Ptolemy’s Theorem
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Abstract
This entry provides an analytic proof to Ptolemy’s Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison’s HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy’s Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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1 Ptolemy’s Theorem
theory Ptolemys-Theorem
imports
  HOL-Analysis.Multivariate-Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory
hide-const (open) normalize
1.1.2 Additions to Transcendental theory

Lemmas about $\arcsin$ and $\arccos$ commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument $y$ is between $-1$ and $1$. As the argumentation for $-1 \leq y$ and $y \leq 1$ is often very similar, we prefer to prove $|y| \leq 1$ to the two goals above.

The lemma for rewriting the term $\cos (\arccos y)$ is already provided in the Isabelle distribution with name $\text{cos-arccos-abs}$. Here, we further provide the analogue on $\arcsin$ for rewriting $\sin (\arcsin y)$.

**lemma** $\text{sin-arcsin-abs}$:

$|y| \leq 1 \Rightarrow \sin (\arcsin y) = y$

by $(\text{simp add: abs-le-iff})$

The further lemmas are the required variants from existing lemmas $\text{arccos-lbound}$ and $\text{arccos-ubound}$.

**lemma** $\text{arccos-lbound-abs}$ [simp]:

$|y| \leq 1 \Rightarrow 0 \leq \arccos y$

by $(\text{simp add: arccos-lbound})$

**lemma** $\text{arccos-ubound-abs}$ [simp]:

$|y| \leq 1 \Rightarrow \arccos y \leq \pi$

by $(\text{simp add: arccos-ubound})$

As we choose angles to be between $0$ between $2 \times \pi$, we need some lemmas to reason about the sign of $\sin x$ for angles $x$.

**lemma** $\text{sin-ge-zero-iff}$:

assumes $0 \leq x < 2 \times \pi$

shows $0 \leq \sin x \longleftrightarrow x \leq \pi$

proof

assume $0 \leq \sin x$

show $x \leq \pi$

proof (rule ccontr)

assume $\neg x \leq \pi$

from this $|x < 2 \times \pi$ have $\sin x < 0$

using $\text{sin-lt-zero}$ by auto

from this $0 \leq \sin x$ show False by auto

qed

next

assume $x \leq \pi$

from this $0 \leq x$ show $0 \leq \sin x$ by $(\text{simp add: sin-ge-zero})$

qed

**lemma** $\text{sin-less-zero-iff}$:

assumes $0 \leq x < 2 \times \pi$

shows $\sin x < 0 \longleftrightarrow \pi < x$

using assms $\text{sin-ge-zero-iff}$ by fastforce
1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

**lemma** *axis-nth-eq-0* [simp]:
- **assumes** $i \neq j$
- **shows** $\langle \text{axis } i \rangle \times \langle \text{axis } j \rangle = 0$

**using** *assms unfolding* *axis-def by simp*

**lemma** *norm-axis*:
- **fixes** $x :: \text{real}$
- **shows** $\|\langle \text{axis } i \rangle \times x\| = \|x\|

**by** (*simp add:* *norm-eq-sqrt-inner inner-axis-axis*)

**lemma** *norm-eq-on-real-2-vec*:
- **fixes** $x :: \text{real }^2$
- **shows** $\|x\| = \sqrt{(\text{axis } 1 \cdot x_1)^2 + (\text{axis } 2 \cdot x_2)^2}$

**by** (*simp add: norm-eq-sqrt-inner inner-vec-def UNIV-2 power2-eq-square*)

**lemma** *dist-eq-on-real-2-vec*:
- **fixes** $a \cdot b :: \text{real }^2$
- **shows** $d \langle a \rangle \cdot \langle b \rangle = \sqrt{(\text{axis } 1 \cdot a - \text{axis } 1 \cdot b)^2 + (\text{axis } 2 \cdot a - \text{axis } 2 \cdot b)^2}$

**unfolding** *dist-norm norm-eq-on-real-2-vec by simp*

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

**definition** *of-radiant* :: $\text{real } \Rightarrow \text{real }^2$

**where**
$\langle \text{of-radiant } \omega \rangle = \langle \text{axis } 1 \rangle (\cos \omega) + \langle \text{axis } 2 \rangle (\sin \omega)$

**definition** *normalize* :: $\text{real }^2 \Rightarrow \text{real }^2$

**where**
$\langle \text{normalize } p \rangle = (\text{if } p = 0 \text{ then } \langle \text{axis } 1 \rangle 1 \text{ else } (1 / \|p\|) \cdot R p)$

**definition** *radiant-of* :: $\text{real }^2 \Rightarrow \text{real}$

**where**
$\langle \text{radiant-of } p \rangle = (\text{THE } \omega. \ 0 \leq \omega \land \omega < 2 \cdot \pi \land \text{of-radiant } \omega = \text{normalize } p)$

The vector *of-radiant* $\omega$ is the vector with length 1 and angle $\omega$ to the first axis. We normalize vectors to length 1 keeping their orientation with the normalize function. Conversely, *radiant-of* $p$ is the angle of vector $p$ to the first axis, where we choose *radiant-of* to return angles between 0 and $2 \cdot \pi$, following the usual high-school convention. With these definitions, we can express the main result $\|p\|^R \cdot \langle \text{of-radiant } \rangle (\text{radiant-of } p) = p$. Note that the main result holds for any definition of *radiant-of* 0. So, we choose to define *normalize* 0 and *radiant-of* 0, such that *radiant-of* 0 = 0.
1.2.2 Lemmas on of-radiant

lemma nth-of-radiant-1 [simp]:
of-radiant $\omega$ $1 = \cos \omega$
unfolding of-radiant-def by simp

lemma nth-of-radiant-2 [simp]:
of-radiant $\omega$ $2 = \sin \omega$
unfolding of-radiant-def by simp

lemma norm-of-radiant:
norm ($\omega$) = 1
unfolding of-radiant-def norm-eq-on-real-2-vec by simp

lemma of-radiant-plus-2pi:
of-radiant ($\omega + 2 \times pi$) = of-radiant $\omega$
unfolding of-radiant-def by simp

lemma of-radiant-minus-2pi:
of-radiant ($\omega - 2 \times pi$) = of-radiant $\omega$
proof
  have of-radiant ($\omega - 2 \times pi$) = of-radiant ($\omega - 2 \times pi + 2 \times pi$)
  by (simp only: of-radiant-plus-2pi)
  also have ... = of-radiant $\omega$ by simp
  finally show ?thesis.
qed

1.2.3 Lemmas on normalize

lemma normalize-eq:
norm $p \times_R$ normalize $p = p$
unfolding normalize-def by simp

lemma norm-normalize:
norm (normalize $p$) = 1
unfolding normalize-def by (auto simp add: norm-axis)

lemma nth-normalize [simp]:
|normalize $p$ |i| |\leq 1
using norm-normalize component-le-norm-cart by metis

lemma normalize-square:
(normalize $p$ |1|^2 + (normalize $p$ |2|^2 = 1
using dot-square-norm[of normalize $p$]
by (simp add: inner-vec-def UNIV-2 power2-eq-square norm-normalize)

lemma nth-normalize-ge-zero-iff:
$0 \leq$ normalize $p$ $i$ $\longleftrightarrow 0 \leq p$ $i$
proof
assume $0 \leq$ normalize $p$ $i$
from this show \( 0 \leq p \$ i \)

unfolding normalize-def by (auto split: if-split-asn simp add: zero-le-divide-iff)

next
assume \( 0 \leq p \$ i \)
have \( 0 \leq \text{axis 1} (1 :: \text{real}) \$ i \)
using exhaust-2[of \( i \)] by auto
from this \( 0 \leq p \$ i \): show \( 0 \leq \text{normalize} p \$ i \)
unfolding normalize-def by auto

qed

lemma nth-normalize-less-zero-iff:
\( \text{normalize} p \$ i < 0 \iff p \$ i < 0 \)
using nth-normalize-ge-zero-iff leD leI by metis

lemma normalize-boundary_iff:
\( |\text{normalize} p \$ 1| = 1 \iff p \$ 2 = 0 \)

proof
assume \( |\text{normalize} p \$ 1| = 1 \)
from this have \( 1: (p \$ 1) \sim 2 = \text{norm} p \sim 2 \)
unfolding normalize-def by (auto split: if-split-asn simp add: power2_eq_iff)
moreover have \( (p \$ 1) \sim 2 + (p \$ 2) \sim 2 = \text{norm} p \sim 2 \)
using norm_eq_on_real_2_vec by auto
ultimately show \( p \$ 2 = 0 \) by simp

next
assume \( p \$ 2 = 0 \)
from this have \( |p \$ 1| = \text{norm} p \)
by (auto simp add: norm_eq_on_real_2_vec)
from this show \( |\text{normalize} p \$ 1| = 1 \)
unfolding normalize_def by simp

qed

lemma between-normalize-if-distant-from-0:
 assumes \( \text{norm} p \geq 1 \)
 shows between \((0, p)\) (normalize p)
using assms by (auto simp add: between_mem_segment closed_segment_def normalize_def)

lemma between-normalize-if-near-0:
 assumes \( \text{norm} p \leq 1 \)
 shows between \((0, \text{normalize} p)\) \( p \)

proof
have \( 0 \leq \text{norm} p \) by simp
from assms have \( p = (\text{norm} p / \text{norm} p) \ast_R p \wedge 0 \leq \text{norm} p \wedge \text{norm} p \leq 1 \) by auto
from this have \( \exists u. p = (u / \text{norm} p) \ast_R p \wedge 0 \leq u \wedge u \leq 1 \) by blast
from this show \( \text{thesis} \)
by (auto simp add: between_mem_segment closed_segment_def normalize_def)

qed
1.2.4 Lemmas on $\text{radiant-of}$

**Lemma** $\text{radiant-of}$:
\[ 0 \leq \text{radiant-of } p \land \text{radiant-of } p < 2 \cdot \pi \land \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p \]

**Proof** –

Let $\ ?a = \text{if } 0 \leq p \cdot 2 \text{ then } \arccos (\text{normalize } p \cdot 1) \text{ else } \arccos (- (\text{normalize } p \cdot 1))$

Have $0 \leq ?a \land ?a < 2 \cdot \pi \land \text{of-radiant } ?a = \text{normalize } p$

**Proof** –

Have $0 \leq ?a$ by auto

Moreover have $?a < 2 \cdot \pi$

**Proof** cases

Assume $0 \leq p \cdot 2$

From this have $?a \leq \pi$ by simp

From this show $?\text{thesis}$

Using $\pi \cdot \text{gt-zero}$ by linarith

Next

Assume $\sim 0 \leq p \cdot 2$

Have $\arccos (- \text{normalize } p \cdot 1) < \pi$

**Proof** –

Have $|\text{normalize } p \cdot 1| \neq 1$

Using $\sim 0 \leq p \cdot 2$ by (simp only: normalize-boundary-iff)

From this have $\arccos (- \text{normalize } p \cdot 1) \neq \pi$

Unfolding $\arccos\text{-minus-}1\text{[symmetric]}$ by (subst $\arccos\text{-iff}$) auto

Moreover have $\arccos (- \text{normalize } p \cdot 1) \leq \pi$ by simp

Ultimately show $\arccos (- \text{normalize } p \cdot 1) < \pi$ by linarith

Qed

From this $\sim 0 \leq p \cdot 2$ show $?\text{thesis}$ by simp

Qed

Moreover have $\text{of-radiant } ?a = \text{normalize } p$

**Proof** –

Have $\text{of-radiant } ?a \cdot i = \text{normalize } p \cdot i$ for $i$

**Proof** –

Have $\text{of-radiant } ?a \cdot i = \text{normalize } p \cdot i$

Unfolding $\text{of-radiant-def}$ by (simp add: cos-arccos-abs)

Moreover have $\text{of-radiant } ?a \cdot 2 = \text{normalize } p \cdot 2$

**Proof** cases

Assume $0 \leq p \cdot 2$

Have $\sin (\arccos (\text{normalize } p \cdot 1)) = \sqrt{1 - (\text{normalize } p \cdot 1)^2}$

Using (simp add: sin-arccos-abs)

Also have $\ldots = \text{normalize } p \cdot 2$

**Proof** –

Have $1 - (\text{normalize } p \cdot 1)^2 = (\text{normalize } p \cdot 2)^2$

Using $\text{normalize-square}[p]$ by auto

From this $0 \leq p \cdot 2$ show $?\text{thesis}$ by (simp add: nth-normalize-ge-zero-iff)

Qed

Finally show $?\text{thesis}$

Using $0 \leq p \cdot 2$ unfolding $\text{of-radiant-def}$ by auto

Next
assume $\neg 0 \leq p \leq 2$

have $- \sin (\arccos (- \text{normalize } p \leq 1)) = - \sqrt{1 - (\text{normalize } p \leq 1)^2}$
by (simp add: sin-arccos-abs)
also have $\ldots = \text{normalize } p \leq 2$
proof$
\quad \text{have } 1 - (\text{normalize } p \leq 1)^2 = (\text{normalize } p \leq 2)^2$
using normalize-square[of p] by auto
from this ($\neg 0 \leq p \leq 2$) show ?thesis
using nth-normalize-ge-zero-iff by fastforce
qed
finally show ?thesis
using ($\neg 0 \leq p \leq 2$) unfolding of-radiant-def by auto
qed
ultimately show ?thesis
unfolding of-radiant-def by (rule theI)
moreover { fix $\omega$
assume $0 \leq \omega$ ∧ $\omega < 2 * \pi$ ∧ of-radiant $\omega = \text{normalize } p$
from this have $0 \leq \omega$ ∧ $\omega < 2 * \pi$ normalize $p = \text{of-radiant } \omega$ by auto
from this have $\cos \omega = \text{normalize } p \leq 1$ $\sin \omega = \text{normalize } p \leq 2$ by auto
have $\omega = ?a$
proof cases
assume $0 \leq p \leq 2$
from this have $\omega \leq \pi$
using ($0 \leq \omega$) $\omega < 2 * \pi$ ($\sin \omega = \text{normalize } p \leq 2$)
by (simp add: sin-ge-zero-iff[ symmetric] nth-normalize-ge-zero-iff)
from ($0 \leq \omega$) this have $\omega = \arccos (\cos \omega)$ by (simp add: arccos-cos)
from $\langle \cos \omega = \text{normalize } p \leq 1$ $\rangle$ this have $\omega = \arccos (\text{normalize } p \leq 1)$
by (simp add: arccos-eq-iff)
from this show $\omega = ?a$ using ($0 \leq p \leq 2$) by auto
next
assume $\neg 0 \leq p \leq 2$
from this have $\omega > \pi$
using ($0 \leq \omega$) $\omega < 2 * \pi$ ($\sin \omega = \text{normalize } p \leq 2$)
by (simp add: sin-less-zero-iff[ symmetric] nth-normalize-less-zero-iff)
from this $\omega < 2 * \pi$ have $\omega - \pi = \arccos (\omega - \pi)$
by (auto simp only: arccos-cos)
from this $\langle \cos \omega = \text{normalize } p \leq 1$ $\rangle$ have $\omega - \pi = \arccos (- \text{normalize } p \leq 1)$
by simp
from this have $\omega = \pi + \arccos (- \text{normalize } p \leq 1)$ by simp
from this show $\omega = ?a$ using ($\neg 0 \leq p \leq 2$) by auto
qed
}
ultimately show ?thesis
unfolding radiant-of-def by (rule theI)
lemma radiant-of-bounds [simp]:
0 ≤ radiant-of p radiant-of p < 2 * pi
using radiant-of by auto

lemma radiant-of-weak-ubound [simp]:
radiant-of p ≤ 2 * pi
using radiant-of-bounds(2)[of p] by linarith

1.2.5 Main Equations for Transforming to Polar Form

lemma polar-form-eq:
norm p * R of-radiant (radiant-of p) = p
using radiant-of normalize-eq by simp

lemma relative-polar-form-eq:
Q + dist P Q * R of-radiant (radiant-of (P − Q)) = P
proof
have norm (P − Q) * R of-radiant (radiant-of (P − Q)) = P − Q
  unfolding polar-form-eq ..
moreover have dist P Q = norm (P − Q) by (simp add: dist-norm)
ultimately show ?thesis by (metis add.commute diff-add-cancel)
qed

1.3 Ptolemy’s Theorem

lemma dist-circle-segment:
assumes 0 ≤ radius 0 ≤ α α ≤ β β ≤ 2 * pi
shows dist (center + radius * R of-radiant α) (center + radius * R of-radiant β)
  = 2 * radius * sin ((β − α) / 2)
(is ?lhs = ?rhs)
proof
have trigonometry: (cos α − cos β)² + (sin α − sin β)² = (2 * sin ((β − α) / 2)²)
  unfolding sin-diff-minus by algebra
also have .. . = (2 * sin ((β − α) / 2))² * ((sin ((α + β) / 2))² + (cos ((α + β) / 2))²)
unfolding sin-diff-minus by algebra
also have .. . = (2 * sin ((β − α) / 2))² by simp
finally show ?thesis .
qed

from assms have 0 ≤ sin ((β − α) / 2) by (simp add: sin-ge-zero)
have ?lhs = sqrt (radius² * ((cos α − cos β)² + (sin α − sin β)²))
unfolding \text{dist-eq-on-real-2-vec} \text{by simp algebra}
also have \ldots = \sqrt{(\text{radius}^2 \ast (2 \ast \sin((\beta - \alpha) / 2)))} \text{by (simp add: trigonometry)}
also have \ldots = \text{?rhs}
using (\text{0} \leq \text{radius}) \text{0} \leq \text{sin((\beta - \alpha) / 2))} \text{by (simp add: real-sqrt-mult)}
finally show \text{?thesis} .
qd

\text{theorem ptolemy-trigonometric}:
fixes \omega_1 \omega_2 \omega_3 :: \text{real}
shows \sin(\omega_1 + \omega_2) \ast \sin(\omega_2 + \omega_3) = \sin(\omega_1 \ast \sin(\omega_3 + \sin(\omega_2 \ast \sin(\omega_1 + \omega_2 + \omega_3)))
proof –
\text{have sin(\omega_1 + \omega_2) \ast sin(\omega_2 + \omega_3) = ((\sin \omega_2)^2 + (\cos \omega_2)^2) \ast \sin \omega_1 \ast sin(\omega_3 + \sin(\omega_2 \ast \sin(\omega_1 + \omega_2 + \omega_3)))}
\text{by (simp only: sin-add cos-add) algebra}
also have \ldots = \text{sin \omega_1} \ast \sin \omega_3 + \sin \omega_2 \ast \sin(\omega_1 + \omega_2 + \omega_3) \text{by simp}
finally show \text{?thesis} .
qd

\text{theorem ptolemy}:
fixes A B C D center :: \text{real} ^{\text{2}}
assumes \text{dist center A = radius and dist center B = radius}
assumes \text{dist center C = radius and dist center D = radius}
assumes \text{ordering-of-points}:
\text{radiant-of (A - center) \leq} \text{radiant-of (B - center)}
\text{radiant-of (B - center) \leq} \text{radiant-of (C - center)}
\text{radiant-of (C - center) \leq} \text{radiant-of (D - center)}
shows \text{dist A C \ast dist B D = dist A B \ast dist C D} + \text{dist A D \ast dist B C}
proof –
\text{from \text{dist center A = radius} have} \text{0} \leq \text{radius by auto}
define \alpha \beta \gamma \delta
\text{where} \alpha = \text{radiant-of (A - center)} \text{and} \beta = \text{radiant-of (B - center)}
\text{and} \gamma = \text{radiant-of (C - center)} \text{and} \delta = \text{radiant-of (D - center)}
from ordering-of-points \text{have angle-basics:}
\alpha \leq \beta \leq \gamma \leq \delta
\text{0} \leq \text{\alpha} \leq \text{2} \ast \text{pi} \text{0} \leq \text{\beta} \leq \text{2} \ast \text{pi}
\text{0} \leq \gamma \leq \text{2} \ast \text{pi} \text{0} \leq \delta \leq \text{2} \ast \text{pi}
\text{unfolding \alpha-def \beta-def \gamma-def \delta-def by auto}
from \text{assms(1-4) have}
\text{A = center + radius \ast R of-radiant \alpha B = center + radius \ast R of-radiant \beta}
\text{C = center + radius \ast R of-radiant \gamma D = center + radius \ast R of-radiant \delta}
\text{unfolding \alpha-def \beta-def \gamma-def \delta-def}
\text{using \text{relative-polar-form-eq dist-commute by metis+}
from this have dist-eqs:
dist A C = 2 \ast \text{radius} \ast \sin((\gamma - \alpha) / 2)
dist B D = 2 \ast \text{radius} \ast \sin((\delta - \beta) / 2)
dist A B = 2 \ast \text{radius} \ast \sin((\beta - \alpha) / 2)
\[ \text{dist } CD = 2 \times \text{radius} \times \sin \left( \frac{\delta - \gamma}{2} \right) \]
\[ \text{dist } AD = 2 \times \text{radius} \times \sin \left( \frac{\delta - \alpha}{2} \right) \]
\[ \text{dist } BC = 2 \times \text{radius} \times \sin \left( \frac{\gamma - \beta}{2} \right) \]

using angle-basics \( \text{radius} \geq 0 \) dist-circle-segment by (auto)

have \( \text{dist } AC \times \text{dist } BD = 4 \times \text{radius}^2 \times \sin \left( \frac{\gamma - \alpha}{2} \right) \times \sin \left( \frac{\delta - \beta}{2} \right) \)

unfolding dist-eqs by (simp add: power2-eq-square)
also have \( \ldots \times 4 \times \text{radius}^2 \times \left( \sin \left( \frac{\beta - \alpha}{2} \right) \times \sin \left( \frac{\delta - \gamma}{2} \right) + \sin \left( \frac{\gamma - \beta}{2} \right) \times \sin \left( \frac{\delta - \alpha}{2} \right) \right) \)
proof
\[ \text{define } \omega_1, \omega_2, \omega_3 \text{ where } \omega_1 = \left( \frac{\beta - \alpha}{2} \right) \text{ and } \omega_2 = \left( \frac{\gamma - \beta}{2} \right) \text{ and } \omega_3 = \left( \frac{\delta - \gamma}{2} \right) \]
have \( \frac{\gamma - \alpha}{2} = \omega_1 + \omega_2 \text{ and } \frac{\delta - \beta}{2} = \omega_2 + \omega_3 \text{ and } \frac{\delta - \alpha}{2} = \omega_1 + \omega_2 + \omega_3 \)
unfolding \( \omega_1 \text{-def } \omega_2 \text{-def } \omega_3 \text{-def } \) by (auto simp add: field-simps)
have \( \sin \left( \frac{\gamma - \alpha}{2} \right) \times \sin \left( \frac{\delta - \beta}{2} \right) = \sin \left( \omega_1 + \omega_2 \right) \times \sin \left( \omega_2 + \omega_3 \right) \)
using \( \frac{\gamma - \alpha}{2} / 2 = \omega_1 + \omega_2 \) \( \frac{\delta - \beta}{2} / 2 = \omega_2 + \omega_3 \) by (simp only:)
also have \( \ldots = \sin \omega_1 \times \sin \omega_3 + \sin \omega_2 \times \sin \left( \omega_1 + \omega_2 + \omega_3 \right) \)
by (rule ptolemy-trigonometric)
also have \( \ldots = \left( \sin \left( \frac{\beta - \alpha}{2} \right) \times \sin \left( \frac{\delta - \gamma}{2} \right) \right) + \sin \left( \frac{\gamma - \beta}{2} \right) \times \sin \left( \frac{\delta - \alpha}{2} \right) \)
using \( \omega_1 \text{-def } \omega_2 \text{-def } \omega_3 \text{-def } \) \( \delta - \alpha \) / 2 = \( \omega_1 + \omega_2 + \omega_3 \) by (simp only:)
finally show ?thesis by simp
qed
also have \( \ldots = \text{dist } AB \times \text{dist } CD + \text{dist } AD \times \text{dist } BC \)
unfolding dist-eqs by (simp add: distrib-left power2-eq-square)
finally show ?thesis .
qed

end

References