

Possibilistic Noninterference*

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Abstract

We formalize a wide variety of Volpano/Smith-style noninterference notions for a while language with parallel composition. We systematize and classify these notions according to compositionality w.r.t. the language constructs. Compositionality yields sound syntactic criteria (a.k.a. type systems) in a uniform way.

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1 Introduction

This is a formalization of the mathematical development presented in the paper [1]:

- a uniform framework where a wide range of language-based noninterference variants from the literature are expressed and compared w.r.t. their *contracts*: the strength of the security properties they ensure weighed against the harshness of the syntactic conditions they enforce;
- syntactic criteria for proving that a program has a specific noninterference property, using only compositionality, which captures uniformly several security type-system results from the literature and suggests a further improved type system.

There are two auxiliary theories:

- MyTactics, introducing a few customized tactics;
- Bisim, describing an abstract notion of bisimilarity relation, namely, the greatest symmetric relation that is a fixpoint of a monotonic operator—this shall be instantiated to several concrete bisimilarity later.

The main theories of the development (shown in Fig. 1) are organized similarly to the sectionwise structure of [1]:

Language_Semantics corresponds to §2 in [1]. It introduces and customizes the syntax and small-step operational semantics of a while language with parallel composition, using notations very similar to the paper.

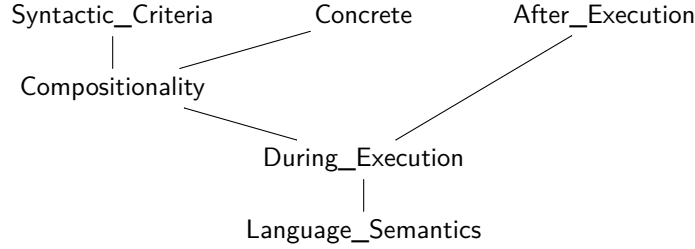


Figure 1: Main Theory Structure

`During_Execution`¹ mainly corresponds to §3 in [1], defining the various coinductive notions from there: self isomorphism, discreteness, variations of strong, weak and 01-bisimilarity. Prop. 1 from the paper, stating implications between these notions, is proved as the theorems `bis_imp` and `siso_bis`.² The bisimilarity inclusions stated in `bis_imp` are slightly more general than those in Prop. 1, in that they employ the binary version of the relation, e.g., $c \approx_s d \implies c \approx_{\text{WT}} d$ instead of $c \approx_s c \implies c \approx_{\text{WT}} c$.

`Compositionality` mainly corresponds to the homonymous §4 in [1]. The paper’s compositionality result, Prop. 2, is scattered through the theory as theorems with self-explanatory names, indicating the compositionality relationship between notions of noninterference and language constructs, e.g., `While_WbisT` (while versus termination-sensitive weak bisimilarity), `Par_ZO_bis` (parallel composition versus 01-bisimilarity).

Theories `During_Execution` and `Compositionality` also include the novel notion of noninterference \approx_{τ} introduced in §5 of [1], based on the “must terminate” condition, which is given the same treatment as the other notions: `bis_imp` in `During_Execution` states the implication relationship between \approx_{τ} and the other bisimilarities (Prop. 3.(1) from [1]), while various intuitively named theorems from `Language_Semantics` state the compositionality properties of \approx_{τ} (Prop. 3.(2) from [1]).

`Syntactic_Criteria` corresponds to the homonymous §6 in [1]. The syntactic analogues of the semantics notions, indicated in the paper by overlining, e.g., `discr`, are in the scripts prefixed by “SC” (from “syntactic criterion”), e.g., `SC_discr`, `SC_WbisT`. Props. 4 and 5 from the paper (stating the relationship between the syntactic and the semantic notions and the implications between the syntactic notions, respectively) are again scattered through the theory under self-explanatory names.

`Concrete` contains an instantiation of the indistinguishability relation \sim from [1] to the standard two-level security setting described in the paper’s Exam-

¹“During-execution” (bisimilarity-based) noninterference should be contrasted with “after-execution” (trace-based) noninterference according to the distinction made in [1] at the beginning of §7.

²To help the reader distinguish the main results from the auxiliary lemmas, the former are marked in the scripts with the keyword “theorem”.

ple 2.

Finally, `After_Execution` corresponds to §7 in [1], dealing with the after-execution guarantees of the during-execution notions of security. Prop. 6 in the paper is stated in the scripts as theorems `Sbis_trace`, `ZObisT_trace` and `WbisT_trace`, Prop. 7 as theorems `ZObis_trace` and `Wbis_trace`, and Prop. 8 as theorem `BisT_trace`.

2 Bisimilarity, abstractly

```
theory Bisim  
imports Interface  
begin
```

```
type-synonym 'a rel = ('a * 'a) set  
type-synonym ('cmd,'state) config = 'cmd * 'state
```

```
definition mono where  
mono Retr  $\equiv$   
 $\forall$  theta theta'. theta  $\leq$  theta'  $\longrightarrow$  Retr theta  $\leq$  Retr theta'
```

```
definition simul where  
simul Retr theta  $\equiv$  theta  $\leq$  Retr theta
```

```
definition bisim where  
bisim Retr theta  $\equiv$  sym theta  $\wedge$  simul Retr theta
```

```
lemma mono-Union:  
assumes mono Retr  
shows Union (Retr ' Theta)  $\leq$  Retr (Union Theta)  
{proof}
```

```
lemma mono-Un:  
assumes mono Retr  
shows Retr theta Un Retr theta'  $\subseteq$  Retr (theta Un theta')  
{proof}
```

```
lemma sym-Union:  
assumes  $\bigwedge$  theta. theta  $\in$  Theta  $\implies$  sym theta  
shows sym (Union Theta)  
{proof}
```

```
lemma sym-Un:  
assumes sym theta1 and sym theta2  
shows sym (theta1 Un theta2)  
{proof}
```

```
lemma simul-Union:  
assumes mono Retr
```

and $\bigwedge \theta. \theta \in \Theta \implies \text{simul Retr } \theta$
shows $\text{simul Retr } (\text{Union } \Theta)$
<proof>

lemma *simul-Un*:
assumes mono Retr **and** $\text{simul Retr } \theta_1$ **and** $\text{simul Retr } \theta_2$
shows $\text{simul Retr } (\theta_1 \text{ Un } \theta_2)$
<proof>

lemma *bisim-Union*:
assumes mono Retr **and** $\bigwedge \theta. \theta \in \Theta \implies \text{bisim Retr } \theta$
shows $\text{bisim Retr } (\text{Union } \Theta)$
<proof>

lemma *bisim-Un*:
assumes mono Retr **and** $\text{bisim Retr } \theta_1$ **and** $\text{bisim Retr } \theta_2$
shows $\text{bisim Retr } (\theta_1 \text{ Un } \theta_2)$
<proof>

definition *bis where*
 $\text{bis Retr} \equiv \text{Union } \{\theta. \text{bisim Retr } \theta\}$

lemma *bisim-bis[simp]*:
assumes mono Retr
shows $\text{bisim Retr } (\text{bis Retr})$
<proof>

corollary *sym-bis[simp]*: $\text{mono Retr} \implies \text{sym } (\text{bis Retr})$
and *simul-bis[simp]*: $\text{mono Retr} \implies \text{simul Retr } (\text{bis Retr})$
<proof>

lemma *bis-raw-coind*:
assumes mono Retr **and** $\text{sym } \theta$ **and** $\theta \subseteq \text{Retr } \theta$
shows $\theta \subseteq \text{bis Retr}$
<proof>

lemma *bis-prefix[simp]*:
assumes mono Retr
shows $\text{bis Retr} \subseteq \text{Retr } (\text{bis Retr})$
<proof>

lemma *bis-coind*:
assumes $*$: mono Retr **and** $\text{sym } \theta$ **and** $*$: $\theta \subseteq \text{Retr } (\theta \text{ Un } (\text{bis Retr}))$
shows $\theta \subseteq \text{bis Retr}$
<proof>

lemma *bis-coind2*:
assumes $*$: mono Retr **and**
 $*$: $\theta \subseteq \text{Retr } (\theta \text{ Un } (\text{bis Retr}))$ **and**

```

***:  $\theta^{-1} \subseteq \text{Retr} ((\theta^{-1}) \text{Un} (\text{bis Retr}))$ 
shows  $\theta \subseteq \text{bis Retr}$ 
⟨proof⟩

```

```

lemma bis-raw-coind2:
assumes *: mono Retr and
**:  $\theta \subseteq \text{Retr} \theta$  and
***:  $\theta^{-1} \subseteq \text{Retr} (\theta^{-1})$ 
shows  $\theta \subseteq \text{bis Retr}$ 
⟨proof⟩

```

```

lemma mono-bis:
assumes mono Retr1 and mono Retr2
and  $\bigwedge \theta. \text{Retr1 } \theta \subseteq \text{Retr2 } \theta$ 
shows  $\text{bis Retr1} \subseteq \text{bis Retr2}$ 
⟨proof⟩

```

end

3 The programming language and its semantics

theory *Language-Semantics* **imports** *Interface* **begin**

3.1 Syntax and operational semantics

```

datatype ('test,'atom) com =
  Atm 'atom |
  Seq ('test,'atom) com ('test,'atom) com
  (- ;; - [60, 61] 60) |
  If 'test ('test,'atom) com ('test,'atom) com
  ((if -/ then -/ else -) [0, 0, 61] 61) |
  While 'test ('test,'atom) com
  ((while -/ do -) [0, 61] 61) |
  Par ('test,'atom) com ('test,'atom) com
  (- | - [60, 61] 60)

```

```

locale PL =
fixes
  tval :: 'test  $\Rightarrow$  'state  $\Rightarrow$  bool and
  aval :: 'atom  $\Rightarrow$  'state  $\Rightarrow$  'state

```

```

context PL
begin

```

Conventions and notations: – suffixes: “C” for “Continuation”, “T” for “termination” – prefix: “M” for multistep – tst, tst’ are tests – atm, atm’ are atoms (atomic commands) – s, s’, t, t’ are states – c, c’, d, d’ are commands – cf, cf’ are configurations, i.e., pairs command-state

inductive *transT* ::
 (('test,'atom)com * 'state) ⇒ 'state ⇒ bool
 (**infix** →t 55)
where
Atm[simp]:
 (Atm atm, s) →t aval atm s
 | *WhileFalse*[simp]:
 ~ tval tst s ⇒ (While tst c, s) →t s

lemmas *trans-Atm* = *Atm*
lemmas *trans-WhileFalse* = *WhileFalse*

inductive *transC* ::
 (('test,'atom)com * 'state) ⇒ (('test,'atom)com * 'state) ⇒ bool
 (**infix** →c 55)
and *MtransC* ::
 (('test,'atom)com * 'state) ⇒ (('test,'atom)com * 'state) ⇒ bool
 (**infix** →*c 55)
where
SeqC[simp]:
 (c1, s) →c (c1', s') ⇒ (c1 ;; c2, s) →c (c1' ;; c2, s')
 | *SeqT*[simp]:
 (c1, s) →t s' ⇒ (c1 ;; c2, s) →c (c2, s')
 | *IfTrue*[simp]:
 tval tst s ⇒ (If tst c1 c2, s) →c (c1, s)
 | *IfFalse*[simp]:
 ~ tval tst s ⇒ (If tst c1 c2, s) →c (c2, s)
 | *WhileTrue*[simp]:
 tval tst s ⇒ (While tst c, s) →c (c ;; (While tst c), s)

| *ParCL*[simp]:
 (c1, s) →c (c1', s') ⇒ (Par c1 c2, s) →c (Par c1' c2, s')
 | *ParCR*[simp]:
 (c2, s) →c (c2', s') ⇒ (Par c1 c2, s) →c (Par c1 c2', s')
 | *ParTL*[simp]:
 (c1, s) →t s' ⇒ (Par c1 c2, s) →c (c2, s')
 | *ParTR*[simp]:
 (c2, s) →t s' ⇒ (Par c1 c2, s) →c (c1, s')
 | *Refl*:
 (c,s) →*c (c,s)
 | *Step*:
 [[(c,s) →*c (c',s'); (c',s') →c (c'',s'')] ⇒ (c,s) →*c (c'',s')

lemmas *trans-SeqC* = *SeqC* **lemmas** *trans-SeqT* = *SeqT*
lemmas *trans-IfTrue* = *IfTrue* **lemmas** *trans-IfFalse* = *IfFalse*
lemmas *trans-WhileTrue* = *WhileTrue*
lemmas *trans-ParCL* = *ParCL* **lemmas** *trans-ParCR* = *ParCR*
lemmas *trans-ParTL* = *ParTL* **lemmas** *trans-ParTR* = *ParTR*

lemmas *trans-Refl* = *Refl* **lemmas** *trans-Step* = *Step*

lemma *MtransC-Refl*[*simp*]: $cf \rightarrow^*c cf$
<*proof*>

lemmas *transC-induct* = *transC-MtransC.inducts*(1)
[*split-format*(*complete*),

where $?P2.0 = \lambda c s c' s'. True$]

lemmas *MtransC-induct-temp* = *transC-MtransC.inducts*(2)[*split-format*(*complete*)]

inductive *MtransT* ::

((*'test, 'atom*)*com* * *'state*) \Rightarrow *'state* \Rightarrow *bool*

(**infix** \rightarrow^*t 55)

where

StepT:

$\llbracket cf \rightarrow^*c cf'; cf' \rightarrow t s'' \rrbracket \Longrightarrow cf \rightarrow^*t s''$

lemma *MtransC-rtranclp-transC*:

$MtransC = transC \hat{\ }^*$

<*proof*>

lemma *transC-MtransC*[*simp*]:

assumes $cf \rightarrow c cf'$

shows $cf \rightarrow^*c cf'$

<*proof*>

lemma *MtransC-Trans*:

assumes $cf \rightarrow^*c cf'$ **and** $cf' \rightarrow^*c cf''$

shows $cf \rightarrow^*c cf''$

<*proof*>

lemma *MtransC-StepC*:

assumes $*$: $cf \rightarrow^*c cf'$ **and** $**$: $cf' \rightarrow c cf''$

shows $cf \rightarrow^*c cf''$

<*proof*>

lemma *MtransC-induct*[*consumes* 1, *case-names* *Refl Trans*]:

assumes $cf \rightarrow^*c cf'$

and $\bigwedge cf. phi\ cf\ cf'$

and

$\bigwedge cf\ cf'\ cf''.$

$\llbracket cf \rightarrow^*c cf'; phi\ cf\ cf'; cf' \rightarrow c cf'' \rrbracket$

$\Longrightarrow phi\ cf\ cf''$

shows $phi\ cf\ cf'$

<*proof*>

lemma *MtransC-induct2*[*consumes* 1, *case-names* *Refl Trans*, *induct* *pred*: *MtransC*]:

assumes $(c,s) \rightarrow^*c (c',s')$

and $\bigwedge c\ s. phi\ c\ s\ c\ s$

and

$\bigwedge c\ s\ c'\ s'\ c''\ s''.$
 $\llbracket (c,s) \rightarrow^* c\ (c',s');\ \text{phi}\ c\ s\ c'\ s';\ (c',s') \rightarrow c\ (c'',s'') \rrbracket$
 $\implies \text{phi}\ c\ s\ c''\ s''$
shows $\text{phi}\ c\ s\ c'\ s'$
 $\langle \text{proof} \rangle$

lemma *transT-MtransT[simp]*:

assumes $cf \rightarrow t\ s'$
shows $cf \rightarrow^* t\ s'$
 $\langle \text{proof} \rangle$

lemma *MtransC-MtransT*:

assumes $cf \rightarrow^* c\ cf'$ **and** $cf' \rightarrow^* t\ cf''$
shows $cf \rightarrow^* t\ cf''$
 $\langle \text{proof} \rangle$

lemma *transC-MtransT[simp]*:

assumes $cf \rightarrow c\ cf'$ **and** $cf' \rightarrow^* t\ s''$
shows $cf \rightarrow^* t\ s''$
 $\langle \text{proof} \rangle$

Inversion rules, nchotomies and such:

lemma *Atm-transC-simp[simp]*:

$\sim (Atm\ atm,\ s) \rightarrow c\ cf$
 $\langle \text{proof} \rangle$

lemma *Atm-transC-invert[elim!]*:

assumes $(Atm\ atm,\ s) \rightarrow c\ cf$
shows phi
 $\langle \text{proof} \rangle$

lemma *Atm-transT-invert[elim!]*:

assumes $(Atm\ atm,\ s) \rightarrow t\ s'$
and $s' = \text{aval}\ atm\ s \implies \text{phi}$
shows phi
 $\langle \text{proof} \rangle$

lemma *Seq-transC-invert[elim!]*:

assumes $(c1\ ;;\ c2,\ s) \rightarrow c\ (c',\ s')$
and $\bigwedge c1'. \llbracket (c1,\ s) \rightarrow c\ (c1',s');\ c' = c1' ;;\ c2 \rrbracket \implies \text{phi}$
and $\llbracket (c1,\ s) \rightarrow t\ s';\ c' = c2 \rrbracket \implies \text{phi}$
shows phi
 $\langle \text{proof} \rangle$

lemma *Seq-transT-invert[simp]*:

$\sim (c1\ ;;\ c2,\ s) \rightarrow t\ s'$
 $\langle \text{proof} \rangle$

lemma *If-transC-invert[elim!]*:
assumes $(\text{If } \text{tst } c1 \ c2, s) \rightarrow c (c', s')$
and $\llbracket \text{tval } \text{tst } s; c' = c1; s' = s \rrbracket \Longrightarrow \text{phi}$
and $\llbracket \sim \text{tval } \text{tst } s; c' = c2; s' = s \rrbracket \Longrightarrow \text{phi}$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *If-transT-simp[simp]*:
 $\sim (\text{If } b \ c1 \ c2, s) \rightarrow t \ s'$
 $\langle \text{proof} \rangle$

lemma *If-transT-invert[elim!]*:
assumes $(\text{If } b \ c1 \ c2, s) \rightarrow t \ s'$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *While-transC-invert[elim]*:
assumes $(\text{While } \text{tst } c1, s) \rightarrow c (c', s')$
and $\llbracket \text{tval } \text{tst } s; c' = c1 ;; (\text{While } \text{tst } c1); s' = s \rrbracket \Longrightarrow \text{phi}$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *While-transT-invert[elim!]*:
assumes $(\text{While } \text{tst } c1, s) \rightarrow t \ s'$
and $\llbracket \sim \text{tval } \text{tst } s; s' = s \rrbracket \Longrightarrow \text{phi}$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *Par-transC-invert[elim]*:
assumes $(\text{Par } c1 \ c2, s) \rightarrow c (c', s')$
and $\bigwedge c1'. \llbracket (c1, s) \rightarrow c (c1', s'); c' = \text{Par } c1' \ c2 \rrbracket \Longrightarrow \text{phi}$
and $\llbracket (c1, s) \rightarrow t \ s'; c' = c2 \rrbracket \Longrightarrow \text{phi}$
and $\bigwedge c2'. \llbracket (c2, s) \rightarrow c (c2', s'); c' = \text{Par } c1 \ c2' \rrbracket \Longrightarrow \text{phi}$
and $\llbracket (c2, s) \rightarrow t \ s'; c' = c1 \rrbracket \Longrightarrow \text{phi}$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *Par-transT-simp[simp]*:
 $\sim (\text{Par } c1 \ c2, s) \rightarrow t \ s'$
 $\langle \text{proof} \rangle$

lemma *Par-transT-invert[elim]*:
assumes $(\text{Par } c1 \ c2, s) \rightarrow t \ s'$
shows *phi*
 $\langle \text{proof} \rangle$

lemma *trans-nchotomy*:
 $(\exists c' \ s'. (c, s) \rightarrow c (c', s')) \vee$
 $(\exists s'. (c, s) \rightarrow t \ s')$

$\langle proof \rangle$

corollary *trans-invert*:

assumes

$\bigwedge c' s'. (c, s) \rightarrow c (c', s') \implies phi$

and $\bigwedge s'. (c, s) \rightarrow t s' \implies phi$

shows *phi*

$\langle proof \rangle$

lemma *not-transC-transT*:

$\llbracket cf \rightarrow c cf'; cf \rightarrow t s' \rrbracket \implies phi$

$\langle proof \rangle$

lemmas *MtransT-invert = MtransT.cases*

lemma *MtransT-invert2*:

assumes $(c, s) \rightarrow *t s''$

and $\bigwedge c' s'. \llbracket (c, s) \rightarrow *c (c', s'); (c', s') \rightarrow t s'' \rrbracket \implies phi$

shows *phi*

$\langle proof \rangle$

lemma *Seq-MtransC-invert[elim!]*:

assumes $(c1 ;; c2, s) \rightarrow *c (d', t')$

and $\bigwedge c1'. \llbracket (c1, s) \rightarrow *c (c1', t'); d' = c1' ;; c2 \rrbracket \implies phi$

and $\bigwedge s'. \llbracket (c1, s) \rightarrow *t s'; (c2, s') \rightarrow *c (d', t') \rrbracket \implies phi$

shows *phi*

$\langle proof \rangle$

lemma *Seq-MtransT-invert[elim!]*:

assumes $*(c1 ;; c2, s) \rightarrow *t s''$

and $** : \bigwedge s'. \llbracket (c1, s) \rightarrow *t s'; (c2, s') \rightarrow *t s'' \rrbracket \implies phi$

shows *phi*

$\langle proof \rangle$

Direct rules for the multi-step relations

lemma *Seq-MtransC[simp]*:

assumes $(c1, s) \rightarrow *c (c1', s')$

shows $(c1 ;; c2, s) \rightarrow *c (c1' ;; c2, s')$

$\langle proof \rangle$

lemma *Seq-MtransT-MtransC[simp]*:

assumes $(c1, s) \rightarrow *t s'$

shows $(c1 ;; c2, s) \rightarrow *c (c2, s')$

$\langle proof \rangle$

lemma *ParCL-MtransC[simp]*:

assumes $(c1, s) \rightarrow *c (c1', s')$

shows $(Par c1 c2, s) \rightarrow *c (Par c1' c2, s')$

$\langle proof \rangle$

lemma *ParCR-MtransC[simp]*:
assumes $(c2, s) \rightarrow^* c (c2', s')$
shows $(\text{Par } c1 \ c2, s) \rightarrow^* c (\text{Par } c1 \ c2', s')$
 $\langle \text{proof} \rangle$

lemma *ParTL-MtransC[simp]*:
assumes $(c1, s) \rightarrow^* t s'$
shows $(\text{Par } c1 \ c2, s) \rightarrow^* c (c2, s')$
 $\langle \text{proof} \rangle$

lemma *ParTR-MtransC[simp]*:
assumes $(c2, s) \rightarrow^* t s'$
shows $(\text{Par } c1 \ c2, s) \rightarrow^* c (c1, s')$
 $\langle \text{proof} \rangle$

3.2 Sublanguages

fun *noWhile where*

$\text{noWhile } (\text{Atm } atm) = \text{True}$
 $\text{noWhile } (c1 ;; c2) = (\text{noWhile } c1 \wedge \text{noWhile } c2)$
 $\text{noWhile } (\text{If } b \ c1 \ c2) = (\text{noWhile } c1 \wedge \text{noWhile } c2)$
 $\text{noWhile } (\text{While } b \ c) = \text{False}$
 $\text{noWhile } (\text{Par } c1 \ c2) = (\text{noWhile } c1 \wedge \text{noWhile } c2)$

fun *seq where*

$\text{seq } (\text{Atm } atm) = \text{True}$
 $\text{seq } (c1 ;; c2) = (\text{seq } c1 \wedge \text{seq } c2)$
 $\text{seq } (\text{If } b \ c1 \ c2) = (\text{seq } c1 \wedge \text{seq } c2)$
 $\text{seq } (\text{While } b \ c) = \text{seq } c$
 $\text{seq } (\text{Par } c1 \ c2) = \text{False}$

lemma *noWhile-transC*:

assumes *noWhile* c **and** $(c, s) \rightarrow c (c', s')$
shows *noWhile* c'
 $\langle \text{proof} \rangle$

lemma *seq-transC*:

assumes *seq* c **and** $(c, s) \rightarrow c (c', s')$
shows *seq* c'
 $\langle \text{proof} \rangle$

abbreviation *wfP-on where*

$\text{wfP-on } phi \ A \equiv \text{wfP } (\lambda a \ b. a \in A \wedge b \in A \wedge phi \ a \ b)$

fun *numSt where*

$\text{numSt } (\text{Atm } atm) = \text{Suc } 0$

$|numSt (c1 ;; c2) = numSt c1 + numSt c2$
 $|numSt (If b c1 c2) = 1 + max (numSt c1) (numSt c2)$
 $|numSt (Par c1 c2) = numSt c1 + numSt c2$

lemma *numSt-gt-0[simp]*:
noWhile c $\implies numSt c > 0$
 <proof>

lemma *numSt-transC*:
assumes *noWhile c* **and** $(c,s) \rightarrow c (c',s')$
shows $numSt c' < numSt c$
 <proof>

corollary *wfP-tranC-noWhile*:
 $wfP (\lambda (c',s') (c,s). noWhile c \wedge (c,s) \rightarrow c (c',s'))$
 <proof>

lemma *noWhile-MtransT*:
assumes *noWhile c*
shows $\exists s'. (c,s) \rightarrow^* t s'$
 <proof>

coinductive *mayDiverge* **where**
intro:
 $\llbracket (c,s) \rightarrow c (c',s') \wedge mayDiverge c' s' \rrbracket$
 $\implies mayDiverge c s$

Coinduction for may-diverge :

lemma *mayDiverge-coind[consumes 1, case-names Hyp, induct pred: mayDiverge]*:
assumes *: *phi c s* **and**
 **: $\bigwedge c s. phi c s \implies$
 $\exists c' s'. (c,s) \rightarrow c (c',s') \wedge (phi c' s' \vee mayDiverge c' s')$
shows $mayDiverge c s$
 <proof>

lemma *mayDiverge-raw-coind[consumes 1, case-names Hyp]*:
assumes *: *phi c s* **and**
 **: $\bigwedge c s. phi c s \implies$
 $\exists c' s'. (c,s) \rightarrow c (c',s') \wedge phi c' s'$
shows $mayDiverge c s$
 <proof>

May-diverge versus transition:

lemma *mayDiverge-transC*:
assumes $mayDiverge c s$
shows $\exists c' s'. (c,s) \rightarrow c (c',s') \wedge mayDiverge c' s'$
 <proof>

lemma *transC-mayDiverge*:
assumes $(c,s) \rightarrow c (c',s')$ **and** *mayDiverge* $c' s'$
shows *mayDiverge* $c s$
 $\langle proof \rangle$

lemma *mayDiverge-not-transT*:
assumes *mayDiverge* $c s$
shows $\neg (c,s) \rightarrow t s'$
 $\langle proof \rangle$

lemma *MtransC-mayDiverge*:
assumes $(c,s) \rightarrow *c (c',s')$ **and** *mayDiverge* $c' s'$
shows *mayDiverge* $c s$
 $\langle proof \rangle$

lemma *not-MtransT-mayDiverge*:
assumes $\bigwedge s'. \neg (c,s) \rightarrow *t s'$
shows *mayDiverge* $c s$
 $\langle proof \rangle$

lemma *not-mayDiverge-Atm[simp]*:
 $\neg \text{mayDiverge } (Atm \text{ atm}) s$
 $\langle proof \rangle$

lemma *mayDiverge-Seq-L*:
assumes *mayDiverge* $c1 s$
shows *mayDiverge* $(c1 ;; c2) s$
 $\langle proof \rangle$

lemma *mayDiverge-Seq-R*:
assumes $c1: (c1, s) \rightarrow *t s'$ **and** $c2: \text{mayDiverge } c2 s'$
shows *mayDiverge* $(c1 ;; c2) s$
 $\langle proof \rangle$

lemma *mayDiverge-If-L*:
assumes *tval tst s* **and** *mayDiverge* $c1 s$
shows *mayDiverge* $(If \text{ tst } c1 \text{ } c2) s$
 $\langle proof \rangle$

lemma *mayDiverge-If-R*:
assumes $\neg \text{tval } \text{tst } s$ **and** *mayDiverge* $c2 s$
shows *mayDiverge* $(If \text{ tst } c1 \text{ } c2) s$
 $\langle proof \rangle$

lemma *mayDiverge-If*:
assumes *mayDiverge* $c1 s$ **and** *mayDiverge* $c2 s$
shows *mayDiverge* $(If \text{ tst } c1 \text{ } c2) s$
 $\langle proof \rangle$

lemma *mayDiverge-Par-L*:
assumes *mayDiverge c1 s*
shows *mayDiverge (Par c1 c2) s*
<proof>

lemma *mayDiverge-Par-R*:
assumes *mayDiverge c2 s*
shows *mayDiverge (Par c1 c2) s*
<proof>

definition *mustT* **where**
mustT c s $\equiv \neg$ *mayDiverge c s*

lemma *mustT-transC*:
assumes *mustT c s* **and** $(c,s) \rightarrow c (c',s')$
shows *mustT c' s'*
<proof>

lemma *transT-not-mustT*:
assumes $(c,s) \rightarrow t s'$
shows *mustT c s*
<proof>

lemma *mustT-MtransC*:
assumes *mustT c s* **and** $(c,s) \rightarrow^* c (c',s')$
shows *mustT c' s'*
<proof>

lemma *mustT-MtransT*:
assumes *mustT c s*
shows $\exists s'. (c,s) \rightarrow^* t s'$
<proof>

lemma *mustT-Atm[simp]*:
mustT (Atm atm) s
<proof>

lemma *mustT-Seq-L*:
assumes *mustT (c1 ;; c2) s*
shows *mustT c1 s*
<proof>

lemma *mustT-Seq-R*:
assumes *mustT (c1 ;; c2) s* **and** $(c1, s) \rightarrow^* t s'$
shows *mustT c2 s'*
<proof>

lemma *mustT-If-L*:
assumes *tval tst s* **and** *mustT (If tst c1 c2) s*
shows *mustT c1 s*
 \langle *proof* \rangle

lemma *mustT-If-R*:
assumes \neg *tval tst s* **and** *mustT (If tst c1 c2) s*
shows *mustT c2 s*
 \langle *proof* \rangle

lemma *mustT-If*:
assumes *mustT (If tst c1 c2) s*
shows *mustT c1 s* \vee *mustT c2 s*
 \langle *proof* \rangle

lemma *mustT-Par-L*:
assumes *mustT (Par c1 c2) s*
shows *mustT c1 s*
 \langle *proof* \rangle

lemma *mustT-Par-R*:
assumes *mustT (Par c1 c2) s*
shows *mustT c2 s*
 \langle *proof* \rangle

definition *determOn* **where**
determOn phi r \equiv
 $\forall a b b'. \text{phi } a \wedge r a b \wedge r a b' \longrightarrow b = b'$

lemma *determOn-seq-transT*:
determOn $(\lambda(c,s). \text{seq } c)$ *transT*
 \langle *proof* \rangle

end

end

4 During-execution security

theory *During-Execution*
imports *Bisim Language-Semantics* **begin**

4.1 Basic setting

locale *PL-Indis = PL tval aval*
for

$tval :: 'test \Rightarrow 'state \Rightarrow bool$ **and**
 $aval :: 'atom \Rightarrow 'state \Rightarrow 'state$
 +
fixes
 $indis :: 'state \text{ rel}$
assumes
 $equiv-indis: equiv \ UNIV \ indis$

context $PL-Indis$
begin

abbreviation $indisAbbrev$ (**infix** ≈ 50)
where $s1 \approx s2 \equiv (s1, s2) \in indis$

definition $indisE$ (**infix** $\approx_e 50$) **where**
 $se1 \approx_e se2 \equiv$
case $(se1, se2)$ *of*
 $(Inl \ s1, Inl \ s2) \Rightarrow s1 \approx s2$
 $|(Inr \ err1, Inr \ err2) \Rightarrow err1 = err2$

lemma $refl-indis: refl \ indis$
and $trans-indis: trans \ indis$
and $sym-indis: sym \ indis$
 $\langle proof \rangle$

lemma $indis-refl[intro]: s \approx s$
 $\langle proof \rangle$

lemma $indis-trans: \llbracket s \approx s'; s' \approx s'' \rrbracket \Longrightarrow s \approx s''$
 $\langle proof \rangle$

lemma $indis-sym: s \approx s' \Longrightarrow s' \approx s$
 $\langle proof \rangle$

4.2 Compatibility and discreteness

definition $compatTst$ **where**
 $compatTst \ tst \equiv$
 $\forall \ s \ t. s \approx t \longrightarrow tval \ tst \ s = tval \ tst \ t$

definition $compatAtm$ **where**
 $compatAtm \ atm \equiv$
 $\forall \ s \ t. s \approx t \longrightarrow aval \ atm \ s \approx aval \ atm \ t$

definition $presAtm$ **where**
 $presAtm \ atm \equiv$

$\forall s. s \approx \text{aval atm } s$

coinductive *discr where*

intro:

$\llbracket \bigwedge s c' s'. (c,s) \rightarrow c (c',s') \implies s \approx s' \wedge \text{discr } c';$
 $\bigwedge s s'. (c,s) \rightarrow t s' \implies s \approx s' \rrbracket$
 $\implies \text{discr } c$

lemma *presAtm-compatAtm[simp]:*

assumes *presAtm atm*

shows *compatAtm atm*

<proof>

Coinduction for discreteness:

lemma *discr-coind:*

assumes *: *phi c and*

** : $\bigwedge c s c' s'. \llbracket \text{phi } c; (c,s) \rightarrow c (c',s') \rrbracket \implies s \approx s' \wedge (\text{phi } c' \vee \text{discr } c')$ **and**

*** : $\bigwedge c s s'. \llbracket \text{phi } c; (c,s) \rightarrow t s' \rrbracket \implies s \approx s'$

shows *discr c*

<proof>

lemma *discr-raw-coind:*

assumes *: *phi c and*

** : $\bigwedge c s c' s'. \llbracket \text{phi } c; (c,s) \rightarrow c (c',s') \rrbracket \implies s \approx s' \wedge \text{phi } c'$ **and**

*** : $\bigwedge c s s'. \llbracket \text{phi } c; (c,s) \rightarrow t s' \rrbracket \implies s \approx s'$

shows *discr c*

<proof>

Discreteness versus transition:

lemma *discr-transC:*

assumes *: *discr c and* ** : $(c,s) \rightarrow c (c',s')$

shows *discr c'*

<proof>

lemma *discr-MtransC:*

assumes *discr c and* $(c,s) \rightarrow *c (c',s')$

shows *discr c'*

<proof>

lemma *discr-transC-indis:*

assumes *: *discr c and* ** : $(c,s) \rightarrow c (c',s')$

shows $s \approx s'$

<proof>

lemma *discr-MtransC-indis:*

assumes *discr c and* $(c,s) \rightarrow *c (c',s')$

shows $s \approx s'$

<proof>

lemma *discr-transT*:
assumes *: *discr c* **and** **: $(c,s) \rightarrow t s'$
shows $s \approx s'$
 $\langle proof \rangle$

lemma *discr-MtransT*:
assumes *: *discr c* **and** **: $(c,s) \rightarrow *t s'$
shows $s \approx s'$
 $\langle proof \rangle$

4.3 Terminating-interactive discreteness

coinductive *discr0* **where**

intro:

$\llbracket \bigwedge s c' s'. \llbracket mustT c s; (c,s) \rightarrow c (c',s') \rrbracket \implies s \approx s' \wedge discr0 c' ;$
 $\bigwedge s s'. \llbracket mustT c s; (c,s) \rightarrow t s \rrbracket \implies s \approx s \rrbracket$
 $\implies discr0 c$

Coinduction for 0-discreteness:

lemma *discr0-coind*[*consumes 1, case-names Cont Term, induct pred: discr0*]:

assumes *: *phi c* **and**

** : $\bigwedge c s c' s'.$

$\llbracket mustT c s; phi c; (c,s) \rightarrow c (c',s') \rrbracket \implies$

$s \approx s' \wedge (phi c' \vee discr0 c') \mathbf{and}$

*** : $\bigwedge c s s'. \llbracket mustT c s; phi c; (c,s) \rightarrow t s \rrbracket \implies s \approx s'$

shows *discr0 c*

$\langle proof \rangle$

lemma *discr0-raw-coind*[*consumes 1, case-names Cont Term*]:

assumes *: *phi c* **and**

** : $\bigwedge c s c' s'. \llbracket mustT c s; phi c; (c,s) \rightarrow c (c',s') \rrbracket \implies s \approx s' \wedge phi c' \mathbf{and}$

*** : $\bigwedge c s s'. \llbracket mustT c s; phi c; (c,s) \rightarrow t s \rrbracket \implies s \approx s'$

shows *discr0 c*

$\langle proof \rangle$

0-Discreteness versus transition:

lemma *discr0-transC*:

assumes *: *discr0 c* **and** **: $mustT c s (c,s) \rightarrow c (c',s')$

shows *discr0 c'*

$\langle proof \rangle$

lemma *discr0-MtransC*:

assumes *discr0 c* **and** $mustT c s (c,s) \rightarrow *c (c',s')$

shows *discr0 c'*

$\langle proof \rangle$

lemma *discr0-transC-indis*:

assumes *: *discr0 c* **and** **: $mustT c s (c,s) \rightarrow c (c',s')$

shows $s \approx s'$

$\langle \text{proof} \rangle$

lemma *discr0-MtransC-indis*:

assumes *discr0 c* **and** *mustT c s (c,s) \rightarrow * c (c',s')*

shows $s \approx s'$

$\langle \text{proof} \rangle$

lemma *discr0-transT*:

assumes $*$: *discr0 c* **and** $**$: *mustT c s (c,s) \rightarrow t s'*

shows $s \approx s'$

$\langle \text{proof} \rangle$

lemma *discr0-MtransT*:

assumes $*$: *discr0 c* **and** $***$: *mustT c s* **and** $**$: $(c,s) \rightarrow$ * t s'

shows $s \approx s'$

$\langle \text{proof} \rangle$

lemma *discr-discr0[simp]*: *discr c \implies discr0 c*

$\langle \text{proof} \rangle$

4.4 Self-isomorphism

coinductive *siso* **where**

intro:

$\llbracket \bigwedge s c' s'. (c,s) \rightarrow c (c',s') \implies \text{siso } c' \rrbracket$;

$\bigwedge s t c' s'. \llbracket s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$;

$\bigwedge s t s'. \llbracket s \approx t; (c,s) \rightarrow t s' \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

$\implies \text{siso } c$

Coinduction for self-isomorphism:

lemma *siso-coind*:

assumes $*$: *phi c* **and**

$**$: $\bigwedge c s c' s'. \llbracket \text{phi } c; (c,s) \rightarrow c (c',s') \rrbracket \implies \text{phi } c' \vee \text{siso } c'$ **and**

$***$: $\bigwedge c s t c' s'. \llbracket \text{phi } c; s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$

and

$****$: $\bigwedge c s t s'. \llbracket \text{phi } c; s \approx t; (c,s) \rightarrow t s' \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

shows *siso c*

$\langle \text{proof} \rangle$

lemma *siso-raw-coind*:

assumes $*$: *phi c* **and**

$**$: $\bigwedge c s c' s'. \llbracket \text{phi } c; (c,s) \rightarrow c (c',s') \rrbracket \implies \text{phi } c'$ **and**

$***$: $\bigwedge c s t c' s'. \llbracket \text{phi } c; s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$

and

$****$: $\bigwedge c s t s'. \llbracket \text{phi } c; s \approx t; (c,s) \rightarrow t s' \rrbracket \implies \exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

shows *siso c*

$\langle \text{proof} \rangle$

Self-Isomorphism versus transition:

lemma *siso-transC*:
assumes *: *siso c* **and** **: $(c,s) \rightarrow c (c',s')$
shows *siso c'*
 $\langle proof \rangle$

lemma *siso-MtransC*:
assumes *siso c* **and** $(c,s) \rightarrow *c (c',s')$
shows *siso c'*
 $\langle proof \rangle$

lemma *siso-transC-indis*:
assumes *: *siso c* **and** **: $(c,s) \rightarrow c (c',s')$ **and** ***: $s \approx t$
shows $\exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$
 $\langle proof \rangle$

lemma *siso-transT*:
assumes *: *siso c* **and** **: $(c,s) \rightarrow t s'$ **and** ***: $s \approx t$
shows $\exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$
 $\langle proof \rangle$

4.5 MustT-interactive self-isomorphism

coinductive *siso0* **where**

intro:

$\llbracket \bigwedge s c' s'. \llbracket mustT c s; (c,s) \rightarrow c (c',s') \rrbracket \implies siso0 c';$
 $\bigwedge s t c' s'.$
 $\llbracket mustT c s; mustT c t; s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies$
 $\exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t');$
 $\bigwedge s t s'.$
 $\llbracket mustT c s; mustT c t; s \approx t; (c,s) \rightarrow t s' \rrbracket \implies$
 $\exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$
 $\implies siso0 c$

Coinduction for self-isomorphism:

lemma *siso0-coind*[*consumes 1, case-names Indef Cont Term, induct pred: discr0*]:

assumes *: *phi c* **and**

** : $\bigwedge c s c' s'. \llbracket phi c; mustT c s; (c,s) \rightarrow c (c',s') \rrbracket \implies phi c' \vee siso0 c'$ **and**

*** : $\bigwedge c s t c' s'.$

$\llbracket phi c; mustT c s; mustT c t; s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies$

$\exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$ **and**

**** : $\bigwedge c s t s'.$

$\llbracket mustT c s; mustT c t; phi c; s \approx t; (c,s) \rightarrow t s' \rrbracket \implies$

$\exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

shows *siso0 c*

$\langle proof \rangle$

lemma *siso0-raw-coind*[*consumes 1, case-names Indef Cont Term*]:

assumes *: *phi c* **and**

** : $\bigwedge c s c' s'. \llbracket phi c; mustT c s; (c,s) \rightarrow c (c',s') \rrbracket \implies phi c'$ **and**

***: $\bigwedge c s t c' s'$.
 $\llbracket \text{phi } c; \text{ mustT } c s; \text{ mustT } c t; s \approx t; (c,s) \rightarrow c (c',s') \rrbracket \implies$
 $\exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$ and
****: $\bigwedge c s t s'$.
 $\llbracket \text{phi } c; \text{ mustT } c s; \text{ mustT } c t; s \approx t; (c,s) \rightarrow t s' \rrbracket \implies$
 $\exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

shows *siso0 c*

<proof>

Self-Isomorphism versus transition:

lemma *siso0-transC*:

assumes *: *siso0 c* and **: *mustT c s (c,s) → c (c',s')*

shows *siso0 c'*

<proof>

lemma *siso0-MtransC*:

assumes *siso0 c* and *mustT c s* and $(c,s) \rightarrow *c (c',s')$

shows *siso0 c'*

<proof>

lemma *siso0-transC-indis*:

assumes *: *siso0 c*

and **: *mustT c s mustT c t (c,s) → c (c',s')*

and ***: $s \approx t$

shows $\exists t'. s' \approx t' \wedge (c,t) \rightarrow c (c',t')$

<proof>

lemma *siso0-transT*:

assumes *: *siso0 c*

and **: *mustT c s mustT c t (c,s) → t s'*

and ***: $s \approx t$

shows $\exists t'. s' \approx t' \wedge (c,t) \rightarrow t t'$

<proof>

4.6 Notions of bisimilarity

Matchers:

definition *matchC-C* where

matchC-C *theta* $c d \equiv$

$\forall s t c' s'$

$s \approx t \wedge (c,s) \rightarrow c (c',s')$

\longrightarrow

$(\exists d' t'. (d,t) \rightarrow c (d',t') \wedge s' \approx t' \wedge (c',d') \in \text{theta})$

definition *matchC-ZOC* where

matchC-ZOC *theta* $c d \equiv$

$\forall s t c' s'$

$s \approx t \wedge (c,s) \rightarrow c (c',s')$

\longrightarrow

$$\begin{aligned}
& (s' \approx t \wedge (c', d) \in \text{theta}) \\
& \vee \\
& (\exists d' t'. (d, t) \rightarrow c (d', t') \wedge s' \approx t' \wedge (c', d') \in \text{theta})
\end{aligned}$$

definition *matchC-ZO* where

$$\text{matchC-ZO } \text{theta } c \ d \equiv$$

$$\begin{aligned}
& \forall s \ t \ c' \ s'. \\
& \quad s \approx t \wedge (c, s) \rightarrow c (c', s') \\
& \quad \longrightarrow \\
& \quad (s' \approx t \wedge (c', d) \in \text{theta}) \\
& \quad \vee \\
& \quad (\exists d' t'. (d, t) \rightarrow c (d', t') \wedge s' \approx t' \wedge (c', d') \in \text{theta}) \\
& \quad \vee \\
& \quad (\exists t'. (d, t) \rightarrow t \ t' \wedge s' \approx t' \wedge \text{discr } c')
\end{aligned}$$

definition *matchT-T* where

$$\text{matchT-T } c \ d \equiv$$

$$\begin{aligned}
& \forall s \ t \ s'. \\
& \quad s \approx t \wedge (c, s) \rightarrow t \ s' \\
& \quad \longrightarrow \\
& \quad (\exists t'. (d, t) \rightarrow t \ t' \wedge s' \approx t')
\end{aligned}$$

definition *matchT-ZO* where

$$\text{matchT-ZO } c \ d \equiv$$

$$\begin{aligned}
& \forall s \ t \ s'. \\
& \quad s \approx t \wedge (c, s) \rightarrow t \ s' \\
& \quad \longrightarrow \\
& \quad (s' \approx t \wedge \text{discr } d) \\
& \quad \vee \\
& \quad (\exists d' t'. (d, t) \rightarrow c (d', t') \wedge s' \approx t' \wedge \text{discr } d') \\
& \quad \vee \\
& \quad (\exists t'. (d, t) \rightarrow t \ t' \wedge s' \approx t')
\end{aligned}$$

definition *matchC-MC* where

$$\text{matchC-MC } \text{theta } c \ d \equiv$$

$$\begin{aligned}
& \forall s \ t \ c' \ s'. \\
& \quad s \approx t \wedge (c, s) \rightarrow c (c', s') \\
& \quad \longrightarrow \\
& \quad (\exists d' t'. (d, t) \rightarrow *c (d', t') \wedge s' \approx t' \wedge (c', d') \in \text{theta})
\end{aligned}$$

definition *matchC-TMC* where

$$\text{matchC-TMC } \text{theta } c \ d \equiv$$

$$\begin{aligned}
& \forall s \ t \ c' \ s'. \\
& \quad \text{mustT } c \ s \wedge \text{mustT } d \ t \wedge s \approx t \wedge (c, s) \rightarrow c (c', s') \\
& \quad \longrightarrow \\
& \quad (\exists d' t'. (d, t) \rightarrow *c (d', t') \wedge s' \approx t' \wedge (c', d') \in \text{theta})
\end{aligned}$$

definition *matchC-M* where

matchC-M θ c $d \equiv$

$\forall s t c' s'.$

$s \approx t \wedge (c,s) \rightarrow c (c',s')$

\longrightarrow

$(\exists d' t'. (d,t) \rightarrow *c (d',t') \wedge s' \approx t' \wedge (c',d') \in \theta)$

\vee

$(\exists t'. (d,t) \rightarrow *t t' \wedge s' \approx t' \wedge \text{discr } c')$

definition *matchT-MT* where

matchT-MT c $d \equiv$

$\forall s t s'.$

$s \approx t \wedge (c,s) \rightarrow t s'$

\longrightarrow

$(\exists t'. (d,t) \rightarrow *t t' \wedge s' \approx t')$

definition *matchT-TMT* where

matchT-TMT c $d \equiv$

$\forall s t s'.$

$\text{mustT } c s \wedge \text{mustT } d t \wedge s \approx t \wedge (c,s) \rightarrow t s'$

\longrightarrow

$(\exists t'. (d,t) \rightarrow *t t' \wedge s' \approx t')$

definition *matchT-M* where

matchT-M c $d \equiv$

$\forall s t s'.$

$s \approx t \wedge (c,s) \rightarrow t s'$

\longrightarrow

$(\exists d' t'. (d,t) \rightarrow *c (d',t') \wedge s' \approx t' \wedge \text{discr } d')$

\vee

$(\exists t'. (d,t) \rightarrow *t t' \wedge s' \approx t')$

lemmas *match-defs* =

matchC-C-def

matchC-ZOC-def matchC-ZO-def

matchT-T-def matchT-ZO-def

matchC-MC-def matchC-M-def

matchT-MT-def matchT-M-def

matchC-TMC-def matchT-TMT-def

lemma *matchC-C-def2*:

matchC-C θ d $c =$

$(\forall s t d' t'.$

$s \approx t \wedge (d,t) \rightarrow c (d',t')$

\longrightarrow

$(\exists c' s'. (c,s) \rightarrow c (c',s') \wedge s' \approx t' \wedge (d',c') \in \theta))$

<proof>

lemma *matchC-ZOC-def2*:
matchC-ZOC *theta* *d* *c* =
 $(\forall s t d' t'.$
 $s \approx t \wedge (d,t) \rightarrow c (d',t')$
 \rightarrow
 $(s \approx t' \wedge (d',c) \in \text{theta})$
 \vee
 $(\exists c' s'. (c,s) \rightarrow c (c',s') \wedge s' \approx t' \wedge (d',c') \in \text{theta}))$
 $\langle \text{proof} \rangle$

lemma *matchC-ZO-def2*:
matchC-ZO *theta* *d* *c* =
 $(\forall s t d' t'.$
 $s \approx t \wedge (d,t) \rightarrow c (d',t')$
 \rightarrow
 $(s \approx t' \wedge (d',c) \in \text{theta})$
 \vee
 $(\exists c' s'. (c,s) \rightarrow c (c',s') \wedge s' \approx t' \wedge (d',c') \in \text{theta})$
 \vee
 $(\exists s'. (c,s) \rightarrow t s' \wedge s' \approx t' \wedge \text{discr } d')$
 $\langle \text{proof} \rangle$

lemma *matchT-T-def2*:
matchT-T *d* *c* =
 $(\forall s t t'.$
 $s \approx t \wedge (d,t) \rightarrow t t'$
 \rightarrow
 $(\exists s'. (c,s) \rightarrow t s' \wedge s' \approx t')$
 $\langle \text{proof} \rangle$

lemma *matchT-ZO-def2*:
matchT-ZO *d* *c* =
 $(\forall s t t'.$
 $s \approx t \wedge (d,t) \rightarrow t t'$
 \rightarrow
 $(s \approx t' \wedge \text{discr } c)$
 \vee
 $(\exists c' s'. (c,s) \rightarrow c (c',s') \wedge s' \approx t' \wedge \text{discr } c')$
 \vee
 $(\exists s'. (c,s) \rightarrow t s' \wedge s' \approx t')$
 $\langle \text{proof} \rangle$

lemma *matchC-MC-def2*:
matchC-MC *theta* *d* *c* =
 $(\forall s t d' t'.$
 $s \approx t \wedge (d,t) \rightarrow c (d',t')$

\longrightarrow
 $(\exists c' s'. (c,s) \rightarrow *c (c',s') \wedge s' \approx t' \wedge (d',c') \in \text{theta}))$
 $\langle \text{proof} \rangle$

lemma *matchC-TMC-def2*:
 $\text{matchC-TMC } \text{theta } d \text{ c} =$
 $(\forall s t d' t'.$
 $\text{mustT } c \ s \wedge \text{mustT } d \ t \wedge s \approx t \wedge (d,t) \rightarrow c (d',t')$
 \longrightarrow
 $(\exists c' s'. (c,s) \rightarrow *c (c',s') \wedge s' \approx t' \wedge (d',c') \in \text{theta}))$
 $\langle \text{proof} \rangle$

lemma *matchC-M-def2*:
 $\text{matchC-M } \text{theta } d \ c =$
 $(\forall s t d' t'.$
 $s \approx t \wedge (d,t) \rightarrow c (d',t')$
 \longrightarrow
 $(\exists c' s'. (c,s) \rightarrow *c (c',s') \wedge s' \approx t' \wedge (d',c') \in \text{theta})$
 \vee
 $(\exists s'. (c,s) \rightarrow *t s' \wedge s' \approx t' \wedge \text{discr } d'))$
 $\langle \text{proof} \rangle$

lemma *matchT-MT-def2*:
 $\text{matchT-MT } d \ c =$
 $(\forall s t t'.$
 $s \approx t \wedge (d,t) \rightarrow t t'$
 \longrightarrow
 $(\exists s'. (c,s) \rightarrow *t s' \wedge s' \approx t'))$
 $\langle \text{proof} \rangle$

lemma *matchT-TMT-def2*:
 $\text{matchT-TMT } d \ c =$
 $(\forall s t t'.$
 $\text{mustT } c \ s \wedge \text{mustT } d \ t \wedge s \approx t \wedge (d,t) \rightarrow t t'$
 \longrightarrow
 $(\exists s'. (c,s) \rightarrow *t s' \wedge s' \approx t'))$
 $\langle \text{proof} \rangle$

lemma *matchT-M-def2*:
 $\text{matchT-M } d \ c =$
 $(\forall s t t'.$
 $s \approx t \wedge (d,t) \rightarrow t t'$
 \longrightarrow
 $(\exists c' s'. (c,s) \rightarrow *c (c',s') \wedge s' \approx t' \wedge \text{discr } c')$
 \vee
 $(\exists s'. (c,s) \rightarrow *t s' \wedge s' \approx t'))$
 $\langle \text{proof} \rangle$

Retracts:

definition *Sretr* **where**

$$\begin{aligned} Sretr\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-C\ theta\ c\ d \wedge \\ &\quad matchT-T\ c\ d\} \end{aligned}$$

definition *ZOretr* **where**

$$\begin{aligned} ZOretr\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-ZO\ theta\ c\ d \wedge \\ &\quad matchT-ZO\ c\ d\} \end{aligned}$$

definition *ZOretrT* **where**

$$\begin{aligned} ZOretrT\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-ZOC\ theta\ c\ d \wedge \\ &\quad matchT-T\ c\ d\} \end{aligned}$$

definition *Wretr* **where**

$$\begin{aligned} Wretr\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-M\ theta\ c\ d \wedge \\ &\quad matchT-M\ c\ d\} \end{aligned}$$

definition *WretrT* **where**

$$\begin{aligned} WretrT\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-MC\ theta\ c\ d \wedge \\ &\quad matchT-MT\ c\ d\} \end{aligned}$$

definition *RetrT* **where**

$$\begin{aligned} RetrT\ theta &\equiv \\ &\{(c,d). \\ &\quad matchC-TMC\ theta\ c\ d \wedge \\ &\quad matchT-TMT\ c\ d\} \end{aligned}$$

lemmas *Retr-defs* =

$$\begin{aligned} &Sretr-def \\ &ZOretr-def\ ZOretrT-def \\ &Wretr-def\ WretrT-def \\ &RetrT-def \end{aligned}$$

The associated bisimilarity relations:

definition *Sbis* **where** $Sbis \equiv bis\ Sretr$

definition $ZObis$ **where** $ZObis \equiv bis\ ZOretr$
definition $ZObisT$ **where** $ZObisT \equiv bis\ ZOretrT$
definition $Wbis$ **where** $Wbis \equiv bis\ Wretr$
definition $WbisT$ **where** $WbisT \equiv bis\ WretrT$
definition $BisT$ **where** $BisT \equiv bis\ RetrT$

lemmas $bis-defs =$
 $Sbis-def$
 $ZObis-def\ ZObisT-def$
 $Wbis-def\ WbisT-def$
 $BisT-def$

abbreviation $Sbis-abbrev$ (**infix** \approx_s 55) **where** $c1 \approx_s c2 \equiv (c1, c2) \in Sbis$
abbreviation $ZObis-abbrev$ (**infix** \approx_{01} 55) **where** $c1 \approx_{01} c2 \equiv (c1, c2) \in ZObis$
abbreviation $ZObisT-abbrev$ (**infix** \approx_{01T} 55) **where** $c1 \approx_{01T} c2 \equiv (c1, c2) \in ZObisT$
abbreviation $Wbis-abbrev$ (**infix** \approx_w 55) **where** $c1 \approx_w c2 \equiv (c1, c2) \in Wbis$
abbreviation $WbisT-abbrev$ (**infix** \approx_{wT} 55) **where** $c1 \approx_{wT} c2 \equiv (c1, c2) \in WbisT$
abbreviation $BisT-abbrev$ (**infix** \approx_T 55) **where** $c1 \approx_T c2 \equiv (c1, c2) \in BisT$

lemma $mono-Retr$:
 $mono\ Sretr$
 $mono\ ZOretr\ mono\ ZOretrT$
 $mono\ Wretr\ mono\ WretrT$
 $mono\ RetrT$
 $\langle proof \rangle$

lemma $Sbis-prefix$:
 $Sbis \subseteq Sretr\ Sbis$
 $\langle proof \rangle$

lemma $Sbis-sym$: $sym\ Sbis$
 $\langle proof \rangle$

lemma $Sbis-Sym$: $c \approx_s d \implies d \approx_s c$
 $\langle proof \rangle$

lemma $Sbis-converse$:
 $((c, d) \in theta^{-1} \cup Sbis) = ((d, c) \in theta \cup Sbis)$
 $\langle proof \rangle$

lemma
 $Sbis-matchC-C$: $\bigwedge s\ t. c \approx_s d \implies matchC-C\ Sbis\ c\ d$
and
 $Sbis-matchT-T$: $\bigwedge c\ d. c \approx_s d \implies matchT-T\ c\ d$
 $\langle proof \rangle$

lemmas $Sbis\text{-step} = Sbis\text{-matchC-C } Sbis\text{-matchT-T}$

lemma

$Sbis\text{-matchC-C-rev}: \bigwedge s t. s \approx s t \implies matchC-C Sbis t s$

and

$Sbis\text{-matchT-T-rev}: \bigwedge s t. s \approx s t \implies matchT-T t s$

$\langle proof \rangle$

lemmas $Sbis\text{-step-rev} = Sbis\text{-matchC-C-rev } Sbis\text{-matchT-T-rev}$

lemma $Sbis\text{-coind}$:

assumes $sym\ theta$ **and** $theta \subseteq Sretr (theta \cup Sbis)$

shows $theta \subseteq Sbis$

$\langle proof \rangle$

lemma $Sbis\text{-raw-coind}$:

assumes $sym\ theta$ **and** $theta \subseteq Sretr\ theta$

shows $theta \subseteq Sbis$

$\langle proof \rangle$

lemma $Sbis\text{-coind2}$:

assumes $theta \subseteq Sretr (theta \cup Sbis)$ **and**

$theta \hat{-} 1 \subseteq Sretr ((theta \hat{-} 1) \cup Sbis)$

shows $theta \subseteq Sbis$

$\langle proof \rangle$

lemma $Sbis\text{-raw-coind2}$:

assumes $theta \subseteq Sretr\ theta$ **and**

$theta \hat{-} 1 \subseteq Sretr (theta \hat{-} 1)$

shows $theta \subseteq Sbis$

$\langle proof \rangle$

lemma $ZObis\text{-prefix}$:

$ZObis \subseteq ZOretr\ ZObis$

$\langle proof \rangle$

lemma $ZObis\text{-sym}$: $sym\ ZObis$

$\langle proof \rangle$

lemma $ZObis\text{-converse}$:

$((c,d) \in theta \hat{-} 1 \cup ZObis) = ((d,c) \in theta \cup ZObis)$

$\langle proof \rangle$

lemma $ZObis\text{-Sym}$: $s \approx 01 t \implies t \approx 01 s$

$\langle proof \rangle$

lemma

ZObis-matchC-ZO: $\bigwedge s t. s \approx 01 t \implies \text{matchC-ZO } ZObis s t$

and

ZObis-matchT-ZO: $\bigwedge s t. s \approx 01 t \implies \text{matchT-ZO } s t$

<proof>

lemmas *ZObis-step* = *ZObis-matchC-ZO* *ZObis-matchT-ZO*

lemma

ZObis-matchC-ZO-rev: $\bigwedge s t. s \approx 01 t \implies \text{matchC-ZO } ZObis t s$

and

ZObis-matchT-ZO-rev: $\bigwedge s t. s \approx 01 t \implies \text{matchT-ZO } t s$

<proof>

lemmas *ZObis-step-rev* = *ZObis-matchC-ZO-rev* *ZObis-matchT-ZO-rev*

lemma *ZObis-coind*:

assumes *sym theta* **and** $theta \subseteq ZOrtr (theta \cup ZObis)$

shows $theta \subseteq ZObis$

<proof>

lemma *ZObis-raw-coind*:

assumes *sym theta* **and** $theta \subseteq ZOrtr theta$

shows $theta \subseteq ZObis$

<proof>

lemma *ZObis-coind2*:

assumes $theta \subseteq ZOrtr (theta \cup ZObis)$ **and**

$theta \hat{-} 1 \subseteq ZOrtr ((theta \hat{-} 1) \cup ZObis)$

shows $theta \subseteq ZObis$

<proof>

lemma *ZObis-raw-coind2*:

assumes $theta \subseteq ZOrtr theta$ **and**

$theta \hat{-} 1 \subseteq ZOrtr (theta \hat{-} 1)$

shows $theta \subseteq ZObis$

<proof>

lemma *ZObisT-prefix*:

$ZObisT \subseteq ZOrtrT ZObisT$

<proof>

lemma *ZObisT-sym*: *sym ZObisT*

<proof>

lemma *ZObisT-Sym*: $s \approx 01T t \implies t \approx 01T s$

<proof>

lemma *ZObisT-converse*:

$((c,d) \in \text{theta}^{\wedge-1} \cup \text{ZObisT}) = ((d,c) \in \text{theta} \cup \text{ZObisT})$
 $\langle \text{proof} \rangle$

lemma

$\text{ZObisT-matchC-ZOC}: \bigwedge s t. s \approx 01T t \implies \text{matchC-ZOC ZObisT } s t$

and

$\text{ZObisT-matchT-T}: \bigwedge s t. s \approx 01T t \implies \text{matchT-T } s t$

$\langle \text{proof} \rangle$

lemmas $\text{ZObisT-step} = \text{ZObisT-matchC-ZOC ZObisT-matchT-T}$

lemma

$\text{ZObisT-matchC-ZOC-rev}: \bigwedge s t. s \approx 01T t \implies \text{matchC-ZOC ZObisT } t s$

and

$\text{ZObisT-matchT-T-rev}: \bigwedge s t. s \approx 01T t \implies \text{matchT-T } t s$

$\langle \text{proof} \rangle$

lemmas $\text{ZObisT-step-rev} = \text{ZObisT-matchC-ZOC-rev ZObisT-matchT-T-rev}$

lemma ZObisT-coind :

assumes $\text{sym } \text{theta}$ **and** $\text{theta} \subseteq \text{ZOretrT } (\text{theta} \cup \text{ZObisT})$

shows $\text{theta} \subseteq \text{ZObisT}$

$\langle \text{proof} \rangle$

lemma ZObisT-raw-coind :

assumes $\text{sym } \text{theta}$ **and** $\text{theta} \subseteq \text{ZOretrT } \text{theta}$

shows $\text{theta} \subseteq \text{ZObisT}$

$\langle \text{proof} \rangle$

lemma ZObisT-coind2 :

assumes $\text{theta} \subseteq \text{ZOretrT } (\text{theta} \cup \text{ZObisT})$ **and**

$\text{theta}^{\wedge-1} \subseteq \text{ZOretrT } ((\text{theta}^{\wedge-1}) \cup \text{ZObisT})$

shows $\text{theta} \subseteq \text{ZObisT}$

$\langle \text{proof} \rangle$

lemma ZObisT-raw-coind2 :

assumes $\text{theta} \subseteq \text{ZOretrT } \text{theta}$ **and**

$\text{theta}^{\wedge-1} \subseteq \text{ZOretrT } (\text{theta}^{\wedge-1})$

shows $\text{theta} \subseteq \text{ZObisT}$

$\langle \text{proof} \rangle$

lemma Wbis-prefix :

$\text{Wbis} \subseteq \text{Wretr } \text{Wbis}$

$\langle \text{proof} \rangle$

lemma Wbis-sym : $\text{sym } \text{Wbis}$

$\langle \text{proof} \rangle$

lemma *Wbis-converse*:
 $((c,d) \in \text{theta}^{-1} \cup \text{Wbis}) = ((d,c) \in \text{theta} \cup \text{Wbis})$
 $\langle \text{proof} \rangle$

lemma *Wbis-Sym*: $c \approx_w d \implies d \approx_w c$
 $\langle \text{proof} \rangle$

lemma
Wbis-matchC-M: $\bigwedge c d. c \approx_w d \implies \text{matchC-M } \text{Wbis } c d$
and
Wbis-matchT-M: $\bigwedge c d. c \approx_w d \implies \text{matchT-M } c d$
 $\langle \text{proof} \rangle$

lemmas $\text{Wbis-step} = \text{Wbis-matchC-M } \text{Wbis-matchT-M}$

lemma
Wbis-matchC-M-rev: $\bigwedge s t. s \approx_w t \implies \text{matchC-M } \text{Wbis } t s$
and
Wbis-matchT-M-rev: $\bigwedge s t. s \approx_w t \implies \text{matchT-M } t s$
 $\langle \text{proof} \rangle$

lemmas $\text{Wbis-step-rev} = \text{Wbis-matchC-M-rev } \text{Wbis-matchT-M-rev}$

lemma *Wbis-coind*:
assumes $\text{sym } \text{theta}$ **and** $\text{theta} \subseteq \text{Wretr } (\text{theta} \cup \text{Wbis})$
shows $\text{theta} \subseteq \text{Wbis}$
 $\langle \text{proof} \rangle$

lemma *Wbis-raw-coind*:
assumes $\text{sym } \text{theta}$ **and** $\text{theta} \subseteq \text{Wretr } \text{theta}$
shows $\text{theta} \subseteq \text{Wbis}$
 $\langle \text{proof} \rangle$

lemma *Wbis-coind2*:
assumes $\text{theta} \subseteq \text{Wretr } (\text{theta} \cup \text{Wbis})$ **and**
 $\text{theta}^{-1} \subseteq \text{Wretr } ((\text{theta}^{-1}) \cup \text{Wbis})$
shows $\text{theta} \subseteq \text{Wbis}$
 $\langle \text{proof} \rangle$

lemma *Wbis-raw-coind2*:
assumes $\text{theta} \subseteq \text{Wretr } \text{theta}$ **and**
 $\text{theta}^{-1} \subseteq \text{Wretr } (\text{theta}^{-1})$
shows $\text{theta} \subseteq \text{Wbis}$
 $\langle \text{proof} \rangle$

lemma *WbisT-prefix*:
 $\text{WbisT} \subseteq \text{WretrT } \text{WbisT}$
 $\langle \text{proof} \rangle$

lemma *WbisT-sym: sym WbisT*

<proof>

lemma *WbisT-Sym: $c \approx_{wT} d \implies d \approx_{wT} c$*

<proof>

lemma *WbisT-converse:*

$((c,d) \in \text{theta}^{-1} \cup \text{WbisT}) = ((d,c) \in \text{theta} \cup \text{WbisT})$

<proof>

lemma

WbisT-matchC-MC: $\bigwedge c d. c \approx_{wT} d \implies \text{matchC-MC } \text{WbisT } c d$

and

WbisT-matchT-MT: $\bigwedge c d. c \approx_{wT} d \implies \text{matchT-MT } c d$

<proof>

lemmas *WbisT-step = WbisT-matchC-MC WbisT-matchT-MT*

lemma

WbisT-matchC-MC-rev: $\bigwedge s t. s \approx_{wT} t \implies \text{matchC-MC } \text{WbisT } t s$

and

WbisT-matchT-MT-rev: $\bigwedge s t. s \approx_{wT} t \implies \text{matchT-MT } t s$

<proof>

lemmas *WbisT-step-rev = WbisT-matchC-MC-rev WbisT-matchT-MT-rev*

lemma *WbisT-coind:*

assumes *sym theta and $\text{theta} \subseteq \text{WretrT } (\text{theta} \cup \text{WbisT})$*

shows *$\text{theta} \subseteq \text{WbisT}$*

<proof>

lemma *WbisT-raw-coind:*

assumes *sym theta and $\text{theta} \subseteq \text{WretrT } \text{theta}$*

shows *$\text{theta} \subseteq \text{WbisT}$*

<proof>

lemma *WbisT-coind2:*

assumes *$\text{theta} \subseteq \text{WretrT } (\text{theta} \cup \text{WbisT})$ and*

$\text{theta}^{-1} \subseteq \text{WretrT } ((\text{theta}^{-1}) \cup \text{WbisT})$

shows *$\text{theta} \subseteq \text{WbisT}$*

<proof>

lemma *WbisT-raw-coind2:*

assumes *$\text{theta} \subseteq \text{WretrT } \text{theta}$ and*

$\text{theta}^{-1} \subseteq \text{WretrT } (\text{theta}^{-1})$

shows *$\text{theta} \subseteq \text{WbisT}$*

<proof>

lemma *WbisT-coinduct*[consumes 1, case-names sym cont term]:

assumes $\varphi: \varphi\ c\ d$
assumes $S: \bigwedge c\ d. \varphi\ c\ d \implies \varphi\ d\ c$
assumes $C: \bigwedge c\ s\ d\ t\ c'\ s'.$
 $\llbracket \varphi\ c\ d ; s \approx t ; (c, s) \rightarrow c\ (c', s') \rrbracket \implies \exists d'\ t'. (d, t) \rightarrow^* c\ (d', t') \wedge s' \approx t' \wedge$
 $(\varphi\ c'\ d' \vee c' \approx_w T\ d')$
assumes $T: \bigwedge c\ s\ d\ t\ s'.$
 $\llbracket \varphi\ c\ d ; s \approx t ; (c, s) \rightarrow t\ s' \rrbracket \implies \exists t'. (d, t) \rightarrow^* t' \wedge s' \approx t'$
shows $c \approx_w T\ d$
 $\langle proof \rangle$

lemma *BisT-prefix*:

$BisT \subseteq RetrT\ BisT$

$\langle proof \rangle$

lemma *BisT-sym*: $sym\ BisT$

$\langle proof \rangle$

lemma *BisT-Sym*: $c \approx T\ d \implies d \approx T\ c$

$\langle proof \rangle$

lemma *BisT-converse*:

$((c, d) \in \widehat{theta^{-1}} \cup BisT) = ((d, c) \in \widehat{theta} \cup BisT)$

$\langle proof \rangle$

lemma

BisT-matchC-TMC: $\bigwedge c\ d. c \approx T\ d \implies matchC-TMC\ BisT\ c\ d$

and

BisT-matchT-TMT: $\bigwedge c\ d. c \approx T\ d \implies matchT-TMT\ c\ d$

$\langle proof \rangle$

lemmas $BisT-step = BisT-matchC-TMC\ BisT-matchT-TMT$

lemma

BisT-matchC-TMC-rev: $\bigwedge c\ d. c \approx T\ d \implies matchC-TMC\ BisT\ d\ c$

and

BisT-matchT-TMT-rev: $\bigwedge c\ d. c \approx T\ d \implies matchT-TMT\ d\ c$

$\langle proof \rangle$

lemmas $BisT-step-rev = BisT-matchC-TMC-rev\ BisT-matchT-TMT-rev$

lemma *BisT-coind*:

assumes $sym\ \widehat{theta}$ **and** $\widehat{theta} \subseteq RetrT\ (\widehat{theta} \cup BisT)$

shows $\widehat{theta} \subseteq BisT$

$\langle proof \rangle$

lemma *BisT-raw-coind*:

assumes $sym\ \widehat{theta}$ **and** $\widehat{theta} \subseteq RetrT\ \widehat{theta}$

shows $\theta \subseteq \text{BisT}$
<proof>

lemma *BisT-coind2*:
assumes $\theta \subseteq \text{RetrT } (\theta \cup \text{BisT})$ **and**
 $\theta^{-1} \subseteq \text{RetrT } ((\theta^{-1}) \cup \text{BisT})$
shows $\theta \subseteq \text{BisT}$
<proof>

lemma *BisT-raw-coind2*:
assumes $\theta \subseteq \text{RetrT } \theta$ **and**
 $\theta^{-1} \subseteq \text{RetrT } (\theta^{-1})$
shows $\theta \subseteq \text{BisT}$
<proof>

Inclusions between bisimilarities:

lemma *match-imp[simp]*:
 $\bigwedge \theta \ c1 \ c2. \text{matchC-C } \theta \ c1 \ c2 \implies \text{matchC-ZOC } \theta \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchC-ZOC } \theta \ c1 \ c2 \implies \text{matchC-ZO } \theta \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchC-ZOC } \theta \ c1 \ c2 \implies \text{matchC-MC } \theta \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchC-ZO } \theta \ c1 \ c2 \implies \text{matchC-M } \theta \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchC-MC } \theta \ c1 \ c2 \implies \text{matchC-M } \theta \ c1 \ c2$

 $\bigwedge \ c1 \ c2. \text{matchT-T } \ c1 \ c2 \implies \text{matchT-ZO } \ c1 \ c2$
 $\bigwedge \ c1 \ c2. \text{matchT-T } \ c1 \ c2 \implies \text{matchT-MT } \ c1 \ c2$
 $\bigwedge \ c1 \ c2. \text{matchT-ZO } \ c1 \ c2 \implies \text{matchT-M } \ c1 \ c2$
 $\bigwedge \ c1 \ c2. \text{matchT-MT } \ c1 \ c2 \implies \text{matchT-M } \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchC-MC } \theta \ c1 \ c2 \implies \text{matchC-TMC } \theta \ c1 \ c2$
 $\bigwedge \theta \ c1 \ c2. \text{matchT-MT } \ c1 \ c2 \implies \text{matchT-TMT } \ c1 \ c2$
<proof>

lemma *Retr-incl*:
 $\bigwedge \theta. \text{Sretr } \theta \subseteq \text{ZOretrT } \theta$
 $\bigwedge \theta. \text{ZOretrT } \theta \subseteq \text{ZOretr } \theta$
 $\bigwedge \theta. \text{ZOretrT } \theta \subseteq \text{WretrT } \theta$
 $\bigwedge \theta. \text{ZOretr } \theta \subseteq \text{Wretr } \theta$

$\bigwedge \text{theta}. \text{Wretr}T \text{ theta} \subseteq \text{Wretr} \text{ theta}$

$\bigwedge \text{theta}. \text{Wretr}T \text{ theta} \subseteq \text{Retr}T \text{ theta}$
 $\langle \text{proof} \rangle$

lemma *bis-incl*:
 $S\text{bis} \subseteq Z\text{Obis}T$

$Z\text{Obis}T \subseteq Z\text{Obis}$

$Z\text{Obis}T \subseteq W\text{bis}T$

$Z\text{Obis} \subseteq W\text{bis}$

$W\text{bis}T \subseteq W\text{bis}$

$W\text{bis}T \subseteq \text{Bis}T$
 $\langle \text{proof} \rangle$

lemma *bis-imp[simp]*:
 $\bigwedge c1 c2. c1 \approx_s c2 \implies c1 \approx_{01T} c2$

$\bigwedge c1 c2. c1 \approx_{01T} c2 \implies c1 \approx_{01} c2$

$\bigwedge c1 c2. c1 \approx_{01T} c2 \implies c1 \approx_{wT} c2$

$\bigwedge c1 c2. c1 \approx_{01} c2 \implies c1 \approx_w c2$

$\bigwedge c1 c2. c1 \approx_{wT} c2 \implies c1 \approx_w c2$

$\bigwedge c1 c2. c1 \approx_{wT} c2 \implies c1 \approx_T c2$
 $\langle \text{proof} \rangle$

Self-isomorphism implies strong bisimilarity:

lemma *iso-Sbis[simp]*:
assumes *iso* *c*
shows $c \approx_s c$
 $\langle \text{proof} \rangle$

0-Self-isomorphism implies weak T 0-bisimilarity:

lemma *iso0-Sbis[simp]*:
assumes *iso0* *c*
shows $c \approx_T c$
 $\langle \text{proof} \rangle$

end

end

5 Compositionality of the during-execution security notions

theory *Compositionality* **imports** *During-Execution* **begin**

context *PL-Indis*
begin

5.1 Discreetness versus language constructs:

theorem *discr-Atm[simp]*:
 $discr (Atm\ atm) = presAtm\ atm$
<proof>

theorem *discr-If[simp]*:
assumes *discr c1* **and** *discr c2*
shows *discr (If\ tst\ c1\ c2)*
<proof>

theorem *discr-Seq[simp]*:
assumes *: *discr c1* **and** **: *discr c2*
shows *discr (c1\ ;;\ c2)*
<proof>

theorem *discr-While[simp]*:
assumes *discr c*
shows *discr (While\ tst\ c)*
<proof>

theorem *discr-Par[simp]*:
assumes *: *discr c1* **and** **: *discr c2*
shows *discr (Par\ c1\ c2)*
<proof>

5.2 Discreetness versus language constructs:

theorem *discr0-Atm[simp]*:
 $discr0 (Atm\ atm) = presAtm\ atm$
<proof>

theorem *discr0-If[simp]*:
assumes *discr0 c1* **and** *discr0 c2*
shows *discr0 (If\ tst\ c1\ c2)*

<proof>

theorem *discr0-Seq[simp]*:
assumes *: *discr0 c1* **and** **: *discr0 c2*
shows *discr0 (c1 ;; c2)*
<proof>

theorem *discr0-While[simp]*:
assumes *discr0 c*
shows *discr0 (While tst c)*
<proof>

theorem *discr0-Par[simp]*:
assumes *: *discr0 c1* **and** **: *discr0 c2*
shows *discr0 (Par c1 c2)*
<proof>

5.3 Self-Isomorphism versus language constructs:

theorem *siso-Atm[simp]*:
siso (Atm atm) = compatAtm atm
<proof>

theorem *siso-If[simp]*:
assumes *compatTst tst* **and** *siso c1* **and** *siso c2*
shows *siso (If tst c1 c2)*
<proof>

theorem *siso-Seq[simp]*:
assumes *: *siso c1* **and** **: *siso c2*
shows *siso (c1 ;; c2)*
<proof>

theorem *siso-While[simp]*:
assumes *compatTst tst* **and** *siso c*
shows *siso (While tst c)*
<proof>

theorem *siso-Par[simp]*:
assumes *: *siso c1* **and** **: *siso c2*
shows *siso (Par c1 c2)*
<proof>

5.4 Self-Isomorphism versus language constructs:

theorem *siso0-Atm[simp]*:
siso0 (Atm atm) = compatAtm atm
<proof>

theorem *siso0-If[simp]*:

assumes *compatTst tst* **and** *siso0 c1* **and** *siso0 c2*
shows *siso0 (If tst c1 c2)*
 \langle *proof* \rangle

theorem *siso0-Seq[simp]*:
assumes *: *siso0 c1* **and** **: *siso0 c2*
shows *siso0 (c1 ;; c2)*
 \langle *proof* \rangle

theorem *siso0-While[simp]*:
assumes *compatTst tst* **and** *siso0 c*
shows *siso0 (While tst c)*
 \langle *proof* \rangle

theorem *siso0-Par[simp]*:
assumes *: *siso0 c1* **and** **: *siso0 c2*
shows *siso0 (Par c1 c2)*
 \langle *proof* \rangle

5.5 Strong bisimilarity versus language constructs

Atomic commands:

definition *thetaAtm* **where**
thetaAtm atm $\equiv \{(Atm\ atm, Atm\ atm)\}$

lemma *thetaAtm-sym*:
sym (thetaAtm atm)
 \langle *proof* \rangle

lemma *thetaAtm-Sretr*:
assumes *compatAtm atm*
shows *thetaAtm atm* \subseteq *Sretr (thetaAtm atm)*
 \langle *proof* \rangle

lemma *thetaAtm-Sbis*:
assumes *compatAtm atm*
shows *thetaAtm atm* \subseteq *Sbis*
 \langle *proof* \rangle

theorem *Atm-Sbis[simp]*:
assumes *compatAtm atm*
shows *Atm atm* \approx_s *Atm atm*
 \langle *proof* \rangle

Sequential composition:

definition *thetaSeq* **where**
thetaSeq \equiv
 $\{(c1\ ;;\ c2, d1\ ;;\ d2) \mid c1\ c2\ d1\ d2. c1\ \approx_s\ d1\ \wedge\ c2\ \approx_s\ d2\}$

lemma *thetaSeq-sym*:

sym thetaSeq

<proof>

lemma *thetaSeq-Sretr*:

thetaSeq \subseteq *Sretr* (*thetaSeq Un Sbis*)

<proof>

lemma *thetaSeq-Sbis*:

thetaSeq \subseteq *Sbis*

<proof>

theorem *Seq-Sbis[simp]*:

assumes *c1* \approx_s *d1* **and** *c2* \approx_s *d2*

shows *c1* ;; *c2* \approx_s *d1* ;; *d2*

<proof>

Conditional:

definition *thetaIf* **where**

thetaIf \equiv

$\{(If\ tst\ c1\ c2,\ If\ tst\ d1\ d2) \mid tst\ c1\ c2\ d1\ d2.\ compatTst\ tst \wedge c1 \approx_s d1 \wedge c2 \approx_s d2\}$

lemma *thetaIf-sym*:

sym thetaIf

<proof>

lemma *thetaIf-Sretr*:

thetaIf \subseteq *Sretr* (*thetaIf Un Sbis*)

<proof>

lemma *thetaIf-Sbis*:

thetaIf \subseteq *Sbis*

<proof>

theorem *If-Sbis[simp]*:

assumes *compatTst tst* **and** *c1* \approx_s *d1* **and** *c2* \approx_s *d2*

shows *If tst c1 c2* \approx_s *If tst d1 d2*

<proof>

While loop:

definition *thetaWhile* **where**

thetaWhile \equiv

$\{(While\ tst\ c,\ While\ tst\ d) \mid tst\ c\ d.\ compatTst\ tst \wedge c \approx_s d\} Un$
 $\{(c1\ ;;\ (While\ tst\ c),\ d1\ ;;\ (While\ tst\ d)) \mid tst\ c1\ d1\ c\ d.\ compatTst\ tst \wedge c1 \approx_s d1 \wedge c \approx_s d\}$

lemma *thetaWhile-sym*:

sym thetaWhile

<proof>

lemma *thetaWhile-Sretr*:

thetaWhile \subseteq *Sretr* (*thetaWhile Un Sbis*)

<proof>

lemma *thetaWhile-Sbis*:

thetaWhile \subseteq *Sbis*

<proof>

theorem *While-Sbis[simp]*:

assumes *compatTst tst* **and** $c \approx_s d$

shows *While tst c* \approx_s *While tst d*

<proof>

Parallel composition:

definition *thetaPar* **where**

thetaPar \equiv

$\{(Par\ c1\ c2, Par\ d1\ d2) \mid c1\ c2\ d1\ d2. c1 \approx_s d1 \wedge c2 \approx_s d2\}$

lemma *thetaPar-sym*:

sym thetaPar

<proof>

lemma *thetaPar-Sretr*:

thetaPar \subseteq *Sretr* (*thetaPar Un Sbis*)

<proof>

lemma *thetaPar-Sbis*:

thetaPar \subseteq *Sbis*

<proof>

theorem *Par-Sbis[simp]*:

assumes $c1 \approx_s d1$ **and** $c2 \approx_s d2$

shows *Par c1 c2* \approx_s *Par d1 d2*

<proof>

5.5.1 01T-bisimilarity versus language constructs

Atomic commands:

theorem *Atm-ZObisT*:

assumes *compatAtm atm*

shows *Atm atm* \approx_{01T} *Atm atm*

<proof>

Sequential composition:

definition *thetaSeqZOT* **where**

thetaSeqZOT \equiv

$\{(c1 ;; c2, d1 ;; d2) \mid c1 \ c2 \ d1 \ d2. \ c1 \approx_{01T} \ d1 \ \wedge \ c2 \approx_{01T} \ d2\}$

lemma *thetaSeqZOT-sym*:
sym thetaSeqZOT
(proof)

lemma *thetaSeqZOT-ZOretrT*:
thetaSeqZOT \subseteq *ZOretrT* (*thetaSeqZOT* *Un* *ZObisT*)
(proof)

lemma *thetaSeqZOT-ZObisT*:
thetaSeqZOT \subseteq *ZObisT*
(proof)

theorem *Seq-ZObisT[simp]*:
assumes *c1* \approx_{01T} *d1* **and** *c2* \approx_{01T} *d2*
shows *c1* ;; *c2* \approx_{01T} *d1* ;; *d2*
(proof)

Conditional:

definition *thetaIfZOT* **where**
thetaIfZOT \equiv
 $\{(If \ tst \ c1 \ c2, \ If \ tst \ d1 \ d2) \mid \ tst \ c1 \ c2 \ d1 \ d2. \ compatTst \ tst \ \wedge \ c1 \approx_{01T} \ d1 \ \wedge \ c2 \approx_{01T} \ d2\}$

lemma *thetaIfZOT-sym*:
sym thetaIfZOT
(proof)

lemma *thetaIfZOT-ZOretrT*:
thetaIfZOT \subseteq *ZOretrT* (*thetaIfZOT* *Un* *ZObisT*)
(proof)

lemma *thetaIfZOT-ZObisT*:
thetaIfZOT \subseteq *ZObisT*
(proof)

theorem *If-ZObisT[simp]*:
assumes *compatTst* *tst* **and** *c1* \approx_{01T} *d1* **and** *c2* \approx_{01T} *d2*
shows *If* *tst* *c1* *c2* \approx_{01T} *If* *tst* *d1* *d2*
(proof)

While loop:

definition *thetaWhileZOT* **where**
thetaWhileZOT \equiv
 $\{(While \ tst \ c, \ While \ tst \ d) \mid \ tst \ c \ d. \ compatTst \ tst \ \wedge \ c \approx_{01T} \ d\} \ Un$
 $\{(c1 ;; (While \ tst \ c), \ d1 ;; (While \ tst \ d)) \mid \ tst \ c1 \ d1 \ c \ d. \ compatTst \ tst \ \wedge \ c1 \approx_{01T} \ d1 \ \wedge \ c \approx_{01T} \ d\}$

lemma *thetaWhileZOT-sym*:
sym thetaWhileZOT
 ⟨*proof*⟩

lemma *thetaWhileZOT-ZOretrT*:
thetaWhileZOT \subseteq *ZOretrT* (*thetaWhileZOT Un ZObisT*)
 ⟨*proof*⟩

lemma *thetaWhileZOT-ZObisT*:
thetaWhileZOT \subseteq *ZObisT*
 ⟨*proof*⟩

theorem *While-ZObisT[simp]*:
assumes *compatTst tst* **and** $c \approx_{01T} d$
shows *While tst c* \approx_{01T} *While tst d*
 ⟨*proof*⟩

Parallel composition:

definition *thetaParZOT* **where**
thetaParZOT \equiv
 $\{(Par\ c1\ c2, Par\ d1\ d2) \mid c1\ c2\ d1\ d2. c1 \approx_{01T} d1 \wedge c2 \approx_{01T} d2\}$

lemma *thetaParZOT-sym*:
sym thetaParZOT
 ⟨*proof*⟩

lemma *thetaParZOT-ZOretrT*:
thetaParZOT \subseteq *ZOretrT* (*thetaParZOT Un ZObisT*)
 ⟨*proof*⟩

lemma *thetaParZOT-ZObisT*:
thetaParZOT \subseteq *ZObisT*
 ⟨*proof*⟩

theorem *Par-ZObisT[simp]*:
assumes $c1 \approx_{01T} d1$ **and** $c2 \approx_{01T} d2$
shows *Par c1 c2* \approx_{01T} *Par d1 d2*
 ⟨*proof*⟩

5.5.2 01-bisimilarity versus language constructs

Discreetness:

theorem *discr-ZObis[simp]*:
assumes $*$: *discr c* **and** $**$: *discr d*
shows $c \approx_{01} d$
 ⟨*proof*⟩

Atomic commands:

theorem *Atm-ZObis[simp]*:

assumes *compatAtm atm*
shows $Atm\ atm \approx_{01}\ Atm\ atm$
 $\langle proof \rangle$

Sequential composition:

definition *thetaSeqZO* where
 $thetaSeqZO \equiv$
 $\{(c1 ;; c2, d1 ;; d2) \mid c1\ c2\ d1\ d2. c1 \approx_{01T}\ d1 \wedge c2 \approx_{01}\ d2\}$

lemma *thetaSeqZO-sym*:
 $sym\ thetaSeqZO$
 $\langle proof \rangle$

lemma *thetaSeqZO-ZOretr*:
 $thetaSeqZO \subseteq ZOretr\ (thetaSeqZO\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaSeqZO-ZObis*:
 $thetaSeqZO \subseteq ZObis$
 $\langle proof \rangle$

theorem *Seq-ZObisT-ZObis[simp]*:
assumes $c1 \approx_{01T}\ d1$ **and** $c2 \approx_{01}\ d2$
shows $c1 ;; c2 \approx_{01}\ d1 ;; d2$
 $\langle proof \rangle$

theorem *Seq-iso-ZObis[simp]*:
assumes *iso e* **and** $c2 \approx_{01}\ d2$
shows $e ;; c2 \approx_{01}\ e ;; d2$
 $\langle proof \rangle$

definition *thetaSeqZOD* where
 $thetaSeqZOD \equiv$
 $\{(c1 ;; c2, d1 ;; d2) \mid c1\ c2\ d1\ d2. c1 \approx_{01}\ d1 \wedge discr\ c2 \wedge discr\ d2\}$

lemma *thetaSeqZOD-sym*:
 $sym\ thetaSeqZOD$
 $\langle proof \rangle$

lemma *thetaSeqZOD-ZOretr*:
 $thetaSeqZOD \subseteq ZOretr\ (thetaSeqZOD\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaSeqZOD-ZObis*:
 $thetaSeqZOD \subseteq ZObis$
 $\langle proof \rangle$

theorem *Seq-ZObis-discr[simp]*:
assumes $c1 \approx_{01} d1$ **and** *discr* $c2$ **and** *discr* $d2$
shows $c1 ;; c2 \approx_{01} d1 ;; d2$
 $\langle proof \rangle$

Conditional:

definition *thetaIfZO* **where**
 $thetaIfZO \equiv$
 $\{(If\ tst\ c1\ c2,\ If\ tst\ d1\ d2) \mid tst\ c1\ c2\ d1\ d2.\ compatTst\ tst \wedge c1 \approx_{01} d1 \wedge c2 \approx_{01} d2\}$

lemma *thetaIfZO-sym*:
sym *thetaIfZO*
 $\langle proof \rangle$

lemma *thetaIfZO-ZOretr*:
 $thetaIfZO \subseteq ZOretr\ (thetaIfZO\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaIfZO-ZObis*:
 $thetaIfZO \subseteq ZObis$
 $\langle proof \rangle$

theorem *If-ZObis[simp]*:
assumes *compatTst* tst **and** $c1 \approx_{01} d1$ **and** $c2 \approx_{01} d2$
shows $If\ tst\ c1\ c2 \approx_{01} If\ tst\ d1\ d2$
 $\langle proof \rangle$

While loop:

01-bisimilarity does not interact with / preserve the While construct in any interesting way.

Parallel composition:

definition *thetaParZOL1* **where**
 $thetaParZOL1 \equiv$
 $\{(Par\ c1\ c2,\ d) \mid c1\ c2\ d.\ c1 \approx_{01} d \wedge \text{discr}\ c2\}$

lemma *thetaParZOL1-ZOretr*:
 $thetaParZOL1 \subseteq ZOretr\ (thetaParZOL1\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaParZOL1-converse-ZOretr*:
 $thetaParZOL1 \hat{\sim}^{-1} \subseteq ZOretr\ (thetaParZOL1 \hat{\sim}^{-1}\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaParZOL1-ZObis*:
 $thetaParZOL1 \subseteq ZObis$
 $\langle proof \rangle$

theorem *Par-ZObis-discrL1*[simp]:
assumes $c1 \approx 01 d$ **and** *discr c2*
shows $Par\ c1\ c2 \approx 01 d$
 $\langle proof \rangle$

theorem *Par-ZObis-discrR1*[simp]:
assumes $c \approx 01 d1$ **and** *discr d2*
shows $c \approx 01 Par\ d1\ d2$
 $\langle proof \rangle$

definition *thetaParZOL2* **where**
 $thetaParZOL2 \equiv$
 $\{(Par\ c1\ c2, d) \mid c1\ c2\ d.\ discr\ c1 \wedge c2 \approx 01 d\}$

lemma *thetaParZOL2-ZOretr*:
 $thetaParZOL2 \subseteq ZOretr\ (thetaParZOL2\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaParZOL2-converse-ZOretr*:
 $thetaParZOL2\ ^{-1} \subseteq ZOretr\ (thetaParZOL2\ ^{-1}\ Un\ ZObis)$
 $\langle proof \rangle$

lemma *thetaParZOL2-ZObis*:
 $thetaParZOL2 \subseteq ZObis$
 $\langle proof \rangle$

theorem *Par-ZObis-discrL2*[simp]:
assumes $c2 \approx 01 d$ **and** *discr c1*
shows $Par\ c1\ c2 \approx 01 d$
 $\langle proof \rangle$

theorem *Par-ZObis-discrR2*[simp]:
assumes $c \approx 01 d2$ **and** *discr d1*
shows $c \approx 01 Par\ d1\ d2$
 $\langle proof \rangle$

definition *thetaParZO* **where**
 $thetaParZO \equiv$
 $\{(Par\ c1\ c2, Par\ d1\ d2) \mid c1\ c2\ d1\ d2.\ c1 \approx 01 d1 \wedge c2 \approx 01 d2\}$

lemma *thetaParZO-sym*:
 $sym\ thetaParZO$
 $\langle proof \rangle$

lemma *thetaParZO-ZOretr*:
thetaParZO \subseteq *ZOretr* (*thetaParZO* *Un* *ZObis*)
 ⟨*proof*⟩

lemma *thetaParZO-ZObis*:
thetaParZO \subseteq *ZObis*
 ⟨*proof*⟩

theorem *Par-ZObis[simp]*:
assumes *c1* \approx_{01} *d1* **and** *c2* \approx_{01} *d2*
shows *Par c1 c2* \approx_{01} *Par d1 d2*
 ⟨*proof*⟩

5.5.3 WT-bisimilarity versus language constructs

Discreetness:

theorem *noWhile-discr-WbisT[simp]*:
assumes *noWhile c1* **and** *noWhile c2*
and *discr c1* **and** *discr c2*
shows *c1* \approx_{wT} *c2*
 ⟨*proof*⟩

Atomic commands:

theorem *Atm-WbisT*:
assumes *compatAtm atm*
shows *Atm atm* \approx_{wT} *Atm atm*
 ⟨*proof*⟩

Sequential composition:

definition *thetaSeqWT* **where**
thetaSeqWT \equiv
 $\{(c1 ;; c2, d1 ;; d2) \mid c1 c2 d1 d2. c1 \approx_{wT} d1 \wedge c2 \approx_{wT} d2\}$

lemma *thetaSeqWT-sym*:
sym thetaSeqWT
 ⟨*proof*⟩

lemma *thetaSeqWT-WretrT*:
thetaSeqWT \subseteq *WretrT* (*thetaSeqWT* *Un* *WbisT*)
 ⟨*proof*⟩

lemma *thetaSeqWT-WbisT*:
thetaSeqWT \subseteq *WbisT*
 ⟨*proof*⟩

theorem *Seq-WbisT[simp]*:
assumes *c1* \approx_{wT} *d1* **and** *c2* \approx_{wT} *d2*
shows *c1 ;; c2* \approx_{wT} *d1 ;; d2*

<proof>

Conditional:

definition *thetaIfWT* **where**

$thetaIfWT \equiv$
 $\{(If\ tst\ c1\ c2, If\ tst\ d1\ d2) \mid tst\ c1\ c2\ d1\ d2.\ compatTst\ tst \wedge c1 \approx_{wT} d1 \wedge c2 \approx_{wT} d2\}$

lemma *thetaIfWT-sym*:

sym thetaIfWT

<proof>

lemma *thetaIfWT-WretrT*:

$thetaIfWT \subseteq WretrT\ (thetaIfWT\ Un\ WbisT)$

<proof>

lemma *thetaIfWT-WbisT*:

$thetaIfWT \subseteq WbisT$

<proof>

theorem *If-WbisT[simp]*:

assumes *compatTst tst* **and** $c1 \approx_{wT} d1$ **and** $c2 \approx_{wT} d2$

shows $If\ tst\ c1\ c2 \approx_{wT} If\ tst\ d1\ d2$

<proof>

While loop:

definition *thetaWhileW* **where**

$thetaWhileW \equiv$
 $\{(While\ tst\ c, While\ tst\ d) \mid tst\ c\ d.\ compatTst\ tst \wedge c \approx_{wT} d\}\ Un$
 $\{(c1\ ;;\ (While\ tst\ c),\ d1\ ;;\ (While\ tst\ d)) \mid tst\ c1\ d1\ c\ d.\$
 $\quad\ compatTst\ tst \wedge c1 \approx_{wT} d1 \wedge c \approx_{wT} d\}$

lemma *thetaWhileW-sym*:

sym thetaWhileW

<proof>

lemma *thetaWhileW-WretrT*:

$thetaWhileW \subseteq WretrT\ (thetaWhileW\ Un\ WbisT)$

<proof>

lemma *thetaWhileW-WbisT*:

$thetaWhileW \subseteq WbisT$

<proof>

theorem *While-WbisT[simp]*:

assumes *compatTst tst* **and** $c \approx_{wT} d$

shows $While\ tst\ c \approx_{wT} While\ tst\ d$

<proof>

Parallel composition:

definition *thetaParWT* where

$$\begin{aligned} & \textit{thetaParWT} \equiv \\ & \{(Par\ c1\ c2, Par\ d1\ d2) \mid c1\ c2\ d1\ d2. c1 \approx_{wT} d1 \wedge c2 \approx_{wT} d2\} \end{aligned}$$

lemma *thetaParWT-sym*:

$$\begin{aligned} & \textit{sym\ thetaParWT} \\ & \langle \textit{proof} \rangle \end{aligned}$$

lemma *thetaParWT-WretrT*:

$$\begin{aligned} & \textit{thetaParWT} \subseteq \textit{WretrT}\ (\textit{thetaParWT}\ \textit{Un}\ \textit{WbisT}) \\ & \langle \textit{proof} \rangle \end{aligned}$$

lemma *thetaParWT-WbisT*:

$$\begin{aligned} & \textit{thetaParWT} \subseteq \textit{WbisT} \\ & \langle \textit{proof} \rangle \end{aligned}$$

theorem *Par-WbisT[simp]*:

$$\begin{aligned} & \textit{assumes}\ c1 \approx_{wT} d1 \ \textit{and}\ c2 \approx_{wT} d2 \\ & \textit{shows}\ Par\ c1\ c2 \approx_{wT} Par\ d1\ d2 \\ & \langle \textit{proof} \rangle \end{aligned}$$

5.5.4 T-bisimilarity versus language constructs

T-Discreetness:

definition *thetaFDW0* where

$$\begin{aligned} & \textit{thetaFDW0} \equiv \\ & \{(c1, c2). \textit{discr0}\ c1 \wedge \textit{discr0}\ c2\} \end{aligned}$$

lemma *thetaFDW0-sym*:

$$\begin{aligned} & \textit{sym\ thetaFDW0} \\ & \langle \textit{proof} \rangle \end{aligned}$$

lemma *thetaFDW0-RetrT*:

$$\begin{aligned} & \textit{thetaFDW0} \subseteq \textit{RetrT}\ \textit{thetaFDW0} \\ & \langle \textit{proof} \rangle \end{aligned}$$

lemma *thetaFDW0-BisT*:

$$\begin{aligned} & \textit{thetaFDW0} \subseteq \textit{BisT} \\ & \langle \textit{proof} \rangle \end{aligned}$$

theorem *discr0-BisT[simp]*:

$$\begin{aligned} & \textit{assumes}\ \textit{discr0}\ c1 \ \textit{and}\ \textit{discr0}\ c2 \\ & \textit{shows}\ c1 \approx_T c2 \\ & \langle \textit{proof} \rangle \end{aligned}$$

Atomic commands:

theorem *Atm-BisT*:

$$\begin{aligned} & \textit{assumes}\ \textit{compatAtm}\ atm \\ & \textit{shows}\ \textit{Atm}\ atm \approx_T \textit{Atm}\ atm \end{aligned}$$

<proof>

Sequential composition:

definition *thetaSeqTT* **where**

thetaSeqTT \equiv

$\{(c1 ;; c2, d1 ;; d2) \mid c1 \ c2 \ d1 \ d2. \ c1 \approx_T \ d1 \ \wedge \ c2 \approx_T \ d2\}$

lemma *thetaSeqTT-sym*:

sym thetaSeqTT

<proof>

lemma *thetaSeqTT-RetrT*:

thetaSeqTT \subseteq *RetrT* (*thetaSeqTT* \cup *BisT*)

<proof>

lemma *thetaSeqTT-BisT*:

thetaSeqTT \subseteq *BisT*

<proof>

theorem *Seq-BisT[simp]*:

assumes *c1* \approx_T *d1* **and** *c2* \approx_T *d2*

shows *c1* ;; *c2* \approx_T *d1* ;; *d2*

<proof>

Conditional:

definition *thetaIfTT* **where**

thetaIfTT \equiv

$\{(If \ tst \ c1 \ c2, \ If \ tst \ d1 \ d2) \mid \ tst \ c1 \ c2 \ d1 \ d2. \ compatTst \ tst \ \wedge \ c1 \approx_T \ d1 \ \wedge \ c2 \approx_T \ d2\}$

lemma *thetaIfTT-sym*:

sym thetaIfTT

<proof>

lemma *thetaIfTT-RetrT*:

thetaIfTT \subseteq *RetrT* (*thetaIfTT* \cup *BisT*)

<proof>

lemma *thetaIfTT-BisT*:

thetaIfTT \subseteq *BisT*

<proof>

theorem *If-BisT[simp]*:

assumes *compatTst* *tst* **and** *c1* \approx_T *d1* **and** *c2* \approx_T *d2*

shows *If* *tst* *c1* *c2* \approx_T *If* *tst* *d1* *d2*

<proof>

While loop:

definition *thetaWhileW0* **where**

$\text{thetaWhile}W0 \equiv$
 $\{(While\ tst\ c,\ While\ tst\ d) \mid \text{tst}\ c\ d.\ \text{compatTst}\ \text{tst} \wedge c \approx^T d\} \cup$
 $\{(c1 \ ;\ ;\ (While\ tst\ c),\ d1 \ ;\ ;\ (While\ tst\ d)) \mid \text{tst}\ c1\ d1\ c\ d.\$
 $\quad \text{compatTst}\ \text{tst} \wedge c1 \approx^T d1 \wedge c \approx^T d\}$

lemma *thetaWhileW0-sym*:

sym thetaWhileW0

<proof>

lemma *thetaWhileW0-RetrT*:

$\text{thetaWhile}W0 \subseteq \text{Retr}T\ (\text{thetaWhile}W0 \cup \text{Bis}T)$

<proof>

lemma *thetaWhileW0-BisT*:

$\text{thetaWhile}W0 \subseteq \text{Bis}T$

<proof>

theorem *While-BisT[simp]*:

assumes $\text{compatTst}\ \text{tst}$ **and** $c \approx^T d$

shows $While\ \text{tst}\ c \approx^T While\ \text{tst}\ d$

<proof>

Parallel composition:

definition *thetaParTT* **where**

$\text{thetaPar}TT \equiv$

$\{(Par\ c1\ c2,\ Par\ d1\ d2) \mid c1\ c2\ d1\ d2.\ c1 \approx^T d1 \wedge c2 \approx^T d2\}$

lemma *thetaParTT-sym*:

sym thetaParTT

<proof>

lemma *thetaParTT-RetrT*:

$\text{thetaPar}TT \subseteq \text{Retr}T\ (\text{thetaPar}TT \cup \text{Bis}T)$

<proof>

lemma *thetaParTT-BisT*:

$\text{thetaPar}TT \subseteq \text{Bis}T$

<proof>

theorem *Par-BisT[simp]*:

assumes $c1 \approx^T d1$ **and** $c2 \approx^T d2$

shows $Par\ c1\ c2 \approx^T Par\ d1\ d2$

<proof>

5.5.5 W-bisimilarity versus language constructs

Atomic commands:

theorem *Atm-Wbis[simp]*:

assumes $\text{compatAtm}\ \text{atm}$

shows $Atm\ atm \approx_w Atm\ atm$
<proof>

Discreetness:

theorem *discr-Wbis[simp]*:
assumes *: *discr c* **and** **: *discr d*
shows $c \approx_w d$
<proof>

Sequential composition:

definition *thetaSeqW* **where**
 $thetaSeqW \equiv$
 $\{(c1 ;; c2, d1 ;; d2) \mid c1\ c2\ d1\ d2. c1 \approx_{wT} d1 \wedge c2 \approx_w d2\}$

lemma *thetaSeqW-sym*:
sym thetaSeqW
<proof>

lemma *thetaSeqW-Wretr*:
 $thetaSeqW \subseteq Wretr\ (thetaSeqW \cup Wbis)$
<proof>

lemma *thetaSeqW-Wbis*:
 $thetaSeqW \subseteq Wbis$
<proof>

theorem *Seq-WbisT-Wbis[simp]*:
assumes $c1 \approx_{wT} d1$ **and** $c2 \approx_w d2$
shows $c1 ;; c2 \approx_w d1 ;; d2$
<proof>

theorem *Seq-iso-Wbis[simp]*:
assumes *iso e* **and** $c2 \approx_w d2$
shows $e ;; c2 \approx_w e ;; d2$
<proof>

definition *thetaSeqWD* **where**
 $thetaSeqWD \equiv$
 $\{(c1 ;; c2, d1 ;; d2) \mid c1\ c2\ d1\ d2. c1 \approx_w d1 \wedge discr\ c2 \wedge discr\ d2\}$

lemma *thetaSeqWD-sym*:
sym thetaSeqWD
<proof>

lemma *thetaSeqWD-Wretr*:
 $thetaSeqWD \subseteq Wretr\ (thetaSeqWD \cup Wbis)$
<proof>

lemma *thetaSeqWD-Wbis*:

thetaSeqWD \subseteq *Wbis*

<proof>

theorem *Seq-Wbis-discr[simp]*:

assumes *c1* \approx_w *d1* **and** *discr c2* **and** *discr d2*

shows *c1* ;; *c2* \approx_w *d1* ;; *d2*

<proof>

Conditional:

definition *thetaIfW* **where**

thetaIfW \equiv

$\{(If\ tst\ c1\ c2,\ If\ tst\ d1\ d2) \mid tst\ c1\ c2\ d1\ d2.\ compatTst\ tst \wedge c1 \approx_w d1 \wedge c2 \approx_w d2\}$

lemma *thetaIfW-sym*:

sym thetaIfW

<proof>

lemma *thetaIfW-Wretr*:

thetaIfW \subseteq *Wretr* (*thetaIfW* \cup *Wbis*)

<proof>

lemma *thetaIfW-Wbis*:

thetaIfW \subseteq *Wbis*

<proof>

theorem *If-Wbis[simp]*:

assumes *compatTst tst* **and** *c1* \approx_w *d1* **and** *c2* \approx_w *d2*

shows *If tst c1 c2* \approx_w *If tst d1 d2*

<proof>

While loop:

Again, w-bisimilarity does not interact with / preserve the While construct in any interesting way.

Parallel composition:

definition *thetaParWL1* **where**

thetaParWL1 \equiv

$\{(Par\ c1\ c2,\ d) \mid c1\ c2\ d.\ c1 \approx_w d \wedge discr\ c2\}$

lemma *thetaParWL1-Wretr*:

thetaParWL1 \subseteq *Wretr* (*thetaParWL1* \cup *Wbis*)

<proof>

lemma *thetaParWL1-converse-Wretr*:

thetaParWL1 $\hat{\sim}_{-1} \subseteq$ *Wretr* (*thetaParWL1* $\hat{\sim}_{-1} \cup$ *Wbis*)

$\langle proof \rangle$

lemma *thetaParWL1-Wbis*:

$thetaParWL1 \subseteq Wbis$

$\langle proof \rangle$

theorem *Par-Wbis-discrL1[simp]*:

assumes $c1 \approx_w d$ **and** $discr\ c2$

shows $Par\ c1\ c2 \approx_w d$

$\langle proof \rangle$

theorem *Par-Wbis-discrR1[simp]*:

assumes $c \approx_w d1$ **and** $discr\ d2$

shows $c \approx_w Par\ d1\ d2$

$\langle proof \rangle$

definition *thetaParWL2* **where**

$thetaParWL2 \equiv$

$\{(Par\ c1\ c2,\ d) \mid c1\ c2\ d.\ discr\ c1 \wedge c2 \approx_w d\}$

lemma *thetaParWL2-Wretr*:

$thetaParWL2 \subseteq Wretr\ (thetaParWL2 \cup Wbis)$

$\langle proof \rangle$

lemma *thetaParWL2-converse-Wretr*:

$thetaParWL2 \hat{-} 1 \subseteq Wretr\ (thetaParWL2 \hat{-} 1 \cup Wbis)$

$\langle proof \rangle$

lemma *thetaParWL2-Wbis*:

$thetaParWL2 \subseteq Wbis$

$\langle proof \rangle$

theorem *Par-Wbis-discrL2[simp]*:

assumes $c2 \approx_w d$ **and** $discr\ c1$

shows $Par\ c1\ c2 \approx_w d$

$\langle proof \rangle$

theorem *Par-Wbis-discrR2[simp]*:

assumes $c \approx_w d2$ **and** $discr\ d1$

shows $c \approx_w Par\ d1\ d2$

$\langle proof \rangle$

definition *thetaParW* **where**

$thetaParW \equiv$

$\{(Par\ c1\ c2,\ Par\ d1\ d2) \mid c1\ c2\ d1\ d2.\ c1 \approx_w d1 \wedge c2 \approx_w d2\}$

lemma *thetaParW-sym*:

sym thetaParW

<proof>

lemma *thetaParW-Wretr*:

thetaParW \subseteq *Wretr* (*thetaParW* \cup *Wbis*)

<proof>

lemma *thetaParW-Wbis*:

thetaParW \subseteq *Wbis*

<proof>

theorem *Par-Wbis[simp]*:

assumes *c1* \approx_w *d1* **and** *c2* \approx_w *d2*

shows *Par c1 c2* \approx_w *Par d1 d2*

<proof>

end

end

theory *Syntactic-Criteria*

imports *Compositionality*

begin

context *PL-Indis*

begin

lemma *noWhile[intro]*:

noWhile (*Atm atm*)

noWhile c1 \implies *noWhile c2* \implies *noWhile* (*Seq c1 c2*)

noWhile c1 \implies *noWhile c2* \implies *noWhile* (*If tst c1 c2*)

noWhile c1 \implies *noWhile c2* \implies *noWhile* (*Par c1 c2*)

<proof>

lemma *discr[intro]*:

presAtm atm \implies *discr* (*Atm atm*)

discr c1 \implies *discr c2* \implies *discr* (*Seq c1 c2*)

discr c1 \implies *discr c2* \implies *discr* (*If tst c1 c2*)

discr c \implies *discr* (*While tst c*)

discr c1 \implies *discr c2* \implies *discr* (*Par c1 c2*)

<proof>

lemma *siso[intro]*:

compatAtm atm \implies *siso* (*Atm atm*)

$siso\ c1 \implies siso\ c2 \implies siso\ (Seq\ c1\ c2)$
 $compatTst\ tst \implies siso\ c1 \implies siso\ c2 \implies siso\ (If\ tst\ c1\ c2)$
 $compatTst\ tst \implies siso\ c \implies siso\ (While\ tst\ c)$
 $siso\ c1 \implies siso\ c2 \implies siso\ (Par\ c1\ c2)$
 <proof>

lemma *Sbis*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_s\ Atm\ atm$
 $c1 \approx_s\ c1 \implies c2 \approx_s\ c2 \implies Seq\ c1\ c2 \approx_s\ Seq\ c1\ c2$
 $compatTst\ tst \implies c1 \approx_s\ c1 \implies c2 \approx_s\ c2 \implies If\ tst\ c1\ c2 \approx_s\ If\ tst\ c1\ c2$
 $compatTst\ tst \implies c \approx_s\ c \implies While\ tst\ c \approx_s\ While\ tst\ c$
 $c1 \approx_s\ c1 \implies c2 \approx_s\ c2 \implies Par\ c1\ c2 \approx_s\ Par\ c1\ c2$
 <proof>

lemma *ZObisT*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_{01T}\ Atm\ atm$
 $c1 \approx_{01T}\ c1 \implies c2 \approx_{01T}\ c2 \implies Seq\ c1\ c2 \approx_{01T}\ Seq\ c1\ c2$
 $compatTst\ tst \implies c1 \approx_{01T}\ c1 \implies c2 \approx_{01T}\ c2 \implies If\ tst\ c1\ c2 \approx_{01T}\ If\ tst\ c1\ c2$
 $compatTst\ tst \implies c \approx_{01T}\ c \implies While\ tst\ c \approx_{01T}\ While\ tst\ c$
 $c1 \approx_{01T}\ c1 \implies c2 \approx_{01T}\ c2 \implies Par\ c1\ c2 \approx_{01T}\ Par\ c1\ c2$
 <proof>

lemma *BisT*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_T\ Atm\ atm$
 $c1 \approx_T\ c1 \implies c2 \approx_T\ c2 \implies Seq\ c1\ c2 \approx_T\ Seq\ c1\ c2$
 $compatTst\ tst \implies c1 \approx_T\ c1 \implies c2 \approx_T\ c2 \implies If\ tst\ c1\ c2 \approx_T\ If\ tst\ c1\ c2$
 $compatTst\ tst \implies c \approx_T\ c \implies While\ tst\ c \approx_T\ While\ tst\ c$
 $c1 \approx_T\ c1 \implies c2 \approx_T\ c2 \implies Par\ c1\ c2 \approx_T\ Par\ c1\ c2$
 <proof>

lemma *WbisT*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_{wT}\ Atm\ atm$
 $c1 \approx_{wT}\ c1 \implies c2 \approx_{wT}\ c2 \implies Seq\ c1\ c2 \approx_{wT}\ Seq\ c1\ c2$
 $compatTst\ tst \implies c1 \approx_{wT}\ c1 \implies c2 \approx_{wT}\ c2 \implies If\ tst\ c1\ c2 \approx_{wT}\ If\ tst\ c1\ c2$
 $compatTst\ tst \implies c \approx_{wT}\ c \implies While\ tst\ c \approx_{wT}\ While\ tst\ c$
 $c1 \approx_{wT}\ c1 \implies c2 \approx_{wT}\ c2 \implies Par\ c1\ c2 \approx_{wT}\ Par\ c1\ c2$
 <proof>

lemma *ZObis*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_{01}\ Atm\ atm$
 $c1 \approx_{01T}\ c1 \implies c2 \approx_{01}\ c2 \implies Seq\ c1\ c2 \approx_{01}\ Seq\ c1\ c2$
 $c1 \approx_{01}\ c1 \implies discr\ c2 \implies Seq\ c1\ c2 \approx_{01}\ Seq\ c1\ c2$
 $compatTst\ tst \implies c1 \approx_{01}\ c1 \implies c2 \approx_{01}\ c2 \implies If\ tst\ c1\ c2 \approx_{01}\ If\ tst\ c1\ c2$
 $c1 \approx_{01}\ c1 \implies c2 \approx_{01}\ c2 \implies Par\ c1\ c2 \approx_{01}\ Par\ c1\ c2$
 <proof>

lemma *Wbis*[intro]:

$compatAtm\ atm \implies Atm\ atm \approx_w\ Atm\ atm$

$c1 \approx_w T c1 \implies c2 \approx_w c2 \implies \text{Seq } c1 \ c2 \approx_w \text{Seq } c1 \ c2$
 $c1 \approx_w c1 \implies \text{discr } c2 \implies \text{Seq } c1 \ c2 \approx_w \text{Seq } c1 \ c2$
 $\text{compatTst } \text{tst} \implies c1 \approx_w c1 \implies c2 \approx_w c2 \implies \text{If } \text{tst } c1 \ c2 \approx_w \text{If } \text{tst } c1 \ c2$
 $c1 \approx_w c1 \implies c2 \approx_w c2 \implies \text{Par } c1 \ c2 \approx_w \text{Par } c1 \ c2$
 ⟨proof⟩

lemma *discr-noWhile-WbisT*[intro]: $\text{discr } c \implies \text{noWhile } c \implies c \approx_w T c$
 ⟨proof⟩

lemma *siso-ZObis*[intro]: $\text{siso } c \implies c \approx_{01} c$
 ⟨proof⟩

lemma *WbisT-Wbis*[intro]: $c \approx_w T c \implies c \approx_w c$
 ⟨proof⟩

lemma *ZObis-Wbis*[intro]: $c \approx_{01} c \implies c \approx_w c$
 ⟨proof⟩

lemma *discr-BisT*[intro]: $\text{discr } c \implies c \approx_T c$
 ⟨proof⟩

lemma *WbisT-BisT*[intro]: $c \approx_w T c \implies c \approx_T c$
 ⟨proof⟩

lemma *ZObisT-ZObis*[intro]: $c \approx_{01T} c \implies c \approx_{01} c$
 ⟨proof⟩

lemma *siso-ZObisT*[intro]: $\text{siso } c \implies c \approx_{01T} c$
 ⟨proof⟩

primrec *SC-discr* where

$\text{SC-discr } (\text{Atm } \text{atm}) \quad \longleftrightarrow \text{presAtm } \text{atm}$
 $| \text{SC-discr } (\text{Seq } c1 \ c2) \quad \longleftrightarrow \text{SC-discr } c1 \ \wedge \ \text{SC-discr } c2$
 $| \text{SC-discr } (\text{If } \text{tst } c1 \ c2) \quad \longleftrightarrow \text{SC-discr } c1 \ \wedge \ \text{SC-discr } c2$
 $| \text{SC-discr } (\text{While } \text{tst } c) \quad \longleftrightarrow \text{SC-discr } c$
 $| \text{SC-discr } (\text{Par } c1 \ c2) \quad \longleftrightarrow \text{SC-discr } c1 \ \wedge \ \text{SC-discr } c2$

primrec *SC-siso* where

$\text{SC-siso } (\text{Atm } \text{atm}) \quad \longleftrightarrow \text{compatAtm } \text{atm}$
 $| \text{SC-siso } (\text{Seq } c1 \ c2) \quad \longleftrightarrow \text{SC-siso } c1 \ \wedge \ \text{SC-siso } c2$
 $| \text{SC-siso } (\text{If } \text{tst } c1 \ c2) \quad \longleftrightarrow \text{compatTst } \text{tst} \ \wedge \ \text{SC-siso } c1 \ \wedge \ \text{SC-siso } c2$
 $| \text{SC-siso } (\text{While } \text{tst } c) \quad \longleftrightarrow \text{compatTst } \text{tst} \ \wedge \ \text{SC-siso } c$
 $| \text{SC-siso } (\text{Par } c1 \ c2) \quad \longleftrightarrow \text{SC-siso } c1 \ \wedge \ \text{SC-siso } c2$

primrec *SC-WbisT* where

$\text{SC-WbisT } (\text{Atm } \text{atm}) \quad \longleftrightarrow \text{compatAtm } \text{atm}$

$$\begin{aligned}
| \text{SC-WbisT } (\text{Seq } c1 \ c2) &\longleftrightarrow (\text{SC-WbisT } c1 \wedge \text{SC-WbisT } c2) \vee \\
&\quad (\text{noWhile } (\text{Seq } c1 \ c2) \wedge \text{SC-discr } (\text{Seq } c1 \ c2)) \vee \\
&\quad \text{SC-siso } (\text{Seq } c1 \ c2) \\
| \text{SC-WbisT } (\text{If } \text{tst } c1 \ c2) &\longleftrightarrow (\text{if compatTst } \text{tst} \\
&\quad \text{then } (\text{SC-WbisT } c1 \wedge \text{SC-WbisT } c2) \\
&\quad \text{else } ((\text{noWhile } (\text{If } \text{tst } c1 \ c2) \wedge \text{SC-discr } (\text{If } \text{tst } c1 \ c2)) \vee \\
&\quad \quad \text{SC-siso } (\text{If } \text{tst } c1 \ c2))) \\
| \text{SC-WbisT } (\text{While } \text{tst } c) &\longleftrightarrow (\text{if compatTst } \text{tst} \\
&\quad \text{then } \text{SC-WbisT } c \\
&\quad \text{else } ((\text{noWhile } (\text{While } \text{tst } c) \wedge \text{SC-discr } (\text{While } \text{tst } c)) \vee \\
&\quad \quad \text{SC-siso } (\text{While } \text{tst } c))) \\
| \text{SC-WbisT } (\text{Par } c1 \ c2) &\longleftrightarrow (\text{SC-WbisT } c1 \wedge \text{SC-WbisT } c2) \vee \\
&\quad (\text{noWhile } (\text{Par } c1 \ c2) \wedge \text{SC-discr } (\text{Par } c1 \ c2)) \vee \\
&\quad \text{SC-siso } (\text{Par } c1 \ c2)
\end{aligned}$$

primrec SC-ZObis where

$$\begin{aligned}
&\text{SC-ZObis } (\text{Atm } \text{atm}) \longleftrightarrow \text{compatAtm } \text{atm} \\
| \text{SC-ZObis } (\text{Seq } c1 \ c2) &\longleftrightarrow (\text{SC-siso } c1 \wedge \text{SC-ZObis } c2) \vee \\
&\quad (\text{SC-ZObis } c1 \wedge \text{SC-discr } c2) \vee \\
&\quad \text{SC-discr } (\text{Seq } c1 \ c2) \vee \\
&\quad \text{SC-siso } (\text{Seq } c1 \ c2) \\
| \text{SC-ZObis } (\text{If } \text{tst } c1 \ c2) &\longleftrightarrow (\text{if compatTst } \text{tst} \\
&\quad \text{then } (\text{SC-ZObis } c1 \wedge \text{SC-ZObis } c2) \\
&\quad \text{else } (\text{SC-discr } (\text{If } \text{tst } c1 \ c2) \vee \\
&\quad \quad \text{SC-siso } (\text{If } \text{tst } c1 \ c2))) \\
| \text{SC-ZObis } (\text{While } \text{tst } c) &\longleftrightarrow \text{SC-discr } (\text{While } \text{tst } c) \vee \\
&\quad \text{SC-siso } (\text{While } \text{tst } c) \\
| \text{SC-ZObis } (\text{Par } c1 \ c2) &\longleftrightarrow (\text{SC-ZObis } c1 \wedge \text{SC-ZObis } c2) \vee \\
&\quad \text{SC-discr } (\text{Par } c1 \ c2) \vee \\
&\quad \text{SC-siso } (\text{Par } c1 \ c2)
\end{aligned}$$

primrec SC-Wbis where

$$\begin{aligned}
&\text{SC-Wbis } (\text{Atm } \text{atm}) \longleftrightarrow \text{compatAtm } \text{atm} \\
| \text{SC-Wbis } (\text{Seq } c1 \ c2) &\longleftrightarrow (\text{SC-WbisT } c1 \wedge \text{SC-Wbis } c2) \vee \\
&\quad (\text{SC-Wbis } c1 \wedge \text{SC-discr } c2) \vee \\
&\quad \text{SC-ZObis } (\text{Seq } c1 \ c2) \vee \\
&\quad \text{SC-WbisT } (\text{Seq } c1 \ c2) \\
| \text{SC-Wbis } (\text{If } \text{tst } c1 \ c2) &\longleftrightarrow (\text{if compatTst } \text{tst} \\
&\quad \text{then } (\text{SC-Wbis } c1 \wedge \text{SC-Wbis } c2) \\
&\quad \text{else } (\text{SC-ZObis } (\text{If } \text{tst } c1 \ c2) \vee \\
&\quad \quad \text{SC-WbisT } (\text{If } \text{tst } c1 \ c2))) \\
| \text{SC-Wbis } (\text{While } \text{tst } c) &\longleftrightarrow \text{SC-ZObis } (\text{While } \text{tst } c) \vee \\
&\quad \text{SC-WbisT } (\text{While } \text{tst } c) \\
| \text{SC-Wbis } (\text{Par } c1 \ c2) &\longleftrightarrow (\text{SC-Wbis } c1 \wedge \text{SC-Wbis } c2) \vee \\
&\quad \text{SC-ZObis } (\text{Par } c1 \ c2) \vee \\
&\quad \text{SC-WbisT } (\text{Par } c1 \ c2)
\end{aligned}$$

primrec SC-BisT where

$$\text{SC-BisT } (\text{Atm } \text{atm}) \longleftrightarrow \text{compatAtm } \text{atm}$$

$| \text{SC-BisT } (\text{Seq } c1 \ c2) \longleftrightarrow (\text{SC-BisT } c1 \wedge \text{SC-BisT } c2) \vee$
 $\text{SC-discr } (\text{Seq } c1 \ c2) \vee$
 $\text{SC-WbisT } (\text{Seq } c1 \ c2)$
 $| \text{SC-BisT } (\text{If } \text{tst } c1 \ c2) \longleftrightarrow (\text{if } \text{compatTst } \text{tst}$
 $\text{then } (\text{SC-BisT } c1 \wedge \text{SC-BisT } c2)$
 $\text{else } (\text{SC-discr } (\text{If } \text{tst } c1 \ c2) \vee$
 $\text{SC-WbisT } (\text{If } \text{tst } c1 \ c2)))$
 $| \text{SC-BisT } (\text{While } \text{tst } c) \longleftrightarrow (\text{if } \text{compatTst } \text{tst}$
 $\text{then } \text{SC-BisT } c$
 $\text{else } (\text{SC-discr } (\text{While } \text{tst } c) \vee$
 $\text{SC-WbisT } (\text{While } \text{tst } c)))$
 $| \text{SC-BisT } (\text{Par } c1 \ c2) \longleftrightarrow (\text{SC-BisT } c1 \wedge \text{SC-BisT } c2) \vee$
 $\text{SC-discr } (\text{Par } c1 \ c2) \vee$
 $\text{SC-WbisT } (\text{Par } c1 \ c2)$

theorem *SC-discr[intro]*: $\text{SC-discr } c \implies \text{discr } c$
<proof>

theorem *SC-iso[intro]*: $\text{SC-iso } c \implies \text{iso } c$
<proof>

theorem *SC-iso-imp-SC-WbisT[intro]*: $\text{SC-iso } c \implies \text{SC-WbisT } c$
<proof>

theorem *SC-discr-imp-SC-WbisT[intro]*: $\text{no While } c \implies \text{SC-discr } c \implies \text{SC-WbisT } c$
<proof>

theorem *SC-WbisT[intro]*: $\text{SC-WbisT } c \implies c \approx_{wT} c$
<proof>

theorem *SC-discr-imp-SC-ZObis[intro]*: $\text{SC-discr } c \implies \text{SC-ZObis } c$
<proof>

theorem *SC-iso-imp-SC-ZObis[intro]*: $\text{SC-iso } c \implies \text{SC-ZObis } c$
<proof>

theorem *SC-ZObis[intro]*: $\text{SC-ZObis } c \implies c \approx_{01} c$
<proof>

theorem *SC-ZObis-imp-SC-Wbis[intro]*: $\text{SC-ZObis } c \implies \text{SC-Wbis } c$
<proof>

theorem *SC-WbisT-imp-SC-Wbis[intro]*: $\text{SC-WbisT } c \implies \text{SC-Wbis } c$
<proof>

theorem *SC-Wbis[intro]*: $\text{SC-Wbis } c \implies c \approx_w c$
<proof>

theorem *SC-discr-imp-SC-BisT[intro]*: $SC-discr\ c \implies SC-BisT\ c$
<proof>

theorem *SC-WbisT-imp-SC-BisT[intro]*: $SC-WbisT\ c \implies SC-BisT\ c$
<proof>

theorem *SC-ZObis-imp-SC-BisT[intro]*: $SC-ZObis\ c \implies SC-BisT\ c$
<proof>

theorem *SC-Wbis-imp-SC-BisT[intro]*: $SC-Wbis\ c \implies SC-BisT\ c$
<proof>

theorem *SC-BisT[intro]*: $SC-BisT\ c \implies c \approx T\ c$
<proof>

theorem *SC-WbisT-While*: $SC-WbisT\ (While\ tst\ c) \longleftrightarrow SC-WbisT\ c \wedge compatTst\ tst$
<proof>

theorem *SC-ZObis-While*: $SC-ZObis\ (While\ tst\ c) \longleftrightarrow (compatTst\ tst \wedge SC-iso\ c) \vee SC-discr\ c$
<proof>

theorem *SC-ZObis-If*: $SC-ZObis\ (If\ tst\ c1\ c2) \longleftrightarrow (if\ compatTst\ tst\ then\ SC-ZObis\ c1 \wedge SC-ZObis\ c2\ else\ SC-discr\ c1 \wedge SC-discr\ c2)$
<proof>

theorem *SC-WbisT-Seq*: $SC-WbisT\ (Seq\ c1\ c2) \longleftrightarrow (SC-WbisT\ c1 \wedge SC-WbisT\ c2)$
<proof>

theorem *SC-ZObis-Seq*: $SC-ZObis\ (Seq\ c1\ c2) \longleftrightarrow (SC-iso\ c1 \wedge SC-ZObis\ c2) \vee (SC-ZObis\ c1 \wedge SC-discr\ c2)$
<proof>

theorem *SC-Wbis-Seq*: $SC-Wbis\ (Seq\ c1\ c2) \longleftrightarrow (SC-WbisT\ c1 \wedge SC-Wbis\ c2) \vee (SC-Wbis\ c1 \wedge SC-discr\ c2)$
<proof>

theorem *SC-BisT-Par*:
 $SC-BisT\ (Par\ c1\ c2) \longleftrightarrow (SC-BisT\ c1 \wedge SC-BisT\ c2)$
<proof>

end

end

6 After-execution security

theory *After-Execution*
imports *During-Execution*
begin

context *PL-Indis*
begin

6.1 Setup for bisimilarities

lemma *Sbis-transC*[*consumes 3, case-names Match*]:
assumes $0: c \approx_s d$ **and** $s \approx t$ **and** $(c,s) \rightarrow_c (c',s')$
obtains $d' t'$ **where**
 $(d,t) \rightarrow_c (d',t')$ **and** $s' \approx t'$ **and** $c' \approx_s d'$
<proof>

lemma *Sbis-transT*[*consumes 3, case-names Match*]:
assumes $0: c \approx_s d$ **and** $s \approx t$ **and** $(c,s) \rightarrow_t s'$
obtains t' **where** $(d,t) \rightarrow_t t'$ **and** $s' \approx t'$
<proof>

lemma *Sbis-transC2*[*consumes 3, case-names Match*]:
assumes $0: c \approx_s d$ **and** $s \approx t$ **and** $(d,t) \rightarrow_c (d',t')$
obtains $c' s'$ **where**
 $(c,s) \rightarrow_c (c',s')$ **and** $s' \approx t'$ **and** $c' \approx_s d'$
<proof>

lemma *Sbis-transT2*[*consumes 3, case-names Match*]:
assumes $0: c \approx_s d$ **and** $s \approx t$ **and** $(d,t) \rightarrow_t t'$
obtains s' **where** $(c,s) \rightarrow_t s'$ **and** $s' \approx t'$
<proof>

lemma *ZObisT-transC*[*consumes 3, case-names Match MatchS*]:
assumes $0: c \approx_{01T} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow_c (c',s')$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow_c (d',t'); s' \approx t'; c' \approx_{01T} d' \rrbracket \implies \textit{thesis}$
and $\llbracket s' \approx t; c' \approx_{01T} d \rrbracket \implies \textit{thesis}$
shows *thesis*
<proof>

lemma *ZObisT-transT*[*consumes 3, case-names Match*]:
assumes $0: c \approx_{01T} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow_t s'$
obtains t' **where** $(d,t) \rightarrow_t t'$ **and** $s' \approx t'$
<proof>

lemma *ZObisT-transC2*[*consumes 3, case-names Match MatchS*]:

assumes 0: $c \approx_{01T} d$ **and** 2: $s \approx t$ **and** 3: $(d,t) \rightarrow c (d',t')$
and 4: $\bigwedge c' s'. \llbracket (c,s) \rightarrow c (c',s'); s' \approx t'; c' \approx_{01T} d' \rrbracket \implies \textit{thesis}$
and 5: $\llbracket s \approx t'; c \approx_{01T} d' \rrbracket \implies \textit{thesis}$
shows *thesis*
 $\langle \textit{proof} \rangle$

lemma *ZObisT-transT2*[*consumes 3, case-names Match*]:
assumes 0: $c \approx_{01T} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow t t'$
obtains s' **where** $(c,s) \rightarrow t s'$ **and** $s' \approx t'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-transC*[*consumes 3, case-names Match*]:
assumes 0: $c \approx_{wT} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow c (c',s')$
obtains $d' t'$ **where**
 $(d,t) \rightarrow *c (d',t')$ **and** $s' \approx t'$ **and** $c' \approx_{wT} d'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-transT*[*consumes 3, case-names Match*]:
assumes 0: $c \approx_{wT} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow t s'$
obtains t' **where** $(d,t) \rightarrow *t t'$ **and** $s' \approx t'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-transC2*[*consumes 3, case-names Match*]:
assumes 0: $c \approx_{wT} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow c (d',t')$
obtains $c' s'$ **where**
 $(c,s) \rightarrow *c (c',s')$ **and** $s' \approx t'$ **and** $c' \approx_{wT} d'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-transT2*[*consumes 3, case-names Match*]:
assumes 0: $c \approx_{wT} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow t t'$
obtains s' **where** $(c,s) \rightarrow *t s'$ **and** $s' \approx t'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-MtransC*[*consumes 3, case-names Match*]:
assumes 1: $c \approx_{wT} d$ **and** 2: $s \approx t$ **and** 3: $(c,s) \rightarrow *c (c',s')$
obtains $d' t'$ **where**
 $(d,t) \rightarrow *c (d',t')$ **and** $s' \approx t'$ **and** $c' \approx_{wT} d'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-MtransT*[*consumes 3, case-names Match*]:
assumes 1: $c \approx_{wT} d$ **and** 2: $s \approx t$ **and** 3: $(c,s) \rightarrow *t s'$
obtains t' **where** $(d,t) \rightarrow *t t'$ **and** $s' \approx t'$
 $\langle \textit{proof} \rangle$

lemma *WbisT-MtransC2*[*consumes 3, case-names Match*]:
assumes $c \approx_{wT} d$ **and** $s \approx t$ **and** 1: $(d,t) \rightarrow *c (d',t')$
obtains $c' s'$ **where**

$(c,s) \rightarrow^* c (c',s')$ **and** $s' \approx t'$ **and** $c' \approx_{wT} d'$
 ⟨proof⟩

lemma *WbisT-MtransT2*[*consumes 3, case-names Match*]:
assumes $c \approx_{wT} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow^* t'$
obtains s' **where** $(c,s) \rightarrow^* t s'$ **and** $s' \approx t'$
 ⟨proof⟩

lemma *ZObis-transC*[*consumes 3, case-names Match MatchO MatchS*]:
assumes $0: c \approx_{01} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow c (c',s')$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow c (d',t'); s' \approx t'; c' \approx_{01} d' \rrbracket \implies \text{thesis}$
and $\bigwedge t'. \llbracket (d,t) \rightarrow t t'; s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
and $\llbracket s' \approx t; c' \approx_{01} d \rrbracket \implies \text{thesis}$
shows *thesis*
 ⟨proof⟩

lemma *ZObis-transT*[*consumes 3, case-names Match MatchO MatchS*]:
assumes $0: c \approx_{01} d$ **and** $s \approx t$ **and** $(c,s) \rightarrow t s'$
and $\bigwedge t'. \llbracket (d,t) \rightarrow t t'; s' \approx t' \rrbracket \implies \text{thesis}$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow c (d',t'); s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
and $\llbracket s' \approx t; \text{discr } d \rrbracket \implies \text{thesis}$
shows *thesis*
 ⟨proof⟩

lemma *ZObis-transC2*[*consumes 3, case-names Match MatchO MatchS*]:
assumes $0: c \approx_{01} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow c (d',t')$
and $\bigwedge c' s'. \llbracket (c,s) \rightarrow c (c',s'); s' \approx t'; c' \approx_{01} d' \rrbracket \implies \text{thesis}$
and $\bigwedge s'. \llbracket (c,s) \rightarrow t s'; s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
and $\llbracket s \approx t'; c \approx_{01} d' \rrbracket \implies \text{thesis}$
shows *thesis*
 ⟨proof⟩

lemma *ZObis-transT2*[*consumes 3, case-names Match MatchO MatchS*]:
assumes $0: c \approx_{01} d$ **and** $s \approx t$ **and** $(d,t) \rightarrow t t'$
and $\bigwedge s'. \llbracket (c,s) \rightarrow t s'; s' \approx t' \rrbracket \implies \text{thesis}$
and $\bigwedge c' s'. \llbracket (c,s) \rightarrow c (c',s'); s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
and $\llbracket s \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
shows *thesis*
 ⟨proof⟩

lemma *Wbis-transC*[*consumes 3, case-names Match MatchO*]:
assumes $0: c \approx_w d$ **and** $s \approx t$ **and** $(c,s) \rightarrow c (c',s')$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow^* c (d',t'); s' \approx t'; c' \approx_w d' \rrbracket \implies \text{thesis}$
and $\bigwedge t'. \llbracket (d,t) \rightarrow^* t t'; s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
shows *thesis*
 ⟨proof⟩

lemma *Wbis-transT*[consumes 3, case-names Match MatchO]:
assumes $0: c \approx_w d$ **and** $s \approx t$ **and** $(c,s) \rightarrow t s'$
and $\bigwedge t'. \llbracket (d,t) \rightarrow^* t t'; s' \approx t' \rrbracket \implies \text{thesis}$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow^* c (d',t'); s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-transC2*[consumes 3, case-names Match MatchO]:
assumes $0: c \approx_w d$ **and** $s \approx t$ **and** $(d,t) \rightarrow c (d',t')$
and $\bigwedge c' s'. \llbracket (c,s) \rightarrow^* c (c',s'); s' \approx t'; c' \approx_w d \rrbracket \implies \text{thesis}$
and $\bigwedge s'. \llbracket (c,s) \rightarrow^* t s'; s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-transT2*[consumes 3, case-names Match MatchO]:
assumes $0: c \approx_w d$ **and** $s \approx t$ **and** $(d,t) \rightarrow t t'$
and $\bigwedge s'. \llbracket (c,s) \rightarrow^* t s'; s' \approx t' \rrbracket \implies \text{thesis}$
and $\bigwedge c' s'. \llbracket (c,s) \rightarrow^* c (c',s'); s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-MtransC*[consumes 3, case-names Match MatchO]:
assumes $c \approx_w d$ **and** $s \approx t$ **and** $(c,s) \rightarrow^* c (c',s')$
and $\bigwedge d' t'. \llbracket (d,t) \rightarrow^* c (d',t'); s' \approx t'; c' \approx_w d \rrbracket \implies \text{thesis}$
and $\bigwedge t'. \llbracket (d,t) \rightarrow^* t t'; s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-MtransT*[consumes 3, case-names Match MatchO]:
assumes $c-d: c \approx_w d$ **and** $st: s \approx t$ **and** $cs: (c,s) \rightarrow^* t s'$
and $1: \bigwedge t'. \llbracket (d,t) \rightarrow^* t t'; s' \approx t' \rrbracket \implies \text{thesis}$
and $2: \bigwedge d' t'. \llbracket (d,t) \rightarrow^* c (d',t'); s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-MtransC2*[consumes 3, case-names Match MatchO]:
assumes $c \approx_w d$ **and** $s \approx t$ **and** $dt: (d,t) \rightarrow^* c (d',t')$
and $1: \bigwedge c' s'. \llbracket (c,s) \rightarrow^* c (c',s'); s' \approx t'; c' \approx_w d \rrbracket \implies \text{thesis}$
and $2: \bigwedge s'. \llbracket (c,s) \rightarrow^* t s'; s' \approx t'; \text{discr } d \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *Wbis-MtransT2*[consumes 3, case-names Match MatchO]:
assumes $c \approx_w d$ **and** $s \approx t$ **and** $dt: (d,t) \rightarrow^* t t'$
and $1: \bigwedge s'. \llbracket (c,s) \rightarrow^* t s'; s' \approx t' \rrbracket \implies \text{thesis}$
and $2: \bigwedge c' s'. \llbracket (c,s) \rightarrow^* c (c',s'); s' \approx t'; \text{discr } c \rrbracket \implies \text{thesis}$
shows *thesis*
<proof>

lemma *BisT-transC*[consumes 5, case-names Match]:
assumes $0: c \approx T d$
and $mustT\ c\ s$ **and** $mustT\ d\ t$
and $s \approx t$ **and** $(c, s) \rightarrow c\ (c', s')$
obtains $d'\ t'$ **where**
 $(d, t) \rightarrow *c\ (d', t')$ **and** $s' \approx t'$ **and** $c' \approx T d'$
 $\langle proof \rangle$

lemma *BisT-transT*[consumes 5, case-names Match]:
assumes $0: c \approx T d$
and $mustT\ c\ s$ **and** $mustT\ d\ t$
and $s \approx t$ **and** $(c, s) \rightarrow t\ s'$
obtains t' **where** $(d, t) \rightarrow *t\ t'$ **and** $s' \approx t'$
 $\langle proof \rangle$

lemma *BisT-transC2*[consumes 5, case-names Match]:
assumes $0: c \approx T d$
and $mustT\ c\ s$ **and** $mustT\ d\ t$
and $s \approx t$ **and** $(d, t) \rightarrow c\ (d', t')$
obtains $c'\ s'$ **where**
 $(c, s) \rightarrow *c\ (c', s')$ **and** $s' \approx t'$ **and** $c' \approx T d'$
 $\langle proof \rangle$

lemma *BisT-transT2*[consumes 5, case-names Match]:
assumes $0: c \approx T d$
and $mustT\ c\ s$ **and** $mustT\ d\ t$
and $s \approx t$ **and** $(d, t) \rightarrow t\ t'$
obtains s' **where** $(c, s) \rightarrow *t\ s'$ **and** $s' \approx t'$
 $\langle proof \rangle$

lemma *BisT-MtransC*[consumes 5, case-names Match]:
assumes $c \approx T d$
and $mustT\ c\ s$ $mustT\ d\ t$
and $s \approx t$ **and** $(c, s) \rightarrow *c\ (c', s')$
obtains $d'\ t'$ **where**
 $(d, t) \rightarrow *c\ (d', t')$ **and** $s' \approx t'$ **and** $c' \approx T d'$
 $\langle proof \rangle$

lemma *BisT-MtransT*[consumes 5, case-names Match]:
assumes $1: c \approx T d$
and $ter: mustT\ c\ s\ mustT\ d\ t$
and $2: s \approx t$ **and** $3: (c, s) \rightarrow *t\ s'$
obtains t' **where** $(d, t) \rightarrow *t\ t'$ **and** $s' \approx t'$
 $\langle proof \rangle$

lemma *BisT-MtransC2*[consumes 3, case-names Match]:

assumes $c \approx T d$
and $ter: mustT c s mustT d t$
and $s \approx t$ **and** $1: (d,t) \rightarrow^* c (d',t')$
obtains $c' s'$ **where**
 $(c,s) \rightarrow^* c (c',s')$ **and** $s' \approx t'$ **and** $c' \approx T d'$
 $\langle proof \rangle$

lemma *BisT-MtransT02*[*consumes 3, case-names Match*]:
assumes $c \approx T d$
and $ter: mustT c s mustT d t$
and $s \approx t$ **and** $(d,t) \rightarrow^* t t'$
obtains s' **where** $(c,s) \rightarrow^* t s'$ **and** $s' \approx t'$
 $\langle proof \rangle$

6.2 Execution traces

primrec *parTrace* **where**
 $parTrace [] \longleftrightarrow False$ |
 $parTrace (cf \# cfl) \longleftrightarrow (cfl \neq [] \longrightarrow parTrace cfl \wedge cf \rightarrow c hd cfl)$

lemma *trans-Step2*:
 $cf \rightarrow^* c cf' \Longrightarrow cf' \rightarrow c cf'' \Longrightarrow cf \rightarrow^* c cf''$
 $\langle proof \rangle$

lemma *parTrace-not-empty*[*simp*]: $parTrace cfl \Longrightarrow cfl \neq []$
 $\langle proof \rangle$

lemma *parTrace-snoc*[*simp*]:
 $parTrace (cfl @ [cf]) \longleftrightarrow (cfl \neq [] \longrightarrow parTrace cfl \wedge last cfl \rightarrow c cf)$
 $\langle proof \rangle$

lemma *MtransC-Ex-parTrace*:
assumes $cf \rightarrow^* c cf'$ **shows** $\exists cfl. parTrace cfl \wedge hd cfl = cf \wedge last cfl = cf'$
 $\langle proof \rangle$

lemma *parTrace-imp-MtransC*:
assumes $pT: parTrace cfl$
shows $(hd cfl) \rightarrow^* c (last cfl)$
 $\langle proof \rangle$

fun *finTrace* **where**
 $finTrace (cfl,s) \longleftrightarrow$
 $parTrace cfl \wedge last cfl \rightarrow t s$

declare *finTrace.simps*[*simp del*]

definition *lengthFT* $tr \equiv Suc (length (fst tr))$

definition $fstate\ tr \equiv snd\ tr$

definition $iconfig\ tr \equiv hd\ (fst\ tr)$

lemma *MtransT-Ex-finTrace*:

assumes $cf \rightarrow *t\ s$ **shows** $\exists tr. finTrace\ tr \wedge iconfig\ tr = cf \wedge fstate\ tr = s$
<proof>

lemma *finTrace-imp-MtransT*:

$finTrace\ tr \implies iconfig\ tr \rightarrow *t\ fstate\ tr$
<proof>

6.3 Relationship between during-execution and after-execution security

lemma *WbisT-trace2*:

assumes $bis: c \approx wT\ d\ s \approx t$
and $tr: finTrace\ tr\ iconfig\ tr = (c, s)$
shows $\exists tr'. finTrace\ tr' \wedge iconfig\ tr' = (d, t) \wedge fstate\ tr \approx fstate\ tr'$
<proof>

theorem *WbisT-trace*:

assumes $c \approx wT\ c$ **and** $s \approx t$
and $finTrace\ tr$ **and** $iconfig\ tr = (c, s)$
shows $\exists tr'. finTrace\ tr' \wedge iconfig\ tr' = (c, t) \wedge fstate\ tr \approx fstate\ tr'$
<proof>

theorem *ZObisT-trace2*:

assumes $bis: c \approx 01T\ d\ s \approx t$
and $tr: finTrace\ tr\ iconfig\ tr = (c, s)$
shows $\exists tr'. finTrace\ tr' \wedge iconfig\ tr' = (d, t) \wedge$
 $fstate\ tr \approx fstate\ tr' \wedge lengthFT\ tr' \leq lengthFT\ tr$
<proof>

theorem *ZObisT-trace*:

assumes $c \approx 01T\ c$ **and** $s \approx t$
and $finTrace\ tr\ iconfig\ tr = (c, s)$
shows $\exists tr'. finTrace\ tr' \wedge iconfig\ tr' = (c, t) \wedge$
 $fstate\ tr \approx fstate\ tr' \wedge lengthFT\ tr' \leq lengthFT\ tr$
<proof>

theorem *Sbis-trace*:

assumes $bis: c \approx s\ d\ s \approx t$
and $tr: finTrace\ tr\ iconfig\ tr = (c, s)$
shows $\exists tr'. finTrace\ tr' \wedge iconfig\ tr' = (d, t) \wedge fstate\ tr \approx fstate\ tr' \wedge$
 $lengthFT\ tr' = lengthFT\ tr$

<proof>

corollary *siso-trace*:

assumes *siso* c **and** $s \approx t$

and *finTrace* tr **and** *iconfig* $tr = (c, s)$

shows

$\exists tr'. \text{finTrace } tr' \wedge \text{iconfig } tr' = (c, t) \wedge \text{fstate } tr \approx \text{fstate } tr'$
 $\wedge \text{lengthFT } tr' = \text{lengthFT } tr$

<proof>

theorem *Wbis-trace*:

assumes $T: \bigwedge s. \text{mustT } c \ s$

and *bis*: $c \approx_w c \ s \approx t$

and $tr: \text{finTrace } tr \ \text{iconfig } tr = (c, s)$

shows $\exists tr'. \text{finTrace } tr' \wedge \text{iconfig } tr' = (c, t) \wedge \text{fstate } tr \approx \text{fstate } tr'$

<proof>

corollary *ZObis-trace*:

assumes $T: \bigwedge s. \text{mustT } c \ s$

and *ZObis*: $c \approx_{01} c \ \text{and} \ \text{indis}: s \approx t$

and $tr: \text{finTrace } tr \ \text{iconfig } tr = (c, s)$

shows $\exists tr'. \text{finTrace } tr' \wedge \text{iconfig } tr' = (c, t) \wedge \text{fstate } tr \approx \text{fstate } tr'$

<proof>

theorem *BisT-trace*:

assumes *bis*: $c \approx_T c \ s \approx t$

and $T: \text{mustT } c \ s \ \text{mustT } c \ t$

and $tr: \text{finTrace } tr \ \text{iconfig } tr = (c, s)$

shows $\exists tr'. \text{finTrace } tr' \wedge \text{iconfig } tr' = (c, t) \wedge \text{fstate } tr \approx \text{fstate } tr'$

<proof>

end

end

7 Concrete setting

theory *Concrete*

imports *Syntactic-Criteria After-Execution*

begin

lemma (in *PL-Indis*) *WbisT-If-cross*:

assumes $c1 \approx_w T \ c2 \ c1 \approx_w T \ c1 \ c2 \approx_w T \ c2$

shows $(\text{If } \text{tst } c1 \ c2) \approx_w T \ (\text{If } \text{tst } c1 \ c2)$

<proof>

We instantiate the following notions, kept generic so far:

- On the language syntax:
 - atoms, tests and states just like at the possibilistic case;
 - choices, to either if-choices (based on tests) or binary fixed-probability choices;
 - the schedulers, to the uniform one
- On the security semantics, the lattice of levels and the indis relation, again, just like at the possibilistic case.

datatype *level* = *Lo* | *Hi*

lemma [*simp*]: $\bigwedge l. l \neq Hi \longleftrightarrow l = Lo$ **and**

[*simp*]: $\bigwedge l. Hi \neq l \longleftrightarrow Lo = l$ **and**

[*simp*]: $\bigwedge l. l \neq Lo \longleftrightarrow l = Hi$ **and**

[*simp*]: $\bigwedge l. Lo \neq l \longleftrightarrow Hi = l$

<proof>

lemma [*dest*]: $\bigwedge l A. [l \in A; Lo \notin A] \Longrightarrow l = Hi$ **and**

[*dest*]: $\bigwedge l A. [l \in A; Hi \notin A] \Longrightarrow l = Lo$

<proof>

declare *level.split*[*split*]

instantiation *level* :: *complete-lattice*

begin

definition *top-level*: *top* \equiv *Hi*

definition *bot-level*: *bot* \equiv *Lo*

definition *inf-level*: *inf* *l1* *l2* \equiv if *Lo* \in {*l1*,*l2*} then *Lo* else *Hi*

definition *sup-level*: *sup* *l1* *l2* \equiv if *Hi* \in {*l1*,*l2*} then *Hi* else *Lo*

definition *less-eq-level*: *less-eq* *l1* *l2* \equiv (*l1* = *Lo* \vee *l2* = *Hi*)

definition *less-level*: *less* *l1* *l2* \equiv *l1* = *Lo* \wedge *l2* = *Hi*

definition *Inf-level*: *Inf* *L* \equiv if *Lo* \in *L* then *Lo* else *Hi*

definition *Sup-level*: *Sup* *L* \equiv if *Hi* \in *L* then *Hi* else *Lo*

instance

<proof>

end

lemma *sup-eq-Lo*[*simp*]: *sup* *a* *b* = *Lo* \longleftrightarrow *a* = *Lo* \wedge *b* = *Lo*

<proof>

datatype *var* = *h* | *h'* | *l* | *l'*

datatype *exp* = *Ct nat* | *Var var* | *Plus exp exp* | *Minus exp exp*

datatype *test* = *Tr* | *Eq exp exp* | *Gt exp exp* | *Non test*

datatype *atom* = *Assign var exp*
type-synonym *state* = *var* \Rightarrow *nat*

syntax
-assign :: '*a* \Rightarrow '*a* \Rightarrow '*a* (- ::= - [1000, 61] 61)

translations
x ::= *expr* == *CONST Atm (CONST Assign x expr)*

primrec *sec* **where**
sec h = *Hi*
| *sec h'* = *Hi*
| *sec l* = *Lo*
| *sec l'* = *Lo*

fun *eval* **where**
eval (Ct n) s = *n*
| *eval (Var x) s* = *s x*
| *eval (Plus e1 e2) s* = *eval e1 s + eval e2 s*
| *eval (Minus e1 e2) s* = *eval e1 s - eval e2 s*

fun *tval* **where**
tval Tr s = *True*
| *tval (Eq e1 e2) s* = (*eval e1 s = eval e2 s*)
| *tval (Gt e1 e2) s* = (*eval e1 s > eval e2 s*)
| *tval (Non e) s* = (\neg *tval e s*)

fun *aval* **where**
aval (Assign x e) s = (*s (x := eval e s)*)

definition *indis* :: (*state * state*) **setwhere**
indis \equiv $\{(s,t). \text{ALL } x. \text{sec } x = \text{Lo} \longrightarrow s \ x = t \ x\}$

interpretation *Example-PL: PL-Indis tval aval indis*
<*proof*>

fun *exprSec* **where**
exprSec (Ct n) = *bot*
| *exprSec (Var x)* = *sec x*
| *exprSec (Plus e1 e2)* = *sup (exprSec e1) (exprSec e2)*
| *exprSec (Minus e1 e2)* = *sup (exprSec e1) (exprSec e2)*

fun *tstSec* **where**
tstSec Tr = *bot*
| *tstSec (Eq e1 e2)* = *sup (exprSec e1) (exprSec e2)*
| *tstSec (Gt e1 e2)* = *sup (exprSec e1) (exprSec e2)*
| *tstSec (Non e)* = *tstSec e*

lemma *exprSec-Lo-eval-eq*: *exprSec expr = Lo* \implies (*s, t*) \in *indis* \implies *eval expr s*

= *eval expr t*
(*proof*)

lemma *compatAtmSyntactic[simp]*: *exprSec expr = Lo* \vee *sec v = Hi* \implies *Example-PL.compatAtm (Assign v expr)*
(*proof*)

lemma *presAtmSyntactic[simp]*: *sec v = Hi* \implies *Example-PL.presAtm (Assign v expr)*
(*proof*)

lemma *compatTstSyntactic[simp]*: *tstSec tst = Lo* \implies *Example-PL.compatTst tst*
(*proof*)

lemma *Example-PL.SC-discr (h ::= Ct 0)*
(*proof*)

abbreviation *siso c* \equiv *Example-PL.siso c*

abbreviation *siso0 c* \equiv *Example-PL.siso0 c*

abbreviation *discr c* \equiv *Example-PL.discr c*

abbreviation *discr0 c* \equiv *Example-PL.discr0 c*

abbreviation *Sbis-abbrev (infix \approx_s 55)* **where** *c1 \approx_s c2* \equiv *(c1,c2) \in Example-PL.Sbis*

abbreviation *ZObis-abbrev (infix \approx_{01} 55)* **where** *c1 \approx_{01} c2* \equiv *(c1,c2) \in Example-PL.ZObis*

abbreviation *ZObisT-abbrev (infix \approx_{01T} 55)* **where** *c1 \approx_{01T} c2* \equiv *(c1,c2) \in Example-PL.ZObisT*

abbreviation *Wbis-abbrev (infix \approx_w 55)* **where** *c1 \approx_w c2* \equiv *(c1,c2) \in Example-PL.Wbis*

abbreviation *WbisT-abbrev (infix \approx_{wT} 55)* **where** *c1 \approx_{wT} c2* \equiv *(c1,c2) \in Example-PL.WbisT*

abbreviation *BisT-abbrev (infix \approx_T 55)* **where** *c1 \approx_T c2* \equiv *(c1,c2) \in Example-PL.BisT*

7.1 Programs from EXAMPLE 1

definition [*simp*]: *c0 = (h ::= Ct 0)*

definition [*simp*]: *c1 = (if Eq (Var l) (Ct 0) then h ::= Ct 1 else l ::= Ct 2)*

definition [*simp*]: *c2 = (if Eq (Var h) (Ct 0) then h ::= Ct 1 else h ::= Ct 2)*

definition [*simp*]: *c3 = (if Eq (Var h) (Ct 0) then h ::= Ct 1 ;; h ::= Ct 2
else h ::= Ct 3)*

definition [*simp*]: *c4 = l ::= Ct 4 ;; c3*

definition [*simp*]: *c5 = c3 ;; l ::= Ct 4*

definition [simp]: $c6 = l ::= \text{Var } h$

definition [simp]: $c7 = l ::= \text{Var } h ;; l ::= \text{Ct } 0$

definition [simp]: $c8 = h' ::= \text{Var } h ;;$
 $\text{while } Gt (\text{Var } h) (\text{Ct } 0) \text{ do } (h ::= \text{Minus } (\text{Var } h) (\text{Ct } 1) ;; h' ::= \text{Plus } (\text{Var } h')$
 $(\text{Ct } 1)) ;;$
 $l ::= \text{Ct } 4$

definition [simp]: $c9 = c7 \mid l' ::= \text{Var } l$

definition [simp]: $c10 = c5 \mid l ::= \text{Ct } 5$

definition [simp]: $c11 = c8 \mid l ::= \text{Ct } 5$

declare *bot-level*[iff]

theorem *c0*: *siso c0 discr c0*
<proof>

theorem *c1*: *siso c1 c1 \approx_s c1*
<proof>

theorem *c2*: *discr c2*
<proof>

theorem *Sbis-c2*: *c2 \approx_s c2*
<proof>

theorem *c3*: *discr c3*
<proof>

theorem *c4*: *c4 \approx_{01} c4*
<proof>

theorem *c5*: *c5 \approx_w c5*
<proof>

Example 4 from the paper

theorem *c3 \approx_{wT} c3* *<proof>*

theorem *c5 \approx_{wT} c5* *<proof>*

corollary *discr (while Eq (Var h) (Ct 0) do h ::= Ct 0)*
<proof>

Example 5 from the paper

definition [simp]: $c12 \equiv h ::= \text{Ct } 4 ;;$
 $\text{while } Gt (\text{Var } h) (\text{Ct } 0)$

$do (h ::= Minus (Var h) (Ct 1) ;; h' ::= Plus (Var h') (Ct 1)) ;;$
 $l ::= Ct 1$

corollary $(c12 \mid l ::= Ct 2) \approx T (c12 \mid l ::= Ct 2)$
 $\langle proof \rangle$

definition $[simp]$: $c13 =$
 $(if Eq (Var h) (Ct 0) then h ::= Ct 1 ;; l ::= Ct 2 else l ::= Ct 2) ;; l' ::= Ct 4$

lemma $c13$ -inner:
 $(h ::= Ct 1 ;; l ::= Ct 2) \approx wT (l ::= Ct 2)$
 $\langle proof \rangle$

theorem $c13 \approx wT c13$
 $\langle proof \rangle$

end

References

- [1] A. Popescu, J. Hölzl, and T. Nipkow. Proving possibilistic, probabilistic noninterference. In *Certified Programs and Proofs (CPP) '12*, 2012.