Probabilistic Primality Testing

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Abstract

The most efficient known primality tests are *probabilistic* in the sense that they use randomness and may, with some probability, mistakenly classify a composite number as prime – but never a prime number as composite. Examples of this are the Miller–Rabin test, the Solovay–Strassen test, and (in most cases) Fermat's test.

This entry defines these three tests and proves their correctness. It also develops some of the number-theoretic foundations, such as Carmichael numbers and the Jacobi symbol with an efficient executable algorithm to compute it.

Contents

1 Additional Facts about the Legendre Symbol

theory *Legendre-Symbol*
imports
HOL−Number-Theory.Number-Theory
begin

lemma *basic-cong*[*simp*]:
fixes *p* :: *int*
assumes *2* < *p*
shows [*−1* ≠ *1*] (*mod p*)
[*1* ≠ *−1*] (*mod p*)
[*0* ≠ *1*] (*mod p*)
[*1* ≠ *0*] (*mod p*)
[*0* ≠ *−1*] (*mod p*)
[*−1* ≠ *0*] (*mod p*)
⟨*proof*⟩

lemma [*simp*]: *0* < *n* ⟹ (*a mod 2*) ^ *n* = *a mod 2* **for** *n* :: *nat* **and** *a* :: *int*
⟨*proof*⟩

lemma *Legendre-in-cong-eq*:
fixes *p* :: *int*
assumes *p* > *2* **and** *b* ∈ {*−1*,*0*,*1*}
shows [*Legendre a m* = *b*] (*mod p*) ⟷ *Legendre a m* = *b*
⟨*proof*⟩

lemma *Legendre-p-eq-2*[*simp*]: *Legendre a 2* = *a mod 2*
⟨*proof*⟩

lemma *Legendre-p-eq-1*[*simp*]: *Legendre a 1* = *0* ⟨*proof*⟩

lemma *euler-criterion-int*:
assumes *prime p* **and** *2* < *p*
shows [*Legendre a p* = *a*^((*nat p−1*) *div 2*)] (*mod p*)
⟨*proof*⟩

lemma *QuadRes-neg*[*simp*]: *QuadRes* (*−p*) *a* = *QuadRes p a* ⟨*proof*⟩

lemma *Legendre-neg*[*simp*]: *Legendre a* (*−p*) = *Legendre a p* ⟨*proof*⟩

lemma *Legendre-mult*[*simp*]:
assumes *prime p*
shows *Legendre* (*a*∗*b*) *p* = *Legendre a p* ∗ *Legendre b p*
⟨*proof*⟩

lemma *QuadRes-mod*[*simp*]: *p dvd n* ⟹ *QuadRes p* (*a mod n*) = *QuadRes p a*
⟨*proof*⟩

lemma *Legendre-mod*[*simp*]: *p dvd n* $\Longrightarrow$ *Legendre* (*a mod n*) *p* = *Legendre a p*
⟨*proof*⟩

lemma *two-cong-0-iff*: [*2* = *0*] (*mod p*) $\longleftrightarrow$ *p* = *1* ∨ *p* = *2* **for** *p* :: *nat*
⟨*proof*⟩

lemma *two-cong-0-iff-nat*: [*2* = *0*] (*mod int p*) $\longleftrightarrow$ *p* = *1* ∨ *p* = *2*
⟨*proof*⟩

lemma *two-cong-0-iff-int*: *p* > *0* $\Longrightarrow$ [*2* = *0*] (*mod p*) $\longleftrightarrow$ *p* = *1* ∨ *p* = *2* **for** *p* :: *int*
⟨*proof*⟩

lemma *QuadRes-2-2* [*simp*, *intro*]: *QuadRes 2 2*
⟨*proof*⟩

lemma *Suc-mod-eq*[*simp*]: [*Suc a* = *Suc b*] (*mod 2*) = [*a* = *b*] (*mod 2*)
⟨*proof*⟩

lemma *div-cancel-aux*: *c dvd a* $\Longrightarrow$ (*d* + *a* ∗ *b*) *div c* = (*d div c*) + *a div c* ∗ *b* **for** *a b c* :: *nat*
⟨*proof*⟩

corollary *div-cancel-Suc*: *c dvd a* $\Longrightarrow$ *1* < *c* $\Longrightarrow$ *Suc* (*a* ∗ *b*) *div c* = *a div c* ∗ *b*
⟨*proof*⟩

lemma *cong-aux-eq-1*: *odd p* $\Longrightarrow$ [(*p* − *1*) *div 2* − *p div 4* = (*p*^*2* − *1*) *div 8*] (*mod 2*) **for** *p* :: *nat*
⟨*proof*⟩

lemma *cong-2-pow*[*intro*]: (−*1* :: *int*)^*a* = (−*1*)^*b* **if** [*a* = *b*] (*mod 2*) **for** *a b* :: *nat*
⟨*proof*⟩

lemma *card-Int*: *card* (*A* ∩ *B*) = *card A* − *card* (*A* − *B*) **if** *finite A*
⟨*proof*⟩

Proofs are inspired by [**?**].

theorem *supplement2-Legendre*:
fixes *p* :: *int*
assumes *p* > *2 prime p*
shows *Legendre 2 p* = (−*1*) ^ (((*nat p*)^*2* − *1*) *div 8*)
⟨*proof*⟩

theorem *supplement1-Legendre*:
prime p $\Longrightarrow$ *2* < *p* $\Longrightarrow$ *Legendre* (−*1*) *p* = (−*1*)^((*p*−*1*) *div 2*)
⟨*proof*⟩

lemma *QuadRes-1-right* [*intro*, *simp*]: *QuadRes p 1*
⟨*proof*⟩

lemma *Legendre-1-left* [*simp*]: *prime p* $\Longrightarrow$ *Legendre 1 p = 1*
⟨*proof*⟩

lemma *cong-eq-0-not-coprime*: *prime p* $\Longrightarrow$ [*a = 0*] (*mod p*) $\Longrightarrow$ ¬*coprime a p* **for** *a p* :: *int*
⟨*proof*⟩

lemma *not-coprime-cong-eq-0*: *prime p* $\Longrightarrow$ ¬*coprime a p* $\Longrightarrow$ [*a = 0*] (*mod p*) **for** *a p* :: *int*
⟨*proof*⟩

lemma *prime-cong-eq-0-iff*: *prime p* $\Longrightarrow$ [*a = 0*] (*mod p*) $\longleftrightarrow$ ¬*coprime a p* **for** *a p* :: *int*
⟨*proof*⟩

lemma *Legendre-eq-0-iff* [*simp*]: *prime p* $\Longrightarrow$ *Legendre a p = 0* $\longleftrightarrow$ ¬*coprime a p*
⟨*proof*⟩

lemma *Legendre-prod-mset* [*simp*]: *prime p* $\Longrightarrow$ *Legendre* (*prod-mset M*) *p* = ($\prod$ *q*∈#*M*. *Legendre q p*)
⟨*proof*⟩

lemma *Legendre-0-eq-0*[*simp*]: *Legendre 0 p = 0* ⟨*proof*⟩

lemma *Legendre-values*: *Legendre p q* ∈ {*1*, − *1*, *0*}
⟨*proof*⟩

end

2 Auxiliary Material

theory *Algebraic-Auxiliaries*
imports
HOL−Algebra.Algebra
HOL−Computational-Algebra.Squarefree
HOL−Number-Theory.Number-Theory
begin

hide-const (**open**) *Divisibility.prime*

lemma *sum-of-bool-eq-card*:
assumes *finite S*
shows ($\sum$ *a* ∈ *S*. *of-bool* (*P a*)) = *real* (*card* {*a* ∈ *S* . *P a* })
⟨*proof*⟩

lemma *mod-natE*:
fixes *a n b* :: *nat*
assumes *a mod n = b*

shows $\exists\ l.\ a = n * l + b$
⟨*proof*⟩

lemma (**in** *group*) *r-coset-is-image*: $H \#> a = (\lambda\ x.\ x \otimes a)\ `\ H$
⟨*proof*⟩

lemma (**in** *group*) *FactGroup-order*:
assumes *subgroup H G finite H*
shows *order G* = *order* (*G Mod H*) ∗ *card H*
⟨*proof*⟩

corollary (**in** *group*) *FactGroup-order-div*:
assumes *subgroup H G finite H*
shows *order* (*G Mod H*) = *order G div card H*
⟨*proof*⟩

lemma *group-hom-imp-group-hom-image*:
assumes *group-hom G G h*
shows *group-hom G* (*G*(|*carrier* := *h* ` *carrier G*|)) *h*
⟨*proof*⟩

theorem *homomorphism-thm*:
assumes *group-hom G G h*
shows *G Mod kernel G* (*G*(|*carrier* := *h* ` *carrier G*|)) *h* ≅ *G* (|*carrier* := *h* ` *carrier G*|)
⟨*proof*⟩

lemma *is-iso-imp-same-card*:
assumes *H* ≅ *G*
shows *order H* = *order G*
⟨*proof*⟩

corollary *homomorphism-thm-order*:
assumes *group-hom G G h*
shows *order* (*G*(|*carrier* := *h* ` *carrier G*|)) ∗ *card* (*kernel G* (*G*(|*carrier* := *h* ` *carrier G*|)) *h*) = *order G*
⟨*proof*⟩

lemma (**in** *group-hom*) *kernel-subset*: *kernel G H h* ⊆ *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *proper-subgroup-imp-bound-on-card*:
assumes *H* ⊂ *carrier G subgroup H G finite* (*carrier G*)
shows *card H* ≤ *order G div 2*
⟨*proof*⟩

lemma *cong-exp-trans*[*trans*]:
[*a* ^ *b* = *c*] (*mod n*) ⟹ [*a* = *d*] (*mod n*) ⟹ [*d* ^ *b* = *c*] (*mod n*)
[*c* = *a* ^ *b*] (*mod n*) ⟹ [*a* = *d*] (*mod n*) ⟹ [*c* = *d* ^ *b*] (*mod n*)

⟨*proof*⟩

lemma *cong-exp-mod*[*simp*]:
[(*a mod n*) ^ *b* = *c*] (*mod n*) ⟷ [*a* ^ *b* = *c*] (*mod n*)
[*c* = (*a mod n*) ^ *b*] (*mod n*) ⟷ [*c* = *a* ^ *b*] (*mod n*)
⟨*proof*⟩

lemma *cong-mult-mod*[*simp*]:
[(*a mod n*) * *b* = *c*] (*mod n*) ⟷ [*a* * *b* = *c*] (*mod n*)
[*a* * (*b mod n*) = *c*] (*mod n*) ⟷ [*a* * *b* = *c*] (*mod n*)
⟨*proof*⟩

lemma *cong-add-mod*[*simp*]:
[(*a mod n*) + *b* = *c*] (*mod n*) ⟷ [*a* + *b* = *c*] (*mod n*)
[*a* + (*b mod n*) = *c*] (*mod n*) ⟷ [*a* + *b* = *c*] (*mod n*)
[$\sum i{\in}A.\ f\ i$ *mod n* = *c*] (*mod n*) ⟷ [$\sum i{\in}A.\ f\ i$ = *c*] (*mod n*)
⟨*proof*⟩

lemma *cong-add-trans*[*trans*]:
[*a* = *b* + *x*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *b* + *y*] (*mod n*)
[*a* = *x* + *b*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *y* + *b*] (*mod n*)
[*b* + *x* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*b* + *y* = *a*] (*mod n*)
[*x* + *b* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*y* + *b* = *a*] (*mod n*)
⟨*proof*⟩

lemma *cong-mult-trans*[*trans*]:
[*a* = *b* * *x*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *b* * *y*] (*mod n*)
[*a* = *x* * *b*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *y* * *b*] (*mod n*)
[*b* * *x* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*b* * *y* = *a*] (*mod n*)
[*x* * *b* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*y* * *b* = *a*] (*mod n*)
⟨*proof*⟩

lemma *cong-diff-trans*[*trans*]:
[*a* = *b* − *x*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *b* − *y*] (*mod n*)
[*a* = *x* − *b*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*a* = *y* − *b*] (*mod n*)
[*b* − *x* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*b* − *y* = *a*] (*mod n*)
[*x* − *b* = *a*] (*mod n*) ⟹ [*x* = *y*] (*mod n*) ⟹ [*y* − *b* = *a*] (*mod n*)
for *a* :: *'a* :: {*unique-euclidean-semiring*, *euclidean-ring-cancel*}
⟨*proof*⟩

lemma *eq-imp-eq-mod-int*: *a* = *b* ⟹ [*a* = *b*] (*mod m*) **for** *a b* :: *int* ⟨*proof*⟩
lemma *eq-imp-eq-mod-nat*: *a* = *b* ⟹ [*a* = *b*] (*mod m*) **for** *a b* :: *nat* ⟨*proof*⟩

lemma *cong-pow-I*: *a* = *b* ⟹ [*x*^*a* = *x*^*b*](*mod n*) ⟨*proof*⟩

lemma *gre1I*: (*n* = *0* ⟹ *False*) ⟹ (*1* :: *nat*) ≤ *n*
⟨*proof*⟩

lemma *gre1I-nat*: (*n* = *0* ⟹ *False*) ⟹ (*Suc 0* :: *nat*) ≤ *n*

⟨*proof*⟩

lemma *totient-less-not-prime*:
assumes ¬ *prime n 1* < *n*
shows *totient n* < *n* − *1*
⟨*proof*⟩

lemma *power2-diff-nat*: $x \geq y \Longrightarrow (x - y)^2 = x^2 + y^2 - 2 * x * y$ **for** *x y* :: *nat*
⟨*proof*⟩

lemma *square-inequality*: *1* < *n* ⟹ (*n* + *n*) ≤ (*n* ∗ *n*) **for** *n* :: *nat*
⟨*proof*⟩

lemma *square-one-cong-one*:
assumes [*x* = *1*](*mod n*)
shows [*x*^*2* = *1*](*mod n*)
⟨*proof*⟩

lemma *cong-square-alt-int*:
prime p ⟹ [*a* ∗ *a* = *1*] (*mod p*) ⟹ [*a* = *1*] (*mod p*) ∨ [*a* = *p* − *1*] (*mod p*)
for *a p* :: *'a* :: {*normalization-semidom*, *linordered-idom*, *unique-euclidean-ring*}
⟨*proof*⟩

lemma *cong-square-alt*:
prime p ⟹ [*a* ∗ *a* = *1*] (*mod p*) ⟹ [*a* = *1*] (*mod p*) ∨ [*a* = *p* − *1*] (*mod p*)
for *a p* :: *nat*
⟨*proof*⟩

lemma *square-minus-one-cong-one*:
fixes *n x* :: *nat*
assumes *1* < *n* [*x* = *n* − *1*](*mod n*)
shows [*x*^*2* = *1*](*mod n*)
⟨*proof*⟩

lemma *odd-prime-gt-2-int*:
2 < *p* **if** *odd p prime p* **for** *p* :: *int*
⟨*proof*⟩

lemma *odd-prime-gt-2-nat*:
2 < *p* **if** *odd p prime p* **for** *p* :: *nat*
⟨*proof*⟩

lemma *gt-one-imp-gt-one-power-if-coprime*:
1 ≤ *x* ⟹ *1* < *n* ⟹ *coprime x n* ⟹ *1* ≤ *x* ^ (*n* − *1*) *mod n*
⟨*proof*⟩

lemma *residue-one-dvd*: *a mod n* = *1* ⟹ *n dvd a* − *1* **for** *a n* :: *nat*
⟨*proof*⟩

lemma *coprimeI-power-mod*:
fixes $x\ r\ n :: nat$
assumes $x\ \hat{}\ r\ mod\ n = 1\ r \neq 0\ n \neq 0$
shows *coprime* $x\ n$
⟨*proof*⟩

lemma *prime-dvd-choose*:
assumes $0 < k\ k < p$ *prime* p
shows p *dvd* $(p\ choose\ k)$
⟨*proof*⟩

lemma *cong-eq-0-I*: $(\forall\, i{\in}A.\ [f\ i\ mod\ n = 0]\ (mod\ n)) \Longrightarrow [\sum i{\in}A.\ f\ i = 0]\ (mod\ n)$
⟨*proof*⟩

lemma *power-mult-cong*:
assumes $[x\hat{}n = a](mod\ m)\ [y\hat{}n = b](mod\ m)$
shows $[(x{*}y)\hat{}n = a{*}b](mod\ m)$
⟨*proof*⟩

lemma
fixes $n :: nat$
assumes $n > 1$
shows *odd-pow-cong*: $odd\ m \Longrightarrow [(n - 1)\ \hat{}\ m = n - 1]\ (mod\ n)$
and *even-pow-cong*: $even\ m \Longrightarrow [(n - 1)\ \hat{}\ m = 1]\ (mod\ n)$
⟨*proof*⟩

lemma *cong-mult-uneq'*:
fixes $a :: 'a{::}\{unique\text{-}euclidean\text{-}ring, ring\text{-}gcd\}$
assumes *coprime* $d\ a$
shows $[b \neq c]\ (mod\ a) \Longrightarrow [d = e]\ (mod\ a) \Longrightarrow [b * d \neq c * e]\ (mod\ a)$
⟨*proof*⟩

lemma *p-coprime-right-nat*: $prime\ p \Longrightarrow coprime\ a\ p = (\neg\ p\ dvd\ a)$ **for** $p\ a :: nat$
⟨*proof*⟩

lemma *squarefree-mult-imp-coprime*:
assumes $squarefree\ (a * b :: 'a :: semiring\text{-}gcd)$
shows *coprime* $a\ b$
⟨*proof*⟩

lemma *prime-divisor-exists-strong*:
fixes $m :: int$
assumes $m > 1\ \neg prime\ m$
shows $\exists\, n\ k.\ m = n * k \wedge 1 < n \wedge n < m \wedge 1 < k \wedge k < m$
⟨*proof*⟩

lemma *prime-divisor-exists-strong-nat*:
fixes $m :: nat$
assumes $1 < m$ $\neg prime\ m$
shows $\exists p\ k.\ m = p * k \wedge 1 < p \wedge p < m \wedge 1 < k \wedge k < m \wedge prime\ p$
⟨*proof*⟩

lemma *prime-factorization-eqI*:
assumes $\bigwedge p.\ p \in\# P \Longrightarrow prime\ p\ prod\text{-}mset\ P = n$
shows *prime-factorization* $n = P$
⟨*proof*⟩

lemma *prime-factorization-prime-elem*:
assumes *prime-elem* p
shows *prime-factorization* $p = \{\#normalize\ p\#\}$
⟨*proof*⟩

lemma *size-prime-factorization-eq-Suc-0-iff* [*simp*]:
fixes $n :: 'a ::$ *factorial-semiring-multiplicative*
shows *size* (*prime-factorization* n) = *Suc* $0 \longleftrightarrow$ *prime-elem* n
⟨*proof*⟩

lemma *squarefree-prime-elem* [*simp*, *intro*]:
fixes $p :: 'a ::$ *algebraic-semidom*
assumes *prime-elem* p
shows *squarefree* p
⟨*proof*⟩

lemma *squarefree-prime* [*simp*, *intro*]: *prime* $p \Longrightarrow$ *squarefree* p
⟨*proof*⟩

lemma *not-squarefree-primepow*:
assumes *primepow* n
shows *squarefree* $n \longleftrightarrow$ *prime* n
⟨*proof*⟩

lemma *prime-factorization-normalize* [*simp*]:
prime-factorization (*normalize* n) = *prime-factorization* n
⟨*proof*⟩

lemma *one-prime-factor-iff-primepow*:
fixes $n :: 'a ::$ *factorial-semiring-multiplicative*
shows *card* (*prime-factors* n) = *Suc* $0 \longleftrightarrow$ *primepow* (*normalize* n)
⟨*proof*⟩

lemma *squarefree-imp-prod-prime-factors-eq*:

fixes $x :: 'a :: \textit{factorial-semiring-multiplicative}$
assumes *squarefree x*
shows $\prod (\textit{prime-factors } x) = \textit{normalize } x$
⟨*proof*⟩

end

3 The Jacobi Symbol

theory *Jacobi-Symbol*
imports
Legendre-Symbol
Algebraic-Auxiliaries
begin

The Jacobi symbol is a generalisation of the Legendre symbol to non-primes [**?**, **?**]. It is defined as

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{k_1} \ldots \left(\frac{a}{p_l}\right)^{k_l}$$

where $(\frac{a}{p})$ denotes the Legendre symbol, a is an integer, n is an odd natural number and $p_1^{k_1} \ldots p_l^{k_l}$ is its prime factorisation.
There is, however, a fairly natural generalisation to all non-zero integers for n. It is less clear what a good choice for $n = 0$ is; Mathematica and Maxima adopt the convention that $(\frac{\pm 1}{0}) = 1$ and $(\frac{a}{0}) = 0$ otherwise. However, we chose the slightly different convention $(\frac{a}{0}) = 0$ for *all* a because then the Jacobi symbol is completely multiplicative in both arguments without any restrictions.

definition *Jacobi* :: *int* ⇒ *int* ⇒ *int* **where**
Jacobi a n = (*if n* = *0 then 0 else*
(∏ *p*∈#*prime-factorization n*. *Legendre a p*))

lemma *Jacobi-0-right* [*simp*]: *Jacobi a 0* = *0*
⟨*proof*⟩

lemma *Jacobi-mult-left* [*simp*]: *Jacobi* (*a* ∗ *b*) *n* = *Jacobi a n* ∗ *Jacobi b n*
⟨*proof*⟩

lemma *Jacobi-mult-right* [*simp*]: *Jacobi a* (*n* ∗ *m*) = *Jacobi a n* ∗ *Jacobi a m*
⟨*proof*⟩

lemma *prime-p-Jacobi-eq-Legendre*[*intro*!]: *prime p* ⟹ *Jacobi a p* = *Legendre a p*
⟨*proof*⟩

lemma *Jacobi-mod* [*simp*]: *Jacobi* (*a mod m*) *n* = *Jacobi a n* **if** *n dvd m*

⟨*proof*⟩

lemma *Jacobi-mod-cong*: $[a = b]\ (mod\ n) \Longrightarrow Jacobi\ a\ n = Jacobi\ b\ n$
⟨*proof*⟩

lemma *Jacobi-1-eq-1* [*simp*]: $p \neq 0 \Longrightarrow Jacobi\ 1\ p = 1$
⟨*proof*⟩

lemma *Jacobi-eq-0-not-coprime*:
assumes $n \neq 0$ $\neg coprime\ a\ n$
shows $Jacobi\ a\ n = 0$
⟨*proof*⟩

lemma *Jacobi-p-eq-2'*[*simp*]: $n > 0 \Longrightarrow Jacobi\ a\ (2\hat{}n) = a\ mod\ 2$
⟨*proof*⟩

lemma *Jacobi-prod-mset*[*simp*]: $n \neq 0 \Longrightarrow Jacobi\ (prod\text{-}mset\ M)\ n = (\prod q{\in}\#M.\ Jacobi\ q\ n)$
⟨*proof*⟩

lemma *non-trivial-coprime-neq*:
$1 < a \Longrightarrow 1 < b \Longrightarrow coprime\ a\ b \Longrightarrow a \neq b$ **for** $a\ b :: int$ ⟨*proof*⟩

lemma *odd-odd-even*:
fixes $a\ b :: int$
assumes $odd\ a\ odd\ b$
shows $even\ ((a*b-1)\ div\ 2) = even\ ((a-1)\ div\ 2 + (b-1)\ div\ 2)$
⟨*proof*⟩

lemma *prime-nonprime-wlog* [*case-names primes nonprime sym*]:
assumes $\bigwedge p\ q.\ prime\ p \Longrightarrow prime\ q \Longrightarrow P\ p\ q$
assumes $\bigwedge p\ q.\ \neg prime\ p \Longrightarrow P\ p\ q$
assumes $\bigwedge p\ q.\ P\ p\ q \Longrightarrow P\ q\ p$
shows $P\ p\ q$
⟨*proof*⟩

lemma *Quadratic-Reciprocity-Jacobi*:
fixes $p\ q :: int$
assumes $coprime\ p\ q$
and $2 < p$ $2 < q$
and $odd\ p\ odd\ q$
shows $Jacobi\ p\ q * Jacobi\ q\ p =$
$(-1)\ \hat{}\ (nat\ ((p-1)\ div\ 2 * ((q-1)\ div\ 2)))$
⟨*proof*⟩

lemma *Jacobi-values*: $Jacobi\ p\ q \in \{1, -1, 0\}$
⟨*proof*⟩

lemma *Quadratic-Reciprocity-Jacobi'*:
fixes *p q* :: *int*
assumes *coprime p q*
and *2 < p 2 < q*
and *odd p odd q*
shows *Jacobi q p* = (*if p mod 4* = *3* ∧ *q mod 4* = *3 then* −*1 else 1*) ∗ *Jacobi p q*
⟨*proof*⟩

lemma *dvd-odd-square*: *8 dvd* a^2 − *1* **if** *odd a* **for** *a* :: *int*
⟨*proof*⟩

lemma *odd-odd-even'*:
fixes *a b* :: *int*
assumes *odd a odd b*
shows *even* (((a ∗ b)^2 − *1*) *div 8*) ⟷ *even* (((a^2 − *1*) *div 8*) + ((b^2 − *1*) *div 8*))
⟨*proof*⟩

lemma *odd-odd-even-nat'*:
fixes *a b* :: *nat*
assumes *odd a odd b*
shows *even* (((a ∗ b)^2 − *1*) *div 8*) ⟷ *even* (((a^2 − *1*) *div 8*) + ((b^2 − *1*) *div 8*))
⟨*proof*⟩

lemma *supplement2-Jacobi*: *odd p* ⟹ *p > 1* ⟹ *Jacobi 2 p* = (− *1*) ^ (((nat p)^2 − *1*) *div 8*)
⟨*proof*⟩

lemma *mod-nat-wlog* [*consumes 1*, *case-names modulo*]:
fixes *P* :: *nat* ⇒ *bool*
assumes *b > 0*
assumes ⋀*k*. *k* ∈ {*0*..<*b*} ⟹ *n mod b* = *k* ⟹ *P n*
shows *P n*
⟨*proof*⟩

lemma *mod-int-wlog* [*consumes 1*, *case-names modulo*]:
fixes *P* :: *int* ⇒ *bool*
assumes *b > 0*
assumes ⋀*k*. *0* ≤ *k* ⟹ *k < b* ⟹ *n mod b* = *k* ⟹ *P n*
shows *P n*
⟨*proof*⟩

lemma *supplement2-Jacobi'*:
assumes *odd p* **and** *p > 1*
shows *Jacobi 2 p* = (*if p mod 8* = *1* ∨ *p mod 8* = *7 then 1 else* −*1*)
⟨*proof*⟩

theorem *supplement1-Jacobi*:
odd p $\Longrightarrow$ *1 < p* $\Longrightarrow$ *Jacobi (−1) p = (−1) ^ (nat ((p − 1) div 2))*
⟨*proof*⟩

theorem *supplement1-Jacobi′*:
odd n $\Longrightarrow$ *1 < n* $\Longrightarrow$ *Jacobi (−1) n = (if n mod 4 = 1 then 1 else −1)*
⟨*proof*⟩

lemma *Jacobi-0-eq-0*: ¬*is-unit n* $\Longrightarrow$ *Jacobi 0 n = 0*
⟨*proof*⟩

lemma *is-unit-Jacobi-aux*: *is-unit x* $\Longrightarrow$ *Jacobi a x = 1*
⟨*proof*⟩

lemma *is-unit-Jacobi*[*simp*]: *Jacobi a 1 = 1 Jacobi a (−1) = 1*
⟨*proof*⟩

lemma *Jacobi-neg-right* [*simp*]:
Jacobi a (−n) = Jacobi a n
⟨*proof*⟩

lemma *Jacobi-neg-left*:
assumes *odd n 1 < n*
shows *Jacobi (−a) n = (if n mod 4 = 1 then 1 else −1) ∗ Jacobi a n*
⟨*proof*⟩

function *jacobi-code* :: *int* ⇒ *int* ⇒ *int* **where**
jacobi-code a n = (
if n = 0 then 0
else if n = 1 then 1
else if a = 1 then 1
else if n < 0 then jacobi-code a (−n)
else if even n then if even a then 0 else jacobi-code a (n div 2)
else if a < 0 then (if n mod 4 = 1 then 1 else −1) ∗ jacobi-code (−a) n
else if a = 0 then 0
else if a ≥ n then jacobi-code (a mod n) n
else if even a then (if n mod 8 ∈ {1, 7} then 1 else −1) ∗ jacobi-code (a div 2) n
else if coprime a n then (if n mod 4 = 3 ∧ a mod 4 = 3 then −1 else 1) ∗ jacobi-code n a
else 0)
⟨*proof*⟩
termination
⟨*proof*⟩

lemmas [*simp del*] = *jacobi-code.simps*

lemma *Jacobi-code* [*code*]: *Jacobi a n = jacobi-code a n*
⟨*proof*⟩

lemma *Jacobi-eq-0-imp-not-coprime*:
assumes $p \neq 0$ $p \neq 1$
shows *Jacobi n p* $= 0 \Longrightarrow \neg$*coprime n p*
⟨*proof*⟩

lemma *Jacobi-eq-0-iff-not-coprime*:
assumes $p \neq 0$ $p \neq 1$
shows *Jacobi n p* $= 0 \longleftrightarrow \neg$*coprime n p*
⟨*proof*⟩

end

4 Residue Rings of Natural Numbers

theory *Residues-Nat*
imports *Algebraic-Auxiliaries*
begin

4.1 The multiplicative group of residues modulo n

definition *Residues-Mult* :: $'a$ :: {*linordered-semidom*, *euclidean-semiring*} ⇒ $'a$ *monoid* **where**
Residues-Mult p =
(|*carrier* = {$x \in \{1..p\}$. *coprime x p*}, *monoid.mult* = $\lambda x\ y.\ x * y \bmod p$, *one* = *1*|)

locale *residues-mult-nat* =
fixes n :: *nat* **and** G
assumes *n-gt-1*: $n > 1$
defines $G \equiv$ *Residues-Mult n*
begin

lemma *carrier-eq* [*simp*]: *carrier G* = *totatives n*
and *mult-eq* [*simp*]: $(x \otimes_G y) = (x * y) \bmod n$
and *one-eq* [*simp*]: $\mathbf{1}_G = 1$
⟨*proof*⟩

lemma *mult-eq'*: $(\otimes_G) = (\lambda x\ y.\ (x * y) \bmod n)$
⟨*proof*⟩

sublocale *group G*
⟨*proof*⟩

sublocale *comm-group*
⟨*proof*⟩

lemma *nat-pow-eq* [*simp*]: $x\ [\hat{}]_G\ (k :: nat) = (x \hat{}\ k) \bmod n$
⟨*proof*⟩

lemma *nat-pow-eq'*: $([\widehat{}]_G) = (\lambda x\ k.\ (x\ \widehat{}\ k)\ mod\ n)$
⟨*proof*⟩

lemma *order-eq*: $order\ G = totient\ n$
⟨*proof*⟩

lemma *order-less*: $\neg prime\ n \Longrightarrow order\ G < n - 1$
⟨*proof*⟩

lemma *ord-residue-mult-group*:
assumes $a \in totatives\ n$
shows $local.ord\ a = Pocklington.ord\ n\ a$
⟨*proof*⟩

end

4.2 The ring of residues modulo n

definition *Residues-nat* :: $nat \Rightarrow nat\ ring$ **where**
$Residues\text{-}nat\ m = (\!|carrier = \{0..<m\},\ monoid.mult = \lambda x\ y.\ (x * y)\ mod\ m,\ one = 1,$
$ring.zero = 0,\ add = \lambda x\ y.\ (x + y)\ mod\ m|\!)$

locale *residues-nat* =
fixes n :: *nat* **and** R
assumes *n-gt-1*: $n > 1$
defines $R \equiv Residues\text{-}nat\ n$
begin

lemma *carrier-eq* [*simp*]: $carrier\ R = \{0..<n\}$
and *mult-eq* [*simp*]: $x \otimes_R y = (x * y)\ mod\ n$
and *add-eq* [*simp*]: $x \oplus_R y = (x + y)\ mod\ n$
and *one-eq* [*simp*]: $\mathbf{1}_R = 1$
and *zero-eq* [*simp*]: $\mathbf{0}_R = 0$
⟨*proof*⟩

lemma *mult-eq'*: $(\otimes_R) = (\lambda x\ y.\ (x * y)\ mod\ n)$
and *add-eq'*: $(\oplus_R) = (\lambda x\ y.\ (x + y)\ mod\ n)$
⟨*proof*⟩

sublocale *abelian-group R*
⟨*proof*⟩

sublocale *comm-monoid R*
⟨*proof*⟩

sublocale *cring R*
⟨*proof*⟩

lemma *Units-eq*: *Units R = totatives n*
⟨*proof*⟩

sublocale *units*: *residues-mult-nat n units-of R*
⟨*proof*⟩

lemma *nat-pow-eq* [*simp*]: *x* [^]$_R$ (*k* :: *nat*) = (*x* ^ *k*) *mod n*
⟨*proof*⟩

lemma *nat-pow-eq'*: ([^]$_R$) = (λ*x k*. (*x* ^ *k*) *mod n*)
⟨*proof*⟩

end

4.3 The ring of residues modulo a prime

locale *residues-nat-prime* =
fixes *p* :: *nat* **and** *R*
assumes *prime-p*: *prime p*
defines *R* ≡ *Residues-nat p*
begin

sublocale *residues-nat p R*
⟨*proof*⟩

lemma *carrier-eq'* [*simp*]: *totatives p* = {*0*<..<*p*}
⟨*proof*⟩

lemma *order-eq*: *order* (*units-of R*) = *p* − *1*
⟨*proof*⟩

lemma *order-eq'* [*simp*]: *totient p* = *p* − *1*
⟨*proof*⟩

sublocale *field R*
⟨*proof*⟩

lemma *residues-prime-cyclic*: ∃ *x*∈{*0*<..<*p*}. {*0*<..<*p*} = {*y*. ∃ *i*. *y* = *x* ^ *i mod p*}
⟨*proof*⟩

lemma *residues-prime-cyclic'*: ∃ *x*∈{*0*<..<*p*}. *units.ord x* = *p* − *1*
⟨*proof*⟩

end

4.4 −*1* in residue rings

lemma *minus-one-cong-solve-weak*:

fixes n x :: *nat*
assumes $1 < n$ $x \in totatives\ n$ $y \in totatives\ n$
and $[x = n - 1]\ (mod\ n)$ $[x * y = 1]\ (mod\ n)$
shows $y = n - 1$
⟨*proof*⟩

lemma *coprime-imp-mod-not-zero*:
fixes n x :: *nat*
assumes $1 < n$ *coprime* x n
shows $0 < x\ mod\ n$
⟨*proof*⟩

lemma *minus-one-cong-solve*:
fixes n x :: *nat*
assumes $1 < n$
and *eq*: $[x = n - 1]\ (mod\ n)$ $[x * y = 1]\ (mod\ n)$
and *coprime*: *coprime* x n *coprime* y n
shows $[y = n - 1](mod\ n)$
⟨*proof*⟩

corollary *square-minus-one-cong-one′*:
fixes n x :: *nat*
assumes $1 < n$
shows $[(n - 1) * (n - 1) = 1](mod\ n)$
⟨*proof*⟩

end

5 Additional Material on Quadratic Residues

theory *QuadRes*
imports
Jacobi-Symbol
Algebraic-Auxiliaries
begin

Proofs are inspired by [**?**].

lemma *inj-on-QuadRes*:
fixes p :: *int*
assumes *prime* p
shows *inj-on* $(\lambda x.\ x\hat{}2\ mod\ p)$ $\{0..(p-1)\ div\ 2\}$
⟨*proof*⟩

lemma *QuadRes-set-prime*:
assumes *prime* p **and** *odd* p
shows $\{x\ .\ QuadRes\ p\ x \wedge x \in \{0..<p\}\} = \{x\hat{}2\ mod\ p \mid x\ .\ x \in \{0..(p-1)\ div\ 2\}\}$
⟨*proof*⟩

corollary *QuadRes-iff*:
assumes *prime p* **and** *odd p*
shows (*QuadRes p x* ∧ *x* ∈ {*0..<p*}) ⟷ (∃ *a* ∈ {*0..*(*p*−*1*) *div 2*}. *a*^*2 mod *p* = *x*)
⟨*proof*⟩

corollary *card-QuadRes-set-prime*:
fixes *p* :: *int*
assumes *prime p* **and** *odd p*
shows *card* {*x*. *QuadRes p x* ∧ *x* ∈ {*0..<p*}} = *nat* (*p*+*1*) *div 2*
⟨*proof*⟩

corollary *card-not-QuadRes-set-prime*:
fixes *p* :: *int*
assumes *prime p* **and** *odd p*
shows *card* {*x*. ¬*QuadRes p x* ∧ *x* ∈ {*0..<p*}} = *nat* (*p*−*1*) *div 2*
⟨*proof*⟩

lemma *not-QuadRes-ex-if-prime*:
assumes *prime p* **and** *odd p*
shows ∃ *x*. ¬*QuadRes p x*
⟨*proof*⟩

lemma *not-QuadRes-ex*:
1 < *p* ⟹ *odd p* ⟹ ∃*x*. ¬*QuadRes p x*
⟨*proof*⟩

end

6 Euler Witnesses

theory *Euler-Witness*
imports
Jacobi-Symbol
Residues-Nat
QuadRes
begin

Proofs are inspired by [**?**, **?**, **?**, **?**].

definition *euler-witness a p* ⟷ [*Jacobi a p* ≠ *a* ^ ((*p* − *1*) *div 2*)] (*mod p*)
abbreviation *euler-liar a p* ≡ ¬ *euler-witness a p*

lemma *euler-witness-mod*[*simp*]: *euler-witness* (*a mod p*) *p* = *euler-witness a p*
⟨*proof*⟩

lemma *euler-liar-mod*: *euler-liar* (*a mod p*) *p* = *euler-liar a p* ⟨*proof*⟩

lemma *euler-liar-cong*:
assumes [*a* = *b*] (*mod p*)

shows *euler-liar a p = euler-liar b p*
⟨*proof*⟩

lemma *euler-witnessI*:
[*x* ^ ((*n* − *1*) *div 2*) = *a*] (*mod int n*) ⟹ [*Jacobi x* (*int n*) ≠ *a*] (*mod int n*)
⟹ *euler-witness x n*
⟨*proof*⟩

lemma *euler-witnessI2*:
fixes *a b* :: *int* **and** *k* :: *nat*
assumes [*a* ≠ *b*] (*mod k*)
and *k dvd n*
and *main*: *euler-liar x n* ⟹ [*Jacobi x n* = *a*] (*mod k*)
euler-liar x n ⟹ [*x* ^ ((*n* − *1*) *div 2*) = *b*] (*mod k*)
shows *euler-witness x n*
⟨*proof*⟩

lemma *euler-witness-exists-weak*:
assumes *odd n* ¬*prime n 2* < *n*
shows ∃ *a*. *euler-witness a n* ∧ *coprime a n*
⟨*proof*⟩

lemma *euler-witness-exists*:
assumes *odd n* ¬*prime n 2* < *n*
shows ∃ *a*. *euler-witness a n* ∧ *coprime a n* ∧ *0* < *a* ∧ *a* < *n*
⟨*proof*⟩

lemma *euler-witness-exists-nat*:
assumes *odd n* ¬*prime n 2* < *n*
shows ∃ *a*. *euler-witness* (*int a*) *n* ∧ *coprime a n* ∧ *0* < *a* ∧ *a* < *n*
⟨*proof*⟩

lemma *euler-liar-1-p*[*intro*, *simp*]: *p* ≠ *0* ⟹ *euler-liar 1 p*
⟨*proof*⟩

lemma *euler-liar-mult*:
shows *euler-liar y n* ⟹ *euler-liar x n* ⟹ *euler-liar* (*x*∗*y*) *n*
⟨*proof*⟩

lemma *euler-liar-mult′*:
assumes *1* < *n coprime y n*
shows *euler-liar y n* ⟹ *euler-witness x n* ⟹ *euler-witness* (*x*∗*y*) *n*
⟨*proof*⟩

lemma *prime-imp-euler-liar*:
assumes *prime p 2* < *p 0* < *x x* < *p*
shows *euler-liar x p*
⟨*proof*⟩

locale *euler-witness-context* =
  fixes p :: *nat*
  assumes *p-gt-1*: $p > 1$ **and** *odd-p*: *odd p*
begin

definition G **where** $G = \mathit{Residues\text{-}Mult}\ p$

sublocale *residues-mult-nat p G*
  ⟨*proof*⟩

definition $H = \{x.\ \mathit{coprime}\ x\ p \wedge \mathit{euler\text{-}liar}\ (\mathit{int}\ x)\ p \wedge x \in \{1..<p\}\}$

sublocale *H*: *subgroup H G*
⟨*proof*⟩

lemma *H-finite*: *finite H*
  ⟨*proof*⟩

lemma *euler-witness-coset*:
  assumes *euler-witness x p*
  shows $y \in H\ \#>_G\ x \Longrightarrow \mathit{euler\text{-}witness}\ y\ p$
  ⟨*proof*⟩

lemma *euler-liar-coset*:
  assumes *euler-liar x p x* ∈ *carrier G*
  shows $y \in H\ \#>_G\ x \Longrightarrow \mathit{euler\text{-}liar}\ y\ p$
  ⟨*proof*⟩

lemma *in-cosets-aux*:
  assumes *euler-witness x p x* ∈ *carrier G*
  shows $H\ \#>_G\ x \in \mathit{rcosets}_G\ H$
  ⟨*proof*⟩

lemma *H-cosets-aux*:
  assumes *euler-witness x p*
  shows $(H\ \#>_G\ x) \cap H = \{\}$
  ⟨*proof*⟩

lemma *H-rcosets-aux*:
  assumes *euler-witness x p x* ∈ *carrier G*
  shows $\{H,\ H\ \#>_G\ x\} \subseteq \mathit{rcosets}_G\ H$
  ⟨*proof*⟩

lemma *H-not-eq-coset*:
  assumes *euler-witness x p*
  shows $H \neq H\ \#>_G\ x$
  ⟨*proof*⟩

lemma *finite-cosets-H*: *finite* $(\mathit{rcosets}_G\ H)$

⟨*proof*⟩

lemma *card-cosets-limit*:
assumes *euler-witness x p x* ∈ *carrier G*
shows *2* ≤ *card* (*rcosets*$_G$ *H*)
⟨*proof*⟩

lemma *card-euler-liars-cosets-limit*:
assumes ¬*prime p 2* < *p*
shows *2* ≤ *card* (*rcosets*$_G$ *H*) *card H* < (*p* − *1*) *div 2*
⟨*proof*⟩

lemma *prime-imp-G-is-H*:
assumes *prime p 2* < *p*
shows *carrier G* = *H*
⟨*proof*⟩

end

end

7 Carmichael Numbers

theory *Carmichael-Numbers*
imports
Residues-Nat
begin

A Carmichael number is a composite number n that Fermat's test incorrectly labels as primes no matter which witness a is chosen (except in the case that a shares a factor with n). [**?**, **?**]

definition *Carmichael-number* :: *nat* ⇒ *bool* **where**
Carmichael-number n ⟷ *n* > *1* ∧ ¬*prime n* ∧ (∀ *a*. *coprime a n* ⟶ [*a* ^ (*n* − *1*) = *1*] (*mod n*))

lemma *Carmichael-number-0*[*simp*, *intro*]: ¬*Carmichael-number 0*
⟨*proof*⟩

lemma *Carmichael-number-1*[*simp*, *intro*]: ¬*Carmichael-number 1*
⟨*proof*⟩

lemma *Carmichael-number-Suc-0*[*simp*, *intro*]: ¬*Carmichael-number* (*Suc 0*)
⟨*proof*⟩

lemma *Carmichael-number-not-prime*: *Carmichael-number n* ⟹ ¬*prime n*
⟨*proof*⟩

lemma *Carmichael-number-gt-3*: *Carmichael-number n* ⟹ *n* > *3*

⟨*proof*⟩

The proofs are inspired by [**?**, **?**].

lemma *Carmichael-number-imp-squarefree-aux*:
assumes *Carmichael-number n*
assumes *n*: $n = p \hat{} r * l$ **and** *prime p* ¬*p dvd l*
assumes $r > 1$
shows *False*
⟨*proof*⟩

theorem *Carmichael-number-imp-squarefree*:
assumes *Carmichael-number n*
shows *squarefree n*
⟨*proof*⟩

corollary *Carmichael-not-primepow*:
assumes *Carmichael-number n*
shows ¬*primepow n*
⟨*proof*⟩

lemma *Carmichael-number-imp-squarefree-alt-weak*:
assumes *Carmichael-number n*
shows $\exists p\ l.\ (n = p * l) \wedge$ *prime p* $\wedge$ ¬*p dvd l*
⟨*proof*⟩

theorem *Carmichael-number-odd*:
assumes *Carmichael-number n*
shows *odd n*
⟨*proof*⟩

lemma *Carmichael-number-imp-squarefree-alt*:
assumes *Carmichael-number n*
shows $\exists p\ l.\ (n = p * l) \wedge$ *prime p* $\wedge$ ¬*p dvd l* $\wedge$ $2 < l$
⟨*proof*⟩

lemma *Carmichael-number-imp-dvd*:
fixes *n* :: *nat*
assumes *Carmichael-number*: *Carmichael-number n* **and** *prime p p dvd n*
shows $p - 1$ *dvd* $n - 1$
⟨*proof*⟩

The following lemma is also called Korselt's criterion.

lemma *Carmichael-numberI*:
fixes *n* :: *nat*
assumes ¬ *prime n squarefree n* $1 < n$ **and**
DIV: $\bigwedge p.\ p \in$ *prime-factors n* $\Longrightarrow p - 1$ *dvd* $n - 1$
shows *Carmichael-number n*
⟨*proof*⟩

theorem *Carmichael-number-iff*:
Carmichael-number n $\longleftrightarrow$
n $\neq$ *1* $\wedge$ $\neg$*prime n* $\wedge$ *squarefree n* $\wedge$ ($\forall$ *p*$\in$*prime-factors n. p* $-$ *1 dvd n* $-$ *1*)
⟨*proof*⟩

Every Carmichael number has at least three distinct prime factors.

theorem *Carmichael-number-card-prime-factors*:
assumes *Carmichael-number n*
shows *card* (*prime-factors n*) $\geq$ *3*
⟨*proof*⟩

lemma *Carmichael-number-iff'*:
fixes *n* :: *nat*
defines *P* $\equiv$ *prime-factorization n*
shows *Carmichael-number n* $\longleftrightarrow$
n $>$ *1* $\wedge$ *size P* $\neq$ *1* $\wedge$ ($\forall$ *p*$\in$#*P. count P p* $=$ *1* $\wedge$ *p* $-$ *1 dvd n* $-$ *1*)
⟨*proof*⟩

The smallest Carmichael number is 561, and it was found and proven so by Carmichael in 1910 [**?**].

lemma *Carmichael-number-561*: *Carmichael-number 561* (**is** *Carmichael-number ?n*)
⟨*proof*⟩

end

8 Fermat Witnesses

theory *Fermat-Witness*
imports *Euler-Witness Carmichael-Numbers*
begin

definition *divide-out* :: *'a* :: *factorial-semiring* $\Rightarrow$ *'a* $\Rightarrow$ *'a* $\times$ *nat* **where**
divide-out p x = (*x div p* $\hat{}$ *multiplicity p x*, *multiplicity p x*)

lemma *fst-divide-out* [*simp*]: *fst* (*divide-out p x*) = *x div p* $\hat{}$ *multiplicity p x*
and *snd-divide-out* [*simp*]: *snd* (*divide-out p x*) = *multiplicity p x*
⟨*proof*⟩

function *divide-out-aux* :: *'a* :: *factorial-semiring* $\Rightarrow$ *'a* $\times$ *nat* $\Rightarrow$ *'a* $\times$ *nat* **where**
divide-out-aux p (*x*, *acc*) =
(*if x* = *0* $\vee$ *is-unit p* $\vee$ $\neg$*p dvd x then* (*x*, *acc*) *else divide-out-aux p* (*x div p*, *acc* + *1*))
⟨*proof*⟩
termination ⟨*proof*⟩

lemmas [*simp del*] = *divide-out-aux.simps*

lemma *divide-out-aux-correct*:
divide-out-aux p z = (*fst z div p* ^ *multiplicity p* (*fst z*), *snd z* + *multiplicity p* (*fst z*))
⟨*proof*⟩

lemma *divide-out-code* [*code*]: *divide-out p x* = *divide-out-aux p* (*x*, *0*)
⟨*proof*⟩

lemma *multiplicity-code* [*code*]: *multiplicity p x* = *snd* (*divide-out-aux p* (*x*, *0*))
⟨*proof*⟩

lemma *multiplicity-times-same-power*:
assumes *x* ≠ *0* ¬*is-unit p p* ≠ *0*
shows *multiplicity p* (*p* ^ *k* ∗ *x*) = *multiplicity p x* + *k*
⟨*proof*⟩

lemma *divide-out-unique-nat*:
fixes *n* :: *nat*
assumes ¬*is-unit p p* ≠ *0* ¬*p dvd m* **and** *n* = *p* ^ *k* ∗ *m*
shows *m* = *n div p* ^ *multiplicity p n* **and** *k* = *multiplicity p n*
⟨*proof*⟩

context
fixes *a n* :: *nat*
begin

definition *fermat-liar* ⟷ [*a*^(*n*−*1*) = *1*] (*mod n*)
definition *fermat-witness* ⟷ [*a*^(*n*−*1*) ≠ *1*] (*mod n*)

definition *strong-fermat-liar* ⟷
(∀ *k m*. *odd m* ⟶ *n* − *1* = *2*^*k* ∗ *m* ⟶
[*a*^*m* = *1*](*mod n*) ∨ (∃ *i* ∈ {*0*..< *k*}. [*a*^(*2* ^ *i* ∗ *m*) = *n*−*1*] (*mod n*)))

definition *strong-fermat-witness* ⟷ ¬ *strong-fermat-liar*

lemma *fermat-liar-witness-of-composition*[*iff*]:
¬*fermat-liar* = *fermat-witness*
¬*fermat-witness* = *fermat-liar*
⟨*proof*⟩

lemma *strong-fermat-liar-code* [*code*]:
strong-fermat-liar ⟷
(*let* (*m*, *k*) = *divide-out 2* (*n* − *1*)
in [*a*^*m* = *1*](*mod n*) ∨ (∃ *i* ∈ {*0*..< *k*}. [*a*^(*2* ^ *i* ∗ *m*) = *n*−*1*] (*mod n*)))
(**is** *?lhs* = *?rhs*)

⟨*proof*⟩

context
  assumes ∗ : $a \in \{1 ..< n\}$
begin

lemma *strong-fermat-witness-iff*:
  strong-fermat-witness =
    $(\exists k\ m.\ odd\ m \wedge n - 1 = 2 \hat{} k * m \wedge [a \hat{} m \neq 1]\ (mod\ n) \wedge$
      $(\forall i{\in}\{0..{<}k\}.\ [a \hat{} (2 \hat{} i * m) \neq n{-}1]\ (mod\ n)))$
  ⟨*proof*⟩

lemma *not-coprime-imp-witness*: ¬*coprime a n* ⟹ *fermat-witness*
  ⟨*proof*⟩

corollary *liar-imp-coprime*: *fermat-liar* ⟹ *coprime a n*
  ⟨*proof*⟩

lemma *fermat-witness-imp-strong-fermat-witness*:
  assumes *1 < n fermat-witness*
  shows *strong-fermat-witness*
⟨*proof*⟩

corollary *strong-fermat-liar-imp-fermat-liar*:
  assumes *1 < n strong-fermat-liar*
  shows *fermat-liar*
    ⟨*proof*⟩

lemma *euler-liar-is-fermat-liar*:
  assumes *1 < n euler-liar a n coprime a n odd n*
  shows *fermat-liar*
⟨*proof*⟩

corollary *fermat-witness-is-euler-witness*:
  assumes *1 < n fermat-witness coprime a n odd n*
  shows *euler-witness a n*
  ⟨*proof*⟩

end
end

lemma *one-is-fermat-liar*[*simp*]: *1 < n* ⟹ *fermat-liar 1 n*
  ⟨*proof*⟩

lemma *one-is-strong-fermat-liar*[*simp*]: *1 < n* ⟹ *strong-fermat-liar 1 n*
  ⟨*proof*⟩

lemma *prime-imp-fermat-liar*:

prime p $\Longrightarrow$ *a* $\in$ {*1 ..< p*} $\Longrightarrow$ *fermat-liar a p*
⟨*proof*⟩

lemma *not-Carmichael-numberD*:
assumes ¬*Carmichael-number n* ¬*prime n* **and** *1 < n*
shows ∃ *a* ∈ {*2 ..< n*} . *fermat-witness a n* ∧ *coprime a n*
⟨*proof*⟩

proposition *prime-imp-strong-fermat-witness*:
fixes *p* :: *nat*
assumes *prime p 2 < p a* ∈ {*1 ..< p*}
shows *strong-fermat-liar a p*
⟨*proof*⟩

lemma *ignore-one*:
fixes *P* :: - ⇒ *nat* ⇒ *bool*
assumes *P 1 n 1 < n*
shows *card* {*a* ∈ {*1..<n*}. *P a n*} = *1* + *card* {*a. 2* ≤ *a* ∧ *a < n* ∧ *P a n*}
⟨*proof*⟩

Proofs in the following section are inspired by [**?**, **?**, **?**].

proposition *not-Carmichael-number-imp-card-fermat-witness-bound*:
fixes *n* :: *nat*
assumes ¬*prime n* ¬*Carmichael-number n odd n 1 < n*
shows (*n−1*) *div 2 < card* {*a* ∈ {*1 ..< n*}. *fermat-witness a n*}
and (*card* {*a. 2* ≤ *a* ∧ *a < n* ∧ *strong-fermat-liar a n*}) < *real* (*n − 2*) / *2*
and (*card* {*a. 2* ≤ *a* ∧ *a < n* ∧ *fermat-liar a n*}) < *real* (*n − 2*) / *2*
⟨*proof*⟩

proposition *Carmichael-number-imp-lower-bound-on-strong-fermat-witness*:
fixes *n* :: *nat*
assumes *Carmichael-number*: *Carmichael-number n*
shows (*n − 1*) *div 2 < card* {*a* ∈ {*1..<n*}. *strong-fermat-witness a n*}
and *real* (*card* {*a . 2* ≤ *a* ∧ *a < n* ∧ *strong-fermat-liar a n*}) < *real* (*n − 2*) / *2*
⟨*proof*⟩

corollary *strong-fermat-witness-lower-bound*:
assumes *odd n n > 2* ¬*prime n*
shows *card* {*a. 2* ≤ *a* ∧ *a < n* ∧ *strong-fermat-liar a n*} < *real* (*n − 2*) / *2*
⟨*proof*⟩

end

9 A Generic View on Probabilistic Prime Tests

theory *Generalized-Primality-Test*
imports

HOL−Probability.Probability
Algebraic-Auxiliaries
begin

definition *primality-test* :: (*nat* ⇒ *nat* ⇒ *bool*) ⇒ *nat* ⇒ *bool pmf* **where**
primality-test P n =
(*if n* < *3* ∨ *even n then return-pmf* (*n* = *2*) *else*
do {
a ← *pmf-of-set* {*2*..< *n*};
return-pmf (*P n a*)
})

lemma *expectation-of-bool-is-pmf*: *measure-pmf.expectation M of-bool* = *pmf M True*
⟨*proof*⟩

lemma *eq-bernoulli-pmfI*:
assumes *pmf p True* = *x*
shows *p* = *bernoulli-pmf x*
⟨*proof*⟩

We require a probabilistic primality test to never classify a prime as composite. It may, however, mistakenly classify composites as primes.

locale *prob-primality-test* =
fixes *P* :: *nat* ⇒ *nat* ⇒ *bool* **and** *n* :: *nat*
assumes *P-works*: *odd n* ⟹ *2* ≤ *a* ⟹ *a* < *n* ⟹ *prime n* ⟹ *P n a*
begin

lemma *FalseD*:
assumes *false*: *False* ∈ *set-pmf* (*primality-test P n*)
shows ¬ *prime n*
⟨*proof*⟩

theorem *prime*:
assumes *odd-prime*: *prime n*
shows *primality-test P n* = *return-pmf True*
⟨*proof*⟩

end

We call a primality test q-good for a fixed positive real number q if the probability that it mistakenly classifies a composite as a prime is less than q.

locale *good-prob-primality-test* = *prob-primality-test* +
fixes *q* :: *real*
assumes *q-pos*: *q* > *0*
assumes *composite-witness-bound*:
¬*prime n* ⟹ *2* < *n* ⟹ *odd n* ⟹

$$real\ (card\ \{a\ .\ 2 \leq a \wedge a < n \wedge P\ n\ a\}) < q * real\ (n - 2)$$

begin

lemma *composite-aux*:
assumes $\neg prime\ n$
shows *measure-pmf.expectation* (*primality-test P n*) *of-bool* $< q$
⟨*proof*⟩

theorem *composite*:
assumes $\neg prime\ n$
shows *pmf* (*primality-test P n*) *True* $< q$
⟨*proof*⟩

end

end

10 Fermat's Test

theory *Fermat-Test*
imports
Fermat-Witness
Generalized-Primality-Test
begin

definition *fermat-test* = *primality-test* ($\lambda n\ a.$ *fermat-liar a n*)

The Fermat test is a good probabilistic primality test on non-Carmichael numbers.

locale *fermat-test-not-Carmichael-number* =
fixes *n* :: *nat*
assumes *not-Carmichael-number*: $\neg$*Carmichael-number n* $\vee$ $n < 3$
begin

sublocale *fermat-test*: *good-prob-primality-test* $\lambda a\ n.$ *fermat-liar n a n 1 / 2*
rewrites *primality-test* ($\lambda\ a\ n.$ *fermat-liar n a*) = *fermat-test*
⟨*proof*⟩

end

lemma *not-coprime-imp-fermat-witness*:
fixes *n* :: *nat*
assumes $n > 1$ $\neg$*coprime a n*
shows *fermat-witness a n*
⟨*proof*⟩

theorem *fermat-test-prime*:
assumes *prime n*
shows *fermat-test n* = *return-pmf True*

⟨*proof*⟩

theorem *fermat-test-composite*:
  assumes ¬*prime n* ¬*Carmichael-number n* ∨ *n* < *3*
  shows *pmf* (*fermat-test n*) *True* < *1* / *2*
⟨*proof*⟩

For a Carmichael number n, Fermat's test as defined above mistakenly returns 'True' with probability $(\varphi(n) - 1)/(n - 2)$. This probability is close to 1 if n has few and big prime factors; it is not quite as bad if it has many and/or small factors, but in that case, simple trial division can also detect compositeness.

Moreover, Fermat's test only succeeds for a Carmichael number if it happens to guess a number that is not coprime to n. In that case, the fact that we have found a number between 2 and n that is not coprime to n alone is proof that n is composite, and indeed we can even find a non-trivial factor by computing the GCD. This means that for Carmichael numbers, Fermat's test is essentially no better than the very crude method of attempting to guess numbers coprime to n.

This means that, in general, Fermat's test is not very helpful for Carmichael numbers.

theorem *fermat-test-Carmichael-number*:
  assumes *Carmichael-number n*
  shows *fermat-test n* = *bernoulli-pmf* (*real* (*totient n* − *1*) / *real* (*n* − *2*))
⟨*proof*⟩

end

11 The Miller–Rabin Test

theory *Miller-Rabin-Test*
imports
  Fermat-Witness
  Generalized-Primality-Test
begin

definition *miller-rabin* = *primality-test* (λ*n a*. *strong-fermat-liar a n*)

The test is actually $\frac{1}{4}$ good, but we only show $\frac{1}{2}$, since the former is much more involved.

interpretation *miller-rabin*: *good-prob-primality-test* λ*n a*. *strong-fermat-liar a n*
n 1 / 2
  rewrites *primality-test* (λ*n a*. *strong-fermat-liar a n*) = *miller-rabin*
⟨*proof*⟩

end

12 The Solovay–Strassen Test

theory *Solovay-Strassen-Test*
imports
Generalized-Primality-Test
Euler-Witness
begin

definition *solovay-strassen-witness* :: *nat* ⇒ *nat* ⇒ *bool* **where**
solovay-strassen-witness n a =
(*let x* = *Jacobi* (*int a*) (*int n*) *in x* ≠ *0* ∧ [*x* = *int a* ^ ((*n* − *1*) *div 2*)] (*mod n*))

definition *solovay-strassen* :: *nat* ⇒ *bool pmf* **where**
solovay-strassen = *primality-test solovay-strassen-witness*

lemma *prime-imp-solovay-strassen-witness*:
assumes *prime p odd p a* ∈ {*2*..<*p*}
shows *solovay-strassen-witness p a*
⟨*proof*⟩

lemma *card-solovay-strassen-liars-composite*:
fixes *n* :: *nat*
assumes ¬*prime n n* > *2 odd n*
shows *card* {*a* ∈ {*2*..<*n*}. *solovay-strassen-witness n a*} < (*n* − *2*) *div 2*
(**is** *card ?A* < -)
⟨*proof*⟩

interpretation *solovay-strassen*: *good-prob-primality-test solovay-strassen-witness n 1 / 2*
rewrites *primality-test solovay-strassen-witness* = *solovay-strassen*
⟨*proof*⟩

end