Two theorems about the geometry of the critical points of a complex polynomial

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Abstract

This entry formalises two well-known results about the geometric relation between the *roots* of a complex polynomial and its *critical points*, i.e. the roots of its derivative.

The first of these is the *GauSS–Lucas Theorem*: The critical points of a complex polynomial lie inside the convex hull of its roots.

The second one is *Jensen's Theorem*: Every non-real critical point of a real polynomial lies inside a disc between two conjugate roots. These discs are called the *Jensen discs*: the Jensen disc of a pair of conjugate roots $a \pm bi$ is the smallest disc that contains both of them, i.e. the disc with centre a and radius b.

Contents

1 Missing Library Material

theory *Polynomial_Crit_Geometry_Library*
imports
  "HOL-Computational_Algebra.Computational_Algebra"
  "HOL-Library.FuncSet"
  "Polynomial_Interpolation.Ring_Hom_Poly"
begin

1.1 Multisets

lemma *size_repeat_mset [simp]: "size (repeat_mset n A) = n * size A"*
  ⟨*proof*⟩

lemma *count_image_mset_inj:*
  "inj f ⟹ count (image_mset f A) (f x) = count A x"
  ⟨*proof*⟩

lemma *count_le_size: "count A x ≤ size A"*
  ⟨*proof*⟩

lemma *image_mset_cong_simp:*
  "M = M' ⟹ (⋀x. x ∈# M =simp=> f x = g x) ⟹ {#f x. x ∈# M#} = {#g x. x ∈# M'#}"
  ⟨*proof*⟩

lemma *sum_mset_nonneg:*
  fixes *A :: "'a :: ordered_comm_monoid_add multiset"*
  assumes *"⋀x. x ∈# A ⟹ x ≥ 0"*
  shows *"sum_mset A ≥ 0"*
  ⟨*proof*⟩

lemma *sum_mset_pos:*
  fixes *A :: "'a :: ordered_comm_monoid_add multiset"*
  assumes *"A ≠ {#}"*
  assumes *"⋀x. x ∈# A ⟹ x > 0"*
  shows *"sum_mset A > 0"*
⟨*proof*⟩

1.2 Polynomials

lemma *order_pos_iff: "p ≠ 0 ⟹ order x p > 0 ⟷ poly p x = 0"*
  ⟨*proof*⟩

lemma *order_prod_mset:*
  "0 ∉# P ⟹ order x (prod_mset P) = sum_mset (image_mset (λp. order x p) P)"
  ⟨*proof*⟩

lemma *order_prod:*

"(⋀x. x ∈ I ⟹ f x ≠ 0) ⟹ order x (prod f I) = (∑i∈I. order x (f i))"
⟨proof⟩

lemma *order_linear_factor:*
assumes "a ≠ 0 ∨ b ≠ 0"
shows "order x [:a, b:] = (if b * x + a = 0 then 1 else 0)"
⟨proof⟩

lemma *order_linear_factor' [simp]:*
assumes "a ≠ 0 ∨ b ≠ 0" "b * x + a = 0"
shows "order x [:a, b:] = 1"
⟨proof⟩

lemma *degree_prod_mset_eq:* "0 ∉# P ⟹ degree (prod_mset P) = (∑p∈#P. degree p)"
for *P* :: "'a::idom poly multiset"
⟨proof⟩

lemma *degree_prod_list_eq:* "0 ∉ set ps ⟹ degree (prod_list ps) = (∑p←ps. degree p)"
for *ps* :: "'a::idom poly list"
⟨proof⟩

lemma *order_conv_multiplicity:*
assumes "p ≠ 0"
shows "order x p = multiplicity [:-x, 1:] p"
⟨proof⟩

1.3 Polynomials over algebraically closed fields

lemma *irreducible_alg_closed_imp_degree_1:*
assumes "irreducible (p :: 'a :: alg_closed_field poly)"
shows "degree p = 1"
⟨proof⟩

lemma *prime_poly_alg_closedE:*
assumes "prime (q :: 'a :: {alg_closed_field, field_gcd} poly)"
obtains *c* **where** "q = [:-c, 1:]" "poly q c = 0"
⟨proof⟩

lemma *prime_factors_alg_closed_poly_bij_betw:*
assumes "p ≠ (0 :: 'a :: {alg_closed_field, field_gcd} poly)"
shows "bij_betw (λx. [:-x, 1:]) {x. poly p x = 0} (prime_factors p)"
⟨proof⟩

lemma *alg_closed_imp_factorization':*
assumes "p ≠ (0 :: 'a :: alg_closed_field poly)"
shows "p = smult (lead_coeff p) (∏x | poly p x = 0. [:-x, 1:] ^ order

x p)"
⟨*proof*⟩

1.4 Complex polynomials and conjugation

lemma *complex_poly_real_coeffsE:*
  assumes *"set (coeffs p) ⊆ ℝ"*
  obtains *p'* **where** *"p = map_poly complex_of_real p'"*
⟨*proof*⟩

lemma *order_map_poly_cnj:*
  assumes *"p ≠ 0"*
  shows *"order x (map_poly cnj p) = order (cnj x) p"*
⟨*proof*⟩

1.5 *n*-ary product rule for the derivative

lemma *has_field_derivative_prod_mset [derivative_intros]:*
  assumes *"⋀x. x ∈# A ⟹ (f x has_field_derivative f' x) (at z)"*
  shows *"((λu. ∏x∈#A. f x u) has_field_derivative (∑x∈#A. f' x * (∏y∈#A-{#x#}. f y z))) (at z)"*
  ⟨*proof*⟩

lemma *has_field_derivative_prod [derivative_intros]:*
  assumes *"⋀x. x ∈ A ⟹ (f x has_field_derivative f' x) (at z)"*
  shows *"((λu. ∏x∈A. f x u) has_field_derivative (∑x∈A. f' x * (∏y∈A-{x}. f y z))) (at z)"*
  ⟨*proof*⟩

lemma *has_field_derivative_prod_mset':*
  assumes *"⋀x. x ∈# A ⟹ f x z ≠ 0"*
  assumes *"⋀x. x ∈# A ⟹ (f x has_field_derivative f' x) (at z)"*
  defines *"P ≡ (λA u. ∏x∈#A. f x u)"*
  shows *"(P A has_field_derivative (P A z * (∑x∈#A. f' x / f x z))) (at z)"*
  ⟨*proof*⟩

lemma *has_field_derivative_prod':*
  assumes *"⋀x. x ∈ A ⟹ f x z ≠ 0"*
  assumes *"⋀x. x ∈ A ⟹ (f x has_field_derivative f' x) (at z)"*
  defines *"P ≡ (λA u. ∏x∈A. f x u)"*
  shows *"(P A has_field_derivative (P A z * (∑x∈A. f' x / f x z))) (at z)"*
⟨*proof*⟩

1.6 Facts about complex numbers

lemma *Re_sum_mset: "Re (sum_mset X) = (∑x∈#X. Re x)"*
  ⟨*proof*⟩

lemma *Im_sum_mset: "Im (sum_mset X) = (∑x∈#X. Im x)"*
⟨*proof*⟩

lemma *Re_sum_mset': "Re (∑x∈#X. f x) = (∑x∈#X. Re (f x))"*
⟨*proof*⟩

lemma *Im_sum_mset': "Im (∑x∈#X. f x) = (∑x∈#X. Im (f x))"*
⟨*proof*⟩

lemma *inverse_complex_altdef: "inverse z = cnj z / norm z ^ 2"*
⟨*proof*⟩

end

theory *Polynomial_Crit_Geometry*
imports
"HOL-Computational_Algebra.Computational_Algebra"
"HOL-Analysis.Analysis"
Polynomial_Crit_Geometry_Library
begin

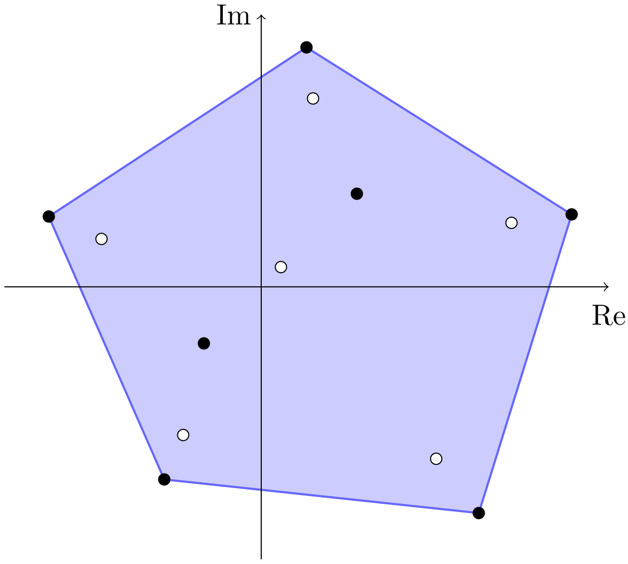

Figure 1: Example for the GauSS–Lucas Theorem: The roots (●) and critical points (○) of $x^7 - 2x^6 + x^5 + x^4 - (1+i)x^3 - 15ix^2 - 4(1-i)x - 7$.
The critical points all lie inside the convex hull of the roots (▢).

2 The GauSS–Lucas Theorem

The following result is known as the *GauSS–Lucas Theorem*: The critical points of a non-constant complex polynomial lie inside the convex hull of its roots.

The proof is relatively straightforward by writing the polynomial in the form

$$p(x) = \prod_{i=1}^{n} (x - x_i)^{a_i} \ ,$$

from which we get the derivative

$$p'(x) = p(x) \cdot \sum_{i=1}^{n} \frac{a_i}{x - x_i} \ .$$

With some more calculations, one can then see that every root x of p' can be written as

$$x = \sum_{i=1}^{n} \frac{u_i}{U} \cdot x_i$$

where $u_i = \frac{a_i}{|x - x_i|^2}$ and $U = \sum_{i=1}^{n} u_i$.

theorem *pderiv_roots_in_convex_hull:*

```
  fixes p :: "complex poly"
  assumes "degree p ≠ 0"
  shows   "{z. poly (pderiv p) z = 0} ⊆ convex hull {z. poly p z = 0}"
⟨proof⟩
```

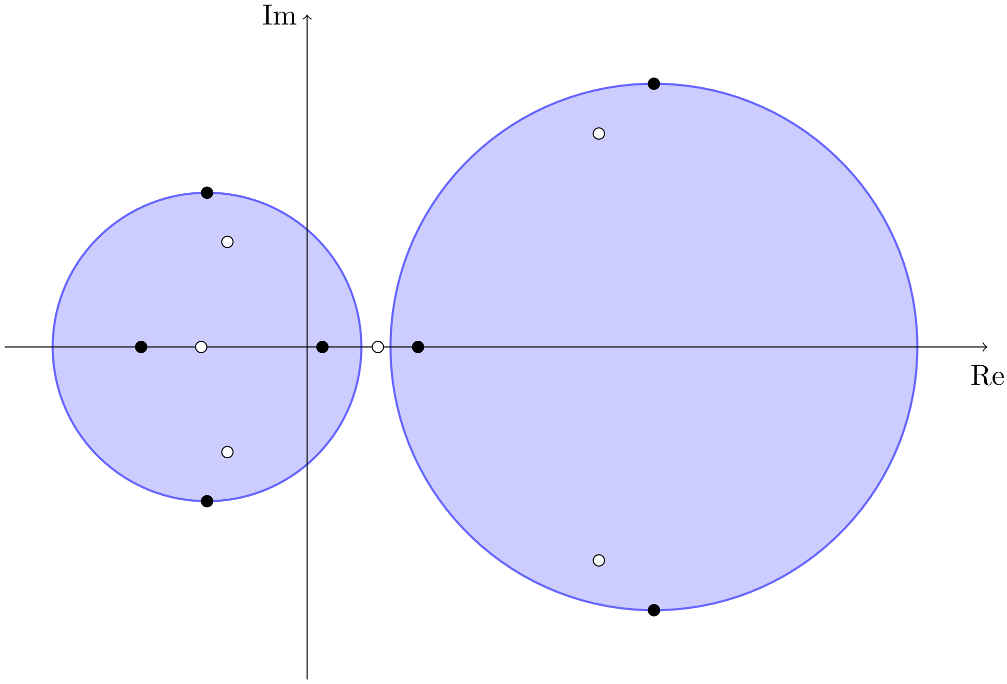

Figure 2: Example for Jensen's Theorem: The roots (•) and critical points (○) of the polynomial $x^7 - 3x^6 + 2x^5 + 8x^4 + 10x^3 - 10x + 1$.
It can be seen that all the non-real critical points lie inside a Jensen disc (○), whereas there can be real critical points that do *not* lie inside a Jensen disc.

3 Jensen's Theorem

For each root `w` of a real polynomial `p`, the Jensen disc of `w` is the smallest disc containing both w and $\overline{w}$, i.e. the disc with centre $\text{Re}(w)$ and radius $|\text{Im}(w)|$.

We now show that if `p` is a real polynomial, every non-real critical point of `p` lies inside a Jensen disc of one of its non-real roots.

```
definition jensen_disc :: "complex ⇒ complex set" where
  "jensen_disc w = cball (of_real (Re w)) |Im w|"

theorem pderiv_root_in_jensen_disc:
  fixes p :: "complex poly"
  assumes "set (coeffs p) ⊆ ℝ" and "degree p ≠ 0"
  assumes "poly (pderiv p) z = 0" and "z ∉ ℝ"
  shows   "∃w. w ∉ ℝ ∧ poly p w = 0 ∧ z ∈ jensen_disc w"
⟨proof⟩
```

end

References

[1] F. Enescu. Math 4444/6444 Polynomials 2, Lecture Notes, Lecture 9. https://math.gsu.edu/fenescu/fall2010/lec9-polyn-2010.pdf, 2010.

[2] M. Marden. *Geometry of Polynomials.* American Mathematical Society Mathematical Surveys. American Mathematical Society, 1966.