

# The Divergence of the Prime Harmonic Series

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## Abstract

In this work, we prove the lower bound  $\ln(H_n) - \ln(\frac{5}{3})$  for the partial sum of the Prime Harmonic series and, based on this, the divergence of the Prime Harmonic Series  $\sum_{p=1}^n [p \text{ prime}] \cdot \frac{1}{p}$ . The proof relies on the unique squarefree decomposition of natural numbers. This proof is similar to Euler's original proof (which was highly informal and morally questionable). Its advantage over proofs by contradiction, like the famous one by Paul Erdős, is that it provides a relatively good lower bound for the partial sums.

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## 1 Auxiliary lemmas

**theory** *Prime-Harmonic-Misc*

**imports**

*Complex-Main*

*HOL-Number-Theory.Number-Theory*

**begin**

**lemma** *sum-list-nonneg*:  $\forall x \in \text{set } xs. x \geq 0 \implies \text{sum-list } xs \geq (0 :: 'a :: \text{ordered-ab-group-add})$   
**by** (*induction xs*) *auto*

**lemma** *sum-telescope'*:

**assumes**  $m \leq n$

**shows**  $(\sum k = \text{Suc } m..n. f k - f (\text{Suc } k)) = f (\text{Suc } m) - (f (\text{Suc } n) :: 'a :: \text{ab-group-add})$

**by** (*rule dec-induct[OF assms]*) (*simp-all add: algebra-simps*)

**lemma** *dvd-prodI*:

**assumes** *finite A x ∈ A*

**shows**  $f\ x\ \text{dvd}\ \text{prod}\ f\ A$

**proof** –

**from** *assms* **have**  $\text{prod}\ f\ A = f\ x * \text{prod}\ f\ (A - \{x\})$

**by** (*intro prod.remove simp-all*)

**thus** *?thesis* **by** *simp*

**qed**

**lemma** *dvd-prodD*:  $\text{finite}\ A \implies \text{prod}\ f\ A\ \text{dvd}\ x \implies a \in A \implies f\ a\ \text{dvd}\ x$

**by** (*erule dvd-trans[OF dvd-prodI]*)

**lemma** *multiplicity-power-nat*:

$\text{prime}\ p \implies n > 0 \implies \text{multiplicity}\ p\ (n \wedge k :: \text{nat}) = k * \text{multiplicity}\ p\ n$

**by** (*induction k*) (*simp-all add: prime-elem-multiplicity-mult-distrib*)

**lemma** *multiplicity-prod-prime-powers-nat'*:

$\text{finite}\ S \implies \forall p \in S. \text{prime}\ p \implies \text{prime}\ p \implies$

$\text{multiplicity}\ p\ (\prod S :: \text{nat}) = (\text{if}\ p \in S\ \text{then}\ 1\ \text{else}\ 0)$

**using** *multiplicity-prod-prime-powers[of S p λ-. 1]* **by** *simp*

**lemma** *prod-prime-subset*:

**assumes** *finite A finite B*

**assumes**  $\bigwedge x. x \in A \implies \text{prime}\ (x :: \text{nat})$

**assumes**  $\bigwedge x. x \in B \implies \text{prime}\ x$

**assumes**  $\prod A\ \text{dvd}\ \prod B$

**shows**  $A \subseteq B$

**proof**

**fix** *x* **assume**  $x \in A$

**from** *assms(4)[of 0]* **have**  $0 \notin B$  **by** *auto*

**with** *assms* **have** *nonzero*:  $\forall z \in B. z \neq 0$  **by** (*intro ballI notI*) *auto*

**from** *x assms* **have**  $1 = \text{multiplicity}\ x\ (\prod A)$

**by** (*subst multiplicity-prod-prime-powers-nat'*) *simp-all*

**also from** *assms nonzero* **have**  $\dots \leq \text{multiplicity}\ x\ (\prod B)$  **by** (*intro dvd-imp-multiplicity-le*) *auto*

**finally have**  $\text{multiplicity}\ x\ (\prod B) > 0$  **by** *simp*

**moreover from** *assms x* **have** *prime x* **by** *simp*

**ultimately show**  $x \in B$  **using** *assms(2,4)*

**by** (*subst (asm) multiplicity-prod-prime-powers-nat'*) (*simp-all split: if-split-asm*)

**qed**

**lemma** *prod-prime-eq*:

**assumes** *finite A finite B*  $\bigwedge x. x \in A \implies \text{prime}\ (x :: \text{nat})$   $\bigwedge x. x \in B \implies \text{prime}\ x$

$\prod A = \prod B$

**shows**  $A = B$

**using** *assms* **by** (*intro equalityI prod-prime-subset*) *simp-all*

```

lemma ln-ln-nonneg:
  assumes  $x: x \geq (3 :: \text{real})$ 
  shows  $\ln (\ln x) \geq 0$ 
proof –
  have  $\exp 1 \leq (3 :: \text{real})$  by (rule exp-le)
  hence  $\ln (\exp 1) \leq \ln (3 :: \text{real})$  by (subst ln-le-cancel-iff) simp-all
  also from  $x$  have  $\dots \leq \ln x$  by (subst ln-le-cancel-iff) simp-all
  finally have  $\ln 1 \leq \ln (\ln x)$  using  $x$  by (subst ln-le-cancel-iff) simp-all
  thus ?thesis by simp
qed

end

```

## 2 Squarefree decomposition of natural numbers

```

theory Squarefree-Nat
imports
  Main
  HOL-Number-Theory.Number-Theory
  Prime-Harmonic-Misc
begin

```

The squarefree part of a natural number is the set of all prime factors that appear with odd multiplicity. The square part, correspondingly, is what remains after dividing by the squarefree part.

```

definition squarefree-part ::  $\text{nat} \Rightarrow \text{nat set}$  where
  squarefree-part  $n = \{p \in \text{prime-factors } n. \text{ odd } (\text{multiplicity } p \ n)\}$ 

```

```

definition square-part ::  $\text{nat} \Rightarrow \text{nat}$  where
  square-part  $n = (\text{if } n = 0 \text{ then } 0 \text{ else } (\prod p \in \text{prime-factors } n. p ^ (\text{multiplicity } p \ n \ \text{div } 2)))$ 

```

```

lemma squarefree-part-0 [simp]: squarefree-part 0 = {}
by (simp add: squarefree-part-def)

```

```

lemma square-part-0 [simp]: square-part 0 = 0
by (simp add: square-part-def)

```

```

lemma squarefree-decompose:  $\prod (\text{squarefree-part } n) * \text{square-part } n ^ 2 = n$ 

```

```

proof (cases n = 0)
  case False
  def  $A \equiv \text{squarefree-part } n$  and  $s \equiv \text{square-part } n$ 
  have  $(\prod A) = (\prod p \in A. p ^ (\text{multiplicity } p \ n \ \text{mod } 2))$ 
  by (intro prod.cong) (auto simp: A-def squarefree-part-def elim!: oddE)
  also have  $\dots = (\prod p \in \text{prime-factors } n. p ^ (\text{multiplicity } p \ n \ \text{mod } 2))$ 
  by (intro prod.mono-neutral-left) (auto simp: A-def squarefree-part-def)
  also from False have  $\dots * s ^ 2 = n$ 
  by (simp add: s-def square-part-def prod.distrib [symmetric] power-add [symmetric])

```

*power-mult [symmetric] prime-factorization-nat [symmetric]*

*algebra-simps*

*prod-power-distrib*)

**finally show**  $\prod A * s^2 = n$  .

**qed** *simp*

**lemma** *squarefree-part-pos [simp]*:  $\prod (\text{squarefree-part } n) > 0$   
**using** *prime-gt-0-nat unfolding squarefree-part-def by auto*

**lemma** *squarefree-part-ge-Suc-0 [simp]*:  $\prod (\text{squarefree-part } n) \geq \text{Suc } 0$   
**using** *squarefree-part-pos[of n] by presburger*

**lemma** *squarefree-part-subset [intro]*:  $\text{squarefree-part } n \subseteq \text{prime-factors } n$   
**unfolding** *squarefree-part-def by auto*

**lemma** *squarefree-part-finite [simp]*: *finite (squarefree-part n)*  
**by** (*rule finite-subset[OF squarefree-part-subset]*) *simp*

**lemma** *squarefree-part-dvd [simp]*:  $\prod (\text{squarefree-part } n) \text{ dvd } n$   
**by** (*subst (2) squarefree-decompose [of n, symmetric]*) *simp*

**lemma** *squarefree-part-dvd' [simp]*:  $p \in \text{squarefree-part } n \implies p \text{ dvd } n$   
**by** (*rule dvd-prodD[OF - squarefree-part-dvd]*) *simp-all*

**lemma** *square-part-dvd [simp]*:  $\text{square-part } n^2 \text{ dvd } n$   
**by** (*subst (2) squarefree-decompose [of n, symmetric]*) *simp*

**lemma** *square-part-dvd' [simp]*:  $\text{square-part } n \text{ dvd } n$   
**by** (*subst (2) squarefree-decompose [of n, symmetric]*) *simp*

**lemma** *squarefree-part-le*:  $p \in \text{squarefree-part } n \implies p \leq n$   
**by** (*cases n = 0*) (*simp-all add: dvd-imp-le*)

**lemma** *square-part-le*:  $\text{square-part } n \leq n$   
**by** (*cases n = 0*) (*simp-all add: dvd-imp-le*)

**lemma** *square-part-le-sqrt*:  $\text{square-part } n \leq \text{nat } \lfloor \text{sqrt } (\text{real } n) \rfloor$

**proof** –

**have**  $1 * \text{square-part } n^2 \leq \prod (\text{squarefree-part } n) * \text{square-part } n^2$   
**by** (*intro mult-right-mono*) *simp-all*

**also have**  $\dots = n$  **by** (*rule squarefree-decompose*)

**finally have**  $\text{real } (\text{square-part } n^2) \leq \text{real } n$  **by** (*subst of-nat-le-iff*) *simp*

**hence**  $\text{sqrt } (\text{real } (\text{square-part } n^2)) \leq \text{sqrt } (\text{real } n)$   
**by** (*subst real-sqrt-le-iff*) *simp-all*

**also have**  $\text{sqrt } (\text{real } (\text{square-part } n^2)) = \text{real } (\text{square-part } n)$  **by** *simp*

**finally show** *?thesis* **by** *linarith*

**qed**

**lemma** *square-part-pos* [*simp*]:  $n > 0 \implies \text{square-part } n > 0$   
**unfolding** *square-part-def* **using** *prime-gt-0-nat* **by** *auto*

**lemma** *square-part-ge-Suc-0* [*simp*]:  $n > 0 \implies \text{square-part } n \geq \text{Suc } 0$   
**using** *square-part-pos*[*of n*] **by** *presburger*

**lemma** *zero-not-in-squarefree-part* [*simp*]:  $0 \notin \text{squarefree-part } n$   
**unfolding** *squarefree-part-def* **by** *auto*

**lemma** *multiplicity-squarefree-part*:  
 $\text{prime } p \implies \text{multiplicity } p (\prod (\text{squarefree-part } n)) = (\text{if } p \in \text{squarefree-part } n \text{ then } 1 \text{ else } 0)$   
**using** *squarefree-part-subset*[*of n*]  
**by** (*intro multiplicity-prod-prime-powers-nat*) *auto*

The squarefree part really is square, its only square divisor is 1.

**lemma** *square-dvd-squarefree-part-iff*:  
 $x^2 \text{ dvd } \prod (\text{squarefree-part } n) \iff x = 1$   
**proof** (*rule iffI*, *rule multiplicity-eq-nat*)  
**assume** *dvd*:  $x^2 \text{ dvd } \prod (\text{squarefree-part } n)$   
**hence**  $x \neq 0$  **using** *squarefree-part-subset*[*of n*] *prime-gt-0-nat* **by** (*intro notI*)  
*auto*  
**thus**  $x: x > 0$  **by** *simp*

**fix**  $p :: \text{nat}$  **assume**  $p$ : *prime p*  
**from**  $p \ x$  **have**  $2 * \text{multiplicity } p \ x = \text{multiplicity } p \ (x^2)$   
**by** (*simp add: multiplicity-power-nat*)  
**also from** *dvd* **have**  $\dots \leq \text{multiplicity } p (\prod (\text{squarefree-part } n))$   
**by** (*intro dvd-imp-multiplicity-le*) *simp-all*  
**also have**  $\dots \leq 1$  **using** *multiplicity-squarefree-part*[*of p n*]  $p$  **by** *simp*  
**finally show**  $\text{multiplicity } p \ x = \text{multiplicity } p \ 1$  **by** *simp*  
**qed** *simp-all*

**lemma** *squarefree-decomposition-unique1*:  
**assumes**  $\text{squarefree-part } m = \text{squarefree-part } n$   
**assumes**  $\text{square-part } m = \text{square-part } n$   
**shows**  $m = n$   
**by** (*subst* (1 2) *squarefree-decompose* [*symmetric*]) (*simp add: assms*)

**lemma** *squarefree-decomposition-unique2*:  
**assumes**  $n: n > 0$   
**assumes** *decomp*:  $n = (\prod A2 * s2^2)$   
**assumes** *prime*:  $\bigwedge x. x \in A2 \implies \text{prime } x$   
**assumes** *fin*: *finite A2*  
**assumes** *s2-nonneg*:  $s2 \geq 0$   
**shows**  $A2 = \text{squarefree-part } n$  **and**  $s2 = \text{square-part } n$   
**proof** –  
**def**  $A1 \equiv \text{squarefree-part } n$  **and**  $s1 \equiv \text{square-part } n$

```

have finite A1 unfolding A1-def by simp
note fin = ⟨finite A1⟩ ⟨finite A2⟩

have A1 ⊆ prime-factors n unfolding A1-def using squarefree-part-subset .
note subset = this prime

have ∏ A1 > 0 ∏ A2 > 0 using subset(1) prime-gt-0-nat
  by (auto intro!: prod-pos dest: prime)
from n have s1 > 0 unfolding s1-def by simp
from assms have s2 ≠ 0 by (intro notI) simp
hence s2 > 0 by simp
note pos = ⟨∏ A1 > 0⟩ ⟨∏ A2 > 0⟩ ⟨s1 > 0⟩ ⟨s2 > 0⟩

have eq': multiplicity p s1 = multiplicity p s2
  multiplicity p (∏ A1) = multiplicity p (∏ A2)
  if p: prime p for p
proof -
  def m ≡ multiplicity p
  from decomp have m (∏ A1 * s1^2) = m (∏ A2 * s2^2) unfolding A1-def
s1-def
  by (simp add: A1-def s1-def squarefree-decompose)
  with p pos have eq: m (∏ A1) + 2 * m s1 = m (∏ A2) + 2 * m s2
  by (simp add: m-def prime-elem-multiplicity-mult-distrib multiplicity-power-nat)
  moreover from fin subset p have m (∏ A1) ≤ 1 m (∏ A2) ≤ 1 unfolding
m-def
  by ((subst multiplicity-prod-prime-powers-nat', auto)[])
  ultimately show m s1 = m s2 by linarith
  with eq show m (∏ A1) = m (∏ A2) by simp
qed

show s2 = square-part n
  by (rule multiplicity-eq-nat) (insert pos eq'(1), auto simp: s1-def)
have ∏ A2 = ∏ (squarefree-part n)
  by (rule multiplicity-eq-nat) (insert pos eq'(2), auto simp: A1-def)
with fin subset show A2 = squarefree-part n unfolding A1-def
  by (intro prod-prime-eq) auto
qed

lemma squarefree-decomposition-unique2':
  assumes decomp: (∏ A1 * s1^2) = (∏ A2 * s2^2 :: nat)
  assumes fin: finite A1 finite A2
  assumes subset: ∧x. x ∈ A1 ⇒ prime x ∧x. x ∈ A2 ⇒ prime x
  assumes pos: s1 > 0 s2 > 0
  defines n ≡ ∏ A1 * s1^2
  shows A1 = A2 s1 = s2
proof -
  from pos have n: n > 0 using prime-gt-0-nat
  by (auto simp: n-def intro!: prod-pos dest: subset)
  have A1 = squarefree-part n s1 = square-part n

```

by ((rule squarefree-decomposition-unique2[of n A1 s1], insert assms n, simp-all)[])+  
 moreover have  $A2 = \text{squarefree-part } n \text{ } s2 = \text{square-part } n$   
 by ((rule squarefree-decomposition-unique2[of n A2 s2], insert assms n, simp-all)[])+  
 ultimately show  $A1 = A2 \text{ } s1 = s2$  by simp-all  
 qed

The following is a nice and simple lower bound on the number of prime numbers less than a given number due to Erdős. In particular, it implies that there are infinitely many primes.

**lemma** *primes-lower-bound*:

fixes  $n :: \text{nat}$   
 assumes  $n > 0$   
 defines  $\pi \equiv \lambda n. \text{card } \{p. \text{prime } p \wedge p \leq n\}$   
 shows  $\text{real } (\pi \ n) \geq \ln (\text{real } n) / \ln 4$   
**proof** –  
 have  $\text{real } n = \text{real } (\text{card } \{1..n\})$  by simp  
 also have  $\{1..n\} = (\lambda(A, b). \prod A * b^2) \text{ ` } (\lambda n. (\text{squarefree-part } n, \text{square-part } n)) \text{ ` } \{1..n\}$   
 unfolding image-comp o-def squarefree-decompose case-prod-unfold fst-conv snd-conv by simp  
 also have  $\text{card } \dots \leq \text{card } ((\lambda n. (\text{squarefree-part } n, \text{square-part } n)) \text{ ` } \{1..n\})$   
 by (rule card-image-le) simp-all  
 also have  $\dots \leq \text{card } (\text{squarefree-part } \text{ ` } \{1..n\} \times \text{square-part } \text{ ` } \{1..n\})$   
 by (rule card-mono) auto  
 also have  $\text{real } \dots = \text{real } (\text{card } (\text{squarefree-part } \text{ ` } \{1..n\})) * \text{real } (\text{card } (\text{square-part } \text{ ` } \{1..n\}))$   
 by simp  
 also have  $\dots \leq 2^{\pi \ n} * \text{sqrt } (\text{real } n)$   
**proof** (rule mult-mono)  
 have  $\text{card } (\text{squarefree-part } \text{ ` } \{1..n\}) \leq \text{card } (\text{Pow } \{p. \text{prime } p \wedge p \leq n\})$   
 using squarefree-part-subset squarefree-part-le by (intro card-mono) force+  
 also have  $\text{real } \dots = 2^{\pi \ n}$  by (simp add:  $\pi$ -def card-Pow)  
 finally show  $\text{real } (\text{card } (\text{squarefree-part } \text{ ` } \{1..n\})) \leq 2^{\pi \ n}$  by – simp-all  
**next**  
 have  $\text{square-part } k \leq \text{nat } \lfloor \text{sqrt } n \rfloor$  if  $k \leq n$  for  $k$   
 by (rule order.trans[OF square-part-le-sqrt])  
 (insert that, auto intro!: nat-mono floor-mono)  
 hence  $\text{card } (\text{square-part } \text{ ` } \{1..n\}) \leq \text{card } \{1..\text{nat } \lfloor \text{sqrt } n \rfloor\}$   
 by (intro card-mono) (auto intro: order.trans[OF square-part-le-sqrt])  
 also have  $\dots = \text{nat } \lfloor \text{sqrt } n \rfloor$  by simp  
 also have  $\text{real } \dots \leq \text{sqrt } n$  by simp  
 finally show  $\text{real } (\text{card } (\text{square-part } \text{ ` } \{1..n\})) \leq \text{sqrt } (\text{real } n)$  by – simp-all  
**qed** simp-all  
 finally have  $\text{real } n \leq 2^{\pi \ n} * \text{sqrt } (\text{real } n)$  by – simp-all  
 with  $\langle n > 0 \rangle$  have  $\ln (\text{real } n) \leq \ln (2^{\pi \ n} * \text{sqrt } (\text{real } n))$   
 by (subst ln-le-cancel-iff) simp-all  
 moreover have  $\ln (4 :: \text{real}) = \text{real } 2 * \ln 2$  by (subst ln-realpow [symmetric])  
 simp-all  
 ultimately show *?thesis* using  $\langle n > 0 \rangle$

by (*simp add: ln-mult ln-realpow[of -  $\pi$  n] ln-sqrt field-simps*)  
 qed  
 end

### 3 The Prime Harmonic Series

theory *Prime-Harmonic*  
 imports  
   *HOL-Analysis.Analysis*  
   *HOL-Number-Theory.Number-Theory*  
   *Prime-Harmonic-Misc*  
   *Squarefree-Nat*  
 begin

#### 3.1 Auxiliary equalities and inequalities

First of all, we prove the following result about rearranging a product over a set into a sum over all subsets of that set.

**lemma** *prime-harmonic-aux1*:  
 fixes  $A :: 'a :: \text{field set}$   
 shows  $\text{finite } A \implies (\prod x \in A. 1 + 1 / x) = (\sum x \in \text{Pow } A. 1 / \prod x)$   
**proof** (*induction rule: finite-induct*)  
 fix  $a :: 'a$  and  $A :: 'a \text{ set}$   
 assume  $a: a \notin A$  and  $\text{fin}: \text{finite } A$   
 assume *IH*:  $(\prod x \in A. 1 + 1 / x) = (\sum x \in \text{Pow } A. 1 / \prod x)$   
 from  $a$  and  $\text{fin}$  have  $(\prod x \in \text{insert } a \ A. 1 + 1 / x) = (1 + 1 / a) * (\prod x \in A. 1 + 1 / x)$  by *simp*  
 also from  $\text{fin}$  have  $\dots = (\sum x \in \text{Pow } A. 1 / \prod x) + (\sum x \in \text{Pow } A. 1 / (a * \prod x))$   
 by (*subst IH*) (*auto simp add: algebra-simps sum-divide-distrib*)  
 also from  $\text{fin } a$  have  $(\sum x \in \text{Pow } A. 1 / (a * \prod x)) = (\sum x \in \text{Pow } A. 1 / \prod (\text{insert } a \ x))$   
 by (*intro sum.cong refl, subst prod.insert*) (*auto dest: finite-subset*)  
 also from  $a$  have  $\dots = (\sum x \in \text{insert } a \ 'Pow \ A. 1 / \prod x)$   
 by (*subst sum.reindex*) (*auto simp: inj-on-def*)  
 also from  $\text{fin } a$  have  $(\sum x \in \text{Pow } A. 1 / \prod x) + \dots = (\sum x \in \text{Pow } A \cup \text{insert } a \ 'Pow \ A. 1 / \prod x)$   
 by (*intro sum.union-disjoint [symmetric]*) (*simp, simp, blast*)  
 also have  $\text{Pow } A \cup \text{insert } a \ 'Pow \ A = \text{Pow } (\text{insert } a \ A)$  by (*simp only: Pow-insert*)  
 finally show  $(\prod x \in \text{insert } a \ A. 1 + 1 / x) = (\sum x \in \text{Pow } (\text{insert } a \ A). 1 / \prod x)$   
 .  
 qed *simp*

Next, we prove a simple and reasonably accurate upper bound for the sum of the squares of any subset of the natural numbers, derived by simple telescoping. Our upper bound is approximately 1.67; the exact value is  $\frac{\pi^2}{6} \approx 1.64$ .

(cf. Basel problem)

**lemma** *prime-harmonic-aux2*:

**assumes** *finite* ( $A :: \text{nat set}$ )

**shows**  $(\sum_{k \in A} 1 / (\text{real } k \wedge 2)) \leq 5/3$

**proof** –

**def**  $n \equiv \text{max } 2 \text{ (Max } A)$

**have**  $n: n \geq \text{Max } A \ n \geq 2$  **by** (*auto simp: n-def*)

**with** *assms* **have**  $A \subseteq \{0..n\}$  **by** (*auto intro: order.trans[OF Max-ge]*)

**hence**  $(\sum_{k \in A} 1 / (\text{real } k \wedge 2)) \leq (\sum_{k=0..n} 1 / (\text{real } k \wedge 2))$  **by** (*intro sum-mono2*) *auto*

**also from**  $n$  **have**  $\dots = 1 + (\sum_{k=\text{Suc } 1..n} 1 / (\text{real } k \wedge 2))$  **by** (*simp add: sum-head-Suc*)

**also have**  $(\sum_{k=\text{Suc } 1..n} 1 / (\text{real } k \wedge 2)) \leq$

$(\sum_{k=\text{Suc } 1..n} 1 / (\text{real } k \wedge 2 - 1/4))$  **unfolding** *power2-eq-square*

**by** (*intro sum-mono divide-left-mono mult-pos-pos*)

(*linarith, simp-all add: field-simps less-1-mult*)

**also have**  $\dots = (\sum_{k=\text{Suc } 1..n} 1 / (\text{real } k - 1/2) - 1 / (\text{real } (\text{Suc } k) - 1/2))$

**by** (*intro sum.cong refl*) (*simp-all add: field-simps power2-eq-square*)

**also from**  $n$  **have**  $\dots = 2 / 3 - 1 / (1 / 2 + \text{real } n)$

**by** (*subst sum-telescope'*) *simp-all*

**also have**  $1 + \dots \leq 5/3$  **by** *simp*

**finally show** *?thesis* **by** – *simp*

**qed**

### 3.2 Estimating the partial sums of the Prime Harmonic Series

We are now ready to show our main result: the value of the partial prime harmonic sum over all primes no greater than  $n$  is bounded from below by the  $n$ -th harmonic number  $H_n$  minus some constant.

In our case, this constant will be  $\frac{5}{3}$ . As mentioned before, using a proof of the Basel problem can improve this to  $\frac{\pi^2}{6}$ , but the improvement is very small and the proof of the Basel problem is a very complex one.

The exact asymptotic behaviour of the partial sums is actually  $\ln(\ln n) + M$ , where  $M$  is the Meissel–Mertens constant (approximately 0.261).

**theorem** *prime-harmonic-lower*:

**assumes**  $n: n \geq 2$

**shows**  $(\sum_{p \leftarrow \text{primes-upto } n} 1 / \text{real } p) \geq \ln (\text{harm } n) - \ln (5/3)$

**proof** –

– the set of primes that we will allow in the squarefree part

**def**  $P \equiv \lambda n. \text{set } (\text{primes-upto } n)$

{  
  **fix**  $n :: \text{nat}$

**have** *finite* ( $P \ n$ ) **by** (*simp add: P-def*)

} **note** [*simp*] = *this*

– The function that combines the squarefree part and the square part

**def**  $f \equiv \lambda(R, s :: \text{nat}). \prod R * s^2$

—  $f$  is injective if the squarefree part contains only primes and the square part is positive.

**have**  $\text{inj}: \text{inj-on } f \text{ (Pow (P n) } \times \{1..n\})$   
**proof** (*rule inj-onI, clarify, rule conjI*)  
**fix**  $A1 A2 :: \text{nat set and } s1 s2 :: \text{nat}$   
**assume**  $A: A1 \subseteq P n A2 \subseteq P n s1 \in \{1..n\} s2 \in \{1..n\} f(A1, s1) = f(A2, s2)$   
**have**  $\text{fin}: \text{finite } A1 \text{ finite } A2$  **by** (*rule A(1,2)[THEN finite-subset], simp*) +  
**show**  $A1 = A2 s1 = s2$   
**by** (*(rule squarefree-decomposition-unique2'[of A1 s1 A2 s2], insert A fin, auto simp: f-def P-def set-primes-upto)[]*) +  
**qed**

—  $f$  hits every number between 1 and  $n$ . It also hits a lot of other numbers, but we do not care about those, since we only need a lower bound.

**have**  $\text{surj}: \{1..n\} \subseteq f' \text{ (Pow (P n) } \times \{1..n\})$   
**proof**  
**fix**  $x$  **assume**  $x: x \in \{1..n\}$   
**have**  $x = f(\text{squarefree-part } x, \text{square-part } x)$  **by** (*simp add: f-def squarefree-decompose*)  
**moreover** **have**  $\text{squarefree-part } x \in \text{Pow (P n)}$  **using** *squarefree-part-subset[of x]*  $x$   
**by** (*auto simp: P-def set-primes-upto intro: order.trans[OF squarefree-part-le[of - x]]*)  
**moreover** **have**  $\text{square-part } x \in \{1..n\}$  **using**  $x$   
**by** (*auto simp: Suc-le-eq intro: order.trans[OF square-part-le[of x]]*)  
**ultimately** **show**  $x \in f' \text{ (Pow (P n) } \times \{1..n\})$  **by** *simp*  
**qed**

— We now show the main result by rearranging the sum over all primes to a product over all all squarefree parts times a sum over all square parts, and then applying some simple-minded approximation

**have**  $\text{harm } n = (\sum_{n=1..n}. 1 / \text{real } n)$  **by** (*simp add: harm-def field-simps*)  
**also** **from**  $\text{surj}$  **have**  $\dots \leq (\sum_{n \in f' \text{ (Pow (P n) } \times \{1..n\})}. 1 / \text{real } n)$   
**by** (*intro sum-mono2 finite-imageI finite-cartesian-product simp-all*)  
**also** **from**  $\text{inj}$  **have**  $\dots = (\sum_{x \in \text{Pow (P n) } \times \{1..n\}}. 1 / \text{real } (f x))$   
**by** (*subst sum.reindex simp-all*)  
**also** **have**  $\dots = (\sum_{A \in \text{Pow (P n)}}. 1 / \text{real } (\prod A)) * (\sum_{k=1..n}. 1 / (\text{real } k)^2)$   
**unfolding** *f-def*  
**by** (*subst sum-product, subst sum.cartesian-product*) (*simp add: case-prod-beta*)  
**also** **have**  $\dots \leq (\sum_{A \in \text{Pow (P n)}}. 1 / \text{real } (\prod A)) * (5/3)$   
**by** (*intro mult-left-mono prime-harmonic-aux2 sum-nonneg*)  
*(auto simp: P-def intro!: prod-nonneg)*  
**also** **have**  $(\sum_{A \in \text{Pow (P n)}}. 1 / \text{real } (\prod A)) = (\sum_{A \in (\text{op}' \text{real})' \text{ Pow (P n)}}. 1 / \prod A)$   
**by** (*subst sum.reindex*) (*auto simp: inj-on-def inj-image-eq-iff prod.reindex*)  
**also** **have**  $(\text{op}' \text{real})' \text{ Pow (P n)} = \text{Pow } (\text{real}' \text{ P n})$  **by** (*intro image-Pow-surj refl*)

**also have**  $(\sum A \in \text{Pow } (\text{real } ' P n). 1 / \prod A) = (\prod x \in \text{real } ' P n. 1 + 1 / x)$   
**by** (*intro prime-harmonic-aux1 [symmetric] finite-imageI*) *simp-all*  
**also have**  $\dots = (\prod i \in P n. 1 + 1 / \text{real } i)$  **by** (*subst prod.reindex*) (*auto simp: inj-on-def*)  
**also have**  $\dots \leq (\prod i \in P n. \text{exp } (1 / \text{real } i))$  **by** (*intro prod-mono*) *auto*  
**also have**  $\dots = \text{exp } (\sum i \in P n. 1 / \text{real } i)$  **by** (*simp add: exp-sum*)  
**finally have**  $\ln (\text{harm } n) \leq \ln (\dots * (5/3))$  **using**  $n$   
**by** (*subst ln-le-cancel-iff*) *simp-all*  
**hence**  $\ln (\text{harm } n) - \ln (5/3) \leq (\sum i \in P n. 1 / \text{real } i)$   
**by** (*subst (asm) ln-mult*) (*simp-all add: algebra-simps*)  
**thus** *?thesis* **unfolding** *P-def*  
**by** (*subst (asm) sum.distinct-set-conv-list*) *simp-all*  
**qed**

We can use the inequality  $\ln(n+1) \leq H_n$  to estimate the asymptotic growth of the partial prime harmonic series. Note that  $H_n \sim \ln n + \gamma$  where  $\gamma$  is the Euler–Mascheroni constant (approximately 0.577), so we lose some accuracy here.

**corollary** *prime-harmonic-lower'*:

**assumes**  $n: n \geq 2$

**shows**  $(\sum p \leftarrow \text{primes-upto } n. 1 / \text{real } p) \geq \ln (\ln (n + 1)) - \ln (5/3)$

**proof** –

**from** *assms ln-le-harm[of n]* **have**  $\ln (\ln (\text{real } n + 1)) \leq \ln (\text{harm } n)$  **by** *simp*

**also from** *assms* **have**  $\dots - \ln (5/3) \leq (\sum p \leftarrow \text{primes-upto } n. 1 / \text{real } p)$

**by** (*rule prime-harmonic-lower*)

**finally show** *?thesis* **by** – *simp*

**qed**

**lemma** *Bseq-eventually-mono*:

**assumes** *eventually*  $(\lambda n. \text{norm } (f n) \leq \text{norm } (g n))$  *sequentially Bseq g*

**shows** *Bseq f*

**proof** –

**from** *assms(1)* **obtain**  $N$  **where**  $N: \bigwedge n. n \geq N \implies \text{norm } (f n) \leq \text{norm } (g n)$

**by** (*auto simp: eventually-at-top-linorder*)

**from** *assms(2)* **obtain**  $K$  **where**  $K: \bigwedge n. \text{norm } (g n) \leq K$  **by** (*blast elim!: BseqE*)

{

**fix**  $n :: \text{nat}$

**have**  $\text{norm } (f n) \leq \max K (\text{Max } \{\text{norm } (f n) \mid n. n < N\})$

**apply** (*cases n < N*)

**apply** (*rule max.coboundedI2, rule Max.coboundedI, auto*) []

**apply** (*rule max.coboundedI1, force intro: order.trans[OF N K]*)

**done**

}

**thus** *?thesis* **by** (*blast intro: BseqI'*)

**qed**

**lemma** *Bseq-add*:  
**assumes** *Bseq* ( $f :: \text{nat} \Rightarrow 'a :: \text{real-normed-vector}$ )  
**shows** *Bseq* ( $\lambda x. f x + c$ )  
**proof** –  
**from** *assms* **obtain**  $K$  **where**  $K: \bigwedge x. \text{norm } (f x) \leq K$  **unfolding** *Bseq-def* **by**  
*blast*  
{  
  **fix**  $x :: \text{nat}$   
  **have**  $\text{norm } (f x + c) \leq \text{norm } (f x) + \text{norm } c$  **by** (*rule norm-triangle-ineq*)  
  **also have**  $\text{norm } (f x) \leq K$  **by** (*rule K*)  
  **finally have**  $\text{norm } (f x + c) \leq K + \text{norm } c$  **by** *simp*  
}  
**thus** *?thesis* **by** (*rule BseqI'*)  
**qed**

**lemma** *convergent-imp-Bseq*: *convergent*  $f \implies \text{Bseq } f$   
**by** (*simp add: Cauchy-Bseq convergent-Cauchy*)

We now use our last estimate to show that the prime harmonic series diverges. This is obvious, since it is bounded from below by  $\ln(\ln(n+1))$  minus some constant, which obviously tends to infinite.

Directly using the divergence of the harmonic series would also be possible and shorten this proof a bit..

**corollary** *prime-harmonic-series-unbounded*:

$\neg \text{Bseq } (\lambda n. \sum p \leftarrow \text{primes-upto } n. 1 / p)$  (**is**  $\neg \text{Bseq } ?f$ )

**proof**

**assume** *Bseq*  $?f$

**hence** *Bseq* ( $\lambda n. ?f n + \ln (5/3)$ ) **by** (*rule Bseq-add*)

**have** *Bseq* ( $\lambda n. \ln (\ln (n + 1))$ )

**proof** (*rule Bseq-eventually-mono*)

**from** *eventually-ge-at-top*[*of*  $2 :: \text{nat}$ ]

**show** *eventually* ( $\lambda n. \text{norm } (\ln (\ln (n + 1))) \leq \text{norm } (?f n + \ln (5/3))$ )

*sequentially*

**proof** *eventually-elim*

**fix**  $n :: \text{nat}$  **assume**  $n: n \geq 2$

**hence**  $\text{norm } (\ln (\ln (\text{real } n + 1))) = \ln (\ln (\text{real } n + 1))$

**using** *ln-ln-nonneg*[*of*  $\text{real } n + 1$ ] **by** *simp*

**also have**  $\dots \leq ?f n + \ln (5/3)$  **using** *prime-harmonic-lower'*[*OF*  $n$ ]

**by** (*simp add: algebra-simps*)

**also have**  $?f n + \ln (5/3) \geq 0$  **by** (*intro add-nonneg-nonneg sum-list-nonneg*)

*simp-all*

**hence**  $?f n + \ln (5/3) = \text{norm } (?f n + \ln (5/3))$  **by** *simp*

**finally show**  $\text{norm } (\ln (\ln (n + 1))) \leq \text{norm } (?f n + \ln (5/3))$

**by** (*simp add: add-ac*)

**qed**

**qed** *fact*

**then obtain**  $k$  **where**  $k: k > 0 \bigwedge n. \text{norm } (\ln (\ln (\text{real } (n :: \text{nat}) + 1))) \leq k$

**by** (*auto elim!: BseqE simp: add-ac*)

```

def  $N \equiv \text{nat } \lceil \text{exp } (\text{exp } k) \rceil$ 
have  $N\text{-pos}: N > 0$  unfolding  $N\text{-def}$  by simp
have  $\text{real } N + 1 > \text{exp } (\text{exp } k)$  unfolding  $N\text{-def}$  by linarith
hence  $\ln (\text{real } N + 1) > \ln (\text{exp } (\text{exp } k))$  by (subst ln-less-cancel-iff) simp-all
with  $N\text{-pos}$  have  $\ln (\ln (\text{real } N + 1)) > \ln (\text{exp } k)$  by (subst ln-less-cancel-iff)
simp-all
hence  $k < \ln (\ln (\text{real } N + 1))$  by simp
also have  $\dots \leq \text{norm } (\ln (\ln (\text{real } N + 1)))$  by simp
finally show False using  $k(2)[\text{of } N]$  by simp
qed

corollary prime-harmonic-series-diverges:
   $\neg \text{convergent } (\lambda n. \sum p \leftarrow \text{primes-upto } n. 1 / p)$ 
  using prime-harmonic-series-unbounded convergent-imp-Bseq by blast

end

```