The Divergence of the Prime Harmonic Series

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Abstract

In this work, we prove the lower bound $\ln(H_n) - \ln(\frac{5}{3})$ for the partial sum of the Prime Harmonic series and, based on this, the divergence of the Prime Harmonic Series $\sum_{p=1}^{n} [p \text{ prime}] \cdot \frac{1}{p}$. The proof relies on the unique squarefree decomposition of natural numbers. This proof is similar to Euler's original proof (which was highly informal and morally questionable). Its advantage over proofs by contradiction, like the famous one by Paul Erdős, is that it provides a relatively good lower bound for the partial sums.

Contents

1 Auxiliary lemmas

theory *Prime-Harmonic-Misc*
imports
Complex-Main
HOL−Number-Theory.Number-Theory
begin

lemma *sum-list-nonneg*: $\forall x{\in}set\ xs.\ x \geq 0 \Longrightarrow sum\text{-}list\ xs \geq (0 :: {'a} :: ordered\text{-}ab\text{-}group\text{-}add)$
$\langle proof \rangle$

lemma *sum-telescope'*:
assumes $m \leq n$
shows $(\sum k = Suc\ m..n.\ f\ k - f\ (Suc\ k)) = f\ (Suc\ m) - (f\ (Suc\ n) :: {'a} :: ab\text{-}group\text{-}add)$

⟨*proof*⟩

lemma *dvd-prodI*:
assumes *finite* A $x \in A$
shows $f\ x$ *dvd prod* $f\ A$
⟨*proof*⟩

lemma *dvd-prodD*: *finite* $A \Longrightarrow$ *prod* $f\ A$ *dvd* $x \Longrightarrow a \in A \Longrightarrow f\ a$ *dvd* x
⟨*proof*⟩

lemma *multiplicity-power-nat*:
prime $p \Longrightarrow n > 0 \Longrightarrow$ *multiplicity* $p\ (n \mathbin{\hat{}} k :: nat) = k * $ *multiplicity* $p\ n$
⟨*proof*⟩

lemma *multiplicity-prod-prime-powers-nat'*:
finite $S \Longrightarrow \forall\, p{\in}S.$ *prime* $p \Longrightarrow$ *prime* $p \Longrightarrow$
multiplicity p $(\prod S :: nat) =$ (*if* $p \in S$ *then* 1 *else* 0)
⟨*proof*⟩

lemma *prod-prime-subset*:
assumes *finite* A *finite* B
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *prime* $(x{::}nat)$
assumes $\bigwedge x.\ x \in B \Longrightarrow$ *prime* x
assumes $\prod A$ *dvd* $\prod B$
shows $A \subseteq B$
⟨*proof*⟩

lemma *prod-prime-eq*:
assumes *finite* A *finite* B $\bigwedge x.\ x \in A \Longrightarrow$ *prime* $(x{::}nat)$ $\bigwedge x.\ x \in B \Longrightarrow$ *prime* x $\prod A = \prod B$
shows $A = B$
⟨*proof*⟩

lemma *ln-ln-nonneg*:
assumes x: $x \geq (3 :: real)$
shows $ln\ (ln\ x) \geq 0$
⟨*proof*⟩

end

2 Squarefree decomposition of natural numbers

theory *Squarefree-Nat*
imports
Main
HOL−Number-Theory.Number-Theory
Prime-Harmonic-Misc
begin

The squarefree part of a natural number is the set of all prime factors that appear with odd multiplicity. The square part, correspondingly, is what remains after dividing by the squarefree part.

definition *squarefree-part* :: *nat* ⇒ *nat set* **where**
  squarefree-part n = {*p*∈*prime-factors n*. *odd* (*multiplicity p n*)}

definition *square-part* :: *nat* ⇒ *nat* **where**
  square-part n = (*if n* = *0 then 0 else* (∏ *p*∈*prime-factors n*. *p* ^ (*multiplicity p n div 2*)))

lemma *squarefree-part-0* [*simp*]: *squarefree-part 0* = {}
  ⟨*proof*⟩

lemma *square-part-0* [*simp*]: *square-part 0* = *0*
  ⟨*proof*⟩

lemma *squarefree-decompose*: ∏ (*squarefree-part n*) ∗ *square-part n* ^ *2* = *n*
⟨*proof*⟩

lemma *squarefree-part-pos* [*simp*]: ∏ (*squarefree-part n*) > *0*
  ⟨*proof*⟩

lemma *squarefree-part-ge-Suc-0* [*simp*]: ∏ (*squarefree-part n*) ≥ *Suc 0*
  ⟨*proof*⟩

lemma *squarefree-part-subset* [*intro*]: *squarefree-part n* ⊆ *prime-factors n*
  ⟨*proof*⟩

lemma *squarefree-part-finite* [*simp*]: *finite* (*squarefree-part n*)
  ⟨*proof*⟩

lemma *squarefree-part-dvd* [*simp*]: ∏ (*squarefree-part n*) *dvd n*
  ⟨*proof*⟩

lemma *squarefree-part-dvd′* [*simp*]: *p* ∈ *squarefree-part n* ⟹ *p dvd n*
  ⟨*proof*⟩

lemma *square-part-dvd* [*simp*]: *square-part n* ^ *2 dvd n*
  ⟨*proof*⟩

lemma *square-part-dvd′* [*simp*]: *square-part n dvd n*
  ⟨*proof*⟩

lemma *squarefree-part-le*: *p* ∈ *squarefree-part n* ⟹ *p* ≤ *n*
  ⟨*proof*⟩

lemma *square-part-le*: *square-part n* ≤ *n*
  ⟨*proof*⟩

lemma *square-part-le-sqrt*: *square-part* $n \leq$ *nat* $\lfloor$*sqrt* (*real* n)$\rfloor$
⟨*proof*⟩

lemma *square-part-pos* [*simp*]: $n > 0 \Longrightarrow$ *square-part* $n > 0$
⟨*proof*⟩

lemma *square-part-ge-Suc-0* [*simp*]: $n > 0 \Longrightarrow$ *square-part* $n \geq$ *Suc 0*
⟨*proof*⟩

lemma *zero-not-in-squarefree-part* [*simp*]: $0 \notin$ *squarefree-part* n
⟨*proof*⟩

lemma *multiplicity-squarefree-part*:
prime $p \Longrightarrow$ *multiplicity* p ($\prod$(*squarefree-part* n)) = (*if* $p \in$ *squarefree-part* n *then* *1 else 0*)
⟨*proof*⟩

The squarefree part really is square, its only square divisor is 1.

lemma *square-dvd-squarefree-part-iff*:
x^2 *dvd* $\prod$(*squarefree-part* n) $\longleftrightarrow x = 1$
⟨*proof*⟩

lemma *squarefree-decomposition-unique1*:
assumes *squarefree-part* m = *squarefree-part* n
assumes *square-part* m = *square-part* n
shows $m = n$
⟨*proof*⟩

lemma *squarefree-decomposition-unique2*:
assumes *n*: $n > 0$
assumes *decomp*: n = ($\prod$ *A2* $*$ *s2*^*2*)
assumes *prime*: $\bigwedge x.\ x \in$ *A2* $\Longrightarrow$ *prime* x
assumes *fin*: *finite A2*
assumes *s2-nonneg*: *s2* ≥ 0
shows *A2* = *squarefree-part* n **and** *s2* = *square-part* n
⟨*proof*⟩

lemma *squarefree-decomposition-unique2'*:
assumes *decomp*: ($\prod$ *A1* $*$ *s1*^*2*) = ($\prod$ *A2* $*$ *s2*^*2* :: *nat*)
assumes *fin*: *finite A1 finite A2*
assumes *subset*: $\bigwedge x.\ x \in$ *A1* $\Longrightarrow$ *prime* x $\bigwedge x.\ x \in$ *A2* $\Longrightarrow$ *prime* x
assumes *pos*: *s1* > 0 *s2* > 0
defines $n \equiv \prod$ *A1* $*$ *s1*^*2*
shows *A1* = *A2* *s1* = *s2*
⟨*proof*⟩

The following is a nice and simple lower bound on the number of prime numbers less than a given number due to Erdős. In particular, it implies that there are infinitely many primes.

lemma *primes-lower-bound*:
  fixes n :: *nat*
  assumes $n > 0$
  defines $\pi \equiv \lambda n.\ card\ \{p.\ prime\ p \wedge p \leq n\}$
  shows $real\ (\pi\ n) \geq ln\ (real\ n)\ /\ ln\ 4$
⟨*proof*⟩

end

3 The Prime Harmonic Series

theory *Prime-Harmonic*
imports
  HOL−Analysis.Analysis
  HOL−Number-Theory.Number-Theory
  Prime-Harmonic-Misc
  Squarefree-Nat
begin

3.1 Auxiliary equalities and inequalities

First of all, we prove the following result about rearranging a product over a set into a sum over all subsets of that set.

lemma *prime-harmonic-aux1*:
  fixes A :: $'a$ :: *field set*
  shows $finite\ A \Longrightarrow (\prod x{\in}A.\ 1 + 1\ /\ x) = (\sum x{\in}Pow\ A.\ 1\ /\ \prod x)$
⟨*proof*⟩

Next, we prove a simple and reasonably accurate upper bound for the sum of the squares of any subset of the natural numbers, derived by simple telescoping. Our upper bound is approximately 1.67; the exact value is $\frac{\pi^2}{6} \approx 1.64$. (cf. Basel problem)

lemma *prime-harmonic-aux2*:
  assumes $finite\ (A :: nat\ set)$
  shows $(\sum k{\in}A.\ 1\ /\ (real\ k\ \hat{}\ 2)) \leq 5/3$
⟨*proof*⟩

3.2 Estimating the partial sums of the Prime Harmonic Series

We are now ready to show our main result: the value of the partial prime harmonic sum over all primes no greater than n is bounded from below by the n-th harmonic number H_n minus some constant.

In our case, this constant will be $\frac{5}{3}$. As mentioned before, using a proof of the Basel problem can improve this to $\frac{\pi^2}{6}$, but the improvement is very small and the proof of the Basel problem is a very complex one.

The exact asymptotic behaviour of the partial sums is actually $\ln(\ln n) + M$, where M is the Meissel–Mertens constant (approximately 0.261).

theorem *prime-harmonic-lower*:
  assumes *n*: $n \geq 2$
  shows $(\sum p \leftarrow \textit{primes-upto } n.\ 1\ /\ \textit{real } p) \geq \textit{ln } (\textit{harm } n) - \textit{ln } (5/3)$
$\langle$*proof*$\rangle$

We can use the inequality $\ln(n+1) \leq H_n$ to estimate the asymptotic growth of the partial prime harmonic series. Note that $H_n \sim \ln n + \gamma$ where γ is the Euler–Mascheroni constant (approximately 0.577), so we lose some accuracy here.

corollary *prime-harmonic-lower'*:
  assumes *n*: $n \geq 2$
  shows $(\sum p \leftarrow \textit{primes-upto } n.\ 1\ /\ \textit{real } p) \geq \textit{ln } (\textit{ln } (n + 1)) - \textit{ln } (5/3)$
$\langle$*proof*$\rangle$

lemma *Bseq-eventually-mono*:
  assumes *eventually* ($\lambda n.$ *norm* (*f n*) $\leq$ *norm* (*g n*)) *sequentially Bseq g*
  shows *Bseq f*
$\langle$*proof*$\rangle$

lemma *Bseq-add*:
  assumes *Bseq* (*f* :: *nat* $\Rightarrow$ *'a* :: *real-normed-vector*)
  shows *Bseq* ($\lambda x.$ *f x* + *c*)
$\langle$*proof*$\rangle$

lemma *convergent-imp-Bseq*: *convergent f* $\Longrightarrow$ *Bseq f*
  $\langle$*proof*$\rangle$

We now use our last estimate to show that the prime harmonic series diverges. This is obvious, since it is bounded from below by $\ln(\ln(n + 1))$ minus some constant, which obviously tends to infinite.

Directly using the divergence of the harmonic series would also be possible and shorten this proof a bit..

corollary *prime-harmonic-series-unbounded*:
  $\neg$*Bseq* ($\lambda n.\ \sum p \leftarrow$ *primes-upto n. 1 / p*) (**is** $\neg$*Bseq ?f*)
$\langle$*proof*$\rangle$

corollary *prime-harmonic-series-diverges*:
  $\neg$*convergent* ($\lambda n.\ \sum p \leftarrow$ *primes-upto n. 1 / p*)
  $\langle$*proof*$\rangle$

end