

# The Polylogarithm Function

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## Abstract

This entry provides a definition of the *Polylogarithm function*, commonly denoted as  $\text{Li}_s(z)$ . Here,  $z$  is a complex number and  $s$  an integer parameter. This function can be defined by the power series expression  $\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$  for  $|z| < 1$  and analytically extended to the entire complex plane, except for a branch cut on  $\mathbb{R}_{\geq 1}$ .

Several basic properties are also proven, such as the relationship to the Eulerian polynomials via  $\text{Li}_{-k}(z) = z(1-z)^{k-1} A_k(z)$  for  $k \geq 0$ , the derivative formula  $\frac{d}{dz} \text{Li}_s(z) = \frac{1}{z} \text{Li}_{s-1}(z)$ , the relation to the “normal” logarithm via  $\text{Li}_1(z) = -\ln(1-z)$ , and the duplication formula  $\text{Li}_s(z) + \text{Li}_s(-z) = 2^{1-s} \text{Li}_s(z^2)$ .

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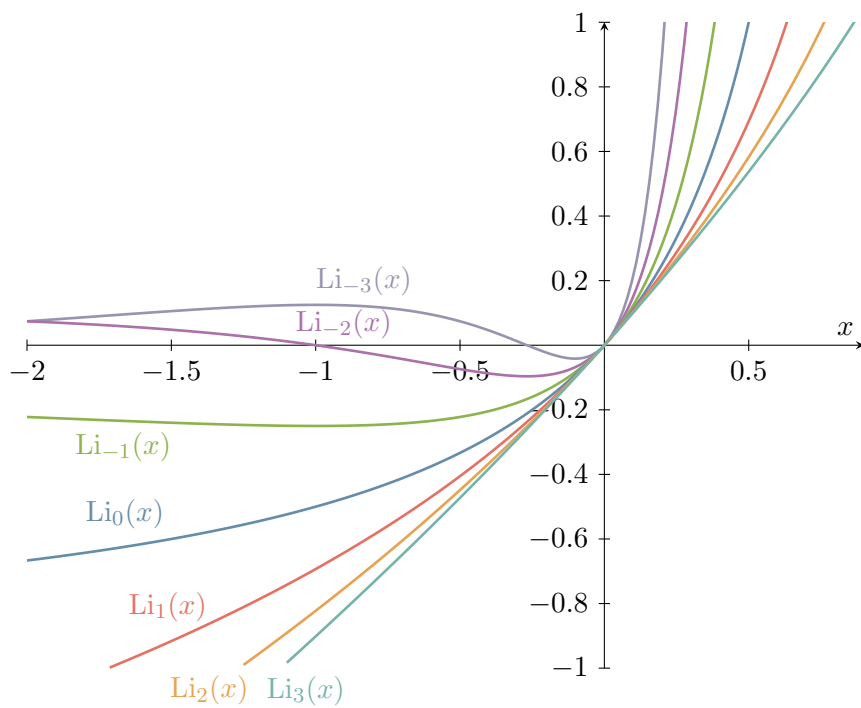


Figure 1: Plots of  $\text{Li}_s(x)$  for  $s = -3, -2, \dots, 3$  and real inputs  $x \in [-2, 1]$

# 1 Auxiliary material

**theory** Polylog\_Library

**imports**

"HOL-Complex\_Analysis.Complex\_Analysis"

"Linear\_Recurrences.Eulerian\_Polynomials"

**begin**

## 1.1 Miscellaneous

**lemma** linear\_of\_real [intro]: "linear of\_real"

*<proof>*

**lemma** fps\_conv\_radius\_fps\_of\_poly [simp]:

fixes  $p :: 'a :: \{\text{banach, real\_normed\_div\_algebra}\}$  poly"

shows "fps\_conv\_radius (fps\_of\_poly p) =  $\infty$ "

*<proof>*

**lemma** eval\_fps\_power:

fixes  $F :: 'a :: \{\text{banach, real\_normed\_div\_algebra, comm\_ring\_1}\}$  fps"

assumes  $z: \text{"norm } z < \text{fps\_conv\_radius } F"$

shows "eval\_fps ( $F \wedge n$ ) z = eval\_fps F z  $\wedge n$ "

*<proof>*

**lemma** eval\_fps\_of\_poly [simp]: "eval\_fps (fps\_of\_poly p) z = poly p z"

*<proof>*

**lemma** poly\_holomorphic\_on [holomorphic\_intros]:

assumes [holomorphic\_intros]: "f holomorphic\_on A"

shows " $(\lambda z. \text{poly } p (f z))$  holomorphic\_on A"

*<proof>*

**lemma** simply\_connected\_eq\_global\_primitive:

assumes "simply\_connected S" "open S" "f holomorphic\_on S"

obtains h where " $\wedge z. z \in S \implies (h \text{ has\_field\_derivative } f z) \text{ (at } z)$ "

*<proof>*

**lemma**

assumes " $x \in \text{closed\_segment } y z$ "

shows in\_closed\_segment\_imp\_Re\_in\_closed\_segment: "Re x  $\in$  closed\_segment (Re y) (Re z)" (is ?th1)

and in\_closed\_segment\_imp\_Im\_in\_closed\_segment: "Im x  $\in$  closed\_segment (Im y) (Im z)" (is ?th2)

*<proof>*

**lemma** linepath\_in\_open\_segment: " $t \in \{0 <..<1\} \implies x \neq y \implies \text{linepath}$

$x y t \in \text{open\_segment } x y$ "

*<proof>*

**lemma** in\_open\_segment\_imp\_Re\_in\_open\_segment:

**assumes** "x ∈ open\_segment y z" "Re y ≠ Re z"  
**shows** "Re x ∈ open\_segment (Re y) (Re z)"  
 ⟨proof⟩

**lemma in\_open\_segment\_imp\_Im\_in\_open\_segment:**  
**assumes** "x ∈ open\_segment y z" "Im y ≠ Im z"  
**shows** "Im x ∈ open\_segment (Im y) (Im z)"  
 ⟨proof⟩

**lemma poly\_eulerian\_poly\_0 [simp]:** "poly (eulerian\_poly n) 0 = 1"  
 ⟨proof⟩

**lemma eulerian\_poly\_at\_1 [simp]:** "poly (eulerian\_poly n) 1 = fact n"  
 ⟨proof⟩

## 1.2 The slotted complex plane

**lemma closed\_slot\_left:** "closed (complex\_of\_real ` {..c})"  
 ⟨proof⟩

**lemma closed\_slot\_right:** "closed (complex\_of\_real ` {c..})"  
 ⟨proof⟩

**lemma complex\_slot\_left\_eq:** "complex\_of\_real ` {..c} = {z. Re z ≤ c  
 ∧ Im z = 0}"  
 ⟨proof⟩

**lemma complex\_slot\_right\_eq:** "complex\_of\_real ` {c..} = {z. Re z ≥ c  
 ∧ Im z = 0}"  
 ⟨proof⟩

**lemma complex\_double\_slot\_eq:**  
 "complex\_of\_real ` ({..c1} ∪ {c2..}) = {z. Im z = 0 ∧ (Re z ≤ c1 ∨  
 Re z ≥ c2)}"  
 ⟨proof⟩

**lemma starlike\_slotted\_complex\_plane\_left\_aux:**  
**assumes** z: "z ∈ -(complex\_of\_real ` {..c})" **and** c: "c < c'"  
**shows** "closed\_segment (complex\_of\_real c') z ⊆ -(complex\_of\_real  
 ` {..c})"  
 ⟨proof⟩

**lemma starlike\_slotted\_complex\_plane\_left:** "starlike (-(complex\_of\_real  
 ` {..c}))"  
 ⟨proof⟩

```

lemma starlike_slotted_complex_plane_right_aux:
  assumes z: "z ∈ -(complex_of_real ` {c..})" and c: "c > c'"
  shows "closed_segment (complex_of_real c') z ⊆ -(complex_of_real `
  {c..})"
  ⟨proof⟩

lemma starlike_slotted_complex_plane_right: "starlike (-(complex_of_real
  ` {c..}))"
  ⟨proof⟩

lemma starlike_doubly_slotted_complex_plane_aux:
  assumes z: "z ∈ -(complex_of_real ` ({..c1} ∪ {c2..}))" and c: "c1
  < c" "c < c2"
  shows "closed_segment (complex_of_real c) z ⊆ -(complex_of_real `
  ({..c1} ∪ {c2..}))"
  ⟨proof⟩

lemma starlike_doubly_slotted_complex_plane:
  assumes "c1 < c2"
  shows "starlike (-(complex_of_real ` ({..c1} ∪ {c2..})))"
  ⟨proof⟩

lemma simply_connected_slotted_complex_plane_left:
  "simply_connected (-(complex_of_real ` {..c}))"
  ⟨proof⟩

lemma simply_connected_slotted_complex_plane_right:
  "simply_connected (-(complex_of_real ` {c..}))"
  ⟨proof⟩

lemma simply_connected_doubly_slotted_complex_plane:
  "c1 < c2 ⇒ simply_connected (-(complex_of_real ` ({..c1} ∪ {c2..})))"
  ⟨proof⟩

end

```

## 2 The Polylogarithm Function

```

theory Polylog
imports
  "HOL-Complex_Analysis.Complex_Analysis"
  "Linear_Recurrences.Eulerian_Polynomials"
  "HOL-Real_Asymp.Real_Asymp"
  Polylog_Library
begin

```

## 2.1 Definition and basic properties

The principal branch of the Polylogarithm function  $\text{Li}_s(z)$  is defined as

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

for  $|z| < 1$  and elsewhere by analytic continuation. For integer  $s \leq 0$  it is holomorphic except for a pole at  $z = 1$ . For other values of  $s$  it is holomorphic except for a branch cut along the line  $[1, \infty)$ .

Special values include  $\text{Li}_0(z) = \frac{z}{1-z}$  and  $\text{Li}_1(z) = -\log(1-z)$ .

One could potentially generalise this to arbitrary  $s \in \mathbb{C}$ , but this makes the analytic continuation somewhat more complicated, so we chosed not to do this at this point.

In the following, we define the principal branch of  $\text{Li}_s(z)$  for integer  $s$ .

```

definition polylog :: "int  $\Rightarrow$  complex  $\Rightarrow$  complex" where
  "polylog k z =
    (if k  $\leq$  0 then z * poly (eulerian_poly (nat (-k))) z * (1 - z) powi
    (k - 1)
    else if z  $\in$  of_real ` {1..} then 0
    else (SOME f. f holomorphic_on -of_real`{1..}  $\wedge$ 
    ( $\forall z \in$  ball 0 1. f z = ( $\sum$  n. of_nat (Suc n) powi (-k)
    * z ^ Suc n))) z)"

```

```

lemma conv_radius_polylog: "conv_radius ( $\lambda$ r. of_nat r powi k :: complex)
= 1"
<proof>

```

```

lemma abs_summable_polylog:
  "norm z < 1  $\implies$  summable ( $\lambda$ r. norm (of_nat r powi k * z ^ r :: complex))"
<proof>

```

Two very central results that characterise the polylogarithm:

$$\text{Li}'_s(z) = \frac{1}{z} \text{Li}_{s-1}(z) \quad \text{and} \quad \text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad \text{for } |z| < 1$$

```

theorem has_field_derivative_polylog [derivative_intros]:
  " $\wedge$ z. z  $\in$  (if k  $\leq$  0 then -{1} else -(of_real ` {1..}))  $\implies$ 
    (polylog k has_field_derivative (if z = 0 then 1 else polylog
    (k - 1) z / z)) (at z within A)"
  and sums_polylog: "norm z < 1  $\implies$  ( $\lambda$ n. of_nat (Suc n) powi (-k) * z
  ^ Suc n) sums polylog k z"
<proof>

```

```

lemma has_field_derivative_polylog' [derivative_intros]:
  assumes "(f has_field_derivative f') (at z within A)"

```

**assumes** "if  $k \leq 0$  then  $f z \neq 1$  else  $\text{Im}(f z) \neq 0 \vee \text{Re}(f z) < 1$ "  
**shows** " $(\lambda z. \text{polylog } k (f z)) \text{ has\_field\_derivative}$   
 $(\text{if } f z = 0 \text{ then } 1 \text{ else } \text{polylog } (k-1) (f z) / f z * f')$ "  
 (at  $z$  within  $A$ )"  
 <proof>

**lemma polylog\_0 [simp]:** "polylog  $k$   $0 = 0$ "  
 <proof>

A simple consequence of the derivative formula is the following recurrence for  $\text{Li}_s$  via a contour integral:

$$\text{Li}_s(z) = \int_0^z \frac{1}{w} \text{Li}_{s-1}(w) dw$$

**theorem polylog\_has\_contour\_integral:**  
**assumes** " $z \notin \text{complex\_of\_real } \setminus (\{..-1\} \cup \{1..})$ "  
**shows** " $(\lambda w. \text{polylog } s w / w) \text{ has\_contour\_integral polylog } (s + 1)$   
 $z) (\text{linepath } 0 z)$ "  
 <proof>

**lemma sums\_polylog':**  
 " $\text{norm } z < 1 \implies k \neq 0 \implies (\lambda n. \text{of\_nat } n \text{ powi } -k * z ^ n) \text{ sums polylog}$   
 $k z$ "  
 <proof>

**lemma polylog\_altdef1:**  
 " $\text{norm } z < 1 \implies \text{polylog } k z = (\sum n. \text{of\_nat } (\text{Suc } n) \text{ powi } -k * z ^ \text{Suc}$   
 $n)$ "  
 <proof>

**lemma polylog\_altdef2:**  
 " $\text{norm } z < 1 \implies k \neq 0 \implies \text{polylog } k z = (\sum n. \text{of\_nat } n \text{ powi } -k * z ^ n)$ "  
 <proof>

**lemma polylog\_at\_pole:** "polylog  $k$   $1 = 0$ "  
 <proof>

**lemma polylog\_at\_branch\_cut:** " $x \geq 1 \implies k > 0 \implies \text{polylog } k (\text{of\_real } x) = 0$ "  
 <proof>

**lemma holomorphic\_on\_polylog [holomorphic\_intros]:**  
**assumes** " $A \subseteq (\text{if } k \leq 0 \text{ then } \{-1\} \text{ else } \text{-of\_real } \setminus \{1..})$ "  
**shows** "polylog  $k$  holomorphic\_on  $A$ "  
 <proof>

**lemmas holomorphic\_on\_polylog' [holomorphic\_intros] =**

holomorphic\_on\_compose\_gen [OF \_ holomorphic\_on\_polylog[OF order.refl],  
unfolded o\_def]

lemma analytic\_on\_polylog [analytic\_intros]:  
  assumes "A  $\subseteq$  (if k  $\leq$  0 then -{1} else -of\_real ` {1..})"  
  shows "polylog k analytic\_on A"  
<proof>

lemmas analytic\_on\_polylog' [analytic\_intros] =  
  analytic\_on\_compose\_gen [OF \_ analytic\_on\_polylog[OF order.refl], unfolded  
  o\_def]

lemma continuous\_on\_polylog [analytic\_intros]:  
  assumes "A  $\subseteq$  (if k  $\leq$  0 then -{1} else -of\_real ` {1..})"  
  shows "continuous\_on A (polylog k)"  
<proof>

lemmas continuous\_on\_polylog' [continuous\_intros] =  
  continuous\_on\_compose2 [OF continuous\_on\_polylog [OF order.refl]]

## 2.2 Special values

lemma polylog\_neg\_int\_left:  
  "k < 0  $\implies$  polylog k z = z \* poly (eulerian\_poly (nat (-k))) z \* (1  
- z) powi (k - 1)"  
<proof>

lemma polylog\_0\_left: "polylog 0 z = z / (1 - z)"  
<proof>

lemma polylog\_neg1\_left: "polylog (-1) x = x / (1 - x) ^ 2"  
<proof>

lemma polylog\_neg2\_left: "polylog (-2) x = x \* (1 + x) / (1 - x) ^ 3"  
<proof>

lemma polylog\_neg3\_left: "polylog (-3) x = x \* (1 + 4 \* x + x<sup>2</sup>) / (1  
- x) ^ 4"  
<proof>

lemma polylog\_1:  
  assumes "z  $\notin$  of\_real ` {1..}"  
  shows "polylog 1 z = -ln (1 - z)"  
<proof>

lemma is\_pole\_polylog\_1:  
  assumes "k  $\leq$  0"  
  shows "is\_pole (polylog k) 1"  
<proof>



```

lemma zorder_polylog_1:
  assumes "k ≤ 0"
  shows "zorder (polylog k) 1 = k - 1"
⟨proof⟩

lemma isolated_singularity_polylog_1:
  assumes "k ≤ 0"
  shows "isolated_singularity_at (polylog k) 1"
⟨proof⟩

lemma not_essential_polylog_1:
  assumes "k ≤ 0"
  shows "not_essential (polylog k) 1"
⟨proof⟩

lemma polylog_meromorphic_on [meromorphic_intros]:
  assumes "k ≤ 0"
  shows "polylog k meromorphic_on UNIV {1}"
⟨proof⟩

```

## 2.3 Duplication formula

Lastly, we prove the following duplication formula that the polylogarithm satisfies:

$$\operatorname{Li}_s(z) + \operatorname{Li}_s(-z) = 2^{1-s} \operatorname{Li}_s(z^2)$$

The proof is a relatively simple manipulation of infinite sum that defines  $\operatorname{Li}_s(z)$  for  $|z| < 1$ , followed by analytic continuation to its full domain.

```

theorem polylog_duplication:
  assumes "if s ≤ 0 then z ∉ {-1, 1} else z ∉ complex_of_real ` ({..-1}
  ∪ {1..})"
  shows "polylog s z + polylog s (-z) = 2 powi (1 - s) * polylog s (z^2)"
⟨proof⟩

```

end

## References

- [1] J. Mason and D. Handscomb. *Chebyshev Polynomials*. CRC Press, 2002.