# The Divergence of the Prime Harmonic Series 

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September 13, 2023


#### Abstract

In this work, we prove the lower bound $\ln \left(H_{n}\right)-\ln \left(\frac{5}{3}\right)$ for the partial sum of the Prime Harmonic series and, based on this, the divergence of the Prime Harmonic Series $\sum_{p=1}^{n}[p$ prime $] \cdot \frac{1}{p}$. The proof relies on the unique squarefree decomposition of natural numbers. This proof is similar to Euler's original proof (which was highly informal and morally questionable). Its advantage over proofs by contradiction, like the famous one by Paul Erdős, is that it provides a relatively good lower bound for the partial sums.


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## 1 Auxiliary lemmas

```
theory Prime-Harmonic-Misc
imports
    Complex-Main
    HOL-Number-Theory.Number-Theory
begin
lemma sum-list-nonneg: \(\forall x \in\) set \(x s . x \geq 0 \Longrightarrow\) sum-list \(x s \geq(0::\) 'a :: or-
dered-ab-group-add)
    \(\langle p r o o f\rangle\)
lemma sum-telescope':
    assumes \(m \leq n\)
    shows \(\left(\sum \bar{k}=\right.\) Suc m..n. \(f k-f(\) Suc \(\left.k)\right)=f(\) Suc \(m)-\left(f(\right.\) Suc \(n)::{ }^{\prime} a::\)
ab-group-add)
```

$$
\langle p r o o f\rangle
$$

```
lemma dvd-prodI:
    assumes finite A x }\in
    shows fx dvd prod f A
<proof>
```

lemma dvd-prodD: finite $A \Longrightarrow \operatorname{prod} f A d v d x \Longrightarrow a \in A \Longrightarrow f a d v d x$
$\langle p r o o f\rangle$
lemma multiplicity-power-nat:
prime $p \Longrightarrow n>0 \Longrightarrow$ multiplicity $p\left(n^{\wedge} k::\right.$ nat $)=k *$ multiplicity $p n$
$\langle p r o o f\rangle$
lemma multiplicity-prod-prime-powers-nat':
finite $S \Longrightarrow \forall p \in S$. prime $p \Longrightarrow$ prime $p \Longrightarrow$
multiplicity $p\left(\prod S::\right.$ nat $)=($ if $p \in S$ then 1 else 0$)$
〈proof〉
lemma prod-prime-subset:
assumes finite $A$ finite $B$
assumes $\bigwedge x . x \in A \Longrightarrow$ prime $(x:: n a t)$
assumes $\bigwedge x . x \in B \Longrightarrow$ prime $x$
assumes $\prod A d v d \prod B$
shows $A \subseteq B$
$\langle p r o o f\rangle$
lemma prod-prime-eq:
assumes finite $A$ finite $B \bigwedge x . x \in A \Longrightarrow$ prime $(x:: n a t) ~ \bigwedge x . x \in B \Longrightarrow$ prime
$x \prod A=\prod B$
shows $A=B$
$\langle p r o o f\rangle$
lemma ln-ln-nonneg:
assumes $x: x \geq(3::$ real $)$
shows $\quad \ln (\ln x) \geq 0$
$\langle p r o o f\rangle$
end

## 2 Squarefree decomposition of natural numbers

theory Squarefree-Nat<br>imports<br>Main<br>HOL-Number-Theory.Number-Theory<br>Prime-Harmonic-Misc<br>begin

The squarefree part of a natural number is the set of all prime factors that appear with odd multiplicity．The square part，correspondingly，is what remains after dividing by the squarefree part．

```
definition squarefree-part :: nat \(\Rightarrow\) nat set where
    squarefree-part \(n=\{p \in\) prime-factors \(n\). odd (multiplicity \(p n)\}\)
definition square-part :: nat \(\Rightarrow\) nat where
    square-part \(n=\left(\right.\) if \(n=0\) then 0 else ( \(\prod p \in\) prime-factors \(n . ~ p \wedge\) (multiplicity \(p n\)
div 2)))
lemma squarefree-part-0 [simp]: squarefree-part \(0=\{ \}\)
    \(\langle p r o o f\rangle\)
lemma square-part- 0 [simp]: square-part \(0=0\)
    \(\langle p r o o f\rangle\)
lemma squarefree-decompose: \(\prod\) (squarefree-part \(\left.n\right) *\) square-part \(n\) へ \(2=n\)
\(\langle p r o o f\rangle\)
lemma squarefree-part-pos \([\) simp \(]: \prod(\) squarefree-part \(n)>0\)
    \(\langle p r o o f\rangle\)
lemma squarefree-part-ge-Suc-0 \([\) simp \(]: \prod\) (squarefree-part \(\left.n\right) \geq\) Suc 0
    \(\langle p r o o f\rangle\)
lemma squarefree-part-subset [intro]: squarefree-part \(n \subseteq\) prime-factors \(n\)
    \(\langle p r o o f\rangle\)
lemma squarefree-part-finite [simp]: finite (squarefree-part \(n\) )
    \(\langle p r o o f\rangle\)
lemma squarefree-part-dvd \([\) simp \(]: \prod\) (squarefree-part \(n\) ) dvd \(n\)
    〈proof〉
lemma squarefree-part-dvd' \([\) simp \(]: p \in\) squarefree-part \(n \Longrightarrow p\) dvd \(n\)
    〈proof〉
lemma square-part-dvd [simp]: square-part \(n\) へ 2 dvd \(n\)
    \(\langle p r o o f\rangle\)
lemma square-part-dvd \({ }^{\prime}[\) simp \(]\) : square-part \(n\) dvd \(n\)
    \(\langle p r o o f\rangle\)
lemma squarefree-part-le: \(p \in\) squarefree-part \(n \Longrightarrow p \leq n\)
    \(\langle p r o o f\rangle\)
lemma square-part-le: square-part \(n \leq n\)
    \(\langle\) proof〉
```

```
lemma square-part-le-sqrt: square-part \(n \leq\) nat \(\lfloor\) sqrt (real \(n\) ) \(\rfloor\)
\(\langle p r o o f\rangle\)
lemma square-part-pos \([\) simp \(]: n>0 \Longrightarrow\) square-part \(n>0\)
    〈proof〉
lemma square-part-ge-Suc-0 [simp]: \(n>0 \Longrightarrow\) square-part \(n \geq\) Suc 0
    〈proof〉
lemma zero-not-in-squarefree-part [simp]: \(0 \notin\) squarefree-part \(n\)
    \(\langle p r o o f\rangle\)
lemma multiplicity-squarefree-part:
    prime \(p \Longrightarrow\) multiplicity \(p(\Pi\) (squarefree-part \(n))=(\) if \(p \in\) squarefree-part \(n\) then
1 else 0)
    〈proof〉
```

The squarefree part really is square，its only square divisor is 1 ．
lemma square－dvd－squarefree－part－iff：
$x^{\wedge}$ 2 dvd $\prod$ (squarefree-part $\left.n\right) \longleftrightarrow x=1$
$\langle p r o o f\rangle$
lemma squarefree-decomposition-unique1:
assumes squarefree-part $m=$ squarefree-part $n$
assumes square-part $m=$ square-part $n$
shows $m=n$
$\langle p r o o f\rangle$
lemma squarefree-decomposition-unique2:
assumes $n$ : $n>0$
assumes decomp: $n=\left(\prod A 2 * s 2\right.$ 2 2$)$
assumes prime: $\backslash x . x \in A \mathcal{Z} \Longrightarrow$ prime $x$
assumes fin: finite A2
assumes s2-nonneg: s2 $\geq 0$
shows $A 2=$ squarefree-part $n$ and $s 2=$ square-part $n$
$\langle p r o o f\rangle$
lemma squarefree-decomposition-unique2':
assumes decomp: $\left(\prod A 1 * s 1^{\wedge} 2\right)=\left(\prod A 2 * s 2\right.$ ^2 $::$ nat $)$
assumes fin: finite A1 finite A2
assumes subset: $\bigwedge x . x \in A 1 \Longrightarrow$ prime $x \bigwedge x . x \in A 2 \Longrightarrow$ prime $x$
assumes pos: s1 > 0 s2 $>0$
defines $n \equiv \prod A 1 * s 1^{\wedge} 2$
shows $A 1=A 2$ s1 $=s 2$
$\langle p r o o f\rangle$

The following is a nice and simple lower bound on the number of prime numbers less than a given number due to Erdős．In particular，it implies
that there are infinitely many primes.

```
lemma primes-lower-bound:
    fixes n :: nat
    assumes n>0
    defines }\pi\equiv\lambdan.card {p.prime p\wedgep\leqn
    shows real (\pi n)\geqln(real n)/ln}
<proof\rangle
end
```


## 3 The Prime Harmonic Series

```
theory Prime-Harmonic
imports
    HOL-Analysis.Analysis
    HOL-Number-Theory.Number-Theory
    Prime-Harmonic-Misc
    Squarefree-Nat
begin
```


### 3.1 Auxiliary equalities and inequalities

First of all, we prove the following result about rearranging a product over a set into a sum over all subsets of that set.

```
lemma prime-harmonic-aux1:
    fixes }A\mathrm{ :: ' }a\mathrm{ :: field set
    shows finite A\Longrightarrow(\prodx\inA.1 + 1/x)=(\sumx\inPow A. 1 / \x)
<proof>
```

Next, we prove a simple and reasonably accurate upper bound for the sum of the squares of any subset of the natural numbers, derived by simple telescoping. Our upper bound is approximately 1.67 ; the exact value is $\frac{\pi^{2}}{6} \approx 1.64$. (cf. Basel problem)
lemma prime-harmonic-aux2:
assumes finite ( $A$ :: nat set)
shows $\left(\sum k \in A .1 /(\right.$ real $\left.k \wedge 2)\right) \leq 5 / 3$
$\langle p r o o f\rangle$

### 3.2 Estimating the partial sums of the Prime Harmonic Series

We are now ready to show our main result: the value of the partial prime harmonic sum over all primes no greater than $n$ is bounded from below by the $n$-th harmonic number $H_{n}$ minus some constant.
In our case, this constant will be $\frac{5}{3}$. As mentioned before, using a proof of the Basel problem can improve this to $\frac{\pi^{2}}{6}$, but the improvement is very
small and the proof of the Basel problem is a very complex one．
The exact asymptotic behaviour of the partial sums is actually $\ln (\ln n)+M$ ， where $M$ is the Meissel－Mertens constant（approximately 0．261）．
theorem prime－harmonic－lower：
assumes $n: n \geq 2$
shows $\left(\sum p \leftarrow\right.$ primes－upto $n .1 /$ real $\left.p\right) \geq \ln ($ harm $n)-\ln (5 / 3)$
〈proof〉
We can use the inequality $\ln (n+1) \leq H_{n}$ to estimate the asymptotic growth of the partial prime harmonic series．Note that $H_{n} \sim \ln n+\gamma$ where $\gamma$ is the Euler－Mascheroni constant（approximately 0.577 ），so we lose some accuracy here．

```
corollary prime-harmonic-lower':
    assumes n: n\geq2
    shows (\sump\leftarrow\mathrm{ primes-upto n. 1 | real p) }\geq\operatorname{ln}(\operatorname{ln}(n+1))-\operatorname{ln}(5/3)
<proof\rangle
```

```
lemma Bseq-eventually-mono:
    assumes eventually ( \(\lambda n\). norm \((f n) \leq \operatorname{norm}(g n)\) ) sequentially Bseq \(g\)
    shows Bseq \(f\)
〈proof〉
lemma Bseq-add:
    assumes Bseq ( \(f::\) nat \(\Rightarrow{ }^{\prime}\) ' \(a::\) real-normed-vector \()\)
    shows Bseq ( \(\lambda x . f x+c\) )
〈proof〉
lemma convergent-imp-Bseq: convergent \(f \Longrightarrow\) Bseq \(f\)
    〈proof〉
```

We now use our last estimate to show that the prime harmonic series di－ verges．This is obvious，since it is bounded from below by $\ln (\ln (n+1))$ minus some constant，which obviously tends to infinite．
Directly using the divergence of the harmonic series would also be possible and shorten this proof a bit．．

```
corollary prime-harmonic-series-unbounded:
    \neg\mathrm{ Beq ( }\lambdan.\sump\leftarrow\mathrm{ primes-upto n. 1/p)(is }\neg\mathrm{ Bseq ?f)})
<proof\rangle
corollary prime-harmonic-series-diverges:
    \negonvergent ( }\lambdan.\sump\leftarrow\mathrm{ primes-upto n. 1/p)
    <proof\rangle
end
```

