

# Ptolemy's Theorem

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December 14, 2021

## Abstract

This entry provides an analytic proof to Ptolemy's Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison's HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy's Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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## 1 Ptolemy's Theorem

```
theory Ptolemys-Theorem
imports
  HOL-Analysis.Multivariate-Analysis
begin
```

### 1.1 Preliminaries

#### 1.1.1 Additions to Rat theory

```
hide-const (open) normalize
```

### 1.1.2 Additions to Transcendental theory

Lemmas about  $\arcsin$  and  $\arccos$  commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument  $y$  is between  $-1$  and  $1$ . As the argumentation for  $-1 \leq y$  and  $y \leq 1$  is often very similar, we prefer to prove  $|y| \leq 1$  to the two goals above.

The lemma for rewriting the term  $\cos (\arccos y)$  is already provided in the Isabelle distribution with name  $\cos\text{-arccos-abs}$ . Here, we further provide the analogue on  $\arcsin$  for rewriting  $\sin (\arcsin y)$ .

**lemma** *sin-arcsin-abs*:  $|y| \leq 1 \implies \sin (\arcsin y) = y$   
**by** (*simp add: abs-le-iff*)

The further lemmas are the required variants from existing lemmas *arccos-lbound* and *arccos-ubound*.

**lemma** *arccos-lbound-abs* [*simp*]:  
 $|y| \leq 1 \implies 0 \leq \arccos y$   
**by** (*simp add: arccos-lbound*)

**lemma** *arccos-ubound-abs* [*simp*]:  
 $|y| \leq 1 \implies \arccos y \leq \pi$   
**by** (*simp add: arccos-ubound*)

As we choose angles to be between  $0$  between  $2 * \pi$ , we need some lemmas to reason about the sign of  $\sin x$  for angles  $x$ .

**lemma** *sin-ge-zero-iff*:  
**assumes**  $0 \leq x < 2 * \pi$   
**shows**  $0 \leq \sin x \longleftrightarrow x \leq \pi$   
**proof**  
**assume**  $0 \leq \sin x$   
**show**  $x \leq \pi$   
**proof** (*rule ccontr*)  
**assume**  $\neg x \leq \pi$   
**from** *this*  $\langle x < 2 * \pi \rangle$  **have**  $\sin x < 0$   
**using** *sin-lt-zero* **by** *auto*  
**from** *this*  $\langle 0 \leq \sin x \rangle$  **show** *False* **by** *auto*  
**qed**  
**next**  
**assume**  $x \leq \pi$   
**from** *this*  $\langle 0 \leq x \rangle$  **show**  $0 \leq \sin x$  **by** (*simp add: sin-ge-zero*)  
**qed**

**lemma** *sin-less-zero-iff*:  
**assumes**  $0 \leq x < 2 * \pi$   
**shows**  $\sin x < 0 \longleftrightarrow \pi < x$   
**using** *assms sin-ge-zero-iff* **by** *fastforce*

### 1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

**lemma** *axis-nth-eq-0* [*simp*]:  
  **assumes**  $i \neq j$   
  **shows**  $\text{axis } i \ x \ \$ \ j = 0$   
**using** *assms unfolding axis-def* **by** *simp*

**lemma** *norm-axis*:  
  **fixes**  $x :: \text{real}$   
  **shows**  $\text{norm } (\text{axis } i \ x) = \text{abs } x$   
**by** (*simp add: norm-eq-sqrt-inner inner-axis-axis*)

**lemma** *norm-eq-on-real-2-vec*:  
  **fixes**  $x :: \text{real}^2$   
  **shows**  $\text{norm } x = \text{sqrt } ((x \ \$ \ 1)^2 + (x \ \$ \ 2)^2)$   
**by** (*simp add: norm-eq-sqrt-inner inner-vec-def UNIV-2 power2-eq-square*)

**lemma** *dist-eq-on-real-2-vec*:  
  **fixes**  $a \ b :: \text{real}^2$   
  **shows**  $\text{dist } a \ b = \text{sqrt } ((a \ \$ \ 1 - b \ \$ \ 1)^2 + (a \ \$ \ 2 - b \ \$ \ 2)^2)$   
**unfolding** *dist-norm norm-eq-on-real-2-vec* **by** *simp*

## 1.2 Polar Form of Two-Dimensional Real-Valued Vectors

### 1.2.1 Definitions to Transfer to Polar Form and Back

**definition** *of-radiant*  $:: \text{real}^2 \Rightarrow \text{real}^2$   
**where**  
   $\text{of-radiant } \omega = \text{axis } 1 \ (\cos \ \omega) + \text{axis } 2 \ (\sin \ \omega)$

**definition** *normalize*  $:: \text{real}^2 \Rightarrow \text{real}^2$   
**where**  
   $\text{normalize } p = (\text{if } p = 0 \ \text{then } \text{axis } 1 \ 1 \ \text{else } (1 / \text{norm } p) *_{\mathbb{R}} p)$

**definition** *radiant-of*  $:: \text{real}^2 \Rightarrow \text{real}$   
**where**  
   $\text{radiant-of } p = (\text{THE } \omega. \ 0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{of-radiant } \omega = \text{normalize } p)$

The vector *of-radiant*  $\omega$  is the vector with length 1 and angle  $\omega$  to the first axis. We normalize vectors to length 1 keeping their orientation with the *normalize* function. Conversely, *radiant-of*  $p$  is the angle of vector  $p$  to the first axis, where we choose *radiant-of* to return angles between 0 and  $2 * \pi$ , following the usual high-school convention. With these definitions, we can express the main result  $\text{norm } p *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } p) = p$ . Note that the main result holds for any definition of *radiant-of* 0. So, we choose to define *normalize* 0 and *radiant-of* 0, such that *radiant-of* 0 = 0.

### 1.2.2 Lemmas on *of-radiant*

**lemma** *nth-of-radiant-1* [*simp*]:

*of-radiant*  $\omega$  \$ 1 =  $\cos \omega$

**unfolding** *of-radiant-def* **by** *simp*

**lemma** *nth-of-radiant-2* [*simp*]:

*of-radiant*  $\omega$  \$ 2 =  $\sin \omega$

**unfolding** *of-radiant-def* **by** *simp*

**lemma** *norm-of-radiant*:

*norm* (*of-radiant*  $\omega$ ) = 1

**unfolding** *of-radiant-def* *norm-eq-on-real-2-vec* **by** *simp*

**lemma** *of-radiant-plus-2pi*:

*of-radiant* ( $\omega + 2 * \pi$ ) = *of-radiant*  $\omega$

**unfolding** *of-radiant-def* **by** *simp*

**lemma** *of-radiant-minus-2pi*:

*of-radiant* ( $\omega - 2 * \pi$ ) = *of-radiant*  $\omega$

**proof** –

**have** *of-radiant* ( $\omega - 2 * \pi$ ) = *of-radiant* ( $\omega - 2 * \pi + 2 * \pi$ )

**by** (*simp only: of-radiant-plus-2pi*)

**also have** ... = *of-radiant*  $\omega$  **by** *simp*

**finally show** *?thesis* .

**qed**

### 1.2.3 Lemmas on *normalize*

**lemma** *normalize-eq*:

*norm*  $p *_R$  *normalize*  $p$  =  $p$

**unfolding** *normalize-def* **by** *simp*

**lemma** *norm-normalize*:

*norm* (*normalize*  $p$ ) = 1

**unfolding** *normalize-def* **by** (*auto simp add: norm-axis*)

**lemma** *nth-normalize* [*simp*]:

$|$ *normalize*  $p$  \$  $i$  $|$   $\leq 1$

**using** *norm-normalize component-le-norm-cart* **by** *metis*

**lemma** *normalize-square*:

(*normalize*  $p$  \$ 1)<sup>2</sup> + (*normalize*  $p$  \$ 2)<sup>2</sup> = 1

**using** *dot-square-norm*[*of normalize p*]

**by** (*simp add: inner-vec-def UNIV-2 power2-eq-square norm-normalize*)

**lemma** *nth-normalize-ge-zero-iff*:

$0 \leq$  *normalize*  $p$  \$  $i$   $\iff 0 \leq p$  \$  $i$

**proof**

**assume**  $0 \leq$  *normalize*  $p$  \$  $i$

**from this show**  $0 \leq p \ \$ \ i$   
**unfolding** *normalize-def* **by** (*auto split: if-split-asm simp add: zero-le-divide-iff*)  
**next**  
**assume**  $0 \leq p \ \$ \ i$   
**have**  $0 \leq \text{axis } 1 \ (1 :: \text{real}) \ \$ \ i$   
**using** *exhaust-2[of i]* **by** *auto*  
**from this**  $\langle 0 \leq p \ \$ \ i \rangle$  **show**  $0 \leq \text{normalize } p \ \$ \ i$   
**unfolding** *normalize-def* **by** *auto*  
**qed**

**lemma** *nth-normalize-less-zero-iff*:  
 $\text{normalize } p \ \$ \ i < 0 \longleftrightarrow p \ \$ \ i < 0$   
**using** *nth-normalize-ge-zero-iff leD leI* **by** *metis*

**lemma** *normalize-boundary-iff*:  
 $|\text{normalize } p \ \$ \ 1| = 1 \longleftrightarrow p \ \$ \ 2 = 0$   
**proof**  
**assume**  $|\text{normalize } p \ \$ \ 1| = 1$   
**from this have**  $1: (p \ \$ \ 1) \ ^2 = \text{norm } p \ ^2$   
**unfolding** *normalize-def* **by** (*auto split: if-split-asm simp add: power2-eq-iff*)  
**moreover have**  $(p \ \$ \ 1) \ ^2 + (p \ \$ \ 2) \ ^2 = \text{norm } p \ ^2$   
**using** *norm-eq-on-real-2-vec* **by** *auto*  
**ultimately show**  $p \ \$ \ 2 = 0$  **by** *simp*  
**next**  
**assume**  $p \ \$ \ 2 = 0$   
**from this have**  $|p \ \$ \ 1| = \text{norm } p$   
**by** (*auto simp add: norm-eq-on-real-2-vec*)  
**from this show**  $|\text{normalize } p \ \$ \ 1| = 1$   
**unfolding** *normalize-def* **by** *simp*  
**qed**

**lemma** *between-normalize-if-distant-from-0*:  
**assumes**  $\text{norm } p \geq 1$   
**shows** *between*  $(0, p)$   $(\text{normalize } p)$   
**using** *assms* **by** (*auto simp add: between-mem-segment closed-segment-def normalize-def*)

**lemma** *between-normalize-if-near-0*:  
**assumes**  $\text{norm } p \leq 1$   
**shows** *between*  $(0, \text{normalize } p)$   $p$   
**proof** –  
**have**  $0 \leq \text{norm } p$  **by** *simp*  
**from** *assms* **have**  $p = (\text{norm } p / \text{norm } p) *_{\mathbb{R}} p \wedge 0 \leq \text{norm } p \wedge \text{norm } p \leq 1$   
**by** *auto*  
**from this have**  $\exists u. p = (u / \text{norm } p) *_{\mathbb{R}} p \wedge 0 \leq u \wedge u \leq 1$  **by** *blast*  
**from this show** *?thesis*  
**by** (*auto simp add: between-mem-segment closed-segment-def normalize-def*)  
**qed**

### 1.2.4 Lemmas on *radiant-of*

**lemma** *radiant-of*:

$0 \leq \text{radiant-of } p \wedge \text{radiant-of } p < 2 * \pi \wedge \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p$

**proof** –

**let**  $?a = \text{if } 0 \leq p \text{ \$ } 2 \text{ then } \arccos (\text{normalize } p \text{ \$ } 1) \text{ else } \pi + \arccos (- (\text{normalize } p \text{ \$ } 1))$

**have**  $0 \leq ?a \wedge ?a < 2 * \pi \wedge \text{of-radiant } ?a = \text{normalize } p$

**proof** –

**have**  $0 \leq ?a$  **by** *auto*

**moreover have**  $?a < 2 * \pi$

**proof cases**

**assume**  $0 \leq p \text{ \$ } 2$

**from this have**  $?a \leq \pi$  **by** *simp*

**from this show** *?thesis*

**using** *pi-gt-zero* **by** *linarith*

**next**

**assume**  $\neg 0 \leq p \text{ \$ } 2$

**have**  $\arccos (- \text{normalize } p \text{ \$ } 1) < \pi$

**proof** –

**have**  $|\text{normalize } p \text{ \$ } 1| \neq 1$

**using**  $\langle \neg 0 \leq p \text{ \$ } 2 \rangle$  **by** (*simp only: normalize-boundary-iff*)

**from this have**  $\arccos (- \text{normalize } p \text{ \$ } 1) \neq \pi$

**unfolding** *arccos-minus-1[symmetric]* **by** (*subst arccos-eq-iff*) *auto*

**moreover have**  $\arccos (- \text{normalize } p \text{ \$ } 1) \leq \pi$  **by** *simp*

**ultimately show**  $\arccos (- \text{normalize } p \text{ \$ } 1) < \pi$  **by** *linarith*

**qed**

**from this**  $\langle \neg 0 \leq p \text{ \$ } 2 \rangle$  **show** *?thesis* **by** *simp*

**qed**

**moreover have** *of-radiant*  $?a = \text{normalize } p$

**proof** –

**have** *of-radiant*  $?a \text{ \$ } i = \text{normalize } p \text{ \$ } i$  **for**  $i$

**proof** –

**have** *of-radiant*  $?a \text{ \$ } 1 = \text{normalize } p \text{ \$ } 1$

**unfolding** *of-radiant-def* **by** (*simp add: cos-arccos-abs*)

**moreover have** *of-radiant*  $?a \text{ \$ } 2 = \text{normalize } p \text{ \$ } 2$

**proof cases**

**assume**  $0 \leq p \text{ \$ } 2$

**have**  $\sin (\arccos (\text{normalize } p \text{ \$ } 1)) = \text{sqrt } (1 - (\text{normalize } p \text{ \$ } 1) ^ 2)$

**by** (*simp add: sin-arccos-abs*)

**also have**  $\dots = \text{normalize } p \text{ \$ } 2$

**proof** –

**have**  $1 - (\text{normalize } p \text{ \$ } 1) ^ 2 = (\text{normalize } p \text{ \$ } 2) ^ 2$

**using** *normalize-square[of p]* **by** *auto*

**from this**  $\langle 0 \leq p \text{ \$ } 2 \rangle$  **show** *?thesis* **by** (*simp add: nth-normalize-ge-zero-iff*)

**qed**

**finally show** *?thesis*

**using**  $\langle 0 \leq p \text{ \$ } 2 \rangle$  **unfolding** *of-radiant-def* **by** *auto*

**next**

```

    assume  $\neg 0 \leq p \ \$ 2$ 
    have  $-\sin(\arccos(-\text{normalize } p \ \$ 1)) = -\sqrt{1 - (\text{normalize } p \ \$ 1)^2}$ 
    by (simp add: sin-arccos-abs)
    also have  $\dots = \text{normalize } p \ \$ 2$ 
    proof -
      have  $1 - (\text{normalize } p \ \$ 1)^2 = (\text{normalize } p \ \$ 2)^2$ 
      using normalize-square[of p] by auto
      from this  $\langle \neg 0 \leq p \ \$ 2 \rangle$  show ?thesis
      using nth-normalize-ge-zero-iff by fastforce
    qed
    finally show ?thesis
      using  $\langle \neg 0 \leq p \ \$ 2 \rangle$  unfolding of-radiant-def by auto
    qed
    ultimately show ?thesis by (metis exhaust-2[of i])
  qed
  from this show ?thesis by (simp add: vec-eq-iff)
  qed
  ultimately show ?thesis by blast
  qed
  moreover {
    fix  $\omega$ 
    assume  $0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{of-radiant } \omega = \text{normalize } p$ 
    from this have  $0 \leq \omega \wedge \omega < 2 * \pi \wedge \text{normalize } p = \text{of-radiant } \omega$  by auto
    from this have  $\cos \omega = \text{normalize } p \ \$ 1 \wedge \sin \omega = \text{normalize } p \ \$ 2$  by auto
    have  $\omega = ?a$ 
    proof cases
      assume  $0 \leq p \ \$ 2$ 
      from this have  $\omega \leq \pi$ 
      using  $\langle 0 \leq \omega \rangle \langle \omega < 2 * \pi \rangle \langle \sin \omega = \text{normalize } p \ \$ 2 \rangle$ 
      by (simp add: sin-ge-zero-iff[symmetric] nth-normalize-ge-zero-iff)
      from  $\langle 0 \leq \omega \rangle$  this have  $\omega = \arccos(\cos \omega)$  by (simp add: arccos-cos)
      from  $\langle \cos \omega = \text{normalize } p \ \$ 1 \rangle$  this have  $\omega = \arccos(\text{normalize } p \ \$ 1)$ 
      by (simp add: arccos-eq-iff)
      from this show  $\omega = ?a$  using  $\langle 0 \leq p \ \$ 2 \rangle$  by auto
    next
      assume  $\neg 0 \leq p \ \$ 2$ 
      from this have  $\omega > \pi$ 
      using  $\langle 0 \leq \omega \rangle \langle \omega < 2 * \pi \rangle \langle \sin \omega = \text{normalize } p \ \$ 2 \rangle$ 
      by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)
      from this  $\langle \omega < 2 * \pi \rangle$  have  $\omega - \pi = \arccos(\cos(\omega - \pi))$ 
      by (auto simp only: arccos-cos)
      from this  $\langle \cos \omega = \text{normalize } p \ \$ 1 \rangle$  have  $\omega - \pi = \arccos(-\text{normalize } p \ \$ 1)$  by simp
      from this have  $\omega = \pi + \arccos(-\text{normalize } p \ \$ 1)$  by simp
      from this show  $\omega = ?a$  using  $\langle \neg 0 \leq p \ \$ 2 \rangle$  by auto
    qed
  }
  ultimately show ?thesis

```

**unfolding** *radiant-of-def* **by** (rule *theI*)  
**qed**

**lemma** *radiant-of-bounds* [*simp*]:  
 $0 \leq \text{radiant-of } p \text{ radiant-of } p < 2 * \pi$   
**using** *radiant-of* **by** *auto*

**lemma** *radiant-of-weak-ubound* [*simp*]:  
 $\text{radiant-of } p \leq 2 * \pi$   
**using** *radiant-of-bounds(2)[of p]* **by** *linarith*

### 1.2.5 Main Equations for Transforming to Polar Form

**lemma** *polar-form-eq*:  
 $\text{norm } p *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } p) = p$   
**using** *radiant-of normalize-eq* **by** *simp*

**lemma** *relative-polar-form-eq*:  
 $Q + \text{dist } P \ Q *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } (P - Q)) = P$   
**proof** –  
**have**  $\text{norm } (P - Q) *_{\mathbb{R}} \text{of-radiant } (\text{radiant-of } (P - Q)) = P - Q$   
**unfolding** *polar-form-eq* ..  
**moreover** **have**  $\text{dist } P \ Q = \text{norm } (P - Q)$  **by** (*simp add: dist-norm*)  
**ultimately show** *?thesis* **by** (*metis add commute diff-add-cancel*)  
**qed**

### 1.3 Ptolemy's Theorem

**lemma** *dist-circle-segment*:  
**assumes**  $0 \leq \text{radius}$   $0 \leq \alpha$   $\alpha \leq \beta$   $\beta \leq 2 * \pi$   
**shows**  $\text{dist } (\text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \alpha) (\text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \beta)$   
 $= 2 * \text{radius} * \sin ((\beta - \alpha) / 2)$   
(is *?lhs = ?rhs*)  
**proof** –  
**have** *trigonometry*:  $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = (2 * \sin ((\beta - \alpha) / 2))^2$   
**proof** –  
**have** *sin-diff-minus*:  $\sin ((\alpha - \beta) / 2) = - \sin ((\beta - \alpha) / 2)$   
**by** (*simp only: sin-minus[symmetric] minus-divide-left minus-diff-eq*)  
**have**  $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 =$   
 $(2 * \sin ((\alpha + \beta) / 2) * \sin ((\beta - \alpha) / 2))^2 + (2 * \sin ((\alpha - \beta) / 2) * \cos$   
 $((\alpha + \beta) / 2))^2$   
**by** (*simp only: cos-diff-cos sin-diff-sin*)  
**also have**  $\dots = (2 * \sin ((\beta - \alpha) / 2))^2 * ((\sin ((\alpha + \beta) / 2))^2 + (\cos ((\alpha$   
 $+ \beta) / 2))^2)$   
**unfolding** *sin-diff-minus* **by** *algebra*  
**also have**  $\dots = (2 * \sin ((\beta - \alpha) / 2))^2$  **by** *simp*  
**finally show** *?thesis* .  
**qed**  
**from** *assms* **have**  $0 \leq \sin ((\beta - \alpha) / 2)$  **by** (*simp add: sin-ge-zero*)



**have**  $?lhs = \text{sqrt}(\text{radius}^2 * ((\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2))$   
**unfolding** *dist-eq-on-real-2-vec* **by** *simp algebra*  
**also have**  $\dots = \text{sqrt}(\text{radius}^2 * (2 * \sin((\beta - \alpha) / 2))^2)$  **by** (*simp add: trigonometry*)  
**also have**  $\dots = ?rhs$   
**using**  $\langle 0 \leq \text{radius} \rangle \langle 0 \leq \sin((\beta - \alpha) / 2) \rangle$  **by** (*simp add: real-sqrt-mult*)  
**finally show** *?thesis* .  
**qed**

**theorem** *ptolemy-trigonometric:*

**fixes**  $\omega_1 \omega_2 \omega_3 :: \text{real}$   
**shows**  $\sin(\omega_1 + \omega_2) * \sin(\omega_2 + \omega_3) = \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin(\omega_1 + \omega_3)$   
**proof** –  
**have**  $\sin(\omega_1 + \omega_2) * \sin(\omega_2 + \omega_3) = ((\sin \omega_2)^2 + (\cos \omega_2)^2) * \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin(\omega_1 + \omega_2 + \omega_3)$   
**by** (*simp only: sin-add cos-add*) *algebra*  
**also have**  $\dots = \sin \omega_1 * \sin \omega_3 + \sin \omega_2 * \sin(\omega_1 + \omega_2 + \omega_3)$  **by** *simp*  
**finally show** *?thesis* .  
**qed**

**theorem** *ptolemy:*

**fixes**  $A B C D \text{ center} :: \text{real}^2$   
**assumes**  $\text{dist center } A = \text{radius}$  **and**  $\text{dist center } B = \text{radius}$   
**assumes**  $\text{dist center } C = \text{radius}$  **and**  $\text{dist center } D = \text{radius}$   
**assumes** *ordering-of-points:*  
 $\text{radiant-of}(A - \text{center}) \leq \text{radiant-of}(B - \text{center})$   
 $\text{radiant-of}(B - \text{center}) \leq \text{radiant-of}(C - \text{center})$   
 $\text{radiant-of}(C - \text{center}) \leq \text{radiant-of}(D - \text{center})$   
**shows**  $\text{dist } A C * \text{dist } B D = \text{dist } A B * \text{dist } C D + \text{dist } A D * \text{dist } B C$   
**proof** –  
**from**  $\langle \text{dist center } A = \text{radius} \rangle$  **have**  $0 \leq \text{radius}$  **by** *auto*  
**define**  $\alpha \beta \gamma \delta$   
**where**  $\alpha = \text{radiant-of}(A - \text{center})$  **and**  $\beta = \text{radiant-of}(B - \text{center})$   
**and**  $\gamma = \text{radiant-of}(C - \text{center})$  **and**  $\delta = \text{radiant-of}(D - \text{center})$   
**from** *ordering-of-points* **have** *angle-basics:*  
 $\alpha \leq \beta \beta \leq \gamma \gamma \leq \delta$   
 $0 \leq \alpha \alpha \leq 2 * \text{pi} \ 0 \leq \beta \beta \leq 2 * \text{pi}$   
 $0 \leq \gamma \gamma \leq 2 * \text{pi} \ 0 \leq \delta \delta \leq 2 * \text{pi}$   
**unfolding**  $\alpha$ -def  $\beta$ -def  $\gamma$ -def  $\delta$ -def **by** *auto*  
**from** *assms(1-4)* **have**  
 $A = \text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \alpha \ B = \text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \beta$   
 $C = \text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \gamma \ D = \text{center} + \text{radius} *_{\mathbb{R}} \text{of-radiant } \delta$   
**unfolding**  $\alpha$ -def  $\beta$ -def  $\gamma$ -def  $\delta$ -def  
**using** *relative-polar-form-eq dist-commute* **by** *metis+*

**from this have** *dist-eqs:*

$$\text{dist } A C = 2 * \text{radius} * \sin((\gamma - \alpha) / 2)$$

$$\text{dist } B D = 2 * \text{radius} * \sin((\delta - \beta) / 2)$$

```

dist A B = 2 * radius * sin ((β - α) / 2)
dist C D = 2 * radius * sin ((δ - γ) / 2)
dist A D = 2 * radius * sin ((δ - α) / 2)
dist B C = 2 * radius * sin ((γ - β) / 2)
using angle-basics ⟨radius ≥ 0⟩ dist-circle-segment by (auto)

have dist A C * dist B D = 4 * radius ^ 2 * sin ((γ - α) / 2) * sin ((δ - β) / 2)
unfolding dist-eqs by (simp add: power2-eq-square)
also have ... = 4 * radius ^ 2 * (sin ((β - α) / 2) * sin ((δ - γ) / 2) + sin ((γ - β) / 2) * sin ((δ - α) / 2))
proof -
  define ω1 ω2 ω3 where ω1 = (β - α) / 2 and ω2 = (γ - β) / 2 and ω3 = (δ - γ) / 2
  have (γ - α) / 2 = ω1 + ω2 and (δ - β) / 2 = ω2 + ω3 and (δ - α) / 2 = ω1 + ω2 + ω3
  unfolding ω1-def ω2-def ω3-def by (auto simp add: field-simps)
  have sin ((γ - α) / 2) * sin ((δ - β) / 2) = sin (ω1 + ω2) * sin (ω2 + ω3)
  using ⟨(γ - α) / 2 = ω1 + ω2⟩ ⟨(δ - β) / 2 = ω2 + ω3⟩ by (simp only:)
  also have ... = sin ω1 * sin ω3 + sin ω2 * sin (ω1 + ω2 + ω3)
  by (rule ptolemy-trigonometric)
  also have ... = (sin ((β - α) / 2) * sin ((δ - γ) / 2) + sin ((γ - β) / 2) * sin ((δ - α) / 2))
  using ω1-def ω2-def ω3-def ⟨(δ - α) / 2 = ω1 + ω2 + ω3⟩ by (simp only:)
  finally show ?thesis by simp
qed
also have ... = dist A B * dist C D + dist A D * dist B C
unfolding dist-eqs by (simp add: distrib-left power2-eq-square)
finally show ?thesis .
qed

end

```

## References

- [1] J. Harrison. Ptolemy's theorem. <https://github.com/jrh13/hol-light/blob/master/100/ptolemy.ml>.
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- [3] Wikipedia. Ptolemy's theorem — wikipedia, the free encyclopedia, 2016. [https://en.wikipedia.org/w/index.php?title=Ptolemy%27s\\_theorem&oldid=727017817](https://en.wikipedia.org/w/index.php?title=Ptolemy%27s_theorem&oldid=727017817) [Online; accessed 6-August-2016].