Abstract
This entry provides an analytic proof to Ptolemy’s Theorem using polar form transformation and trigonometric identities. In this formalization, we use ideas from John Harrison’s HOL Light formalization [1] and the proof sketch on the Wikipedia entry of Ptolemy’s Theorem [3]. This theorem is the 95th theorem of the Top 100 Theorems list [2].

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1.3 Ptolemy’s Theorem

1 Ptolemy’s Theorem

theory Ptolemy’s-Theorem
imports
  HOL-Analysis.Multivariate-Analysis
begin

1.1 Preliminaries

1.1.1 Additions to Rat theory

hide-const (open) normalize
1.1.2 Additions to Transcendental theory

Lemmas about \( \text{arcsin} \) and \( \text{arccos} \) commonly involve to show that their argument is in the domain of those partial functions, i.e., the argument \( y \) is between \(-1\) and \(1\). As the argumentation for \(-1 \leq y \) and \(y \leq 1\) is often very similar, we prefer to prove \( |y| \leq 1 \) to the two goals above.
The lemma for rewriting the term \( \cos (\text{arccos} \ y) \) is already provided in the Isabelle distribution with name \( \text{cos-arccos-abs} \). Here, we further provide the analogue on \( \text{arcsin} \) for rewriting \( \sin (\text{arcsin} \ y) \).

**lemma** sin-arcsin-abs:
\[
|y| \leq 1 =\Rightarrow \sin (\text{arcsin} \ y) = y
\]
 by \( (\text{simp add: abs-le-iff}) \)

The further lemmas are the required variants from existing lemmas \( \text{arccos-lbound} \) and \( \text{arccos-ubound} \).

**lemma** arccos-lbound-abs [simp]:
\[
|y| \leq 1 =\Rightarrow 0 \leq \arccos y
\]
 by \( (\text{simp add: arccos-lbound}) \)

**lemma** arccos-ubound-abs [simp]:
\[
|y| \leq 1 =\Rightarrow \arccos y \leq \pi
\]
 by \( (\text{simp add: arccos-ubound}) \)

As we choose angles to be between \(0\) between \(2\pi\), we need some lemmas to reason about the sign of \( \sin x \) for angles \( x \).

**lemma** sin-ge-zero-iff:
\[
\text{assumes } 0 \leq x < 2 \ast \pi \text{ shows } 0 \leq \sin x =\iff x \leq \pi
\]
 proof
 assume \(0 \leq \sin x\)
 show \(x \leq \pi\)
 proof (rule ccontr)
 assume \(\neg x \leq \pi\)
 from this \(\langle x < 2 \ast \pi \rangle\) have \(\sin x < 0\)
 using sin-lt-zero by auto
 from this \(\langle 0 \leq \sin x \rangle\) show False by auto
 qed
 next
 assume \(x \leq \pi\)
 from this \(\langle 0 \leq x \rangle\) show \(0 \leq \sin x\) by \( (\text{simp add: sin-ge-zero}) \)
 qed

**lemma** sin-less-zero-iff:
\[
\text{assumes } 0 \leq x < 2 \ast \pi \text{ shows } \sin x < 0 =\iff \pi < x
\]
 using assms sin-ge-zero-iff by fastforce
1.1.3 Addition to Finite-Cartesian-Product theory

Here follow generally useful additions and specialised equations for two-dimensional real-valued vectors.

**Lemma** \( axis\text{-}nth\text{-eq}\text{-}0 \) [simp]:

- **Assumes** \( i \neq j \)
- **Shows** \( axis\_i\ x \ \&\ \&\ \&\ j = 0 \)

**Using** \text{assms unfolding axis-def by simp}

**Lemma** \( norm\text{-}axis \):

- **Fixes** \( x :: \text{real} \)
- **Shows** \( norm\ (axis\_i\ x) = abs\ x \)

**By** \( \text{simp add: norm-eq-sqrt-inner inner-axis-axis} \)

**Lemma** \( norm\text{-eq\text{-}on\text{-}real\text{-}2\text{-}vec} \):

- **Fixes** \( x :: \text{real} \ ^\ast\ 2 \)
- **Shows** \( norm\ x = \sqrt{((x \ \&\ 1) \ ^\ast\ 2 + (x \ \&\ 2) \ ^\ast\ 2)} \)

**By** \( \text{simp add: norm-eq-sqrt-inner inner-vec-def UNIV\text{-}2 power2\text{-eq-square}} \)

**Lemma** \( dist\text{-eq\text{-}on\text{-}real\text{-}2\text{-}vec} \):

- **Fixes** \( a\ b :: \text{real} \ ^\ast\ 2 \)
- **Shows** \( dist\ a\ b = \sqrt{((a \ \&\ 1 - b \ \&\ 1) \ ^\ast\ 2 + (a \ \&\ 2 - b \ \&\ 2) \ ^\ast\ 2)} \)

**Unfolding** \( dist\text{-norm norm\text{-eq\text{-}on\text{-}real\text{-}2\text{-}vec} by simp \)

1.2 Polar Form of Two-Dimensional Real-Valued Vectors

1.2.1 Definitions to Transfer to Polar Form and Back

**Definition** \( of\text{-}radiant :: real \Rightarrow real \ ^\ast\ 2 \)

**Where**

\( of\text{-}radiant\ \omega = axis\ 1\ (cos\ \omega) + axis\ 2\ (sin\ \omega) \)

**Definition** \( normalize :: real \ ^\ast\ 2 \Rightarrow real \ ^\ast\ 2 \)

**Where**

\( normalize\ p = (if\ p = 0\ then\ axis\ 1\ 1\ else\ (1 / norm\ p) \ast_R p) \)

**Definition** \( radiant\text{-}of :: real \ ^\ast\ 2 \Rightarrow real \)

**Where**

\( radiant\text{-}of\ p = (THE\ \omega.\ 0 \leq \omega \land \omega < 2 \ast pi \land of\text{-}radiant\ \omega = normalize\ p) \)

The vector \( of\text{-}radiant\ \omega \) is the vector with length 1 and angle \( \omega \) to the first axis. We normalize vectors to length 1 keeping their orientation with the normalize function. Conversely, \( radiant\text{-}of\ p \) is the angle of vector \( p \) to the first axis, where we choose \( radiant\text{-}of \) to return angles between 0 and \( 2 \ast pi \), following the usual high-school convention. With these definitions, we can express the main result \( norm\ p \ast_R of\text{-}radiant\ (radiant\text{-}of\ p) = p \). Note that the main result holds for any definition of \( radiant\text{-}of \) 0. So, we choose to define \( normalize\ 0 \) and \( radiant\text{-}of\ 0 \), such that \( radiant\text{-}of\ 0 = 0 \).
1.2.2 Lemmas on of-radiant

lemma nth-of-radiant-1 [simp]:
  of-radiant $\omega \$ 1 = \cos \omega
unfolding of-radiant-def by simp

lemma nth-of-radiant-2 [simp]:
  of-radiant $\omega \$ 2 = \sin \omega
unfolding of-radiant-def by simp

lemma norm-of-radiant:
  \( \|\text{of-radiant } \omega \| = 1 \)
unfolding of-radiant-def norm-eq-on-real-2-vec by simp

lemma of-radiant-plus-2pi:
  of-radiant ($\omega + 2 \pi$) = of-radiant $\omega$
unfolding of-radiant-def by simp

lemma of-radiant-minus-2pi:
  of-radiant ($\omega - 2 \pi$) = of-radiant $\omega$
proof
  have of-radiant ($\omega - 2 \pi$) = of-radiant ($\omega - 2 \pi + 2 \pi$)
    by (simp only: of-radiant-plus-2pi)
  also have \ldots = of-radiant $\omega$ by simp
finally show \$thesis .
qed

1.2.3 Lemmas on normalize

lemma normalize-eq:
  \( \|\text{normalize } p \| R \text{ normalize } p = p \)
unfolding normalize-def by simp

lemma norm-normalize:
  \( \|\text{normalize } p \| = 1 \)
unfolding normalize-def by (auto simp add: norm-axis)

lemma nth-normalize [simp]:
  \( \|\text{normalize } p \$ i \| \leq 1 \)
using norm-normalize component-le-norm-cart by metis

lemma normalize-square:
  \( (\text{normalize } p \$ 1)^2 + (\text{normalize } p \$ 2)^2 = 1 \)
using dot-square-norm[of normalize p]
by (simp add: inner-vec-def UNIV-2 power2-eq-square norm-normalize)

lemma nth-normalize-ge-zero-iff:
  \( 0 \leq \text{normalize } p \$ i \iff 0 \leq p \$ i \)
proof
  assume \( 0 \leq \text{normalize } p \$ i \)
from this show $0 \leq p \is i$

unfolding normalize-def by (auto split: if-split-asm simp add: zero-le-divide-iff)

next

assume $0 \leq p \is i$

have $0 \leq \text{axis 1} \is (1 :: \text{real}) \is i$

using exhaust-2[of i] by auto

from this ::$0 \leq p \is i$ show $0 \leq \text{normalize} p \is i$

unfolding normalize-def by auto

qed

lemma nth-normalize-less-zero-iff:

normalize $p \is i < 0 \iff p \is i < 0$

using nth-normalize-ge-zero-iff leD leI by metis

lemma normalize-boundary-iff:

$|\text{normalize} p \is 1| = 1 \iff p \is 2 = 0$

proof

assume $|\text{normalize} p \is 1| = 1$

from this have $1: (p \is 1) \is ^2 = \text{norm} p \is ^2$

unfolding normalize-def by (auto split: if-split-asm simp add: power2-eq-iff)

moreover have $(p \is 1) \is ^2 + (p \is 2) \is ^2 = \text{norm} p \is ^2$

using norm-eq-on-real-2-vec by auto

ultimately show $p \is 2 = 0$ by simp

next

assume $p \is 2 = 0$

from this have $|p \is 1| = \text{norm} p$

by (auto simp add: norm-eq-on-real-2-vec)

from this show $|\text{normalize} p \is 1| = 1$

unfolding normalize-def by simp

qed

lemma between-normalize-if-distant-from-0:

assumes $\text{norm} p \geq 1$

shows between $(0, p)$ (normalize $p$)

using assms by (auto simp add: between-mem-segment closed-segment-def normalize-def)

lemma between-normalize-if-near-0:

assumes $\text{norm} p \leq 1$

shows between $(0, \text{normalize} p)$

proof

have $0 \leq \text{norm} p$ by simp

from assms have $p = (\text{norm} p / \text{norm} p) \is_R p \land 0 \leq \text{norm} p \land \text{norm} p \leq 1$

by auto

from this have $\exists u. p = (u / \text{norm} p) \is_R p \land 0 \leq u \land u \leq 1$ by blast

from this show ?thesis

by (auto simp add: between-mem-segment closed-segment-def normalize-def)

qed
1.2.4 Lemmas on radiant-of

**Lemma radiant-of:**

\[ 0 \leq \text{radiant-of } p \land \text{radiant-of } p < 2 \ast \pi \land \text{of-radiant } (\text{radiant-of } p) = \text{normalize } p \]

**Proof:**

\[ \text{let } ?a = \text{if } 0 \leq p \mid 2 \text{ then } \arccos (\text{normalize } p \mid 1) \text{ else } \arccos (- (\text{normalize } p \mid 1)) \]

\[ \text{have } 0 \leq ?a \land ?a < 2 \ast \pi \land \text{of-radiant } ?a = \text{normalize } p \]

**Proof:**

\[ \text{have } 0 \leq ?a \text{ by auto} \]

**Moreover have** \(?a < 2 \ast \pi\)

**Proof**

- **Cases**
  - **Assume** \(0 \leq p \mid 2\)
    - **From this** \(?a \leq \pi\) by simp
    - **From this** show \(?\text{thesis}\)
      - **Using** \(\pi\text{-gt-zero}\) by linarith

**Next**

- **Assume** \(\neg 0 \leq p \mid 2\)
  - **Have** \(\arccos (- (\text{normalize } p \mid 1)) < \pi\)
    - **Proof**
      - **Have** \(|\text{normalize } p \mid 1| \neq 1\)
        - **Using** \((\neg 0 \leq p \mid 2)\) by (simp only: normalize-boundary-iff)
      - **From this** \(\arccos (- (\text{normalize } p \mid 1)) \neq \pi\)
        - **Unfolding** \(\arccos\text{-minus-1}\) by (subst \(\arccos\text{-eq-iff}\) auto)
      - **Moreover have** \(\arccos (- (\text{normalize } p \mid 1)) \leq \pi\) by simp
      - **Ultimately show** \(\arccos (- (\text{normalize } p \mid 1)) < \pi\) by linarith

**QED**

- **From this** \((\neg 0 \leq p \mid 2);\) **show** \(?\text{thesis}\) by simp

**QED**

**Moreover have** \(\text{of-radiant } ?a = \text{normalize } p\)

**Proof**

- **Have** \(\text{of-radiant } ?a \mid i = \text{normalize } p \mid i\) for \(i\)
  - **Proof**
    - **Have** \(\text{of-radiant } ?a \mid 1 = \text{normalize } p \mid 1\)
      - **Unfolding** \(\text{of-radiant-def}\) by (simp add: \(\cos\text{-arccos-abs}\))
    - **Moreover have** \(\text{of-radiant } ?a \mid 2 = \text{normalize } p \mid 2\)
  - **Proof**
    - **Assume** \(0 \leq p \mid 2\)
      - **Have** \(\sin (\arccos (\text{normalize } p \mid 1)) = \sqrt{1 - (\text{normalize } p \mid 1)^2}\)
        - **By** (simp add: \(\sin\text{-arccos-approx}\))
      - **Also have** \dots = \(\text{normalize } p \mid 2\)
    - **Proof**
      - **Have** \(1 - (\text{normalize } p \mid 1)^2 = (\text{normalize } p \mid 2)^2\)
        - **Using** \(\text{normalize-square[of p]}\) by auto
      - **From this** \(0 \leq p \mid 2;\) **show** \(?\text{thesis}\) by (simp add: \(\text{n-th-normalize-ge-zero-iff}\))
    - **QED**
  - **Using** \(0 \leq p \mid 2;\) **unfolding** \(\text{of-radiant-def}\) by auto

**Next**
assume \( \neg 0 \leq p \leq 2 \)

have \(\sin (\arccos (- \text{normalize } p)) = - \sqrt{1 - (\text{normalize } p)^2}\)

by (simp add: sin-arccos-abs)
also have \(\ldots = \text{normalize } p\)

proof
have \(1 - (\text{normalize } p)^2 = (\text{normalize } p)^2\)
using normalize-square[of p] by auto
from this \((\neg 0 \leq p \leq 2)\) show \(?\)thesis
using nth-normalize-ge-zero-iff by fastforce
qed

finally show \(?\)thesis
using \((\neg 0 \leq p \leq 2)\) unfolding of-radiant-def by auto

qed

ultimately show \(?\)thesis by blast

qed

moreover {
fix \(\omega\)
assume \(0 \leq \omega < 2 \pi \land \omega < 2 \pi \land \text{normalize } \omega = \text{normalize } p\)
from this have \(0 \leq \omega \leq 2 \pi \land \text{normalize } p = \text{of-radiant } \omega \) by auto
from this have \(\cos \omega = \text{normalize } p \land \sin \omega = \text{normalize } p\)

have \(\omega = ?a\)

proof cases
assumption \(0 \leq \omega\)

using \(\langle 0 \leq \omega \rangle \omega < 2 \pi\) (\(\sin \omega = \text{normalize } p\))

by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)

from this \(\omega \leq \pi\) this have \(\omega = \arccos (\cos \omega)\) by (simp add: arccos-cos)
from \(\cos \omega = \text{normalize } p\) this have \(\omega = \arccos (\text{normalize } p)\)

by (simp add: arccos-eq-iff)

from this show \(\omega = ?a\) using \(0 \leq p \leq 2\) by auto

next
assume \(\neg 0 \leq p \leq 2\)
from this have \(\omega > \pi\)

using \(\langle 0 \leq \omega \rangle \omega < 2 \pi^2\) (\(\sin \omega = \text{normalize } p\))

by (simp add: sin-less-zero-iff[symmetric] nth-normalize-less-zero-iff)

from this \(\omega < 2 \pi\) this have \(\omega - \pi = \arccos (\cos (\omega - \pi))\)

by (auto simp only: arccos-cos)

from this \(\cos \omega = \text{normalize } p\) this have \(\omega - \pi = \arccos (\text{normalize } p)\)

\(\land \omega = ?a\) using \(\neg 0 \leq p \leq 2\) by auto

qed

} ultimately show \(?\)thesis
unfolding radiant-of-def by (rule theI)

qed

lemma radiant-of-bounds [simp]:
\[ 0 \leq \text{radiant-of } p < 2 \times \pi \]
using radiant-of by auto

lemma radiant-of-weak-ubound [simp]:
\[ \text{radiant-of } p \leq 2 \times \pi \]
using radiant-of-bounds(2)[of p] by linarith

1.2.5 Main Equations for Transforming to Polar Form

lemma polar-form-eq:
\[ \| p \| \times R \text{of-radiant } (\text{radiant-of } p) = p \]
using radiant-of normalize-eq by simp

lemma relative-polar-form-eq:
\[ Q + \text{dist } P Q \times R \text{of-radiant } (\text{radiant-of } (P - Q)) = P \]
proof
  have \[ \| (P - Q) \| \times R \text{of-radiant } (\text{radiant-of } (P - Q)) = P - Q \]
  unfolding polar-form-eq ..
  moreover have \[ \text{dist } P Q = \| (P - Q) \| \]
  by (simp add: dist-norm)
  ultimately show \[ \text{thesis} \]
  by (metis add.commute diff-add-cancel)
qed

1.3 Ptolemy’s Theorem

lemma dist-circle-segment:
\[ \text{assumes } 0 \leq \text{radius } 0 \leq \alpha \leq \beta \leq 2 \times \pi \]
\[ \text{shows } \text{dist } (\text{center } + \text{radius } \times R \text{of-radiant } \alpha) (\text{center } + \text{radius } \times R \text{of-radiant } \beta) \]
\[ = 2 \times \text{radius } \times \sin \left( \frac{(\beta - \alpha)}{2} \right) \]
(is \[ \text{lhs} = \text{rhs} \])
proof
  have \[ \text{trigonometry: } (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = (2 \times \sin \left( \frac{(\beta - \alpha)}{2} \right))^2 \]
  proof
    have \[ \text{sin-diff-minus: } \sin \left( \frac{(\alpha - \beta)}{2} \right) = - \sin \left( \frac{(\beta - \alpha)}{2} \right) \]
    by (simp only: sin-minus[symmetric] minus-divide-left minus-diff-eq)
    have \[ (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = \]
    \[ (2 \times \sin \left( \frac{(\alpha + \beta)}{2} \right) \times \sin \left( \frac{(\beta - \alpha)}{2} \right))^2 + (2 \times \sin \left( \frac{(\alpha - \beta)}{2} \right) \times \cos \]
    \[ \left( \frac{(\alpha + \beta)}{2} \right))^2 \]
    by (simp only: cos-diff-cos sin-diff-sin)
    also have \[ \ldots = (2 \times \sin \left( \frac{(\beta - \alpha)}{2} \right))^2 \times (\sin \left( \frac{(\alpha + \beta)}{2} \right)^2 + (\cos \left( \frac{(\alpha + \beta)}{2} \right))^2) \]
    unfolding sin-diff-minus by algebra
    also have \[ \ldots = (2 \times \sin \left( \frac{(\beta - \alpha)}{2} \right))^2 \]
    by simp
    finally show \[ \text{thesis} \]
  qed
from \[ \text{assms} \]
\[ 0 \leq \sin \left( \frac{(\beta - \alpha)}{2} \right) \]
by (simp add: sin-ge-zero)


have \( ?\text{rhs} = \sqrt{\text{radius}^2 * ((\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2)} \)

unfolding dist-eq-on-real-2-vec by simp algebra
also have \( \ldots = \sqrt{\text{radius}^2 * (2 * \sin ((\beta - \alpha) / 2))^2} \) by (simp add: trigonometry)
also have \( \ldots = ?\text{rhs} \)
using \( 0 \leq \text{radius} \) \( 0 \leq \sin ((\beta - \alpha) / 2) \) by (simp add: real-sqrt-mult)
finally show \( ?\text{thesis} \).
qed

theorem ptoley-trigonometric:
fixes \( \omega_1 \omega_2 \omega_3 :: \text{real} \)
shows \( \sin (\omega_1 + \omega_2) * \sin (\omega_2 + \omega_3) = \sin \omega_1 \sin \omega_3 + \sin \omega_2 \sin (\omega_1 + \omega_2 + \omega_3) \)
proof –
have \( \sin (\omega_1 + \omega_2) * \sin (\omega_2 + \omega_3) = ((\sin \omega_2)^2 + (\cos \omega_2)^2) * \sin \omega_1 \sin \omega_3 + \sin \omega_2 \sin (\omega_1 + \omega_2 + \omega_3) \)
by (simp only: sin-add cos-add) algebra
also have \( \ldots = \sin \omega_1 \sin \omega_3 + \sin \omega_2 \sin (\omega_1 + \omega_2 + \omega_3) \) by simp
finally show \( ?\text{thesis} \).
qed

theorem ptoley:
fixes \( A B C D \) center :: \( \text{real} * 2 \)
assumes \( \text{dist center} A = \text{radius} \) and \( \text{dist center} B = \text{radius} \)
assumes \( \text{dist center} C = \text{radius} \) and \( \text{dist center} D = \text{radius} \)
assumes ordering-of-points:
\( \text{radiant-of} (A - \text{center}) \leq \text{radiant-of} (B - \text{center}) \)
\( \text{radiant-of} (B - \text{center}) \leq \text{radiant-of} (C - \text{center}) \)
\( \text{radiant-of} (C - \text{center}) \leq \text{radiant-of} (D - \text{center}) \)
shows \( \text{dist} A C * \text{dist} B D = \text{dist} A B * \text{dist} C D + \text{dist} A D * \text{dist} B C \)
proof –
from \( \text{dist center} A = \text{radius} \) have \( 0 \leq \text{radius} \) by auto
define \( \alpha \beta \gamma \delta \)
where \( \alpha = \text{radiant-of} (A - \text{center}) \) and \( \beta = \text{radiant-of} (B - \text{center}) \)
and \( \gamma = \text{radiant-of} (C - \text{center}) \) and \( \delta = \text{radiant-of} (D - \text{center}) \)
from ordering-of-points have angle-basics:
\( \alpha \leq \beta \beta \leq \gamma \gamma \leq \delta \)
\( 0 \leq \alpha \alpha \leq 2 * \pi 0 \leq \beta \beta \leq 2 * \pi \)
\( 0 \leq \gamma \gamma \leq 2 * \pi 0 \leq \delta \delta \leq 2 * \pi \)
unfolding \( \alpha \text{-def} \beta \text{-def} \gamma \text{-def} \delta \text{-def} \) by auto
from assms(1-4) have
\( A = \text{center} + \text{radius} * \text{R of-radiant} \alpha B = \text{center} + \text{radius} * \text{R of-radiant} \beta \)
\( C = \text{center} + \text{radius} * \text{R of-radiant} \gamma D = \text{center} + \text{radius} * \text{R of-radiant} \delta \)
unfolding \( \alpha \text{-def} \beta \text{-def} \gamma \text{-def} \delta \text{-def} \)
using relative-polar-form-eq dist-commute by metis+

from this have dist-eqs:
\( \text{dist} A C = 2 * \text{radius} * \sin ((\gamma - \alpha) / 2) \)
\( \text{dist} B D = 2 * \text{radius} * \sin ((\delta - \beta) / 2) \)
dist $A\ B = 2 \cdot \text{radius} \cdot \sin ((\beta - \alpha) / 2)$

dist $C\ D = 2 \cdot \text{radius} \cdot \sin ((\delta - \gamma) / 2)$

dist $A\ D = 2 \cdot \text{radius} \cdot \sin ((\delta - \alpha) / 2)$

dist $B\ C = 2 \cdot \text{radius} \cdot \sin ((\gamma - \beta) / 2)$

using angle-basics : (radius $\geq 0$) dist-circle-segment by (auto)

have dist $A\ C \ast$ dist $B\ D = 4 \cdot \text{radius} \cdot 2 \cdot \sin ((\gamma - \alpha) / 2) \cdot \sin ((\delta - \beta) / 2)$

unfolding dist-eqs by (simp add: power2-eq-square)

also have $\ldots = 4 \cdot \text{radius} \cdot 2 \cdot (\sin ((\beta - \alpha) / 2) \cdot \sin ((\delta - \gamma) / 2) + \sin ((\gamma - \beta) / 2) \cdot \sin ((\delta - \alpha) / 2))$

proof -

define $\omega_1\ \omega_2\ \omega_3$ where $\omega_1 = (\beta - \alpha) / 2$ and $\omega_2 = (\gamma - \beta) / 2$ and $\omega_3 = (\delta - \gamma) / 2$

have $(\gamma - \alpha) / 2 = \omega_1 + \omega_2$ and $(\delta - \beta) / 2 = \omega_2 + \omega_3$ and $(\delta - \alpha) / 2 = \omega_1 + \omega_2 + \omega_3$

unfolding $\omega_1$-def $\omega_2$-def $\omega_3$-def by (auto simp add: field-simps)

have $\sin ((\gamma - \alpha) / 2) \cdot \sin ((\delta - \beta) / 2) = \sin (\omega_1 + \omega_2) \cdot \sin (\omega_2 + \omega_3)$

using $(\gamma - \alpha) / 2 = \omega_1 + \omega_2$ and $(\delta - \beta) / 2 = \omega_2 + \omega_3$ by (simp only:)

also have $\ldots = \sin (\omega_1) \cdot \sin (\omega_2) \cdot \sin (\omega_3) \cdot \sin (\omega_1 + \omega_2 + \omega_3)$

by (rule ptolemy-trigonometric)

also have $\ldots = (\sin ((\beta - \alpha) / 2) \cdot \sin ((\delta - \gamma) / 2) + \sin ((\gamma - \beta) / 2) \cdot \sin ((\delta - \alpha) / 2))$

unfolding $\omega_1$-def $\omega_2$-def $\omega_3$-def by (simp only:)

finally show ?thesis by simp

qed

also have $\ldots = \text{dist } A\ B \ast \text{dist } C\ D \ast \text{dist } A\ D \ast \text{dist } B\ C$

unfolding dist-eqs by (simp add: distrib-left power2-eq-square)

finally show ?thesis .

qed

end

References


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