# Projective Geometry

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**Abstract**

We formalize the basics of projective geometry.
In particular, we give a proof of the so-called Hessenberg's theorem in projective plane geometry (see [1] for an alternative proof using a Coherent Logic prover in Prolog which generates Coq proof scripts).
We also provide a proof of the so-called Desargues's theorem based on an axiomatization [2] of (higher) projective space geometry using the notion of rank of a matroid. This last approach allows to handle incidence relations in an homogeneous way dealing only with points and without the need of talking explicitly about lines, planes or any higher entity.

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**theory** *Projective-Plane-Axioms*
**imports** *Main*
**begin**

Contents:

- We introduce the types of points and lines and an incidence relation between them.
- A set of axioms for the projective plane (the models of these axioms are n-dimensional with $n \geq 2$).

# 1 The Axioms of the Projective Plane

**locale** *projective-plane* =

**fixes** *incid* :: *'point* ⇒ *'line* ⇒ *bool*

**assumes** *ax1*: ∃ *l*. *incid P l* ∧ *incid Q l*

**assumes** *ax2*: ∃ *P*. *incid P l* ∧ *incid P m*

**assumes** *ax-uniqueness*: ⟦*incid P l*; *incid Q l*; *incid P m*; *incid Q m*⟧ ⟹ *P* = *Q* ∨ *l* = *m*

**assumes** *ax3*: ∃ *A B C D*. *distinct* [*A*,*B*,*C*,*D*] ∧ (∀ *l*.
(*incid A l* ∧ *incid B l* ⟶ ¬(*incid C l*) ∧ ¬(*incid D l*)) ∧
(*incid A l* ∧ *incid C l* ⟶ ¬(*incid B l*) ∧ ¬(*incid D l*)) ∧
(*incid A l* ∧ *incid D l* ⟶ ¬(*incid B l*) ∧ ¬(*incid C l*)) ∧
(*incid B l* ∧ *incid C l* ⟶ ¬(*incid A l*) ∧ ¬(*incid D l*)) ∧
(*incid B l* ∧ *incid D l* ⟶ ¬(*incid A l*) ∧ ¬(*incid C l*)) ∧
(*incid C l* ∧ *incid D l* ⟶ ¬(*incid A l*) ∧ ¬(*incid B l*)))

**end**
**theory** *Pappus-Property*
**imports** *Main Projective-Plane-Axioms*
**begin**

Contents:

- We give two formulations of Pappus's property for a configuration of nine points *is-pappus1 is-pappus2*.
- We prove the equivalence of these two formulations *pappus-equiv*.
- We state Pappus property for a plane *is-pappus*.

## 2 Pappus's Property

**context** *projective-plane*
**begin**

**definition** *col* :: ['*point*, '*point*, '*point*] ⇒ *bool* **where**
*col A B C* ≡ ∃ *l*. *incid A l* ∧ *incid B l* ∧ *incid C l*

**lemma** *distinct6-def*:
*distinct* [*A*,*B*,*C*,*D*,*E*,*F*] ≡ (*A* ≠ *B*) ∧ (*A* ≠ *C*) ∧ (*A* ≠ *D*) ∧ (*A* ≠ *E*) ∧ (*A* ≠ *F*)
∧
(*B* ≠ *C*) ∧ (*B* ≠ *D*) ∧ (*B* ≠ *E*) ∧ (*B* ≠ *F*) ∧
(*C* ≠ *D*) ∧ (*C* ≠ *E*) ∧ (*C* ≠ *F*) ∧
(*D* ≠ *E*) ∧ (*D* ≠ *F*) ∧
(*E* ≠ *F*)
**by** *auto*

**definition** *lines* :: '*point* ⇒ '*point* ⇒ '*line set* **where**
*lines P Q* ≡ {*l*. *incid P l* ∧ *incid Q l*}

**lemma** *uniq-line*:
**assumes** *P* ≠ *Q* **and** *l* ∈ *lines P Q* **and** *m* ∈ *lines P Q*
**shows** *l* = *m*
**using** *assms lines-def ax-uniqueness*
**by** *blast*

**definition** *line* :: '*point* ⇒ '*point* ⇒ '*line* **where**
*line P Q* ≡ @*l*. *incid P l* ∧ *incid Q l*

**definition** *is-a-proper-intersec* :: ['*point*, '*point*, '*point*, '*point*, '*point*] ⇒ *bool* **where**
*is-a-proper-intersec P A B C D* ≡ (*A* ≠ *B*) ∧ (*C* ≠ *D*) ∧ (*line A B* ≠ *line C D*)
∧ *col P A B* ∧ *col P C D*

**definition** *is-pappus1* ::
['*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*] => *bool* **where**
*is-pappus1 A B C A′ B′ C′ P Q R* ≡
*distinct*[*A*,*B*,*C*,*A′*,*B′*,*C′*] ⟶ *col A B C* ⟶ *col A′ B′ C′*
⟶ *is-a-proper-intersec P A B′ A′ B* ⟶ *is-a-proper-intersec Q B C′ B′ C*

$\longrightarrow$ *is-a-proper-intersec R A C′ A′ C*
$\longrightarrow$ *col P Q R*

**definition** *is-a-intersec* :: [*'point*, *'point*, *'point*, *'point*, *'point*] $\Rightarrow$ *bool* **where**
*is-a-intersec P A B C D* $\equiv$ *col P A B* $\wedge$ *col P C D*

**definition** *is-pappus2* ::
[*'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*] $\Rightarrow$ *bool* **where**
*is-pappus2 A B C A′ B′ C′ P Q R* $\equiv$
(*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*] $\vee$ (*A* $\neq$ *B′* $\wedge$ *A′*$\neq$ *B* $\wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$
*B* $\neq$ *C′* $\wedge$ *B′* $\neq$ *C* $\wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
*A* $\neq$ *C′* $\wedge$ *A′* $\neq$ *C* $\wedge$ *line A C′* $\neq$ *line A′ C*))
$\longrightarrow$ *col A B C* $\longrightarrow$ *col A′ B′ C′* $\longrightarrow$ *is-a-intersec P A B′ A′ B*
$\longrightarrow$ *is-a-intersec Q B C′ B′ C* $\longrightarrow$ *is-a-intersec R A C′ A′ C*
$\longrightarrow$ *col P Q R*

**lemma** *is-a-proper-intersec-is-a-intersec*:
**assumes** *is-a-proper-intersec P A B C D*
**shows** *is-a-intersec P A B C D*
**using** *assms is-a-intersec-def is-a-proper-intersec-def*
**by** *auto*

**lemma** *pappus21*:
**assumes** *is-pappus2 A B C A′ B′ C′ P Q R*
**shows** *is-pappus1 A B C A′ B′ C′ P Q R*
**using** *assms is-pappus2-def is-pappus1-def is-a-proper-intersec-is-a-intersec*
**by** *auto*

**lemma** *col-AAB*: *col A A B*
**by** (*simp add*: *ax1 col-def*)

**lemma** *col-ABA*: *col A B A*
**using** *ax1 col-def*
**by** *blast*

**lemma** *col-ABB*: *col A B B*
**by** (*simp add*: *ax1 col-def*)

**lemma** *incidA-lAB*: *incid A* (*line A B*)
**by** (*metis* (*no-types*, *lifting*) *ax1 line-def someI-ex*)

**lemma** *incidB-lAB*: *incid B* (*line A B*)
**by** (*metis* (*no-types*, *lifting*) *ax1 line-def someI-ex*)

**lemma** *degenerate-hexagon-is-pappus*:
**assumes** *distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*] **and** *col A B C* **and** *col A′ B′ C′* **and**
*is-a-intersec P A B′ A′ B* **and** *is-a-intersec Q B C′ B′ C* **and** *is-a-intersec R A*

$C'\ A'\ C$
**and** *line A B′ = line A′ B ∨ line B C′ = line B′ C ∨ line A C′ = line A′ C*
**shows** *col P Q R*
**proof** −
**have** *col P Q R* **if** *line A B′ = line A′ B*
**by** (*smt assms(1) assms(3) assms(4) assms(5) assms(6) ax-uniqueness col-def distinct6-def*
*incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** *line B C′ = line B′ C*
**by** (*smt ‹line A B′ = line A′ B ⟹ col P Q R› assms(1) assms(2) assms(3) ax-uniqueness col-def*
*distinct6-def incidA-lAB incidB-lAB that*)
**have** *col P Q R* **if** *line A C′ = line A′ C*
**by** (*smt ‹line B C′ = line B′ C ⟹ col P Q R› assms(1) assms(2) assms(3) assms(7) ax-uniqueness*
*col-def distinct6-def incidA-lAB incidB-lAB*)
**show** *col P Q R*
**using** ‹*line A B′ = line A′ B ⟹ col P Q R*› ‹*line A C′ = line A′ C ⟹ col P Q R*›
‹*line B C′ = line B′ C ⟹ col P Q R*› *assms(7)*
**by** *blast*
**qed**

**lemma** *pappus12*:
**assumes** *is-pappus1 A B C A′ B′ C′ P Q R*
**shows** *is-pappus2 A B C A′ B′ C′ P Q R*
**proof** −
**have** *col P Q R* **if** (*A ≠ B′ ∧ A′≠ B ∧ line A B′ ≠ line A′ B ∧*
*B ≠ C′ ∧ B′ ≠ C ∧ line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C*) **and** *h1*:*col A B C* **and** *h2*:*col A′ B′ C′*
**and** *is-a-intersec P A B′ A′ B* **and** *is-a-intersec Q B C′ B′ C*
**and** *is-a-intersec R A C′ A′ C*
**proof** −
**have** *col P Q R* **if** *A = B*
**proof** −
**have** *P = A*
**by** (*metis ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C*
*∧ line B C′ ≠ line B′ C*
*∧ A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec P A B′ A′ B› ax-uniqueness col-def*
*incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P A C′ ∧ col Q A C′ ∧ col R A C′*
**using** ‹*P = A*› ‹*is-a-intersec Q B C′ B′ C*› ‹*is-a-intersec R A C′ A′ C*› *col-AAB*
*is-a-intersec-def that*
**by** *auto*
**then show** *col P Q R*
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C*

$\wedge$ *line B C′ ≠ line B′ C*
*∧ A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ax-uniqueness col-def*)
**qed**
**have** *col P Q R* **if** $A = C$
**proof** −
**have** $R = A$
**by** (*metis ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C*
$\wedge$ *line B C′ ≠ line B′ C*
*∧ A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec R A C′ A′ C›*
*ax-uniqueness col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P A B′* $\wedge$ *col Q A B′* $\wedge$ *col R A B′*
**using** *‹R = A› ‹is-a-intersec P A B′ A′ B› ‹is-a-intersec Q B C′ B′ C› col-def*
*is-a-intersec-def that*
**by** *auto*
**then show** *col P Q R*
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C*
*∧ A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ax-uniqueness col-def*)
**qed**
**have** *col P Q R* **if** $A = A'$
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec P A B′ A′ B›*
*‹is-a-intersec R A C′ A′ C›*
*ax-uniqueness col-ABA col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $B = C$
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec P A B′ A′ B›*
*‹is-a-intersec Q B C′ B′ C›*
*‹is-a-intersec R A C′ A′ C› ax-uniqueness col-def incidA-lAB incidB-lAB*
*is-a-intersec-def that*)
**have** *col P Q R* **if** $B = B'$
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec P A B′ A′ B›*
*‹is-a-intersec Q B C′ B′ C›*
*ax-uniqueness col-AAB col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $C = C'$
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec Q B C′ B′ C›*
*‹is-a-intersec R A C′ A′ C›*
*ax-uniqueness col-ABB col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $A' = B'$
**by** (*smt ‹A ≠ B′ ∧ A′ ≠ B ∧ line A B′ ≠ line A′ B ∧ B ≠ C′ ∧ B′ ≠ C ∧*
*line B C′ ≠ line B′ C ∧*
*A ≠ C′ ∧ A′ ≠ C ∧ line A C′ ≠ line A′ C› ‹is-a-intersec P A B′ A′ B›*

‹*is-a-intersec Q B C′ B′ C*›
‹*is-a-intersec R A C′ A′ C*› *ax-uniqueness col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $A' = C'$
**by** (*smt* ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› ‹*is-a-intersec P A B′ A′ B*› ‹*is-a-intersec Q B C′ B′ C*›
‹*is-a-intersec R A C′ A′ C*› *ax-uniqueness col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $B' = C'$
**by** (*smt* ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› ‹*is-a-intersec P A B′ A′ B*› ‹*is-a-intersec Q B C′ B′ C*›
‹*is-a-intersec R A C′ A′ C*› *ax-uniqueness col-def incidA-lAB incidB-lAB is-a-intersec-def that*)
**have** *col P Q R* **if** $A \neq B \wedge A \neq C \wedge A \neq A' \wedge B \neq C \wedge B \neq B' \wedge C \neq C' \wedge A' \neq B'$
$\wedge\ A' \neq C' \wedge B' \neq C'$
**proof** −
**have** *a1*:*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]
**using** ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› *distinct6-def that*
**by** *auto*
**have** *is-a-proper-intersec P A B′ A′ B*
**using** ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› ‹*is-a-intersec P A B′ A′ B*› *is-a-intersec-def*
*is-a-proper-intersec-def*
**by** *auto*
**have** *is-a-proper-intersec Q B C′ B′ C*
**using** ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› ‹*is-a-intersec Q B C′ B′ C*› *is-a-intersec-def*
*is-a-proper-intersec-def*
**by** *auto*
**have** *is-a-proper-intersec R A C′ A′ C*
**using** ‹$A \neq B' \wedge A' \neq B \wedge$ *line A B′* $\neq$ *line A′ B* $\wedge$ $B \neq C' \wedge B' \neq C \wedge$ *line B C′* $\neq$ *line B′ C* $\wedge$
$A \neq C' \wedge A' \neq C \wedge$ *line A C′* $\neq$ *line A′ C*› ‹*is-a-intersec R A C′ A′ C*› *is-a-intersec-def*
*is-a-proper-intersec-def*
**by** *auto*
**show** *col P Q R*
**using** ‹*is-a-proper-intersec P A B′ A′ B*› ‹*is-a-proper-intersec Q B C′ B′ C*›

‹*is-a-proper-intersec R A C′ A′ C*› *a1 assms h1 h2 is-pappus1-def*
**by** *blast*
**qed**
**show** *col P Q R*
**using** ‹$A = A' \Longrightarrow$ *col P Q R*› ‹$A = B \Longrightarrow$ *col P Q R*› ‹$A = C \Longrightarrow$ *col P Q R*›
‹$A \neq B \wedge A \neq C \wedge A \neq A' \wedge B \neq C \wedge B \neq B' \wedge C \neq C' \wedge A' \neq B' \wedge A' \neq C' \wedge B' \neq C' \Longrightarrow$ *col P Q R*›
‹$A' = B' \Longrightarrow$ *col P Q R*› ‹$A' = C' \Longrightarrow$ *col P Q R*› ‹$B = B' \Longrightarrow$ *col P Q R*› ‹$B = C \Longrightarrow$ *col P Q R*›
‹$B' = C' \Longrightarrow$ *col P Q R*› ‹$C = C' \Longrightarrow$ *col P Q R*›
**by** *blast*
**qed**
**have** *col P Q R* **if** *distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*] **and** *col A B C* **and** *col A′ B′ C′*
**and** *is-a-intersec P A B′ A′ B* **and** *is-a-intersec Q B C′ B′ C* **and** *is-a-intersec R A C′ A′ C*
**proof** −
**have** *col P Q R* **if** *line A B′* = *line A′ B*
**using** ‹*col A B C*› ‹*col A′ B′ C′*› ‹*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]› ‹*is-a-intersec P A B′ A′ B*›
‹*is-a-intersec Q B C′ B′ C*› ‹*is-a-intersec R A C′ A′ C*› *degenerate-hexagon-is-pappus that*
**by** *blast*
**have** *col P Q R* **if** *line B C′* = *line B′ C*
**using** ‹*col A B C*› ‹*col A′ B′ C′*› ‹*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]› ‹*is-a-intersec P A B′ A′ B*›
‹*is-a-intersec Q B C′ B′ C*› ‹*is-a-intersec R A C′ A′ C*› *degenerate-hexagon-is-pappus that*
**by** *blast*
**have** *col P Q R* **if** *line A′ C* = *line A C′*
**using** ‹*col A B C*› ‹*col A′ B′ C′*› ‹*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]› ‹*is-a-intersec P A B′ A′ B*›
‹*is-a-intersec Q B C′ B′ C*› ‹*is-a-intersec R A C′ A′ C*› *degenerate-hexagon-is-pappus that*
**by** *auto*
**have** *col P Q R* **if** *line A B′* ≠ *line A′ B* **and** *line B C′* ≠ *line B′ C* **and**
*line A C′* ≠ *line A′ C*
**proof** −
**have** *is-a-proper-intersec P A B′ A′ B*
**using** ‹*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]› ‹*is-a-intersec P A B′ A′ B*› *distinct6-def is-a-intersec-def*
*is-a-proper-intersec-def that*(*1*)
**by** *auto*
**have** *is-a-proper-intersec Q B C′ B′ C*
**using** ‹*distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*]› ‹*is-a-intersec Q B C′ B′ C*› *distinct6-def is-a-intersec-def*
*is-a-proper-intersec-def that*(*2*)
**by** *auto*
**have** *is-a-proper-intersec R A C′ A′ C*

**using** ‹*distinct* [*A*,*B*,*C*,*A'*,*B'*,*C'*]› ‹*is-a-intersec R A C' A' C*› *distinct6-def is-a-intersec-def*
*is-a-proper-intersec-def that*(*3*)
**by** *auto*
**show** *col P Q R*
**using** ‹*col A B C*› ‹*col A' B' C'*› ‹*distinct* [*A*,*B*,*C*,*A'*,*B'*,*C'*]› ‹*is-a-proper-intersec P A B' A' B*›
‹*is-a-proper-intersec Q B C' B' C*› ‹*is-a-proper-intersec R A C' A' C*› *assms is-pappus1-def*
**by** *blast*
**qed**
**show** *col P Q R*
**using** ‹⟦*line A B' ≠ line A' B*; *line B C' ≠ line B' C*; *line A C' ≠ line A' C*⟧ ⟹ *col P Q R*›
‹*line A B' = line A' B* ⟹ *col P Q R*› ‹*line A' C = line A C'* ⟹ *col P Q R*›
‹*line B C' = line B' C* ⟹ *col P Q R*›
**by** *fastforce*
**qed**
**show** *is-pappus2 A B C A' B' C' P Q R*
**by** (*simp add*: ‹⟦*A ≠ B' ∧ A' ≠ B ∧ line A B' ≠ line A' B ∧ B ≠ C' ∧ B' ≠ C ∧ line B C' ≠ line B' C*
*∧ A ≠ C' ∧ A' ≠ C ∧ line A C' ≠ line A' C*; *col A B C*; *col A' B' C'*; *is-a-intersec P A B' A' B*; *is-a-intersec Q B C' B' C*; *is-a-intersec R A C' A' C*⟧ ⟹ *col P Q R*›
‹⟦*distinct* [*A*,*B*,*C*,*A'*,*B'*,*C'*]; *col A B C*; *col A' B' C'*; *is-a-intersec P A B' A' B*; *is-a-intersec Q B C' B' C*; *is-a-intersec R A C' A' C*⟧ ⟹ *col P Q R*›
*is-pappus2-def*)
**qed**

**lemma** *pappus-equiv*: *is-pappus1 A B C A' B' C' P Q R* = *is-pappus2 A B C A' B' C' P Q R*
**using** *pappus12 pappus21*
**by** *blast*

**definition** *is-pappus* :: *bool* **where**
*is-pappus* ≡ ∀ *A B C D E F P Q R*. *is-pappus2 A B C D E F P Q R*

**end**

**end**
**theory** *Pascal-Property*
**imports** *Main Projective-Plane-Axioms Pappus-Property*
**begin**

Contents:

- A hexagon is pascal if its three opposite sides meet in collinear points

*is-pascal*.

- A plane is pascal, or has Pascal's property, if for every hexagon of that plane Pascal property is stable under any permutation of that hexagon.

# 3 Pascal's Property

**context** *projective-plane*
**begin**

**definition** *inters* :: *'line* ⇒ *'line* ⇒ *'point set* **where**
*inters l m* ≡ {*P*. *incid P l* ∧ *incid P m*}

**lemma** *inters-is-singleton*:
**assumes** *l* ≠ *m* **and** *P* ∈ *inters l m* **and** *Q* ∈ *inters l m*
**shows** *P* = *Q*
**using** *assms ax-uniqueness inters-def*
**by** *blast*

**definition** *inter* :: *'line* ⇒ *'line* ⇒ *'point* **where**
*inter l m* ≡ @*P*. *P* ∈ *inters l m*

**lemma** *uniq-inter*:
**assumes** *l* ≠ *m* **and** *incid P l* **and** *incid P m*
**shows** *inter l m* = *P*
**proof** −
**have** *P* ∈ *inters l m*
**by** (*simp add*: *assms*(*2*) *assms*(*3*) *inters-def*)
**have** ∀ *Q*. *Q* ∈ *inters l m* ⟶ *Q* = *P*
**using** ‹*P* ∈ *inters l m*› *assms*(*1*) *inters-is-singleton*
**by** *blast*
**show** *inter l m* = *P*
**using** ‹*P* ∈ *inters l m*› *assms*(*1*) *inter-def inters-is-singleton*
**by** *auto*
**qed**

**definition** *is-pascal* :: [*'point*, *'point*, *'point*, *'point*, *'point*, *'point*] ⇒ *bool* **where**
*is-pascal A B C D E F* ≡ *distinct* [*A*,*B*,*C*,*D*,*E*,*F*] ⟶ *line B C* ≠ *line E F* ⟶
*line C D* ≠ *line A F*
⟶ *line A B* ≠ *line D E* ⟶
(*let P* = *inter* (*line B C*) (*line E F*) *in*
*let Q* = *inter* (*line C D*) (*line A F*) *in*
*let R* = *inter* (*line A B*) (*line D E*) *in*
*col P Q R*)

**lemma** *col-rot-CW*:
**assumes** *col P Q R*
**shows** *col R P Q*
**using** *assms col-def*
**by** *auto*

**lemma** *col-2cycle*:
**assumes** *col P Q R*
**shows** *col P R Q*
**using** *assms col-def*
**by** *auto*

**lemma** *distinct6-rot-CW*:
**assumes** *distinct* [*A*,*B*,*C*,*D*,*E*,*F*]
**shows** *distinct* [*F*,*A*,*B*,*C*,*D*,*E*]
**using** *assms distinct6-def*
**by** *auto*

**lemma** *lines-comm*: *lines P Q* = *lines Q P*
**using** *lines-def*
**by** *auto*

**lemma** *line-comm*:
**assumes** *P* ≠ *Q*
**shows** *line P Q* = *line Q P*
**by** (*metis ax-uniqueness incidA-lAB incidB-lAB*)

**lemma** *inters-comm*: *inters l m* = *inters m l*
**using** *inters-def*
**by** *auto*

**lemma** *inter-comm*: *inter l m* = *inter m l*
**by** (*simp add*: *inter-def inters-comm*)

**lemma** *inter-line-line-comm*:
**assumes** *C* ≠ *D*
**shows** *inter* (*line A B*) (*line C D*) = *inter* (*line A B*) (*line D C*)
**using** *assms line-comm*
**by** *auto*

**lemma** *inter-line-comm-line*:
**assumes** *A* ≠ *B*
**shows** *inter* (*line A B*) (*line C D*) = *inter* (*line B A*) (*line C D*)
**using** *assms line-comm*
**by** *auto*

**lemma** *inter-comm-line-line-comm*:
**assumes** *C* ≠ *D* **and** *line A B* ≠ *line C D*
**shows** *inter* (*line A B*) (*line C D*) = *inter* (*line D C*) (*line A B*)

**by** (*metis inter-comm line-comm*)

**lemma** *is-pascal-rot-CW*:
**assumes** *is-pascal A B C D E F*
**shows** *is-pascal F A B C D E*
**proof** −
**define** *P Q R* **where** *P* = *inter* (*line A B*) (*line D E*) **and** *Q* = *inter* (*line B C*) (*line E F*) **and**
*R* = *inter* (*line F A*) (*line C D*)
**have** *col P Q R* **if** *distinct* [*F*,*A*,*B*,*C*,*D*,*E*] **and** *line A B* ≠ *line D E* **and** *line B C* ≠ *line E F*
**and** *line F A* ≠ *line C D*
**using** *P-def Q-def R-def assms col-rot-CW distinct6-def inter-comm is-pascal-def line-comm*
*that*(*1*) *that*(*2*) *that*(*3*) *that*(*4*)
**by** *auto*
**then show** *is-pascal F A B C D E*
**by** (*metis P-def Q-def R-def is-pascal-def line-comm*)
**qed**

**lemma** *incid-C-AB*:
**assumes** *A* ≠ *B* **and** *incid A l* **and** *incid B l* **and** *incid C l*
**shows** *incid C* (*line A B*)
**using** *assms ax-uniqueness incidA-lAB incidB-lAB*
**by** *blast*

**lemma** *incid-inters-left*:
**assumes** *P* ∈ *inters l m*
**shows** *incid P l*
**using** *assms inters-def*
**by** *auto*

**lemma** *incid-inters-right*:
**assumes** *P* ∈ *inters l m*
**shows** *incid P m*
**using** *assms incid-inters-left inters-comm*
**by** *blast*

**lemma** *inter-in-inters*: *inter l m* ∈ *inters l m*
**proof** −
**have** ∃ *P*. *P* ∈ *inters l m*
**using** *inters-def ax2*
**by** *auto*
**show** *inter l m* ∈ *inters l m*

**by** (*metis* ‹∃ *P*. *P* ∈ *inters l m*› *inter-def some-eq-ex*)
**qed**

**lemma** *incid-inter-left*: *incid* (*inter l m*) *l*
**using** *incid-inters-left inter-in-inters*
**by** *blast*

**lemma** *incid-inter-right*: *incid* (*inter l m*) *m*
**using** *incid-inter-left inter-comm*
**by** *fastforce*

**lemma** *col-A-B-ABl*: *col A B* (*inter* (*line A B*) *l*)
**using** *col-def incidA-lAB incidB-lAB incid-inter-left*
**by** *blast*

**lemma** *col-A-B-lAB*: *col A B* (*inter l* (*line A B*))
**using** *col-A-B-ABl inter-comm*
**by** *auto*

**lemma** *inter-is-a-intersec*: *is-a-intersec* (*inter* (*line A B*) (*line C D*)) *A B C D*
**by** (*simp add*: *col-A-B-ABl col-A-B-lAB col-rot-CW is-a-intersec-def*)

**definition** *line-ext* :: *'line* ⇒ *'point set* **where**
*line-ext l* ≡ {*P*. *incid P l*}

**lemma** *line-left-inter-1*:
**assumes** *P* ∈ *line-ext l* **and** *P* ∉ *line-ext m*
**shows** *line* (*inter l m*) *P* = *l*
**by** (*metis CollectD CollectI assms*(*1*) *assms*(*2*) *incidA-lAB incidB-lAB incid-inter-left*

*incid-inter-right line-ext-def uniq-inter*)

**lemma** *line-left-inter-2*:
**assumes** *P* ∈ *line-ext m* **and** *P* ∉ *line-ext l*
**shows** *line* (*inter l m*) *P* = *m*
**using** *assms inter-comm line-left-inter-1*
**by** *fastforce*

**lemma** *line-right-inter-1*:
**assumes** *P* ∈ *line-ext l* **and** *P* ∉ *line-ext m*
**shows** *line P* (*inter l m*) = *l*
**by** (*metis assms line-comm line-left-inter-1*)

**lemma** *line-right-inter-2*:
**assumes** *P* ∈ *line-ext m* **and** *P* ∉ *line-ext l*
**shows** *line P* (*inter l m*) = *m*
**by** (*metis assms inter-comm line-comm line-left-inter-1*)

**lemma** *inter-ABC-1*:

**assumes** *line A B* ≠ *line C A*
**shows** *inter* (*line A B*) (*line C A*) = *A*
**using** *assms ax-uniqueness incidA-lAB incidB-lAB incid-inter-left incid-inter-right*

**by** *blast*

**lemma** *line-inter-2*:
**assumes** *inter l m* ≠ *inter l′ m*
**shows** *line* (*inter l m*) (*inter l′ m*) = *m*
**using** *assms ax-uniqueness incidA-lAB incidB-lAB incid-inter-right*
**by** *blast*

**lemma** *col-line-ext-1*:
**assumes** *col A B C* **and** *A* ≠ *C*
**shows** *B* ∈ *line-ext* (*line A C*)
**by** (*metis CollectI assms ax-uniqueness col-def incidA-lAB incidB-lAB line-ext-def*)

**lemma** *inter-line-ext-1*:
**assumes** *inter l m* ∈ *line-ext n* **and** *l* ≠ *m* **and** *l* ≠ *n*
**shows** *inter l m* = *inter l n*
**using** *assms*(*1*) *assms*(*3*) *ax-uniqueness incid-inter-left incid-inter-right line-ext-def*

**by** *blast*

**lemma** *inter-line-ext-2*:
**assumes** *inter l m* ∈ *line-ext n* **and** *l* ≠ *m* **and** *m* ≠ *n*
**shows** *inter l m* = *inter m n*
**by** (*metis assms inter-comm inter-line-ext-1*)

**definition** *pascal-prop* :: *bool* **where**
*pascal-prop* ≡ ∀ *A B C D E F*. *is-pascal A B C D E F* ⟶ *is-pascal B A C D E F*

**lemma** *pappus-pascal*:
**assumes** *is-pappus*
**shows** *pascal-prop*
**proof**−
**have** *is-pascal B A C D E F* **if** *is-pascal A B C D E F* **for** *A B C D E F*
**proof**−
**define** *X Y Z* **where** *X* = *inter* (*line A C*) (*line E F*) **and** *Y* = *inter* (*line C D*) (*line B F*)
**and** *Z* = *inter* (*line B A*) (*line D E*)
**have** *col X Y Z* **if** *distinct* [*B*,*A*,*C*,*D*,*E*,*F*] **and** *line A C* ≠ *line E F* **and** *line C D* ≠ *line B F*
**and** *line B A* ≠ *line D E* **and** *line B C* = *line E F*
**by** (*smt X-def Y-def ax-uniqueness col-ABA col-rot-CW distinct6-def incidB-lAB incid-inter-left*
*incid-inter-right line-comm that*(*1*) *that*(*2*) *that*(*3*) *that*(*5*))
**have** *col X Y Z* **if** *distinct* [*B*,*A*,*C*,*D*,*E*,*F*] **and** *line A C* ≠ *line E F* **and** *line C D* ≠ *line B F*

**and** *line B A* $\neq$ *line D E* **and** *line C D* = *line A F*
**by** (*metis X-def Y-def col-ABA col-rot-CW distinct6-def inter-ABC-1 line-comm that(1) that(2)*
*that(3) that(5)*)
**have** *col X Y Z* **if** *distinct* [*B*,*A*,*C*,*D*,*E*,*F*] **and** *line A C* $\neq$ *line E F* **and** *line C D* $\neq$ *line B F*
**and** *line B A* $\neq$ *line D E* **and** *line B C* $\neq$ *line E F* **and** *line C D* $\neq$ *line A F*
**proof**−
**define** *W* **where** *W* = *inter* (*line A C*) (*line E F*)
**have** *col A C W*
**by** (*simp add*: *col-A-B-ABl W-def*)
**define** *P Q R* **where** *P* = *inter* (*line B C*) (*line E F*)
**and** *Q* = *inter* (*line A B*) (*line D E*)
**and** *R* = *inter* (*line C D*) (*line A F*)
**have** *col P Q R*
**using** *P-def Q-def R-def* ‹*is-pascal A B C D E F*› *col-2cycle distinct6-def is-pascal-def*
*line-comm that(1) that(4) that(5) that(6)*
**by** *auto*

**have** *col X Y Z* **if** *P* = *Q*
**by** (*smt P-def Q-def X-def Y-def Z-def* ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› *ax-uniqueness col-ABA col-def*
*distinct6-def incidA-lAB incidB-lAB incid-inter-left inter-comm that*)
**have** *col X Y Z* **if** *P* = *R*
**by** (*smt P-def R-def X-def Y-def Z-def* ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› ‹*line A C* $\neq$ *line E F*›
‹*line C D* $\neq$ *line B F*› *col-2cycle col-A-B-ABl col-rot-CW distinct6-def incidA-lAB*
*incidB-lAB incid-inter-left incid-inter-right that uniq-inter*)
**have** *col X Y Z* **if** *P* = *A*
**by** (*smt P-def Q-def R-def X-def Y-def Z-def* ‹*P* = *Q* $\Longrightarrow$ *col X Y Z*› ‹*P* = *R* $\Longrightarrow$ *col X Y Z*›
‹*col P Q R*› ‹*line B C* $\neq$ *line E F*› *ax-uniqueness col-def incidA-lAB incid-inter-left*
*incid-inter-right line-comm that*)
**have** *col X Y Z* **if** *P* = *C*
**by** (*smt P-def Q-def R-def X-def Y-def Z-def* ‹*P* = *R* $\Longrightarrow$ *col X Y Z*› ‹*col P Q R*›
‹*line A C* $\neq$ *line E F*› *ax-uniqueness col-def incidA-lAB incid-inter-left*
*incid-inter-right line-comm that*)
**have** *col X Y Z* **if** *P* = *W*
**by** (*smt P-def Q-def R-def W-def X-def Y-def Z-def* ‹*P* = *C* $\Longrightarrow$ *col X Y Z*› ‹*P* = *Q* $\Longrightarrow$ *col X Y Z*›
‹*col P Q R*› ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› *ax-uniqueness col-def distinct6-def incidB-lAB*
*incid-inter-left incid-inter-right line-comm that*)
**have** *col X Y Z* **if** *Q* = *R*
**by** (*smt Q-def R-def X-def Y-def Z-def* ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› *ax-uniqueness*

*col-A-B-lAB*
*col-rot-CW distinct6-def incidB-lAB incid-inter-right inter-comm line-comm that*)
**have** *col X Y Z* **if** *Q = A*
**by** (*smt P-def Q-def R-def X-def Y-def Z-def ‹col P Q R› ‹distinct [B,A,C,D,E,F]›*
*‹line C D ≠ line B F› ax-uniqueness col-ABA col-def distinct6-def incidA-lAB incidB-lAB*
*incid-inter-left incid-inter-right that*)
**have** *col X Y Z* **if** *Q = C*
**by** (*metis P-def Q-def W-def ‹P = W ⟹ col X Y Z› ‹distinct [B,A,C,D,E,F]› ax-uniqueness*
*distinct6-def incidA-lAB incid-inter-left line-comm that*)
**have** *col X Y Z* **if** *Q = W*
**by** (*metis Q-def W-def X-def Z-def col-ABA line-comm that*)
**have** *col X Y Z* **if** *R = A*
**by** (*smt P-def Q-def R-def W-def X-def Y-def ‹P = W ⟹ col X Y Z› ‹Q = A ⟹ col X Y Z›*
*‹col P Q R› ‹distinct [B,A,C,D,E,F]› ax-uniqueness col-ABA col-def col-rot-CW distinct6-def*
*incidA-lAB incidB-lAB incid-inter-right inter-comm that*)
**have** *col X Y Z* **if** *R = C*
**by** (*smt P-def Q-def R-def X-def Y-def Z-def ‹col P Q R› ‹distinct [B,A,C,D,E,F]›*
*‹line A C ≠ line E F› ax-uniqueness col-def distinct6-def incidA-lAB incidB-lAB*
*incid-inter-left inter-comm that*)
**have** *col X Y Z* **if** *R = W*
**by** (*metis R-def W-def ‹R = A ⟹ col X Y Z› ‹R = C ⟹ col X Y Z› ‹line C D ≠ line A F›*
*ax-uniqueness incidA-lAB incidB-lAB incid-inter-left incid-inter-right that*)
**have** *col X Y Z* **if** *A = W*
**by** (*smt P-def Q-def R-def W-def X-def Y-def Z-def ‹P = R ⟹ col X Y Z› ‹Q = A ⟹ col X Y Z›*
*‹col P Q R› ‹distinct [B,A,C,D,E,F]› ax-uniqueness col-def distinct6-def incidA-lAB*
*incidB-lAB incid-inter-left incid-inter-right that*)
**have** *col X Y Z* **if** *C = W*
**by** (*metis P-def W-def ‹P = C ⟹ col X Y Z› ‹line B C ≠ line E F› ax-uniqueness incidB-lAB*
*incid-inter-left incid-inter-right that*)
**have** *f1*:*col* (*inter* (*line P C*) (*line A Q*)) (*inter* (*line Q W*) (*line C R*))
(*inter* (*line P W*) (*line A R*)) **if** *distinct* [*P*,*Q*,*R*,*A*,*C*,*W*]
**using** *assms*(*1*) *is-pappus-def is-pappus2-def ‹distinct [P,Q,R,A,C,W]› ‹col P Q R›*
*‹col A C W› inter-is-a-intersec inter-line-line-comm*
**by** *presburger*
**have** *col X Y Z* **if** *C ∈ line-ext* (*line E F*)

**using** *P-def* ‹$P = C \Longrightarrow$ *col X Y Z*› ‹*line B C* $\neq$ *line E F*› *incidB-lAB line-ext-def that uniq-inter*
**by** *auto*
**have** *col X Y Z* **if** $A \in$ *line-ext* (*line D E*)
**by** (*metis Q-def* ‹$Q = A \Longrightarrow$ *col X Y Z*› ‹*line B A* $\neq$ *line D E*› *ax-uniqueness incidA-lAB*
*incid-inter-left incid-inter-right line-comm line-ext-def mem-Collect-eq that*)
**have** *col X Y Z* **if** *line B C* = *line A B*
**by** (*metis P-def W-def* ‹$P = W \Longrightarrow$ *col X Y Z*› ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› *ax-uniqueness*
*distinct6-def incidA-lAB incidB-lAB that*)

**have** *f2*:*inter* (*line P C*) (*line A Q*) = *B* **if**
$C \notin$ *line-ext* (*line E F*) **and** $A \notin$ *line-ext* (*line D E*) **and** *line B C* $\neq$ *line A B*
**by** (*smt CollectI P-def Q-def ax-uniqueness incidA-lAB incidB-lAB incid-inter-left*
*incid-inter-right line-ext-def that*(*1*) *that*(*2*) *that*(*3*))

**have** *col X Y Z* **if** *line E F* = *line A F*
**by** (*metis W-def* ‹$A = W \Longrightarrow$ *col X Y Z*› ‹*line A C* $\neq$ *line E F*› *inter-ABC-1 inter-comm that*)
**have** *col X Y Z* **if** $A \in$ *line-ext* (*line C D*)
**using** *R-def* ‹$R = A \Longrightarrow$ *col X Y Z*› ‹*line C D* $\neq$ *line A F*› *ax-uniqueness incidA-lAB*
*incid-inter-left incid-inter-right line-ext-def that*
**by** *blast*
**have** *col X Y Z* **if** *inter* (*line B C*) (*line E F*) = *inter* (*line A C*) (*line E F*)
**by** (*simp add*: *P-def W-def* ‹$P = W \Longrightarrow$ *col X Y Z*› *that*)

**have** *f3*:*inter* (*line P W*) (*line A R*) = *F* **if** *line E F* $\neq$ *line A F* **and** $A \notin$ *line-ext* (*line C D*)
**and** *inter* (*line B C*) (*line E F*) $\neq$ *inter* (*line A C*) (*line E F*)
**by** (*smt CollectI P-def R-def W-def ax-uniqueness incidA-lAB incidB-lAB incid-inter-left*
*incid-inter-right line-ext-def that*(*1*) *that*(*2*) *that*(*3*))

**have** *col X Y Z* **if** $C \in$ *line-ext* (*line A F*)
**using** *R-def* ‹$R = C \Longrightarrow$ *col X Y Z*› ‹*line C D* $\neq$ *line A F*› *ax-uniqueness incidA-lAB*
*incid-inter-left incid-inter-right line-ext-def that*
**by** *blast*
**have** *f4*:*inter* (*line Q W*) (*line C R*) = *inter* (*line Q W*) (*line C D*) **if** $C \notin$ *line-ext* (*line A F*)
**using** *R-def incidA-lAB line-ext-def line-right-inter-1 that*
**by** *auto*
**then have** *inter* (*line Q W*) (*line C D*) $\in$ *line-ext* (*line B F*) **if** *distinct* [*P*,*Q*,*R*,*A*,*C*,*W*]

**and** $C \notin$ *line-ext* (*line E F*) **and** $A \notin$ *line-ext* (*line D E*) **and** *line B C* $\neq$ *line A B*
**and** *line E F* $\neq$ *line A F* **and** $A \notin$ *line-ext* (*line C D*)
**and** *inter* (*line B C*) (*line E F*) $\neq$ *inter* (*line A C*) (*line E F*)
**by** (*smt R-def* ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› *ax-uniqueness col-line-ext-1 distinct6-def f1 f2 f3*
*incidA-lAB incidB-lAB incid-inter-left that*(*1*) *that*(*2*) *that*(*3*) *that*(*5*) *that*(*6*) *that*(*7*))
**then have** *inter* (*line Q W*) (*line C D*) = *inter* (*line C D*) (*line B F*) **if** *distinct* [*P*,*Q*,*R*,*A*,*C*,*W*]
**and** $C \notin$ *line-ext* (*line E F*) **and** $A \notin$ *line-ext* (*line D E*) **and** *line B C* $\neq$ *line A B*
**and** *line E F* $\neq$ *line A F* **and** $A \notin$ *line-ext* (*line C D*)
**and** *inter* (*line B C*) (*line E F*) $\neq$ *inter* (*line A C*) (*line E F*)
**by** (*smt W-def* ‹*distinct* [*B*,*A*,*C*,*D*,*E*,*F*]› ‹*line C D* $\neq$ *line B F*› *ax-uniqueness distinct6-def f2*
*incidA-lAB incidB-lAB incid-inter-left incid-inter-right inter-line-ext-2 that*(*1*) *that*(*2*)
*that*(*3*) *that*(*5*) *that*(*6*) *that*(*7*))
**moreover have** *inter* (*line C D*) (*line B F*) $\in$ *line-ext* (*line Q W*) **if** *distinct* [*P*,*Q*,*R*,*A*,*C*,*W*]
**and** $C \notin$ *line-ext* (*line E F*) **and** $A \notin$ *line-ext* (*line D E*) **and** *line B C* $\neq$ *line A B*
**and** *line E F* $\neq$ *line A F* **and** $A \notin$ *line-ext* (*line C D*)
**and** *inter* (*line B C*) (*line E F*) $\neq$ *inter* (*line A C*) (*line E F*)
**by** (*metis calculation col-2cycle col-A-B-ABl col-line-ext-1 distinct6-def that*(*1*) *that*(*2*)
*that*(*3*) *that*(*4*) *that*(*5*) *that*(*6*) *that*(*7*))
**ultimately have** *col* (*inter* (*line A C*) (*line E F*)) (*inter* (*line C D*) (*line B F*))
(*inter* (*line A B*) (*line D E*)) **if** *distinct* [*P*,*Q*,*R*,*A*,*C*,*W*]
**and** $C \notin$ *line-ext* (*line E F*) **and** $A \notin$ *line-ext* (*line D E*) **and** *line B C* $\neq$ *line A B*
**and** *line E F* $\neq$ *line A F* **and** $A \notin$ *line-ext* (*line C D*)
**and** *inter* (*line B C*) (*line E F*) $\neq$ *inter* (*line A C*) (*line E F*)
**by** (*metis Q-def W-def col-A-B-ABl col-rot-CW that*(*1*) *that*(*2*) *that*(*3*) *that*(*4*) *that*(*5*) *that*(*6*)
*that*(*7*))
**show** *col X Y Z*
**by** (*metis P-def W-def X-def Y-def Z-def* ‹*A* = *W* $\Longrightarrow$ *col X Y Z*› ‹*A* $\in$ *line-ext* (*line C D*) $\Longrightarrow$ *col X Y Z*›
‹*A* $\in$ *line-ext* (*line D E*) $\Longrightarrow$ *col X Y Z*› ‹*C* = *W* $\Longrightarrow$ *col X Y Z*› ‹*C* $\in$ *line-ext* (*line E F*) $\Longrightarrow$ *col X Y Z*›
‹*P* = *A* $\Longrightarrow$ *col X Y Z*› ‹*P* = *C* $\Longrightarrow$ *col X Y Z*› ‹*P* = *Q* $\Longrightarrow$ *col X Y Z*› ‹*P* = *R* $\Longrightarrow$ *col X Y Z*›
‹*inter* (*line B C*) (*line E F*) = *inter* (*line A C*) (*line E F*) $\Longrightarrow$ *col X Y Z*›
‹*Q* = *A* $\Longrightarrow$ *col X Y Z*› ‹*Q* = *C* $\Longrightarrow$ *col X Y Z*› ‹*Q* = *R* $\Longrightarrow$ *col X Y Z*› ‹*Q* = *W* $\Longrightarrow$ *col X Y Z*› ‹*R* = *A* $\Longrightarrow$ *col X Y Z*›

‹$R = C \Longrightarrow col\ X\ Y\ Z$› ‹$R = W \Longrightarrow col\ X\ Y\ Z$› ‹⟦*distinct* [$P$,$Q$,$R$,$A$,$C$,$W$]; $C \notin$ *line-ext* (*line* $E\ F$); $A \notin$ *line-ext* (*line* $D\ E$); *line* $B\ C \neq$ *line* $A\ B$; *line* $E\ F \neq$ *line* $A\ F$; $A \notin$ *line-ext* (*line* $C\ D$); *inter* (*line* $B\ C$) (*line* $E\ F$) $\neq$ *inter* (*line* $A\ C$) (*line* $E\ F$)⟧ $\Longrightarrow$ *col* (*inter* (*line* $A\ C$) (*line* $E\ F$)) (*inter* (*line* $C\ D$) (*line* $B\ F$)) (*inter* (*line* $A\ B$) (*line* $D\ E$))›

‹*line* $B\ C$ = *line* $A\ B \Longrightarrow col\ X\ Y\ Z$› ‹*line* $E\ F$ = *line* $A\ F \Longrightarrow col\ X\ Y\ Z$› *distinct6-def line-comm*)

**qed**

**show** *is-pascal* $B\ A\ C\ D\ E\ F$

**using** *X-def Y-def Z-def* ‹⟦*distinct* [$B$,$A$,$C$,$D$,$E$,$F$]; *line* $A\ C \neq$ *line* $E\ F$; *line* $C\ D \neq$ *line* $B\ F$; *line* $B\ A \neq$ *line* $D\ E$; *line* $B\ C$ = *line* $E\ F$⟧ $\Longrightarrow col\ X\ Y\ Z$›

‹⟦*distinct* [$B$,$A$,$C$,$D$,$E$,$F$]; *line* $A\ C \neq$ *line* $E\ F$; *line* $C\ D \neq$ *line* $B\ F$; *line* $B\ A \neq$ *line* $D\ E$; *line* $B\ C \neq$ *line* $E\ F$; *line* $C\ D \neq$ *line* $A\ F$⟧ $\Longrightarrow col\ X\ Y\ Z$›

‹⟦*distinct* [$B$,$A$,$C$,$D$,$E$,$F$]; *line* $A\ C \neq$ *line* $E\ F$; *line* $C\ D \neq$ *line* $B\ F$; *line* $B\ A \neq$ *line* $D\ E$; *line* $C\ D$ = *line* $A\ F$⟧ $\Longrightarrow col\ X\ Y\ Z$›

*is-pascal-def*

**by** *force*

**qed**

**thus** *pascal-prop* **using** *pascal-prop-def*

**by** *auto*

**qed**

**lemma** *is-pascal-under-alternate-vertices*:

**assumes** *pascal-prop* **and** *is-pascal* $A\ B\ C\ A'\ B'\ C'$

**shows** *is-pascal* $A\ B'\ C\ A'\ B\ C'$

**using** *assms pascal-prop-def is-pascal-rot-CW*

**by** *presburger*

**lemma** *col-inter*:

**assumes** *distinct* [$A$,$B$,$C$,$D$,$E$,$F$] **and** *col* $A\ B\ C$ **and** *col* $D\ E\ F$

**shows** *inter* (*line* $B\ C$) (*line* $E\ F$) = *inter* (*line* $A\ B$) (*line* $D\ E$)

**by** (*smt assms ax-uniqueness col-def distinct6-def incidA-lAB incidB-lAB*)

**lemma** *pascal-pappus1*:

**assumes** *pascal-prop*

**shows** *is-pappus1* $A\ B\ C\ A'\ B'\ C'\ P\ Q\ R$

**proof**−

**define** *a1 a2 a3 a4 a5 a6* **where** *a1* = *distinct* [$A$,$B$,$C$,$A'$,$B'$,$C'$] **and** *a2* = *col* $A\ B\ C$ **and**

*a3* = *col* $A'\ B'\ C'$ **and** *a4* = *is-a-proper-intersec* $P\ A\ B'\ A'\ B$ **and** *a5* = *is-a-proper-intersec* $Q\ B\ C'\ B'\ C$

**and** *a6* = *is-a-proper-intersec* $R\ A\ C'\ A'\ C$

**have** *inter* (*line* $B\ C$) (*line* $B'\ C'$) = *inter* (*line* $A\ B$) (*line* $A'\ B'$) **if** *a1 a2 a3 a4 a5 a6*

**using** *a1-def a2-def a3-def col-inter that*(1) *that*(2) *that*(3)

**by** *blast*

**then have** *is-pascal* $A\ B\ C\ A'\ B'\ C'$ **if** *a1 a2 a3 a4 a5 a6*

**using** *a1-def col-ABA is-pascal-def that*(1) *that*(2) *that*(3) *that*(4) *that*(5)

*that(6)*
**by** *auto*
**then have** *is-pascal A B′ C A′ B C′* **if** *a1 a2 a3 a4 a5 a6*
**using** *assms is-pascal-under-alternate-vertices that(1) that(2) that(3) that(4) that(5) that(6)*
**by** *blast*
**then have** *col P Q R* **if** *a1 a2 a3 a4 a5 a6*
**by** (*smt a1-def a4-def a5-def a6-def ax-uniqueness col-def distinct6-def incidB-lAB incid-inter-left incid-inter-right is-a-proper-intersec-def is-pascal-def line-comm that(1) that(2) that(3) that(4) that(5) that(6)*)
**show** *is-pappus1 A B C A′ B′ C′ P Q R*
**by** (*simp add*: ‹⟦*a1*; *a2*; *a3*; *a4*; *a5*; *a6*⟧ ⟹ *col P Q R*› *a1-def a2-def a3-def a4-def a5-def a6-def is-pappus1-def*)
**qed**

**lemma** *pascal-pappus*:
**assumes** *pascal-prop*
**shows** *is-pappus*
**by** (*simp add*: *assms is-pappus-def pappus12 pascal-pappus1*)

**theorem** *pappus-iff-pascal*: *is-pappus* = *pascal-prop*
**using** *pappus-pascal pascal-pappus*
**by** *blast*

**end**

**end**
**theory** *Desargues-Property*
**imports** *Main Projective-Plane-Axioms Pappus-Property Pascal-Property*
**begin**

Contents:

- We formalize Desargues's property, *desargues-prop*, that states that if two triangles are perspective from a point, then they are perspective from a line. Note that some planes satisfy that property and some others don't, hence Desargues's property is not a theorem though it is a theorem in projective space geometry.

# 4 Desargues's Property

**context** *projective-plane*
**begin**

**lemma** *distinct3-def*:
*distinct* [*A*, *B*, *C*] = (*A* ≠ *B* ∧ *A* ≠ *C* ∧ *B* ≠ *C*)
**by** *auto*

**definition** *triangle* :: ['*point*, '*point*, '*point*] ⇒ *bool* **where**
*triangle A B C* ≡ *distinct* [*A*,*B*,*C*] ∧ (*line A B* ≠ *line A C*)

**definition** *meet-in* :: '*line* ⇒ '*line* => '*point* => *bool* **where**
*meet-in l m P* ≡ *incid P l* ∧ *incid P m*

**lemma** *meet-col-1*:
**assumes** *meet-in* (*line A B*) (*line C D*) *P*
**shows** *col A B P*
**using** *assms col-def incidA-lAB incidB-lAB meet-in-def*
**by** *blast*

**lemma** *meet-col-2*:
**assumes** *meet-in* (*line A B*) (*line C D*) *P*
**shows** *col C D P*
**using** *assms meet-col-1 meet-in-def*
**by** *auto*

**definition** *meet-3-in* :: ['*line*, '*line*, '*line*, '*point*] ⇒ *bool* **where**
*meet-3-in l m n P* ≡ *meet-in l m P* ∧ *meet-in l n P*

**lemma** *meet-all-3*:
**assumes** *meet-3-in l m n P*
**shows** *meet-in m n P*
**using** *assms meet-3-in-def meet-in-def*
**by** *auto*

**lemma** *meet-comm*:
**assumes** *meet-in l m P*
**shows** *meet-in m l P*
**using** *assms meet-in-def*
**by** *auto*

**lemma** *meet-3-col-1*:
**assumes** *meet-3-in* (*line A B*) *m n P*
**shows** *col A B P*
**using** *assms meet-3-in-def meet-col-2 meet-in-def*
**by** *auto*

**lemma** *meet-3-col-2*:
**assumes** *meet-3-in l* (*line A B*) *n P*
**shows** *col A B P*
**using** *assms col-def incidA-lAB incidB-lAB meet-3-in-def meet-in-def*
**by** *blast*

**lemma** *meet-3-col-3*:
**assumes** *meet-3-in l m* (*line A B*) *P*
**shows** *col A B P*
**using** *assms meet-3-col-2 meet-3-in-def*
**by** *auto*

**lemma** *distinct7-def*: *distinct* [*A*,*B*,*C*,*D*,*E*,*F*,*G*] = ((*A* ≠ *B*) ∧ (*A* ≠ *C*) ∧ (*A* ≠ *D*) ∧ (*A* ≠ *E*) ∧ (*A* ≠ *F*) ∧ (*A* ≠ *G*) ∧
(*B* ≠ *C*) ∧ (*B* ≠ *D*) ∧ (*B* ≠ *E*) ∧ (*B* ≠ *F*) ∧ (*B* ≠ *G*) ∧
(*C* ≠ *D*) ∧ (*C* ≠ *E*) ∧ (*C* ≠ *F*) ∧ (*C* ≠ *G*) ∧
(*D* ≠ *E*) ∧ (*D* ≠ *F*) ∧ (*D* ≠ *G*) ∧
(*E* ≠ *F*) ∧ (*E* ≠ *G*) ∧
(*F* ≠ *G*))
**by** *auto*

**definition** *desargues-config* ::
['*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*] => *bool*
**where**
*desargues-config A B C A′ B′ C′ M N P R* ≡ *distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*,*R*] ∧ ¬ *col A B C*
∧ ¬ *col A′ B′ C′* ∧ *distinct* [(*line A A′*),(*line B B′*),(*line C C′*)] ∧
*meet-3-in* (*line A A′*) (*line B B′*) (*line C C′*) *R* ∧ (*line A B*) ≠ (*line A′ B′*) ∧
(*line B C*) ≠ (*line B′ C′*) ∧ (*line A C*) ≠ (*line A′ C′*) ∧ *meet-in* (*line B C*) (*line B′ C′*) *M* ∧
*meet-in* (*line A C*) (*line A′ C′*) *N* ∧ *meet-in* (*line A B*) (*line A′ B′*) *P*

**lemma** *distinct7-rot-CW*:
**assumes** *distinct* [*A*,*B*,*C*,*D*,*E*,*F*,*G*]
**shows** *distinct* [*C*,*A*,*B*,*F*,*D*,*E*,*G*]
**using** *assms distinct7-def*
**by** *auto*

**lemma** *desargues-config-rot-CW*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** *desargues-config C A B C′ A′ B′ P M N R*
**by** (*smt assms col-rot-CW desargues-config-def distinct3-def distinct7-rot-CW line-comm*
*meet-3-in-def meet-all-3 meet-comm*)

**lemma** *desargues-config-rot-CCW*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** *desargues-config B C A B′ C′ A′ N P M R*
**by** (*simp add*: *assms desargues-config-rot-CW*)

**definition** *are-perspective-from-point* ::
[*'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*] ⇒ *bool* **where**
*are-perspective-from-point A B C A′ B′ C′ R* ≡ *distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*,*R*] ∧
*triangle A B C* ∧
*triangle A′ B′ C′* ∧ *distinct* [(*line A A′*),(*line B B′*),(*line C C′*)] ∧
*meet-3-in* (*line A A′*) (*line B B′*) (*line C C′*) *R*

**definition** *are-perspective-from-line* ::
[*'point*, *'point*, *'point*, *'point*, *'point*, *'point*] ⇒ *bool* **where**
*are-perspective-from-line A B C A′ B′ C′* ≡ *distinct* [*A*,*B*,*C*,*A′*,*B′*,*C′*] ⟶ *triangle A B C* ⟶
*triangle A′ B′ C′* ⟶ *line A B* ≠ *line A′ B′* ⟶ *line A C* ≠ *line A′ C′* ⟶ *line B C* ≠ *line B′ C′* ⟶
*col* (*inter* (*line A B*) (*line A′ B′*)) (*inter* (*line A C*) (*line A′ C′*)) (*inter* (*line B C*) (*line B′ C′*))

**lemma** *meet-in-inter*:
**assumes** *l* ≠ *m*
**shows** *meet-in l m* (*inter l m*)
**by** (*simp add*: *incid-inter-left incid-inter-right meet-in-def*)

**lemma** *perspective-from-point-desargues-config*:
**assumes** *are-perspective-from-point A B C A′ B′ C′ R* **and** *line A B* ≠ *line A′ B′* **and**
*line A C* ≠ *line A′ C′* **and** *line B C* ≠ *line B′ C′*
**shows** *desargues-config A B C A′ B′ C′* (*inter* (*line B C*) (*line B′ C′*)) (*inter* (*line A C*) (*line A′ C′*))
(*inter* (*line A B*) (*line A′ B′*)) *R*
**unfolding** *desargues-config-def distinct7-def distinct3-def*

**using** *assms are-perspective-from-point-def* **apply** *auto*
**apply** (*smt* (*z3*) *ax-uniqueness col-2cycle col-line-ext-1 incidB-lAB line-ext-def mem-Collect-eq triangle-def*)
**apply** (*smt* (*z3*) *ax-uniqueness col-def incidA-lAB line-comm triangle-def*)
**using** *meet-in-inter* **apply** *presburger+*
**done**

**definition** *desargues-prop* :: *bool* **where**
*desargues-prop* ≡
∀ *A B C A′ B′ C′ P*.
*are-perspective-from-point A B C A′ B′ C′ P* ⟶ *are-perspective-from-line A B C A′ B′ C′*

**end**

**end**
**theory** *Pappus-Desargues*

**imports** *Main Projective-Plane-Axioms Pappus-Property Pascal-Property Desargues-Property*
**begin**

Contents:

- We prove Hessenberg's theorem *hessenberg-theorem*: Pappus's property implies Desargues's property in a projective plane.

# 5 Hessenberg's Theorem

**context** *projective-plane*
**begin**

**lemma** *col-ABC-ABD-1*:
**assumes** $A \neq B$ **and** *col A B C* **and** *col A B D*
**shows** *col B C D*
**by** (*metis assms ax-uniqueness col-def*)

**lemma** *col-ABC-ABD-2*:
**assumes** $A \neq B$ **and** *col A B C* **and** *col A B D*
**shows** *col A C D*
**by** (*metis assms ax-uniqueness col-def*)

**lemma** *col-line-eq-1*:
**assumes** $A \neq B$ **and** $B \neq C$**and** *col A B C*
**shows** *line A B = line B C*
**by** (*metis assms ax-uniqueness col-def incidA-lAB incidB-lAB*)

**lemma** *col-line-eq-2*:
**assumes** $A \neq B$ **and** $A \neq C$ **and** *col A B C*
**shows** *line A B = line A C*
**by** (*metis assms col-line-eq-1 col-rot-CW line-comm*)

**lemma** *desargues-config-not-col-1*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col A A′ B′*
**proof**
**assume** *a1*:*col A A′ B′*
**have** *f1*:$A \neq A'$
**using** *assms desargues-config-def distinct7-def*
**by** *force*
**have** *f2*:*col A A′ R*
**using** *assms desargues-config-def meet-3-col-1*
**by** *blast*
**from** *f1* **and** *f2* **and** *a1* **have** *f3*:*col A′ B′ R*
**using** *col-ABC-ABD-1*
**by** *blast*

**from** *a1* **have** *f4*:*line A A′ = line A′ B′*
**by** (*metis assms ax-uniqueness col-def desargues-config-def f1 incidA-lAB incidB-lAB*)
**have** *f5*:*A′ ≠ B′*
**using** *assms desargues-config-def distinct7-def*
**by** *force*
**have** *f6*:*B′ ≠ R*
**using** *assms desargues-config-def distinct7-def*
**by** *force*
**from** *f3* **and** *f5* **and** *f6* **have** *f7*:*line A′ B′ = line B′ R*
**using** *col-line-eq-1*
**by** *blast*
**have** *line B′ R = line B B′*
**by** (*metis a1 assms col-2cycle col-AAB col-line-eq-1 desargues-config-def f6 meet-3-col-2*)
**have** *line A A′ = line B B′*
**by** (*simp add*: ‹*line B′ R = line B B′*› *f4 f7*)
**then have** *False*
**using** *assms desargues-config-def distinct3-def*
**by** *auto*
**then show** *f8*:*col A A′ B′ ⟹ False*
**by** *simp*
**qed**

**lemma** *desargues-config-not-col-2*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col B B′ C′*
**using** *assms desargues-config-not-col-1 desargues-config-rot-CCW*
**by** *blast*

**lemma** *desargues-config-not-col-3*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col C C′ B′*
**by** (*smt assms col-line-eq-1 col-rot-CW desargues-config-def desargues-config-not-col-2*

*desargues-config-rot-CW distinct3-def meet-3-in-def meet-col-1 meet-col-2*)

**lemma** *desargues-config-not-col-4*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col A A′ C′*
**using** *assms desargues-config-not-col-3 desargues-config-rot-CCW*
**by** *blast*

**lemma** *desargues-config-not-col-5*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col B B′ A′*
**using** *assms desargues-config-not-col-3 desargues-config-rot-CW*
**by** *blast*

**lemma** *desargues-config-not-col-6*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col C C′ A′*
**using** *assms desargues-config-not-col-1 desargues-config-rot-CW*
**by** *blast*

**lemma** *desargues-config-not-col-7*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col A B B′*
**proof**
**assume** *a1*:*col A B B′*
**have** *f1*:*col A B R*
**by** (*metis a1 assms col-ABB col-ABC-ABD-2 col-rot-CW desargues-config-def desargues-config-not-col-5*
*meet-3-col-2*)
**have** *f2*:*col A A′ R*
**using** *assms desargues-config-def meet-3-col-1*
**by** *blast*
**have** *f3*:*A* ≠ *A′*
**using** *assms col-AAB desargues-config-not-col-4*
**by** *blast*
**have** *f4*:*A* ≠ *R* **using** *assms desargues-config-def distinct7-def*
**by** *auto*
**from** *f2* **and** *f3* **and** *f4* **have** *f5*:*line A A′* = *line A R*
**using** *col-line-eq-2*
**by** *blast*
**from** *f1* **have** *f6*:*line A R* = *line B R*
**by** (*metis a1 assms col-2cycle col-ABC-ABD-2 desargues-config-not-col-1 f2 f4*)
**have** *f7*:*line B R* = *line B B′*
**by** (*metis ‹line A R = line B R› a1 assms col-AAB col-line-eq-1 desargues-config-def*
*desargues-config-not-col-2 f1*)
**from** *f5* **and** *f6* **and** *f7* **have** *line A A′* = *line B B′*
**by** *simp*
**then have** *False*
**using** *assms desargues-config-def distinct3-def*
**by** *auto*
**thus** *col A B B′* ⟹ *False*
**by** *simp*
**qed**

**lemma** *desargues-config-not-col-8*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col A C C′*

**by** (*smt* (*z3*) *assms col-ABA col-line-eq-1 col-rot-CW desargues-config-def desargues-config-not-col-6 desargues-config-not-col-7 distinct3-def meet-3-col-1 meet-3-col-3 projective-plane.meet-all-3 projective-plane.meet-col-1 projective-plane-axioms*)

**lemma** *desargues-config-not-col-9*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col B A A′*
**using** *assms desargues-config-not-col-8 desargues-config-rot-CCW*
**by** *blast*

**lemma** *desargues-config-not-col-10*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col B C C′*
**using** *assms desargues-config-not-col-7 desargues-config-rot-CCW*
**by** *blast*

**lemma** *desargues-config-not-col-11*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col C A A′*
**using** *assms desargues-config-not-col-7 desargues-config-rot-CW*
**by** *blast*

**lemma** *desargues-config-not-col-12*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R*
**shows** ¬ *col C B B′*
**using** *assms desargues-config-not-col-8 desargues-config-rot-CW*
**by** *blast*

**lemma** *col-inter*:
**assumes** *A* ≠ *C* **and** *B* ≠ *C* **and** *col A B C*
**shows** *inter l* (*line B C*) = *inter l* (*line A C*)
**by** (*metis assms col-line-eq-1 col-line-eq-2*)

**lemma** *lemma-1*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R* **and** *is-pappus*
**shows** *col M N P* ∨ *incid A* (*line B′ C′*) ∨ *incid C′* (*line A B*)
**proof**−
**have** *?thesis* **if** *incid A* (*line B′ C′*) ∨ *incid C′* (*line A B*)
**by** (*simp add*: *that*)

**define** *Q E X F* **where** *Q* = *inter* (*line A B*) (*line B′ C′*) **and** *E* = *inter* (*line A C*) (*line R Q*)
**and** *X* = *inter* (*line A C′*) (*line R B*) **and** *F* = *inter* (*line A′ C′*) (*line R Q*)
**have** *col X E M* **if** ¬ *incid A* (*line B′ C′*) **and** ¬ *incid C′* (*line A B*)
**proof**−
**have** *f1*:*distinct* [*C*,*C′*,*R*,*Q*,*B*,*A*]
**by** (*smt Q-def* ‹¬ *incid A* (*line B′ C′*)› ‹¬ *incid C′* (*line A B*)› *assms*(*1*) *col-ABB col-A-B-ABl*
*col-A-B-lAB col-line-eq-2 col-rot-CW desargues-config-def desargues-config-not-col-12*

*desargues-config-not-col-2 desargues-config-not-col-3 desargues-config-not-col-7*

*desargues-config-not-col-9 distinct6-def incidA-lAB line-comm meet-3-col-1 meet-3-col-2*)
**have** *f2*: *col C C′ R*
**using** *assms*(*1*) *desargues-config-def meet-3-col-3*
**by** *blast*
**have** *f3*: *col Q B A*
**using** *Q-def col-2cycle col-A-B-ABl col-rot-CW*
**by** *blast*
**have** *f4*: *is-a-intersec E C A Q R*
**using** *E-def col-2cycle inter-is-a-intersec is-a-intersec-def*
**by** *auto*
**have** *f5*:*is-a-intersec M C B Q C′*
**by** (*metis Q-def assms*(*1*) *col-2cycle col-ABB col-ABC-ABD-1 col-A-B-lAB col-rot-CW*
*desargues-config-def is-a-intersec-def meet-col-1 meet-col-2*)
**have** *f6*:*is-a-intersec X C′ A B R*
**using** *X-def col-2cycle inter-is-a-intersec is-a-intersec-def*
**by** *auto*
**from** *f1* **and** *f2* **and** *f3* **and** *f4* **and** *f5* **and** *f6* **have** *col M X E*
**using** *assms*(*2*) *is-pappus2-def is-pappus-def*
**by** *blast*
**then show** *col X E M*
**using** *col-def*
**by** *auto*
**qed**
**have** *col P X F* **if** ¬ *incid A* (*line B′ C′*) **and** ¬ *incid C′* (*line A B*)
**proof**−
**have** *f1*:*distinct* [*A′*,*A*,*R*,*Q*,*B′*,*C′*]
**by** (*smt Q-def* ‹¬ *incid A* (*line B′ C′*)› ‹¬ *incid C′* (*line A B*)› *assms*(*1*) *col-AAB col-A-B-ABl*
*col-A-B-lAB col-line-eq-1 col-rot-CW desargues-config-def desargues-config-not-col-2*
*desargues-config-not-col-3 desargues-config-not-col-4 desargues-config-not-col-6*
*desargues-config-not-col-7 distinct6-def incidB-lAB meet-3-col-2 meet-3-col-3*)
**have** *f2*:*col A′ A R*
**by** (*metis assms*(*1*) *col-ABA col-line-eq-1 desargues-config-def meet-3-col-1*)
**have** *f3*:*col Q B′ C′*
**by** (*simp add*: *Q-def col-A-B-lAB col-rot-CW*)
**have** *is-a-intersec* (*inter* (*line A B*) (*line A′ B′*)) *A′ B′ Q A*
**by** (*metis Q-def col-def incidA-lAB incid-inter-left inter-is-a-intersec is-a-intersec-def*)
**then have** *f4*:*is-a-intersec P A′ B′ Q A*
**using** *assms*(*1*) *desargues-config-def meet-in-def uniq-inter*
**by** *auto*
**have** *f5*:*is-a-intersec X A C′ B′ R*
**by** (*metis X-def assms*(*1*) *col-def col-line-eq-2 desargues-config-def desargues-config-not-col-9*
*f2 inter-is-a-intersec is-a-intersec-def line-comm meet-3-col-2*)
**have** *f6*:*is-a-intersec F A′ C′ Q R*

**by** (*metis F-def inter-is-a-intersec inter-line-line-comm*)
**from** *f1* **and** *f2* **and** *f3* **and** *f4* **and** *f5* **and** *f6* **and** *assms*(*2*)
**show** *col P X F*
**using** *is-pappus2-def is-pappus-def*
**by** *blast*
**qed**
**have** *col M N P* **if** ¬ *incid A* (*line B′ C′*) **and** ¬ *incid C′* (*line A B*)
**proof** −
**have** *f1*:*Q* ≠ *C′* ∧ *X* ≠ *E* ∧ *line Q C′* ≠ *line X E*
**by** (*smt E-def Q-def X-def* ‹¬ *incid A* (*line B′ C′*)› ‹¬ *incid C′* (*line A B*)› *assms*(*1*) *col-ABB*
*col-A-B-ABl col-A-B-lAB col-line-eq-2 col-rot-CW desargues-config-def*
*desargues-config-not-col-10 desargues-config-not-col-2 desargues-config-not-col-8*
*incidB-lAB incid-C-AB line-comm meet-3-col-1 meet-3-col-2 meet-3-col-3*)
**have** *f2*:*E* ≠ *A* ∧ *C′* ≠ *F* ∧ *line E A* ≠ *line C′ F*
**by** (*smt E-def F-def Q-def X-def* ‹⟦¬ *incid A* (*line B′ C′*); ¬ *incid C′* (*line A B*)⟧ ⟹ *col X E M*› *assms*(*1*) *ax-uniqueness col-def desargues-config-def desargues-config-not-col-10*
*desargues-config-not-col-3 f1 incidA-lAB incidB-lAB incid-inter-left incid-inter-right*
*meet-in-def that*(*1*))
**have** *f3*:*Q* ≠ *A* ∧ *X* ≠ *F* ∧ *line Q A* ≠ *line X F*
**by** (*smt F-def Q-def X-def* ‹¬ *incid A* (*line B′ C′*)› ‹¬ *incid C′* (*line A B*)› *assms*(*1*)
*ax-uniqueness col-def desargues-config-def desargues-config-not-col-10*
*desargues-config-not-col-2 desargues-config-not-col-7 incidA-lAB incidB-lAB incid-inter-left*
*incid-inter-right meet-3-col-2 meet-3-col-3*)
**have** *f4*:*col Q E F*
**using** *E-def F-def col-def incidB-lAB incid-inter-right*
**by** *blast*
**have** *f5*:*col X C′ A*
**using** *X-def col-2cycle col-A-B-ABl col-rot-CW*
**by** *blast*
**have** *f6*:*is-a-intersec M Q C′ X E*
**by** (*metis Q-def* ‹⟦¬ *incid A* (*line B′ C′*); ¬ *incid C′* (*line A B*)⟧ ⟹ *col X E M*› *assms*(*1*)
*col-ABB col-A-B-lAB col-def col-line-eq-1 desargues-config-def incidB-lAB is-a-intersec-def*
*meet-in-def that*(*1*) *that*(*2*))
**have** *f7*:*is-a-intersec N E A C′ F*
**by** (*metis E-def F-def assms*(*1*) *ax-uniqueness col-rot-CW desargues-config-def f2 incidA-lAB*
*incidB-lAB incid-inter-left is-a-intersec-def meet-col-1 meet-col-2*)
**have** *f8*:*is-a-intersec P Q A X F*
**by** (*metis Q-def* ‹⟦¬ *incid A* (*line B′ C′*); ¬ *incid C′* (*line A B*)⟧ ⟹ *col P*

*X F*› *assms*(*1*) *col-AAB col-A-B-ABl col-line-eq-2 col-rot-CW desargues-config-def is-a-intersec-def meet-col-1 that*(*1*) *that*(*2*))
**from** *f1* **and** *f2* **and** *f3* **and** *f4* **and** *f5* **and** *f6* **and** *f7* **and** *f8* **and** *assms*(*2*)
**show** *col M N P*
**using** *is-pappus2-def is-pappus-def*
**by** *blast*
**qed**
**show** *col M N P* ∨ *incid A* (*line B′ C′*) ∨ *incid C′* (*line A B*)
**using** ‹⟦¬ *incid A* (*line B′ C′*); ¬ *incid C′* (*line A B*)⟧ ⟹ *col M N P*›
**by** *auto*
**qed**

**corollary** *corollary-1*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R* **and** *is-pappus*
**shows** *col M N P* ∨ ((*incid A* (*line B′ C′*) ∨ *incid C′* (*line A B*)) ∧
(*incid C* (*line A′ B′*) ∨ *incid B′* (*line A C*)) ∧ (*incid B* (*line A′ C′*) ∨ *incid A′* (*line B C*)))
**by** (*metis assms*(*1*) *assms*(*2*) *col-rot-CW desargues-config-rot-CCW lemma-1 line-comm*)

**definition** *triangle-circumscribes-triangle* ::
['*point*, '*point*, '*point*, '*point*, '*point*, '*point*] ⇒ *bool* **where**
*triangle-circumscribes-triangle A′ B′ C′ A B C* ≡ *incid A* (*line B′ C′*) ∧ *incid C* (*line A′ B′*) ∧
*incid B* (*line A′ C′*)

**lemma** *lemma-2*:
**assumes** *desargues-config A B C A′ B′ C′ M N P R* **and** *incid A* (*line B′ C′*) ∨ *incid C′* (*line A B*)
**and** *incid C* (*line A′ B′*) ∨ *incid B′* (*line A C*) **and** *incid B* (*line A′ C′*) ∨ *incid A′* (*line B C*)
**shows** *col M N P* ∨ *triangle-circumscribes-triangle A B C A′ B′ C′* ∨ *triangle-circumscribes-triangle A′ B′ C′ A B C*
**by** (*smt assms ax-uniqueness col-def desargues-config-not-col-1*
*desargues-config-not-col-11 desargues-config-not-col-12 desargues-config-not-col-2*

*desargues-config-not-col-3 desargues-config-not-col-9 incidA-lAB incidB-lAB*
*triangle-circumscribes-triangle-def*)

**lemma** *lemma-3*:
**assumes** *is-pappus* **and** *desargues-config A B C A′ B′ C′ M N P R* **and**
*triangle-circumscribes-triangle A′ B′ C′ A B C*
**shows** *col M N P*
**proof**−
**define** *S T* **where** *S* = *inter* (*line C′ P*) (*line R A*) **and** *T* = *inter* (*line C′ P*) (*line R B*)

**have** *col N S B′*
**proof**−

**have** *f1*:*distinct* [*R*,*C*,*C'*,*P*,*B*,*A*]
**by** (*smt assms*(*2*) *col-AAB col-line-eq-2 col-rot-CW desargues-config-def desargues-config-not-col-1 desargues-config-not-col-12 desargues-config-not-col-2 desargues-config-not-col-5 desargues-config-not-col-7 desargues-config-not-col-8 desargues-config-not-col-9 distinct6-def line-comm meet-3-col-1 meet-3-col-2 meet-col-1 meet-col-2*)
**have** *f2*:*col R C C'*
**using** *assms*(*2*) *col-rot-CW desargues-config-def meet-3-col-3*
**by** *blast*
**have** *f3*:*col P B A*
**by** (*metis assms*(*2*) *col-rot-CW desargues-config-def line-comm meet-col-1*)
**have** *f4*:*is-a-intersec B' R B P C*
**by** (*metis assms*(*2*) *assms*(*3*) *col-def desargues-config-def incidB-lAB is-a-intersec-def meet-3-col-2 meet-in-def triangle-circumscribes-triangle-def*)
**have** *f5*:*is-a-intersec S R A P C'*
**using** *S-def col-2cycle inter-is-a-intersec is-a-intersec-def*
**by** *auto*
**have** *line B C' = line A' C'*
**by** (*metis* ‹*distinct* [*R*,*C*,*C'*,*P*,*B*,*A*]› *assms*(*3*) *ax-uniqueness distinct6-def incidA-lAB incidB-lAB triangle-circumscribes-triangle-def*)
**then have** *f6*:*is-a-intersec N C A B C'*
**by** (*metis assms*(*2*) *desargues-config-def inter-is-a-intersec line-comm meet-in-def uniq-inter*)
**from** *f1* **and** *f2* **and** *f3* **and** *f4* **and** *f5* **and** *f6* **and** *assms*(*1*) **have** *col B' N S*
**using** *is-pappus2-def is-pappus-def*
**by** *blast*
**then show** *col N S B'*
**by** (*simp add*: *col-rot-CW*)
**qed**
**have** *col M T A'*
**proof**−
**have** *f1*:*distinct* [*R*,*C*,*C'*,*P*,*A*,*B*]
**by** (*smt assms*(*2*) *col-ABA col-line-eq-2 col-rot-CW desargues-config-def desargues-config-not-col-1 desargues-config-not-col-12 desargues-config-not-col-2 desargues-config-not-col-5 desargues-config-not-col-7 desargues-config-not-col-8 desargues-config-not-col-9 distinct6-def line-comm meet-3-col-1 meet-3-col-2 meet-col-1 meet-col-2*)
**have** *f2*:*col R C C'*
**using** *assms*(*2*) *col-rot-CW desargues-config-def meet-3-col-3*
**by** *blast*
**have** *f3*:*col P A B*
**using** *assms*(*2*) *col-rot-CW desargues-config-def meet-col-1*

**by** *blast*
**have** *f4*:*line P C = line A′ B′*
**by** (*metis ‹distinct [R,C,C′,P,A,B]› assms(2) assms(3) ax-uniqueness desargues-config-def*
*distinct6-def incidA-lAB incidB-lAB meet-in-def triangle-circumscribes-triangle-def*)
**have** *f5*:*line R A = line A A′*
**by** (*metis ‹distinct [R,C,C′,P,A,B]› assms(2) col-AAB col-line-eq-2 desargues-config-def*
*desargues-config-not-col-1 distinct6-def line-comm meet-3-col-1*)
**from** *f4* **and** *f5* **have** *f6*:*is-a-intersec A′ R A P C*
**by** (*metis col-def incidA-lAB incidB-lAB is-a-intersec-def*)
**have** *line A C′ = line B′ C′*
**by** (*metis assms(3) ax-uniqueness distinct6-def f1 incidA-lAB incidB-lAB*
*triangle-circumscribes-triangle-def*)
**then have** *f7*:*is-a-intersec M C B A C′*
**by** (*metis assms(2) col-rot-CW desargues-config-def is-a-intersec-def line-comm meet-col-1*
*meet-col-2*)
**have** *f8*:*is-a-intersec T R B P C′*
**by** (*metis T-def distinct6-def f1 inter-comm-line-line-comm inter-is-a-intersec line-comm*)
**from** *f1* **and** *f2* **and** *f3* **and** *f6* **and** *f7* **and** *f8* **and** *assms(1)* **have** *col A′ M T*
**using** *is-pappus2-def is-pappus-def*
**by** *blast*
**thus** *col M T A′*
**by** (*simp add*: *col-rot-CW*)
**qed**
**then show** *col M N P*
**proof**−
**have** *f1*:*S ≠ T ∧ B ≠ A ∧ line S T ≠ line B A*
**by** (*smt T-def ‹col N S B′› assms(2) assms(3) ax-uniqueness col-AAB col-line-eq-2 col-rot-CW*
*desargues-config-def desargues-config-not-col-10 desargues-config-not-col-7*
*desargues-config-not-col-9 incidB-lAB incid-inter-left incid-inter-right*
*line-comm meet-3-col-2 meet-3-col-3 meet-col-1 meet-col-2 triangle-circumscribes-triangle-def*)
**have** *f2*:*A ≠ B′ ∧ T ≠ A′ ∧ line A B′ ≠ line T A′*
**by** (*smt* (*verit*) *S-def T-def assms(2) assms(3) col-A-B-ABl col-line-eq-2 desargues-config-def desargues-config-not-col-1 desargues-config-not-col-9 f1 incidA-lAB inter-comm line-comm meet-3-col-1 meet-col-2 projective-plane.col-def projective-plane-axioms triangle-circumscribes-triangle-def*)
**have** *f3*:*S ≠ B′ ∧ B ≠ A′*
**by** (*smt S-def assms(2) assms(3) ax-uniqueness col-A-B-ABl col-line-eq-2 col-rot-CW*
*desargues-config-def desargues-config-not-col-2 desargues-config-not-col-5*
*desargues-config-not-col-7 incidA-lAB incidB-lAB incid-inter-right inter-comm line-comm*
*meet-3-col-2 meet-in-def triangle-circumscribes-triangle-def*)
**then have** *f4*:*line S B′ ≠ line B A′*

**by** (*metis assms(2) col-def desargues-config-not-col-5 incidA-lAB incidB-lAB*)
**have** *f5*:*col S A A'*
**by** (*metis S-def assms(2) col-ABC-ABD-1 col-A-B-lAB col-rot-CW desargues-config-def*
*desargues-config-not-col-8 meet-3-col-1 meet-3-col-3*)
**have** *f6*:*col B T B'*
**by** (*metis T-def assms(2) col-def col-line-eq-2 desargues-config-def desargues-config-not-col-10*
*incidB-lAB incid-inter-right line-comm meet-3-col-2 meet-3-col-3*)
**have** *f7*:*is-a-intersec P S T B A*
**by** (*metis S-def T-def assms(2) col-ABC-ABD-1 col-A-B-ABl col-def desargues-config-def incidA-lAB*
*incidB-lAB is-a-intersec-def meet-in-def*)
**have** *f8*:*is-a-intersec M A B' T A'*
**by** (*smt* (*verit*, *del-insts*) ‹*col M T A'*› *assms(2) assms(3) col-rot-CW desargues-config-def f2 incidA-lAB incidB-lAB is-a-intersec-def meet-col-2 projective-plane.ax-uniqueness projective-plane-axioms triangle-circumscribes-triangle-def*)
**have** *f9*:*is-a-intersec N S B' B A'*
**using** ‹*col N S B'*› *assms(2) assms(3) col-def desargues-config-def incidA-lAB is-a-intersec-def*
*meet-in-def triangle-circumscribes-triangle-def*
**by** *auto*
**from** *f1* **and** *f2* **and** *f3* **and** *f4* **and** *f5* **and** *f6* **and** *f7* **and** *f8* **and** *f9* **and** *assms(1)* **have** *col P M N*
**using** *is-pappus2-def is-pappus-def*
**by** *blast*
**thus** *col M N P*
**by** (*simp add*: *col-rot-CW*)
**qed**
**qed**

**theorem** *pappus-desargues*:
**assumes** *is-pappus* **and** *desargues-config A B C A' B' C' M N P R*
**shows** *col M N P*
**proof**−
**have** *f1*:*col M N P* ∨ ((*incid A* (*line B' C'*) ∨ *incid C'* (*line A B*)) ∧
(*incid C* (*line A' B'*) ∨ *incid B'* (*line A C*)) ∧ (*incid B* (*line A' C'*) ∨ *incid A'* (*line B C*)))
**using** *assms corollary-1*
**by** *auto*
**have** *f2*:*col M N P* ∨ *triangle-circumscribes-triangle A B C A' B' C'* ∨ *triangle-circumscribes-triangle A' B' C' A B C*
**if** (*incid A* (*line B' C'*) ∨ *incid C'* (*line A B*)) ∧ (*incid C* (*line A' B'*) ∨ *incid B'* (*line A C*))
∧ (*incid B* (*line A' C'*) ∨ *incid A'* (*line B C*))
**using** *assms(2) lemma-2 that*
**by** *auto*
**have** *f3*:*col M N P* **if** *triangle-circumscribes-triangle A' B' C' A B C*
**using** *assms lemma-3 that*

```
    by auto
  have f4:col M N P if triangle-circumscribes-triangle A B C A' B' C'
  proof−
    have desargues-config A' B' C' A B C M N P R
    proof−
      have f1:distinct [A',B',C',A,B,C,R]
        using assms(2) desargues-config-def distinct7-def
        by auto
      have f2:¬ col A' B' C'
        using assms(2) desargues-config-def
        by blast
      have f3:¬ col A B C
        using assms(2) desargues-config-def
        by blast
      have f4:distinct [(line A' A),(line B' B),(line C' C)]
        by (metis assms(2) desargues-config-def line-comm)
      have f5:meet-3-in (line A' A) (line B' B) (line C' C) R
        by (metis assms(2) desargues-config-def line-comm)
     have f6:(line A' B') ≠ (line A B) ∧ (line B' C') ≠ (line B C) ∧ (line A' C')
≠ (line A C)
        using assms(2) desargues-config-def
        by auto
     have f7:meet-in (line B' C') (line B C) M ∧ meet-in (line A' C') (line A C)
N ∧
      meet-in (line A' B') (line A B) P
        using assms(2) desargues-config-def meet-comm
        by blast
     from f1 and f2 and f3 and f4 and f5 and f6 and f7 show desargues-config
A' B' C' A B C M N P R
        by (simp add: desargues-config-def)
    qed
    then show col M N P
      using assms(1) lemma-3 that
      by blast
  qed
  from f1 and f2 and f3 and f4 show col M N P
    by blast
qed

theorem hessenberg-thereom:
  assumes is-pappus
  shows desargues-prop
  by (smt are-perspective-from-line-def assms col-def desargues-prop-def pappus-desargues

      perspective-from-point-desargues-config)

corollary pascal-desargues:
  assumes pascal-prop
  shows desargues-prop
```

**by** (*simp add*: *assms hessenberg-thereom pascal-pappus*)

**end**

**end**
**theory** *Higher-Projective-Space-Rank-Axioms*
**imports** *Main*
**begin**

Contents:

- Following [2] we introduce a set of axioms for projective space geometry based on the notions of matroid and rank.

# 6 A Based-rank Set of Axioms for Projective Space Geometry

**locale** *higher-projective-space-rank* =

**fixes** *rk* :: *'point set* ⇒ *nat*

**assumes**
*matroid-ax-1a*: *rk X* ≥ *0* **and**
*matroid-ax-1b*: *rk X* ≤ *card X* **and**
*matroid-ax-2*: *X* ⊆ *Y* ⟶ *rk X* ≤ *rk Y* **and**
*matroid-ax-3*: *rk* (*X* ∪ *Y*) + *rk* (*X* ∩ *Y*) ≤ *rk X* + *rk Y*

**assumes**
*rk-ax-singleton*: *rk* {*P*} ≥ *1* **and**
*rk-ax-couple*: *P* ≠ *Q* ⟶ *rk* {*P*,*Q*} ≥ *2* **and**
*rk-ax-pasch*: *rk* {*A*,*B*,*C*,*D*} ≤ *3* ⟶ (∃ *J*. *rk* {*A*,*B*,*J*} = *2* ∧ *rk* {*C*,*D*,*J*} = *2*)
**and**
*rk-ax-3-pts*: ∃ *C*. *rk* {*A*,*B*,*C*} = *2* ∧ *rk* {*B*,*C*} = *2* ∧ *rk* {*A*,*C*} = *2* **and**
*rk-ax-dim*: ∃ *A B C D*. *rk* {*A*,*B*,*C*,*D*} ≥ *4*

**end**
**theory** *Matroid-Rank-Properties*
**imports** *Main Higher-Projective-Space-Rank-Axioms*
**begin**

Contents:

- In this file we introduce the basic lemmas and properties derived from our based-rank axioms that will allow us to simplify our future proofs.

# 7 Proof Techniques Using Ranks

**context** *higher-projective-space-rank*
**begin**

**lemma** *matroid-ax-3-alt*:
**assumes** $I \subseteq X \cap Y$
**shows** $rk\ (X \cup Y) + rk\ I \leq rk\ X + rk\ Y$
**by** (*metis add.commute add-le-cancel-right assms matroid-ax-2 matroid-ax-3 order-trans*)

**lemma** *rk-uniqueness*:
**assumes** $rk\ \{A,B\} = 2$ **and** $rk\ \{C,D\} = 2$ **and** $rk\ \{A,B,M\} \leq 2$ **and** $rk\ \{C,D,M\} \leq 2$ **and**
$rk\ \{A,B,P\} \leq 2$ **and** $rk\ \{C,D,P\} \leq 2$ **and** $rk\ \{A,B,C,D\} \geq 3$
**shows** $rk\ \{M,P\} = 1$
**proof**−
**have** $\{A,B,M\} \cup \{A,B,P\} = \{A,B,M,P\}$
**by** *auto*
**have** $rk\ \{A,B,M,P\} + rk\ \{A,B\} \leq rk\ \{A,B,M\} + rk\ \{A,B,P\}$
**by** (*metis* (*full-types*) ‹$\{A, B, M\} \cup \{A, B, P\} = \{A, B, M, P\}$› *insert-commute le-inf-iff*
*matroid-ax-3-alt subset-insertI*)
**then have** $rk\ \{A,B,M,P\} = 2$
**by** (*metis add-diff-cancel-right' antisym assms*(1) *assms*(3) *assms*(5) *insert-commute le-diff-conv matroid-ax-2 subset-insertI*)
**have** $\{C,D,M\} \cup \{C,D,P\} = \{C,D,M,P\}$
**by** *auto*
**have** $rk\ \{C,D,M,P\} + rk\ \{C,D\} \leq rk\ \{C,D,M\} + rk\ \{C,D,P\}$
**by** (*metis Un-insert-left Un-upper1* ‹$\{C, D, M\} \cup \{C, D, P\} = \{C, D, M, P\}$› *insert-is-Un le-inf-iff*
*matroid-ax-3-alt*)
**then have** *i1*:$rk\ \{C,D,M,P\} + 2 \leq 4$
**using** *assms*(2) *assms*(4) *assms*(6)
**by** *linarith*
**moreover have** *i2*:$rk\ \{C,D,M,P\} \geq 2$
**by** (*metis assms*(2) *insertI1 insert-subset matroid-ax-2 subset-insertI*)
**from** *i1* **and** *i2* **have** $rk\ \{C,D,M,P\} = 2$
**by** *linarith*
**have** $rk\ \{A,B,C,D,M,P\} \geq 3$
**by** (*metis Un-insert-right Un-upper2 assms*(7) *matroid-ax-2 order-trans sup-bot.right-neutral*)
**have** $\{A,B,M,P\} \cup \{C,D,M,P\} = \{A,B,C,D,M,P\}$
**by** *auto*
**then have** $rk\ \{A,B,C,D,M,P\} + rk\ \{M,P\} \leq rk\ \{A,B,M,P\} + rk\ \{C,D,M,P\}$

**by** (*smt le-inf-iff matroid-ax-3-alt order-trans subset-insertI*)
**then have** *i3*:*rk* {*A*,*B*,*C*,*D*,*M*,*P*} + *rk* {*M*,*P*} ≤ *4*
**using** ‹*rk* {*A*, *B*, *M*, *P*} = *2*› ‹*rk* {*C*, *D*, *M*, *P*} = *2*›
**by** *linarith*
**have** *i4*:*rk* {*A*,*B*,*C*,*D*,*M*,*P*} + *rk* {*M*,*P*} ≥ *3* + *rk*{*M*,*P*}
**by** (*simp add*: ‹*3* ≤ *rk* {*A*, *B*, *C*, *D*, *M*, *P*}›)
**from** *i3* **and** *i4* **show** *rk* {*M*,*P*} = *1*
**by** (*metis* (*no-types*, *lifting*) ‹*rk* {*A*, *B*, *C*, *D*, *M*, *P*} + *rk* {*M*, *P*} ≤ *rk* {*A*, *B*, *M*, *P*} + *rk* {*C*, *D*, *M*, *P*}›
‹*rk* {*A*, *B*, *M*, *P*} = *2*› ‹*rk* {*C*, *D*, *M*, *P*} = *2*› *add-le-cancel-left add-numeral-left antisym*
*insert-absorb2 numeral-Bit1 numeral-One numeral-plus-one one-add-one one-le-numeral*
*order-trans rk-ax-couple rk-ax-singleton*)
**qed**

**lemma** *rk-ax-dim-alt*: ∃ *A B C D*. ∀ *M*. *rk* {*A*,*B*,*M*} ≠ *2* ∨ *rk* {*C*,*D*,*M*} ≠ *2*
**proof**−
**obtain** *A B C D* **where** *f1*:*rk* {*A*,*B*,*C*,*D*} ≥ *4*
**using** *rk-ax-dim*
**by** *auto*
**have** ∀ *M*. *rk* {*A*,*B*,*M*} ≠ *2* ∨ *rk* {*C*,*D*,*M*} ≠ *2*
**proof**
**fix** *M*
**have** {*A*,*B*,*C*,*D*,*M*} = {*A*,*B*,*M*} ∪ {*C*,*D*,*M*}
**by** *auto*
**then have** *rk* {*A*,*B*,*C*,*D*,*M*} + *rk* {*M*} ≤ *rk* {*A*,*B*,*M*} + *rk* {*C*,*D*,*M*}
**by** (*smt le-inf-iff matroid-ax-3-alt order-trans subset-insertI*)
**then have** *rk* {*A*,*B*,*C*,*D*,*M*} ≤ *3* **if** *rk* {*A*,*B*,*M*} = *2* **and** *rk* {*C*,*D*,*M*} = *2*
**by** (*smt* (*z3*) *One-nat-def Suc-le-eq Suc-numeral add-Suc-right add-le-same-cancel1 nat-1-add-1 not-less numeral-Bit1 numerals*(*1*) *order-trans rk-ax-singleton semiring-norm*(*2*) *that*(*1*) *that*(*2*))
**then have** *rk* {*A*,*B*,*C*,*D*} ≤ *3* **if** *rk* {*A*,*B*,*M*} = *2* **and** *rk* {*C*,*D*,*M*} = *2*
**by** (*smt insert-commute matroid-ax-2 order-trans subset-insertI that*(*1*) *that*(*2*))
**thus** *rk* {*A*, *B*, *M*} ≠ *2* ∨ *rk* {*C*, *D*, *M*} ≠ *2*
**using** ‹*4* ≤ *rk* {*A*, *B*, *C*, *D*}›
**by** *linarith*
**qed**
**thus** ∃ *A B C D*. ∀ *M*. *rk* {*A*, *B*, *M*} ≠ *2* ∨ *rk* {*C*, *D*, *M*} ≠ *2*
**by** *blast*
**qed**

**lemma** *rk-empty*: *rk* {} = *0*
**proof**−
**have** *rk* {} ≥ *0*
**by** *simp*
**have** *rk* {} ≤ *0*
**by** (*metis card.empty matroid-ax-1b*)

**thus** *rk* {} = *0*
**by** *blast*
**qed**

**lemma** *matroid-ax-2-alt*: *rk X* ≤ *rk* (*X* ∪ {*x*}) ∧ *rk* (*X* ∪ {*x*}) ≤ *rk X* + *1*
**proof**
**have** *X* ⊆ *X* ∪ {*x*}
**by** *auto*
**thus** *rk X* ≤ *rk* (*X* ∪ {*x*})
**by** (*simp add*: *matroid-ax-2*)
**have** *rk* {*x*} ≤ *1*
**by** (*metis One-nat-def card.empty card-Suc-eq insert-absorb insert-not-empty matroid-ax-1b*)
**thus** *rk* (*X* ∪ {*x*}) ≤ *rk X* + *1*
**by** (*metis add-leD1 le-antisym matroid-ax-3 rk-ax-singleton*)
**qed**

**lemma** *matroid-ax-3-alt'*: *rk* (*X* ∪ {*y*}) = *rk* (*X* ∪ {*z*}) ⟶ *rk* (*X* ∪ {*z*}) = *rk X* ⟶ *rk X* = *rk* (*X* ∪ {*y*,*z*})
**proof**−
**have** *i1*:*rk X* ≤ *rk* (*X* ∪ {*y*,*z*})
**using** *matroid-ax-2*
**by** *blast*
**have** *i2*:*rk X* ≥ *rk* (*X* ∪ {*y*,*z*}) **if** *rk* (*X* ∪ {*y*}) = *rk* (*X* ∪ {*z*}) **and** *rk* (*X* ∪ {*z*}) = *rk X*
**proof**−
**have** (*X* ∪ {*y*}) ∪ (*X* ∪ {*z*}) = *X* ∪ {*y*,*z*}
**by** *blast*
**then have** *rk* (*X* ∪ {*y*,*z*}) + *rk X* ≤ *rk X* + *rk X*
**by** (*metis* ‹*rk* (*X* ∪ {*y*}) = *rk* (*X* ∪ {*z*})› ‹*rk* (*X* ∪ {*z*}) = *rk X*› *inf-sup-ord*(*3*) *le-inf-iff*
*matroid-ax-3-alt*)
**thus** *rk* (*X* ∪ {*y*,*z*}) ≤ *rk X*
**by** *simp*
**qed**
**thus** *rk* (*X* ∪ {*y*}) = *rk* (*X* ∪ {*z*}) ⟶ *rk* (*X* ∪ {*z*}) = *rk X* ⟶ *rk X* = *rk* (*X* ∪ {*y*, *z*})
**using** *antisym i1*
**by** *blast*
**qed**

**lemma** *rk-ext*:
**assumes** *rk X* ≤ *3*
**shows** ∃ *P*. *rk*(*X* ∪ {*P*}) = *rk X* + *1*
**proof**−
**obtain** *A B C D* **where** *rk* {*A*,*B*,*C*,*D*} ≥ *4*
**using** *rk-ax-dim*
**by** *auto*
**have** *f1*:*rk* (*X* ∪ {*A*, *B*, *C*, *D*}) ≥ *4*

**by** (*metis Un-upper2* ‹*4* ≤ *rk* {*A*, *B*, *C*, *D*}› *matroid-ax-2 sup.coboundedI2 sup.orderE*)
**have** *rk* (*X* ∪ {*A*, *B*, *C*, *D*}) = *rk X* **if** *rk*(*X* ∪ {*A*}) = *rk*(*X* ∪ {*B*}) **and** *rk*(*X* ∪ {*B*}) = *rk*(*X* ∪ {*C*})
**and** *rk*(*X* ∪ {*C*}) = *rk*(*X* ∪ {*D*}) **and** *rk*(*X* ∪ {*D*}) = *rk X*
**using** *matroid-ax-3-alt′ that*(*1*) *that*(*2*) *that*(*3*) *that*(*4*)
**by** *auto*
**then have** *f2*:*rk* (*X* ∪ {*A*, *B*, *C*, *D*}) ≤ *3* **if** *rk*(*X* ∪ {*A*}) = *rk*(*X* ∪ {*B*}) **and** *rk*(*X* ∪ {*B*}) = *rk*(*X* ∪ {*C*})
**and** *rk*(*X* ∪ {*C*}) = *rk*(*X* ∪ {*D*}) **and** *rk*(*X* ∪ {*D*}) = *rk X*
**using** *assms that*(*1*) *that*(*2*) *that*(*3*) *that*(*4*)
**by** *linarith*
**from** *f1* **and** *f2* **have** *False* **if** *rk*(*X* ∪ {*A*}) = *rk*(*X* ∪ {*B*}) **and** *rk*(*X* ∪ {*B*}) = *rk*(*X* ∪ {*C*})
**and** *rk*(*X* ∪ {*C*}) = *rk*(*X* ∪ {*D*}) **and** *rk*(*X* ∪ {*D*}) = *rk X*
**using** *that*(*1*) *that*(*2*) *that*(*3*) *that*(*4*)
**by** *linarith*
**then have** *rk* (*X* ∪ {*A*}) = *rk X* + *1* ∨ *rk* (*X* ∪ {*B*}) = *rk X* + *1* ∨ *rk* (*X* ∪ {*C*}) = *rk X* + *1* ∨
*rk* (*X* ∪ {*D*}) = *rk X* + *1*
**by** (*smt One-nat-def Suc-le-eq Suc-numeral Un-upper2* ‹*4* ≤ *rk* {*A*, *B*, *C*, *D*}›
‹⟦*rk* (*X* ∪ {*A*}) = *rk* (*X* ∪ {*B*}); *rk* (*X* ∪ {*B*}) = *rk* (*X* ∪ {*C*}); *rk* (*X* ∪ {*C*}) = *rk* (*X* ∪ {*D*}); *rk* (*X* ∪ {*D*}) = *rk X*⟧ ⟹ *rk* (*X* ∪ {*A*, *B*, *C*, *D*}) = *rk X*›
*add.right-neutral add-Suc-right assms antisym-conv1 matroid-ax-2 matroid-ax-2-alt*
*not-less semiring-norm*(*2*) *semiring-norm*(*8*) *sup.coboundedI2 sup.orderE*)
**thus** ∃ *P* . *rk* (*X* ∪ {*P*}) = *rk X* + *1*
**by** *blast*
**qed**

**lemma** *rk-singleton* : ∀ *P*. *rk* {*P*} = *1*
**proof**
**fix** *P*
**have** *f1*:*rk* {*P*} ≤ *1*
**by** (*metis One-nat-def card.empty card-Suc-eq insert-absorb insert-not-empty matroid-ax-1b*)
**have** *f2*:*rk* {*P*} ≥ *1*
**using** *rk-ax-singleton*
**by** *auto*
**from** *f1* **and** *f2* **show** *rk* {*P*} = *1*
**using** *antisym*
**by** *blast*
**qed**

**lemma** *rk-singleton-bis* :
**assumes** *A* = *B*
**shows** *rk* {*A*, *B*} = *1*
**by** (*simp add*: *assms rk-singleton*)

**lemma** *rk-couple* :
**assumes** $A \neq B$
**shows** $rk\ \{A,\ B\} = 2$
**proof**−
**have** *f1*:$rk\ \{A,\ B\} \leq 2$
**by** (*metis insert-is-Un matroid-ax-2-alt one-add-one rk-singleton*)
**have** *f2*:$rk\ \{A,\ B\} \geq 2$
**by** (*simp add*: *assms rk-ax-couple*)
**from** *f1* **and** *f2* **show** *?thesis*
**by** (*simp add*: *f1 le-antisym*)
**qed**

**lemma** *rk-triple-le* : $rk\ \{A,\ B,\ C\} \leq 3$
**by** (*metis Suc-numeral Un-commute insert-absorb2 insert-is-Un linear matroid-ax-2-alt numeral-2-eq-2*
*numeral-3-eq-3 numeral-le-one-iff numeral-plus-one rk-couple rk-singleton semiring-norm*(*70*))

**lemma** *rk-couple-to-singleton* :
**assumes** $rk\ \{A,\ B\} = 1$
**shows** $A = B$
**proof**−
**have** $rk\ \{A,\ B\} = 2$ **if** $A \neq B$
**using** *rk-couple*
**by** (*simp add*: *that*)
**thus** $A = B$
**using** *assms*
**by** *auto*
**qed**

**lemma** *rk-triple-to-rk-couple* :
**assumes** $rk\ \{A,\ B,\ C\} = 3$
**shows** $rk\ \{A,\ B\} = 2$
**proof**−
**have** $rk\ \{A,\ B\} \leq 2$
**using** *matroid-ax-1b*
**by** (*metis one-le-numeral rk-ax-couple rk-couple rk-singleton-bis*)
**have** $rk\ \{A,\ B,\ C\} \leq 2$ **if** $rk\ \{A,\ B\} = 1$
**using** *matroid-ax-2-alt*[*of* $\{A,\ B\}$ $C$]
**by** (*simp add*: *insert-commute that*)
**then have** $rk\ \{A,\ B\} \geq 2$
**using** *assms rk-ax-couple rk-singleton-bis*
**by** *force*
**thus** $rk\ \{A,\ B\} = 2$
**by** (*simp add*: ‹$rk\ \{A,\ B\} \leq 2$› *le-antisym*)
**qed**

**end**

**end**
**theory** *Desargues-2D*
**imports** *Main Higher-Projective-Space-Rank-Axioms Matroid-Rank-Properties*
**begin**

Contents:

- We prove Desargues's theorem: if two triangles ABC and A'B'C' are perspective from a point P (ie. the lines AA', BB' and CC' are concurrent in P), then they are perspective from a line (ie. the points $\alpha = BC \cap B'C'$, $\beta = AC \cap A'C'$ and $\gamma = AB \cap A'B'$ are collinear). In this file we restrict ourself to the case where the two triangles ABC and A'B'C' are coplanar.

# 8 Desargues's Theorem: The Coplanar Case

**context** *higher-projective-space-rank*
**begin**

**definition** *desargues-config-2D* ::
[*'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*, *'point*] ⇒ *bool*
**where** *desargues-config-2D A B C A' B' C' P α β γ* ≡ *rk* {*A*, *B*, *C*} = *3* ∧ *rk* {*A'*, *B'*, *C'*} = *3* ∧
*rk* {*A*, *A'*, *P*} = *2* ∧ *rk* {*B*, *B'*, *P*} = *2* ∧ *rk* {*C*, *C'*, *P*} = *2* ∧ *rk* {*A*, *B*, *γ*} = *2* ∧ *rk* {*A'*, *B'*, *γ*} = *2* ∧
*rk* {*A*, *C*, *β*} = *2* ∧ *rk* {*A'*, *C'*, *β*} = *2* ∧ *rk* {*B*, *C*, *α*} = *2* ∧ *rk* {*B'*, *C'*, *α*} = *2* ∧
*rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*} = *3* ∧
— We add the following non-degeneracy conditions
*rk* {*A*, *B*, *P*} = *3* ∧ *rk* {*A*, *C*, *P*} = *3* ∧ *rk* {*B*, *C*, *P*} = *3* ∧
*rk* {*A*, *A'*} = *2* ∧ *rk* {*B*, *B'*} = *2* ∧ *rk* {*C*, *C'*} = *2*

**lemma** *coplanar-ABCA'B'C'P* :
**assumes** *rk* {*A*, *A'*} = *2* **and** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*} = *3* **and** *rk* {*A*, *A'*, *P*} = *2*
**shows** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*} = *3*
**proof**−
**have** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*} + *rk* {*A*, *A'*} ≤ *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*} + *rk* {*A*, *A'*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A*, *A'*} {*A*, *B*, *C*, *A'*, *B'*, *C'*} {*A*, *A'*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*} ≤ *3*
**using** *assms*(*1*) *assms*(*2*) *assms*(*3*)
**by** *linarith*
**then show** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*} = *3*

**using** *assms(2) matroid-ax-2*
**by** (*metis Un-insert-right Un-upper2 le-antisym sup-bot.right-neutral*)
**qed**

**lemma** *non-colinear-A′B′P* :
**assumes** *rk* {*A*, *B*, *P*} = *3* **and** *rk* {*A*, *A′*, *P*} = *2* **and** *rk* {*B*, *B′*, *P*} = *2* **and** *rk* {*A′*, *P*} = *2*
**and** *rk* {*B′*, *P*} = *2*
**shows** *rk* {*A′*, *B′*, *P*} = *3*
**proof**−
**have** *f1*:*rk* {*A′*, *B′*, *P*} ≤ *3*
**using** *rk-triple-le* **by** *auto*
**have** *rk* {*A*, *B*, *A′*, *B′*, *P*} ≥ *3*
**using** *assms(1) matroid-ax-2*
**by** (*metis insert-mono insert-subset subset-insertI*)
**then have** *f2*:*rk* {*A*, *B*, *A′*, *B′*, *P*} = *3*
**using** *matroid-ax-3-alt*[*of* {*P*} {*A*, *A′*, *P*} {*B*, *B′*, *P*}] *assms(2) assms(3)*
**by** (*simp add*: *insert-commute rk-singleton*)
**have** *rk* {*A*, *B*, *A′*, *B′*, *P*} + *rk* {*B′*, *P*} ≤ *rk* {*A*, *A′*, *B′*, *P*} + *rk* {*B*, *B′*, *P*}
**using** *matroid-ax-3-alt*[*of* {*B′*, *P*} {*A*, *A′*, *B′*, *P*} {*B*, *B′*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A*, *A′*, *B′*, *P*} ≥ *3*
**using** *f2 assms(3) assms(5)* **by** *linarith*
**then have** *f3*:*rk* {*A*, *A′*, *B′*, *P*} = *3*
**using** *f2 matroid-ax-2*
**by** (*metis eq-iff insert-commute subset-insertI*)
**have** *rk* {*A*, *A′*, *B′*, *P*} + *rk* {*A′*, *P*} ≤ *rk* {*A′*, *B′*, *P*} + *rk* {*A*, *A′*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A′*, *P*} {*A′*, *B′*, *P*} {*A*, *A′*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A′*, *B′*, *P*} ≥ *3*
**using** *f3 assms(2) assms(4)* **by** *linarith*
**thus** *rk* {*A′*, *B′*, *P*} = *3*
**using** *f1* **by** *auto*
**qed**

**lemma** *desargues-config-2D-non-collinear-P* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A′*, *P*} = *2* **and** *rk* {*B′*, *P*} = *2*
**and** *rk* {*C′*, *P*} = *2*
**shows** *rk* {*A′*, *B′*, *P*} = *3* **and** *rk* {*A′*, *C′*, *P*} = *3* **and** *rk* {*B′*, *C′*, *P*} = *3*
**proof**−
**show** *rk* {*A′*, *B′*, *P*} = *3*
**using** *non-colinear-A′B′P assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2)*
*assms(3)*
**by** *blast*
**show** *rk* {*A′*, *C′*, *P*} = *3*
**using** *non-colinear-A′B′P assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2)*

*assms*(*4*)
**by** *blast*
**show** *rk* {*B'*, *C'*, *P*} = *3*
**using** *non-colinear-A'B'P assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$] *assms*(*3*)
*assms*(*4*)
**by** *blast*
**qed**

**lemma** *rk-A'B'PQ* :
**assumes** *rk* {*A*, *A'*} = *2* **and** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*} = *3* **and** *rk* {*A*, *A'*, *P*} = *2* **and**
*rk* {*A*, *B*, *P*} = *3* **and** *rk* {*B*, *B'*, *P*} = *2* **and** *rk* {*A'*, *P*} = *2* **and** *rk* {*B'*, *P*} = *2* **and**
*rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*, *Q*} ≥ *4*
**shows** *rk* {*A'*, *B'*, *P*, *Q*} = *4*
**proof**−
**have** *card* {*A'*, *B'*, *P*, *Q*} ≤ *4*
**by** (*smt One-nat-def Suc-numeral card.insert card.empty finite.emptyI finite-insert insert-absorb*
*insert-not-empty linear nat-add-left-cancel-le numeral-3-eq-3 numeral-Bit0 numeral-code*(*3*)
*numeral-le-one-iff numerals*(*1*) *one-plus-numeral semiring-norm*(*4*) *semiring-norm*(*69*)
*semiring-norm*(*70*) *semiring-norm*(*8*))
**then have** *f1*:*rk* {*A'*, *B'*, *P*, *Q*} ≤ *4*
**using** *matroid-ax-1b dual-order.trans* **by** *blast*
**have** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*, *Q*} + *rk* {*A'*, *B'*, *P*} ≤ *rk* {*A'*, *B'*, *P*, *Q*} + *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A'*, *B'*, *P*} {*A'*, *B'*, *P*, *Q*} {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A'*, *B'*, *P*, *Q*} ≥ *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*, *Q*} + *rk* {*A'*, *B'*, *P*} − *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*}
**using** *le-diff-conv*
**by** *blast*
**then have** *f2*:*rk* {*A'*, *B'*, *P*, *Q*} ≥ *4*
**using** *assms non-colinear-A'B'P coplanar-ABCA'B'C'P*
**by** (*smt diff-add-inverse2 le-trans*)
**from** *f1* **and** *f2* **show** *rk* {*A'*, *B'*, *P*, *Q*} = *4*
**by** (*simp add*: *f1 eq-iff*)
**qed**

**lemma** *desargues-config-2D-rkA'B'PQ-rkA'C'PQ-rkB'C'PQ* :
**assumes** *desargues-config-2D A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ **and** *rk* {*A'*, *P*} = *2* **and** *rk* {*B'*, *P*} = *2*
**and** *rk* {*C'*, *P*} = *2* **and** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*, *Q*} ≥ *4*
**shows** *rk* {*A'*, *B'*, *P*, *Q*} = *4* **and** *rk* {*A'*, *C'*, *P*, *Q*} = *4* **and** *rk* {*B'*, *C'*, *P*, *Q*} = *4*

**proof**−
**show** *rk* {*A′*, *B′*, *P*, *Q*} = *4*
**using** *rk-A′B′PQ*[*of A A′ B C B′ C′ P Q*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*2*) *assms*(*3*) *assms*(*5*)
**by** *blast*
**show** *rk* {*A′*, *C′*, *P*, *Q*} = *4*
**using** *rk-A′B′PQ*[*of A A′ C B C′ B′ P Q*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*2*) *assms*(*4*) *assms*(*5*)
**by** (*metis insert-commute*)
**show** *rk* {*B′*, *C′*, *P*, *Q*} = *4*
**using** *rk-A′B′PQ*[*of B B′ C A C′ A′ P Q*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*3*) *assms*(*4*) *assms*(*5*)
**by** (*metis insert-commute*)
**qed**

**lemma** *rk-A′B′PR* :
**assumes** *rk* {*P*, *Q*, *R*} = *2* **and** *rk* {*P*, *R*} = *2* **and** *rk* {*A′*, *B′*, *P*, *Q*} = *4*
**shows** *rk* {*A′*, *B′*, *P*, *R*} = *4*
**proof**−
**have** *card* {*A′*, *B′*, *P*, *R*} ≤ *4*
**by** (*smt Suc-numeral assms*(*2*) *card.empty card-insert-disjoint dual-order.trans finite.emptyI*
*finite-insert insert-absorb linear nat-add-left-cancel-le numeral-2-eq-2 numeral-3-eq-3*
*numeral-Bit0 numeral-code*(*3*) *numeral-le-one-iff rk-singleton rk-triple-le semiring-norm*(*2*)
*semiring-norm*(*69*) *semiring-norm*(*8*))
**then have** *f1*:*rk* {*A′*, *B′*, *P*, *R*} ≤ *4*
**using** *dual-order.trans matroid-ax-1b*
**by** *auto*
**have** *f2*:*rk* {*A′*, *B′*, *P*, *Q*, *R*} + *rk* {*P*, *R*} ≤ *rk* {*A′*, *B′*, *P*, *R*} + *rk* {*P*, *Q*, *R*}
**using** *matroid-ax-3-alt*[*of* {*P*, *R*} {*A′*, *B′*, *P*, *R*} {*P*, *Q*, *R*}]
**by** (*simp add*: *insert-commute*)
**have** *f3*:*rk* {*A′*, *B′*, *P*, *Q*, *R*} ≥ *4*
**using** *matroid-ax-2 assms*(*3*)
**by** (*metis insert-mono subset-insertI*)
**from** *f2* **and** *f3* **have** *f4*:*rk* {*A′*, *B′*, *P*, *R*} ≥ *4*
**using** *assms*(*1*) *assms*(*2*)
**by** *linarith*
**thus** *rk* {*A′*, *B′*, *P*, *R*} = *4*
**using** *f1 f4*
**by** (*simp add*: *f1 le-antisym*)
**qed**

**lemma** *rk-A′C′PR* :
**assumes** *rk* {*P*, *Q*, *R*} = *2* **and** *rk* {*P*, *R*} = *2* **and** *rk* {*A′*, *C′*, *P*, *Q*} = *4*

**shows** $rk\ \{A',\ C',\ P,\ R\} = 4$
**using** *assms(1) assms(2) assms(3) rk-A'B'PR*
**by** *blast*

**lemma** *rk-B'C'PR* :
**assumes** $rk\ \{P,\ Q,\ R\} = 2$ **and** $rk\ \{P,\ R\} = 2$ **and** $rk\ \{B',\ C',\ P,\ Q\} = 4$
**shows** $rk\ \{B',\ C',\ P,\ R\} = 4$
**using** *assms(1) assms(2) assms(3) rk-A'C'PR*
**by** *blast*

**lemma** *rk-ABA'* :
**assumes** $rk\ \{A,\ B,\ P\} = 3$ **and** $rk\ \{A,\ A'\} = 2$ **and** $rk\ \{A,\ A',\ P\} = 2$
**shows** $rk\ \{A,\ B,\ A'\} = 3$
**proof**−
**have** $rk\ \{A,\ B,\ A',\ P\} + rk\ \{A,\ A'\} \leq rk\ \{A,\ B,\ A'\} + rk\ \{A,\ A',\ P\}$
**using** *matroid-ax-3-alt*[*of* $\{A,\ A'\}\ \{A,\ B,\ A'\}\ \{A,\ A',\ P\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A,\ B,\ A'\} \geq 3$
**using** *assms matroid-ax-2*
**by** (*smt eq-iff insert-absorb2 insert-commute non-colinear-A'B'P rk-couple*)
**thus** $rk\ \{A,\ B,\ A'\} = 3$
**by** (*simp add*: *le-antisym rk-triple-le*)
**qed**

**lemma** *desargues-config-2D-non-collinear* :
**assumes** *desargues-config-2D A B C A' B' C' P* $\alpha\ \beta\ \gamma$
**shows** $rk\ \{A,\ B,\ A'\} = 3$ **and** $rk\ \{A,\ B,\ B'\} = 3$ **and** $rk\ \{A,\ C,\ C'\} = 3$
**proof**−
**show** $rk\ \{A,\ B,\ A'\} = 3$
**using** *assms desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *rk-ABA'*
**by** *auto*
**show** $rk\ \{A,\ B,\ B'\} = 3$
**using** *assms desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *rk-ABA'*
**by** (*smt insert-commute*)
**show** $rk\ \{A,\ C,\ C'\} = 3$
**using** *assms desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *rk-ABA'*
**by** (*smt insert-commute*)
**qed**

**lemma** *rk-Aa* :
**assumes** $rk\ \{A,\ B,\ P\} = 3$ **and** $rk\ \{A,\ A'\} = 2$ **and** $rk\ \{A,\ A',\ P\} = 2$ **and**
$rk\ \{Q,\ A',\ a\} = 2$
**and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} \geq 4$ **and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C'\} \leq 3$
**shows** $rk\ \{A,\ a\} = 2$
**proof**−
**have** $rk\ \{Q,\ A',\ A,\ a\} + rk\ \{a\} \leq rk\ \{Q,\ A',\ a\} + rk\ \{A,\ a\}$
**using** *matroid-ax-3-alt*[*of* $\{a\}\ \{Q,\ A',\ a\}\ \{A,\ a\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{Q,\ A',\ A,\ a\} \leq rk\ \{Q,\ A',\ a\} + rk\ \{A,\ a\} - rk\ \{a\}$

**using** *add-le-imp-le-diff*
**by** *blast*
**then have** $rk\ \{Q,\ A',\ A,\ a\} \leq 2$ **if** $rk\ \{A,\ a\} = 1$
**using** *assms*(4)
**by** (*simp add*: *rk-singleton that*)
**then have** $rk\ \{Q,\ A',\ A\} \leq 2$ **if** $rk\ \{A,\ a\} = 1$
**using** *matroid-ax-2*
**by** (*metis One-nat-def assms*(4) *le-numeral-extra*(4) *nat-add-left-cancel-le numeral-2-eq-2*
*numeral-3-eq-3 one-add-one rk-couple rk-triple-le that*)
**then have** $rk\ \{Q,\ A',\ A\} = 2$ **if** $rk\ \{A,\ a\} = 1$
**using** *assms*(2) *matroid-ax-2*
**by** (*metis assms*(4) *numeral-eq-one-iff rk-couple semiring-norm*(85) *that*)
**then have** $rk\ \{A,\ A',\ P,\ Q\} = 2$ **if** $rk\ \{A,\ a\} = 1$
**using** *assms*(3) *matroid-ax-3-alt'*[*of* $\{A,\ A'\}\ P\ Q$]
**by** (*simp add*: *assms*(2) *insert-commute that*)
**then have** *f1*:$rk\ \{A,\ A',\ B,\ P,\ Q\} \leq 3$ **if** $rk\ \{A,\ a\} = 1$
**by** (*metis One-nat-def Un-insert-right add.right-neutral add-Suc-right insert-commute matroid-ax-2-alt*
*numeral-2-eq-2 numeral-3-eq-3 sup-bot.right-neutral that*)
**have** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} + rk\ \{A,\ B,\ A'\} \leq rk\ \{A,\ A',\ B,\ P,\ Q\} + rk\ \{A,\ B,\ C,\ A',\ B',\ C'\}$
**using** *matroid-ax-3-alt*[*of* $\{A,\ B,\ A'\}\ \{A,\ A',\ B,\ P,\ Q\}\ \{A,\ B,\ C,\ A',\ B',\ C'\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} \leq rk\ \{A,\ A',\ B,\ P,\ Q\} + rk\ \{A,\ B,\ C,\ A',\ B',\ C'\} - rk\ \{A,\ B,\ A'\}$
**by** *linarith*
**then have** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} \leq 3$ **if** $rk\ \{A,\ a\} = 1$
**using** *assms*(1) *assms*(2) *assms*(3) *assms*(6) *f1 rk-ABA'*
**by** (*smt* ‹$rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} + rk\ \{A,\ B,\ A'\} \leq rk\ \{A,\ A',\ B,\ P,\ Q\} + rk\ \{A,\ B,\ C,\ A',\ B',\ C'\}$›
*add-diff-cancel-right' add-leD2 le-less-trans not-le*
*ordered-cancel-comm-monoid-diff-class.add-diff-inverse*
*ordered-cancel-comm-monoid-diff-class.le-add-diff that*)
**then have** $\neg\ (rk\ \{A,\ a\} = 1)$
**using** *assms*(5)
**by** *linarith*
**thus** $rk\ \{A,\ a\} = 2$
**using** *rk-couple rk-singleton-bis*
**by** *blast*
**qed**

**lemma** *desargues-config-2D-rkAa-rkBb-rkCc* :
**assumes** *desargues-config-2D A B C A' B' C' P* $\alpha\ \beta\ \gamma$ **and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} \geq 4$
**and** $rk\ \{Q,\ A',\ a\} = 2$ **and** $rk\ \{Q,\ B',\ b\} = 2$ **and** $rk\ \{Q,\ C',\ c\} = 2$
**shows** $rk\ \{A,\ a\} = 2$ **and** $rk\ \{B,\ b\} = 2$ **and** $rk\ \{C,\ c\} = 2$
**proof**−
**show** $rk\ \{A,\ a\} = 2$

**using** *rk-Aa assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2) assms(3)*
**by** (*metis rk-triple-le*)
**show** *rk* {*B*, *b*} = *2*
**using** *rk-Aa assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2) assms(4)*
**by** (*smt insert-commute rk-triple-le*)
**show** *rk* {*C*, *c*} = *2*
**using** *rk-Aa*[*of C A P C′ Q c B B′ A′*] *assms(1)*
*desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2) assms(5)*
**by** (*metis insert-commute rk-triple-le*)
**qed**

**lemma** *rk-ABPRa* :
**assumes** *rk* {*A*, *B*, *P*} = *3* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3* **and** *rk* {*P*, *Q*, *R*} = *2*
**and** *rk* {*P*, *R*} = *2* **and** *rk* {*A′*, *B′*, *P*, *Q*} = *4*
**shows** *rk* {*A*, *B*, *P*, *R*, *a*} ≥ *4*
**proof**−
**have** *rk* {*A′*, *B′*, *P*, *R*, *a*, *A*, *B*} ≥ *rk* {*A′*, *B′*, *P*, *R*}
**using** *matroid-ax-2*
**by** *auto*
**then have** *f1*:*rk* {*A′*, *B′*, *P*, *R*, *a*, *A*, *B*} ≥ *4*
**using** *rk-A′B′PR assms(3) assms(4) assms(5)*
**by** *auto*
**have** *f2*:*rk* {*A′*, *B′*, *A*, *B*, *P*} ≤ *3*
**using** *matroid-ax-2 assms(2)*
**by** (*smt insertI1 insert-subset subset-insertI*)
**have** *rk* {*A′*, *B′*, *P*, *R*, *a*, *A*, *B*} + *rk* {*A*, *B*, *P*} ≤ *rk* {*A*, *B*, *P*, *R*, *a*} + *rk* {*A′*, *B′*, *A*, *B*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A*, *B*, *P*} {*A*, *B*, *P*, *R*, *a*} {*A′*, *B′*, *A*, *B*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A*, *B*, *P*, *R*, *a*} ≥ *rk* {*A′*, *B′*, *P*, *R*, *a*, *A*, *B*} + *rk* {*A*, *B*, *P*} − *rk* {*A′*, *B′*, *A*, *B*, *P*}
**by** *linarith*
**thus** *rk* {*A*, *B*, *P*, *R*, *a*} ≥ *4*
**using** *f1 assms(1) f2*
**by** *linarith*
**qed**

**lemma** *rk-ABPa* :
**assumes** *rk* {*A*, *B*, *P*} = *3* **and** *rk* {*A*, *A′*} = *2* **and** *rk* {*A*, *A′*, *P*} = *2* **and** *rk* {*Q*, *A′*, *a*} = *2*
**and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3* **and** *rk* {*P*, *Q*, *R*} = *2*
**and** *rk* {*P*, *R*} = *2* **and** *rk* {*A′*, *B′*, *P*, *Q*} = *4* **and** *rk* {*R*, *A*, *a*} = *2*
**shows** *rk* {*A*, *B*, *P*, *a*} ≥ *4*
**proof**−
**have** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*} ≤ *3*

**using** *matroid-ax-2 assms(6)*
**by** (*smt insert-iff subsetI*)
**then have** *f1*:*rk* $\{A, a\} = 2$
**using** *assms(1) assms(2) assms(3) assms(4) assms(5) rk-Aa*
**by** *blast*
**have** *f2*:*rk* $\{A, B, P, R, a\} \geq 4$
**using** *assms(1) assms(6) assms(7) assms(8) assms(9) rk-ABPRa*
**by** *blast*
**have** *rk* $\{A, B, P, R, a\} + rk\ \{A, a\} \leq rk\ \{A, B, P, a\} + rk\ \{R, A, a\}$
**using** *matroid-ax-3-alt*[*of* $\{A, a\}$ $\{A, B, P, a\}$ $\{R, A, a\}$]
**by** (*simp add*: *insert-commute*)
**thus** *rk* $\{A, B, P, a\} \geq 4$
**using** *f1 f2 assms(10)*
**by** (*smt add-le-imp-le-diff diff-add-inverse2 order-trans*)
**qed**

**lemma** *desargues-config-2D-rkABPa-rkABPb-rkABPc* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* $\{A, B, C, A', B', C', P, Q\} \geq 4$
**and** *rk* $\{P, Q, R\} = 2$ **and** *rk* $\{P, R\} = 2$ **and** *rk* $\{A', P\} = 2$ **and** *rk* $\{B', P\} = 2$ **and**
*rk* $\{Q, A', a\} = 2$ **and** *rk* $\{R, A, a\} = 2$ **and** *rk* $\{Q, B', b\} = 2$ **and** *rk* $\{R, B, b\} = 2$ **and**
*rk* $\{Q, C', c\} = 2$ **and** *rk* $\{R, C, c\} = 2$
**shows** *rk* $\{A, B, P, a\} \geq 4$ **and** *rk* $\{A, B, P, b\} \geq 4$ **and** *rk* $\{A, B, P, c\} \geq 4$
**proof**−
**have** *f1*:*rk* $\{A, B, C, A', B', C', P\} = 3$
**using** *assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-ABCA′B′C′P*
**by** *auto*
**have** *f2*:*rk* $\{A', B', P, Q\} = 4$
**using** *assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(2) assms(5) assms(6)*
*rk-A′B′PQ*[*of A A′ B C B′ C′ P Q*]
**by** *auto*
**show** *rk* $\{A, B, P, a\} \geq 4$
**using** *f1 f2 assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(7) assms(2) assms(3)*
*assms(4) assms(8) rk-ABPa*
**by** *auto*
**show** *rk* $\{A, B, P, b\} \geq 4$
**using** *f1 f2 assms(1) desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms(9) assms(2) assms(3)*
*assms(4) assms(10) rk-ABPa*[*of B A P B′ Q b C A′ C′ R*]
**by** (*metis insert-commute*)
**show** *rk* $\{A, B, P, c\} \geq 4$
**proof**−
**have** *f3*:*rk* $\{A, B, P, R, c\} \geq 4$
**using** *rk-ABPRa*[*of A B P C A′ B′ C′ Q R c*] *assms(1) desargues-config-2D-def*[*of*

*A B C A′ B′ C′ P α β γ*]
*f1 assms*(*3*) *assms*(*4*) *f2*
**by** *auto*
**have** {*A*, *B*, *P*, *R*, *c*} ⊆ {*A*, *B*, *C*, *P*, *R*, *c*}
**by** *auto*
**then have** *f4*:*rk* {*A*, *B*, *C*, *P*, *R*, *c*} ≥ *4*
**using** *matroid-ax-2 f3*
**by** (*meson dual-order.trans*)
**have** *rk* {*A*, *B*, *C*, *P*, *R*, *c*} + *rk* {*C*, *c*} ≤ *rk* {*A*, *B*, *C*, *P*, *c*} + *rk* {*R*, *C*, *c*}
**using** *matroid-ax-3-alt*[*of* {*C*,*c*} {*A*, *B*, *C*, *P*, *c*} {*R*, *C*, *c*}]
**by** (*simp add*: *insert-commute*)
**then have** *f5*:*rk* {*A*, *B*, *C*, *P*, *c*} ≥ *4*
**using** *f4 assms*(*12*) *desargues-config-2D-rkAa-rkBb-rkCc assms*(*1*) *assms*(*9*) *assms*(*11*) *assms*(*2*) *assms*(*7*)
**by** *auto*
**have** *f6*:*rk* {*A*, *B*, *C*, *P*} ≤ *3*
**using** *matroid-ax-2 f1*
**by** (*smt insert-iff subsetI*)
**have** *rk* {*A*, *B*, *C*, *P*, *c*} + *rk* {*A*, *B*, *P*} ≤ *rk* {*A*, *B*, *P*, *c*} + *rk* {*A*, *B*, *C*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A*, *B*, *P*} {*A*, *B*, *P*, *c*} {*A*, *B*, *C*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*A*, *B*, *P*, *c*} ≥ *rk* {*A*, *B*, *C*, *P*, *c*}
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *f6*
**by** *linarith*
**thus** *rk* {*A*, *B*, *P*, *c*} ≥ *4*
**using** *f5*
**by** *linarith*
**qed**
**qed**

**lemma** *rk-AA′C* :
**assumes** *rk* {*A*, *C*, *P*} = *3* **and** *rk* {*A*, *A′*} = *2* **and** *rk* {*A*, *A′*, *P*} = *2*
**shows** *rk* {*A*, *A′*, *C*} ≥ *3*
**proof**−
**have** *f1*:*rk* {*A*, *C*, *A′*, *P*} ≥ *3*
**using** *assms*(*1*) *matroid-ax-2*
**by** (*metis insert-commute subset-insertI*)
**have** *rk* {*A*, *C*, *A′*, *P*} + *rk* {*A*, *A′*} ≤ *rk* {*A*, *A′*, *C*} + *rk* {*A*, *A′*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A*, *A′*} {*A*, *A′*, *C*} {*A*, *A′*, *P*}]
**by** (*simp add*: *insert-commute*)
**thus** *rk* {*A*, *A′*, *C*} ≥ *3*
**using** *f1 assms*(*2*) *assms*(*3*)
**by** *linarith*
**qed**

**lemma** *rk-AA′C′* :
**assumes** *rk* {*A′*, *C′*, *P*} = *3* **and** *rk* {*A*, *A′*} = *2* **and** *rk* {*A*, *A′*, *P*} = *2*

**shows** $rk\ \{A,\ A',\ C'\} \geq 3$
**by** (*smt assms(1) assms(2) assms(3) insert-commute rk-AA'C*)

**lemma** *rk-AA'Ca* :
**assumes** $rk\ \{A,\ A',\ C\} \geq 3$ **and** $rk\ \{A,\ B,\ P,\ a\} \geq 4$ **and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P\} = 3$
**shows** $rk\ \{A,\ A',\ C,\ a\} \geq 4$
**proof**−
**have** *f1*:$rk\ \{A,\ A',\ C,\ a,\ B,\ P\} \geq 4$
**using** *assms(2) matroid-ax-2*
**by** (*smt dual-order.trans insert-commute insert-mono insert-subset subset-insertI*)
**have** *f2*:$rk\ \{A,\ B,\ C,\ P,\ A'\} \leq 3$
**using** *assms(3) matroid-ax-2*
**by** (*smt empty-subsetI insert-commute insert-mono semiring-norm(3)*)
**have** $rk\ \{A,\ A',\ C,\ a,\ B,\ P\} + rk\ \{A,\ A',\ C\} \leq rk\ \{A,\ A',\ C,\ a\} + rk\ \{A,\ B,\ C,\ P,\ A'\}$
**using** *matroid-ax-3-alt*[*of* $\{A,\ A',\ C\}$ $\{A,\ A',\ C,\ a\}$ $\{A,\ B,\ C,\ P,\ A'\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A,\ A',\ C,\ a\} \geq rk\ \{A,\ A',\ C,\ a,\ B,\ P\} + rk\ \{A,\ A',\ C\} - rk\ \{A,\ B,\ C,\ P,\ A'\}$
**using** *le-diff-conv*
**by** *blast*
**thus** $rk\ \{A,\ A',\ C,\ a\} \geq 4$
**using** *f1 assms(1) f2*
**by** *linarith*
**qed**

**lemma** *rk-AA'C'a* :
**assumes** $rk\ \{A,\ A',\ C'\} \geq 3$ **and** $rk\ \{A,\ B,\ P,\ a\} \geq 4$ **and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P\} = 3$
**shows** $rk\ \{A,\ A',\ C',\ a\} \geq 4$
**by** (*smt assms(1) assms(2) assms(3) insert-commute rk-AA'Ca*)

**lemma** *rk-Ra* :
**assumes** $rk\ \{Q,\ A',\ a\} = 2$ **and** $rk\ \{P,\ Q,\ R\} = 2$ **and** $rk\ \{R,\ Q\} = 2$ **and** $rk\ \{A,\ A',\ P\} = 2$
**and** $rk\ \{A',\ P\} = 2$ **and** $rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P\} = 3$ **and** $rk\ \{A,\ B,\ A'\} = 3$ **and**
$rk\ \{A,\ B,\ C,\ A',\ B',\ C',\ P,\ Q\} \geq 4$
**shows** $rk\ \{R,\ a\} = 2$
**proof**−
**have** $R = a$ **if** $rk\ \{R,\ a\} = 1$
**using** *rk-couple-to-singleton*
**by** (*simp add*: *that*)
**then have** $rk\ \{R,\ Q,\ A'\} = 2$ **if** $rk\ \{R,\ a\} = 1$
**using** *assms(1)*
**by** (*simp add*: *insert-commute that*)
**then have** *f1*:$rk\ \{P,\ Q,\ R,\ A'\} = 2$ **if** $rk\ \{R,\ a\} = 1$
**using** *assms(2) assms(3) matroid-ax-3-alt'*

**by** (*metis Un-empty-right Un-insert-right insert-commute that*)
**have** *rk* {*P*, *Q*, *R*, *A′*, *A*} + *rk* {*A′*, *P*} ≤ *rk* {*A*, *A′*, *P*} + *rk* {*P*, *Q*, *R*, *A′*}
**using** *matroid-ax-3-alt*[*of* {*A′*, *P*} {*A*, *A′*, *P*} {*P*, *Q*, *R*, *A′*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*P*, *Q*, *R*, *A′*, *A*} = *2* **if** *rk* {*R*, *a*} = *1*
**using** *assms*(*4*) *f1 assms*(*5*)
**by** (*metis Un-insert-right add-le-cancel-right insert-is-Un le-antisym matroid-ax-2 subset-insertI that*)
**then have** *f2*:*rk* {*P*, *Q*, *R*, *A′*, *A*, *B*} ≤ *3* **if** *rk* {*R*, *a*} = *1*
**using** *matroid-ax-2-alt*[*of* {*P*, *Q*, *R*, *A′*, *A*} *B*]
**by** (*simp add*: *insert-commute that*)
**have** *f3*:*rk* {*A*, *B*, *A′*, *P*} ≥ *3*
**using** *assms*(*7*) *matroid-ax-2*
**by** (*metis insert-commute subset-insertI*)
**have** *rk* {*P*, *Q*, *R*, *A′*, *A*, *B*, *C*, *B′*, *C′*} + *rk* {*A*, *B*, *A′*, *P*} ≤ *rk* {*P*, *Q*, *R*, *A′*, *A*, *B*} + *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*}
**using** *matroid-ax-3-alt*[*of* {*A*, *B*, *A′*, *P*} {*P*, *Q*, *R*, *A′*, *A*, *B*} {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *f4*:*rk* {*P*, *Q*, *R*, *A′*, *A*, *B*, *C*, *B′*, *C′*} ≤ *3* **if** *rk* {*R*, *a*} = *1*
**using** *f2 f3 assms*(*6*) *that*
**by** *linarith*
**have** *f5*:*rk* {*P*, *Q*, *R*, *A′*, *A*, *B*, *C*, *B′*, *C′*} ≥ *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*}
**using** *matroid-ax-2*
**by** *auto*
**thus** *rk* {*R*, *a*} = *2* **using** *f4 f5 assms*(*8*)
**by** (*smt Suc-1 Suc-le-eq add-Suc add-Suc-right nat-1-add-1 not-le numeral-2-eq-2 numeral-3-eq-3*
*numeral-Bit0 order.trans rk-couple rk-singleton-bis*)
**qed**

**lemma** *desargues-config-2D-rkRa-rkRb-rkRc* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**and** *rk* {*P*, *Q*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2* **and** *rk* {*Q*, *A′*, *a*} = *2* **and** *rk* {*Q*, *B′*, *b*} = *2* **and**
*rk* {*Q*, *C′*, *c*} = *2* **and** *rk* {*A′*, *P*} = *2* **and** *rk* {*B′*, *P*} = *2* **and** *rk* {*C′*, *P*} = *2*
**shows** *rk* {*R*, *a*} = *2* **and** *rk* {*R*, *b*} = *2* **and** *rk* {*R*, *c*} = *2*
**proof**−
**have** *f1*:*rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3*
**using** *coplanar-ABCA′B′C′P assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *blast*
**have** *f2*:*rk* {*A*, *B*, *A′*} = *3*
**using** *desargues-config-2D-non-collinear assms*(*1*)
**by** *auto*
**have** *f3*:*rk* {*A*, *B*, *B′*} = *3*
**using** *desargues-config-2D-non-collinear assms*(*1*)
**by** *auto*

**have** *f4*:*rk* $\{A,\ C,\ C'\} = 3$
**using** *desargues-config-2D-non-collinear assms*(*1*)
**by** *auto*
**show** *rk* $\{R,\ a\} = 2$
**using** *f1 f2 rk-Ra*[*of Q A' a P R A B C B' C'*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P α β γ*]
*assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*) *assms*(*8*)
**by** (*metis insert-commute*)
**show** *rk* $\{R,\ b\} = 2$
**using** *f1 f3 rk-Ra*[*of Q B' b P R B A C A' C'*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P α β γ*]
*assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*6*) *assms*(*9*)
**by** (*metis insert-commute*)
**show** *rk* $\{R,\ c\} = 2$
**using** *f1 f4 rk-Ra*[*of Q C' c P R C A B A' B'*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P α β γ*]
*assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*7*) *assms*(*10*)
**by** (*metis insert-commute*)
**qed**

**lemma** *rk-acACβ* :
**assumes** *rk* $\{R,\ A,\ a\} = 2$ **and** *rk* $\{R,\ C,\ c\} = 2$ **and** *rk* $\{A,\ C\} = 2$ **and** *rk* $\{A,\ C,\ \beta\} = 2$
**and** *rk* $\{Q,\ A',\ a\} = 2$ **and** *rk* $\{A,\ A',\ C,\ a\} \geq 4$
**shows** *rk* $\{a,\ c,\ A,\ C,\ \beta\} = 3$
**proof**−
**have** *rk* $\{a,\ c,\ A,\ C,\ R\} + rk\ \{R\} \leq rk\ \{R,\ A,\ a\} + rk\ \{R,\ C,\ c\}$
**using** *matroid-ax-3-alt*[*of* $\{R\}$ $\{R,\ A,\ a\}$ $\{R,\ C,\ c\}$]
**by** (*simp add*: *insert-commute*)
**then have** *f1*:*rk* $\{a,\ c,\ A,\ C,\ R\} \leq 3$
**using** *assms*(*1*) *assms*(*2*)
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono add-Suc-right numeral-3-eq-3 numeral-nat*(*1*) *numerals*(*1*)
*rk-singleton*)
**have** *rk* $\{a,\ c,\ A,\ C,\ R,\ \beta\} + rk\ \{A,\ C\} \leq rk\ \{a,\ c,\ A,\ C,\ R\} + rk\ \{A,\ C,\ \beta\}$
**using** *matroid-ax-3-alt*[*of* $\{A,\ C\}$ $\{a,\ c,\ A,\ C,\ R\}$ $\{A,\ C,\ \beta\}$]
**by** (*simp add*: *insert-commute*)
**then have** *f2*:*rk* $\{a,\ c,\ A,\ C,\ R,\ \beta\} \leq 3$
**using** *f1 assms*(*3*) *assms*(*4*)
**by** *linarith*
**have** $\{a,\ c,\ A,\ C,\ \beta\} \subseteq \{a,\ c,\ A,\ C,\ R,\ \beta\}$
**by** *auto*
**then have** *f3*:*rk* $\{a,\ c,\ A,\ C,\ \beta\} \leq 3$
**using** *matroid-ax-2 f2*
**by** (*meson order-trans*)
**have** *f4*:*rk* $\{A,\ A',\ C,\ a,\ c,\ Q\} \geq 4$
**using** *matroid-ax-2 assms*(*6*)
**by** (*smt dual-order.trans insert-commute insert-mono insert-subset subset-insertI*)
**have** *rk* $\{A,\ A',\ C,\ a,\ c,\ Q\} + rk\ \{a\} \leq rk\ \{a,\ c,\ A,\ C\} + rk\ \{Q,\ A',\ a\}$

**using** *matroid-ax-3-alt*[*of* {*a*} {*a*, *c*, *A*, *C*} {*Q*, *A*′, *a*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*a*, *c*, *A*, *C*} ≥ *rk* {*A*, *A*′, *C*, *a*, *c*, *Q*} + *rk* {*a*} − *rk* {*Q*, *A*′, *a*}
**using** *le-diff-conv*
**by** *blast*
**then have** *rk* {*a*, *c*, *A*, *C*} ≥ *3*
**using** *f4 assms*(*5*)
**by** (*smt One-nat-def* ‹*rk* {*A*, *A*′, *C*, *a*, *c*, *Q*} + *rk* {*a*} ≤ *rk* {*a*, *c*, *A*, *C*} + *rk* {*Q*, *A*′, *a*}›
*ab-semigroup-add-class.add-ac*(*1*) *add-2-eq-Suc*′ *add-diff-cancel-right*′ *add-le-imp-le-diff*
*card.empty card.insert dual-order.antisym finite.intros*(*1*) *insert-absorb insert-not-empty*
*matroid-ax-1b nat-1-add-1 numeral-3-eq-3 numeral-Bit0 order.trans rk-ax-singleton*)
**then have** *rk* {*a*, *c*, *A*, *C*, *β*} ≥ *3*
**using** *matroid-ax-2*
**by** (*metis Un-insert-right Un-upper2 dual-order.trans sup-bot.comm-neutral*)
**thus** *rk* {*a*, *c*, *A*, *C*, *β*} = *3*
**using** *f3 le-antisym*
**by** *blast*
**qed**

**lemma** *rk-acA*′*C*′*β* :
**assumes** *rk* {*Q*, *A*′, *a*} = *2* **and** *rk* {*Q*, *C*′, *c*} = *2* **and** *rk* {*A*′, *C*′} = *2* **and** *rk* {*A*′, *C*′, *β*} = *2*
**and** *rk* {*R*, *A*, *a*} = *2* **and** *rk* {*A*′, *A*, *C*′, *a*} ≥ *4*
**shows** *rk* {*a*, *c*, *A*′, *C*′, *β*} = *3*
**using** *assms rk-acACβ*
**by** *blast*

**lemma** *plane-representation-change* :
**assumes** *rk* {*A*, *B*, *C*, *P*} = *3* **and** *rk* {*B*, *C*, *P*} = *3* **and** *rk* {*A*, *B*, *C*, *Q*} = *4*
**shows** *rk* {*P*, *B*, *C*, *Q*} = *4*
**proof**−
**have** *rk* {*P*, *B*, *C*, *Q*} ≤ *4* **using** *assms*(*2*) *matroid-ax-2-alt*[*of* {*B*, *C*, *P*} *Q*]
**by** (*simp add*: *insert-commute*)
**have** *rk* {*A*, *B*, *C*, *Q*, *P*} + *rk* {*B*, *C*, *P*} ≤ *rk* {*P*, *B*, *C*, *Q*} + *rk* {*A*, *B*, *C*, *P*}
**using** *matroid-ax-3-alt*[*of* {*B*, *C*, *P*} {*P*, *B*, *C*, *Q*} {*A*, *B*, *C*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*P*, *B*, *C*, *Q*} ≥ *4*
**using** *assms*
**by** (*smt add.commute dual-order.trans insert-commute matroid-ax-2 nat-add-left-cancel-le*
*subset-insertI*)
**thus** *rk* {*P*, *B*, *C*, *Q*} = *4*
**by** (*simp add*: ‹*rk* {*P*, *B*, *C*, *Q*} ≤ *4*› *antisym*)
**qed**

**lemma** *desargues-config-2D-rkABCP* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*A*, *B*, *C*, *P*} = *3*
**proof**−
**have** *rk* {*A*, *B*, *C*} = *3*
**using** *assms desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**then have** *f1*:*rk* {*A*, *B*, *C*, *P*} ≥ *3*
**using** *matroid-ax-2*
**by** (*metis empty-subsetI insert-mono*)
**have** *f2*:*rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3*
**using** *assms desargues-config-2D-def*[*of A B C A′ B′ C′ P*] *coplanar-ABCA′B′C′P*

**by** *auto*
**have** {*A*, *B*, *C*, *P*} ⊆ {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*}
**by** *auto*
**then have** *rk* {*A*, *B*, *C*, *P*} ≤ *3*
**using** *matroid-ax-2 f2*
**by** *metis*
**thus** *rk* {*A*, *B*, *C*, *P*} = *3*
**using** *f1 antisym*
**by** *blast*
**qed**

**lemma** *desargues-config-2D-rkABCabc* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**and** *rk* {*Q*, *A′*, *a*} = *2* **and** *rk* {*P*, *Q*, *R*} = *2* **and** *rk* {*P*, *R*} = *2* **and** *rk* {*R*, *A*, *a*} = *2* **and**
*rk* {*A′*, *P*} = *2* **and** *rk* {*B′*, *P*} = *2*
**shows** *rk* {*A*, *B*, *C*, *a*, *b*, *c*} ≥ *4*
**proof**−
**have** *f1*:*rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-ABCA′B′C′P*
**by** *auto*
**have** *f2*:*rk* {*A′*, *B′*, *P*, *Q*} = *4*
**using** *rk-A′B′PQ*[*of A A′ B C B′ C′ P Q*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*2*) *assms*(*7*) *assms*(*8*)
**by** *auto*
**from** *f1* **and** *f2* **have** *f3*:*rk* {*A*, *B*, *P*, *a*} ≥ *4*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms*(*2*) *assms*(*3*) *assms*(*4*)
*assms*(*5*) *assms*(*6*) *rk-ABPa*
**by** *auto*
**have** {*A*, *B*, *P*, *a*} ⊆ {*A*, *B*, *C*, *a*, *b*, *c*, *P*}
**by** *auto*

**then have** *f4*:*rk* $\{A, B, C, a, b, c, P\} \geq 4$
**using** *matroid-ax-2 f3*
**by** (*meson dual-order.trans*)
**have** *f5*:*rk* $\{A, B, C, P\} = 3$
**using** *assms*(*1*) *desargues-config-2D-rkABCP*
**by** *auto*
**have** *rk* $\{A, B, C, a, b, c, P\} + rk\ \{A, B, C\} \leq rk\ \{A, B, C, a, b, c\} + rk\ \{A, B, C, P\}$
**using** *matroid-ax-3-alt*[*of* $\{A, B, C\}$ $\{A, B, C, a, b, c\}$ $\{A, B, C, P\}$]
**by** (*simp add*: *insert-commute*)
**thus** *rk* $\{A, B, C, a, b, c\} \geq 4$
**using** *f4 assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *f5*
**by** *linarith*
**qed**

**lemma** *rk-abc* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* $\{A, B, C, A', B', C', P, Q\} \geq 4$
**and** *rk* $\{Q, A', a\} = 2$ **and** *rk* $\{Q, B', b\} = 2$ **and** *rk* $\{Q, C', c\} = 2$ **and** *rk* $\{P, Q, R\} = 2$ **and**
*rk* $\{P, R\} = 2$ **and** *rk* $\{Q, R\} = 2$ **and** *rk* $\{R, A, a\} = 2$ **and** *rk* $\{R, B, b\} = 2$ **and**
*rk* $\{R, C, c\} = 2$ **and** *rk* $\{A', P\} = 2$ **and** *rk* $\{B', P\} = 2$ **and** *rk* $\{C', P\} = 2$
**shows** *rk* $\{a, b, c\} = 3$
**proof**−
**have** *rk* $\{a, b, c\} \leq 3$
**by** (*simp add*: *rk-triple-le*)
**have** *rk* $\{A, B, C, a, b, c\} \geq 4$
**using** *desargues-config-2D-rkABCabc assms*(*1*) *assms*(*13*) *assms*(*2*) *assms*(*3*) *assms*(*6*) *assms*(*7*) *assms*(*9*)
*assms*(*12*)
**by** *auto*
**then have** *f1*:*rk* $\{A, B, C, R, a, b, c\} \geq 4$
**using** *matroid-ax-2*
**by** (*smt dual-order.trans insert-commute subset-insertI*)
**have** *f2*:*rk* $\{A, B, C, A', B', C', P\} = 3$
**using** *coplanar-ABCA′B′C′P assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**have** *f3*:*rk* $\{A, C, C'\} = 3$
**using** *assms*(*1*) *desargues-config-2D-non-collinear*(*3*)
**by** *auto*
**from** *f2* **and** *f3* **have** *f4*:*rk* $\{R, c\} = 2$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *assms*(*2*) *assms*(*5*) *assms*(*6*)
*assms*(*8*) *assms*(*14*) *rk-Ra*[*of Q C′ c P R C A B A′ B′*]
**by** (*metis insert-commute*)
**have** *rk* $\{A, B, C, R, a, b, c\} + rk\ \{R, c\} \leq rk\ \{a, b, c, R, A, B\} + rk\ \{R, C, c\}$

**using** *matroid-ax-3-alt*[*of* {*R*, *c*} {*a*, *b*, *c*, *R*, *A*, *B*} {*R*, *C*, *c*}]
**by** (*simp add*: *insert-commute*)
**then have** *f5*:*rk* {*a*, *b*, *c*, *R*, *A*, *B*} ≥ *4*
**using** *f1 f4 assms*(*11*)
**by** *linarith*
**have** *rk* {*A*, *B*, *B'*} = *3*
**using** *assms*(*1*) *desargues-config-2D-non-collinear*(*2*)
**by** *auto*
**then have** *f6*:*rk* {*R*, *b*} = *2*
**using** *f2 assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P α β γ*]
*rk-Ra*[*of Q B' b P R B A C A' C'*] *assms*(*2*) *assms*(*4*) *assms*(*6*) *assms*(*8*)
*assms*(*13*)
**by** (*metis insert-commute*)
**have** *rk* {*a*, *b*, *c*, *R*, *A*, *B*} + *rk* {*R*, *b*} ≤ *rk* {*a*, *b*, *c*, *R*, *A*} + *rk* {*R*, *B*, *b*}
**using** *matroid-ax-3-alt*[*of* {*R*, *b*} {*a*, *b*, *c*, *R*, *A*} {*R*, *B*, *b*}]
**by** (*simp add*: *insert-commute*)
**then have** *f7*:*rk* {*a*, *b*, *c*, *R*, *A*} ≥ *4*
**using** *f5 f6 assms*(*10*)
**by** *linarith*
**have** *rk* {*a*, *b*, *c*, *R*, *A*} + *rk* {*a*} ≤ *rk* {*a*, *b*, *c*} + *rk* {*R*, *A*, *a*}
**using** *matroid-ax-3-alt*[*of* {*a*} {*a*, *b*, *c*} {*R*, *A*, *a*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*a*, *b*, *c*} ≥ *3*
**using** *f7 assms*(*9*)
**by** (*smt One-nat-def Suc-eq-plus1 Suc-le-mono Suc-numeral add.assoc card.empty card.insert*
*dual-order.trans finite.intros*(*1*) *insert-absorb insert-not-empty le-antisym matroid-ax-1b*
*one-add-one rk-ax-singleton semiring-norm*(*2*) *semiring-norm*(*8*))
**thus** *rk* {*a*, *b*, *c*} = *3*
**by** (*simp add*: ‹*rk* {*a*, *b*, *c*} ≤ *3*› *le-antisym*)
**qed**

**lemma** *rk-acβ* :
**assumes** *desargues-config-2D A B C A' B' C' P α β γ* **and** *rk* {*A*, *B*, *C*, *A'*, *B'*, *C'*, *P*, *Q*} ≥ *4*
**and** *rk* {*Q*, *A'*, *a*} = *2* **and** *rk* {*Q*, *B'*, *b*} = *2* **and** *rk* {*Q*, *C'*, *c*} = *2* **and** *rk* {*P*, *Q*, *R*} = *2* **and**
*rk* {*P*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2* **and** *rk* {*R*, *A*, *a*} = *2* **and** *rk* {*R*, *B*, *b*} = *2* **and**
*rk* {*R*, *C*, *c*} = *2* **and** *rk* {*A'*, *P*} = *2* **and** *rk* {*B'*, *P*} = *2* **and** *rk* {*C'*, *P*} = *2*
**shows** *rk* {*a*, *c*, *β*} = *2*
**proof**−
**have** *rk* {*a*, *b*, *c*} = *3*
**using** *rk-abc assms*
**by** *auto*
**then have** *rk* {*a*, *c*} = *2*
**by** (*metis insert-commute rk-triple-to-rk-couple*)
**then have** *rk* {*a*, *c*, *β*} ≥ *2*

**using** *matroid-ax-2*
**by** (*metis empty-subsetI insert-mono*)
**have** *f1*:*rk* {*a*, *c*, *A*, *C*, *A′*, *C′*, *β*} + *rk* {*a*, *c*, *β*} ≤ *rk* {*a*, *c*, *A*, *C*, *β*} + *rk* {*a*, *c*, *A′*, *C′*, *β*}
**using** *matroid-ax-3-alt*[*of* {*a*, *c*, *β*} {*a*, *c*, *A*, *C*, *β*} {*a*, *c*, *A′*, *C′*, *β*}]
**by** (*simp add*: *insert-commute*)
**have** *f2*:*rk* {*A*, *A′*, *C*} ≥ *3*
**using** *rk-AA′C assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**have** *f3*:*rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3*
**using** *coplanar-ABCA′B′C′P assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**then have** *f4*:*rk* {*A*, *B*, *P*, *a*} ≥ *4*
**using** *rk-ABPa assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** (*meson assms*(*12*) *assms*(*13*) *assms*(*14*) *assms*(*2*) *assms*(*3*) *assms*(*6*) *assms*(*7*) *assms*(*9*)
*desargues-config-2D-rkA′B′PQ-rkA′C′PQ-rkB′C′PQ*(*1*))
**have** *rk* {*A*, *A′*, *C*, *a*} ≥ *4*
**using** *f2 f3 f4 rk-AA′Ca*[*of A A′ C B P a B′ C′*]
**by** *auto*
**then have** *f5*:*rk* {*a*, *c*, *A*, *C*, *β*} = *3*
**using** *rk-acACβ*[*of R A a C c β Q A′*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*9*) *assms*(*11*) *assms*(*3*) *rk-triple-to-rk-couple*
**by** *blast*
**have** *rk* {*A′*, *A*, *C′*, *a*} ≥ *4*
**using** *rk-AA′C′a*[*of A A′ C′ B P a C B′*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** (*smt assms*(*12*) *assms*(*13*) *assms*(*14*) *desargues-config-2D-non-collinear-P*(*2*) *f3 f4 insert-commute*
*rk-AA′C*)
**then have** *f6*:*rk* {*a*, *c*, *A′*, *C′*, *β*} = *3*
**using** *rk-acA′C′β*[*of Q A′ a C′ c β R A*] *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*assms*(*3*) *assms*(*5*) *assms*(*9*)
**by** (*metis* (*full-types*) *insert-commute rk-triple-to-rk-couple*)
**have** {*A*, *A′*, *C*, *a*} ⊆ {*a*, *c*, *A*, *C*, *A′*, *C′*, *β*}
**by** *auto*
**then have** *f7*:*rk* {*a*, *c*, *A*, *C*, *A′*, *C′*, *β*} ≥ *4*
**using** *matroid-ax-2*
**by** (*meson* ‹*4* ≤ *rk* {*A*, *A′*, *C*, *a*}› *dual-order.trans*)
**then have** *rk* {*a*, *c*, *β*} ≤ *2*
**using** *f1 f5 f6*
**by** *linarith*
**thus** *rk* {*a*, *c*, *β*} = *2*
**using** ‹*2* ≤ *rk* {*a*, *c*, *β*}› *le-antisym*
**by** *blast*
**qed**

**lemma** *rk-abγ* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**and** *rk* {*Q*, *A′*, *a*} = *2* **and** *rk* {*Q*, *B′*, *b*} = *2* **and** *rk* {*Q*, *C′*, *c*} = *2* **and** *rk* {*P*, *Q*, *R*} = *2* **and**
*rk* {*P*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2* **and** *rk* {*R*, *A*, *a*} = *2* **and** *rk* {*R*, *B*, *b*} = *2* **and**
*rk* {*R*, *C*, *c*} = *2* **and** *rk* {*A′*, *P*} = *2* **and** *rk* {*B′*, *P*} = *2* **and** *rk* {*C′*, *P*} = *2*
**shows** *rk* {*a*, *b*, *γ*} = *2*
**proof**−
**have** *desargues-config-2D A C B A′ C′ B′ P α γ β*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*desargues-config-2D-def*[*of A C B A′ C′ B′ P α γ β*]
**by** (*metis insert-commute*)
**thus** *rk* {*a*, *b*, *γ*} = *2*
**using** *rk-acβ*[*of A C B A′ C′ B′ P α γ β Q a c b R*]
**by** (*metis assms*(*10*) *assms*(*11*) *assms*(*12*) *assms*(*13*) *assms*(*14*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*)
*assms*(*6*) *assms*(*7*) *assms*(*8*) *assms*(*9*) *insert-commute*)
**qed**

**lemma** *rk-bcα* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**and** *rk* {*Q*, *A′*, *a*} = *2* **and** *rk* {*Q*, *B′*, *b*} = *2* **and** *rk* {*Q*, *C′*, *c*} = *2* **and** *rk* {*P*, *Q*, *R*} = *2* **and**
*rk* {*P*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2* **and** *rk* {*R*, *A*, *a*} = *2* **and** *rk* {*R*, *B*, *b*} = *2* **and**
*rk* {*R*, *C*, *c*} = *2* **and** *rk* {*A′*, *P*} = *2* **and** *rk* {*B′*, *P*} = *2* **and** *rk* {*C′*, *P*} = *2*
**shows** *rk* {*b*, *c*, *α*} = *2*
**proof**−
**have** *desargues-config-2D B A C B′ A′ C′ P β α γ*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*desargues-config-2D-def*[*of B A C B′ A′ C′ P β α γ*]
**by** (*metis insert-commute*)
**thus** *rk* {*b*, *c*, *α*} = *2*
**using** *rk-acβ*[*of B A C B′ A′ C′ P β α γ Q b a c R*]
**by** (*metis assms*(*10*) *assms*(*11*) *assms*(*12*) *assms*(*13*) *assms*(*14*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*)
*assms*(*6*) *assms*(*7*) *assms*(*8*) *assms*(*9*) *insert-commute*)
**qed**

**lemma** *rk-abcαβγ* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**and** *rk* {*Q*, *A′*, *a*} = *2* **and** *rk* {*Q*, *B′*, *b*} = *2* **and** *rk* {*Q*, *C′*, *c*} = *2* **and** *rk* {*P*, *Q*, *R*} = *2* **and**
*rk* {*P*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2* **and** *rk* {*R*, *A*, *a*} = *2* **and** *rk* {*R*, *B*, *b*} = *2*

**and**
$rk\ \{R,\ C,\ c\} = 2$ **and** $rk\ \{A',\ P\} = 2$ **and** $rk\ \{B',\ P\} = 2$ **and** $rk\ \{C',\ P\} = 2$
**shows** $rk\ \{a,\ b,\ c,\ \alpha,\ \beta,\ \gamma\} = 3$
**proof**−
**have** *f1*:$rk\ \{a,\ b,\ \gamma\} = 2$
**using** *rk-ab*$\gamma$[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ *Q a b c R*] *assms*
**by** *auto*
**have** *f2*:$rk\ \{a,\ c,\ \beta\} = 2$
**using** *rk-ac*$\beta$[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ *Q a b c R*] *assms*
**by** *auto*
**have** *f3*:$rk\ \{b,\ c,\ \alpha\} = 2$
**using** *rk-bc*$\alpha$[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ *Q a b c R*] *assms*
**by** *auto*
**have** $rk\ \{a,\ b,\ c,\ \beta,\ \gamma\} + rk\ \{a\} \leq rk\ \{a,\ b,\ \gamma\} + rk\ \{a,\ c,\ \beta\}$
**using** *matroid-ax-3-alt*[*of* $\{a\}$ $\{a,\ b,\ \gamma\}$ $\{a,\ c,\ \beta\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{a,\ b,\ c,\ \beta,\ \gamma\} \leq 3$
**using** *f1 f2*
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono add-Suc-right numeral-3-eq-3 numeral-nat*(*1*) *numerals*(*1*)
*rk-singleton*)
**then have** *f4*:$rk\ \{a,\ b,\ c,\ \beta,\ \gamma\} = 3$
**using** *matroid-ax-2 rk-abc*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ *Q a b c R*]
**by** (*metis Un-upper2 assms*(*1*) *assms*(*10*) *assms*(*11*) *assms*(*12*) *assms*(*13*) *assms*(*14*) *assms*(*2*) *assms*(*3*)
*assms*(*4*) *assms*(*5*) *assms*(*6*) *assms*(*7*) *assms*(*8*) *assms*(*9*) *insert-mono le-antisym sup-bot.comm-neutral*)
**have** $rk\ \{a,\ b,\ c\} = 3$
**using** *rk-abc*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ *Q a b c R*] *assms*(*1*) *assms*(*10*) *assms*(*11*) *assms*(*12*) *assms*(*13*)
*assms*(*14*) *assms*(*2*) *assms*(*3*) *assms*(*4*) *assms*(*5*) *assms*(*6*) *assms*(*7*) *assms*(*8*) *assms*(*9*)
**by** *blast*
**then have** *f5*:$rk\ \{b,\ c\} = 2$
**using** *rk-triple-to-rk-couple rk-couple*
**by** *force*
**have** $rk\ \{a,\ b,\ c,\ \alpha,\ \beta,\ \gamma\} + rk\ \{b,\ c\} \leq rk\ \{a,\ b,\ c,\ \beta,\ \gamma\} + rk\ \{b,\ c,\ \alpha\}$
**using** *matroid-ax-3-alt*[*of* $\{b,\ c\}$ $\{a,\ b,\ c,\ \beta,\ \gamma\}$ $\{b,\ c,\ \alpha\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{a,\ b,\ c,\ \alpha,\ \beta,\ \gamma\} \leq 3$
**using** *f3 f4 f5*
**by** *linarith*
**thus** $rk\ \{a,\ b,\ c,\ \alpha,\ \beta,\ \gamma\} = 3$
**using** *matroid-ax-2*
**by** (*metis* ‹$rk\ \{a,\ b,\ c\} = 3$› *empty-subsetI insert-mono le-antisym*)
**qed**

**lemma** *rk-ABC*$\alpha\beta\gamma$ :
**assumes** *desargues-config-2D A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$ **and** $rk\ \{A,\ B,\ C,\ A',$

$B'$, $C'$, $P$, $Q\} \geq 4$
**and** $rk\ \{Q,\ A',\ a\} = 2$ **and** $rk\ \{Q,\ B',\ b\} = 2$ **and** $rk\ \{Q,\ C',\ c\} = 2$ **and** $rk$ $\{P,\ Q,\ R\} = 2$ **and**
$rk\ \{P,\ R\} = 2$ **and** $rk\ \{Q,\ R\} = 2$ **and** $rk\ \{R,\ A,\ a\} = 2$ **and** $rk\ \{R,\ B,\ b\} = 2$ **and**
$rk\ \{R,\ C,\ c\} = 2$ **and** $rk\ \{A',\ P\} = 2$ **and** $rk\ \{B',\ P\} = 2$ **and** $rk\ \{C',\ P\} = 2$
**shows** $rk\ \{A,\ B,\ C,\ \alpha,\ \beta,\ \gamma\} = 3$
**proof**−
**have** *f1*:$rk\ \{A,\ B,\ \gamma\} = 2$
**using** *assms*(1) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**have** *f2*:$rk\ \{A,\ C,\ \beta\} = 2$
**using** *assms*(1) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**have** *f3*:$rk\ \{B,\ C,\ \alpha\} = 2$
**using** *assms*(1) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**have** $rk\ \{A,\ B,\ C,\ \beta,\ \gamma\} + rk\ \{A\} \leq rk\ \{A,\ B,\ \gamma\} + rk\ \{A,\ C,\ \beta\}$
**using** *matroid-ax-3-alt*[*of* $\{A\}$ $\{A,\ B,\ \gamma\}$ $\{A,\ C,\ \beta\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A,\ B,\ C,\ \beta,\ \gamma\} \leq 3$
**using** *f1 f2*
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono add-Suc-right numeral-3-eq-3 numeral-nat*(1) *numerals*(1)
*rk-singleton*)
**have** $rk\ \{A,\ B,\ C\} = 3$
**using** *assms*(1) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** *auto*
**then have** *f4*:$rk\ \{A,\ B,\ C,\ \beta,\ \gamma\} = 3$
**using** *matroid-ax-2*
**by** (*metis* ‹$rk\ \{A,\ B,\ C,\ \beta,\ \gamma\} \leq 3$› *empty-subsetI insert-mono le-antisym*)
**have** *f5*:$rk\ \{B,\ C\} = 2$
**using** ‹$rk\ \{A,\ B,\ C\} = 3$› *rk-couple rk-triple-to-rk-couple*
**by** *force*
**have** $rk\ \{A,\ B,\ C,\ \alpha,\ \beta,\ \gamma\} + rk\ \{B,\ C\} \leq rk\ \{A,\ B,\ C,\ \beta,\ \gamma\} + rk\ \{B,\ C,\ \alpha\}$
**using** *matroid-ax-3-alt*[*of* $\{B,\ C\}$ $\{A,\ B,\ C,\ \beta,\ \gamma\}$ $\{B,\ C,\ \alpha\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A,\ B,\ C,\ \alpha,\ \beta,\ \gamma\} \leq 3$
**using** *f3 f4 f5*
**by** *linarith*
**thus** $rk\ \{A,\ B,\ C,\ \alpha,\ \beta,\ \gamma\} = 3$
**using** *matroid-ax-2*
**by** (*metis* ‹$rk\ \{A,\ B,\ C\} = 3$› *empty-subsetI insert-mono le-antisym*)
**qed**

**lemma** *rk-αβγ* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** $rk\ \{A,\ B,\ C,\ A',$ $B',\ C',\ P,\ Q\} \geq 4$
**and** $rk\ \{Q,\ A',\ a\} = 2$ **and** $rk\ \{Q,\ B',\ b\} = 2$ **and** $rk\ \{Q,\ C',\ c\} = 2$ **and** $rk$

$\{P, Q, R\} = 2$ **and**
$rk\ \{P, R\} = 2$ **and** $rk\ \{Q, R\} = 2$ **and** $rk\ \{R, A, a\} = 2$ **and** $rk\ \{R, B, b\} = 2$ **and**
$rk\ \{R, C, c\} = 2$ **and** $rk\ \{A', P\} = 2$ **and** $rk\ \{B', P\} = 2$ **and** $rk\ \{C', P\} = 2$
**shows** $rk\ \{\alpha, \beta, \gamma\} \leq 2$
**proof**−
**have** $rk\ \{A, B, C, a, b, c\} \geq 4$
**using** *desargues-config-2D-rkABCabc*[*of* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma\ Q\ a\ R\ b\ c$] *assms*
**by** *auto*
**have** $\{A, B, C, a, b, c\} \subseteq \{A, B, C, a, b, c, \alpha, \beta, \gamma\}$
**by** *auto*
**then have** *f1*:$rk\ \{A, B, C, a, b, c, \alpha, \beta, \gamma\} \geq 4$
**using** *matroid-ax-2*
**by** (*meson* ‹$4 \leq rk\ \{A, B, C, a, b, c\}$› *dual-order.trans*)
**have** $rk\ \{A, B, C, a, b, c, \alpha, \beta, \gamma\} + rk\ \{\alpha, \beta, \gamma\} \leq rk\ \{A, B, C, \alpha, \beta, \gamma\} + rk\ \{a, b, c, \alpha, \beta, \gamma\}$
**using** *matroid-ax-3-alt*[*of* $\{\alpha, \beta, \gamma\}\ \{A, B, C, \alpha, \beta, \gamma\}\ \{a, b, c, \alpha, \beta, \gamma\}$]
**by** (*simp add*: *insert-commute*)
**thus** $rk\ \{\alpha, \beta, \gamma\} \leq 2$
**using** *f1 rk-ABCαβγ*[*of* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma\ Q\ a\ b\ c\ R$] *rk-abcαβγ*[*of* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma\ Q\ a\ b\ c\ R$]
**by** (*smt One-nat-def Suc-1 Suc-le-eq add-Suc-right add-le-imp-le-diff assms*(1) *assms*(10) *assms*(11)
*assms*(12) *assms*(13) *assms*(14) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *assms*(7) *assms*(8)
*assms*(9) *diff-add-inverse2 le-antisym nat-1-add-1 not-less numeral-3-eq-3 one-plus-numeral*
*rk-triple-le semiring-norm*(2) *semiring-norm*(4))
**qed**

**lemma** *rk-αβγ-special-case-1* :
**assumes** *desargues-config-2D* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma$ **and** $rk\ \{A', P\} = 1$
**shows** $rk\ \{\alpha, \beta, \gamma\} \leq 2$
**proof**−
**have** $A' = P$
**by** (*simp add*: *assms*(2) *rk-couple-to-singleton*)
**then have** $rk\ \{A', C', C, \beta\} + rk\ \{C', A'\} \leq rk\ \{C, C', P\} + rk\ \{A', C', \beta\}$
**using** *matroid-ax-3-alt*[*of* $\{C', A'\}\ \{C, C', P\}\ \{A', C', \beta\}$]
**by** (*simp add*: *insert-commute*)
**then have** $rk\ \{A', C', C, \beta\} \leq rk\ \{A', C', \beta\}$
**using** *assms*(1) *desargues-config-2D-def*[*of* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma$]
**by** (*metis* (*full-types*) *add-le-cancel-right insert-commute rk-triple-to-rk-couple*)
**then have** *f5*:$rk\ \{A', C', C, \beta\} = 2$
**using** *assms*(1) *desargues-config-2D-def*[*of* $A\ B\ C\ A'\ B'\ C'\ P\ \alpha\ \beta\ \gamma$]
**by** (*metis insert-commute le-antisym matroid-ax-2 subset-insertI*)
**have** $rk\ \{A, A', C, a, C', \beta\} + rk\ \{A, C, \beta\} \leq rk\ \{A, A', C', C, \beta\} + rk\ \{a, A, C, \beta\}$ **for** $a$
**using** *matroid-ax-3-alt*[*of* $\{A, C, \beta\}\ \{A, A', C', C, \beta\}\ \{a, A, C, \beta\}$]

**by** (*simp add*: *insert-commute*)
**then have** *f6*:*rk* $\{A, A', C', C, \beta\} \geq 3$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$] *rk-AA'Ca rk-acAC*$\beta$
**by** (*metis Un-insert-right Un-upper2* ‹$A' = P$› *insert-commute matroid-ax-2 sup-bot.right-neutral*)
**have** *rk* $\{A, A', C', C, \beta\}$ + *rk* $\{\beta, C\} \leq$ *rk* $\{A, C, \beta\}$ + *rk* $\{A', C', C, \beta\}$
**using** *matroid-ax-3-alt*[*of* $\{\beta, C\}$ $\{A, C, \beta\}$ $\{A', C', C, \beta\}$]
**by** (*simp add*: *insert-commute*)
**then have** *rk* $\{\beta, C\} \leq 1$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$] *f5 f6*
**by** *linarith*
**then have** $\beta = C$
**using** *rk-couple*
**by** *force*
**have** *rk* $\{A', B', B, \gamma\}$ + *rk* $\{B', A'\} \leq$ *rk* $\{B, B', P\}$ + *rk* $\{A', B', \gamma\}$
**using** *matroid-ax-3-alt*[*of* $\{B', A'\}$ $\{B, B', P\}$ $\{A', B', \gamma\}$]
**by** (*simp add*: ‹$A' = P$› *insert-commute*)
**then have** *rk* $\{A', B', B, \gamma\} \leq$ *rk* $\{A', B', \gamma\}$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$]
**by** (*metis* (*full-types*) *add-le-cancel-right insert-commute rk-triple-to-rk-couple*)
**then have** *f7*:*rk* $\{A', B', B, \gamma\} = 2$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$]
**by** (*metis insert-commute le-antisym matroid-ax-2 subset-insertI*)
**have** *rk* $\{A, A', B, a, B', \gamma\}$ + *rk* $\{A, B, \gamma\} \leq$ *rk* $\{A, A', B', B, \gamma\}$ + *rk* $\{a, A, B, \gamma\}$ **for** *a*
**using** *matroid-ax-3-alt*[*of* $\{A, B, \gamma\}$ $\{A, A', B', B, \gamma\}$ $\{a, A, B, \gamma\}$]
**by** (*simp add*: *insert-commute*)
**then have** *f8*:*rk* $\{A, A', B', B, \gamma\} \geq 3$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$] *rk-AA'Ca rk-acAC*$\beta$
**by** (*metis Un-insert-right Un-upper2* ‹$A' = P$› *insert-commute matroid-ax-2 sup-bot.right-neutral*)
**have** *rk* $\{A, A', B', B, \gamma\}$ + *rk* $\{\gamma, B\} \leq$ *rk* $\{A, B, \gamma\}$ + *rk* $\{A', B', B, \gamma\}$
**using** *matroid-ax-3-alt*[*of* $\{\gamma, B\}$ $\{A, B, \gamma\}$ $\{A', B', B, \gamma\}$]
**by** (*simp add*: *insert-commute*)
**then have** *rk* $\{\gamma, B\} \leq 1$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$] *f7 f8*
**by** *linarith*
**then have** $\gamma = B$
**using** *rk-couple*
**by** *force*
**then have** *rk* $\{\alpha, \beta, \gamma\}$ = *rk* $\{\alpha, C, B\}$
**using** ‹$\beta = C$ ›
**by** *auto*
**thus** *rk* $\{\alpha, \beta, \gamma\} \leq 2$
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A' B' C' P* $\alpha$ $\beta$ $\gamma$]
**by** (*metis insert-commute order-refl*)
**qed**

**lemma** *rk-αβγ-special-case-2* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*B′*, *P*} = *1*
**shows** *rk* {*α*, *β*, *γ*} ≤ *2*
**proof**−
**have** *desargues-config-2D B A C B′ A′ C′ P β α γ*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*desargues-config-2D-def*[*of B A C B′ A′ C′ P β α γ*]
**by** (*metis insert-commute*)
**thus** *rk* {*α*, *β*, *γ*} ≤ *2*
**using** *rk-αβγ-special-case-1*[*of B A C B′ A′ C′ P β α γ*] *assms*(*2*)
**by** (*simp add*: *insert-commute*)
**qed**

**lemma** *rk-αβγ-special-case-3* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ* **and** *rk* {*C′*, *P*} = *1*
**shows** *rk* {*α*, *β*, *γ*} ≤ *2*
**proof**−
**have** *desargues-config-2D C B A C′ B′ A′ P γ β α*
**using** *assms*(*1*) *desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
*desargues-config-2D-def*[*of C B A C′ B′ A′ P γ β α*]
**by** (*metis insert-commute*)
**thus** *rk* {*α*, *β*, *γ*} ≤ *2*
**using** *rk-αβγ-special-case-1*[*of C B A C′ B′ A′ P γ β α*] *assms*(*2*)
**by** (*simp add*: *insert-commute*)
**qed**

**theorem** *desargues-2D* :
**assumes** *desargues-config-2D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*α*, *β*, *γ*} ≤ *2*
**proof**−
**have** *f1*:*rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*} = *3*
**using** *assms desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-ABCA′B′C′P*
**by** *auto*
**obtain** *Q* **where** *rk* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*, *Q*} ≥ *4*
**using** *f1 rk-ext*[*of* {*A*, *B*, *C*, *A′*, *B′*, *C′*, *P*}]
**by** (*metis Un-insert-left add.commute one-plus-numeral order-refl semiring-norm*(*2*) *semiring-norm*(*4*)
*sup-bot.left-neutral*)
**obtain** *R* **where** *rk* {*P*, *Q*, *R*} = *2* **and** *rk* {*P*, *R*} = *2* **and** *rk* {*Q*, *R*} = *2*
**using** *rk-ax-3-pts*
**by** *auto*
**have** *rk* {*Q*, *A′*, *R*, *A*, *P*} + *rk* {*P*} ≤ *rk* {*P*, *Q*, *R*} + *rk* {*A*, *A′*, *P*}
**using** *matroid-ax-3-alt*[*of* {*P*} {*P*, *Q*, *R*} {*A*, *A′*, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*Q*, *A′*, *R*, *A*, *P*} ≤ *3*
**using** *rk-singleton assms desargues-config-2D-def*[*of A B C A′ B′ C′ P α β γ*]
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono* ‹*rk* {*P*, *Q*, *R*} = *2*› *add-Suc-right*

*eval-nat-numeral*(*3*))
**then have** *f2*:*rk* {*Q*, *A*′, *R*, *A*} ≤ *3*
**using** *matroid-ax-2*
**by** (*metis* (*no-types*, *opaque-lifting*) *dual-order.trans insert-commute subset-insertI*)
**obtain** *a* **where** *rk* {*Q*, *A*′, *a*} = *2* **and** *rk* {*R*, *A*, *a*} = *2*
**using** *f2 rk-ax-pasch*
**by** *blast*
**have** *rk* {*Q*, *B*′, *R*, *B*, *P*} + *rk* {*P*} ≤ *rk* {*P*, *Q*, *R*} + *rk* {*B*, *B*′, *P*}
**using** *matroid-ax-3-alt*[*of* {*P*} {*P*, *Q*, *R*} {*B*, *B*′, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*Q*, *B*′, *R*, *B*, *P*} ≤ *3*
**using** *rk-singleton assms desargues-config-2D-def*[*of A B C A*′ *B*′ *C*′ *P* α β γ]
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono* ‹*rk* {*P*, *Q*, *R*} = *2*› *add-Suc-right*
*eval-nat-numeral*(*3*))
**then have** *f3*:*rk* {*Q*, *B*′, *R*, *B*} ≤ *3*
**using** *matroid-ax-2*
**by** (*metis* (*no-types*, *opaque-lifting*) *dual-order.trans insert-commute subset-insertI*)
**obtain** *b* **where** *rk* {*Q*, *B*′, *b*} = *2* **and** *rk* {*R*, *B*, *b*} = *2*
**using** *f3 rk-ax-pasch*
**by** *blast*
**have** *rk* {*Q*, *C*′, *R*, *C*, *P*} + *rk* {*P*} ≤ *rk* {*P*, *Q*, *R*} + *rk* {*C*, *C*′, *P*}
**using** *matroid-ax-3-alt*[*of* {*P*} {*P*, *Q*, *R*} {*C*, *C*′, *P*}]
**by** (*simp add*: *insert-commute*)
**then have** *rk* {*Q*, *C*′, *R*, *C*, *P*} ≤ *3*
**using** *rk-singleton assms desargues-config-2D-def*[*of A B C A*′ *B*′ *C*′ *P* α β γ]
**by** (*metis Suc-1 Suc-eq-plus1 Suc-le-mono* ‹*rk* {*P*, *Q*, *R*} = *2*› *add-Suc-right*
*eval-nat-numeral*(*3*))
**then have** *f4*:*rk* {*Q*, *C*′, *R*, *C*} ≤ *3*
**using** *matroid-ax-2*
**by** (*metis* (*no-types*, *opaque-lifting*) *dual-order.trans insert-commute subset-insertI*)
**obtain** *c* **where** *rk* {*Q*, *C*′, *c*} = *2* **and** *rk* {*R*, *C*, *c*} = *2*
**using** *f4 rk-ax-pasch*
**by** *blast*
**then have** *rk* {α, β, γ} ≤ *2* **if** *rk* {*A*′, *P*} = *2* **and** *rk* {*B*′, *P*} = *2* **and** *rk* {*C*′,
*P*} = *2*
**using** *rk-αβγ*[*of A B C A*′ *B*′ *C*′ *P* α β γ *Q a b c R*] ‹*4* ≤ *rk* {*A*, *B*, *C*, *A*′,
*B*′, *C*′, *P*, *Q*}›
‹*rk* {*P*, *Q*, *R*} = *2*› ‹*rk* {*P*, *R*} = *2*› ‹*rk* {*Q*, *A*′, *a*} = *2*› ‹*rk* {*Q*, *B*′, *b*} =
*2*› ‹*rk* {*Q*, *R*} = *2*›
‹*rk* {*R*, *A*, *a*} = *2*› ‹*rk* {*R*, *B*, *b*} = *2*› *assms that*(*1*) *that*(*2*) *that*(*3*)
**by** *blast*
**have** *rk* {α, β, γ} ≤ *2* **if** *rk* {*A*′, *P*} = *1*
**using** *rk-αβγ-special-case-1 assms that*
**by** *auto*
**have** *rk* {α, β, γ} ≤ *2* **if** *rk* {*B*′, *P*} = *1*
**using** *rk-αβγ-special-case-2 assms that*
**by** *auto*
**have** *rk* {α, β, γ} ≤ *2* **if** *rk* {*C*′, *P*} = *1*
**using** *rk-αβγ-special-case-3 assms that*

**by** *auto*
**thus** *rk* $\{\alpha, \beta, \gamma\} \leq 2$
**using** ‹$[\![rk\ \{A', P\} = 2; rk\ \{B', P\} = 2; rk\ \{C', P\} = 2]\!] \Longrightarrow rk\ \{\alpha, \beta, \gamma\} \leq 2$›
‹$rk\ \{A', P\} = 1 \Longrightarrow rk\ \{\alpha, \beta, \gamma\} \leq 2$› ‹$rk\ \{B', P\} = 1 \Longrightarrow rk\ \{\alpha, \beta, \gamma\} \leq 2$›
*rk-couple rk-singleton-bis*
**by** *blast*
**qed**
**end**

**end**
**theory** *Desargues-3D*
**imports** *Main Higher-Projective-Space-Rank-Axioms Matroid-Rank-Properties*
**begin**

Contents:

- We prove Desargues's theorem: if two triangles ABC and A'B'C' are perspective from a point P (ie. the lines AA', BB' and CC' are concurrent in P), then they are perspective from a line (ie. the points $\alpha = BC \cap B'C'$, $\beta = AC \cap A'C'$ and $\gamma = AB \cap A'B'$ are collinear). In this file we restrict ourself to the case where the two triangles ABC and A'B'C' are not coplanar.

# 9 Desargues's Theorem: The Non-coplanar Case

**context** *higher-projective-space-rank*
**begin**

**definition** *desargues-config-3D* ::
['*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*, '*point*] => *bool*
**where** *desargues-config-3D A B C A' B' C' P* $\alpha\ \beta\ \gamma \equiv$ *rk* $\{A, B, C\} = 3 \wedge$ *rk* $\{A', B', C'\} = 3 \wedge$
*rk* $\{A, A', P\} = 2 \wedge$ *rk* $\{B, B', P\} = 2 \wedge$ *rk* $\{C, C', P\} = 2 \wedge$ *rk* $\{A, B, C, A', B', C'\} \geq 4 \wedge$
*rk* $\{B, C, \alpha\} = 2 \wedge$ *rk* $\{B', C', \alpha\} = 2 \wedge$ *rk* $\{A, C, \beta\} = 2 \wedge$ *rk* $\{A', C', \beta\} = 2 \wedge$ *rk* $\{A, B, \gamma\} = 2 \wedge$
*rk* $\{A', B', \gamma\} = 2$

**lemma** *coplanar-4* :
**assumes** *rk* $\{A, B, C\} = 3$ **and** *rk* $\{B, C, \alpha\} = 2$
**shows** *rk* $\{A, B, C, \alpha\} = 3$
**proof**−
**have** *f1*:*rk* $\{A, B, C, \alpha\} \geq 3$
**using** *matroid-ax-2*
**by** (*metis assms*(*1*) *empty-subsetI insert-mono*)

**have** *rk* $\{A,\ B,\ C,\ \alpha\} + rk\ \{B,\ C\} \leq rk\ \{A,\ B,\ C\} + rk\ \{B,\ C,\ \alpha\}$
**using** *matroid-ax-3-alt*
**by** (*metis Un-insert-right add-One-commute add-mono assms(1) assms(2) matroid-ax-2-alt*
*nat-1-add-1 numeral-plus-one rk-singleton semiring-norm(3) sup-bot.right-neutral*)
**then have** *f2*:*rk* $\{A,\ B,\ C,\ \alpha\} \leq 3$
**by** (*metis Un-insert-right add-One-commute assms(2) matroid-ax-2-alt numeral-plus-one*
*semiring-norm(3) sup-bot.right-neutral*)
**from** *f1* **and** *f2* **show** *rk* $\{A,\ B,\ C,\ \alpha\} = 3$
**by** *auto*
**qed**

**lemma** *desargues-config-3D-coplanar-4* :
**assumes** *desargues-config-3D A B C A' B' C' P* $\alpha\ \beta\ \gamma$
**shows** *rk* $\{A,\ B,\ C,\ \alpha\} = 3$ **and** *rk* $\{A',\ B',\ C',\ \alpha\} = 3$
**proof**−
**show** *rk* $\{A,\ B,\ C,\ \alpha\} = 3$
**using** *assms desargues-config-3D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *coplanar-4*
**by** *blast*
**show** *rk* $\{A',\ B',\ C',\ \alpha\} = 3$
**using** *assms desargues-config-3D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *coplanar-4*
**by** *blast*
**qed**

**lemma** *coplanar-4-bis* :
**assumes** *rk* $\{A,\ B,\ C\} = 3$ **and** *rk* $\{A,\ C,\ \beta\} = 2$
**shows** *rk* $\{A,\ B,\ C,\ \beta\} = 3$
**by** (*smt assms(1) assms(2) coplanar-4 insert-commute*)

**lemma** *desargues-config-3D-coplanar-4-bis* :
**assumes** *desargues-config-3D A B C A' B' C' P* $\alpha\ \beta\ \gamma$
**shows** *rk* $\{A,\ B,\ C,\ \beta\} = 3$ **and** *rk* $\{A',\ B',\ C',\ \beta\} = 3$
**proof**−
**show** *rk* $\{A,\ B,\ C,\ \beta\} = 3$
**using** *assms desargues-config-3D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *coplanar-4-bis*

**by** *auto*
**show** *rk* $\{A',\ B',\ C',\ \beta\} = 3$
**using** *assms desargues-config-3D-def*[*of A B C A' B' C' P* $\alpha\ \beta\ \gamma$] *coplanar-4-bis*

**by** *auto*
**qed**

**lemma** *coplanar-4-ter* :
**assumes** *rk* $\{A,\ B,\ C\} = 3$ **and** *rk* $\{A,\ B,\ \gamma\} = 2$
**shows** *rk* $\{A,\ B,\ C,\ \gamma\} = 3$
**by** (*smt assms(1) assms(2) coplanar-4 insert-commute*)

**lemma** *desargues-config-3D-coplanar-4-ter* :
**assumes** *desargues-config-3D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*A*, *B*, *C*, *γ*} = *3* **and** *rk* {*A′*, *B′*, *C′*, *γ*} = *3*
**proof**−
**show** *rk* {*A*, *B*, *C*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-4-ter*

**by** *auto*
**show** *rk* {*A′*, *B′*, *C′*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-4-ter*

**by** *auto*
**qed**

**lemma** *coplanar-5* :
**assumes** *rk* {*A*, *B*, *C*} = *3* **and** *rk* {*B*, *C*, *α*} = *2* **and** *rk* {*A*, *C*, *β*} = *2*
**shows** *rk* {*A*, *B*, *C*, *α*, *β*} = *3*
**proof**−
**have** *f1*:*rk* {*A*, *B*, *C*, *α*} = *3*
**using** *coplanar-4*
**by** (*smt One-nat-def Un-assoc Un-commute add.commute add-Suc-right assms*(*1*) *assms*(*2*) *insert-is-Un*
*le-antisym matroid-ax-2-alt numeral-2-eq-2 numeral-3-eq-3 one-add-one*)
**have** *f2*:*rk* {*A*, *B*, *C*, *β*} = *3*
**using** *coplanar-4-bis*
**by** (*smt One-nat-def Un-assoc Un-commute add.commute add-Suc-right assms*(*1*) *assms*(*3*) *insert-is-Un*
*le-antisym matroid-ax-2-alt numeral-2-eq-2 numeral-3-eq-3 one-add-one*)
**from** *f1* **and** *f2* **show** *rk* {*A*, *B*, *C*, *α*, *β*} = *3*
**using** *matroid-ax-3-alt′*
**by** (*metis Un-assoc assms*(*1*) *insert-is-Un*)
**qed**

**lemma** *desargues-config-3D-coplanar-5* :
**assumes** *desargues-config-3D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*A*, *B*, *C*, *α*, *β*} = *3* **and** *rk* {*A′*, *B′*, *C′*, *α*, *β*} = *3*
**proof**−
**show** *rk* {*A*, *B*, *C*, *α*, *β*} = *3*
**using** *assms desargues-config-3D-def coplanar-5*
**by** *auto*
**show** *rk* {*A′*, *B′*, *C′*, *α*, *β*} = *3*
**using** *assms desargues-config-3D-def coplanar-5*
**by** *auto*
**qed**

**lemma** *coplanar-5-bis* :
**assumes** *rk* {*A*, *B*, *C*} = *3* **and** *rk* {*B*, *C*, *α*} = *2* **and** *rk* {*A*, *B*, *γ*} = *2*
**shows** *rk* {*A*, *B*, *C*, *α*, *γ*} = *3*

**by** (*smt assms coplanar-5 insert-commute*)

**lemma** *desargues-config-3D-coplanar-5-bis* :
**assumes** *desargues-config-3D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*A*, *B*, *C*, *α*, *γ*} = *3* **and** *rk* {*A′*, *B′*, *C′*, *α*, *γ*} = *3*
**proof**−
**show** *rk* {*A*, *B*, *C*, *α*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-5-bis*

**by** *auto*
**show** *rk* {*A′*, *B′*, *C′*, *α*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-5-bis*

**by** *auto*
**qed**

**lemma** *coplanar-6* :
**assumes** *rk* {*A*, *B*, *C*} = *3* **and** *rk* {*B*, *C*, *α*} = *2* **and** *rk* {*A*, *B*, *γ*} = *2* **and** *rk* {*A*, *C*, *β*} = *2*
**shows** *rk* {*A*, *B*, *C*, *α*, *β*, *γ*} = *3*
**proof**−
**have** *f1*:*rk* {*A*, *B*, *C*, *α*, *γ*} = *3*
**using** *coplanar-5-bis assms*(*1*) *assms*(*2*) *assms*(*3*)
**by** *auto*
**have** *f2*:*rk* {*A*, *B*, *C*, *α*, *β*} = *3*
**using** *coplanar-5 assms*(*1*) *assms*(*2*) *assms*(*4*)
**by** *auto*
**have** *f3*:*rk* {*A*, *B*, *C*, *α*} = *3*
**using** *coplanar-4 assms*(*1*) *assms*(*2*)
**by** *auto*
**from** *f1* **and** *f2* **and** *f3* **show** *rk* {*A*, *B*, *C*, *α*, *β*, *γ*} = *3*
**using** *matroid-ax-3-alt′*[*of* {*A*, *B*, *C*, *α*} *β γ*]
**by** (*metis Un-insert-left sup-bot.left-neutral*)
**qed**

**lemma** *desargues-config-3D-coplanar-6* :
**assumes** *desargues-config-3D A B C A′ B′ C′ P α β γ*
**shows** *rk* {*A*, *B*, *C*, *α*, *β*, *γ*} = *3* **and** *rk* {*A′*, *B′*, *C′*, *α*, *β*, *γ*} = *3*
**proof**−
**show** *rk* {*A*, *B*, *C*, *α*, *β*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-6*
**by** *auto*
**show** *rk* {*A′*, *B′*, *C′*, *α*, *β*, *γ*} = *3*
**using** *assms desargues-config-3D-def*[*of A B C A′ B′ C′ P α β γ*] *coplanar-6*
**by** *auto*
**qed**

**lemma** *desargues-config-3D-non-coplanar* :
**assumes** *desargues-config-3D A B C A′ B′ C′ P α β γ*

**shows** *rk* $\{A, B, C, A', B', C', \alpha, \beta, \gamma\} \geq 4$
**proof**−
**have** *rk* $\{A, B, C, A', B', C'\} \leq$ *rk* $\{A, B, C, A', B', C', \alpha, \beta, \gamma\}$
**using** *matroid-ax-2*
**by** *auto*
**thus** $4 \leq$ *rk* $\{A, B, C, A', B', C', \alpha, \beta, \gamma\}$
**using** *matroid-ax-2 assms desargues-config-3D-def*[*of A B C A' B' C' P α β γ*]
**by** *linarith*
**qed**

**theorem** *desargues-3D* :
**assumes** *desargues-config-3D A B C A' B' C' P α β γ*
**shows** *rk* $\{\alpha, \beta, \gamma\} \leq 2$
**proof**−
**have** *rk* $\{A, B, C, A', B', C', \alpha, \beta, \gamma\}$ + *rk* $\{\alpha, \beta, \gamma\} \leq$ *rk* $\{A, B, C, \alpha, \beta, \gamma\}$ + *rk* $\{A', B', C', \alpha, \beta, \gamma\}$
**using** *matroid-ax-3-alt*[*of* $\{\alpha, \beta, \gamma\}$ $\{A, B, C, \alpha, \beta, \gamma\}$ $\{A', B', C', \alpha, \beta, \gamma\}$]
**by** (*simp add*: *insert-commute*)
**then have** *rk* $\{\alpha, \beta, \gamma\} \leq$ *rk* $\{A, B, C, \alpha, \beta, \gamma\}$ + *rk* $\{A', B', C', \alpha, \beta, \gamma\}$ − *rk* $\{A, B, C, A', B', C', \alpha, \beta, \gamma\}$
**by** *linarith*
**thus** *rk* $\{\alpha, \beta, \gamma\} \leq 2$
**using** *assms desargues-config-3D-coplanar-6 desargues-config-3D-non-coplanar*
**by** *fastforce*
**qed**

**end**

**end**
**theory** *Projective-Space-Axioms*
**imports** *Main*
**begin**

Contents:

- We introduce the types *'point* of points and *'line* of lines and an incidence relation between them.
- A set of axioms for the (3-dimensional) projective space. An alternative set of axioms could use planes as basic objects in addition to points and lines

# 10 The axioms of the Projective Space

**lemma** *distinct4-def*:
*distinct* [*A*,*B*,*C*,*D*] = $((A \neq B) \wedge (A \neq C) \wedge (A \neq D) \wedge (B \neq C) \wedge (B \neq D) \wedge (C \neq D))$

**by** *auto*

**lemma** *distinct3-def*:
*distinct* [*A*, *B*, *C*] = (*A* ≠ *B* ∧ *A* ≠ *C* ∧ *B* ≠ *C*)
**by** *auto*

**locale** *projective-space* =

**fixes** *incid* :: *'point* ⇒ *'line* ⇒ *bool*
**fixes** *meet* :: *'line* ⇒ *'line* ⇒ *'point*
**assumes** *meet-def*: (*incid* (*meet l m*) *l* ∧ *incid* (*meet l m*) *m*)

**assumes** *incid-dec*: (*incid P l*) ∨ ¬(*incid P l*)

**assumes** *ax1-existence*: ∃ *l*. (*incid P l*) ∧ (*incid M l*)
**assumes** *ax1-uniqueness*: (*incid P k*) ⟶ (*incid M k*) ⟶ (*incid P l*) ⟶ (*incid M l*) ⟶ (*P* = *M*) ∨ (*k* = *l*)

**assumes** *ax2*: *distinct* [*A*,*B*,*C*,*D*] ⟶ (*incid A lAB* ∧ *incid B lAB*)
⟶ (*incid C lCD* ∧ *incid D lCD*) ⟶ (*incid A lAC* ∧ *incid C lAC*) ⟶
(*incid B lBD* ∧ *incid D lBD*) ⟶ (∃ *I*.(*incid I lAB* ∧ *incid I lCD*)) ⟶
(∃ *J*.(*incid J lAC* ∧ *incid J lBD*))

**assumes** *ax3*: ∃ *A B C*. *distinct3 A B C* ∧ (*incid A l*) ∧ (*incid B l*) ∧ (*incid C l*)

**assumes** *ax4*: ∃ *l m*.∀ *P*. ¬(*incid P l* ∧ *incid P m*)

**assumes** *ax5*: *distinct* [*l1*,*l2*,*l3*] ⟶ (∃ *l4 J1 J2 J3*. *distinct* [*J1*,*J2*,*J3*] ∧
*meet l1 l4* = *J1* ∧ *meet l2 l4* = *J2* ∧ *meet l3 l4* = *J3*)

**end**
**theory** *Higher-Projective-Space-Axioms*
**imports** *Main*

**begin**

Contents:

- We introduce the types of 'point and 'line and an incidence relation between them.
- A set of axioms for higher projective spaces, i.e. we allow models of dimension $> 3$.

# 11 The axioms for Higher Projective Geometry

**lemma** *distinct4-def*:
*distinct* [A,B,C,D] = ((A ≠ B) ∧ (A ≠ C) ∧ (A ≠ D) ∧ (B ≠ C) ∧ (B ≠ D) ∧ (C ≠ D))
**by** *auto*

**lemma** *distinct3-def*:
*distinct* [A,B,C] = ((A ≠ B) ∧ (A ≠ C) ∧ (B ≠ C))
**by** *auto*

**locale** *higher-projective-space* =

**fixes** *incid* :: *'point* ⇒ *'line* ⇒ *bool*

**assumes** *ax1-existence*: ∃ *l*. (*incid P l*) ∧ (*incid M l*)
**assumes** *ax1-uniqueness*: (*incid P k*) ⟶ (*incid M k*) ⟶ (*incid P l*) ⟶ (*incid M l*) ⟶ (*P* = *M*) ∨ (*k* = *l*)

**assumes** *ax2*: *distinct* [A,B,C,D] ⟶ (*incid A lAB* ∧ *incid B lAB*) ⟶ (*incid C lCD* ∧ *incid D lCD*) ⟶ (*incid A lAC* ∧ *incid C lAC*) ⟶ (*incid B lBD* ∧ *incid D lBD*) ⟶ (∃ *I*.(*incid I lAB* ∧ *incid I lCD*)) ⟶ (∃ *J*.(*incid J lAC* ∧ *incid J lBD*))

**assumes** *ax3*: ∃ *A B C*. *distinct* [A,B,C] ∧ (*incid A l*) ∧ (*incid B l*) ∧ (*incid C l*)

**assumes** $ax4$: $\exists\, l\ m. \forall\, P.\ \neg(incid\ P\ l \wedge incid\ P\ m)$

**end**

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