0.
1.2.2 Lemmas on of-radiant

lemma nth-of-radiant-1 [simp]:
of-radiant $\omega$ $1 = \cos \omega$
unfolding of-radiant-def by simp

lemma nth-of-radiant-2 [simp]:
of-radiant $\omega$ $2 = \sin \omega$
unfolding of-radiant-def by simp

lemma norm-of-radiant:
norm (of-radiant $\omega$) = 1
unfolding of-radiant-def norm-eq-on-real-2-vec by simp

lemma of-radiant-plus-2pi:
of-radiant ($\omega + 2 * pi$) = of-radiant $\omega$
unfolding of-radiant-def by simp

lemma of-radiant-minus-2pi:
of-radiant ($\omega - 2 * pi$) = of-radiant $\omega$
proof –
  have of-radiant ($\omega - 2 * pi$) = of-radiant ($\omega - 2 * pi + 2 * pi$)
    by (simp only: of-radiant-plus-2pi)
  also have ... = of-radiant $\omega$ by simp
finally show ?thesis .
qed

1.2.3 Lemmas on normalize

lemma normalize-eq:
norm p $R$ normalize p = p
unfolding normalize-def by simp

lemma norm-normalize:
norm (normalize p) = 1
unfolding normalize-def by (auto simp add: norm-axis)

lemma nth-normalize [simp]:
|normalize p $i| \leq 1
using norm-normalize component-le-norm-cart by metis

lemma normalize-square:
(normalize p $1)^2 + (normalize p $ 2)^2 = 1
using dot-square-norm[of normalize p]
by (simp add: inner-vec-def UNIV-2 power2-eq-square norm-normalize)

lemma nth-normalize-ge-zero-iff:
$0 \leq$ normalize p $i$ $\longleftrightarrow$ $0 \leq$ p $i$
proof
assume $0 \leq$ normalize p $i$
from this show \( 0 \leq p \$ i \)

unfolding normalize-def by (auto split: if-split-asm simp add: zero-le-divide-iff)

next

assume \( 0 \leq p \$ i \)
have \( 0 \leq \text{axis 1} \ (1 :: \text{real}) \$ i \)
  using exhaust-2[of i] by auto
from this \( 0 \leq p \$ i \) show \( 0 \leq \text{normalize} \ p \$ i \)
  unfolding normalize-def by auto
qed

lemma nth-normalize-less-zero-iff:
  \( \text{normalize} \ p \$ i < 0 \longleftrightarrow p \$ i < 0 \)
using nth-normalize-ge-zero-iff leD leI by blast

lemma normalize-boundary-iff:
  \( |\text{normalize} \ p \$ 1| = 1 \longleftrightarrow p \$ 2 = 0 \)
proof
  assume \( |\text{normalize} \ p \$ 1| = 1 \)
  from this have 1: \((p \$ 1) ^ 2 = \text{norm} \ p \ ^ 2\)
  unfolding normalize-def by (auto split: if-split-asm simp add: power2-eq-iff)
moreover have \((p \$ 1) ^ 2 + (p \$ 2) ^ 2 = \text{norm} \ p \ ^ 2\)
  using norm-eq-on-real-2-vec by auto
ultimately show \( p \$ 2 = 0 \) by simp
next
  assume \( p \$ 2 = 0 \)
  from this have \( |p \$ 1| = \text{norm} \ p \)
    by (auto simp add: norm-eq-on-real-2-vec)
  from this show \( |\text{normalize} \ p \$ 1| = 1 \)
    unfolding normalize-def by simp
qed

lemma between-normalize-if-distant-from-0:
  assumes \( \text{norm} \ p \geq 1 \)
  shows \( \text{between} \ (0, p) \ (\text{normalize} \ p) \)
using assms by (auto simp add: between-mem-segment closed-segment-def normalize-def)

lemma between-normalize-if-near-0:
  assumes \( \text{norm} \ p \leq 1 \)
  shows \( \text{between} \ (0, \text{normalize} \ p) \ p \)
proof
  have \( 0 \leq \text{norm} \ p \) by simp
  from assms have \( p = (\text{norm} \ p / \text{norm} \ p) \ * _R \ p \land 0 \leq \text{norm} \ p \land \text{norm} \ p \leq 1 \)
    by auto
  from this have \( \exists \ u. \ p = (u/\text{norm} \ p) \ * _R \ p \land 0 \leq u \land u \leq 1 \) by blast
  from this show \( ? \text{thesis} \)
    by (auto simp add: between-mem-segment closed-segment-def normalize-def)
qed
1.2.4 Lemmas on radiant-of

**Lemma radiant-of:**

\[ 0 \leq \text{radiant-of } p \land \text{radiant-of } p < 2 \ast \pi \land \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p \]

**Proof:**

- Let \( ?a = \text{if } 0 \leq p \land p < 2 \ast \pi \text{ then } \arccos(\text{normalize } p \land 1) \text{ else } \arccos\left(\sim (\text{normalize } p \land 1)\right) \)

  - Have \( 0 \leq ?a \land ?a < 2 \ast \pi \land \text{of-radiant } ?a = \text{normalize } p \)

  **Proof:**
  - Have \( 0 \leq ?a \) by auto
  - Moreover have \( ?a < 2 \ast \pi \)
  - **Proof** cases
    - Assume \( 0 \leq p \land 2 \)
      - From this have \( ?a \leq \pi \) by simp
      - From this show \( ?\text{thesis} \)
        - Using \( \pi \gt \text{zero} \) by linarith
  - Next
    - Assume \( \sim 0 \leq p \land 2 \)
      - Have \( \arccos\left(\sim \text{normalize } p \land 1\right) < \pi \)
        - **Proof**
          - Have \( \lnot \sim 0 \leq p \land 2 \)
            - Show \( ?\text{thesis} \) by simp
        - Qed
    - Moreover have \( \text{of-radiant } ?a = \text{normalize } p \)
      - **Proof**
        - Have \( \text{of-radiant } ?a \land i = \text{normalize } p \land i \) for \( i \)
          - **Proof**
            - Have \( \text{of-radiant } ?a \land 1 = \text{normalize } p \land 1 \)
              - Unfolding \( \text{of-radiant-def} \) by \( \text{simp add: cos-arccos-abs} \)
              - Moreover have \( \text{of-radiant } ?a \land 2 = \text{normalize } p \land 2 \)
                - **Proof** cases
                  - Assume \( 0 \leq p \land 2 \)
                    - Have \( \sin\left(\arccos\left(\text{normalize } p \land 1\right)\right) = \sqrt{(1 - (\text{normalize } p \land 1) \ast 2)} \)
                      - By \( \text{simp add: sin-arccos-abs} \)
                    - Also have \( \ldots = \text{normalize } p \land 2 \)
                      - **Proof**
                        - Have \( 1 - (\text{normalize } p \land 1)^2 = (\text{normalize } p \land 2)^2 \)
                          - Using \( \text{normalize-square[of p]} \) by auto
                        - From this \( \sim 0 \leq p \land 2 \) show \( ?\text{thesis} \) by \( \text{simp add: nth-normalize-ge-zero-iff} \)
                    - Qed
                  - Finally show \( ?\text{thesis} \)
                    - Using \( \sim 0 \leq p \land 2 \), unfolding \( \text{of-radiant-def} \) by auto
              - Next
assume \(\neg 0 \leq p \leq 2\)

have \(-\sin(\arccos(\neg \text{normalize } p \leq 1)) = -\sqrt{1 - (\text{normalize } p \leq 1)^2}\)

by (simp add: sin-arccos-abs)
also have \(\ldots = \text{normalize } p \leq 2\)
proof
  have \(1 - (\text{normalize } p \leq 1)^2 = (\text{normalize } p \leq 2)^2\)
  using normalize-square[of p] by auto
from this \((\neg 0 \leq p \leq 2)\) show \(\text{thesis}\)
  using nth-normalize-ge-zero-iff by fastforce
qed

finally show \(\text{thesis}\)
  using \((\neg 0 \leq p \leq 2)\) unfolding of-radiant-def by auto
qed

ultimately show \(\text{thesis}\)
  using \((\neg 0 \leq p \leq 2)\) blast
qed

moreover {
  fix \(\omega\)
  assume \(0 \leq \omega \land \omega < 2 \pi \land \text{of-radiant } \omega = \text{normalize } p\)
  from this have \(0 \leq \omega \omega < 2 \pi \text{ normalize } p = \text{of-radiant } \omega\) by auto
  from this have \(\cos \omega = \text{normalize } p \leq 1 \sin \omega = \text{normalize } p \leq 2\) by auto
  have \(\omega = ?a\)
  proof cases
    assume \(0 \leq p \leq 2\)
    from this have \(\omega \leq \pi\)
    using \((0 \leq \omega \omega < 2 \pi) \sin \omega = \text{normalize } p \leq 2\)
    by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)
    from \((0 \leq \omega)\) this have \(\omega = \arccos(\cos \omega)\) by (simp add: arccos-cos)
    from \((\cos \omega = \text{normalize } p \leq 1)\) this have \(\omega = \arccos(\text{normalize } p \leq 1)\)
    by (simp add: arccos-eq-iff)
    from this show \(\omega = ?a\) using \((0 \leq p \leq 2)\) by auto
  next
    assume \(\neg 0 \leq p \leq 2\)
    from this have \(\omega > \pi\)
    using \((0 \leq \omega \omega < 2 \pi) \sin \omega = \text{normalize } p \leq 2\)
    by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)
    from this \((\omega < 2 \pi)\) have \(\omega - \pi = \arccos(\cos(\omega - \pi))\)
    by (auto simp only: arccos-cos)
    from this \((\cos \omega = \text{normalize } p \leq 1)\) have \(\omega - \pi = \arccos(-\text{normalize } p \leq 1)\)
    by simp
    from this have \(\omega = \pi + \arccos(-\text{normalize } p \leq 1)\) by simp
    from this show \(\omega = ?a\) using \((\neg 0 \leq p \leq 2)\) by auto
  qed
}

ultimately show \(\text{thesis}\)

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unfolding radiant-of-def by (rule theI)
qed

lemma radiant-of-bounds [simp]:
\[ 0 \leq \text{radiant-of } p < 2 \cdot \pi \]
using radiant-of by auto

lemma radiant-of-weak-ubound [simp]:
\[ \text{radiant-of } p \leq 2 \cdot \pi \]
using radiant-of-bounds (2)[of p] by linarith

1.2.5 Main Equations for Transforming to Polar Form

lemma polar-form-eq:
\[ \text{norm } p \cdot R \cdot \text{of-radiant } (\text{radiant-of } p) = p \]
using radiant-of normalize-eq by simp

lemma relative-polar-form-eq:
\[ \text{Q} + \text{dist } P \cdot Q \cdot R \cdot \text{of-radiant } (\text{radiant-of } (P - Q)) = P \]
proof --
  have \[ \text{norm } (P - Q) \cdot R \cdot \text{of-radiant } (\text{radiant-of } (P - Q)) = P - Q \]
  unfolding polar-form-eq ..
  moreover have \[ \text{dist } P \cdot Q = \text{norm } (P - Q) \]
  by (simp add: dist-norm)
  ultimately show \?thesis by (metis add.commute diff-add-cancel)
qed

1.3 Ptolemy’s Theorem

lemma dist-circle-segment:
assumes \[ 0 \leq \text{radius } 0 \leq \alpha, \beta \leq 2 \cdot \pi \]
shows \[ \text{dist } (\text{center} + \text{radius} \cdot R \cdot \text{of-radiant } \alpha) (\text{center} + \text{radius} \cdot R \cdot \text{of-radiant } \beta) = 2 \cdot \text{radius} \cdot \text{sin } ((\beta - \alpha) / 2) \]
(is \?lhs = \?rhs)
proof --
  have \[ \text{trigonometry} : (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = (2 \cdot \sin ((\beta - \alpha) / 2))^2 \]
  proof --
    have \[ \text{sin-diff-minus} : \sin ((\alpha - \beta) / 2) = -\sin ((\beta - \alpha) / 2) \]
    by (simp only: sin-minus[symmetric] minus-divide-left minus-diff-eq)
    have \[ (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 
           (2 \cdot \sin ((\alpha + \beta) / 2) \cdot \sin ((\beta - \alpha) / 2))^2 + (2 \cdot \sin ((\alpha - \beta) / 2) \cdot \cos ((\alpha + \beta) / 2))^2 \]
    by (simp only: cos-diff-cos sin-diff-sin)
    also have \[ \ldots = (2 \cdot \sin ((\beta - \alpha) / 2))^2 \cdot ((\sin ((\alpha + \beta) / 2))^2 + (\cos ((\alpha + \beta) / 2))^2) \]
    unfolding \text{sin-diff-minus} by algebra
    also have \[ \ldots = (2 \cdot \sin ((\beta - \alpha) / 2))^2 \]
    by simp
    finally show \?thesis .
qed
from assms have \[ 0 \leq \sin ((\beta - \alpha) / 2) \]
by (simp add: sin-ge-zero)
\[\text{have } \frac{\text{lhs}}{\text{rhs}} = \sqrt{\text{radius}^2 \ast ((\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2)}\]

**unfolding** \text{dist-eq-on-real-2-vec} \text{by simp algebra}

\text{also have } \ldots = \text{sqrt} \left(\text{radius}^2 \ast (2 \ast \sin ((\beta - \alpha) / 2))^2\right) \text{by (simp add: trigonometry)}

\text{also have } \ldots = \frac{\text{lhs}}{\text{rhs}}

\text{using } 0 \leq \text{radius} : 0 \leq \sin ((\beta - \alpha) / 2) \text{by (simp add: real-sqrt-mult)}

finally show \text{thesis}.

qed

**theorem** \text{ptolemy-trigonometric}:

\text{fixes } \omega_1 \omega_2 \omega_3 :: \text{real}

\text{shows } \sin (\omega_1 + \omega_2) \ast \sin (\omega_2 + \omega_3) = \sin \omega_1 \ast \sin \omega_3 \ast \sin \omega_2 \ast \sin (\omega_1 + \omega_2 + 3)

\text{proof} –

\text{have } \sin (\omega_1 + \omega_2) \ast \sin (\omega_2 + \omega_3) = ((\sin \omega_2)^2 + (\cos \omega_2)^2) \ast \sin \omega_1 \ast \sin \omega_3 \ast \sin (\omega_1 + \omega_2 + \omega_3)

\text{by (simp only: sin-add cos-add) algebra}

\text{also have } \ldots = \sin \omega_1 \ast \sin \omega_3 \ast \sin \omega_2 \ast \sin (\omega_1 + \omega_2 + \omega_3) \text{by simp}

finally show \text{thesis}.

qed

**theorem** \text{ptolemy}:

\text{fixes } A B C D \text{ center } :: \text{real} \ast 2

\text{assumes } \text{dist center } A = \text{radius} \text{ and } \text{dist center } B = \text{radius}

\text{assumes } \text{dist center } C = \text{radius} \text{ and } \text{dist center } D = \text{radius}

\text{assumes } \text{ordering-of-points:}

\text{radiant-of } (A \text{ - center}) \leq \text{radiant-of } (B \text{ - center})

\text{radiant-of } (B \text{ - center}) \leq \text{radiant-of } (C \text{ - center})

\text{radiant-of } (C \text{ - center}) \leq \text{radiant-of } (D \text{ - center})

\text{shows } \text{dist A C } \ast \text{dist B D } = \text{dist A B } \ast \text{dist C D } + \text{dist A D } \ast \text{dist B C}

\text{proof} –

\text{from (dist center } A = \text{radius } \text{ have } 0 \leq \text{radius } \text{ by auto)}

\text{def } \alpha \equiv \text{radiant-of } (A \text{ - center}) \text{ and } \beta \equiv \text{radiant-of } (B \text{ - center})

\text{and } \gamma \equiv \text{radiant-of } (C \text{ - center}) \text{ and } \delta \equiv \text{radiant-of } (D \text{ - center})

\text{from } \text{ordering-of-points } \text{have } \text{angle-basics:}

\begin{align*}
\alpha & \leq \beta \leq \gamma \leq \delta \\
0 & \leq \alpha \leq 2 \ast \pi 0 \leq \beta \leq 2 \ast \pi \\
0 & \leq \gamma \leq 2 \ast \pi 0 \leq \delta \leq 2 \ast \pi
\end{align*}

\text{unfolding } \alpha \text{-def } \beta \text{-def } \gamma \text{-def } \delta \text{-def by auto}

\text{from } \text{assms } (1-4) \text{ have}

\begin{align*}
A & = \text{center } + \text{radius } \ast_R \text{ of-radiant } \alpha B = \text{center } + \text{radius } \ast_R \text{ of-radiant } \beta \\
C & = \text{center } + \text{radius } \ast_R \text{ of-radiant } \gamma D = \text{center } + \text{radius } \ast_R \text{ of-radiant } \delta
\end{align*}

\text{unfolding } \alpha \text{-def } \beta \text{-def } \gamma \text{-def } \delta \text{-def}

\text{using } \text{relative-polar-form-eq dist-commute } \text{by metis+}

\text{from this have } \text{dist-eqs:}

\begin{align*}
\text{dist A C } & = 2 \ast \text{radius } \ast \sin ((\gamma - \alpha) / 2) \\
\text{dist B D } & = 2 \ast \text{radius } \ast \sin ((\delta - \beta) / 2) \\
\text{dist A B } & = 2 \ast \text{radius } \ast \sin ((\beta - \alpha) / 2)
\end{align*}
\[
\begin{align*}
\text{dist } C D &= 2 \ast \text{radius} \ast \sin \left(\frac{\delta - \gamma}{2}\right) \\
\text{dist } A D &= 2 \ast \text{radius} \ast \sin \left(\frac{\delta - \alpha}{2}\right) \\
\text{dist } B C &= 2 \ast \text{radius} \ast \sin \left(\frac{\gamma - \beta}{2}\right)
\end{align*}
\]

**have** \( \text{dist } A C \ast \text{dist } B D = 4 \ast \text{radius} \ast \sin \left(\frac{\gamma - \alpha}{2}\right) \ast \sin \left(\frac{\delta - \beta}{2}\right) \)

**unfolding** \( \text{dist-eqs} \) **by** \((\text{simp add: power2-eq-square})\)

**also have** \( \ldots = 4 \ast \text{radius} \ast \sin \left(\frac{\beta - \alpha}{2}\right) \ast \sin \left(\frac{\delta - \gamma}{2}\right) + \sin \left(\frac{\gamma - \beta}{2}\right) \ast \sin \left(\frac{\delta - \alpha}{2}\right) \)

**proof**

\(\text{def } \omega_1 \equiv \frac{\beta - \alpha}{2} \text{ and } \omega_2 \equiv \frac{\gamma - \beta}{2} \text{ and } \omega_3 \equiv \frac{\delta - \gamma}{2} \)

\(\text{have } \left(\frac{\gamma - \alpha}{2}\right) = \omega_1 + \omega_2 \text{ and } \left(\frac{\delta - \beta}{2}\right) = \omega_2 + \omega_3 \text{ and } \left(\frac{\delta - \alpha}{2}\right) \)

\(= \omega_1 + \omega_2 + \omega_3 \)

**unfolding** \( \omega_1\text{-def } \omega_2\text{-def } \omega_3\text{-def by} \) \((\text{auto simp add: field-simps})\)

**have** \( \sin \left(\frac{\gamma - \alpha}{2}\right) \ast \sin \left(\frac{\delta - \beta}{2}\right) = \sin \left(\omega_1 + \omega_2 \right) \ast \sin \left(\omega_2 + \omega_3 \right) \)

**using** \( \left(\frac{\gamma - \alpha}{2}\right) = \omega_1 + \omega_2 \text{ and } \left(\frac{\delta - \beta}{2}\right) = \omega_2 + \omega_3 \) **by** \((\text{simp only:})\)

**also have** \( \ldots = \sin \omega_1 \ast \sin \omega_3 \ast \sin \omega_2 \ast \sin \left(\omega_1 + \omega_2 + \omega_3 \right) \)

**by** \((\text{rule ptolemy-trigonometric})\)

**also have** \( \ldots = \sin \left(\frac{\beta - \alpha}{2}\right) \ast \sin \left(\frac{\delta - \gamma}{2}\right) + \sin \left(\frac{\gamma - \beta}{2}\right) \ast \sin \left(\frac{\delta - \alpha}{2}\right) \)

**finally show** \(?\text{thesis by simp}\)

**qed**

**also have** \( \ldots = \text{dist } A B \ast \text{dist } C D \ast \text{dist } A D \ast \text{dist } B C \)

**unfolding** \( \text{dist-eqs by} \) \((\text{simp add: distrib-left power2-eq-square})\)

**finally show** \(?\text{thesis} .\)

**qed**

end

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**References**

