Graph Theory

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Abstract

This development provides a formalization of planarity based on combinatorial maps and proves that Kuratowski's theorem implies combinatorial planarity. Moreover, it contains verified implementations of programs checking certificates for planarity (i.e., a combinatorial map) or non-planarity (i.e., a Kuratowski subgraph).

The development is described in [1].

Contents

theory *Graph-Genus*
imports
HOL−Combinatorics.Permutations
Graph-Theory.Graph-Theory
begin

lemma *nat-diff-mod-right*:
fixes *a b c* :: *nat*
assumes *b* < *a*
shows (*a* − *b*) *mod c* = (*a* − *b mod c*) *mod c*
proof −
from *assms* **have** *b-mod*: *b mod c* ≤ *a*
by (*metis mod-less-eq-dividend linear not-le order-trans*)
have *int* ((*a* − *b*) *mod c*) = (*int a* − *int b mod int c*) *mod int c*
using *assms* **by** (*simp add*: *zmod-int of-nat-diff mod-simps*)
also have . . . = *int* ((*a* − *b mod c*) *mod c*)
using *assms b-mod* **by** (*simp add*: *zmod-int*)
finally show *?thesis* **by** *simp*
qed

lemma *inj-on-f-imageI*:
assumes *inj-on f S* ⋀*t*. *t* ∈ *T* ⟹ *t* ⊆ *S*
shows *inj-on* ((‘) *f*) *T*
using *assms* **by** (*auto simp*: *inj-on-image-eq-iff intro*: *inj-onI*)

1 Combinatorial Maps

lemma (**in** *bidirected-digraph*) *has-dom-arev*:
has-dom arev (*arcs G*)
using *arev-dom* **by** (*auto simp*: *has-dom-def*)

record ′*b pre-map* =
edge-rev :: ′*b* ⇒ ′*b*
edge-succ :: ′*b* ⇒ ′*b*

definition *edge-pred* :: ′*b pre-map* ⇒ ′*b* ⇒ ′*b* **where**
edge-pred M = *inv* (*edge-succ M*)

locale *pre-digraph-map* = *pre-digraph* + **fixes** *M* :: ′*b pre-map*

locale *digraph-map* = *fin-digraph G*
+ *pre-digraph-map G M*
+ *bidirected-digraph G edge-rev M* **for** *G M* +
assumes *edge-succ-permutes*: *edge-succ M permutes arcs G*
assumes *edge-succ-cyclic*: $\bigwedge v.\ v \in$ *verts G* $\Longrightarrow$ *out-arcs G v* $\neq \{\} \Longrightarrow$ *cyclic-on* (*edge-succ M*) (*out-arcs G v*)

lemma (**in** *fin-digraph*) *digraph-mapI*:
assumes *bidi*: $\bigwedge a.\ a \notin$ *arcs G* $\Longrightarrow$ *edge-rev M a* = *a*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *edge-rev M a* $\neq$ *a*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *edge-rev M* (*edge-rev M a*) = *a*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *tail G* (*edge-rev M a*) = *head G a*
assumes *edge-succ-permutes*: *edge-succ M permutes arcs G*
assumes *edge-succ-cyclic*: $\bigwedge v.\ v \in$ *verts G* $\Longrightarrow$ *out-arcs G v* $\neq \{\} \Longrightarrow$ *cyclic-on* (*edge-succ M*) (*out-arcs G v*)
shows *digraph-map G M*
using *assms* **by** *unfold-locales auto*

lemma (**in** *fin-digraph*) *digraph-mapI-permutes*:
assumes *bidi*: *edge-rev M permutes arcs G*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *edge-rev M a* $\neq$ *a*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *edge-rev M* (*edge-rev M a*) = *a*
$\bigwedge a.\ a \in$ *arcs G* $\Longrightarrow$ *tail G* (*edge-rev M a*) = *head G a*
assumes *edge-succ-permutes*: *edge-succ M permutes arcs G*
assumes *edge-succ-cyclic*: $\bigwedge v.\ v \in$ *verts G* $\Longrightarrow$ *out-arcs G v* $\neq \{\} \Longrightarrow$ *cyclic-on* (*edge-succ M*) (*out-arcs G v*)
shows *digraph-map G M*
proof −
interpret *bidirected-digraph G edge-rev M* **using** *bidi* **by** *unfold-locales* (*auto simp*: *permutes-def*)
show *?thesis*
using *edge-succ-permutes edge-succ-cyclic* **by** *unfold-locales*
qed

context *digraph-map*
begin

lemma *digraph-map*[*intro*]: *digraph-map G M* **by** *unfold-locales*

lemma *permutation-edge-succ*: *permutation* (*edge-succ M*)
by (*metis edge-succ-permutes finite-arcs permutation-permutes*)

lemma *edge-pred-succ*[*simp*]: *edge-pred M* (*edge-succ M a*) = *a*
by (*metis edge-pred-def edge-succ-permutes permutes-inverses*(*2*))

lemma *edge-succ-pred*[*simp*]: *edge-succ M* (*edge-pred M a*) = *a*
by (*metis edge-pred-def edge-succ-permutes permutes-inverses*(*1*))

lemma *edge-pred-permutes*: *edge-pred M permutes arcs G*
  unfolding *edge-pred-def* **using** *edge-succ-permutes* **by** (*rule permutes-inv*)

lemma *permutation-edge-pred*: *permutation* (*edge-pred M*)
  by (*metis edge-pred-permutes finite-arcs permutation-permutes*)

lemma *edge-succ-eq-iff* [*simp*]: $\bigwedge x\ y.$ *edge-succ M x* = *edge-succ M y* $\longleftrightarrow$ *x* = *y*
  by (*metis edge-pred-succ*)

lemma *edge-rev-in-arcs*[*simp*]: *edge-rev M a* $\in$ *arcs G* $\longleftrightarrow$ *a* $\in$ *arcs G*
  by (*metis arev-arev arev-permutes-arcs permutes-not-in*)

lemma *edge-succ-in-arcs*[*simp*]: *edge-succ M a* $\in$ *arcs G* $\longleftrightarrow$ *a* $\in$ *arcs G*
  by (*metis edge-pred-succ edge-succ-permutes permutes-not-in*)

lemma *edge-pred-in-arcs*[*simp*]: *edge-pred M a* $\in$ *arcs G* $\longleftrightarrow$ *a* $\in$ *arcs G*
  by (*metis edge-succ-pred edge-pred-permutes permutes-not-in*)

lemma *tail-edge-succ*[*simp*]: *tail G* (*edge-succ M a*) = *tail G a*
proof *cases*
  assume *a* $\in$ *arcs G*
  then have *tail G a* $\in$ *verts G* **by** *auto*
  moreover
  then have *out-arcs G* (*tail G a*) $\neq$ {}
    using ‹*a* $\in$ *arcs G*› **by** *auto*
  ultimately
  have *cyclic-on* (*edge-succ M*) (*out-arcs G* (*tail G a*))
    by (*rule edge-succ-cyclic*)
  moreover
  have *a* $\in$ *out-arcs G* (*tail G a*)
    using ‹*a* $\in$ *arcs G*› **by** *simp*
  ultimately
  have *edge-succ M a* $\in$ *out-arcs G* (*tail G a*)
    by (*rule cyclic-on-inI*)
  then show *?thesis* **by** *simp*
next
  assume *a* $\notin$ *arcs G* **then show** *?thesis* **using** *edge-succ-permutes* **by** (*simp add*: *permutes-not-in*)
qed

lemma *tail-edge-pred*[*simp*]: *tail G* (*edge-pred M a*) = *tail G a*
by (*metis edge-succ-pred tail-edge-succ*)

lemma *bij-edge-succ*[*intro*]: *bij* (*edge-succ M*)
  using *edge-succ-permutes* **by** (*simp add*: *permutes-conv-has-dom*)

lemma *edge-pred-cyclic*:
  assumes *v* $\in$ *verts G out-arcs G v* $\neq$ {}
  shows *cyclic-on* (*edge-pred M*) (*out-arcs G v*)

proof −
  obtain *a* **where** *orb-a-eq*: *orbit* (*edge-succ M*) *a* = *out-arcs G v*
    using *edge-succ-cyclic*[*OF assms*] **by** (*auto simp*: *cyclic-on-def*)
  have *cyclic-on* (*edge-pred M*) (*orbit* (*edge-pred M*) *a*)
    using *permutation-edge-pred* **by** (*rule cyclic-on-orbit′*)
  also have *orbit* (*edge-pred M*) *a* = *orbit* (*edge-succ M*) *a*
    unfolding *edge-pred-def* **using** *permutation-edge-succ* **by** (*rule orbit-inv-eq*)
  finally show *cyclic-on* (*edge-pred M*) (*out-arcs G v*) **by** (*simp add*: *orb-a-eq*)
qed

definition (**in** *pre-digraph-map*) *face-cycle-succ* :: ′*b* ⇒ ′*b* **where**
  face-cycle-succ ≡ *edge-succ M o edge-rev M*

definition (**in** *pre-digraph-map*) *face-cycle-pred* :: ′*b* ⇒ ′*b* **where**
  face-cycle-pred ≡ *edge-rev M o edge-pred M*

lemma *face-cycle-pred-succ*[*simp*]:
  shows *face-cycle-pred* (*face-cycle-succ a*) = *a*
  unfolding *face-cycle-pred-def face-cycle-succ-def* **by** *simp*

lemma *face-cycle-succ-pred*[*simp*]:
  shows *face-cycle-succ* (*face-cycle-pred a*) = *a*
  unfolding *face-cycle-pred-def face-cycle-succ-def* **by** *simp*

lemma *tail-face-cycle-succ*: *a* ∈ *arcs G* ⟹ *tail G* (*face-cycle-succ a*) = *head G*
a
  by (*auto simp*: *face-cycle-succ-def*)

lemma *funpow-prop*:
  assumes $\bigwedge x$. *P* (*f x*) ⟷ *P x*
  shows *P* ((*f* ^^ *n*) *x*) ⟷ *P x*
  using *assms* **by** (*induct n*) *auto*

lemma *face-cycle-succ-no-arc*[*simp*]: *a* ∉ *arcs G* ⟹ *face-cycle-succ a* = *a*
  by (*auto simp*: *face-cycle-succ-def permutes-not-in*[*OF arev-permutes-arcs*]
    permutes-not-in[*OF edge-succ-permutes*])

lemma *funpow-face-cycle-succ-no-arc*[*simp*]:
  assumes *a* ∉ *arcs G* **shows** (*face-cycle-succ* ^^ *n*) *a* = *a*
  using *assms* **by** (*induct n*) *auto*

lemma *funpow-face-cycle-pred-no-arc*[*simp*]:
  assumes *a* ∉ *arcs G* **shows** (*face-cycle-pred* ^^ *n*) *a* = *a*
  using *assms*
 by (*induct n*) (*auto simp*: *face-cycle-pred-def permutes-not-in*[*OF arev-permutes-arcs*]
    permutes-not-in[*OF edge-pred-permutes*])

lemma *face-cycle-succ-closed*[*simp*]:
  face-cycle-succ a ∈ *arcs G* ⟷ *a* ∈ *arcs G*

by (*metis comp-apply edge-rev-in-arcs edge-succ-in-arcs face-cycle-succ-def*)

lemma *face-cycle-pred-closed*[*simp*]:
face-cycle-pred a $\in$ *arcs G* $\longleftrightarrow$ *a* $\in$ *arcs G*
by (*metis face-cycle-succ-closed face-cycle-succ-pred*)

lemma *face-cycle-succ-permutes*:
face-cycle-succ permutes arcs G
unfolding *face-cycle-succ-def*
using *arev-permutes-arcs edge-succ-permutes* **by** (*rule permutes-compose*)

lemma *permutation-face-cycle-succ*: *permutation face-cycle-succ*
using *face-cycle-succ-permutes finite-arcs* **by** (*metis permutation-permutes*)

lemma *bij-face-cycle-succ*: *bij face-cycle-succ*
using *face-cycle-succ-permutes* **by** (*simp add*: *permutes-conv-has-dom*)

lemma *face-cycle-pred-permutes*:
face-cycle-pred permutes arcs G
unfolding *face-cycle-pred-def*
using *edge-pred-permutes arev-permutes-arcs* **by** (*rule permutes-compose*)

definition (**in** *pre-digraph-map*) *face-cycle-set* :: *'b* $\Rightarrow$ *'b set* **where**
face-cycle-set a = *orbit face-cycle-succ a*

definition (**in** *pre-digraph-map*) *face-cycle-sets* :: *'b set set* **where**
face-cycle-sets = *face-cycle-set* ‘ *arcs G*

lemma *face-cycle-set-altdef*: *face-cycle-set a* = $\{$(*face-cycle-succ* ^^ *n*) *a* | *n*. *True*$\}$
unfolding *face-cycle-set-def*
by (*intro orbit-altdef-self-in permutation-self-in-orbit permutation-face-cycle-succ*)

lemma *face-cycle-set-self*[*simp*, *intro*]: *a* $\in$ *face-cycle-set a*
unfolding *face-cycle-set-def* **using** *permutation-face-cycle-succ* **by** (*rule permutation-self-in-orbit*)

lemma *empty-not-in-face-cycle-sets*: $\{\}$ $\notin$ *face-cycle-sets*
by (*auto simp*: *face-cycle-sets-def*)

lemma *finite-face-cycle-set*[*simp*, *intro*]: *finite* (*face-cycle-set a*)
using *face-cycle-set-self* **unfolding** *face-cycle-set-def* **by** (*simp add*: *finite-orbit*)

lemma *finite-face-cycle-sets*[*simp*, *intro*]: *finite face-cycle-sets*
by (*auto simp*: *face-cycle-sets-def*)

lemma *face-cycle-set-induct*[*case-names base step*, *induct set*: *face-cycle-set*]:
assumes *consume*: *a* $\in$ *face-cycle-set x*
and *ih-base*: *P x*

and *ih-step*: $\bigwedge y.\ y \in$ *face-cycle-set* $x \Longrightarrow P\ y \Longrightarrow P$ (*face-cycle-succ* y)
shows $P\ a$
using *consume* **unfolding** *face-cycle-set-def*
by *induct* (*auto simp*: *ih-step face-cycle-set-def*[*symmetric*] *ih-base*)

lemma *face-cycle-succ-cyclic*:
cyclic-on face-cycle-succ (*face-cycle-set a*)
unfolding *face-cycle-set-def* **using** *permutation-face-cycle-succ* **by** (*rule cyclic-on-orbit′*)

lemma *face-cycle-eq*:
assumes $b \in$ *face-cycle-set a* **shows** *face-cycle-set b* = *face-cycle-set a*
using *assms* **unfolding** *face-cycle-set-def*
by (*auto intro*: *orbit-swap orbit-trans permutation-face-cycle-succ permutation-self-in-orbit*)

lemma *face-cycle-succ-in-arcsI*: $\bigwedge a.\ a \in$ *arcs* $G \Longrightarrow$ *face-cycle-succ* $a \in$ *arcs* G
by (*auto simp*: *face-cycle-succ-def*)

lemma *face-cycle-succ-inI*: $\bigwedge x\ y.\ x \in$ *face-cycle-set* $y \Longrightarrow$ *face-cycle-succ* $x \in$ *face-cycle-set* y
by (*metis face-cycle-succ-cyclic cyclic-on-inI*)

lemma *face-cycle-succ-inD*: $\bigwedge x\ y.$ *face-cycle-succ* $x \in$ *face-cycle-set* $y \Longrightarrow x \in$ *face-cycle-set* y
by (*metis face-cycle-eq face-cycle-set-self face-cycle-succ-inI*)

lemma *face-cycle-set-parts*:
face-cycle-set a = *face-cycle-set b* $\vee$ *face-cycle-set a* $\cap$ *face-cycle-set b* = {}
by (*metis disjoint-iff-not-equal face-cycle-eq*)

definition *fc-equiv* :: $'b \Rightarrow 'b \Rightarrow$ *bool* **where**
fc-equiv a b $\equiv a \in$ *face-cycle-set b*

lemma *reflp-fc-equiv*: *reflp fc-equiv*
by (*rule reflpI*) (*simp add*: *fc-equiv-def*)

lemma *symp-fc-equiv*: *symp fc-equiv*
using *face-cycle-set-parts*
by (*intro sympI*) (*auto simp*: *fc-equiv-def*)

lemma *transp-fc-equiv*: *transp fc-equiv*
using *face-cycle-set-parts*
by (*intro transpI*) (*auto simp*: *fc-equiv-def*)

lemma *equivp fc-equiv*
by (*intro equivpI reflp-fc-equiv symp-fc-equiv transp-fc-equiv*)

lemma *in-face-cycle-setD*:
assumes $y \in$ *face-cycle-set* x $x \in$ *arcs* G **shows** $y \in$ *arcs* G

using *assms*
by (*auto simp*: *face-cycle-set-def dest*: *permutes-orbit-subset*[*OF face-cycle-succ-permutes*])

lemma *in-face-cycle-setsD*:
assumes $x \in$ *face-cycle-sets* **shows** $x \subseteq$ *arcs G*
using *assms* **by** (*auto simp*: *face-cycle-sets-def dest*: *in-face-cycle-setD*)

end

definition (**in** *pre-digraph*) *isolated-verts* :: *'a set* **where**
isolated-verts $\equiv \{v \in$ *verts G*. *out-arcs G v* $= \{\}\}$

definition (**in** *pre-digraph-map*) *euler-char* :: *int* **where**
euler-char $\equiv$ *int* (*card* (*verts G*)) $-$ *int* (*card* (*arcs G*) *div 2*) $+$ *int* (*card face-cycle-sets*)

definition (**in** *pre-digraph-map*) *euler-genus* :: *int* **where**
euler-genus $\equiv$ (*int* (*2* $*$ *card sccs*) $-$ *int* (*card isolated-verts*) $-$ *euler-char*) *div 2*

definition *comb-planar* :: (*'a*,*'b*) *pre-digraph* $\Rightarrow$ *bool* **where**
comb-planar G $\equiv \exists M$. *digraph-map G M* $\wedge$ *pre-digraph-map.euler-genus G M* $=$ *0*

Number of isolated vertices is a graph invariant

context
fixes *G hom* **assumes** *hom*: *pre-digraph.digraph-isomorphism G hom*
begin

interpretation *wf-digraph G* **using** *hom* **by** (*auto simp*: *pre-digraph.digraph-isomorphism-def*)

lemma *isolated-verts-app-iso*[*simp*]:
pre-digraph.isolated-verts (*app-iso hom G*) $=$ *iso-verts hom* ‘ *isolated-verts*
using *hom*
by (*auto simp*: *pre-digraph.isolated-verts-def iso-verts-tail inj-image-mem-iff out-arcs-app-iso-eq*)

lemma *card-isolated-verts-iso*[*simp*]:
card (*iso-verts hom* ‘ *pre-digraph.isolated-verts G*) $=$ *card isolated-verts*
apply (*rule card-image*)
using *hom* **apply** (*rule digraph-isomorphism-inj-on-verts*[*THEN subset-inj-on*])
apply (*auto simp*: *isolated-verts-def*)
done

end

context *digraph-map* **begin**

lemma *face-cycle-succ-neq*:
assumes $a \in$ *arcs G tail G a* $\neq$ *head G a* **shows** *face-cycle-succ a* $\neq a$
proof −
from *assms* **have** *edge-rev M a* $\in$ *arcs G*
by (*subst edge-rev-in-arcs*) *simp*
then have *cyclic-on* (*edge-succ M*) (*out-arcs G* (*tail G* (*edge-rev M a*)))
by (*intro edge-succ-cyclic*) (*auto dest*: *tail-in-verts simp*: *out-arcs-def intro*: *exI*[**where** *x*=*edge-rev M a*])
then have *edge-succ M* (*edge-rev M a*) $\in$ (*out-arcs G* (*tail G* (*edge-rev M a*)))
by (*rule cyclic-on-inI*) (*auto simp*: ‹*edge-rev M a* $\in$ -›[*simplified*])
moreover have *tail G* (*edge-succ M* (*edge-rev M a*)) = *head G a*
using *assms* **by** *auto*
then have *edge-succ M* (*edge-rev M a*) $\neq a$ **using** *assms* **by** *metis*
ultimately show *?thesis*
using *assms* **by** (*auto simp*: *face-cycle-succ-def*)
qed

end

2 Maps and Isomorphism

definition (**in** *pre-digraph*)
wrap-iso-arcs hom f = *perm-restrict* (*iso-arcs hom o f o iso-arcs* (*inv-iso hom*)) (*arcs* (*app-iso hom G*))

definition (**in** *pre-digraph-map*) *map-iso* :: ('*a*,'*b*,'*a2*,'*b2*) *digraph-isomorphism* ⇒ '*b2 pre-map* **where**
map-iso f ≡
⦇ *edge-rev* = *wrap-iso-arcs f* (*edge-rev M*)
, *edge-succ* = *wrap-iso-arcs f* (*edge-succ M*)
⦈

lemma *funcsetI-permutes*:
assumes *f permutes S* **shows** $f \in S \to S$
by (*metis assms funcsetI permutes-in-image*)

context
fixes *G hom* **assumes** *hom*: *pre-digraph.digraph-isomorphism G hom*
begin

interpretation *wf-digraph G* **using** *hom* **by** (*auto simp*: *pre-digraph.digraph-isomorphism-def*)

lemma *wrap-iso-arcs-iso-arcs*[*simp*]:
assumes $x \in$ *arcs G*
shows *wrap-iso-arcs hom f* (*iso-arcs hom x*) = *iso-arcs hom* (*f x*)
using *assms hom* **by** (*auto simp*: *wrap-iso-arcs-def perm-restrict-def*)

lemma *inj-on-wrap-iso-arcs*:
assumes *dom*: $\bigwedge f.\ f \in F \Longrightarrow$ *has-dom f* (*arcs G*)

assumes *funcset*: $F \subseteq$ *arcs* $G \to$ *arcs* G
shows *inj-on* (*wrap-iso-arcs hom*) F
proof (*rule inj-onI*)
fix f g **assume** F: $f \in F$ $g \in F$ **and** *eq*: *wrap-iso-arcs hom f* = *wrap-iso-arcs hom g*
{ **fix** x **assume** $x \notin$ *arcs* G
then have $f\ x = x$ $g\ x = x$ **using** F *dom* **by** (*auto simp*: *has-dom-def*)
then have $f\ x = g\ x$ **by** *simp*
}
moreover
{ **fix** x **assume** $x \in$ *arcs* G
then have $f\ x \in$ *arcs* G $g\ x \in$ *arcs* G **using** F *funcset* **by** *auto*
with *digraph-isomorphism-inj-on-arcs*[*OF hom*] -
have *iso-arcs hom* ($f\ x$) = *iso-arcs hom* ($g\ x$) $\Longrightarrow f\ x = g\ x$
by (*rule inj-onD*)
then have $f\ x = g\ x$
using *assms hom* ‹$x \in$ *arcs* G› *eq*
by (*auto simp*: *wrap-iso-arcs-def fun-eq-iff perm-restrict-def split*: *if-splits*)
}
ultimately show $f = g$ **by** *auto*
qed

lemma *inj-on-wrap-iso-arcs-f*:
assumes $A \subseteq$ *arcs* G $f \in A \to A$ B = *iso-arcs hom* ‘ A
assumes *inj-on f A* **shows** *inj-on* (*wrap-iso-arcs hom f*) B
proof (*rule inj-onI*)
fix x y
assume *in-hom-A*: $x \in B$ $y \in B$
and *wia-eq*: *wrap-iso-arcs hom f x* = *wrap-iso-arcs hom f y*
from *in-hom-A* ‹B = -› **obtain** *x0* **where** *x0*: x = *iso-arcs hom x0* *x0* $\in A$ **by** *auto*
from *in-hom-A* ‹B = -› **obtain** *y0* **where** *y0*: y = *iso-arcs hom y0* *y0* $\in A$ **by** *auto*
have *arcs-0*: *x0* $\in$ *arcs* G *y0* $\in$ *arcs* G f *x0* $\in$ *arcs* G f *y0* $\in$ *arcs* G
using *x0 y0* ‹$A \subseteq$ -› ‹$f \in$ -› **by** *auto*

have (*iso-arcs hom o f o iso-arcs* (*inv-iso hom*)) x = (*iso-arcs hom o f o iso-arcs* (*inv-iso hom*)) y
using *in-hom-A wia-eq assms*(*1*) ‹B = -› **by** (*auto simp*: *wrap-iso-arcs-def perm-restrict-def split*: *if-splits*)
then show $x = y$
using *hom assms digraph-isomorphism-inj-on-arcs*[*OF hom*] *x0 y0 arcs-0* ‹*inj-on f A*› ‹$A \subseteq$ -›
by (*auto dest*!: *inj-onD*)
qed

lemma *wrap-iso-arcs-in-funcsetI*:
assumes $A \subseteq$ *arcs* G $f \in A \to A$
shows *wrap-iso-arcs hom f* $\in$ *iso-arcs hom* ‘ A $\to$ *iso-arcs hom* ‘ A

proof
fix *x* **assume** $x \in$ *iso-arcs hom* ‘ *A*
then obtain *x0* **where** $x =$ *iso-arcs hom x0 x0* $\in A$ **by** *blast*
then have *f x0* $\in A$ **using** ‹*f* $\in$ -› **by** *auto*
then show *wrap-iso-arcs hom f x* $\in$ *iso-arcs hom* ‘ *A*
unfolding ‹$x =$ -› **using** ‹*x0* $\in A$› *assms hom* **by** (*auto simp*: *wrap-iso-arcs-def perm-restrict-def*)
qed

lemma *wrap-iso-arcs-permutes*:
assumes $A \subseteq$ *arcs G f permutes A*
shows *wrap-iso-arcs hom f permutes* (*iso-arcs hom* ‘ *A*)
proof −
{ **fix** *x* **assume** *A*: $x \notin$ *iso-arcs hom* ‘ *A*
have *wrap-iso-arcs hom f x* $= x$
proof *cases*
assume $x \in$ *iso-arcs hom* ‘ *arcs G*
then have *iso-arcs* (*inv-iso hom*) $x \notin A$ $x \in$ *arcs* (*app-iso hom G*)
using *A hom* **by** (*metis arcs-app-iso image-eqI pre-digraph.iso-arcs-iso-inv*, *simp*)
then have *f* (*iso-arcs* (*inv-iso hom*) *x*) $=$ (*iso-arcs* (*inv-iso hom*) *x*)
using ‹*f permutes A*› **by** (*simp add*: *permutes-not-in*)
then show *?thesis* **using** *hom assms* ‹$x \in$ *arcs* -›
by (*simp add*: *wrap-iso-arcs-def perm-restrict-def*)
next
assume $x \notin$ *iso-arcs hom* ‘ *arcs G*
then show *?thesis*
by (*simp add*: *wrap-iso-arcs-def perm-restrict-def*)
qed
} **note** *not-in-id* = *this*

have $f \in A \to A$ **using** *assms* **by** (*intro funcsetI-permutes*)
have *inj-on-wrap*: *inj-on* (*wrap-iso-arcs hom f*) (*iso-arcs hom* ‘ *A*)
using *assms* ‹$f \in A \to A$› **by** (*intro inj-on-wrap-iso-arcs-f*) (*auto intro*: *subset-inj-on permutes-inj*)
have *woa-in-fs*: *wrap-iso-arcs hom f* $\in$ *iso-arcs hom* ‘ $A \to$ *iso-arcs hom* ‘ *A*
using *assms* ‹$f \in A \to A$› **by** (*intro wrap-iso-arcs-in-funcsetI*)

{ **fix** *x y* **assume** *wrap-iso-arcs hom f x* $=$ *wrap-iso-arcs hom f y*
then have $x = y$
apply (*cases* $x \in$ *iso-arcs hom* ‘ *A*; *cases* $y \in$ *iso-arcs hom* ‘ *A*)
using *woa-in-fs inj-on-wrap* **by** (*auto dest*: *inj-onD simp*: *not-in-id*)
} **note** *uniqueD* = *this*

note ‹*f permutes A*›
moreover
note *not-in-id*
moreover
{ **fix** *y* **have** $\exists x.$ *wrap-iso-arcs hom f x* $= y$

proof *cases*
assume $y \in$ *iso-arcs hom* ‘ A
then obtain *y0* **where** *y0* $\in A$ *iso-arcs hom y0* $= y$ **by** *blast*
with ‹*f permutes A*› **obtain** *x0* **where** *x0* $\in A$ *f x0* $=$ *y0* **unfolding** *permutes-def* **by** *metis*
moreover
then have $\bigwedge x.\ x \in$ *arcs G* $\Longrightarrow$ *iso-arcs hom x0* $=$ *iso-arcs hom x* $\Longrightarrow$ *x0* $= x$
using *assms hom* **by** (*auto simp*: *digraph-isomorphism-def dest*: *inj-onD*)
ultimately
have *wrap-iso-arcs hom f* (*iso-arcs hom x0*) $= y$
using ‹- $= y$› *assms hom* **by** (*auto simp*: *wrap-iso-arcs-def perm-restrict-def*)
then show *?thesis* **..**
qed (*metis not-in-id*)
}
ultimately
show *?thesis* **unfolding** *permutes-def* **by** (*auto simp*: *dest*: *uniqueD*)
qed

end

lemma (**in** *digraph-map*) *digraph-map-isoI*:
assumes *digraph-isomorphism hom* **shows** *digraph-map* (*app-iso hom G*) (*map-iso hom*)
proof −
interpret *iG*: *fin-digraph app-iso hom G* **using** *assms* **by** (*rule fin-digraphI-app-iso*)
show *?thesis*
proof (*rule iG.digraph-mapI-permutes*)
show *edge-rev* (*map-iso hom*) *permutes arcs* (*app-iso hom G*)
using *assms* **unfolding** *map-iso-def* **by** (*simp add*: *wrap-iso-arcs-permutes arev-permutes-arcs*)
next
show *edge-succ* (*map-iso hom*) *permutes arcs* (*app-iso hom G*)
using *assms* **unfolding** *map-iso-def* **by** (*simp add*: *wrap-iso-arcs-permutes edge-succ-permutes*)
next
fix *a* **assume** *A*: $a \in$ *arcs* (*app-iso hom G*)
show *tail* (*app-iso hom G*) (*edge-rev* (*map-iso hom*) *a*) $=$ *head* (*app-iso hom G*) *a*
using *A assms*
by (*cases rule*: *in-arcs-app-iso-cases*) (*auto simp*: *map-iso-def iso-verts-tail iso-verts-head*)
show *edge-rev* (*map-iso hom*) (*edge-rev* (*map-iso hom*) *a*) $= a$
using *A assms* **by** (*cases rule*: *in-arcs-app-iso-cases*) (*auto simp*: *map-iso-def*)
show *edge-rev* (*map-iso hom*) *a* $\neq a$
using *A assms* **by** (*auto simp*: *map-iso-def arev-neq*)
next
fix *v* **assume** $v \in$ *verts* (*app-iso hom G*) **and** *oa-hom*: *out-arcs* (*app-iso hom G*) $v \neq \{\}$

```
    then obtain v0 where v0 ∈ verts G v = iso-verts hom v0 by auto
    moreover
    then have oa: out-arcs G v0 ≠ {}
      using assms oa-hom by (auto simp: out-arcs-def iso-verts-tail)
    ultimately
    have cyclic-on-v0: cyclic-on (edge-succ M) (out-arcs G v0)
      by (intro edge-succ-cyclic)

    from oa-hom obtain a where a ∈ out-arcs (app-iso hom G) v by blast
    then obtain a0 where a0 ∈ arcs G a = iso-arcs hom a0 by auto
    then have a0 ∈ out-arcs G v0
      using ‹v = -› ‹v0 ∈ -› ‹a ∈ -› assms by (simp add: iso-verts-tail)

    show cyclic-on (edge-succ (map-iso hom)) (out-arcs (app-iso hom G) v)
    proof (rule cyclic-on-singleI)
      show a ∈ out-arcs (app-iso hom G) v by fact
    next
      have out-arcs (app-iso hom G) v = iso-arcs hom ‘ out-arcs G v0
        unfolding ‹v = -› by (rule out-arcs-app-iso-eq) fact+
      also have out-arcs G v0 = orbit (edge-succ M) a0
          using cyclic-on-v0 ‹a0 ∈ out-arcs G v0› unfolding cyclic-on-alldef by
simp
      also have iso-arcs hom ‘ . . . = orbit (edge-succ (map-iso hom)) a
      proof −
        have ⋀x. x ∈ orbit (edge-succ M) a0 ⟹ x ∈ arcs G
          using ‹out-arcs G v0 = -› by auto
        then show ?thesis using ‹out-arcs G v0 = -›
          unfolding ‹a = -› using ‹a0 ∈ out-arcs G v0› assms
          by (intro orbit-inverse) (auto simp: map-iso-def)
      qed
      finally show out-arcs (app-iso hom G) v = orbit (edge-succ (map-iso hom))
a .
    qed
  qed
qed

end
theory List-Aux
imports
  List−Index.List-Index
begin
```

3 Auxiliary List Lemmas

```
lemma nth-rotate-conv-nth1-conv-nth:
  assumes m < length xs
  shows rotate1 xs ! m = xs ! (Suc m mod length xs)
  using assms
proof (induction xs arbitrary: m)
```

case (*Cons x xs*)
show *?case*
proof (*cases m* < *length xs*)
case *False*
with *Cons.prems* **have** *m* = *length xs* **by** *force*
then show *?thesis* **by** (*auto simp*: *nth-append*)
qed (*auto simp*: *nth-append*)
qed *simp*

lemma *nth-rotate-conv-nth*:
assumes *m* < *length xs*
shows *rotate n xs* ! *m* = *xs* ! ((*m* + *n*) *mod length xs*)
using *assms*
proof (*induction n arbitrary*: *m*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
show *?case*
proof *cases*
assume *m* + *1* < *length xs*
with *Suc* **show** *?thesis* **using** *Suc* **by** (*auto simp*: *nth-rotate-conv-nth1-conv-nth*)
next
assume ¬(*m* + *1* < *length xs*)
with *Suc* **have** *m* + *1* = *length xs 0* < *length xs* **by** *auto*
moreover
{ have *Suc* (*m* + *n*) *mod length xs* = (*Suc m* + *n*) *mod length xs*
by *auto*
also have ... = *n mod length xs* **using** ‹*m* + *1* = -› **by** *simp*
finally have *Suc* (*m* + *n*) *mod length xs* = *n mod length xs* **.}**
ultimately
show *?thesis* **by** (*auto simp*: *nth-rotate-conv-nth1-conv-nth Suc.IH*)
qed
qed

lemma *not-nil-if-in-set*:
assumes *x* ∈ *set xs* **shows** *xs* ≠ []
using *assms* **by** *auto*

lemma *length-fold-remove1-le*:
length (*fold remove1 ys xs*) ≤ *length xs*
proof (*induct ys arbitrary*: *xs*)
case (*Cons y ys*)
then have *length* (*fold remove1 ys* (*remove1 y xs*)) ≤ *length* (*remove1 y xs*) **by** *auto*
also have ... ≤ *length xs* **by** (*auto simp*: *length-remove1*)
finally show *?case* **by** *simp*
qed *simp*

lemma *set-fold-remove1′*:

assumes *x* ∈ *set xs* − *set ys* **shows** *x* ∈ *set* (*fold remove1 ys xs*)
using *assms* **by** (*induct ys arbitrary*: *xs*) *auto*

lemma *set-fold-remove1*:
set (*fold remove1 xs ys*) ⊆ *set ys*
by (*induct xs arbitrary*: *ys*) (*auto*, *metis notin-set-remove1 subsetCE*)

lemma *set-fold-remove1-distinct*:
assumes *distinct xs* **shows** *set* (*fold remove1 ys xs*) = *set xs* − *set ys*
using *assms* **by** (*induct ys arbitrary*: *xs*) *auto*

lemma *distinct-fold-remove1*:
assumes *distinct xs*
shows *distinct* (*fold remove1 ys xs*)
using *assms* **by** (*induct ys arbitrary*: *xs*) *auto*

end

4 Permutations as Products of Disjoint Cycles

theory *Executable-Permutations*
imports
HOL−Combinatorics.Permutations
Graph-Theory.Auxiliary
List-Aux
begin

4.1 Cyclic Permutations

definition *list-succ* :: *'a list* ⇒ *'a* ⇒ *'a* **where**
list-succ xs x = (*if x* ∈ *set xs then xs* ! ((*index xs x* + *1*) *mod length xs*) *else x*)

We demonstrate the functions on the following simple lemmas

list-succ [*1*, *2*, *3*] *1* = *2* *list-succ* [*1*, *2*, *3*] *2* = *3* *list-succ* [*1*, *2*, *3*] *3* = *1*

lemma *list-succ-altdef*:
list-succ xs x = (*let n* = *index xs x in if n* + *1* = *length xs then xs* ! *0 else if n* + *1* < *length xs then xs* ! (*n* + *1*) *else x*)
using *index-le-size*[*of xs x*] **unfolding** *list-succ-def index-less-size-conv*[*symmetric*]
by (*auto simp*: *Let-def*)

lemma *list-succ-Nil*:
list-succ [] = *id*
by (*simp add*: *list-succ-def fun-eq-iff*)

lemma *list-succ-singleton*:
list-succ [*x*] = *list-succ* []
by (*simp add*: *fun-eq-iff list-succ-def*)

lemma *list-succ-short*:
  assumes *length xs* < *2* **shows** *list-succ xs* = *id*
  using *assms*
  by (*cases xs*) (*rename-tac* [*2*] *y ys*, *case-tac* [*2*] *ys*, *auto simp*: *list-succ-Nil list-succ-singleton*)

lemma *list-succ-simps*:
  index xs x + *1* = *length xs* ⟹ *list-succ xs x* = *xs* ! *0*
  index xs x + *1* < *length xs* ⟹ *list-succ xs x* = *xs* ! (*index xs x* + *1*)
  length xs ≤ *index xs x* ⟹ *list-succ xs x* = *x*
  by (*auto simp*: *list-succ-altdef*)

lemma *list-succ-not-in*:
  assumes *x* ∉ *set xs* **shows** *list-succ xs x* = *x*
  using *assms* **by** (*auto simp*: *list-succ-def*)

lemma *list-succ-list-succ-rev*:
  assumes *distinct xs* **shows** *list-succ* (*rev xs*) (*list-succ xs x*) = *x*
proof −
  { assume *index xs x* + *1* < *length xs*
    moreover then have *length xs* − *Suc* (*Suc* (*length xs* − *Suc* (*Suc* (*index xs x*)))) = *index xs x*
      by *linarith*
    ultimately have *?thesis* **using** *assms*
      by (*simp add*: *list-succ-def index-rev index-nth-id rev-nth*)
  }
  moreover
  { assume *A*: *index xs x* + *1* = *length xs*
    moreover
    from *A* **have** *xs* ≠ [] **by** *auto*
    moreover
    with *A* **have** *last xs* = *xs* ! *index xs x* **by** (*cases length xs*) (*auto simp*: *last-conv-nth*)
    ultimately
    have *?thesis*
      using *assms*
      by (*auto simp add*: *list-succ-def rev-nth index-rev index-nth-id last-conv-nth*)
  }
  moreover
  { assume *A*: *index xs x* ≥ *length xs*
    then have *x* ∉ *set xs* **by** (*metis index-less less-irrefl*)
    then have *?thesis* **by** (*auto simp*: *list-succ-def*) **}**
  ultimately show *?thesis*
    by *linarith*
qed

lemma *inj-list-succ*: *distinct xs* ⟹ *inj* (*list-succ xs*)
  by (*metis injI list-succ-list-succ-rev*)

lemma *inv-list-succ-eq*: *distinct xs* $\Longrightarrow$ *inv* (*list-succ xs*) = *list-succ* (*rev xs*)
  by (*metis distinct-rev inj-imp-inv-eq inj-list-succ list-succ-list-succ-rev*)

lemma *bij-list-succ*: *distinct xs* $\Longrightarrow$ *bij* (*list-succ xs*)
  by (*metis bij-def inj-list-succ distinct-rev list-succ-list-succ-rev surj-def*)

lemma *list-succ-permutes*:
  assumes *distinct xs* **shows** *list-succ xs permutes set xs*
   using *assms* **by** (*auto simp*: *permutes-conv-has-dom bij-list-succ has-dom-def list-succ-def*)

lemma *permutation-list-succ*:
  assumes *distinct xs* **shows** *permutation* (*list-succ xs*)
  using *list-succ-permutes*[*OF assms*] **by** (*auto simp*: *permutation-permutes*)

lemma *list-succ-nth*:
  assumes *distinct xs n* < *length xs* **shows** *list-succ xs* (*xs* ! *n*) = *xs* ! (*Suc n mod length xs*)
  using *assms* **by** (*auto simp*: *list-succ-def index-nth-id*)

lemma *list-succ-last*[*simp*]:
  assumes *distinct xs xs* ≠ [] **shows** *list-succ xs* (*last xs*) = *hd xs*
  using *assms* **by** (*auto simp*: *list-succ-def hd-conv-nth*)

lemma *list-succ-rotate1*[*simp*]:
  assumes *distinct xs* **shows** *list-succ* (*rotate1 xs*) = *list-succ xs*
proof (*rule ext*)
  fix *y* **show** *list-succ* (*rotate1 xs*) *y* = *list-succ xs y*
    using *assms*
  proof (*induct xs*)
    case *Nil* **then show** *?case* **by** *simp*
  next
    case (*Cons x xs*)
    show *?case*
    proof (*cases x* = *y*)
      case *True*
      then have *index* (*xs* @ [*y*]) *y* = *length xs*
        using ‹*distinct* (*x* # *xs*)› **by** (*simp add*: *index-append*)
     with *True* **show** *?thesis* **by** (*cases xs*=[]) (*auto simp*: *list-succ-def nth-append*)
    next
      case *False*
      then show *?thesis*
        apply (*cases index xs y* + *1* < *length xs*)
        apply (*auto simp*:*list-succ-def index-append nth-append*)
          by (*metis Suc-lessI index-less-size-conv mod-self nth-Cons-0 nth-append nth-append-length*)
    qed
  qed
qed

lemma *list-succ-rotate*[*simp*]:
  assumes *distinct xs* **shows** *list-succ* (*rotate n xs*) = *list-succ xs*
  using *assms* **by** (*induct n*) *auto*

lemma *list-succ-in-conv*:
  list-succ xs x ∈ *set xs* ⟷ *x* ∈ *set xs*
  by (*auto simp*: *list-succ-def not-nil-if-in-set*)

lemma *list-succ-in-conv1*:
  assumes *A* ∩ *set xs* = {}
  shows *list-succ xs x* ∈ *A* ⟷ *x* ∈ *A*
  by (*metis assms disjoint-iff-not-equal list-succ-in-conv list-succ-not-in*)

lemma *list-succ-commute*:
  assumes *set xs* ∩ *set ys* = {}
  shows *list-succ xs* (*list-succ ys x*) = *list-succ ys* (*list-succ xs x*)
proof −
  have ⋀*x*. *x* ∈ *set xs* ⟹ *list-succ ys x* = *x*
    ⋀*x*. *x* ∈ *set ys* ⟹ *list-succ xs x* = *x*
    using *assms* **by** (*blast intro*: *list-succ-not-in*)+
  then show *?thesis*
    by (*cases x* ∈ *set xs* ∪ *set ys*) (*auto simp*: *list-succ-in-conv list-succ-not-in*)
qed

4.2 Arbitrary Permutations

fun *lists-succ* :: *'a list list* ⇒ *'a* ⇒ *'a* **where**
  lists-succ [] *x* = *x*
| *lists-succ* (*xs* # *xss*) *x* = *list-succ xs* (*lists-succ xss x*)

definition *distincts* :: *'a list list* ⇒ *bool* **where**
  distincts xss ≡ *distinct xss* ∧ (∀ *xs* ∈ *set xss*. *distinct xs* ∧ *xs* ≠ []) ∧ (∀ *xs* ∈ *set xss*. ∀ *ys* ∈ *set xss*. *xs* ≠ *ys* ⟶ *set xs* ∩ *set ys* = {})

lemma *distincts-distinct*: *distincts xss* ⟹ *distinct xss*
  by (*auto simp*: *distincts-def*)

lemma *distincts-Nil*[*simp*]: *distincts* []
  by (*simp add*: *distincts-def*)

lemma *distincts-single*: *distincts* [*xs*] ⟷ *distinct xs* ∧ *xs* ≠ []
  by (*auto simp add*: *distincts-def*)

lemma *distincts-Cons*: *distincts* (*xs* # *xss*)
  ⟷ *xs* ≠ [] ∧ *distinct xs* ∧ *distincts xss* ∧ (*set xs* ∩ (⋃ *ys* ∈ *set xss*. *set ys*)) = {} (**is** *?L* ⟷ *?R*)
proof
  assume *?L* **then show** *?R* **by** (*auto simp*: *distincts-def*)

next
  assume *?R*
  then have *distinct* (*xs* # *xss*)
    apply (*auto simp*: *disjoint-iff-not-equal distincts-distinct*)
    apply (*metis length-greater-0-conv nth-mem*)
    done
  moreover
  from ‹*?R*› **have** $\forall$ *xs* $\in$ *set* (*xs* # *xss*). *distinct xs* $\wedge$ *xs* $\neq$ []
    by (*auto simp*: *distincts-def*)
  moreover
  from ‹*?R*› **have** $\forall$ *xs'* $\in$ *set* (*xs* # *xss*). $\forall$ *ys* $\in$ *set* (*xs* # *xss*). *xs'* $\neq$ *ys* $\longrightarrow$ *set xs'* $\cap$ *set ys* = {}
    by (*simp add*: *distincts-def*) *blast*
  ultimately show *?L* **unfolding** *distincts-def* **by** (*intro conjI*)
qed

lemma *distincts-Cons'*: *distincts* (*xs* # *xss*)
  $\longleftrightarrow$ *xs* $\neq$ [] $\wedge$ *distinct xs* $\wedge$ *distincts xss* $\wedge$ ($\forall$ *ys* $\in$ *set xss*. *set xs* $\cap$ *set ys* = {})
(**is** *?L* $\longleftrightarrow$ *?R*)
 unfolding *distincts-Cons* **by** *blast*

lemma *distincts-rev*:
  distincts (*map rev xss*) $\longleftrightarrow$ *distincts xss*
  by (*simp add*: *distincts-def distinct-map*)

lemma *length-distincts*:
  assumes *distincts xss*
  shows *length xss* = *card* (*set* ‘ *set xss*)
  using *assms*
proof (*induct xss*)
  case *Nil* **then show** *?case* **by** *simp*
next
  case (*Cons xs xss*)
  then have *set xs* $\notin$ *set* ‘ *set xss*
    using *equals0I*[*of set xs*] **by** (*auto simp*: *distincts-Cons disjoint-iff-not-equal*)
  with *Cons* **show** *?case* **by** (*auto simp add*: *distincts-Cons*)
qed

lemma *distincts-remove1*: *distincts xss* $\Longrightarrow$ *distincts* (*remove1 xs xss*)
  by (*auto simp*: *distincts-def*)

lemma *distinct-Cons-remove1*:
  x $\in$ *set xs* $\Longrightarrow$ *distinct* (*x* # *remove1 x xs*) = *distinct xs*
  by (*induct xs*) *auto*

lemma *set-Cons-remove1*:
  x $\in$ *set xs* $\Longrightarrow$ *set* (*x* # *remove1 x xs*) = *set xs*
  by (*induct xs*) *auto*

lemma *distincts-Cons-remove1*:
xs ∈ *set xss* ⟹ *distincts* (*xs* # *remove1 xs xss*) = *distincts xss*
by (*simp only*: *distinct-Cons-remove1 set-Cons-remove1 distincts-def*)

lemma *distincts-inj-on-set*:
assumes *distincts xss* **shows** *inj-on set* (*set xss*)
by (*rule inj-onI*) (*metis assms distincts-def inf.idem set-empty*)

lemma *distincts-distinct-set*:
assumes *distincts xss* **shows** *distinct* (*map set xss*)
using *assms* **by** (*auto simp*: *distinct-map distincts-distinct distincts-inj-on-set*)

lemma *distincts-distinct-nth*:
assumes *distincts xss* $n <$ *length xss* **shows** *distinct* (*xss* ! *n*)
using *assms* **by** (*auto simp*: *distincts-def*)

lemma *lists-succ-not-in*:
assumes $x \notin (\bigcup xs{\in}set\ xss.\ set\ xs)$ **shows** *lists-succ xss x* = *x*
using *assms* **by** (*induct xss*) (*auto simp*: *list-succ-not-in*)

lemma *lists-succ-in-conv*:
lists-succ xss x $\in (\bigcup xs{\in}set\ xss.\ set\ xs) \longleftrightarrow x \in (\bigcup xs{\in}set\ xss.\ set\ xs)$
by (*induct xss*) (*auto simp*: *list-succ-in-conv lists-succ-not-in list-succ-not-in*)

lemma *lists-succ-in-conv1*:
assumes $A \cap (\bigcup xs{\in}set\ xss.\ set\ xs) = \{\}$
shows *lists-succ xss x* ∈ *A* ⟷ *x* ∈ *A*
by (*metis Int-iff assms emptyE lists-succ-in-conv lists-succ-not-in*)

lemma *lists-succ-Cons-pf*: *lists-succ* (*xs* # *xss*) = *list-succ xs o lists-succ xss*
by *auto*

lemma *lists-succ-Nil-pf*: *lists-succ* [] = *id*
by (*simp add*: *fun-eq-iff*)

lemmas *lists-succ-simps-pf* = *lists-succ-Cons-pf lists-succ-Nil-pf*

lemma *lists-succ-permutes*:
assumes *distincts xss*
shows *lists-succ xss permutes* $(\bigcup xs \in set\ xss.\ set\ xs)$
using *assms*
proof (*induction xss*)
case *Nil* **then show** *?case* **by** *auto*
next
case (*Cons xs xss*)
have *list-succ xs permutes* (*set xs*)
using *Cons* **by** (*intro list-succ-permutes*) (*simp add*: *distincts-def in-set-member*)
moreover
have *lists-succ xss permutes* $(\bigcup ys \in set\ xss.\ set\ ys)$

```
    using Cons by (auto simp: Cons distincts-def)
  ultimately show lists-succ (xs # xss) permutes (⋃ ys ∈ set (xs # xss). set ys)
     using Cons by (auto simp: lists-succ-Cons-pf intro: permutes-compose per-
mutes-subset)
qed

lemma bij-lists-succ: distincts xss ⟹ bij (lists-succ xss)
 by (induct xss) (auto simp: lists-succ-simps-pf bij-comp bij-list-succ distincts-Cons)

lemma lists-succ-snoc: lists-succ (xss @ [xs]) = lists-succ xss o list-succ xs
  by (induct xss) auto

lemma inv-lists-succ-eq:
  assumes distincts xss
  shows inv (lists-succ xss) = lists-succ (rev (map rev xss))
proof −
  have *: ⋀f g. inv (λb. f (g b)) = inv (f o g) by (simp add: o-def)
  have **: lists-succ [] = id by auto
  show ?thesis
    using assms by (induct xss) (auto simp: * ** lists-succ-snoc lists-succ-Cons-pf
o-inv-distrib
      inv-list-succ-eq distincts-Cons bij-list-succ bij-lists-succ)
qed

lemma lists-succ-remove1:
  assumes distincts xss xs ∈ set xss
  shows lists-succ (xs # remove1 xs xss) = lists-succ xss
  using assms
proof (induct xss)
  case Nil then show ?case by simp
next
  case (Cons ys xss)
  show ?case
  proof cases
    assume xs = ys then show ?case by simp
  next
    assume xs ≠ ys
    with Cons.prems have inter: set xs ∩ set ys = {} and xs ∈ set xss
      by (auto simp: distincts-Cons)
    have dists:
        distincts (xs # remove1 xs xss)
        distincts (xs # ys # remove1 xs xss)
      using ‹distincts (ys # xss)› ‹xs ∈ set xss› by (auto simp: distincts-def)

    have list-succ xs ∘ (list-succ ys ∘ lists-succ (remove1 xs xss))
        = list-succ ys ∘ (list-succ xs ∘ lists-succ (remove1 xs xss))
      using inter unfolding fun-eq-iff comp-def
      by (subst list-succ-commute) auto
    also have … = list-succ ys o (lists-succ (xs # remove1 xs xss))
```

using *dists* **by** (*simp add*: *lists-succ-Cons-pf distincts-Cons*)
also have ... = *list-succ ys o lists-succ xss*
using ‹*xs* ∈ *set xss*› ‹*distincts* (*ys* # *xss*)›
by (*simp add*: *distincts-Cons Cons.hyps*)
finally
show *lists-succ* (*xs* # *remove1 xs* (*ys* # *xss*)) = *lists-succ* (*ys* # *xss*)
using *Cons dists* **by** (*auto simp*: *lists-succ-Cons-pf distincts-Cons*)
qed
qed

lemma *lists-succ-no-order*:
assumes *distincts xss distincts yss set xss* = *set yss*
shows *lists-succ xss* = *lists-succ yss*
using *assms*
proof (*induct xss arbitrary*: *yss*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons xs xss*)
have *xs* ∉ *set xss xs* ∈ *set yss* **using** *Cons.prems*
by (*auto dest*: *distincts-distinct*)
have *lists-succ xss* = *lists-succ* (*remove1 xs yss*)
using *Cons.prems* ‹*xs* ∉ -›
by (*intro Cons.hyps*) (*auto simp add*: *distincts-Cons distincts-remove1 distincts-distinct*)
then have *lists-succ* (*xs* # *xss*) = *lists-succ* (*xs* # *remove1 xs yss*)
using *Cons.prems* ‹*xs* ∈ -›
by (*simp add*: *lists-succ-Cons-pf distincts-Cons-remove1*)
then show *?case*
using *Cons.prems* ‹*xs* ∈ -› **by** (*simp add*: *lists-succ-remove1*)
qed

5 List Orbits

Computes the orbit of x under f

definition *orbit-list* :: ($'a \Rightarrow 'a$) $\Rightarrow$ $'a \Rightarrow 'a$ *list* **where**
orbit-list f x ≡ *iterate 0* (*funpow-dist1 f x x*) *f x*

partial-function (*tailrec*)
orbit-list-impl :: ($'a \Rightarrow 'a$) $\Rightarrow$ $'a \Rightarrow 'a$ *list* $\Rightarrow$ $'a \Rightarrow 'a$ *list*
where
orbit-list-impl f s acc x = (*let x′* = *f x in if x′* = *s then rev* (*x* # *acc*) *else orbit-list-impl f s* (*x* # *acc*) *x′*)

context notes [*simp*] = *length-fold-remove1-le* **begin**

Computes the list of orbits

fun *orbits-list* :: ($'a \Rightarrow 'a$) $\Rightarrow$ $'a$ *list* $\Rightarrow$ $'a$ *list list* **where**
orbits-list f [] = []

| *orbits-list f* $(x \# xs) =$
  orbit-list f x $\#$ *orbits-list f* (*fold remove1* (*orbit-list f x*) *xs*)

fun *orbits-list-impl* :: $('a \Rightarrow 'a) \Rightarrow 'a$ *list* $\Rightarrow 'a$ *list list* **where**
  orbits-list-impl f [] = []
| *orbits-list-impl f* $(x \# xs) =$
  (*let fc* = *orbit-list-impl f x* [] *x in fc* $\#$ *orbits-list-impl f* (*fold remove1 fc xs*))

declare *orbit-list-impl.simps*[*code*]
end

abbreviation *sset* :: $'a$ *list list* $\Rightarrow 'a$ *set set* **where**
  sset xss $\equiv$ *set* ' *set xss*

lemma *iterate-funpow-step*:
  assumes $f\ x \neq y$ $y \in$ *orbit f x*
  shows *iterate 0* (*funpow-dist1 f x y*) *f x* = *x* $\#$ *iterate 0* (*funpow-dist1 f* (*f x*) *y*) *f* (*f x*)
proof −
  from *assms* **have** *A*: $y \in$ *orbit f* (*f x*) **by** (*simp add*: *orbit-step*)
  have *iterate 0* (*funpow-dist1 f x y*) *f x* = *x* $\#$ *iterate 1* (*funpow-dist1 f x y*) *f x* (**is** - = - $\#$ *?it*)
    unfolding *iterate-def* **by** (*simp add*: *upt-rec*)
  also have *?it* = *map* (λn. (f ^^ n) x) (*map Suc* [*0..<funpow-dist f* (*f x*) *y*])
    unfolding *iterate-def map-Suc-upt* **by** *simp*
  also have ... = *map* (λn. (f ^^ n) ($f\ x$)) [*0..<funpow-dist f* (*f x*) *y*]
    by (*simp add*: *funpow-swap1*)
  also have ... = *iterate 0* (*funpow-dist1 f* (*f x*) *y*) *f* (*f x*)
    unfolding *iterate-def*
    unfolding *iterate-def* **by** (*simp add*: *funpow-dist-step*[*OF assms*(*1*) *A*])
  finally show *?thesis* **.**
qed

lemma *orbit-list-impl-conv*:
  assumes $y \in$ *orbit f x*
  shows *orbit-list-impl f y acc x* = *rev acc* @ *iterate 0* (*funpow-dist1 f x y*) *f x*
  using *assms*
proof (*induct n*$\equiv$*funpow-dist1 f x y arbitrary*: *x acc*)
  case (*Suc x*)

  show *?case*
  proof *cases*
    assume $f\ x = y$
    then show *?thesis* **by** (*subst orbit-list-impl.simps*) (*simp add*: *Let-def iterate-def funpow-dist-0*)
  next
    assume *not-y* :$f\ x \neq y$

    have *y-in-succ*: $y \in$ *orbit f* (*f x*)

    by (*intro orbit-step Suc.prems not-y*)

   have *orbit-list-impl f y acc x* = *orbit-list-impl f y* (*x* # *acc*) (*f x*)
    using *not-y* **by** (*subst orbit-list-impl.simps*) *simp*
   also have ... = *rev* (*x* # *acc*) @ *iterate 0* (*funpow-dist1 f* (*f x*) *y*) *f* (*f x*) (**is** - = *?rev* @ *?it*)
    by (*intro Suc funpow-dist-step not-y y-in-succ*)
   also have ... = *rev acc* @ *iterate 0* (*funpow-dist1 f x y*) *f x*
    using *not-y Suc.prems* **by** (*simp add*: *iterate-funpow-step*)
   finally show *?thesis* **.**
  qed
qed

lemma *orbit-list-conv-impl*:
  assumes *x* ∈ *orbit f x*
  shows *orbit-list f x* = *orbit-list-impl f x* [] *x*
  unfolding *orbit-list-impl-conv*[*OF assms*] *orbit-list-def* **by** *simp*

lemma *set-orbit-list*:
  assumes *x* ∈ *orbit f x*
  shows *set* (*orbit-list f x*) = *orbit f x*
  by (*simp add*: *orbit-list-def orbit-conv-funpow-dist1*[*OF assms*] *set-iterate*)

lemma *set-orbit-list'*:
  assumes *permutation f* **shows** *set* (*orbit-list f x*) = *orbit f x*
  using *assms* **by** (*simp add*: *permutation-self-in-orbit set-orbit-list*)

lemma *distinct-orbit-list*:
  assumes *x* ∈ *orbit f x*
  shows *distinct* (*orbit-list f x*)
 by (*simp del*: *upt-Suc add*: *orbit-list-def iterate-def distinct-map inj-on-funpow-dist1*[*OF assms*])

lemma *distinct-orbit-list'*:
  assumes *permutation f* **shows** *distinct* (*orbit-list f x*)
  using *assms* **by** (*simp add*: *permutation-self-in-orbit distinct-orbit-list*)

lemma *orbits-list-conv-impl*:
  assumes *permutation f*
  shows *orbits-list f xs* = *orbits-list-impl f xs*
proof (*induct length xs arbitrary*: *xs rule*: *less-induct*)
  case *less* **show** *?case*
   using *assms* **by** (*cases xs*) (*auto simp*: *assms less less-Suc-eq-le length-fold-remove1-le orbit-list-conv-impl permutation-self-in-orbit Let-def*)
qed

lemma *orbit-list-not-nil*[*simp*]: *orbit-list f x* ≠ []
  by (*simp add*: *orbit-list-def*)

lemma *sset-orbits-list*:
  assumes *permutation f* **shows** *sset* (*orbits-list f xs*) = (*orbit f*) ‘ *set xs*
proof (*induct length xs arbitrary*: *xs rule*: *less-induct*)
  case *less*
  show *?case*
  proof (*cases xs*)
    case *Nil* **then show** *?thesis* **by** *simp*
  next
    case (*Cons x′ xs′*)
    let *?xs′′* = *fold remove1* (*orbit-list f x′*) *xs′*
    have *A*: *sset* (*orbits-list f ?xs′′*) = *orbit f* ‘ *set ?xs′′*
      using *Cons* **by** (*simp add*: *less-Suc-eq-le length-fold-remove1-le less.hyps*)
    have *B*: *set* (*orbit-list f x′*) = *orbit f x′*
      by (*rule set-orbit-list*) (*simp add*: *permutation-self-in-orbit assms*)

    have *orbit f* ‘ *set* (*fold remove1* (*orbit-list f x′*) *xs′*) ⊆ *orbit f* ‘ *set xs′*
      using *set-fold-remove1*[*of - xs′*] **by** *auto*
    moreover
    have *orbit f* ‘ *set xs′* − {*orbit f x′*} ⊆ (*orbit f* ‘ *set* (*fold remove1* (*orbit-list f x′*) *xs′*)) (**is** *?L* ⊆ *?R*)
    proof
      fix *A* **assume** *A* ∈ *?L*
      then obtain *y* **where** *A* = *orbit f y y* ∈ *set xs′* **by** *auto*
      have *A* ≠ *orbit f x′* **using** ‹*A* ∈ *?L*› **by** *auto*
      from ‹*A* = -› ‹*A* ≠ -› **have** *y* ∉ *orbit f x′*
        by (*meson assms cyclic-on-orbit orbit-cyclic-eq3 permutation-permutes*)
      with ‹*y* ∈ -› **have** *y* ∈ *set* (*fold remove1* (*orbit-list f x′*) *xs′*)
      by (*auto simp*: *set-fold-remove1′ set-orbit-list permutation-self-in-orbit assms*)
      then show *A* ∈ *?R* **using** ‹*A* = -› **by** *auto*
    qed
    ultimately
    show *?thesis* **by** (*auto simp*: *A B Cons*)
  qed
qed

5.1 Relation to *cyclic-on*

lemma *list-succ-orbit-list*:
  assumes *s* ∈ *orbit f s* ⋀*x*. *x* ∉ *orbit f s* ⟹ *f x* = *x*
  shows *list-succ* (*orbit-list f s*) = *f*
proof −
  have *distinct* (*orbit-list f s*) ⋀*x*. *x* ∉ *set* (*orbit-list f s*) ⟹ *x* = *f x*
    using *assms* **by** (*simp-all add*: *distinct-orbit-list set-orbit-list*)
  moreover
  have ⋀*i*. *i* < *length* (*orbit-list f s*) ⟹ *orbit-list f s* ! (*Suc i mod length* (*orbit-list f s*)) = *f* (*orbit-list f s* ! *i*)
    using *funpow-dist1-prop*[*OF* ‹*s* ∈ *orbit f s*›] **by** (*auto simp*: *orbit-list-def funpow-mod-eq*)

ultimately show *?thesis*
  by (*auto simp*: *list-succ-def fun-eq-iff*)
qed

lemma *list-succ-funpow-conv*:
  assumes *A*: *distinct xs* $x \in$ *set xs*
  shows (*list-succ xs* ^^ *n*) x = *xs* ! ((*index xs x* + *n*) *mod length xs*)
proof −
  have *xs* $\neq$ [] **using** *assms* **by** *auto*
  then show *?thesis*
    by (*induct n*) (*auto simp*: *hd-conv-nth A index-nth-id list-succ-def mod-simps*)
qed

lemma *orbit-list-succ*:
  assumes *distinct xs* $x \in$ *set xs*
  shows *orbit* (*list-succ xs*) x = *set xs*
proof (*intro set-eqI iffI*)
  fix *y* **assume** $y \in$ *orbit* (*list-succ xs*) *x*
  then show $y \in$ *set xs*
    by *induct* (*auto simp*: *list-succ-in-conv* ‹$x \in$ *set xs*›)
next
  fix *y* **assume** $y \in$ *set xs*
  moreover
  { **fix** *i j* **have** $i <$ *length xs* $\Longrightarrow$ $j <$ *length xs* $\Longrightarrow$ $\exists$ *n*. *xs* ! *j* = *xs* ! ((*i* + *n*) *mod length xs*)
      using *assms* **by** (*auto simp*: *exI*[**where** *x*=*j* + (*length xs* − *i*)])
  }
  ultimately
  show $y \in$ *orbit* (*list-succ xs*) *x*
    using *assms* **by** (*auto simp*: *orbit-altdef-permutation permutation-list-succ list-succ-funpow-conv index-nth-id in-set-conv-nth*)
qed

lemma *cyclic-on-list-succ*:
  assumes *distinct xs xs* $\neq$ [] **shows** *cyclic-on* (*list-succ xs*) (*set xs*)
  using *assms last-in-set* **by** (*auto simp*: *cyclic-on-def orbit-list-succ*)

lemma *obtain-orbit-list-func*:
  assumes $s \in$ *orbit f s* $\bigwedge x.\ x \notin$ *orbit f s* $\Longrightarrow$ *f x* = *x*
  obtains *xs* **where** *f* = *list-succ xs set xs* = *orbit f s distinct xs hd xs* = *s*
proof −
  { **from** *assms* **have** *f* = *list-succ* (*orbit-list f s*) **by** (*simp add*: *list-succ-orbit-list*)
    moreover
    have *set* (*orbit-list f s*) = *orbit f s distinct* (*orbit-list f s*)
      by (*auto simp*: *set-orbit-list distinct-orbit-list assms*)
    moreover have *hd* (*orbit-list f s*) = *s*
      by (*simp add*: *orbit-list-def iterate-def hd-map del*: *upt-Suc*)
    ultimately have $\exists$ *xs*. *f* = *list-succ xs* $\wedge$ *set xs* = *orbit f s* $\wedge$ *distinct xs* $\wedge$ *hd xs* = *s* **by** *blast*

} **then show** *?thesis* **by** (*metis that*)
qed

lemma *cyclic-on-obtain-list-succ*:
assumes *cyclic-on f S* $\bigwedge x.\ x \notin S \Longrightarrow f\ x = x$
obtains *xs* **where** f = *list-succ xs set xs* = *S distinct xs*
proof −
from *assms* **obtain** *s* **where** *s*: $s \in$ *orbit f s* $\bigwedge x.\ x \notin$ *orbit f s* $\Longrightarrow f\ x = x$ *S* = *orbit f s*
by (*auto simp*: *cyclic-on-def*)
then show *?thesis* **by** (*metis that obtain-orbit-list-func*)
qed

lemma *cyclic-on-obtain-list-succ'*:
assumes *cyclic-on f S f permutes S*
obtains *xs* **where** f = *list-succ xs set xs* = *S distinct xs*
using *assms* **unfolding** *permutes-def* **by** (*metis cyclic-on-obtain-list-succ*)

lemma *list-succ-unique*:
assumes $s \in$ *orbit f s* $\bigwedge x.\ x \notin$ *orbit f s* $\Longrightarrow f\ x = x$
shows $\exists!$*xs*. f = *list-succ xs* $\wedge$ *distinct xs* $\wedge$ *hd xs* = *s* $\wedge$ *set xs* = *orbit f s*
proof −
from *assms* **obtain** *xs* **where** *xs*: f = *list-succ xs distinct xs hd xs* = *s set xs* = *orbit f s*
by (*rule obtain-orbit-list-func*)
moreover
{ **fix** *zs*
assume *A*: f = *list-succ zs distinct zs hd zs* = *s set zs* = *orbit f s*
then have *zs* $\neq$ [] **using** ‹$s \in$ *orbit f s*› **by** *auto*
from ‹*distinct xs*› ‹*distinct zs*› ‹*set xs* = *orbit f s*› ‹*set zs* = *orbit f s*›
have *len*: *length xs* = *length zs* **by** (*metis distinct-card*)

{ **fix** *n* **assume** $n <$ *length xs*
then have *zs* ! *n* = *xs* ! *n*
proof (*induct n*)
case *0* **with** *A xs* ‹*zs* $\neq$ []› **show** *?case* **by** (*simp add*: *hd-conv-nth nth-rotate-conv-nth*)
next
case (*Suc n*)
then have *list-succ zs* (*zs* ! *n*) = *list-succ xs* (*xs*! *n*)
using ‹f = *list-succ xs*› ‹f = *list-succ zs*› **by** *simp*
with ‹*Suc n* < -› **show** *?case*
by (*simp add*:*list-succ-nth len* ‹*distinct xs*› ‹*distinct zs*›)
qed }
then have *zs* = *xs* **by** (*metis len nth-equalityI*) }
ultimately show *?thesis* **by** *metis*
qed

lemma *distincts-orbits-list*:

assumes *distinct as permutation f*
shows *distincts* (*orbits-list f as*)
using *assms*(*1*)
proof (*induct length as arbitrary*: *as rule*: *less-induct*)
case *less*
show *?case*
proof (*cases as*)
case *Nil* **then show** *?thesis* **by** *simp*
next
case (*Cons a as′*)
let *?as′* = *fold remove1* (*orbit-list f a*) *as′*
from *Cons less.prems* **have** *A*: *distincts* (*orbits-list f* (*fold remove1* (*orbit-list f a*) *as′*))
by (*intro less*) (*auto simp*: *distinct-fold-remove1 length-fold-remove1-le less-Suc-eq-le*)

have *B*: *set* (*orbit-list f a*) $\cap$ $\bigcup$(*sset* (*orbits-list f* (*fold remove1* (*orbit-list f a*) *as′*))) = {}
proof −
have *orbit f a* $\cap$ *set* (*fold remove1* (*orbit-list f a*) *as′*) = {}
using *assms less.prems Cons* **by** (*simp add*: *set-fold-remove1-distinct set-orbit-list′*)
then have *orbit f a* $\cap$ $\bigcup$ (*orbit f ‘ set* (*fold remove1* (*orbit-list f a*) *as′*)) = {}
by *auto* (*metis assms*(*2*) *cyclic-on-orbit disjoint-iff-not-equal permutation-self-in-orbit*[*OF assms*(*2*)] *orbit-cyclic-eq3 permutation-permutes*)
then show *?thesis* **using** *assms*
by (*auto simp*: *set-orbit-list′ sset-orbits-list disjoint-iff-not-equal*)
qed
show *?thesis*
using *A B assms* **by** (*auto simp*: *distincts-Cons Cons distinct-orbit-list′*)
qed
qed

lemma *cyclic-on-lists-succ′*:
assumes *distincts xss*
shows $A \in$ *sset xss* $\Longrightarrow$ *cyclic-on* (*lists-succ xss*) *A*
using *assms*
proof (*induction xss arbitrary*: *A*)
case *Nil* **then show** *?case* **by** *auto*
next
case (*Cons xs xss A*)
then have *inter*: *set xs* $\cap$ ($\bigcup$*ys*$\in$*set xss. set ys*) = {} **by** (*auto simp*: *distincts-Cons*)

note *pcp*[*OF - - inter*] = *permutes-comp-preserves-cyclic1 permutes-comp-preserves-cyclic2*
from *Cons* **show** *cyclic-on* (*lists-succ* (*xs # xss*)) *A*
by (*cases A* = *set xs*)
(*auto intro*: *pcp simp*: *cyclic-on-list-succ list-succ-permutes*
lists-succ-permutes lists-succ-Cons-pf distincts-Cons)
qed

lemma *cyclic-on-lists-succ*:
  assumes *distincts xss*
  shows $\bigwedge xs.\ xs \in set\ xss \Longrightarrow cyclic\text{-}on\ (lists\text{-}succ\ xss)\ (set\ xs)$
  using *assms* **by** (*auto intro*: *cyclic-on-lists-succ′*)

lemma *permutes-as-lists-succ*:
  assumes *distincts xss*
  assumes *ls-eq*: $\bigwedge xs.\ xs \in set\ xss \Longrightarrow list\text{-}succ\ xs = perm\text{-}restrict\ f\ (set\ xs)$
  assumes *f permutes* $(\bigcup (sset\ xss))$
  shows $f = lists\text{-}succ\ xss$
  using *assms*
proof (*induct xss arbitrary*: *f*)
  case *Nil* **then show** *?case* **by** *simp*
next
  case (*Cons xs xss*)
  let $?sets = \lambda xss.\ \bigcup ys \in set\ xss.\ set\ ys$

  have *xs*: *distinct xs* $xs \neq []$ **using** *Cons* **by** (*auto simp*: *distincts-Cons*)

  have *f-xs*: $perm\text{-}restrict\ f\ (set\ xs) = list\text{-}succ\ xs$
    using *Cons* **by** *simp*

  have *co-xs*: $cyclic\text{-}on\ (perm\text{-}restrict\ f\ (set\ xs))\ (set\ xs)$
    unfolding *f-xs* **using** *xs* **by** (*rule cyclic-on-list-succ*)

  have *perm-xs*: *perm-restrict f* (*set xs*) *permutes set xs*
    unfolding *f-xs* **using** ‹*distinct xs*› **by** (*rule list-succ-permutes*)

  have *perm-xss*: *perm-restrict f* (*?sets xss*) *permutes* (*?sets xss*)
  proof −
    have $perm\text{-}restrict\ f\ (?sets\ (xs\ \#\ xss) - set\ xs)\ permutes\ (?sets\ (xs\ \#\ xss) -$
$set\ xs)$
     using *Cons co-xs* **by** (*intro perm-restrict-diff-cyclic*) (*auto simp*: *cyclic-on-perm-restrict*)
    also have $?sets\ (xs\ \#\ xss) - set\ xs = ?sets\ xss$
      using *Cons* **by** (*auto simp*: *distincts-Cons*)
    finally show *?thesis* **.**
  qed

  have *f-xss*: $perm\text{-}restrict\ f\ (?sets\ xss) = lists\text{-}succ\ xss$
  proof −
    have $*$: $\bigwedge xs.\ xs \in set\ xss \Longrightarrow ((\bigcup x{\in}set\ xss.\ set\ x) \cap set\ xs) = set\ xs$
      by *blast*
    with *perm-xss Cons.prems* **show** *?thesis*
     by (*intro Cons.hyps*) (*auto simp*: *distincts-Cons perm-restrict-perm-restrict* $*$)
  qed

  from *Cons.prems* **show** $f = lists\text{-}succ\ (xs\ \#\ xss)$
    by (*simp add*: *lists-succ-Cons-pf distincts-Cons f-xss*[*symmetric*]

perm-restrict-union perm-xs perm-xss)
qed

lemma *cyclic-on-obtain-lists-succ*:
assumes
permutes: *f permutes S* **and**
S: $S = \bigcup(sset\ css)$ **and**
dists: *distincts css* **and**
cyclic: $\bigwedge cs.\ cs \in set\ css \Longrightarrow cyclic\text{-}on\ f\ (set\ cs)$
obtains *xss* **where** $f = lists\text{-}succ\ xss$ *distincts xss* $map\ set\ xss = map\ set\ css$ $map\ hd\ xss = map\ hd\ css$
proof −
let $?fc = \lambda cs.\ perm\text{-}restrict\ f\ (set\ cs)$
define *some-list* **where** $some\text{-}list\ cs = (SOME\ xs.\ ?fc\ cs = list\text{-}succ\ xs \wedge set\ xs = set\ cs \wedge distinct\ xs \wedge hd\ xs = hd\ cs)$ **for** *cs*
{ fix *cs* **assume** $cs \in set\ css$
then have $cyclic\text{-}on\ (?fc\ cs)\ (set\ cs)$ $\bigwedge x.\ x \notin set\ cs \Longrightarrow ?fc\ cs\ x = x$ $hd\ cs \in set\ cs$
using *cyclic dists* **by** (*auto simp add*: *cyclic-on-perm-restrict perm-restrict-def distincts-def*)
then have $hd\ cs \in orbit\ (?fc\ cs)\ (hd\ cs)$ $\bigwedge x.\ x \notin orbit\ (?fc\ cs)\ (hd\ cs) \Longrightarrow ?fc\ cs\ x = x$ $hd\ cs \in set\ cs$ $set\ cs = orbit\ (?fc\ cs)\ (hd\ cs)$
by (*auto simp*: *cyclic-on-alldef*)
then have $\exists xs.\ ?fc\ cs = list\text{-}succ\ xs \wedge set\ xs = set\ cs \wedge distinct\ xs \wedge hd\ xs = hd\ cs$
by (*metis obtain-orbit-list-func*)
then have $?fc\ cs = list\text{-}succ\ (some\text{-}list\ cs) \wedge set\ (some\text{-}list\ cs) = set\ cs \wedge distinct\ (some\text{-}list\ cs) \wedge hd\ (some\text{-}list\ cs) = hd\ cs$
unfolding *some-list-def* **by** (*rule someI-ex*)
then have $?fc\ cs = list\text{-}succ\ (some\text{-}list\ cs)$ $set\ (some\text{-}list\ cs) = set\ cs$ $distinct\ (some\text{-}list\ cs)$ $hd\ (some\text{-}list\ cs) = hd\ cs$
by *auto*
} note *sl-cs* = *this*

have $\bigwedge cs.\ cs \in set\ css \Longrightarrow cs \neq []$ **using** *dists* **by** (*auto simp*: *distincts-def*)
then have *some-list-ne*: $\bigwedge cs.\ cs \in set\ css \Longrightarrow some\text{-}list\ cs \neq []$
by (*metis set-empty sl-cs*(*2*))

have *set*: $map\ set\ (map\ some\text{-}list\ css) = map\ set\ css$ $map\ hd\ (map\ some\text{-}list\ css) = map\ hd\ css$
using *sl-cs*(*2*,*4*) **by** (*auto simp add*: *map-idI*)

have *distincts*: *distincts* (*map some-list css*)
proof −
have *c-dist*: $\bigwedge xs\ ys.\ [\![xs{\in}set\ css;\ ys{\in}set\ css;\ xs \neq ys]\!] \Longrightarrow set\ xs \cap set\ ys = \{\}$
using *dists* **by** (*auto simp*: *distincts-def*)

have *distinct* (*map some-list css*)
proof −

have *inj-on some-list* (*set css*)
using *sl-cs*(*2*) *c-dist* **by** (*intro inj-onI*) (*metis inf.idem set-empty*)
with ‹*distincts css*› **show** *?thesis*
by (*auto simp*: *distincts-distinct distinct-map*)
qed
moreover
have $\forall xs \in set$ (*map some-list css*). *distinct xs* $\wedge$ *xs* $\neq$ []
using *sl-cs*(*3*) *some-list-ne* **by** *auto*
moreover
from *c-dist* **have** ($\forall xs \in set$ (*map some-list css*). $\forall ys \in set$ (*map some-list css*). *xs* $\neq$ *ys* $\longrightarrow$ *set xs* $\cap$ *set ys* = {})
using *sl-cs*(*2*) **by** *auto*
ultimately
show *?thesis* **by** (*simp add*: *distincts-def*)
qed

have *f*: *f* = *lists-succ* (*map some-list css*)
using *distincts*
proof (*rule permutes-as-lists-succ*)
fix *xs* **assume** *xs* $\in$ *set* (*map some-list css*)
then show *list-succ xs* = *perm-restrict f* (*set xs*)
using *sl-cs*(*1*) *sl-cs*(*2*) **by** *auto*
next
have *S* = ($\bigcup xs \in set$ (*map some-list css*). *set xs*)
using *S sl-cs*(*2*) **by** *auto*
with *permutes* **show** *f permutes* $\bigcup$ (*sset* (*map some-list css*))
by *simp*
qed

from *f distincts set* **show** *?thesis* **..**
qed

5.2 Permutations of a List

lemma *length-remove1-less*:
assumes *x* $\in$ *set xs* **shows** *length* (*remove1 x xs*) < *length xs*
proof −
from *assms* **have** *0* < *length xs* **by** *auto*
with *assms* **show** *?thesis* **by** (*auto simp*: *length-remove1*)
qed
context notes [*simp*] = *length-remove1-less* **begin**
fun *permutations* :: *'a list* $\Rightarrow$ *'a list list* **where**
permutations-Nil: *permutations* [] = [[]]
| *permutations-Cons*:
permutations xs = [*y* # *ys*. *y* <− *xs*, *ys* <− *permutations* (*remove1 y xs*)]
end

declare *permutations-Cons*[*simp del*]

The function above returns all permutations of a list. The function below computes only those which yield distinct cyclic permutation functions (cf. *list-succ*).

fun *cyc-permutations* :: *'a list* ⇒ *'a list list* **where**
  cyc-permutations [] = [[]]
| *cyc-permutations* (*x* # *xs*) = *map* (*Cons x*) (*permutations xs*)

lemma *nil-in-permutations*[*simp*]: [] ∈ *set* (*permutations xs*) ⟷ *xs* = []
  by (*induct xs*) (*auto simp*: *permutations-Cons*)

lemma *permutations-not-nil*:
  assumes *xs* ≠ []
  shows *permutations xs* = *concat* (*map* (λ*x*. *map* ((#) *x*) (*permutations* (*remove1* *x xs*))) *xs*)
  using *assms* **by** (*cases xs*) (*auto simp*: *permutations-Cons*)

lemma *set-permutations-step*:
  assumes *xs* ≠ []
  shows *set* (*permutations xs*) = (⋃*x* ∈ *set xs*. *Cons x* ` *set* (*permutations* (*remove1* *x xs*)))
  using *assms* **by** (*cases xs*) (*auto simp*: *permutations-Cons*)

lemma *in-set-permutations*:
  assumes *distinct xs*
  shows *ys* ∈ *set* (*permutations xs*) ⟷ *distinct ys* ∧ *set xs* = *set ys* (**is** *?L xs ys* ⟷ *?R xs ys*)
  using *assms*
proof (*induct length xs arbitrary*: *xs ys*)
  case *0* **then show** *?case* **by** *auto*
next
  case (*Suc n*)
  then have *xs* ≠ [] **by** *auto*

  show *?case*
  proof
    assume *?L xs ys*
    then obtain *y ys'* **where** *ys* = *y* # *ys'* *y* ∈ *set xs* *ys'* ∈ *set* (*permutations* (*remove1* (*hd ys*) *xs*))
      using ‹*xs* ≠ []› **by** (*auto simp*: *permutations-not-nil*)
    moreover
    then have *?R* (*remove1 y xs*) *ys'*
      using *Suc.prems Suc.hyps*(*2*) **by** (*intro Suc.hyps*(*1*)[*THEN iffD1*]) (*auto simp*: *length-remove1*)
    ultimately show *?R xs ys*
      using *Suc* **by** *auto*
  next
    assume *?R xs ys*
    with ‹*xs* ≠ []› **obtain** *y ys'* **where** *ys* = *y* # *ys'* *y* ∈ *set xs* **by** (*cases ys*) *auto*

moreover
then have *ys'* ∈ *set* (*permutations* (*remove1 y xs*))
using *Suc* ‹*?R xs ys*› **by** (*intro Suc.hyps*(*1*)[*THEN iffD2*]) (*auto simp*: *length-remove1*)
ultimately
show *?L xs ys*
using ‹*xs* ≠ []› **by** (*auto simp*: *permutations-not-nil*)
qed
qed

lemma *in-set-cyc-permutations*:
assumes *distinct xs*
shows *ys* ∈ *set* (*cyc-permutations xs*) ⟷ *distinct ys* ∧ *set xs* = *set ys* ∧ *hd ys* = *hd xs* (**is** *?L xs ys* ⟷ *?R xs ys*)
proof (*cases xs*)
case (*Cons x xs*) **with** *assms* **show** *?thesis*
by (*cases ys*) (*auto simp*: *in-set-permutations intro*!: *imageI*)
qed *auto*

lemma *in-set-cyc-permutations-obtain*:
assumes *distinct xs distinct ys set xs* = *set ys*
obtains *n* **where** *rotate n ys* ∈ *set* (*cyc-permutations xs*)
proof (*cases xs*)
case *Nil* **with** *assms* **have** *rotate 0 ys* ∈ *set* (*cyc-permutations xs*) **by** *auto*
then show *?thesis* **..**
next
case (*Cons x xs'*)
let *?ys'* = *rotate* (*index ys x*) *ys*
have *ys* ≠ [] *x* ∈ *set ys*
using *Cons assms* **by** *auto*
then have *distinct ?ys'* ∧ *set xs* = *set ?ys'* ∧ *hd ?ys'* = *hd xs*
using *assms Cons* **by** (*auto simp add*: *hd-rotate-conv-nth*)
with ‹*distinct xs*› **have** *?ys'* ∈ *set* (*cyc-permutations xs*)
by (*rule in-set-cyc-permutations*[*THEN iffD2*])
then show *?thesis* **..**
qed

lemma *list-succ-set-cyc-permutations*:
assumes *distinct xs xs* ≠ []
shows *list-succ* ‘ *set* (*cyc-permutations xs*) = {*f*. *f permutes set xs* ∧ *cyclic-on f* (*set xs*)} (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
fix *f* **assume** *f* ∈ *?L*
moreover have ⋀*ys*. *set xs* = *set ys* ⟹ *xs* ≠ [] ⟹ *ys* ≠ [] **by** *auto*
ultimately show *f* ∈ *?R*
using *assms* **by** (*auto simp*: *in-set-cyc-permutations list-succ-permutes cyclic-on-list-succ*)
next
fix *f* **assume** *f* ∈ *?R*
then obtain *ys* **where** *ys*: *list-succ ys* = *f distinct ys set ys* = *set xs*

by (*auto elim*: *cyclic-on-obtain-list-succ'*)
moreover
with ‹*distinct xs*› **obtain** *n* **where** *rotate n ys* $\in$ *set* (*cyc-permutations xs*)
by (*auto elim*: *in-set-cyc-permutations-obtain*)
then have *list-succ* (*rotate n ys*) $\in$ *?L* **by** *simp*
ultimately
show *f* $\in$ *?L* **by** *simp*
qed

5.3 Enumerating Permutations from List Orbits

definition *cyc-permutationss* :: *'a list list* $\Rightarrow$ *'a list list list* **where**
cyc-permutationss = *product-lists o map cyc-permutations*

lemma *cyc-permutationss-Nil*[*simp*]: *cyc-permutationss* [] = [[]]
by (*auto simp*: *cyc-permutationss-def*)

lemma *in-set-cyc-permutationss*:
assumes *distincts xss*
shows *yss* $\in$ *set* (*cyc-permutationss xss*) $\longleftrightarrow$ *distincts yss* $\wedge$ *map set xss* = *map set yss* $\wedge$ *map hd xss* = *map hd yss*
proof −
{ assume *A*: *list-all2* (λ*x ys*. *x* $\in$ *set ys*) *yss* (*map cyc-permutations xss*)
then have *length yss* = *length xss* **by** (*auto simp*: *list-all2-lengthD*)
then have $\bigcup$(*sset xss*) = $\bigcup$(*sset yss*) *distincts yss map set xss* = *map set yss map hd xss* = *map hd yss*
using *A assms*
by (*induct yss xss rule*: *list-induct2*) (*auto simp*: *distincts-Cons in-set-cyc-permutations*)
} note *X* = *this*
{ assume *A*: *distincts yss map set xss* = *map set yss map hd xss* = *map hd yss*
then have *length yss* = *length xss* **by** (*auto dest*: *map-eq-imp-length-eq*)
then have *list-all2* (λ*x ys*. *x* $\in$ *set ys*) *yss* (*map cyc-permutations xss*)
using *A assms*
by (*induct yss xss rule*: *list-induct2*) (*auto simp*: *distincts-Cons in-set-cyc-permutations*)
} note *Y* = *this*
show *?thesis*
unfolding *cyc-permutationss-def*
by (*auto simp*: *product-lists-set intro*: *X Y*)
qed

lemma *lists-succ-set-cyc-permutationss*:
assumes *distincts xss*
shows *lists-succ* ‘ *set* (*cyc-permutationss xss*) = {*f*. *f permutes* $\bigcup$(*sset xss*) $\wedge$ ($\forall$ *c* $\in$ *sset xss*. *cyclic-on f c*)} (**is** *?L* = *?R*)
using *assms*
proof (*intro set-eqI iffI*)
fix *f* **assume** *f* $\in$ *?L*
then obtain *yss* **where** *yss* $\in$ *set* (*cyc-permutationss xss*) *f* = *lists-succ yss* **by** (*rule imageE*)

moreover
from ‹*yss* ∈ -› *assms* **have** *set* (*map set xss*) = *set* (*map set yss*)
by (*auto simp*: *in-set-cyc-permutationss*)
then have *sset xss* = *sset yss* **by** *simp*
ultimately
show *f* ∈ *?R*
using *assms*
by (*auto simp*: *in-set-cyc-permutationss cyclic-on-lists-succ′*) (*metis lists-succ-permutes*)
next
fix *f* **assume** *f* ∈ *?R*
then have *f permutes* $\bigcup$(*sset xss*) $\bigwedge$*cs. cs* ∈ *set xss* ⟹ *cyclic-on f* (*set cs*)
by *auto*
from *this*(*1*) *refl assms this*(*2*)
obtain *yss* **where** *f* = *lists-succ yss distincts yss map set yss* = *map set xss map hd yss* = *map hd xss*
by (*rule cyclic-on-obtain-lists-succ*)
with *assms* **show** *f* ∈ *?L* **by** (*auto intro*!: *imageI simp*: *in-set-cyc-permutationss*)
qed

5.4 Lists of Permutations

definition *permutationss* :: *′a list list* ⇒ *′a list list list* **where**
permutationss = *product-lists o map permutations*

lemma *permutationss-Nil*[*simp*]: *permutationss* [] = [[]]
by (*auto simp*: *permutationss-def*)

lemma *permutationss-Cons*:
permutationss (*xs* # *xss*) = *concat* (*map* (*λys. map* (*Cons ys*) (*permutationss xss*)) (*permutations xs*))
by (*auto simp*: *permutationss-def*)

lemma *in-set-permutationss*:
assumes *distincts xss*
shows *yss* ∈ *set* (*permutationss xss*) ⟷ *distincts yss* ∧ *map set xss* = *map set yss*
proof −
{ assume *A*: *list-all2* (*λx ys. x* ∈ *set ys*) *yss* (*map permutations xss*)
then have *length yss* = *length xss* **by** (*auto simp*: *list-all2-lengthD*)
then have $\bigcup$(*sset xss*) = $\bigcup$(*sset yss*) *distincts yss map set xss* = *map set yss*
using *A assms*
by (*induct yss xss rule*: *list-induct2*) (*auto simp*: *distincts-Cons in-set-permutations*)
} note *X* = *this*
{ assume *A*: *distincts yss map set xss* = *map set yss*
then have *length yss* = *length xss* **by** (*auto dest*: *map-eq-imp-length-eq*)
then have *list-all2* (*λx ys. x* ∈ *set ys*) *yss* (*map permutations xss*)
using *A assms*
by (*induct yss xss rule*: *list-induct2*) (*auto simp*: *in-set-permutations distincts-Cons*)

} **note** *Y* = *this*
show *?thesis*
unfolding *permutationss-def*
by (*auto simp*: *product-lists-set intro*: *X Y*)
qed

lemma *set-permutationss*:
assumes *distincts xss*
shows *set* (*permutationss xss*) = {*yss*. *distincts yss* ∧ *map set xss* = *map set yss*}
using *in-set-permutationss*[*OF assms*] **by** *blast*

lemma *permutationss-complete*:
assumes *distincts xss distincts yss xss* ≠ []
and *set* ' *set xss* = *set* ' *set yss*
shows *set yss* ∈ *set* ' *set* (*permutationss xss*)
proof −
have *length xss* = *length yss*
using *assms* **by** (*simp add*: *length-distincts*)
from ‹*sset xss* = -›
have ∃ *yss'*. *set yss'* = *set yss* ∧ *map set yss'* = *map set xss*
using *assms*(*1*−*2*)
proof (*induct xss arbitrary*: *yss*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons xs xss*)
from ‹*sset* (*xs* # *xss*) = *sset yss*›
obtain *ys* **where** *ys*: *ys* ∈ *set yss set ys* = *set xs*
by *auto* (*metis imageE insertI1*)
with ‹*distincts yss*› **have** *set ys* ∉ *sset* (*remove1 ys yss*)
by (*fastforce simp*: *distincts-def*)
moreover
from ‹*distincts* (*xs* # *xss*)› **have** *set xs* ∉ *sset xss*
by (*fastforce simp*: *distincts-def*)
ultimately have *sset xss* = *sset* (*remove1 ys yss*)
using ‹*distincts yss*› ‹*sset* (*xs* # *xss*) = *sset yss*›
apply (*auto simp*: *distincts-distinct* ‹*set ys* = *set xs*›[*symmetric*])
apply (*smt Diff-insert-absorb* ‹*ys* ∈ *set yss*› *image-insert insert-Diff rev-image-eqI*)
by (*metis* ‹*ys* ∈ *set yss*› *image-eqI insert-Diff insert-iff*)
then obtain *yss'* **where** *set yss'* = *set* (*remove1 ys yss*) ∧ *map set yss'* = *map set xss*
using *Cons* **by** *atomize-elim* (*auto simp*: *distincts-Cons distincts-remove1*)
then have *set* (*ys* # *yss'*) = *set yss* ∧ *map set* (*ys* # *yss'*) = *map set* (*xs* # *xss*)
using *ys set-remove1-eq* ‹*distincts yss*› **by** (*auto simp*: *distincts-distinct*)
then show *?case* **..**
qed
then obtain *yss'* **where** *set yss'* = *set yss map set yss'* = *map set xss* **by** *blast*
then have *distincts yss'* **using** ‹*distincts xss*› ‹*distincts yss*›

unfolding *distincts-def*
by *simp-all* (*metis* ‹*length xss* = *length yss*› *card-distinct distinct-card length-map*)
then have *set yss'* ∈ *set* ‘ *set* (*permutationss xss*)
using ‹*distincts xss*› ‹*map set yss'* = -›
by (*auto simp*: *set-permutationss*)
then show *?thesis* **using** ‹*set yss'* = -› **by** *auto*
qed

lemma *permutations-complete*:
assumes *distinct xs distinct ys set xs* = *set ys*
shows *ys* ∈ *set* (*permutations xs*)
using *assms*
proof (*induct length xs arbitrary*: *xs ys*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
from *Suc.hyps* **have** *xs* ≠ [] **by** *auto*
then obtain *y ys'* **where** [*simp*]: *ys* = *y* # *ys' y* ∈ *set xs* **using** *Suc.prems* **by** (*cases ys*) *auto*
have *ys'* ∈ *set* (*permutations* (*remove1 y xs*))
using *Suc.prems* ‹*Suc n* = -› **by** (*intro Suc.hyps*) (*simp-all add*: *length-remove1*)
then show *?case* **using** ‹*xs* ≠ []› **by** (*auto simp*: *set-permutations-step*)
qed

end
theory *Digraph-Map-Impl*
imports
Graph-Genus
Executable-Permutations
Transitive−Closure.Transitive-Closure-Impl
begin

6 Enumerating Maps

definition *grouped-by-fst* :: (*'a* × *'b*) *list* ⇒ (*'a* × *'b*) *list list* **where**
grouped-by-fst xs = *map* (λ*u*. *filter* (λ*x*. *fst x* = *u*) *xs*) (*remdups* (*map fst xs*))

fun *grouped-out-arcs* :: *'a list* × (*'a* × *'a*) *list* ⇒ (*'a* × *'a*) *list list* **where**
grouped-out-arcs (*vs*,*as*) = *grouped-by-fst as*

definition *all-maps-list* :: (*'a list* × (*'a* × *'a*) *list*) ⇒ (*'a* × *'a*) *list list list* **where**
all-maps-list G-list = (*cyc-permutationss o grouped-out-arcs*) *G-list*

definition *list-digraph-ext ext G-list* ≡ (| *pverts* = *set* (*fst G-list*), *parcs* = *set* (*snd G-list*), . . . = *ext* |)
abbreviation *list-digraph* ≡ *list-digraph-ext* ()

code-datatype *list-digraph-ext*

lemma *list-digraph-simps*:
pverts (*list-digraph G-list*) = *set* (*fst G-list*)
parcs (*list-digraph G-list*) = *set* (*snd G-list*)
by (*auto simp*: *list-digraph-ext-def*)

lemma *union-grouped-by-fst*:
($\bigcup$ *xs* ∈ *set* (*grouped-by-fst ys*). *set xs*) = *set ys*
by (*auto simp*: *grouped-by-fst-def*)

lemma *union-grouped-out-arcs*:
($\bigcup$ *xs* ∈ *set* (*grouped-out-arcs G-list*). *set xs*) = *set* (*snd G-list*)
by (*cases G-list*) (*simp add*: *union-grouped-by-fst*)

lemma *nil-not-in-grouped-out-arcs*: [] ∉ *set* (*grouped-out-arcs G-list*)
apply (*cases G-list*) **apply** (*auto simp*: *grouped-by-fst-def*)
by (*metis* (*mono-tags*) *filter-empty-conv fst-conv*)

lemma *set-grouped-out-arcs*:
assumes *pair-wf-digraph* (*list-digraph G-list*)
shows *set* ‘ *set* (*grouped-out-arcs G-list*) = {*out-arcs* (*list-digraph G-list*) *v* | *v*. *v* ∈ *pverts* (*list-digraph G-list*) ∧ *out-arcs* (*list-digraph G-list*) *v* ≠ {} }
(**is** *?L* = *?R*)
proof −
interpret *pair-wf-digraph list-digraph G-list* **by** *fact*
define *vs* **where** *vs* = *remdups* (*map fst* (*snd G-list*))
have *set vs* = {*v*. *out-arcs* (*list-digraph G-list*) *v* ≠ {}}
by (*auto simp*: *out-arcs-def list-digraph-ext-def vs-def intro*: *rev-image-eqI*)
then have *vs*: *set vs* = {*v* ∈ *pverts* (*list-digraph G-list*). *out-arcs* (*list-digraph G-list*) *v* ≠ {}}
by (*auto dest*: *in-arcsD1*)
have *goa*: *grouped-out-arcs G-list* = *map* (λ*u*. *filter* (λ*x*. *fst x* = *u*) (*snd G-list*)) *vs*
by (*cases G-list*) (*auto simp*: *grouped-by-fst-def vs-def*)
have *filter*: *set* ∘ (λ*u*. *filter* (λ*x*. *fst x* = *u*) (*snd G-list*)) = *out-arcs* (*list-digraph G-list*)
by (*rule ext*) (*auto simp*: *list-digraph-ext-def*)

have *set* (*map set* (*grouped-out-arcs G-list*)) = *?R* **by** (*auto simp add*: *goa filter vs*)
then show *?thesis* **by** *simp*
qed

lemma *distincts-grouped-by-fst*:
assumes *distinct xs* **shows** *distincts* (*grouped-by-fst xs*)

proof −
  have *list-eq-setD*: $\bigwedge xs\ ys.\ xs = ys \Longrightarrow set\ xs = set\ ys$ **by** *auto*
  have *inj*: *inj-on* ($\lambda u.$ *filter* ($\lambda x.$ *fst* $x = u$) *xs*) (*fst* ‘ *set xs*)
    by (*rule inj-onI*) (*drule list-eq-setD*, *auto*)
  with *assms* **show** *?thesis*
    by (*auto simp*: *grouped-by-fst-def distincts-def distinct-map filter-empty-conv*)
qed

lemma *distincts-grouped-arcs*:
  assumes *distinct* (*snd G-list*) **shows** *distincts* (*grouped-out-arcs G-list*)
  using *assms* **by** (*cases G-list*) (*simp add*: *distincts-grouped-by-fst*)

lemma *distincts-in-all-maps-list*:
  distinct (*snd X*) $\Longrightarrow$ *xss* $\in$ *set* (*all-maps-list X*) $\Longrightarrow$ *distincts xss*
  by (*simp add*: *all-maps-list-def distincts-grouped-arcs in-set-cyc-permutationss*)

definition *to-map* :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a \Rightarrow 'a \times 'a) \Rightarrow ('a \times 'a)$ *pre-map*
where
  to-map A f = (| *edge-rev* = *swap-in A*, *edge-succ* = *f* |)

abbreviation *to-map'* *as xss* $\equiv$ *to-map* (*set as*) (*lists-succ xss*)

definition *all-maps* :: *'a pair-pre-digraph* $\Rightarrow$ $('a \times 'a)$ *pre-map set* **where**
  all-maps G $\equiv$ *to-map* (*arcs G*) ‘ $\{f.$ *f permutes arcs G* $\wedge$ $(\forall v \in$ *verts G*. *out-arcs G v* $\neq \{\} \longrightarrow$ *cyclic-on f* (*out-arcs G v*))$\}$

definition *maps-all-maps-list* :: $('a$ *list* $\times$ $('a \times 'a)$ *list*$)$ $\Rightarrow$ $('a \times 'a)$ *pre-map list*
where
  maps-all-maps-list G-list = *map* (*to-map* (*set* (*snd G-list*)) *o lists-succ*) (*all-maps-list G-list*)

lemma (**in** *pair-graph*) *all-maps-correct*:
  shows *all-maps G* = $\{M.$ *digraph-map G M*$\}$
proof (*intro set-eqI iffI*)
  fix *M* **assume** *A*:$M \in$ *all-maps G*
  then have [*simp*]: *edge-rev M* = *swap-in* (*arcs G*) *edge-succ M permutes parcs G*
    by (*auto simp*: *all-maps-def to-map-def*)

  have *digraph-map G M*
  proof (*rule digraph-mapI*)
    fix *a* **assume** $a \notin$ *parcs G* **then show** *edge-rev M a* = *a* **by** (*auto simp*: *swap-in-def*)
  next
    fix *a* **assume** $a \in$ *parcs G*
    then show *edge-rev M* (*edge-rev M a*) = *a fst* (*edge-rev M a*) = *snd a edge-rev*

M a ≠ *a*
  by (*case-tac* [!] *a*) (*auto intro*: *arcs-symmetric simp*: *swap-in-def dest*: *no-loops*)
 next
  show *edge-succ M permutes parcs G* **by** *simp*
 next
  fix *v* **assume** *v* ∈ *pverts G out-arcs* (*with-proj G*) *v* ≠ {}
  then show *cyclic-on* (*edge-succ M*) (*out-arcs* (*with-proj G*) *v*)
   using *A* **unfolding** *all-maps-def* **by** (*auto simp*: *to-map-def*)
 qed
 then show *M* ∈ {*M*. *digraph-map G M*} **by** *simp*
next
 fix *M* **assume** *A*: *M* ∈ {*M*. *digraph-map G M*}
 then interpret *M*: *digraph-map G M* **by** *simp*
 from *A* **have** ⋀*x*. *fst* (*edge-rev M x*) = *fst* (*swap-in* (*arcs G*) *x*)
   ⋀*x*. *snd* (*edge-rev M x*) = *snd* (*swap-in* (*arcs G*) *x*)
  using *M.tail-arev M.head-arev* **by** (*auto simp*: *fun-eq-iff swap-in-def M.arev-eq*)
 then have *edge-rev M* = *swap-in* (*arcs G*)
  by (*metis prod.collapse fun-eq-iff*)
 then show *M* ∈ *all-maps G*
  using *M.edge-succ-permutes M.edge-succ-cyclic*
  unfolding *all-maps-def*
  by (*auto simp*: *to-map-def intro*!: *image-eqI*[**where** *x*=*edge-succ M*])
qed

lemma *set-maps-all-maps-list*:
 assumes *pair-wf-digraph* (*list-digraph G-list*) *distinct* (*snd G-list*)
 shows *all-maps* (*list-digraph G-list*) = *set* (*maps-all-maps-list G-list*)
proof −
 let *?G* = *list-digraph G-list*

 { fix *f*
  have (∀ *x*∈*set* (*grouped-out-arcs G-list*). *cyclic-on f* (*set x*))
    ⟷ (∀ *x*∈*set* ‘ *set* (*grouped-out-arcs G-list*). *cyclic-on f x*) (**is** *?all1* = -)
   by *simp*
  also have . . . ⟷ (∀ *v*∈*pverts ?G*. *out-arcs ?G v* ≠ {} ⟶ *cyclic-on f* (*out-arcs ?G v*)) (**is** - = *?all2*)
   using *assms* **by** (*auto simp*: *set-grouped-out-arcs*)
  finally have *?all1* = *?all2* **.**
 } **note** *all-eq* = *this*

 have *lists-succ* ‘ *set* (*all-maps-list G-list*)
   = {*f*. *f permutes arcs ?G* ∧ (∀ *v* ∈ *pverts ?G*. *out-arcs ?G v* ≠ {} ⟶ *cyclic-on f* (*out-arcs ?G v*))}
  unfolding *all-maps-list-def* **using** *assms all-eq*
  by (*simp add*: *lists-succ-set-cyc-permutationss distincts-grouped-arcs union-grouped-out-arcs list-digraph-simps*)
 then have ∗: *lists-succ* ‘ *set* (*all-maps-list G-list*) = {*f*. *f permutes set* (*snd G-list*) ∧ (∀ *v*∈*set* (*fst G-list*). *out-arcs* (*with-proj* (|*pverts* = *set* (*fst G-list*), *parcs*

= *set* (*snd G-list*)⦈) *v* ≠ {} ⟶ *cyclic-on f* (*out-arcs* (*with-proj* ⦇*pverts* = *set* (*fst G-list*), *parcs* = *set* (*snd G-list*)⦈) *v*))}
  by (*auto simp add*: *maps-all-maps-list-def all-maps-def list-digraph-simps list-digraph-ext-def*)
  then have ∗∗: $\bigwedge f$. ¬ (*f permutes set* (*snd G-list*) ∧ (∀ *a*. *a* ∈ *set* (*fst G-list*) ⟶ *out-arcs* (*with-proj* ⦇*pverts* = *set* (*fst G-list*), *parcs* = *set* (*snd G-list*)⦈) *a* ≠ {} ⟶ *cyclic-on f* (*out-arcs* (*with-proj* ⦇*pverts* = *set* (*fst G-list*), *parcs* = *set* (*snd G-list*)⦈) *a*))) ∨ *f* ∈ *lists-succ* ‘ *set* (*all-maps-list G-list*)
    by *force*
  from ∗ **show** *?thesis*
  by (*auto simp add*: *maps-all-maps-list-def all-maps-def list-digraph-simps list-digraph-ext-def*) (*use* ∗∗ **in** *blast*)
qed

7 Compute Face Cycles

definition *lists-fc-succ* :: (′*a* × ′*a*) *list list* ⇒ (′*a* × ′*a*) ⇒ (′*a* × ′*a*) **where**
  lists-fc-succ xss = (*let sxss* = $\bigcup$(*sset xss*) *in* (λ*x*. *lists-succ xss* (*swap-in sxss x*)))

locale *lists-digraph-map* =
  fixes *G-list* :: ′*b list* × (′*b* × ′*b*) *list*
    and *xss* :: (′*b* × ′*b*) *list list*
  assumes *digraph-map*: *digraph-map* (*list-digraph G-list*) (*to-map*′ (*snd G-list*) *xss*)
  assumes *no-loops*: $\bigwedge a$. *a* ∈ *parcs* (*list-digraph G-list*) ⟹ *fst a* ≠ *snd a*
  assumes *distincts-xss*: *distincts xss*
  assumes *parcs-xss*: *parcs* (*list-digraph G-list*) = $\bigcup$(*sset xss*)
begin

abbreviation (*input*) *G* ≡ *list-digraph G-list*
abbreviation (*input*) *M* ≡ *to-map*′ (*snd G-list*) *xss*

lemma *edge-rev-simps*:
  assumes (*u*,*v*) ∈ *parcs G* **shows** *edge-rev M* (*u*,*v*) = (*v*,*u*)
  using *assms*
  unfolding *to-map-def list-digraph-ext-def* **by** (*auto simp*: *swap-in-def to-map-def*)

end

sublocale *lists-digraph-map* ⊆ *digraph-map G M* **by** (*rule local.digraph-map*)

sublocale *lists-digraph-map* ⊆ *pair-graph G*
proof
  fix *e* **assume** *e* ∈ *parcs G*
  then have *e* ∈ *arcs G* **by** *simp*
  then have *head G e* ∈ *verts G tail G e* ∈ *verts G* **by** (*blast dest*: *wellformed*)+
  then show *fst e* ∈ *pverts G snd e* ∈ *pverts G* **by** *auto*
next
  fix *e* **assume** *e* ∈ *parcs G* **then show** *fst e* ≠ *snd e* **using** *no-loops* **by** *simp*
next

```
  show finite (pverts G) finite (parcs G)
    unfolding list-digraph-ext-def by simp-all
next
  { fix u v assume (u,v) ∈ parcs G
    then have edge-rev M (u,v) ∈ parcs G
      using edge-rev-in-arcs by simp
    then have (v,u) ∈ parcs G using ‹(u,v) ∈ - › by (simp add: edge-rev-simps) }
  then show symmetric G
    unfolding symmetric-def by (auto intro: symI)
qed

context lists-digraph-map begin

definition lists-fcs ≡ orbits-list (lists-fc-succ xss)

lemma M-simps:
  edge-succ M = lists-succ xss
  unfolding to-map-def by (cases G-list) auto

lemma lists-fc-succ-permutes: lists-fc-succ xss permutes (⋃(sset xss))
proof −
  have ∀(u,v) ∈ ⋃(sset xss). (v,u) ∈ ⋃(sset xss)
    using sym-arcs unfolding parcs-xss[symmetric] symmetric-def by (auto elim:
symE)
  then have swap-in (⋃(sset xss)) permutes ⋃(sset xss)
    using distincts-xss
    apply (auto simp: permutes-def split: if-splits)
    unfolding swap-in-def
    apply (simp-all split: if-splits prod.splits)
    apply metis+
    done
  moreover
  have lists-succ xss permutes (⋃(sset xss))
    using lists-succ-permutes[OF distincts-xss] by simp
  moreover
  have lists-fc-succ xss = lists-succ xss o swap-in (⋃(sset xss))
    by (simp add: fun-eq-iff lists-fc-succ-def)
  ultimately
  show ?thesis by (metis permutes-compose)
qed

lemma permutation-lists-fc-succ[intro, simp]: permutation (lists-fc-succ xss)
  using lists-fc-succ-permutes by (auto simp: permutation-permutes)

lemma face-cycle-succ-conv: face-cycle-succ = lists-fc-succ xss
  using parcs-xss unfolding face-cycle-succ-def
  by (simp add: fun-eq-iff to-map-def lists-fc-succ-def swap-in-def list-digraph-ext-def)

lemma sset-lists-fcs:
```

sset (*lists-fcs as*) = {*face-cycle-set a* | *a*. *a* ∈ *set as*}
by (*auto simp*: *lists-fcs-def sset-orbits-list face-cycle-set-def face-cycle-succ-conv*)

lemma *distincts-lists-fcs*: *distinct as* ⟹*distincts* (*lists-fcs as*)
by (*simp add*: *lists-fcs-def distincts-orbits-list*)

lemma *face-cycle-set-ss*: *a* ∈ *parcs G* ⟹ *face-cycle-set a* ⊆ *parcs G*
using *in-face-cycle-setD with-proj-simps*(*2*) **by** *blast*

lemma *face-cycle-succ-neq*:
assumes *a* ∈ *parcs G* **shows** *face-cycle-succ a* ≠ *a*
using *assms no-loops* **by** (*intro face-cycle-succ-neq*) *auto*

lemma *card-face-cycle-sets-conv*:
shows *card* (*pre-digraph-map.face-cycle-sets G M*) = *length* (*lists-fcs* (*remdups* (*snd G-list*)))
proof −
interpret *digraph-map G M* **by** (*rule digraph-map*)

have *face-cycle-sets* = {*face-cycle-set a* | *a*. *a* ∈ *parcs G*}
by (*auto simp*: *face-cycle-sets-def*)
also have . . . = *sset* (*lists-fcs* (*remdups* (*snd G-list*)))
unfolding *sset-lists-fcs* **by** (*simp add*: *list-digraph-simps*)
also have *card* . . . = *length* (*lists-fcs* (*remdups* (*snd G-list*)))
by (*simp add*: *card-image distincts-inj-on-set distinct-card distincts-distinct distincts-lists-fcs*)
finally show *?thesis* **.**
qed

end

definition *gen-succ* ≡ *λas xs*. [*b*. (*a*,*b*) <− *as*, *a* ∈ *set xs*]
interpretation *RTLI*: *set-access-gen set λx xs*. *x* ∈ *set xs* [] *λxs ys*. *remdups* (*xs* @ *ys*) *gen-succ*
by *standard* (*auto simp*: *gen-succ-def*)
hide-const (**open**) *gen-succ*

It would suffice to check that *set* (*RTLI.rtrancl-i A* [*u*]) = *set V*. We don't do this here, since it makes the proof more complicated (and is not necessary for the graphs we care about

definition *sccs-verts-impl* :: *'a list* × (*'a* × *'a*) *list* ⇒ *'a set set* **where**
sccs-verts-impl G ≡ *set* ‘ (*λx*. *RTLI.rtrancl-i* (*snd G*) [*x*]) ‘ *set* (*fst G*)

definition *isolated-verts-impl* :: *'a list* × (*'a* × *'a*) *list* ⇒ *'a list* **where**
isolated-verts-impl G = [*v* ← (*fst G*). ¬(∃ *e* ∈ *set* (*snd G*). *fst e* = *v*)]

definition *pair-graph-impl* :: *'a list* × (*'a* × *'a*) *list* ⇒ *bool* **where**
pair-graph-impl G ≡ *case G of* (*V*,*A*) ⇒ (∀ (*u*,*v*) ∈ *set A*. *u* ≠ *v* ∧ *u* ∈ *set V* ∧

```
v ∈ set V ∧ (v,u) ∈ set A)

definition genus-impl :: 'a list × ('a × 'a) list ⇒ ('a × 'a) list list ⇒ int where
  genus-impl G M ≡ case G of (V,A) ⇒
    (int (2∗card (sccs-verts-impl G)) − int (length (isolated-verts-impl G))
      − (int (length V) − int (length A) div 2
      + int (length (orbits-list-impl (lists-fc-succ M) A)))) div 2

definition comb-planar-impl :: 'a list × ('a × 'a) list ⇒ bool where
  comb-planar-impl G ≡ case G of (V,A) ⇒
    let i = int (2∗card (sccs-verts-impl G)) − int (length (isolated-verts-impl G))
      − int (length V) + int (length A) div 2
    in (∃ M∈set (all-maps-list G). (i − int (length (orbits-list-impl (lists-fc-succ M)
A))) div 2 = 0)



lemma sccs-verts-impl-correct:
  assumes pair-pseudo-graph (list-digraph G)
  shows pre-digraph.sccs-verts (list-digraph G) = sccs-verts-impl G
proof −
  interpret pair-pseudo-graph list-digraph G by fact
  { fix u assume u ∈ set (fst G)
    then have ⋀x. (u, x) ∈ (set (snd G))^* ⟹ x ∈ set (fst G)
      by (metis in-arcsD2 list-digraph-simps rtrancl.cases)
    then have set (RTLI.rtrancl-i (snd G) [u]) = {v. u →^*_{list-digraph G} v }
     unfolding RTLI.rtrancl-impl reachable-conv by (auto simp: list-digraph-simps
‹u ∈ -›)
    also have . . . = scc-of u
      unfolding scc-of-def by (auto intro: symmetric-reachable')
    finally have scc-of u = set (RTLI.rtrancl-i (snd G) [u]) by simp
  }
   then have pre-digraph.sccs-verts (list-digraph G) = set ‘ (λx. RTLI.rtrancl-i
(snd G) [x]) ‘ set (fst G)
    unfolding sccs-verts-conv-scc-of list-digraph-simps
    by (force intro: rev-image-eqI)
  then show ?thesis unfolding sccs-verts-impl-def by simp
qed

lemma isolated-verts-impl-correct:
  pre-digraph.isolated-verts (list-digraph G) = set (isolated-verts-impl G)
 by (auto simp: pre-digraph.isolated-verts-def isolated-verts-impl-def list-digraph-simps
out-arcs-def)

lemma pair-graph-impl-correct[code]:
  pair-graph (list-digraph G) = pair-graph-impl G (is ?L = ?R)
  unfolding pair-graph-def pair-digraph-def pair-fin-digraph-def pair-wf-digraph-def
   pair-fin-digraph-axioms-def pair-loopfree-digraph-def pair-loopfree-digraph-axioms-def
    pair-sym-digraph-def pair-sym-digraph-axioms-def pair-pseudo-graph-def
```

pair-graph-impl-def
by (*auto simp*: *pair-graph-impl-def list-digraph-simps symmetric-def intro*: *symI dest*: *symD split*: *prod.splits*)

lemma *genus-impl-correct*:
assumes *dist-V*: *distinct* (*fst G*) **and** *dist-A*: *distinct* (*snd G*)
assumes *lists-digraph-map G M*
shows *pre-digraph-map.euler-genus* (*list-digraph G*) (*to-map′* (*snd G*) *M*) = *genus-impl G M*
proof −
interpret *lists-digraph-map G M* **by** *fact*
obtain *V A* **where** *G-eq*: *G* = (*V*,*A*) **by** (*cases G*)
moreover
have *distinct* (*isolated-verts-impl G*)
using *dist-V* **by** (*auto simp*: *isolated-verts-impl-def*)
moreover
have *faces*: *card face-cycle-sets* = *length* (*orbits-list-impl* (*lists-fc-succ M*) (*snd G*))
using *dist-A*
by (*simp add*: *card-face-cycle-sets-conv lists-fcs-def orbits-list-conv-impl distinct-remdups-id*)
ultimately show *?thesis*
using *pair-pseudo-graph dist-V dist-A*
unfolding *euler-genus-def euler-char-def genus-impl-def card-sccs-verts*[*symmetric*]

by (*simp add*: *sccs-verts-impl-correct isolated-verts-impl-correct distinct-card list-digraph-simps zdiv-int*)
qed

lemma *elems-all-maps-list*:
assumes *M* ∈ *set* (*all-maps-list G*) *distinct* (*snd G*)
shows ⋃(*sset M*) = *set* (*snd G*)
using *assms*
by (*simp add*: *all-maps-list-def in-set-cyc-permutationss distincts-grouped-arcs union-grouped-out-arcs*[*symmetric*])
(*metis set-map*)

lemma *comb-planar-impl-altdef*: *comb-planar-impl G* = (∃ *M*∈*set* (*all-maps-list G*). *genus-impl G M* = *0*)
unfolding *comb-planar-impl-def Let-def genus-impl-def* **by** (*cases G*) (*simp add*: *algebra-simps*)

lemma *comb-planar-impl-correct*:
assumes *pair-graph* (*list-digraph G*)
assumes *dist-V*: *distinct* (*fst G*) **and** *dist-A*: *distinct* (*snd G*)
shows *comb-planar* (*list-digraph G*) = *comb-planar-impl G* (**is** *?L* = *?R*)
proof −
interpret *G*: *pair-graph list-digraph G* **by** *fact*
let *?G* = *list-digraph G*

have ∗: *all-maps* (*list-digraph G*) = *set* (*maps-all-maps-list G*)
  by (*rule set-maps-all-maps-list*) (*unfold-locales*, *simp add*: *dist-A*)

obtain *V A* where $G = (V,A)$ by (*cases G*)

{ fix *M* assume $M \in$ *set* (*all-maps-list G*)
  have *digraph-map* (*list-digraph G*) (*to-map′* (*snd G*) *M*)
    using ‹*M* ∈ -› *G.all-maps-correct* by (*auto simp*: ∗ *maps-all-maps-list-def*)
  then interpret *G*: *digraph-map list-digraph G to-map′* (*snd G*) *M* .

  have *distincts M* using ‹*M* ∈ -›
    using *dist-A distincts-in-all-maps-list* by *blast*

  have *lists-digraph-map G M*
    using *elems-all-maps-list*[*OF* ‹*M* ∈ -› ‹*distinct* (*snd G*)›]
    apply *unfold-locales*
  by (*auto intro*: ‹*distincts M*› *dest*: *G.adj-not-same*) (*auto simp*: *list-digraph-simps*)
} note *ldm* = *this*

have *comb-planar ?G* = (∃ *M* ∈ {*M*. *digraph-map ?G M*}. *pre-digraph-map.euler-genus ?G M* = *0*)
  unfolding *comb-planar-def* by *simp*
also have ... = (∃ *M*∈*set* (*all-maps-list G*). *pre-digraph-map.euler-genus* (*list-digraph G*)
    (*to-map* (*set* (*snd G*)) (*lists-succ M*)) = *0*)
  unfolding *comb-planar-def comb-planar-impl-def Let-def G.all-maps-correct*[*symmetric*]
    set-maps-all-maps-list[*OF G.pair-wf-digraph dist-A*] *maps-all-maps-list-def* by *simp*
also have ... = (∃ *M*∈*set* (*all-maps-list G*). *genus-impl G M* = *0*)
  using *ldm assms* by (*simp add*: *genus-impl-correct*)
also have ... = *comb-planar-impl G*
  unfolding *comb-planar-impl-def genus-impl-def Let-def* by (*simp add*: ‹$G = (V,A)$› *algebra-simps*)
finally show *?thesis* .
qed

end
theory *Planar-Complete*
imports
  Digraph-Map-Impl
begin

8 Kuratowski Graphs are not Combinatorially Planar

8.1 A concrete K5 graph

definition *c-K5-list* ≡ ([*0..4*], [(*x*,*y*). *x* <− [*0..4*], *y* <− [*0..4*], $x \neq y$])

abbreviation *c-K5* :: *int pair-pre-digraph* **where**
c-K5 ≡ *list-digraph c-K5-list*

lemma *c-K5-not-comb-planar*: ¬*comb-planar c-K5*
by (*subst comb-planar-impl-correct*) *eval*+

lemma *pverts-c-K5*: *pverts c-K5* = {*0..4*}
by (*simp add*: *c-K5-list-def list-digraph-ext-def*)

lemma *parcs-c-K5*: *parcs c-K5* = {(*u*,*v*). *u* ∈ {*0..4*} ∧ *v* ∈ {*0..4*} ∧ *u* ≠ *v*}
by (*auto simp*: *c-K5-list-def list-digraph-ext-def*)

lemmas *c-K5-simps* = *pverts-c-K5 parcs-c-K5*

lemma *complete-c-K5*: K_5 *c-K5*
proof −
interpret *K5*: *pair-graph c-K5* **by** *eval*
show *?thesis* **unfolding** *complete-digraph-def* **by** (*auto simp*: *c-K5-simps*)
qed

8.2 A concrete K33 graph

definition *c-K33-list* ≡ ([*0..5*], [(*x*,*y*). *x* <− [*0..5*], *y* <− [*0..5*], *even x* ⟷ *odd y*])

abbreviation *c-K33* :: *int pair-pre-digraph* **where**
c-K33 ≡ *list-digraph c-K33-list*

lemma *c-K33-not-comb-planar*: ¬*comb-planar c-K33*
by (*subst comb-planar-impl-correct*) *eval*+

lemma *complete-c-K33*: $K_{3,3}$ *c-K33*
proof −
interpret *K33*: *pair-graph c-K33* **by** *eval*
show *?thesis*
unfolding *complete-bipartite-digraph-def*
apply (*intro conjI*)
apply *unfold-locales*
apply (*rule exI*[*of* - {*0*,*2*,*4*}])
apply (*rule exI*[*of* - {*1*,*3*,*5*}])
unfolding *c-K33-list-def list-digraph-simps with-proj-simps*
apply *eval*
done
qed

8.3 Generalization to arbitrary Kuratowski Graphs

8.3.1 Number of Face Cycles is a Graph Invariant

lemma (**in** *digraph-map*) *wrap-wrap-iso*:

assumes *hom*: *digraph-isomorphism hom*
assumes *f*: $f \in$ *arcs G* $\to$ *arcs G* **and** *g*: $g \in$ *arcs G* $\to$ *arcs G*
shows *wrap-iso-arcs hom f* (*wrap-iso-arcs hom g x*) = *wrap-iso-arcs hom* (*f o g*) *x*
proof −
have $\bigwedge x.$ $x \in$ *arcs G* $\Longrightarrow$ $g\ x \in$ *arcs G* **using** *g* **by** *auto*
with *hom f* **show** *?thesis*
by (*cases* $x \in$ *iso-arcs hom* ' *arcs G*) (*auto simp*: *wrap-iso-arcs-def perm-restrict-simps*)
qed

lemma (**in** *digraph-map*) *face-cycle-succ-iso*:
assumes *hom*: *digraph-isomorphism hom* $x \in$ *iso-arcs hom* ' *arcs G*
shows *pre-digraph-map.face-cycle-succ* (*map-iso hom*) *x* = *wrap-iso-arcs hom face-cycle-succ x*
using *assms* **by** (*simp add*: *pre-digraph-map.face-cycle-succ-def map-iso-def wrap-wrap-iso*)

lemma (**in** *digraph-map*) *face-cycle-set-iso*:
assumes *hom*: *digraph-isomorphism hom* $x \in$ *iso-arcs hom* ' *arcs G*
shows *pre-digraph-map.face-cycle-set* (*map-iso hom*) *x* = *iso-arcs hom* ' *face-cycle-set* (*iso-arcs* (*inv-iso hom*) *x*)
proof −
have ∗: $\bigwedge x\ y.$ $x \in$ *orbit face-cycle-succ y* $\Longrightarrow$ $y \in$ *arcs G* $\Longrightarrow$ $x \in$ *arcs G*
$\bigwedge x.$ $x \in$ *arcs G* $\Longrightarrow$ $x \in$ *orbit face-cycle-succ x*
using *face-cycle-set-def* **by** (*auto simp*: *in-face-cycle-setD*)
show *?thesis*
using *assms* **unfolding** *pre-digraph-map.face-cycle-set-def*
by (*subst orbit-inverse* [**where** *g'*=*pre-digraph-map.face-cycle-succ* (*map-iso hom*)])
(*auto simp*: ∗ *face-cycle-succ-iso*)
qed

lemma (**in** *digraph-map*) *face-cycle-sets-iso*:
assumes *hom*: *digraph-isomorphism hom*
shows *pre-digraph-map.face-cycle-sets* (*app-iso hom G*) (*map-iso hom*) = ($\lambda x.$ *iso-arcs hom* ' *x*) ' *face-cycle-sets*
using *assms* **by** (*auto simp*: *pre-digraph-map.face-cycle-sets-def face-cycle-set-iso*) (*auto simp*: *face-cycle-set-iso intro*: *rev-image-eqI*)

lemma (**in** *digraph-map*) *card-face-cycle-sets-iso*:
assumes *hom*: *digraph-isomorphism hom*
shows *card* (*pre-digraph-map.face-cycle-sets* (*app-iso hom G*) (*map-iso hom*)) = *card face-cycle-sets*
proof −
have *inj-on* ((') (*iso-arcs hom*)) *face-cycle-sets*
by (*rule inj-on-f-imageI digraph-isomorphism-inj-on-arcs hom in-face-cycle-setsD*)+
then show *?thesis* **using** *hom* **by** (*simp add*: *face-cycle-sets-iso card-image*)
qed

8.3.2 Combinatorial planarity is a Graph Invariant

lemma (**in** *digraph-map*) *euler-char-iso*:
assumes *digraph-isomorphism hom*
shows *pre-digraph-map.euler-char* (*app-iso hom G*) (*map-iso hom*) = *euler-char*
using *assms* **by** (*auto simp*: *pre-digraph-map.euler-char-def card-face-cycle-sets-iso*)

lemma (**in** *digraph-map*) *euler-genus-iso*:
assumes *digraph-isomorphism hom*
shows *pre-digraph-map.euler-genus* (*app-iso hom G*) (*map-iso hom*) = *euler-genus*
using *assms* **by** (*auto simp*: *pre-digraph-map.euler-genus-def euler-char-iso*)

lemma (**in** *wf-digraph*) *comb-planar-iso*:
assumes *digraph-isomorphism hom*
shows *comb-planar* (*app-iso hom G*) ⟷ *comb-planar G*
proof
assume *comb-planar G*
then obtain *M* **where** *digraph-map G M pre-digraph-map.euler-genus G M* = *0*
by (*auto simp*: *comb-planar-def*)
then have *digraph-map* (*app-iso hom G*) (*pre-digraph-map.map-iso G M hom*)
pre-digraph-map.euler-genus (*app-iso hom G*) (*pre-digraph-map.map-iso G M hom*) = *0*
using *assms* **by** (*auto intro*: *digraph-map.digraph-map-isoI simp*: *digraph-map.euler-genus-iso*)
then show *comb-planar* (*app-iso hom G*)
by (*metis comb-planar-def*)
next
let *?G* = *app-iso hom G*
assume *comb-planar ?G*
then obtain *M* **where** *digraph-map ?G M*
pre-digraph-map.euler-genus ?G M = *0*
by (*auto simp*: *comb-planar-def*)
moreover
have *pre-digraph.digraph-isomorphism ?G* (*inv-iso hom*)
using *assms* **by** (*rule digraph-isomorphism-invI*)
ultimately
have *digraph-map* (*app-iso* (*inv-iso hom*) *?G*) (*pre-digraph-map.map-iso ?G M* (*inv-iso hom*))
pre-digraph-map.euler-genus (*app-iso* (*inv-iso hom*) *?G*) (*pre-digraph-map.map-iso ?G M* (*inv-iso hom*)) = *0*
using *assms* **by** (*auto intro*: *digraph-map.digraph-map-isoI simp only*: *digraph-map.euler-genus-iso*)
then show *comb-planar G*
using *assms* **by** (*auto simp*: *comb-planar-def*)
qed

8.3.3 Completeness is a Graph Invariant

lemma (**in** *loopfree-digraph*) *loopfree-digraphI-app-iso*:
assumes *digraph-isomorphism hom*

shows *loopfree-digraph* (*app-iso hom G*)
proof −
interpret *iG*: *wf-digraph app-iso hom G* **using** *assms* **by** (*rule wf-digraphI-app-iso*)
show *?thesis*
using *assms digraph-isomorphism-inj-on-verts*[*OF assms*]
by *unfold-locales* (*auto simp*: *iso-verts-tail iso-verts-head dest*: *inj-onD no-loops*)
qed

lemma (**in** *nomulti-digraph*) *nomulti-digraphI-app-iso*:
assumes *digraph-isomorphism hom*
shows *nomulti-digraph* (*app-iso hom G*)
proof −
interpret *iG*: *wf-digraph app-iso hom G* **using** *assms* **by** (*rule wf-digraphI-app-iso*)
show *?thesis*
using *assms*
by *unfold-locales* (*auto simp*: *iso-verts-tail iso-verts-head arc-to-ends-def no-multi-arcs dest*: *inj-onD*)
qed

lemma (**in** *pre-digraph*) *symmetricI-app-iso*:
assumes *digraph-isomorphism hom*
assumes *symmetric G*
shows *symmetric* (*app-iso hom G*)
proof −
let *?G* = *app-iso hom G*
have *sym* (*arcs-ends ?G*)
proof (*rule symI*)
fix *u v* **assume** $u \to_{?G} v$
then obtain *a* **where** *a*: *a* ∈ *arcs ?G tail ?G a* = *u head ?G a* = *v* **by** *auto*
then obtain *a0* **where** *a0*: *a0* ∈ *arcs G a* = *iso-arcs hom a0* **by** *auto*
with ‹*symmetric G*› **obtain** *b0* **where** *b0* ∈ *arcs G tail G b0* = *head G a0 head G b0* = *tail G a0*
by (*auto simp*: *symmetric-def arcs-ends-conv elim*: *symE*)
moreover
define *b* **where** *b* = *iso-arcs hom b0*
ultimately
have *b* ∈ *iso-arcs hom* ‘ *arcs G tail ?G b* = *v head ?G b* = *u*
using *a a0 assms* **by** (*auto simp*: *iso-verts-tail iso-verts-head*)
then show $v \to_{?G} u$ **by** (*auto simp*: *arcs-ends-conv*)
qed
then show *?thesis* **by** (*simp add*: *symmetric-def*)
qed

lemma (**in** *sym-digraph*) *sym-digraphI-app-iso*:
assumes *digraph-isomorphism hom*
shows *sym-digraph* (*app-iso hom G*)
proof −
interpret *iG*: *wf-digraph app-iso hom G* **using** *assms* **by** (*rule wf-digraphI-app-iso*)
show *?thesis* **using** *assms* **by** *unfold-locales* (*intro symmetricI-app-iso sym-arcs*)

qed

lemma (**in** *graph*) *graphI-app-iso*:
  assumes *digraph-isomorphism hom*
  shows *graph* (*app-iso hom G*)
proof –
  interpret *iG*: *fin-digraph app-iso hom G*
    using *assms* **by** (*rule fin-digraphI-app-iso*)
  interpret *iG*: *loopfree-digraph app-iso hom G*
    using *assms* **by** (*rule loopfree-digraphI-app-iso*)
  interpret *iG*: *nomulti-digraph app-iso hom G*
    using *assms* **by** (*rule nomulti-digraphI-app-iso*)
  interpret *iG*: *sym-digraph app-iso hom G*
    using *assms* **by** (*rule sym-digraphI-app-iso*)
  show *?thesis* **by** *intro-locales*
qed

lemma (**in** *wf-digraph*) *graph-app-iso-eq*:
  assumes *digraph-isomorphism hom*
  shows *graph* (*app-iso hom G*) $\longleftrightarrow$ *graph G*
 using *assms* **by** (*metis app-iso-inv digraph-isomorphism-invI graph.graphI-app-iso*)

lemma (**in** *pre-digraph*) *arcs-ends-iso*:
  assumes *digraph-isomorphism hom*
  shows *arcs-ends* (*app-iso hom G*) = (λ(*u*,*v*). (*iso-verts hom u*, *iso-verts hom v*)) ‘ *arcs-ends G*
  using *assms*
  by (*auto simp*: *arcs-ends-conv image-image iso-verts-tail iso-verts-head cong*: *image-cong*)

lemma *inj-onI-pair*:
  assumes *inj-on f S* $T \subseteq S \times S$
  shows *inj-on* (λ(*u*,*v*). (*f u*, *f v*)) *T*
  using *assms* **by** (*intro inj-onI*) (*auto dest*: *inj-onD*)

lemma (**in** *wf-digraph*) *complete-digraph-iso*:
  assumes *digraph-isomorphism hom*
  shows K_n (*app-iso hom G*) $\longleftrightarrow$ K_n *G* (**is** *?L* $\longleftrightarrow$ *?R*)
proof
  assume *?L*
  then interpret *iG*: *graph app-iso hom G* **by** (*simp add*: *complete-digraph-def*)
  { have {(*u*, *v*). *u* $\in$ *iso-verts hom* ‘ *verts G* $\wedge$ *v* $\in$ *iso-verts hom* ‘ *verts G* $\wedge$ *u* $\neq$ *v*}
        = (λ(*u*,*v*). (*iso-verts hom u*, *iso-verts hom v*)) ‘ {(*u*,*v*). *u* $\in$ *verts G* $\wedge$ *v* $\in$ *verts G* $\wedge$ *iso-verts hom u* $\neq$ *iso-verts hom v*} (**is** *?L* = -)
      by *auto*
    also have ... = (λ(*u*,*v*). (*iso-verts hom u*, *iso-verts hom v*)) ‘ {(*u*,*v*). *u* $\in$ *verts G* $\wedge$ *v* $\in$ *verts G* $\wedge$ *u* $\neq$ *v*}

using *digraph-isomorphism-inj-on-verts*[*OF assms*] **by** (*auto dest*: *inj-onD*)
finally have *?L* = ...
} **note** *X* = *this*

{ **fix** *A* **assume** *A*: $A \subseteq$ *verts G* $\times$ *verts G*
then have *inj-on* (λ(*u*, *v*). (*iso-verts hom u*, *iso-verts hom v*)) *A*
using *A digraph-isomorphism-inj-on-verts*[*OF assms*] **by** (*intro inj-onI-pair*)
} **note** *Y* = *this*
have (*arcs-ends G* $\cup$ {(*u*, *v*). *u* $\in$ *verts G* $\wedge$ *v* $\in$ *verts G* $\wedge$ *u* $\neq$ *v*}) $\subseteq$ *verts G* $\times$ *verts G*
by *auto*
note *Y'* = *Y*[*OF this*]

show *?R* **using** *assms* ‹*?L*›
by (*simp add*: *complete-digraph-def X arcs-ends-iso graph-app-iso-eq inj-on-Un-image-eq-iff Y'*)
next
assume *?R* **then show** *?L* **using** *assms*
by (*fastforce simp add*: *complete-digraph-def arcs-ends-iso graph-app-iso-eq*)
qed

8.3.4 Conclusion

definition (**in** *pre-digraph*)
mk-iso :: ($'a \Rightarrow 'c$) $\Rightarrow$ ($'b \Rightarrow 'd$) $\Rightarrow$ ($'a$, $'b$, $'c$, $'d$) *digraph-isomorphism*
where
mk-iso fv fa $\equiv$ (| *iso-verts* = *fv*, *iso-arcs* = *fa*,
iso-head = *fv o head G o the-inv-into* (*arcs G*) *fa*,
iso-tail = *fv o tail G o the-inv-into* (*arcs G*) *fa* |)

lemma (**in** *pre-digraph*) *mk-iso-simps*[*simp*]:
iso-verts (*mk-iso fv fa*) = *fv*
iso-arcs (*mk-iso fv fa*) = *fa*
by (*auto simp*: *mk-iso-def*)

lemma (**in** *wf-digraph*) *digraph-isomorphism-mk-iso*:
assumes *inj-on fv* (*verts G*) *inj-on fa* (*arcs G*)
shows *digraph-isomorphism* (*mk-iso fv fa*)
using *assms* **by** (*auto simp*: *digraph-isomorphism-def mk-iso-def the-inv-into-f-f wf-digraph*)

definition *pairself f* $\equiv$ λx. *case x of* (*u*,*v*) $\Rightarrow$ (*f u*, *f v*)

lemma *inj-on-pairself*:
assumes *inj-on f S* **and** $T \subseteq S \times S$
shows *inj-on* (*pairself f*) *T*
using *assms* **unfolding** *pairself-def* **by** (*rule inj-onI-pair*)

definition

mk-iso-nomulti :: ('*a*,'*b*) *pre-digraph* ⇒ ('*c*,'*d*) *pre-digraph* ⇒ ('*a* ⇒ '*c*) ⇒ ('*a*, '*b*, '*c*, '*d*) *digraph-isomorphism*
where
mk-iso-nomulti G H fv ≡ (|
iso-verts = *fv*,
iso-arcs = *the-inv-into* (*arcs H*) (*arc-to-ends H*) *o pairself fv o arc-to-ends G*,
iso-head = *head H*,
iso-tail = *tail H*
|)

lemma (**in** *pre-digraph*) *mk-iso-simps-nomulti*[*simp*]:
iso-verts (*mk-iso-nomulti G H fv*) = *fv*
iso-head (*mk-iso-nomulti G H fv*) = *head H*
iso-tail (*mk-iso-nomulti G H fv*) = *tail H*
by (*auto simp*: *mk-iso-nomulti-def*)

lemma (**in** *nomulti-digraph*)
assumes *nomulti-digraph H*
assumes *fv*: *inj-on fv* (*verts G*) *verts H* = *fv* ' *verts G* **and** *arcs-ends*: *arcs-ends H* = *pairself fv* ' *arcs-ends G*
shows *digraph-isomorphism-mk-iso-nomulti*: *digraph-isomorphism* (*mk-iso-nomulti G H fv*) (**is** *?t-multi*)
and *ap-iso-mk-iso-nomulti-eq*: *app-iso* (*mk-iso-nomulti G H fv*) *G* = *H* (**is** *?t-app*)
and *digraph-iso-mk-iso-nomulti*: *digraph-iso G H* (**is** *?t-iso*)
using *assms*
proof −
interpret *H*: *nomulti-digraph H* **by** *fact*
let *?fa* = *iso-arcs* (*mk-iso-nomulti G H fv*)

have *fa*: *bij-betw ?fa* (*arcs G*) (*arcs H*)
proof −
have *bij-betw* (*arc-to-ends G*) (*arcs G*) (*arcs-ends G*)
by (*auto simp*: *bij-betw-def inj-on-arc-to-ends arcs-ends-def*)
also have *bij-betw* (*pairself fv*) (*arcs-ends G*) (*arcs-ends H*)
using *arcs-ends* **by** (*auto simp*: *bij-betw-def arcs-ends-def arc-to-ends-def intro*: *fv inj-on-pairself*)
also (*bij-betw-trans*) **have** *bij-betw* (*the-inv-into* (*arcs H*) (*arc-to-ends H*)) (*arcs-ends H*) (*arcs H*)
by (*auto simp*: *bij-betw-def the-inv-into-into H.inj-on-arc-to-ends arcs-ends-def inj-on-the-inv-into*)
finally (*bij-betw-trans*) **show** *?thesis*
by (*simp add*: *mk-iso-nomulti-def o-assoc*)
qed
moreover
{ fix *a* **assume** *a* ∈ *arcs G*
then have *pairself fv* (*arc-to-ends G a*) ∈ *arcs-ends H*
using *arcs-ends* **by** (*auto simp*: *arcs-ends-def*)
then obtain *b* **where** (*pairself fv* (*arc-to-ends G a*)) = *arc-to-ends H b b* ∈

```
arcs H
      by (auto simp: arcs-ends-def)
    then have fv (tail G a) = tail H (?fa a) fv (head G a) = head H (?fa a)
      by (auto simp: mk-iso-nomulti-def the-inv-into-f-f H.inj-on-arc-to-ends)
        (auto simp: pairself-def arc-to-ends-def)
  }
  ultimately
 show ?t-multi ?t-app using fv by (auto simp: digraph-isomorphism-def bij-betw-def
wf-digraph)
  then show ?t-iso by (auto simp: digraph-iso-def)
qed

lemma complete-digraph-are-iso:
  assumes K_n G K_n H shows digraph-iso G H
proof −
  interpret G: graph G using assms by (simp add: complete-digraph-def)
  interpret H: graph H using assms by (simp add: complete-digraph-def)

  from assms have card (verts G) = n card (verts H) = n
    by (auto simp: complete-digraph-def)
   with G.finite-verts H.finite-verts obtain fv where bij-betw fv (verts G) (verts
H)
    by (metis finite-same-card-bij)
   then have fv: inj-on fv (verts G) verts H = fv ‘ verts G by (auto simp:
bij-betw-def)

  have arcs-ends H = {(u,v). u ∈ verts H ∧ v ∈ verts H ∧ u ≠ v}
    using ‹K_n H› by (auto simp: complete-digraph-def)
  also have ... = pairself fv ‘ {(u,v). u ∈ verts G ∧ v ∈ verts G ∧ u ≠ v} (is ?L
= ?R)
  proof (intro set-eqI iffI)
    fix x assume x ∈ ?L
    then have fst x ∈ fv ‘ verts G snd x ∈ fv ‘ verts G fst x ≠ snd x
      using fv by auto
    then obtain u v where fst x = fv u snd x = fv v u ∈ verts G v ∈ verts G by
auto
    then have (fst x, snd x) ∈ ?R using ‹x ∈ ?L› by (auto simp: pairself-def)
    then show x ∈ ?R by auto
  next
    fix x assume x ∈ ?R then show x ∈ ?L
    using fv by (auto simp: pairself-def dest: inj-onD)
  qed
  also have ... = pairself fv ‘ arcs-ends G
    using ‹K_n G› by (auto simp: complete-digraph-def)
  finally have arcs-ends: arcs-ends H = pairself fv ‘ arcs-ends G .

 show ?thesis using H.nomulti-digraph fv arcs-ends by (rule G.digraph-iso-mk-iso-nomulti)
qed
```

lemma *pairself-image-prod*:
pairself f ‘ $(A \times B) = f$ ‘ $A \times f$ ‘ B
by (*auto simp*: *pairself-def*)

lemma *complete-bipartite-digraph-are-iso*:
assumes $K_{m,n}$ G $K_{m,n}$ H **shows** *digraph-iso* G H
proof −
interpret G: *graph* G **using** *assms* **by** (*simp add*: *complete-bipartite-digraph-def*)
interpret H: *graph* H **using** *assms* **by** (*simp add*: *complete-bipartite-digraph-def*)

from *assms* **obtain** GU GV **where** *G-parts*: *verts* $G = GU \cup GV$ $GU \cap GV = \{\}$
card $GU = m$ *card* $GV = n$ *arcs-ends* $G = GU \times GV \cup GV \times GU$
by (*auto simp*: *complete-bipartite-digraph-def*)
from *assms* **obtain** HU HV **where** *H-parts*: *verts* $H = HU \cup HV$ $HU \cap HV = \{\}$
card $HU = m$ *card* $HV = n$ *arcs-ends* $H = HU \times HV \cup HV \times HU$
by (*auto simp*: *complete-bipartite-digraph-def*)

have *fin*: *finite* GU *finite* GV *finite* HU *finite* HV
using *G-parts* *H-parts* *G.finite-verts* *H.finite-verts* **by** *simp-all*

obtain *fv-U* **where** *fv-U*: *bij-betw fv-U GU HU*
using ‹*card* GU = -› ‹*card* HU = -› ‹*finite* GU› ‹*finite* HU› **by** (*metis finite-same-card-bij*)
obtain *fv-V* **where** *fv-V*: *bij-betw fv-V GV HV*
using ‹*card* GV = -› ‹*card* HV = -› ‹*finite* GV› ‹*finite* HV› **by** (*metis finite-same-card-bij*)

define *fv* **where** *fv* x = (*if* $x \in GU$ *then* *fv-U* x *else* *fv-V* x) **for** x
have $\bigwedge x.\ x \in GV \Longrightarrow x \notin GU$ **using** ‹$GU \cap GV = \{\}$› **by** *blast*
then have *bij-fv-UV*: *bij-betw fv GU HU* *bij-betw fv GV HV*
using *fv-U* *fv-V* **by** (*auto simp*: *fv-def* *cong*: *bij-betw-cong*)
then have *bij-betw fv* (*verts* G) (*verts* H)
unfolding ‹*verts* G = -› ‹*verts* H = -› **using** ‹$HU \cap$ - $= \{\}$› **by** (*rule bij-betw-combine*)
then have *fv*: *inj-on fv* (*verts* G) *verts* H = *fv* ‘ *verts* G **by** (*auto simp*: *bij-betw-def*)

have *arcs-ends* $H = HU \times HV \cup HV \times HU$
using ‹$K_{m,n}$ H› *H-parts* **by** (*auto simp*: *complete-digraph-def*)
also have … = *pairself fv* ‘ $(GU \times GV \cup GV \times GU)$ (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
fix x **assume** $x \in$ *?L*
then have (*fst* $x \in$ *fv* ‘ $GU \land$ *snd* $x \in$ *fv* ‘ GV) $\lor$ (*fst* $x \in$ *fv* ‘ $GV \land$ *snd* $x \in$ *fv* ‘ GU)
using *bij-fv-UV* **by** (*auto simp*: *bij-betw-def*)
then show $x \in$ *?R*
by (*cases* x) (*auto simp*: *pairself-image-prod image-Un*)

next
fix *x* **assume** $x \in$ *?R* **then show** $x \in$ *?L*
using *bij-fv-UV* **by** (*auto simp*: *pairself-image-prod image-Un bij-betw-def*)
qed
also have ... = *pairself fv* ‘ *arcs-ends G*
using ‹$K_{m,n}$ *G*› *G-parts* **by** (*auto simp*: *complete-bipartite-digraph-def*)
finally have *arcs-ends*: *arcs-ends H* = *pairself fv* ‘ *arcs-ends G* **.**

show *?thesis* **using** *H.nomulti-digraph fv arcs-ends* **by** (*rule G.digraph-iso-mk-iso-nomulti*)
qed

lemma *K5-not-comb-planar*:
assumes K_5 *G* **shows** ¬*comb-planar G*
proof −
interpret *graph G* **using** *assms* **by** (*auto simp*: *complete-digraph-def*)
have *digraph-iso G c-K5*
using *assms complete-c-K5* **by** (*rule complete-digraph-are-iso*)
then obtain *hom* **where** *hom*: *digraph-isomorphism hom app-iso hom G* = *c-K5*
by (*auto simp*: *digraph-iso-def*)
then show *?thesis* **using** *c-K5-not-comb-planar comb-planar-iso* **by** *fastforce*
qed

lemma *K33-not-comb-planar*:
assumes $K_{3,3}$ *G* **shows** ¬*comb-planar G*
proof −
interpret *graph G* **using** *assms* **by** (*auto simp*: *complete-bipartite-digraph-def*)
have *digraph-iso G c-K33*
using *assms complete-c-K33* **by** (*rule complete-bipartite-digraph-are-iso*)
then obtain *hom* **where** *hom*: *digraph-isomorphism hom app-iso hom G* = *c-K33*
by (*auto simp*: *digraph-iso-def*)
then show *?thesis* **using** *c-K33-not-comb-planar comb-planar-iso* **by** *fastforce*
qed

end

9 n-step reachability

theory *Reachablen*
imports
Graph-Theory.Graph-Theory
begin

inductive
ntrancl-onp :: *'a set* ⇒ *'a rel* ⇒ *nat* ⇒ *'a* ⇒ *'a* ⇒ *bool*
for *F* :: *'a set* **and** *r* :: *'a rel*
where
ntrancl-on-0: *a* = *b* ⟹ *a* ∈ *F* ⟹ *ntrancl-onp F r 0 a b*
| *ntrancl-on-Suc*: (*a*,*b*) ∈ *r* ⟹ *ntrancl-onp F r n b c* ⟹ *a* ∈ *F* ⟹ *ntrancl-onp F r* (*Suc n*) *a c*

lemma *ntrancl-onpD-rtrancl-on*:
  assumes *ntrancl-onp F r n a b* **shows** $(a,b) \in$ *rtrancl-on F r*
  using *assms* **by** *induct* (*auto intro*: *converse-rtrancl-on-into-rtrancl-on*)

lemma *rtrancl-onE-ntrancl-onp*:
  assumes $(a,b) \in$ *rtrancl-on F r* **obtains** *n* **where** *ntrancl-onp F r n a b*
proof *atomize-elim*
  from *assms* **show** $\exists n.$ *ntrancl-onp F r n a b*
  proof *induct*
    case *base*
    then have *ntrancl-onp F r 0 b b* **by** (*auto intro*: *ntrancl-onp.intros*)
    then show *?case* **..**
  next
    case (*step a c*)
    from ‹$\exists n.$ -› **obtain** *n* **where** *ntrancl-onp F r n c b* **..**
    with ‹$(a,c) \in r$› **have** *ntrancl-onp F r* (*Suc n*) *a b* **using** ‹$a \in F$› **by** (*rule ntrancl-onp.intros*)
    then show *?case* **..**
  qed
qed

lemma *rtrancl-on-conv-ntrancl-onp*: $(a,b) \in$ *rtrancl-on F r* $\longleftrightarrow$ ($\exists n.$ *ntrancl-onp F r n a b*)
  by (*metis ntrancl-onpD-rtrancl-on rtrancl-onE-ntrancl-onp*)

definition *nreachable* :: $('a,'b)$ *pre-digraph* $\Rightarrow$ $'a \Rightarrow$ *nat* $\Rightarrow$ $'a \Rightarrow$ *bool* (‹- $\to$¯1 -› [*100*,*100*] *40*) **where**
  nreachable G u n v $\equiv$ *ntrancl-onp* (*verts G*) (*arcs-ends G*) *n u v*

context *wf-digraph* **begin**

lemma *reachableE-nreachable*:
  assumes $u \to^* v$ **obtains** *n* **where** $u \to^n v$
  using *assms* **by** (*auto simp*: *reachable-def nreachable-def elim*: *rtrancl-onE-ntrancl-onp*)

lemma *converse-nreachable-cases*[*cases pred*: *nreachable*]:
  assumes $u \to^n v$
  obtains (*ntrancl-on-0*) $u = v$ $n = 0$ $u \in$ *verts G*
    | (*ntrancl-on-Suc*) *w m* **where** $u \to w$ $n = Suc\ m$ $w \to^m v$
  using *assms* **unfolding** *nreachable-def* **by** *cases auto*

lemma *converse-nreachable-induct*[*consumes 1*, *case-names base step*, *induct pred*: *reachable*]:
  assumes *major*: $u \to^n{}_G v$
    and *cases*: $v \in$ *verts G* $\Longrightarrow$ *P 0 v*
      $\bigwedge n\ x\ y.\ [\![x \to_G y;\ y \to^n{}_G v;\ P\ n\ y]\!] \Longrightarrow P\ (Suc\ n)\ x$
  shows *P n u*

using *assms* **unfolding** *nreachable-def* **by** *induct auto*

lemma *converse-nreachable-induct-less*[*consumes 1*, *case-names base step*, *induct pred*: *reachable*]:
assumes *major*: $u \to^n{}_G\ v$
and *cases*: $v \in verts\ G \Longrightarrow P\ 0\ v$
$\bigwedge n\ x\ y.\ [\![x \to_G y;\ y \to^n{}_G\ v;\ \bigwedge z\ m.\ m \leq n \Longrightarrow (z \to^m{}_G\ v) \Longrightarrow P\ m\ z]\!] \Longrightarrow P\ (Suc\ n)\ x$
shows $P\ n\ u$
proof −
have $\bigwedge q\ u.\ q \leq n \Longrightarrow (u \to^q{}_G\ v) \Longrightarrow P\ q\ u$
proof (*induction n arbitrary*: *u rule*: *less-induct*)
case (*less n*)
show *?case*
proof (*cases q*)
case *0* **with** *less* **show** *?thesis* **by** (*auto intro*: *cases elim*: *converse-nreachable-cases*)
next
case (*Suc q′*)
with ‹$u \to^q v$› **obtain** *w* **where** $u \to w\ w \to^{q'} v$ **by** (*auto elim*: *converse-nreachable-cases*)
then show *?thesis*
unfolding ‹q = -› **using** *Suc less* **by** (*auto intro*!: *less.IH cases*)
qed
qed
with *major* **show** *?thesis* **by** *auto*
qed

end

end
theory *Permutations-2*
imports
HOL−Combinatorics.Permutations
Graph-Theory.Auxiliary
Executable-Permutations
begin

10 More

abbreviation *funswapid* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\rightleftharpoons_F$› *90*) **where**
$x \rightleftharpoons_F y \equiv transpose\ x\ y$

lemma *in-funswapid-image-iff*: $x \in (a \rightleftharpoons_F b)$ ‘ $S \longleftrightarrow (a \rightleftharpoons_F b)\ x \in S$
by (*fact in-transpose-image-iff*)

lemma *bij-swap-compose*: $bij\ (x \rightleftharpoons_F y \circ f) \longleftrightarrow bij\ f$
by (*metis UNIV-I bij-betw-comp-iff2 bij-betw-id bij-swap-iff subsetI*)

lemma *bij-eq-iff*:

assumes *bij f* **shows** $f\ x = f\ y \longleftrightarrow x = y$
using *assms* **by** (*auto simp add*: *bij-iff*)

lemma *swap-swap-id*[*simp*]: $(x \rightleftharpoons_F y)\ ((x \rightleftharpoons_F y)\ z) = z$
by (*fact transpose-involutory*)

11 Modifying Permutations

definition *perm-swap* :: $'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a)$ **where**
perm-swap x y f $\equiv x \rightleftharpoons_F y\ o\ f\ o\ x \rightleftharpoons_F y$

definition *perm-rem* :: $'a \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a)$ **where**
perm-rem x f $\equiv$ *if* $f\ x \neq x$ *then* $x \rightleftharpoons_F f\ x\ o\ f$ *else f*

An example:

perm-rem 2 (*list-succ* $[1, 2, 3, 4]$) x = *list-succ* $[1, 3, 4]$ x

lemma *perm-swap-id*[*simp*]: *perm-swap a b id* = *id*
by (*auto simp*: *perm-swap-def*)

lemma *perm-rem-permutes*:
assumes *f permutes* $S \cup \{x\}$
shows *perm-rem x f permutes S*
using *assms* **by** (*auto simp*: *permutes-def perm-rem-def*) (*metis transpose-def*)+

lemma *perm-rem-same*:
assumes *bij f* $f\ y = y$ **shows** *perm-rem x f y* = $f\ y$
using *assms* **by** (*auto simp*: *perm-rem-def bij-iff transpose-def*)

lemma *perm-rem-simps*:
assumes *bij f*
shows
$x = y \Longrightarrow$ *perm-rem x f y* = x
$f\ y = x \Longrightarrow$ *perm-rem x f y* = $f\ x$
$y \neq x \Longrightarrow f\ y \neq x \Longrightarrow$ *perm-rem x f y* = $f\ y$
using *assms* **by** (*auto simp*: *perm-rem-def transpose-def bij-iff*)

lemma *bij-perm-rem*[*simp*]: *bij* (*perm-rem x f*) $\longleftrightarrow$ *bij f*
by (*simp add*: *perm-rem-def bij-swap-compose*)

lemma *perm-rem-conv*: $\bigwedge f\ x\ y.$ *bij f* $\Longrightarrow$ *perm-rem x f y* = (
if $x = y$ *then* x
else if $f\ y = x$ *then* $f\ (f\ y)$
else $f\ y$)
by (*auto simp*: *perm-rem-simps*)

lemma *perm-rem-commutes*:
assumes *bij f* **shows** *perm-rem a* (*perm-rem b f*) = *perm-rem b* (*perm-rem a f*)
proof –

have *bij-simp*: $\bigwedge x\ y.\ f\ x = f\ y \longleftrightarrow x = y$
using *assms* **by** (*auto simp*: *bij-iff*)
show *?thesis* **using** *assms* **by** (*auto simp*: *perm-rem-conv bij-simp fun-eq-iff*)
qed

lemma *perm-rem-id*[*simp*]: *perm-rem a id = id*
by (*simp add*: *perm-rem-def*)

lemma *perm-swap-comp*: *perm-swap a b* $(f \circ g)$ *x = perm-swap a b f* (*perm-swap a b g x*)
by (*auto simp*: *perm-swap-def*)

lemma *bij-perm-swap-iff*[*simp*]: *bij* (*perm-swap a b f*) $\longleftrightarrow$ *bij f*
by (*simp add*: *bij-swap-compose bij-swap-iff perm-swap-def*)

lemma *funpow-perm-swap*: *perm-swap a b f* $\widehat{\ }\widehat{\ }$ *n = perm-swap a b* (*f* $\widehat{\ }\widehat{\ }$ *n*)
by (*induct n*) (*auto simp*: *perm-swap-def fun-eq-iff*)

lemma *orbit-perm-swap*: *orbit* (*perm-swap a b f*) *x* = $(a \rightleftharpoons_F b)$ ‘ *orbit f* $((a \rightleftharpoons_F b)\ x)$
by (*auto simp*: *orbit-altdef funpow-perm-swap*) (*auto simp*: *perm-swap-def*)

lemma *has-dom-perm-swap*: *has-dom* (*perm-swap a b f*) *S = has-dom f* $((a \rightleftharpoons_F b)$ ‘ $S)$
by (*auto simp*: *has-dom-def perm-swap-def inj-image-mem-iff*) (*metis image-iff swap-swap-id*)

lemma *perm-restrict-dom-subset*:
assumes *has-dom f A* **shows** *perm-restrict f A = f*
proof −
from *assms* **have** $\bigwedge x.\ x \notin A \Longrightarrow f\ x = x$ **by** (*auto simp*: *has-dom-def*)
then show *?thesis* **by** (*auto simp*: *perm-restrict-def fun-eq-iff*)
qed

lemma *perm-swap-permutes2*:
assumes *f permutes* $((x \rightleftharpoons_F y)$ ‘ $S)$
shows *perm-swap x y f permutes S*
using *assms*
by (*auto simp*: *perm-swap-def permutes-conv-has-dom has-dom-perm-swap* [*unfolded perm-swap-def*] *intro*!: *bij-comp*)

12 Cyclic Permutations

lemma *cyclic-on-perm-swap*:
assumes *cyclic-on f S* **shows** *cyclic-on* (*perm-swap x y f*) $((x \rightleftharpoons_F y)$ ‘ $S)$
using *assms* **by** (*rule cyclic-on-image*) (*auto simp*: *perm-swap-def*)

lemma *orbit-perm-rem*:
assumes *bij f* $x \neq y$ **shows** *orbit* (*perm-rem y f*) *x = orbit f x* − $\{y\}$ (**is** *?L =*

?R)
proof (*intro set-eqI iffI*)
  fix *z* **assume** *z* ∈ *?L*
  then show *z* ∈ *?R*
    using *assms* **by** *induct* (*auto simp*: *perm-rem-conv bij-iff intro*: *orbit.intros*)
next
  fix *z* **assume** *A*: *z* ∈ *?R*

  { **assume** *z* ∈ *orbit f x*
    then have (*z* ≠ *y* ⟶ *z* ∈ *?L*) ∧ (*z* = *y* ⟶ *f z* ∈ *?L*)
    proof *induct*
    case *base* **with** *assms* **show** *?case* **by** (*auto intro*: *orbit-eqI*(*1*) *simp*: *perm-rem-conv*)
    next
      case (*step z*) **then show** *?case*
        using *assms* **by** (*cases y* = *z*) (*auto intro*: *orbit-eqI simp*: *perm-rem-conv*)
    qed
  } **with** *A* **show** *z* ∈ *?L* **by** *auto*
qed

lemma *orbit-perm-rem-eq*:
  assumes *bij f* **shows** *orbit* (*perm-rem y f*) *x* = (*if x* = *y then* {*y*} *else orbit f x*
− {*y*})
  using *assms* **by** (*simp add*: *orbit-eq-singleton-iff orbit-perm-rem perm-rem-simps*)

lemma *cyclic-on-perm-rem*:
  assumes *cyclic-on f S bij f S* ≠ {*x*} **shows** *cyclic-on* (*perm-rem x f*) (*S* − {*x*})
  using *assms*[*unfolded cyclic-on-alldef*] **by** (*simp add*: *cyclic-on-def orbit-perm-rem-eq*)
auto

end
theory *Planar-Subdivision*
imports
  Graph-Genus
  Reachablen
  Permutations-2
begin

13 Combinatorial Planarity and Subdivisions

locale *subdiv1-contr* = *subdiv-step* +
  fixes *HM*
  assumes *H-map*: *digraph-map H HM*
  assumes *edge-rev-conv*: *edge-rev HM* = *rev-H*

sublocale *subdiv1-contr* ⊆ *H*: *digraph-map H HM*
  rewrites *edge-rev HM* = *rev-H* **by** (*intro H-map edge-rev-conv*)+

sublocale *subdiv1-contr* ⊆ *G*: *fin-digraph G*
  by *unfold-locales* (*auto simp*: *arcs-G verts-G*)

context *subdiv1-contr* **begin**

definition *GM* :: *'b pre-map* **where**
GM ≡
(| *edge-rev* = *rev-G*
, *edge-succ* = *perm-swap uw uv* (*perm-swap vw vu* (*fold perm-rem* [*wu*, *wv*] (*edge-succ HM*)))
|)

lemma *edge-rev-GM*: *edge-rev GM* = *rev-G*
by (*simp add*: *GM-def*)

lemma *edge-succ-GM*: *edge-succ GM* = *perm-swap uw uv* (*perm-swap vw* (*rev-G uv*) (*fold perm-rem* [*wu*, *wv*] (*edge-succ HM*)))
by (*simp add*: *GM-def*)

lemma *rev-H-eq-rev-G*:
assumes $x \in$ *arcs G* $-$ {*uv*,*vu*} **shows** *rev-H x* = *rev-G x*
proof −
have *perm-restrict rev-H* (*arcs G*) = *perm-restrict rev-G* (*arcs H*)
using *subdiv-step* **by** (*auto simp*: *subdivision-step-def*)
with *assms* **show** *?thesis*
unfolding *arcs-H* **by** (*auto simp*: *perm-restrict-def fun-eq-iff split*: *if-splits*)
qed

lemma *edge-succ-permutes*: *edge-succ GM permutes arcs G*
proof −
have *arcs H* ⊆ (*vw* $\rightleftharpoons_F$ *rev-G uv*) ‘ (*uw* $\rightleftharpoons_F$ *uv*) ‘ *arcs G* ∪ {*wv*} ∪ {*wu*}
using *subdiv-distinct-arcs in-arcs-G*
by (*auto simp*: *arcs-H in-funswapid-image-iff transpose-def split*: *if-splits*)
then have *perm-swap uw uv* (*perm-swap vw* (*rev-G uv*) (*perm-rem* (*wv*) (*perm-rem* (*wu*) (*edge-succ HM*)))) *permutes arcs G*
by (*blast intro*: *perm-rem-permutes perm-swap-permutes2 permutes-subset H.edge-succ-permutes*)
then show *?thesis* **by** (*auto simp*: *edge-succ-GM*)
qed

lemma *out-arcs-empty*:
assumes $x \in$ *verts G*
shows *out-arcs G x* = {} ⟷ *out-arcs H x* = {}
proof
assume *A*: *out-arcs H x* = {}
have *tail-eqI*: ⋀*a*. *tail H a* = *tail G a* **by** (*simp only*: *tail-eq*)

{ **fix** *a* **assume** $a \in$ *out-arcs G x*
moreover have $a \in$ *arcs H* ⟹ $a \neq$ *uv* $a \in$ *arcs H* ⟹ $a \neq$ *vu*
using *not-in-arcs-H* **by** *auto*
ultimately have (*uw* $\rightleftharpoons_F$ *uv*) ((*vw* $\rightleftharpoons_F$ *vu*) *a*) $\in$ *out-arcs H x*

using *subdiv-distinct-arcs in-arcs-H not-in-arcs-H*
by (*auto simp*: *arcs-G intro*: *tail-eqI*)
}
then show *out-arcs G x* = {}
using *A* **by** (*auto simp del*: *in-out-arcs-conv*)
next
assume *A*: *out-arcs G x* = {}
have *tail-eqI*: $\bigwedge a$. *tail H a* = *tail G a* **by** (*simp only*: *tail-eq*)

{ fix *a* **assume** $a \in$ *out-arcs H x*
moreover have $x \neq w$ **using** *assms not-in-verts-G* **by** *blast*
ultimately have $(uw \rightleftharpoons_F uv)$ $((vw \rightleftharpoons_F vu)\ a) \in$ *out-arcs G x*
using *subdiv-distinct-arcs in-arcs-G not-in-arcs-G*
by (*auto simp*: *arcs-H*) (*auto simp*: *transpose-def intro*: *tail-eqI*[*symmetric*])
}
then show *out-arcs H x* = {}
using *A* **by** (*auto simp del*: *in-out-arcs-conv*)
qed

lemma *cyclic-on-edge-succ*:
assumes $x \in$ *verts G out-arcs G x* $\neq$ {}
shows *cyclic-on* (*edge-succ GM*) (*out-arcs G x*)
proof −
have *oa-Gx*: *out-arcs G x* = $(uw \rightleftharpoons_F uv)$ ‘ $(vw \rightleftharpoons_F vu)$ ‘ (*out-arcs H x* − {*wu*} − {*wv*})
using *subdiv-distinct-arcs not-in-arcs-G in-arcs-G*
by (*auto simp*: *in-funswapid-image-iff arcs-H transpose-def tail-eq*[*symmetric*] *split*: *if-splits*)

have *cyclic-on* (*perm-swap uw uv* (*perm-swap vw* (*rev-G uv*) (*perm-rem* (*wv*) (*perm-rem* (*wu*) (*edge-succ HM*))))) (*out-arcs G x*)
unfolding *oa-Gx*
proof (*intro cyclic-on-perm-swap cyclic-on-perm-rem*)
show *cyclic-on* (*edge-succ HM*) (*out-arcs H x*)
using *assms* **by** (*auto simp*: *out-arcs-empty verts-H intro*: *H.edge-succ-cyclic*)
show *bij* (*edge-succ HM*) **by** (*simp add*: *H.bij-edge-succ*)
show *bij* (*perm-rem* (*wu*) (*edge-succ HM*)) **by** (*simp add*: *H.bij-edge-succ*)

have $x \neq w$ **using** *assms not-in-verts-G* **by** *auto*
then have *wu* $\notin$ *out-arcs H x wv* $\notin$ *out-arcs H x*
by (*auto simp*: *arc-to-ends-def*)
then show *out-arcs H x* − {*wu*} $\neq$ {*wv*} *out-arcs H x* $\neq$ {*wu*}
by *blast+*
qed
then show *?thesis* **by** (*simp add*: *edge-succ-GM*)
qed

lemma *digraph-map-GM*:
shows *digraph-map G GM*

by *unfold-locales* (*auto simp*: *edge-rev-GM G.arev-dom edge-succ-permutes cyclic-on-edge-succ verts-G*)

end

sublocale *subdiv1-contr* $\subseteq$ *GM*: *digraph-map G GM* **by** (*rule digraph-map-GM*)

context *subdiv1-contr* **begin**

lemma *reachableGD*:
assumes $x \to^*{}_G\ y$ **shows** $x \to^*{}_H\ y$
using *assms*
proof *induct*
case *base* **then show** *?case* **by** (*auto simp*: *verts-H*)
next
case (*step x z*)
moreover
have $u \to^*{}_H\ v$ $v \to^*{}_H\ u$ **using** *adj-with-w* **by** *auto*
moreover
{ assume *A*: $(x,z) \neq (u,v)$ $(x,z) \neq (v,u)$
from ‹$x \to_G z$› **obtain** *a* **where** $a \in$ *arcs G* *tail G a* = *x* *head G a* = *z*
by *auto*
with *A* **have** $a \in$ *arcs H* *arc-to-ends H a* = (*x*,*z*) *tail H a* = *x* *head G a* = *z*
by (*auto simp*: *arcs-H tail-eq head-eq arc-to-ends-def fun-eq-iff*)
then have $x \to_H z$ **by** (*auto simp*: *arcs-ends-def intro*: *rev-image-eqI*)
}
ultimately
show *?case* **by** (*auto intro*: *H.reachable-trans*)
qed

definition *proj-verts-H* :: *'a* ⇒ *'a* **where**
proj-verts-H x ≡ *if x* = *w then u else x*

lemma *proj-verts-H-in-G*: $x \in$ *verts H* ⟹ *proj-verts-H x* $\in$ *verts G*
using *in-verts-G* **by** (*auto simp*: *proj-verts-H-def verts-H*)

lemma *dominatesHD*:
assumes $x \to_H y$ **shows** *proj-verts-H x* $\to^*{}_G$ *proj-verts-H y*
proof −
have *X1*: $\bigwedge a.$ (*w*, *y*) = *arc-to-ends G a* ⟹ $a \notin$ *arcs G*
by (*metis G.adj-in-verts*(*1*) *G.dominatesI not-in-verts-G*)
have *X2*: $\bigwedge a.$ (*x*, *w*) = *arc-to-ends G a* ⟹ $a \notin$ *arcs G*
by (*metis G.adj-in-verts*(*2*) *G.dominatesI not-in-verts-G*)
show *?thesis*
using *assms subdiv-ate-H-rev subdiv-ate in-verts-G*
by (*auto simp*: *arcs-ends-def arcs-H arc-to-ends-eq proj-verts-H-def G-reach dest*: *X1 X2*)
qed

lemma *reachableHD*:
assumes *reach*:$x \to^*{}_H\ y$ **shows** *proj-verts-H* $x \to^*{}_G$ *proj-verts-H* y
using *assms* **by** *induct* (*blast intro*: *proj-verts-H-in-G G.reachable-trans dominatesHD*)+

lemma *H-reach-conv*: $\bigwedge x\ y.\ x \to^*{}_H\ y \longleftrightarrow$ *proj-verts-H* $x \to^*{}_G$ *proj-verts-H* y
using *w-reach* **by** (*auto simp*: *reachableHD*)
(*auto simp*: *proj-verts-H-def verts-H split*: *if-splits dest*: *reachableGD intro*: *H.reachable-trans*)

lemma *sccs-eq*: *G.sccs-verts* = (‘) *proj-verts-H* ‘ *H.sccs-verts* (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
fix *S* **assume** $S \in$ *?L*
then have $w \notin S$ **using** *G.sccs-verts-subsets not-in-verts-G* **by** *blast*
then have *S-eq*: *proj-verts-H* ‘ *proj-verts-H* −‘ *S* = *S*
by (*auto simp*: *proj-verts-H-def intro*: *range-eqI*)
then have *proj-verts-H* −‘ $S \neq \{\}$ **using** ‹$S \in$ *?L*› **by** *safe* (*auto simp*: *G.sccs-verts-def*)
with ‹$S \in$ *?L*› **have** *proj-verts-H* −‘ $S \in$ *H.sccs-verts*
by (*auto simp*: *G.in-sccs-verts-conv-reachable H.in-sccs-verts-conv-reachable H-reach-conv*)
then have *proj-verts-H* ‘ *proj-verts-H* −‘ $S \in$ (‘) *proj-verts-H* ‘ *H.sccs-verts* **by** (*rule imageI*)
then show $S \in$ *?R* **by** (*simp only*: *S-eq*)
next
fix *S* **assume** $S \in$ *?R*
have *X*: $\bigwedge v\ x.\ v \notin$ *proj-verts-H* ‘ $x \Longrightarrow v = w \vee (\exists\, y.\ v =$ *proj-verts-H* $y \wedge y \notin x)$
by (*auto simp*: *proj-verts-H-def split*: *if-splits*)
from ‹$S \in$ *?R*› **show** $S \in$ *?L*
using *not-in-verts-G* **by** (*fastforce simp*: *G.reachable-in-verts G.in-sccs-verts-conv-reachable H.in-sccs-verts-conv-reachable H-reach-conv dest*: *X*)
qed

lemma *inj-on-proj-verts-H*: *inj-on* ((‘) *proj-verts-H*) (*pre-digraph.sccs-verts H*)
proof (*rule inj-onI*)
fix *S T* **assume** *A*: $S \in$ *H.sccs-verts* $T \in$ *H.sccs-verts proj-verts-H* ‘ *S* = *proj-verts-H* ‘ *T*
have $\bigwedge x.\ w \notin x \Longrightarrow$ *proj-verts-H* ‘ $x = x$ **by** (*auto simp*: *proj-verts-H-def*)
with *A* **have** $S \neq T \Longrightarrow S \cap T \neq \{\}$
by (*metis H.in-sccs-verts-conv-reachable Int-iff empty-iff image-eqI proj-verts-H-def w-reach*(*1*,*2*))
then show $S = T$ **using** *H.sccs-verts-disjoint*[*OF A*(*1*,*2*)] **by** *metis*
qed

lemma *card-sccs-verts*: *card G.sccs-verts* = *card H.sccs-verts*
unfolding *sccs-eq* **by** (*intro card-image inj-on-proj-verts-H*)

lemma *card-sccs-eq*: *card G.sccs* = *card H.sccs*
using *card-sccs-verts G.inj-on-verts-sccs H.inj-on-verts-sccs*
by (*auto simp*: *G.sccs-verts-conv H.sccs-verts-conv card-image*)

lemma *isolated-verts-eq*: *G.isolated-verts* = *H.isolated-verts*
by (*auto simp*: *G.isolated-verts-def H.isolated-verts-def verts-H out-arcs-w dest*: *out-arcs-empty*)

lemma *card-verts*: *card* (*verts H*) = *card* (*verts G*) + *1*
unfolding *verts-H* **using** *not-in-verts-G* **by** *auto*

lemma *card-arcs*: *card* (*arcs H*) = *card* (*arcs G*) + *2*
unfolding *arcs-H* **using** *not-in-arcs-G subdiv-distinct-arcs in-arcs-G* **by** (*auto simp*: *card-insert-if*)

lemma *edge-succ-wu*: *edge-succ HM wu* = *wv*
using *out-arcs-w out-degree-w edge-succ-permutes H.edge-succ-cyclic*[*of w*]
by (*auto elim*: *eq-on-cyclic-on-iff1* [**where** *x*=*wu*] *simp*: *verts-H out-degree-def*)

lemma *edge-succ-wv*: *edge-succ HM wv* = *wu*
using *out-arcs-w out-degree-w edge-succ-permutes H.edge-succ-cyclic*[*of w*]
by (*auto elim*: *eq-on-cyclic-on-iff1* [**where** *x*=*wv*] *simp*: *verts-H out-degree-def*)

lemmas *edge-succ-w* = *edge-succ-wu edge-succ-wv*

lemma *H-face-cycle-succ*:
H.face-cycle-succ uw = *wv*
H.face-cycle-succ vw = *wu*
unfolding *H.face-cycle-succ-def* **by** (*auto simp*: *edge-succ-w*)

lemma *H-edge-succ-tail-eqD*:
assumes *edge-succ HM a* = *b* **shows** *tail H a* = *tail H b*
using *assms H.tail-edge-succ*[*of a*] **by** *auto*

lemma *YYY*:
($wu \rightleftharpoons_F wv$) (*edge-succ HM vw*) = (*edge-succ HM vw*)
($wu \rightleftharpoons_F wv$) (*edge-succ HM uw*) = (*edge-succ HM uw*)
using *H.edge-succ-cyclic*[*of w*] *subdiv-distinct-verts0*
by (*auto simp*: *Transposition.transpose-def dest*: *H-edge-succ-tail-eqD*)

Project arcs of H to corresponding arcs of G

definition *proj-arcs-H* :: $'b \Rightarrow 'b$ **where**
proj-arcs-H x ≡
if x = *uw* ∨ *x* = *wv then uv*
else if x = *vw* ∨ *x* = *wu then vu*
else x

Project arcs of G to corresponding arcs of H

definition *proj-arcs-G* :: $'b \Rightarrow 'b$ **where**

proj-arcs-G x ≡
if x = uv then uw
else if x = vu then vw
else x

lemma *proj-arcs-H-simps*[*simp*]:
proj-arcs-H uw = uv
proj-arcs-H wv = uv
proj-arcs-H vw = vu
proj-arcs-H wu = vu
x ∉ {*uw*,*vw*,*wu*,*wv*} ⟹ *proj-arcs-H x = x*
a ∈ *arcs G* ⟹ *proj-arcs-H a = a*
using *subdiv-distinct-arcs not-in-arcs-G* **by** (*auto simp*: *proj-arcs-H-def*)

lemma *proj-arcs-H-in-arcs-G*: *a* ∈ *arcs H* ⟹ *proj-arcs-H a* ∈ *arcs G*
using *subdiv-distinct-arcs in-arcs-G* **by** (*auto simp*: *proj-arcs-H-def arcs-H*)

lemma *proj-arcs-eq-swap*:
assumes *a* ∉ {*uv*,*vu*,*wu*,*wv*}
shows *proj-arcs-H a* = (*uw* $\rightleftharpoons_F$ *uv o vw* $\rightleftharpoons_F$ *vu*) *a*
using *assms subdiv-distinct-arcs* **by** (*cases a* ∈ {*uw*,*vw*}) *auto*

lemma *proj-arcs-G-simps*:
proj-arcs-G uv = uw
proj-arcs-G vu = vw
a ∉ {*uv*,*vu*} ⟹ *proj-arcs-G a = a*
using *subdiv-distinct-arcs not-in-arcs-G* **by** (*auto simp*: *swap-id-eq proj-arcs-G-def*)

lemma *proj-arcs-G-in-arcs-H*:
assumes *a* ∈ *arcs G* **shows** *proj-arcs-G a* ∈ *arcs H*
using *assms subdiv-distinct-arcs* **by** (*auto simp*: *proj-arcs-G-def arcs-H*)

lemma *proj-arcs-HG*: *a* ∈ *arcs G* ⟹ *proj-arcs-H* (*proj-arcs-G a*) = *a*
by (*auto simp*: *proj-arcs-G-def*)

lemma *fcs-proj-arcs-GH*:
assumes *a* ∈ *arcs H* **shows** *H.face-cycle-set* (*proj-arcs-G* (*proj-arcs-H a*)) = *H.face-cycle-set a*
proof −
have *H.face-cycle-set vw* = *H.face-cycle-set wu H.face-cycle-set uw* = *H.face-cycle-set wv*
unfolding *H.face-cycle-set-def* **by** (*auto simp add*: *H-face-cycle-succ*[*symmetric*]
self-in-orbit-step H.permutation-face-cycle-succ permutation-self-in-orbit)
then show *?thesis*
using *assms not-in-arcs-H* **by** (*cases a* ∈ {*uv*,*vu*,*uw*,*wu*,*vw*,*wv*}) (*auto simp*: *proj-arcs-G-simps*)
qed

lemma *H-face-cycle-succ-neq-uv*:

$a \notin \{uv,vu\} \Longrightarrow H.face\text{-}cycle\text{-}succ\ a \notin \{uv,vu\}$
using *not-in-arcs-H* **by** (*cases* $a \in arcs\ H$) (*auto dest*: *H.face-cycle-succ-in-arcsI*)

lemma *face-cycle-succ-choose-inter*:
$\{H.face\text{-}cycle\text{-}succ\ uw, H.face\text{-}cycle\text{-}succ\ vw, H.face\text{-}cycle\text{-}succ\ wu, H.face\text{-}cycle\text{-}succ\ wv\} \cap \{uv,vu\} = \{\}$
using *subdiv-distinct-arcs H-face-cycle-succ-neq-uv* **by** *safe* (*simp-all*, *metis+*)

lemma *face-cycle-succ-choose-neq*:
$H.face\text{-}cycle\text{-}succ\ wu \notin \{wu,wv\}$
$H.face\text{-}cycle\text{-}succ\ wv \notin \{wu,wv\}$
using *subdiv-distinct-verts0 in-arcs-H*
by (*auto simp del*: *H.edge-rev-in-arcs dest*: *H.tail-face-cycle-succ*)

lemma *H-face-cycle-succ-G-not-in*:
assumes $a \in arcs\ G$ **shows** $H.face\text{-}cycle\text{-}succ\ a \notin \{wu,wv\}$
proof (*cases* $a \in \{uv,vu\}$)
case *True* **with** *assms* **show** *?thesis* **using** *subdiv-distinct-arcs* **by** (*auto simp*: *arcs-H*)
next
case *False* **with** *assms* **have** $a \in arcs\ H$ **by** (*auto simp*: *arcs-H*)
from *assms* **have** $head\ H\ a \neq w$ **by** (*auto simp*: *head-eq verts-G arcs-H dest*: *G.head-in-verts*)
then show *?thesis* **using** *H.tail-face-cycle-succ*[*OF* ‹$a \in arcs\ H$›] **by** *auto*
qed

lemma
face-cycle-succ-uv: $GM.face\text{-}cycle\text{-}succ\ uv = proj\text{-}arcs\text{-}H\ (H.face\text{-}cycle\text{-}succ\ wv)$
and
face-cycle-succ-vu: $GM.face\text{-}cycle\text{-}succ\ vu = proj\text{-}arcs\text{-}H\ (H.face\text{-}cycle\text{-}succ\ wu)$
unfolding *GM.face-cycle-succ-def edge-rev-GM edge-succ-GM*
using *face-cycle-succ-choose-neq face-cycle-succ-choose-inter subdiv-distinct-arcs*
apply (*auto simp*: *fun-eq-iff perm-swap-def*)
apply (*auto simp*: *perm-rem-def edge-succ-w H.face-cycle-succ-def YYY proj-arcs-H-def*)
done

lemma *face-cycle-succ-not-uv*:
assumes $a \in arcs\ G\ a \notin \{uv,vu\}$
shows $GM.face\text{-}cycle\text{-}succ\ a = proj\text{-}arcs\text{-}H\ (H.face\text{-}cycle\text{-}succ\ a)$
proof −
have $GM.face\text{-}cycle\text{-}succ\ a = (uw \rightleftharpoons_F uv)\ ((vw \rightleftharpoons_F rev\text{-}G\ uv)\ (perm\text{-}rem\ (wv)\ (perm\text{-}rem\ (wu)\ (edge\text{-}succ\ HM))\ (((vw \rightleftharpoons_F vu)\ ((uw \rightleftharpoons_F uv)\ (rev\text{-}G\ a))))))$
by (*simp add*: *GM.face-cycle-succ-def perm-swap-def edge-succ-GM edge-rev-GM*)
also have $(vw \rightleftharpoons_F vu)\ ((uw \rightleftharpoons_F uv)\ (rev\text{-}G\ a)) = rev\text{-}G\ a$
using *assms not-in-arcs-G* **by** (*auto simp*: *transpose-def G.arev-eq-iff*)
also have $perm\text{-}rem\ (wv)\ (perm\text{-}rem\ (wu)\ (edge\text{-}succ\ HM))\ (rev\text{-}G\ a) = edge\text{-}succ\ HM\ (rev\text{-}G\ a)$
proof −
have $*$: $\bigwedge a.\ tail\ H\ a \neq w \Longrightarrow (wu \rightleftharpoons_F wv)\ a = a$ **by** (*auto simp*: *transpose-def*)

from *assms* **have** *head H a* $\neq$ *w tail H* (*rev-G a*) = *head H a*
by (*auto simp*: *tail-eq head-eq verts-G dest*: *G.head-in-verts*)
then have ((*wu* $\rightleftharpoons_F$ *wv*) (*edge-succ HM* (*rev-G a*))) = *edge-succ HM* (*rev-G a*)
by (*intro* ∗) *auto*
then show *?thesis* **by** (*auto simp*: *perm-rem-def edge-succ-w*)
qed
also have *edge-succ HM* (*rev-G a*) = *H.face-cycle-succ a*
using *assms* **unfolding** *H.face-cycle-succ-def* **by** (*simp add*: *rev-H-eq-rev-G*)
also have (*uw* $\rightleftharpoons_F$ *uv*) ((*vw* $\rightleftharpoons_F$ *rev-G uv*) (*H.face-cycle-succ a*)) = *proj-arcs-H* (*H.face-cycle-succ a*)
proof −
from *assms* **have** *a* ∈ *arcs H* **by** (*auto simp*: *arcs-H*)
then have *fcs-not-in*: *H.face-cycle-succ a* ∉ {*uv*, *vu*, *wu*, *wv*}
using *assms H-face-cycle-succ-G-not-in in-arcs-G not-in-arcs-H*
by (*auto simp del*: *G.arev-in-arcs dest*: *H.face-cycle-succ-closed*[*THEN iffD2*])
then show *?thesis* **by** (*auto simp add*: *proj-arcs-eq-swap*)
qed
finally show *?thesis* **.**
qed

lemmas *G-face-cycle-succ* = *face-cycle-succ-uv face-cycle-succ-vu face-cycle-succ-not-uv*

lemma *in-G-fcs-in-H-fcs*:
assumes *a* ∈ *arcs G*
assumes *x* ∈ *GM.face-cycle-set a*
shows *x* ∈ *proj-arcs-H* ‘ *H.face-cycle-set* (*proj-arcs-G a*)
using ‹*x* ∈ -›
proof *induct*
case *base* **show** *?case*
by (*rule rev-image-eqI*[**where** *x*=*proj-arcs-G a*]) (*auto simp*: ‹*a* ∈ *arcs G*› *proj-arcs-G-def*)
next
case (*step b*)
{ fix *x* **assume** *x* ∈ *H.face-cycle-set* (*proj-arcs-G a*)
then have *x* ∈ *arcs H*
using ‹*a* ∈ *arcs G*› **by** (*auto dest*: *H.in-face-cycle-setD simp*: *proj-arcs-G-in-arcs-H*)
moreover
then have *x* ∉ {*uv*,*vu*} *x* ∉ {*uw*,*wu*,*vw*,*wv*} ⟹ *x* ∈ *arcs G*
using ‹*a* ∈ *arcs G*› **by** (*auto simp*: *arcs-H dest*: *H.in-face-cycle-setD*)
ultimately
have *GM.face-cycle-succ* (*proj-arcs-H x*) ∈ {*proj-arcs-H* (*H.face-cycle-succ x*),
proj-arcs-H (*H.face-cycle-succ* (*H.face-cycle-succ x*))}
by (*cases x* ∈ {*uw*,*vw*,*wu*,*wv*}) (*auto simp*: *G-face-cycle-succ H-face-cycle-succ*)
}
moreover
have *b* ∈ *arcs G*

using *step*(*1*) ‹$a \in$ *arcs G*› **by** (*simp add*: *GM.in-face-cycle-setD GM.face-cycle-set-def*)
ultimately
show *?case* **using** ‹$b \in$ *arcs G*› *step*(*2*) **by** (*auto intro*: *H.face-cycle-succ-inI*)
qed

lemma *in-H-fcs-in-G-fcs*:
assumes $a \in$ *arcs H*
assumes $x \in$ *H.face-cycle-set a*
shows $x \in$ *proj-arcs-H* $-$‘ *GM.face-cycle-set* (*proj-arcs-H a*)
using ‹$x \in$ -›
proof *induct*
case *base* **then show** *?case* **by** *auto*
next
case (*step y*)
then have $y \in$ *arcs H* **using** ‹$a \in$ *arcs H*› **by** (*auto dest*: *H.in-face-cycle-setD*)
moreover then have $y \notin \{uv,vu\}$ **by** (*fastforce simp*: *arcs-H*)
ultimately have *proj-arcs-H* (*H.face-cycle-succ y*) = *GM.face-cycle-succ* (*proj-arcs-H y*)
$\vee$ *proj-arcs-H* (*H.face-cycle-succ y*) = *proj-arcs-H y*
by (*cases* $y \in \{uw,vw,wv,wu\}$) (*auto simp*: *H-face-cycle-succ G-face-cycle-succ arcs-G*)
with *step* **show** *?case* **by** (*auto intro*: *GM.face-cycle-succ-inI*)
qed

lemma *G-fcs-eq*:
assumes $a \in$ *arcs G*
shows *GM.face-cycle-set a* = *proj-arcs-H* ‘ *H.face-cycle-set* (*proj-arcs-G a*) (**is** *?L* = *?R*)
using *assms* **by** (*auto dest*: *in-H-fcs-in-G-fcs*[*rotated*] *in-G-fcs-in-H-fcs*[*rotated*]
simp: *proj-arcs-G-in-arcs-H proj-arcs-HG*)

lemma *H-fcs-eq*:
assumes $a \in$ *arcs H*
shows *proj-arcs-H* ‘ *H.face-cycle-set a* = *GM.face-cycle-set* (*proj-arcs-H a*)
using *assms* **by** (*auto dest*: *in-H-fcs-in-G-fcs*[*rotated*] *in-G-fcs-in-H-fcs*[*rotated*]
simp: *proj-arcs-H-in-arcs-G fcs-proj-arcs-GH*)

lemma *face-cycle-sets*:
shows *GM.face-cycle-sets* = (‘) *proj-arcs-H* ‘ *H.face-cycle-sets* (**is** *?L* = *?R*)
unfolding *GM.face-cycle-sets-def H.face-cycle-sets-def*
by (*blast intro*!: *H-fcs-eq G-fcs-eq proj-arcs-G-in-arcs-H proj-arcs-H-in-arcs-G*)

lemma *inj-on-proj-arcs-H*: *inj-on* ((‘) *proj-arcs-H*) *H.face-cycle-sets*
proof (*rule inj-onI*)
fix *A B* **assume** *fcs*: $A \in$ *H.face-cycle-sets* $B \in$ *H.face-cycle-sets*
and *pa-eq*: *proj-arcs-H* ‘ *A* = *proj-arcs-H* ‘ *B*

have *xw-iff-wy*:
$\bigwedge X.\ X \in$ *H.face-cycle-sets* $\Longrightarrow uw \in X \longleftrightarrow wv \in X$

$\bigwedge X.\ X \in H.face\text{-}cycle\text{-}sets \Longrightarrow vw \in X \longleftrightarrow wu \in X$
using *H-face-cycle-succ* **by** (*auto simp*: *H.face-cycle-sets-def dest*: *H.face-cycle-succ-inI intro*: *H.face-cycle-succ-inD*)

have *not-in-A*: $uv \notin A$ $vu \notin A$ **and** *not-in-B*: $vu \notin B$ $uv \notin B$
using *fcs not-in-arcs-H* **by** (*auto dest*: *H.in-face-cycle-setsD*)

have $A = proj\text{-}arcs\text{-}H$ −‘ ($proj\text{-}arcs\text{-}H$ ‘ A) − {uv,vu}
using *subdiv-distinct-arcs not-in-A* **by** (*auto simp*: *proj-arcs-H-def xw-iff-wy*[*OF fcs*(*1*)] *split*: *if-splits*)
also have ... = $proj\text{-}arcs\text{-}H$ −‘ ($proj\text{-}arcs\text{-}H$ ‘ B) − {uv,vu} **by** (*simp add*: *pa-eq*)
also have ... = B
using *subdiv-distinct-arcs not-in-B* **by** (*auto simp*: *proj-arcs-H-def xw-iff-wy*[*OF fcs*(*2*)] *split*: *if-splits*)
finally show $A = B$ **.**
qed

lemma *card-face-cycle-sets*: *card GM.face-cycle-sets* = *card H.face-cycle-sets*
unfolding *face-cycle-sets* **using** *inj-on-proj-arcs-H* **by** (*rule card-image*)

lemma *euler-char-eq*: *GM.euler-char* = *H.euler-char*
by (*auto simp*: *GM.euler-char-def H.euler-char-def card-verts card-arcs card-face-cycle-sets*)

lemma *euler-genus-eq*: *GM.euler-genus* = *H.euler-genus*
by (*auto simp*: *GM.euler-genus-def H.euler-genus-def euler-char-eq card-sccs-eq isolated-verts-eq*)

end

lemma *subdivision-genus-same-rev*:
assumes *subdivision* (*G*, *rev-G*) (*H*, *edge-rev HM*) *digraph-map H HM pre-digraph-map.euler-genus H HM* = *m*
shows ∃ *GM*. *digraph-map G GM* ∧ *pre-digraph-map.euler-genus G GM* = *m* ∧ *edge-rev GM* = *rev-G*
proof −
from *assms* **show** *?thesis*
proof (*induction rev-H≡edge-rev HM arbitrary*: *HM*)
case *base* **then show** *?case* **by** *auto*
next
case (*divide I rev-I H u v w uv uw vw*)
then interpret *subdiv-step I rev-I H edge-rev HM u v w uv uw vw*
by *unfold-locales simp*

interpret *H*: *digraph-map H HM* **using** ‹*digraph-map H HM*› **.**
interpret *IH*: *subdiv1-contr I rev-I H edge-rev HM u v w uv uw vw HM*
by *unfold-locales simp*

have *eulerI*: *IH.GM.euler-genus* = *m* **by** (*auto simp*: *IH.euler-genus-eq divide*)

with - *IH.digraph-map-GM* **show** *?case* **by** (*rule divide*) (*simp add*: *IH.edge-rev-GM*)
qed
qed

lemma *subdivision-genus*:
assumes *subdivision* (*G*, *rev-G*) (*H*, *rev-H*) *digraph-map H HM pre-digraph-map.euler-genus H HM* = *m*
shows ∃ *GM*. *digraph-map G GM* ∧ *pre-digraph-map.euler-genus G GM* = *m*
proof −
interpret *H*: *digraph-map H HM* **by** *fact*
show *?thesis*
using *subdivision-genus-same-rev subdivision-choose-rev assms H.bidirected-digraph* **by** *metis*
qed

lemma *subdivision-comb-planar*:
assumes *subdivision* (*G*, *rev-G*) (*H*, *rev-H*) *comb-planar H* **shows** *comb-planar G*
using *assms* **unfolding** *comb-planar-def* **by** (*metis subdivision-genus*)

end
theory *Planar-Subgraph*
imports
Graph-Genus
Permutations-2
HOL−Library.FuncSet
HOL−Library.Simps-Case-Conv
begin

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lemma *out-arcs-emptyD-dominates*:
assumes *out-arcs G x* = {} **shows** $\neg x \rightarrow_G y$
using *assms* **by** (*auto simp*: *out-arcs-def*)

lemma (**in** *wf-digraph*) *reachable-refl-iff*: $u \rightarrow^* u \longleftrightarrow u \in$ *verts G*
by (*auto simp*: *reachable-in-verts*)

context *digraph-map* **begin**

lemma *face-cycle-set-succ*[*simp*]: *face-cycle-set* (*face-cycle-succ a*) = *face-cycle-set a*
by (*metis face-cycle-eq face-cycle-set-self face-cycle-succ-inD*)

lemma *face-cycle-succ-funpow-in*[*simp*]:
(*face-cycle-succ* ^^ *n*) *a* ∈ *arcs G* ⟷ *a* ∈ *arcs G*
by (*induct n*) *auto*

lemma *segment-face-cycle-x-x-eq*:

segment face-cycle-succ x x = *face-cycle-set x* − {*x*}
unfolding *face-cycle-set-def* **using** *face-cycle-succ-permutes finite-arcs permutation-permutes*
by (*intro segment-x-x-eq*) *blast*

lemma *fcs-x-eq-x*: *face-cycle-succ x* = *x* ⟷ *face-cycle-set x* = {*x*} (**is** *?L* ⟷ *?R*)
unfolding *face-cycle-set-def orbit-eq-singleton-iff* **..**

end

lemma (**in** *bidirected-digraph*) *bidirected-digraph-del-arc*:
bidirected-digraph (*pre-digraph.del-arc* (*pre-digraph.del-arc G* (*arev a*)) *a*) (*perm-restrict arev* (*arcs G* − {*a* , *arev a*}))
proof *unfold-locales*
fix *b* **assume** *A*: *b* ∈ *arcs* (*pre-digraph.del-arc* (*del-arc* (*arev a*)) *a*)
have *arev b* ≠ *b* ⟹ *b* ≠ *arev a* ⟹ *b* ≠ *a* ⟹ *perm-restrict arev* (*arcs G* − {*a*, *arev a*}) (*arev b*) = *b*
using *bij-arev arev-dom* **by** (*subst perm-restrict-simps*) (*auto simp*: *bij-iff*)
then show *perm-restrict arev* (*arcs G* − {*a*, *arev a*}) (*perm-restrict arev* (*arcs G* − {*a*, *arev a*}) *b*) = *b*
using *A*
by (*auto simp*: *pre-digraph.del-arc-simps perm-restrict-simps arev-dom*)
qed (*auto simp*: *pre-digraph.del-arc-simps perm-restrict-simps arev-dom*)

lemma (**in** *bidirected-digraph*) *bidirected-digraph-del-vert*: *bidirected-digraph* (*del-vert u*) (*perm-restrict arev* (*arcs* (*del-vert u*)))
by *unfold-locales* (*auto simp*: *del-vert-simps perm-restrict-simps arev-dom*)

lemma (**in** *pre-digraph*) *ends-del-arc*: *arc-to-ends* (*del-arc u*) = *arc-to-ends G*
by (*simp add*: *arc-to-ends-def fun-eq-iff*)

lemma (**in** *pre-digraph*) *dominates-arcsD*:
assumes $v \rightarrow_{\textit{del-arc } u} w$ **shows** $v \rightarrow_G w$
using *assms* **by** (*auto simp*: *arcs-ends-def ends-del-arc*)

lemma (**in** *wf-digraph*) *reachable-del-arcD*:
assumes $v \rightarrow^*{}_{\textit{del-arc } u}\ w$ **shows** $v \rightarrow^*{}_G\ w$
proof −
interpret *H*: *wf-digraph del-arc u* **by** (*rule wf-digraph-del-arc*)
from *assms* **show** *?thesis*
by (*induct*) (*auto dest*: *dominates-arcsD intro*: *adj-reachable-trans*)
qed

lemma (**in** *fin-digraph*) *finite-isolated-verts*[*intro*!]: *finite isolated-verts*
by (*auto simp*: *isolated-verts-def*)

lemma (**in** *wf-digraph*) *isolated-verts-in-sccs*:

assumes $u \in$ *isolated-verts* **shows** $\{u\} \in$ *sccs-verts*
proof $-$
have $v = u$ **if** $u \rightarrow^*{}_G\ v$ **for** v
using *that assms* **by** *induct* (*auto simp*: *arcs-ends-def arc-to-ends-def isolated-verts-def*)
with *assms* **show** *?thesis* **by** (*auto simp*: *sccs-verts-def isolated-verts-def*)
qed

lemma (**in** *digraph-map*) *in-face-cycle-sets*:
$a \in$ *arcs G* $\Longrightarrow$ *face-cycle-set a* $\in$ *face-cycle-sets*
by (*auto simp*: *face-cycle-sets-def*)

lemma (**in** *digraph-map*) *heads-face-cycle-set*:
assumes $a \in$ *arcs G*
shows *head G ‘ face-cycle-set a* $=$ *tail G ‘ face-cycle-set a* (**is** *?L* $=$ *?R*)
proof (*intro set-eqI iffI*)
fix u **assume** $u \in$ *?L*
then obtain b **where** $b \in$ *face-cycle-set a head G b* $= u$ **by** *blast*
then have *face-cycle-succ b* $\in$ *face-cycle-set a tail G* (*face-cycle-succ b*) $= u$
using *assms* **by** (*auto simp*: *tail-face-cycle-succ face-cycle-succ-inI in-face-cycle-setD*)
then show $u \in$ *?R* **by** *auto*
next
fix u **assume** $u \in$ *?R*
then obtain b **where** $b \in$ *face-cycle-set a tail G b* $= u$ **by** *blast*
moreover
then obtain c **where** $b =$ *face-cycle-succ c* **by** (*metis face-cycle-succ-pred*)
ultimately
have $c \in$ *face-cycle-set a head G c* $= u$
by (*auto dest*: *face-cycle-succ-inD*) (*metis assms face-cycle-succ-no-arc in-face-cycle-setD tail-face-cycle-succ*)
then show $u \in$ *?L* **by** *auto*
qed

lemma (**in** *pre-digraph*) *casI-nth*:
assumes $p \neq []$ $u =$ *tail G* (*hd p*) $v =$ *head G* (*last p*) $\bigwedge i.$ *Suc i* $<$ *length p* $\Longrightarrow$ *head G* (*p* ! *i*) $=$ *tail G* (*p* ! *Suc i*)
shows *cas u p v*
using *assms*
proof (*induct p arbitrary*: *u*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons a p*)
have *cas* (*head G a*) *p v*
proof (*cases* $p = []$)
case *False* **then show** *?thesis*
using *Cons.prems*(*1−3*) *Cons.prems*(*4*)[*of 0*] *Cons.prems*(*4*)[*of Suc i* **for** *i*]
by (*intro Cons*) (*simp-all add*: *hd-conv-nth*)
qed (*simp add*: *Cons*)
with *Cons* **show** *?case* **by** *simp*

qed

lemma (**in** *digraph-map*) *obtain-trail-in-fcs*:
  assumes $a \in$ *arcs G a0* $\in$ *face-cycle-set a an* $\in$ *face-cycle-set a*
  obtains *p* **where** *trail* (*tail G a0*) *p* (*head G an*) $p \neq []$ *hd p* = *a0* *last p* = *an*
    set p $\subseteq$ *face-cycle-set a*
proof −
  have *fcs-a*: *face-cycle-set a* = *orbit face-cycle-succ a0*
    using *assms face-cycle-eq* **by** (*simp add*: *face-cycle-set-def*)
  have *a0* = (*face-cycle-succ* ^^ *0*) *a0* **by** *simp*
  have *an* = (*face-cycle-succ* ^^ *funpow-dist face-cycle-succ a0 an*) *a0*
    using *assms* **by** (*simp add*: *fcs-a funpow-dist-prop*)

  define *p* **where** *p* = *map* (λ*n*. (*face-cycle-succ* ^^ *n*) *a0*) [*0*..<*Suc* (*funpow-dist face-cycle-succ a0 an*)]
  have *p-nth*: $\bigwedge$*i*. *i* < *length p* $\Longrightarrow$ *p* ! *i* = (*face-cycle-succ* ^^ *i*) *a0*
    by (*auto simp*: *p-def simp del*: *upt-Suc*)

  have *P2*: $p \neq []$ **by** (*simp add*: *p-def*)
  have *P3*: *hd p* = *a0* **using** ‹*a0* = -› **by** (*auto simp*: *p-def hd-map simp del*: *upt-Suc*)
  have *P4*: *last p* = *an* **using** ‹*an* = -› **by** (*simp add*: *p-def*)
  have *P5*: *set p* $\subseteq$ *face-cycle-set a*
    unfolding *p-def fcs-a orbit-altdef-permutation*[*OF permutation-face-cycle-succ*] **by** *auto*

  have *P1*: *trail* (*tail G a0*) *p* (*head G an*)
  proof −
    have *distinct p*
    proof −
      have *an* $\in$ *orbit face-cycle-succ a0* **using** *assms* **by** (*simp add*: *fcs-a*)
      then have *inj-on* (λ*n*. (*face-cycle-succ* ^^ *n*) *a0*) {*0*..*funpow-dist face-cycle-succ a0 an*}
        by (*rule inj-on-funpow-dist*)
      also have {*0*..*funpow-dist face-cycle-succ a0 an*} = (*set* [*0*..<*Suc* (*funpow-dist face-cycle-succ a0 an*)])
        by *auto*
      finally have *inj-on* (λ*n*. (*face-cycle-succ* ^^ *n*) *a0*) (*set* [*0*..<*Suc* (*funpow-dist face-cycle-succ a0 an*)]) **.**
      then show *distinct p* **by** (*simp add*: *distinct-map p-def*)
    qed
    moreover
    have *a0* $\in$ *arcs G* **by** (*metis assms*(*1*−*2*) *in-face-cycle-setD*)
    then have *tail G a0* $\in$ *verts G* **by** *simp*
    moreover
    have *set p* $\subseteq$ *arcs G* **using** *P5*
      by (*metis assms*(*1*) *in-face-cycle-setD subset-code*(*1*))
    moreover
    then have $\bigwedge$*i*. *Suc i* < *length p* $\Longrightarrow$ *p* ! *Suc i* $\in$ *arcs G* **by** *auto*

then have $\bigwedge i.\ Suc\ i < length\ p \Longrightarrow head\ G\ (p\ !\ i) = tail\ G\ (p\ !\ Suc\ i)$
  by (*auto simp*: *p-nth tail-face-cycle-succ*)
ultimately
show *?thesis*
  using *P2 P3 P4* **unfolding** *trail-def awalk-def* **by** (*auto intro*: *casI-nth*)
qed

from *P1 P2 P3 P4 P5* **show** *?thesis* **..**
qed

lemma (**in** *digraph-map*) *obtain-trail-in-fcs'*:
assumes $a \in arcs\ G\ u \in tail\ G\ `\ face\text{-}cycle\text{-}set\ a\ v \in tail\ G\ `\ face\text{-}cycle\text{-}set\ a$
obtains *p* **where** $trail\ u\ p\ v\ set\ p \subseteq face\text{-}cycle\text{-}set\ a$
proof −
from *assms* **obtain** *a0* **where** $tail\ G\ a0 = u\ a0 \in face\text{-}cycle\text{-}set\ a$ **by** *auto*
moreover
from *assms* **obtain** *an* **where** $head\ G\ an = v\ an \in face\text{-}cycle\text{-}set\ a$
  by (*auto simp*: *heads-face-cycle-set*[*symmetric*])
ultimately obtain *p* **where** $trail\ u\ p\ v\ set\ p \subseteq face\text{-}cycle\text{-}set\ a$
  using ‹$a \in arcs\ G$› **by** (*metis obtain-trail-in-fcs*)
then show *?thesis* **..**
qed

14.1 Deleting an isolated vertex

locale *del-vert-props* = *digraph-map* +
fixes *u*
assumes *u-in*: $u \in verts\ G$
assumes *u-isolated*: $out\text{-}arcs\ G\ u = \{\}$

begin

lemma *u-isolated-in*: $in\text{-}arcs\ G\ u = \{\}$
  using *u-isolated* **by** (*simp add*: *in-arcs-eq*)

lemma *arcs-dv*: $arcs\ (del\text{-}vert\ u) = arcs\ G$
  using *u-isolated u-isolated-in* **by** (*auto simp*: *del-vert-simps*)

lemma *out-arcs-dv*: $out\text{-}arcs\ (del\text{-}vert\ u) = out\text{-}arcs\ G$
  by (*auto simp*: *fun-eq-iff arcs-dv tail-del-vert*)

lemma *digraph-map-del-vert*:
  shows *digraph-map* (*del-vert u*) *M*
proof −
  have $perm\text{-}restrict\ (edge\text{-}rev\ M)\ (arcs\ (del\text{-}vert\ u)) = edge\text{-}rev\ M$
    using *has-dom-arev arcs-dv* **by** (*auto simp*: *perm-restrict-dom-subset*)
  then interpret *H*: *bidirected-digraph del-vert u edge-rev M*
    using *bidirected-digraph-del-vert*[*of u*] **by** *simp*
  show *?thesis*

by *unfold-locales* (*auto simp*: *arcs-dv edge-succ-permutes out-arcs-dv edge-succ-cyclic verts-del-vert*)
qed

end

sublocale *del-vert-props* $\subseteq$ *H*: *digraph-map del-vert u M* **by** (*rule digraph-map-del-vert*)

context *del-vert-props* **begin**

lemma *card-verts-dv*: *card* (*verts G*) = *Suc* (*card* (*verts* (*del-vert u*)))
by (*auto simp*: *verts-del-vert*) (*rule card.remove*, *auto simp*: *u-in*)

lemma *card-arcs-dv*: *card* (*arcs* (*del-vert u*)) = *card* (*arcs G*)
using *u-isolated* **by** (*auto simp add*: *arcs-dv in-arcs-eq*)

lemma *isolated-verts-dv*: *H.isolated-verts* = *isolated-verts* − {*u*}
by (*auto simp*: *isolated-verts-def H.isolated-verts-def verts-del-vert out-arcs-dv*)

lemma *u-in-isolated-verts*: *u* $\in$ *isolated-verts*
using *u-in u-isolated* **by** (*auto simp*: *isolated-verts-def*)

lemma *card-isolated-verts-dv*: *card isolated-verts* = *Suc* (*card H.isolated-verts*)
by (*simp add*: *isolated-verts-dv*) (*rule card.remove*, *auto simp*: *u-in-isolated-verts*)

lemma *face-cycles-dv*: *H.face-cycle-sets* = *face-cycle-sets*
unfolding *H.face-cycle-sets-def face-cycle-sets-def arcs-dv* **..**

lemma *euler-char-dv*: *euler-char* = *1* + *H.euler-char*
by (*auto simp*: *euler-char-def H.euler-char-def card-arcs-dv card-verts-dv face-cycles-dv*)

lemma *adj-dv*: $v \rightarrow_{\textit{del-vert } u} w \longleftrightarrow v \rightarrow_G w$
by (*auto simp*: *arcs-ends-def arcs-dv ends-del-vert*)

lemma *reachable-del-vertD*:
assumes $v \rightarrow^*_{\textit{del-vert } u} w$ **shows** $v \rightarrow^*_G w$
using *assms* **by** *induct* (*auto simp*: *verts-del-vert adj-dv intro*: *adj-reachable-trans*)

lemma *reachable-del-vertI*:
assumes $v \rightarrow^*_G w$ $u \neq v \lor u \neq w$ **shows** $v \rightarrow^*_{\textit{del-vert } u} w$
using *assms*
proof *induct*
case (*step x y*)
from ‹$x \rightarrow_G y$› **obtain** *a* **where** $a \in$ *arcs G head G a* = *y* **by** *auto*
then have $a \in$ *in-arcs G y* **by** *auto*
then have $y \neq u$ **using** *u-isolated in-arcs-eq*[*of u*] **by** *auto*
with *step* **show** *?case* **by** (*auto simp*: *adj-dv intro*: *H.adj-reachable-trans*)
qed (*auto simp*: *verts-del-vert*)

lemma *G-reach-conv*: $v \rightarrow^*{}_G\ w \longleftrightarrow v \rightarrow^*{}_{del\text{-}vert\ u}\ w \vee (v = u \wedge w = u)$
by (*auto dest*: *reachable-del-vertI reachable-del-vertD intro*: *u-in*)

lemma *sccs-verts-dv*: *H.sccs-verts* = *sccs-verts* − {{*u*}} (**is** *?L* = *?R*)
proof −
have ∗:⋀*S x*. *S* ∈ *sccs-verts* ⟹ *S* ∉ *H.sccs-verts* ⟹ *x* ∈ *S* ⟹ *x* = *u*
by (*simp add*: *H.in-sccs-verts-conv-reachable in-sccs-verts-conv-reachable G-reach-conv*)
(*meson H.reachable-trans*)
show *?thesis*
by (*auto dest*: ∗) (*auto simp*: *H.in-sccs-verts-conv-reachable in-sccs-verts-conv-reachable G-reach-conv H.reachable-refl-iff verts-del-vert*)
qed

lemma *card-sccs-verts-dv*: *card sccs-verts* = *Suc* (*card H.sccs-verts*)
unfolding *sccs-verts-dv*
by (*rule card.remove*) (*auto simp*: *isolated-verts-in-sccs u-in-isolated-verts finite-sccs-verts*)

lemma *card-sccs-dv*: *card sccs* = *Suc* (*card H.sccs*)
using *card-sccs-verts-dv* **by** (*simp add*: *card-sccs-verts H.card-sccs-verts*)

lemma *euler-genus-eq*: *H.euler-genus* = *euler-genus*
by (*auto simp*: *pre-digraph-map.euler-genus-def card-sccs-dv card-isolated-verts-dv euler-char-dv*)

end

14.2 Deleting an arc pair

locale *bidel-arc* = *G*: *digraph-map* +
fixes *a*
assumes *a-in*: *a* ∈ *arcs G*

begin

abbreviation *a′* ≡ *edge-rev M a*

definition *H* :: (′*a*,′*b*) *pre-digraph* **where**
H ≡ *pre-digraph.del-arc* (*pre-digraph.del-arc G a′*) *a*

definition *HM* :: ′*b pre-map* **where**
HM =
⦇ *edge-rev* = *perm-restrict* (*edge-rev M*) (*arcs G* − {*a*, *a′*})
, *edge-succ* = *perm-rem a* (*perm-rem a′* (*edge-succ M*))
⦈

lemma
verts-H: *verts H* = *verts G* **and**

arcs-H: *arcs H* = *arcs G* − {*a*, *a'*} **and**
tail-H: *tail H* = *tail G* **and**
head-H: *head H* = *head G* **and**
ends-H: *arc-to-ends H* = *arc-to-ends G***and**
arcs-in: {*a*,*a'*} ⊆ *arcs G* **and**
ends-in: {*tail G a*, *head G a*} ⊆ *verts G*
by (*auto simp*: *H-def pre-digraph.del-arc-simps a-in arc-to-ends-def*)

lemma *cyclic-on-edge-succ*:
assumes *x* ∈ *verts H out-arcs H x* ≠ {}
shows *cyclic-on* (*edge-succ HM*) (*out-arcs H x*)
proof −
have *oa-H*: *out-arcs H x* = (*out-arcs G x* − {*a'*}) − {*a*} **by** (*auto simp*: *arcs-H tail-H*)
have *cyclic-on* (*perm-rem a* (*perm-rem a'* (*edge-succ M*))) (*out-arcs G x* − {*a'*} − {*a*})
using *assms*
by (*intro cyclic-on-perm-rem G.edge-succ-cyclic*) (*auto simp*: *oa-H G.bij-edge-succ G.edge-succ-cyclic*)
then show *?thesis* **by** (*simp add*: *HM-def oa-H*)
qed

lemma *digraph-map*: *digraph-map H HM*
proof −
interpret *fin-digraph H* **unfolding** *H-def*
by (*rule fin-digraph.fin-digraph-del-arc*) (*rule G.fin-digraph-del-arc*)
interpret *bidirected-digraph H edge-rev HM* **unfolding** *H-def*
using *G.bidirected-digraph-del-arc*[*of a*] **by** (*auto simp*: *HM-def*)
have ∗: *insert a'* (*insert a* (*arcs H*)) = *arcs G* **using** *a-in* **by** (*auto simp*: *arcs-H*)
have *edge-succ HM permutes arcs H*
unfolding *HM-def* **by** (*auto simp*: ∗ *intro*!: *perm-rem-permutes G.edge-succ-permutes*)
moreover
{ fix *v* **assume** *v* ∈ *verts H out-arcs H v* ≠ {}
then have *cyclic-on* (*edge-succ HM*) (*out-arcs H v*) **by** (*rule cyclic-on-edge-succ*)
}
ultimately
show *?thesis* **by** *unfold-locales*
qed

lemma *rev-H*: *bidel-arc.H G M a'* = *H* (**is** *?t1*)
and *rev-HM*: *bidel-arc.HM G M a'* = *HM* (**is** *?t2*)
proof −
interpret *rev*: *bidel-arc G M a'* **using** *a-in* **by** *unfold-locales simp*
show *?t1*
by (*rule pre-digraph.equality*) (*auto simp*: *rev.verts-H verts-H rev.arcs-H arcs-H rev.tail-H tail-H rev.head-H head-H*)
show *?t2* **using** *G.edge-succ-permutes*
by (*intro pre-map.equality*) (*auto simp*: *HM-def rev.HM-def insert-commute*

perm-rem-commutes permutes-conv-has-dom)
qed

end

sublocale *bidel-arc* $\subseteq$ *H*: *digraph-map H HM* **by** (*rule digraph-map*)

context *bidel-arc* **begin**

lemma *a-neq-a'*: $a \neq a'$
by (*metis G.arev-neq a-in*)

lemma
arcs-G: *arcs G* = *insert a* (*insert a'* (*arcs H*)) **and**
arcs-not-in: $\{a,a'\} \cap$ *arcs H* = {}
using *arcs-in* **by** (*auto simp*: *arcs-H*)

lemma *card-arcs-da*: *card* (*arcs G*) = *2* + *card* (*arcs H*)
using *arcs-G arcs-not-in a-neq-a'* **by** (*auto simp*: *card-insert-if*)

lemma *cas-da*: *H.cas* = *G.cas*
proof −
{ fix *u p v* **have** *H.cas u p v* = *G.cas u p v*
by (*induct p arbitrary*: *u*) (*simp-all add*: *tail-H head-H*)
} then show *?thesis* **by** (*simp add*: *fun-eq-iff*)
qed

lemma *reachable-daD*:
assumes $v \rightarrow^*{}_H w$ **shows** $v \rightarrow^*{}_G w$
apply (*rule G.reachable-del-arcD*)
apply (*rule wf-digraph.reachable-del-arcD*)
apply (*rule G.wf-digraph-del-arc*)
using *assms* **unfolding** *H-def* **by** *assumption*

lemma *not-G-isolated-a*: {*tail G a*, *head G a*} $\cap$ *G.isolated-verts* = {}
using *a-in G.in-arcs-eq*[*of head G a*] **by** (*auto simp*: *G.isolated-verts-def*)

lemma *isolated-other-da*:
assumes $u \notin$ {*tail G a*, *head G a*} **shows** $u \in$ *H.isolated-verts* $\longleftrightarrow$ $u \in$ *G.isolated-verts*
using *assms* **by** (*auto simp*: *pre-digraph.isolated-verts-def verts-H arcs-H tail-H out-arcs-def*)

lemma *isolated-da-pre*: *H.isolated-verts* = *G.isolated-verts* $\cup$
(*if tail G a* $\in$ *H.isolated-verts then* {*tail G a*} *else* {}) $\cup$
(*if head G a* $\in$ *H.isolated-verts then* {*head G a*} *else* {}) (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
fix *x* **assume** $x \in$ *?L* **then show** $x \in$ *?R*
by (*cases* $x \in$ {*tail G a*, *head G a*}) (*auto simp*:*isolated-other-da*)

next
  fix x **assume** $x \in$ *?R* **then show** $x \in$ *?L* **using** *not-G-isolated-a*
   by (*cases* $x \in \{$*tail G a*, *head G a*$\}$) (*auto simp*:*isolated-other-da split*: *if-splits*)
qed

lemma *card-isolated-verts-da0*:
  card H.isolated-verts = *card G.isolated-verts* + *card* ($\{$*tail G a*, *head G a*$\} \cap$ *H.isolated-verts*)
   using *not-G-isolated-a* **by** (*subst isolated-da-pre*) (*auto simp*: *card-insert-if G.finite-isolated-verts*)

lemma *segments-neq*:
  assumes *segment G.face-cycle-succ* a' $a \neq \{\} \vee$ *segment G.face-cycle-succ* a a' $\neq \{\}$
  shows *segment G.face-cycle-succ* a $a' \neq$ *segment G.face-cycle-succ* a' a
proof −
  have *bij-fcs*: *bij G.face-cycle-succ*
    using *G.face-cycle-succ-permutes* **by** (*auto simp*: *permutes-conv-has-dom*)
  show *?thesis* **using** *segment-disj*[*OF a-neq-a′ bij-fcs*] *assms* **by** *auto*
qed

lemma *H-fcs-eq-G-fcs*:
  assumes $b \in$ *arcs G* $\{b$,*G.face-cycle-succ* $b\} \cap \{a,a'\} = \{\}$
  shows *H.face-cycle-succ* b = *G.face-cycle-succ* b
proof −
  have *edge-rev M* $b \notin \{a,a'\}$
    using *assms* **by** *auto* (*metis G.arev-arev*)
  then show *?thesis*
    using *assms* **unfolding** *G.face-cycle-succ-def H.face-cycle-succ-def*
    by (*auto simp*: *HM-def perm-restrict-simps perm-rem-simps G.bij-edge-succ*)
qed

lemma *face-cycle-set-other-da*:
  assumes $\{a,\ a'\} \cap$ *G.face-cycle-set* $b = \{\}$ $b \in$ *arcs G*
  shows *H.face-cycle-set* b = *G.face-cycle-set* b
proof −
  have $\bigwedge s.\ s \in$ *G.face-cycle-set* $b \Longrightarrow b \in$ *arcs G* $\Longrightarrow a \notin$ *G.face-cycle-set* $b \Longrightarrow$ $a' \notin$ *G.face-cycle-set* b
      $\Longrightarrow$ *pre-digraph-map.face-cycle-succ HM* s = *G.face-cycle-succ* s
   by (*subst H-fcs-eq-G-fcs*) (*auto simp*: *G.in-face-cycle-setD G.face-cycle-succ-inI*)
  then show *?thesis*
    using *assms* **unfolding** *pre-digraph-map.face-cycle-set-def*
   by (*intro orbit-cong*) (*auto simp add*: *pre-digraph-map.face-cycle-set-def*[*symmetric*])
qed

lemma *in-face-cycle-set-other*:
  assumes $S \in$ *G.face-cycle-sets* $\{a,\ a'\} \cap S = \{\}$
  shows $S \in$ *H.face-cycle-sets*
proof −

from *assms* **obtain** b **where** $S = G.face\text{-}cycle\text{-}set\ b\ b \in arcs\ G$
by (*auto simp*: *G.face-cycle-sets-def*)
with *assms* **have** $S = H.face\text{-}cycle\text{-}set\ b$ **by** (*simp add*: *face-cycle-set-other-da*)
moreover
with *assms* **have** $b \in arcs\ H$ **using** ‹$b \in arcs\ G$› **by** (*auto simp*: *arcs-H*)
ultimately show *?thesis* **by** (*auto simp*: *H.face-cycle-sets-def*)
qed

lemma *H-fcs-in-G-fcs*:
assumes $b \in arcs\ H - (G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a')$
shows $H.face\text{-}cycle\text{-}set\ b \in G.face\text{-}cycle\text{-}sets - \{G.face\text{-}cycle\text{-}set\ a,\ G.face\text{-}cycle\text{-}set\ a'\}$
proof −
have $H.face\text{-}cycle\text{-}set\ b = G.face\text{-}cycle\text{-}set\ b$
using *assms* **by** (*intro face-cycle-set-other-da*) (*auto simp*: *arcs-H G.face-cycle-eq*)
moreover have $G.face\text{-}cycle\text{-}set\ b \notin \{G.face\text{-}cycle\text{-}set\ a,\ G.face\text{-}cycle\text{-}set\ a'\}$ $b \in arcs\ G$
using *G.face-cycle-eq assms* **by** (*auto simp*: *arcs-H*)
ultimately show *?thesis* **by** (*auto simp*: *G.face-cycle-sets-def*)
qed

lemma *face-cycle-sets-da0*:
$H.face\text{-}cycle\text{-}sets = G.face\text{-}cycle\text{-}sets - \{G.face\text{-}cycle\text{-}set\ a,\ G.face\text{-}cycle\text{-}set\ a'\}$
$\cup\ H.face\text{-}cycle\text{-}set$ ‘ $((G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a') - \{a,a'\})$ (**is** $?L = ?R$)
proof (*intro set-eqI iffI*)
fix S **assume** $S \in ?L$
then obtain b **where** $S = H.face\text{-}cycle\text{-}set\ b\ b \in arcs\ H$ **by** (*auto simp*: *H.face-cycle-sets-def*)
then show $S \in ?R$
using *arcs-not-in H-fcs-in-G-fcs* **by** (*cases* $b \in G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a'$) *auto*
next
fix S **assume** $S \in ?R$
show $S \in ?L$
proof (*cases* $S \in G.face\text{-}cycle\text{-}sets - \{G.face\text{-}cycle\text{-}set\ a,\ G.face\text{-}cycle\text{-}set\ a'\}$)
case *True*
then have $S \cap \{a,a'\} = \{\}$ **using** *G.face-cycle-set-parts* **by** (*auto simp*: *G.face-cycle-sets-def*)
with *True* **show** *?thesis* **by** (*intro in-face-cycle-set-other*) *auto*
next
case *False*
then have $S \in H.face\text{-}cycle\text{-}set$ ‘ $((G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a') - \{a,a'\})$
using ‹$S \in ?R$› **by** *blast*
moreover have $(G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a') - \{a,a'\} \subseteq arcs\ H$
using *a-in* **by** (*auto simp*: *arcs-H dest*: *G.in-face-cycle-setD*)
ultimately show *?thesis* **by** (*auto simp*: *H.face-cycle-sets-def*)
qed

qed

lemma *card-fcs-aa′-le*: *card* $\{$*G.face-cycle-set a*, *G.face-cycle-set a′*$\}$ $\leq$ *card G.face-cycle-sets*
using *a-in* **by** (*intro card-mono*) (*auto simp*: *G.face-cycle-sets-def*)

lemma *card-face-cycle-sets-da0*:
card H.face-cycle-sets = *card G.face-cycle-sets* − *card* $\{$*G.face-cycle-set a*, *G.face-cycle-set a′*$\}$
+ *card* (*H.face-cycle-set* ‘ ((*G.face-cycle-set a* $\cup$ *G.face-cycle-set a′*) − $\{$*a*,*a′*$\}$))
proof −
have *face-cycle-sets-inter*:
(*G.face-cycle-sets* − $\{$*G.face-cycle-set a*, *G.face-cycle-set a′*$\}$) $\cap$ *H.face-cycle-set* ‘ ((*G.face-cycle-set a* $\cup$ *G.face-cycle-set a′*) − $\{$*a*, *a′*$\}$) = $\{\}$ (**is** *?L* $\cap$ *?R* = -)
proof −
define *L R P*
where *L* = *?L* **and** *R* = *?R* **and** *P x* $\longleftrightarrow$ *x* $\cap$ (*G.face-cycle-set a* $\cup$ *G.face-cycle-set a′*) = $\{\}$
for *x*
then have $\bigwedge x.\ x \in L \Longrightarrow P\ x$ $\bigwedge x.\ x \in R \Longrightarrow \neg P\ x$
using *G.face-cycle-set-parts* **by** (*auto simp*: *G.face-cycle-sets-def*)
then have $L \cap R = \{\}$ **by** *blast*
then show *?thesis* **unfolding** *L-def R-def* **.**
qed
then show *?thesis* **using** *arcs-G*
by (*simp add*: *card-Diff-subset*[*symmetric*] *card-Un-disjoint*[*symmetric*]
G.in-face-cycle-sets face-cycle-sets-da0)
qed

end

locale *bidel-arc-same-face* = *bidel-arc* +
assumes *same-face*: *G.face-cycle-set a′* = *G.face-cycle-set a*
begin
lemma *a-in-o*: *a* $\in$ *orbit G.face-cycle-succ a′*
unfolding *G.face-cycle-set-def*[*symmetric*] **by** (*simp add*: *same-face*)

lemma *segment-a′-a-in*: *segment G.face-cycle-succ a′ a* $\subseteq$ *arcs H* (**is** *?seg* $\subseteq$ -)
proof −
have *?seg* $\subseteq$ *G.face-cycle-set a′* **by** (*auto simp*: *G.face-cycle-set-def segmentD-orbit*)
moreover have *G.face-cycle-set a′* $\subseteq$ *arcs G* **by** (*auto simp*: *G.face-cycle-set-altdef a-in*)
ultimately show *?thesis* **using** *a-in-o* **by** (*auto simp*: *arcs-H a-in not-in-segment1 not-in-segment2*)
qed

lemma *segment-a′-a-neD*:
assumes *segment G.face-cycle-succ a′ a* $\neq$ $\{\}$

shows *segment G.face-cycle-succ a′ a* $\in$ *H.face-cycle-sets* (**is** *?seg* $\in$ -)
proof –
let *?b* = *G.face-cycle-succ a′*

have *fcs-a-neq-a′*: *G.face-cycle-succ a′* $\neq$ *a* **by** (*metis assms segment1-empty*)

have *in-aG*: $\bigwedge$*x*. *x* $\in$ *segment G.face-cycle-succ a′ a* $\Longrightarrow$ *x* $\in$ *arcs G* – {*a*,*a′*}
using *not-in-segment1 not-in-segment2 segment-a′-a-in* **by** (*auto simp*: *arcs-H*)

{ **fix** *x* **assume** *A*: *x* $\in$ *segment G.face-cycle-succ a′ a* **and** *B*: *G.face-cycle-succ x* $\neq$ *a*
from *A* **have** *G.face-cycle-succ x* $\neq$ *a′*
proof *induct*
case *base* **then show** *?case*
by (*metis a-neq-a′ G.face-cycle-set-self not-in-segment1 G.face-cycle-set-def same-face segment.intros*)
next
case *step* **then show** *?case* **by** (*metis a-in-o a-neq-a′ not-in-segment1 segment.step*)
qed
with *A B* **have** {*x*, *G.face-cycle-succ x*} $\cap$ {*a*, *a′*} = {}
using *not-in-segment1*[*OF a-in-o*] *not-in-segment2*[*of a G.face-cycle-succ a′*] **by** *safe*
with *in-aG* **have** *H.face-cycle-succ x* = *G.face-cycle-succ x* **by** (*intro H-fcs-eq-G-fcs*) (*auto intro*: *A*)
} **note** *fcs-x-eq* = *this*

{ **fix** *x* **assume** *A*: *x* $\in$ *segment G.face-cycle-succ a′ a* **and** *B*: *G.face-cycle-succ x* = *a*
have *G.face-cycle-succ a* $\neq$ *a* **using** *B in-aG*[*OF A*] *G.bij-face-cycle-succ* **by** (*auto simp*: *bij-eq-iff*)
then have *edge-succ M a* $\neq$ *edge-rev M a*
by (*metis a-in-o G.arev-arev comp-apply G.face-cycle-succ-def not-in-segment1 segment.base*)
then have *H.face-cycle-succ x* = *G.face-cycle-succ a′*
using *in-aG*[*OF A*] *B G.bij-edge-succ* **unfolding** *H.face-cycle-succ-def G.face-cycle-succ-def*
by (*auto simp*: *HM-def perm-restrict-simps perm-rem-conv G.arev-eq-iff*)
} **note** *fcs-last-x-eq* = *this*

have *segment G.face-cycle-succ a′ a* = *H.face-cycle-set ?b*
proof (*intro set-eqI iffI*)
fix *x* **assume** *x* $\in$ *segment G.face-cycle-succ a′ a*
then show *x* $\in$ *H.face-cycle-set ?b*
proof *induct*
case *base* **then show** *?case* **by** *auto*
next
case (*step x*) **then show** *?case* **by** (*subst fcs-x-eq*[*symmetric*]) (*auto simp*:

H.face-cycle-succ-inI)
      qed
    next
      fix *x* **assume** *A*: $x \in$ *H.face-cycle-set ?b*
      then show $x \in$ *segment G.face-cycle-succ a′ a*
      proof *induct*
        case *base* **then show** *?case* **by** (*intro segment.base fcs-a-neq-a′*)
      next
        case (*step x*) **then show** *?case* **using** *fcs-a-neq-a′*
          by (*cases G.face-cycle-succ x = a*) (*auto simp*: *fcs-last-x-eq fcs-x-eq intro*: *segment.intros*)
      qed
    qed
   then show *?thesis* **using** *segment-a′-a-in* **by** (*auto simp*: *H.face-cycle-sets-def*)
  qed

  lemma *segment-a-a′-neD*:
    assumes *segment G.face-cycle-succ a a′* $\neq$ {}
    shows *segment G.face-cycle-succ a a′* $\in$ *H.face-cycle-sets*
  proof −
    interpret *rev*: *bidel-arc-same-face G M a′*
      using *a-in same-face* **by** *unfold-locales simp-all*
     from *assms* **show** *?thesis* **using** *rev.segment-a′-a-neD* **by** (*simp add*: *rev-H rev-HM*)
  qed

  lemma *H-fcs-full*:
    assumes *SS* $\subseteq$ *H.face-cycle-sets* **shows** *H.face-cycle-set* ‘ ($\bigcup$ *SS*) = *SS*
  proof −

    { **fix** *x* **assume** $x \in \bigcup$ *SS*
      then obtain *S* **where** $S \in SS$ $x \in S$ $S \in$ *H.face-cycle-sets* **using** *assms* **by** *auto*
      then have *H.face-cycle-set x* = *S*
        using *H.face-cycle-set-parts* **by** (*auto simp*: *H.face-cycle-sets-def*)
      then have *H.face-cycle-set x* $\in$ *SS* **using** ‹$S \in SS$› **by** *auto*
    }
    moreover
    { **fix** *S* **assume** $S \in SS$
      then obtain *x* **where** $x \in$ *arcs H* *S* = *H.face-cycle-set x* $x \in S$
        using *assms* **by** (*auto simp*: *H.face-cycle-sets-def*)
      then have $S \in$ *H.face-cycle-set* ‘ $\bigcup$ *SS*
        using ‹$S \in SS$› **by** *auto*
    }
    ultimately show *?thesis* **by** *auto*
  qed

  lemma *card-fcs-gt-0*: *0* < *card G.face-cycle-sets*
    using *a-in* **by** (*auto simp*: *card-gt-0-iff dest*: *G.in-face-cycle-sets*)

lemma *card-face-cycle-sets-da'*:
card H.face-cycle-sets = *card G.face-cycle-sets* − *1*
+ *card* ({*segment G.face-cycle-succ a a'*, *segment G.face-cycle-succ a' a*, {}}
− {{}})
proof −
have *G.face-cycle-set a*
= {*a*,*a'*} ∪ *segment G.face-cycle-succ a a'* ∪ *segment G.face-cycle-succ a' a*
using *a-neq-a' same-face* **by** (*intro cyclic-split-segment*) (*auto simp*: *G.face-cycle-succ-cyclic*)
then have ∗: *G.face-cycle-set a* ∪ *G.face-cycle-set a'* − {*a*, *a'*} = *segment G.face-cycle-succ a a'* ∪ *segment G.face-cycle-succ a' a*
by (*auto simp*: *same-face G.face-cycle-set-def*[*symmetric*] *not-in-segment1 not-in-segment2*)

have ∗∗: *H.face-cycle-set* ‘ (*G.face-cycle-set a* ∪ *G.face-cycle-set a'* − {*a*, *a'*})
= (*if segment G.face-cycle-succ a a'* ≠ {} *then* {*segment G.face-cycle-succ a a'*} *else* {})
∪ (*if segment G.face-cycle-succ a' a* ≠ {} *then* {*segment G.face-cycle-succ a' a*} *else* {})
unfolding ∗
using *H-fcs-full*[*of* {*segment G.face-cycle-succ a a'*, *segment G.face-cycle-succ a' a*}]
using *H-fcs-full*[*of* {*segment G.face-cycle-succ a a'*}]
using *H-fcs-full*[*of* {*segment G.face-cycle-succ a' a*}]
by (*auto simp add*: *segment-a-a'-neD segment-a'-a-neD*)
show *?thesis*
unfolding *card-face-cycle-sets-da0* ∗∗ **by** (*simp add*: *same-face card-insert-if*)
qed

end

locale *bidel-arc-diff-face* = *bidel-arc* +
assumes *diff-face*: *G.face-cycle-set a'* ≠ *G.face-cycle-set a*
begin

definition *S* :: *'b set* **where**
S ≡ *segment G.face-cycle-succ a a* ∪ *segment G.face-cycle-succ a' a'*

lemma *diff-face-not-in*: *a* ∉ *G.face-cycle-set a' a'* ∉ *G.face-cycle-set a*
using *diff-face G.face-cycle-eq* **by** *auto*

lemma *H-fcs-eq-for-a*:
assumes *b* ∈ *arcs H* ∩ *G.face-cycle-set a*
shows *H.face-cycle-set b* = *S* (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
fix *c* **assume** *c* ∈ *?L*
then have *c* ∈ *arcs H* **using** *assms* **by** (*auto dest*: *H.in-face-cycle-setD*)
moreover
have *c* ∈ *G.face-cycle-set a* ∪ *G.face-cycle-set a'*

proof (*rule ccontr*)
assume *A*: ¬*?thesis*
then have *G.face-cycle-set c* ∩ (*G.face-cycle-set a* ∪ *G.face-cycle-set a*′) = {}
using *G.face-cycle-set-parts* **by** (*auto simp*: *arcs-H*)
also then have *G.face-cycle-set c* = *H.face-cycle-set c*
using ‹*c* ∈ *arcs H*› **by** (*subst face-cycle-set-other-da*) (*auto simp*: *arcs-H*)
also have ... = *H.face-cycle-set b*
using ‹*c* ∈ *?L*› **using** *H.face-cycle-set-parts* **by** *auto*
finally show *False* **using** *assms* **by** *auto*
qed
ultimately show *c* ∈ *?R* **unfolding** *S-def arcs-H G.segment-face-cycle-x-x-eq* **by** *auto*

next
fix *x* **assume** *x* ∈ *?R*

from *assms* **have** *a* ≠ *b* **by** (*auto simp*: *arcs-H*)
from *assms* **have** *b-in*: *b* ∈ *segment G.face-cycle-succ a a*
using *G.segment-face-cycle-x-x-eq* **by** (*auto simp*: *arcs-H*)
have *fcs-a-neq-a*: *G.face-cycle-succ a* ≠ *a*
using *assms* ‹*a* ≠ *b*› **by** (*auto simp add*: *G.segment-face-cycle-x-x-eq G.fcs-x-eq-x*)

have *split-seg*: *segment G.face-cycle-succ a a* = *segment G.face-cycle-succ a b* ∪ {*b*}
∪ *segment G.face-cycle-succ b a*
using *b-in* **by** (*intro segment-split*)

have *a-in-orb-a*: *a* ∈ *orbit G.face-cycle-succ a* **by** (*simp add*: *G.face-cycle-set-def*[*symmetric*])

define *c* **where** *c* = *inv G.face-cycle-succ a*
have *c-succ*: *G.face-cycle-succ c* = *a* **unfolding** *c-def*
by (*meson bij-inv-eq-iff permutation-bijective G.permutation-face-cycle-succ*)
have *c-in-aa*: *c* ∈ *segment G.face-cycle-succ a a*
unfolding *G.segment-face-cycle-x-x-eq c-def* **using** *fcs-a-neq-a c-succ c-def* **by** *force*
have *c-in*: *c* ∈ {*b*} ∪ *segment G.face-cycle-succ b a*
using *split-seg b-in c-succ c-in-aa*
by (*auto dest*: *not-in-segment1*[*OF segmentD-orbit*] *intro*: *segment.intros*)
from *c-in-aa* **have** *c* ∈ *arcs H* **unfolding** *G.segment-face-cycle-x-x-eq*
using *arcs-in c-succ diff-face* **by** (*auto simp*: *arcs-H G.face-cycle-eq*[*of a*′])

have *b-in-L*: *b* ∈ *?L* **by** *auto*
moreover
{ fix *x* **assume** *x* ∈ *segment G.face-cycle-succ b a* **then have** *x* ∈ *?L*
proof *induct*
case *base* **then show** *?case*
using *assms diff-face-not-in*(*2*) **by** (*subst H-fcs-eq-G-fcs*[*symmetric*])
(*auto simp*: *arcs-H intro*: *H.face-cycle-succ-inI G.face-cycle-succ-inI*)

next
case (*step x*)
have *G.face-cycle-succ x* $\notin$ *G.face-cycle-set a* $\Longrightarrow$ *b* $\in$ *G.face-cycle-set a* $\Longrightarrow$ *False*
using *step*(*1*) **by** (*metis G.face-cycle-eq G.face-cycle-succ-inI pre-digraph-map.face-cycle-set-def segmentD-orbit*)
moreover
have *x* $\in$ *arcs G*
using *step assms H.in-face-cycle-setD arcs-H* **by** *auto*
moreover
then have (*G.face-cycle-succ x* $\notin$ *G.face-cycle-set a* $\Longrightarrow$ *b* $\in$ *G.face-cycle-set a* $\Longrightarrow$ *False*) $\Longrightarrow$ $\{x,$ *G.face-cycle-succ x*$\} \cap \{a, a'\} = \{\}$
using *step*(*2*,*3*) *assms diff-face-not-in*(*2*) *H.in-face-cycle-setD arcs-H* **by** *safe auto*
ultimately show *?case* **using** *step*
by (*subst H-fcs-eq-G-fcs*[*symmetric*]) (*auto intro*: *H.face-cycle-succ-inI*)
qed
} **note** *sba-in-L* = *this*
moreover
{ **fix** *x* **assume** *A*: *x* $\in$ *segment G.face-cycle-succ a' a'* **then have** *x* $\in$ *?L*
proof −
from *c-in* **have** *c* $\in$ *?L* **using** *b-in-L sba-in-L* **by** *blast*

have *G.face-cycle-succ a'* $\neq$ *a'*
using *A* **by** (*auto simp add*: *G.segment-face-cycle-x-x-eq G.fcs-x-eq-x*)
then have $*$: *G.face-cycle-succ a'* = *H.face-cycle-succ c*
using *a-neq-a' c-succ* ‹*c* $\in$ *arcs H*› **unfolding** *G.face-cycle-succ-def H.face-cycle-succ-def arcs-H*
by (*auto simp*: *HM-def perm-restrict-simps perm-rem-conv G.bij-edge-succ G.arev-eq-iff*)

from *A* **have** *x* $\in$ *H.face-cycle-set c*
proof *induct*
case *base* **then show** *?case* **by** (*simp add*: $*$ *H.face-cycle-succ-inI*)
next
case (*step x*)
have *x* $\in$ *arcs G*
using‹*c* $\in$ *arcs H*› *step.hyps*(*2*) **by** (*auto simp*: *arcs-H dest*: *H.in-face-cycle-setD*)
moreover
have *G.face-cycle-succ x* $\neq$ *a'* $\Longrightarrow$ $\{x,$ *G.face-cycle-succ x*$\} \cap \{a, a'\} = \{\}$
using *step*(*1*) *diff-face-not-in*(*1*) *G.face-cycle-succ-inI G.segment-face-cycle-x-x-eq*
by (*auto simp*: *not-in-segment2*)
ultimately
show *?case* **using** *step* **by** (*subst H-fcs-eq-G-fcs*[*symmetric*]) (*auto intro*: *H.face-cycle-succ-inI*)
qed
also have *H.face-cycle-set c* = *?L*
using ‹*c* $\in$ *?L*› *H.face-cycle-set-parts* **by** *auto*

finally show *?thesis* **.**
qed
} **note** *sa′a′-in-L* = *this*
moreover
{ **assume** *A*: $x \in$ *segment G.face-cycle-succ a b*

obtain *d* **where** $d \in$ *?L* **and** *d-succ*: *H.face-cycle-succ d* = *G.face-cycle-succ a*
proof (*cases G.face-cycle-succ a′* = *a′*)
case *True*
from *c-in* **have** $c \in$ *?L* **using** *b-in-L sba-in-L* **by** *blast*
moreover
have *H.face-cycle-succ c* = *G.face-cycle-succ a*
using *fcs-a-neq-a c-succ a-neq-a′ True* ‹$c \in$ *arcs H*›
unfolding *G.face-cycle-succ-def H.face-cycle-succ-def arcs-H*
by (*auto simp*: *HM-def perm-restrict-simps arcs-H perm-rem-conv G.bij-edge-succ G.arev-eq-iff*)
ultimately show *?thesis* **..**
next
case *False*

define *d* **where** *d* = *inv G.face-cycle-succ a′*
have *d-succ*: *G.face-cycle-succ d* = *a′* **unfolding** *d-def*
by (*meson bij-inv-eq-iff permutation-bijective G.permutation-face-cycle-succ*)
have ∗: $d \in$ *?L*
using *sa′a′-in-L False*
by (*metis DiffI d-succ empty-iff G.face-cycle-set-self G.face-cycle-set-succ insert-iff G.permutation-face-cycle-succ pre-digraph-map.face-cycle-set-def segment-x-x-eq*)
then have $d \in$ *arcs H* **using** *assms* **by** (*auto dest*: *H.in-face-cycle-setD*)
have *H.face-cycle-succ d* = *G.face-cycle-succ a*
using *fcs-a-neq-a a-neq-a′* ‹$d \in$ *arcs H*› *d-succ*
unfolding *G.face-cycle-succ-def H.face-cycle-succ-def arcs-H*
by (*auto simp*: *HM-def perm-restrict-simps arcs-H perm-rem-conv G.bij-edge-succ G.arev-eq-iff*)
with ∗ **show** *?thesis* **..**
qed
then have $d \in$ *arcs H* **using** *assms*
by − (*drule H.in-face-cycle-setD*, *auto*)

from *A* **have** $x \in$ *H.face-cycle-set d*
proof *induct*
case *base* **then show** *?case* **by** (*simp add*: *d-succ*[*symmetric*] *H.face-cycle-succ-inI*)
next
case (*step x*)
moreover
have $x \in$ *arcs G*
using ‹$d \in$ *arcs H*› *arcs-H digraph-map.in-face-cycle-setD step.hyps*(*2*) **by** *fastforce*
moreover

have $\{x,\ G.face\text{-}cycle\text{-}succ\ x\} \cap \{a,\ a'\} = \{\}$
proof −
have $a \neq x$ **using** *step*(*2*) *H.in-face-cycle-setD* ‹$d \in arcs\ H$› *arcs-not-in* **by** *blast*
moreover
have $a \neq G.face\text{-}cycle\text{-}succ\ x$
by (*metis b-in not-in-segment1 segment.step segmentD-orbit step*(*1*))
moreover
have $a' \neq x$ $a' \neq G.face\text{-}cycle\text{-}succ\ x$
using *step*(*1*) *diff-face-not-in*(*2*) **by** (*auto simp*: *G.face-cycle-set-def dest*!: *segmentD-orbit intro*: *orbit.step*)
ultimately
show *?thesis* **by** *auto*
qed
ultimately
show *?case* **using** *step*
by (*subst H-fcs-eq-G-fcs*[*symmetric*]) (*auto intro*: *H.face-cycle-succ-inI*)
qed
also have $H.face\text{-}cycle\text{-}set\ d = ?L$
using ‹$d \in ?L$› *H.face-cycle-set-parts* **by** *auto*
finally have $x \in ?L$ **.**
}
ultimately show $x \in ?L$
using ‹$x \in ?R$› **unfolding** *S-def split-seg* **by** *blast*
qed

lemma *HJ-fcs-eq-for-a*′:
assumes $b \in arcs\ H \cap G.face\text{-}cycle\text{-}set\ a'$
shows $H.face\text{-}cycle\text{-}set\ b = S$
proof −
interpret *rev*: *bidel-arc-diff-face G M a*′
using *arcs-in diff-face* **by** *unfold-locales simp-all*
show *?thesis* **using** *rev.H-fcs-eq-for-a assms* **by** (*auto simp*: *rev-H rev-HM S-def rev.S-def*)
qed

lemma *card-face-cycle-sets-da*′:
$card\ H.face\text{-}cycle\text{-}sets = card\ G.face\text{-}cycle\text{-}sets - card\ \{G.face\text{-}cycle\text{-}set\ a,\ G.face\text{-}cycle\text{-}set\ a'\} + (if\ S = \{\}\ then\ 0\ else\ 1)$
proof −
have $S = G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a' - \{a,\ a'\}$
unfolding *S-def* **using** *diff-face-not-in*
by (*auto simp*: *segment-x-x-eq G.permutation-face-cycle-succ G.face-cycle-set-def*)
moreover
{ fix *x* **assume** $x \in S$
then have $x \in arcs\ H \cap (G.face\text{-}cycle\text{-}set\ a \cup G.face\text{-}cycle\text{-}set\ a' - \{a,\ a'\})$
unfolding ‹$S = \text{-}$› *arcs-H* **using** *a-in* **by** (*auto intro*: *G.in-face-cycle-setD*)
then have $H.face\text{-}cycle\text{-}set\ x = S$ **using** *H-fcs-eq-for-a HJ-fcs-eq-for-a*′ **by** *blast*

}
then have *H.face-cycle-set* ‘ $S = ($*if* $S = \{\}$ *then* $\{\}$ *else* $\{S\})$
by *auto*
ultimately show *?thesis* **by** (*simp add*: *card-face-cycle-sets-da0*)
qed

end

locale *bidel-arc-biconnected* = *bidel-arc* +
assumes *reach-a*: *tail G a* $\to^*{}_H$ *head G a*
begin

lemma *reach-a'*: *tail G a'* $\to^*{}_H$ *head G a'*
using *reach-a a-in* **by** (*simp add*: *symmetric-reachable H.sym-arcs*)

lemma
tail-a': *tail G a'* = *head G a* **and**
head-a': *head G a'* = *tail G a*
using *a-in* **by** *simp-all*

lemma *reachable-daI*:
assumes $v \to^*{}_G w$ **shows** $v \to^*{}_H w$
proof −
have ∗: $\bigwedge v\ w.\ v \to_G w \Longrightarrow v \to^*{}_H w$
using *reach-a reach-a'* **by** (*auto simp*: *arcs-ends-def ends-H arcs-G arc-to-ends-def tail-a'*)
show *?thesis* **using** *assms* **by** *induct* (*auto simp*: *verts-H intro*: ∗ *H.reachable-trans*)
qed

lemma *reachable-da*: $v \to^*{}_H w \longleftrightarrow v \to^*{}_G w$
by (*metis reachable-daD reachable-daI*)

lemma *sccs-verts-da*: *H.sccs-verts* = *G.sccs-verts*
by (*auto simp*: *G.in-sccs-verts-conv-reachable H.in-sccs-verts-conv-reachable reachable-da*)

lemma *card-sccs-da*: *card H.sccs* = *card G.sccs*
by (*simp add*: *G.card-sccs-verts*[*symmetric*] *H.card-sccs-verts*[*symmetric*] *sccs-verts-da*)

end

locale *bidel-arc-not-biconnected* = *bidel-arc* +
assumes *not-reach-a*: ¬*tail G a* $\to^*{}_H$ *head G a*
begin

lemma *H-awalkI*: *G.awalk u p v* $\Longrightarrow \{a,a'\} \cap$ *set p* $= \{\} \Longrightarrow$ *H.awalk u p v*
by (*auto simp*: *pre-digraph.apath-def pre-digraph.awalk-def verts-H arcs-H cas-da*)

lemma *tail-neq-head*: *tail G a* $\neq$ *head G a*
  using *not-reach-a a-in* **by** (*metis H.reachable-refl G.head-in-verts verts-H*)

lemma *scc-of-tail-neq-head*: *H.scc-of* (*tail G a*) $\neq$ *H.scc-of* (*head G a*)
proof −
  have *tail G a* $\in$ *H.scc-of* (*tail G a*) *head G a* $\in$ *H.scc-of* (*head G a*)
    using *ends-in* **by** (*auto simp*: *H.in-scc-of-self verts-H*)
  with *not-reach-a* **show** *?thesis* **by** (*auto simp*: *H.scc-of-def*)
qed

lemma *scc-of-G-tail*:
  assumes *u* $\in$ *G.scc-of* (*tail G a*)
  shows *H.scc-of u* = *H.scc-of* (*tail G a*) $\vee$ *H.scc-of u* = *H.scc-of* (*head G a*)
proof −
  from *assms* **have** $u \rightarrow^*{}_G$ *tail G a* **by** (*auto simp*: *G.scc-of-def*)
  then obtain *p* **where** *p*: *G.apath u p* (*tail G a*) **by** (*auto simp*: *G.reachable-apath*)

  show *?thesis*
  proof (*cases head G a* $\in$ *set* (*G.awalk-verts u p*))
    case *True*
    with *p* **obtain** *p′ q* **where** *p* = *p′* @ *q G.awalk* (*head G a*) *q* (*tail G a*)
        and *p′*: *G.awalk u p′* (*head G a*)
      unfolding *G.apath-def* **by** (*metis G.awalk-decomp*)
    moreover
    then have *tail G a* $\in$ *set* (*tl* (*G.awalk-verts* (*head G a*) *q*))
      using *tail-neq-head*
      apply (*cases q*)
      apply (*simp add*: *G.awalk-Nil-iff*)
      apply (*simp add*: *G.awalk-Cons-iff*)
      by (*metis G.awalkE G.awalk-verts-non-Nil last-in-set*)
    ultimately
    have *tail G a* $\notin$ *set* (*G.awalk-verts u p′*)
      using *G.apath-decomp-disjoint*[*OF p*, *of p′ q tail G a*] **by** *auto*
    with *p′* **have** {*a*,*a′*} $\cap$ *set p′* = {}
    by (*auto simp*: *G.set-awalk-verts G.apath-def*) (*metis a-in imageI G.head-arev*)
      with *p′* **show** *?thesis* **unfolding** *G.apath-def* **by** (*metis H.scc-ofI-awalk H.scc-of-eq H-awalkI*)
  next
    case *False*
    with *p* **have** {*a*,*a′*} $\cap$ *set p* = {}
    by (*auto simp*: *G.set-awalk-verts G.apath-def*) (*metis a-in imageI G.tail-arev*)
      with *p* **show** *?thesis* **unfolding** *G.apath-def* **by** (*metis H.scc-ofI-awalk H.scc-of-eq H-awalkI*)
  qed
qed

lemma *scc-of-other*:
  assumes *u* $\notin$ *G.scc-of* (*tail G a*)

shows $H.scc\text{-}of\ u = G.scc\text{-}of\ u$
using *assms*
proof (*intro set-eqI iffI*)
fix v **assume** $v \in H.scc\text{-}of\ u$ **then show** $v \in G.scc\text{-}of\ u$
by (*auto simp*: *H.scc-of-def G.scc-of-def intro*: *reachable-daD*)
next
fix v **assume** $v \in G.scc\text{-}of\ u$
then obtain p **where** p: *G.awalk u p v* **by** (*auto simp*: *G.scc-of-def G.reachable-awalk*)
moreover
have $\{a,a'\} \cap set\ p = \{\}$
proof −
have $\neg u \rightarrow^*{}_G\ tail\ G\ a$ **using** *assms* **by** (*metis G.scc-ofI-reachable*)
then have $\bigwedge p.\ \neg G.awalk\ u\ p\ (tail\ G\ a)$ **by** (*metis G.reachable-awalk*)
then have $tail\ G\ a \notin set\ (G.awalk\text{-}verts\ u\ p)$
using p **by** (*auto dest*: *G.awalk-decomp*)
with p **show** *?thesis*
by (*auto simp*: *G.set-awalk-verts G.apath-def*) (*metis a-in imageI G.head-arev*)
qed
ultimately have *H.awalk u p v* **by** (*rule H-awalkI*)
then show $v \in H.scc\text{-}of\ u$ **by** (*metis H.scc-ofI-reachable′ H.reachable-awalk*)
qed

lemma *scc-of-tail-inter*:
$tail\ G\ a \in G.scc\text{-}of\ (tail\ G\ a) \cap H.scc\text{-}of\ (tail\ G\ a)$
using *ends-in* **by** (*auto simp*: *G.in-scc-of-self H.in-scc-of-self verts-H*)

lemma *scc-of-head-inter*:
$head\ G\ a \in G.scc\text{-}of\ (tail\ G\ a) \cap H.scc\text{-}of\ (head\ G\ a)$
proof −
have $tail\ G\ a \rightarrow_G head\ G\ a$ $head\ G\ a \rightarrow_G tail\ G\ a$
by (*metis a-in G.in-arcs-imp-in-arcs-ends*) (*metis a-in G.graph-symmetric G.in-arcs-imp-in-arcs-ends*)
then have $tail\ G\ a \rightarrow^*{}_G\ head\ G\ a$ $head\ G\ a \rightarrow^*{}_G\ tail\ G\ a$ **by** *auto*
then show *?thesis* **using** *ends-in* **by** (*auto simp*: *G.scc-of-def H.in-scc-of-self verts-H*)
qed

lemma *G-scc-of-tail-not-in*: $G.scc\text{-}of\ (tail\ G\ a) \notin H.sccs\text{-}verts$
proof
assume A: $G.scc\text{-}of\ (tail\ G\ a) \in H.sccs\text{-}verts$
from A *scc-of-tail-inter* **have** $G.scc\text{-}of\ (tail\ G\ a) = H.scc\text{-}of\ (tail\ G\ a)$
by (*metis H.scc-of-in-sccs-verts H.sccs-verts-disjoint a-in empty-iff G.tail-in-verts verts-H*)
moreover
from A *scc-of-head-inter* **have** $G.scc\text{-}of\ (tail\ G\ a) = H.scc\text{-}of\ (head\ G\ a)$
by (*metis H.scc-of-in-sccs-verts H.sccs-verts-disjoint a-in empty-iff G.head-in-verts verts-H*)
ultimately show *False* **using** *scc-of-tail-neq-head* **by** *blast*
qed

lemma *H-scc-of-a-not-in*:
  H.scc-of (*tail G a*) ∉ *G.sccs-verts H.scc-of* (*head G a*) ∉ *G.sccs-verts*
proof *safe*
  assume *H.scc-of* (*tail G a*) ∈ *G.sccs-verts*
  with *scc-of-tail-inter* **have** *G.scc-of* (*tail G a*) = *H.scc-of* (*tail G a*)
  by (*metis G.scc-of-in-sccs-verts G.sccs-verts-disjoint a-in empty-iff G.tail-in-verts*)
  with *G-scc-of-tail-not-in* **show** *False*
    using *ends-in* **by** (*auto simp*: *H.scc-of-in-sccs-verts verts-H*)
next
  assume *H.scc-of* (*head G a*) ∈ *G.sccs-verts*
  with *scc-of-head-inter* **have** *G.scc-of* (*tail G a*) = *H.scc-of* (*head G a*)
  by (*metis G.scc-of-in-sccs-verts G.sccs-verts-disjoint a-in empty-iff G.tail-in-verts*)
  with *G-scc-of-tail-not-in* **show** *False*
    using *ends-in* **by** (*auto simp*: *H.scc-of-in-sccs-verts verts-H*)
qed

lemma *scc-verts-da*:
  H.sccs-verts = (*G.sccs-verts* − {*G.scc-of* (*tail G a*)}) ∪ {*H.scc-of* (*tail G a*), *H.scc-of* (*head G a*)} (**is** *?L* = *?R*)
proof (*intro set-eqI iffI*)
  fix *S* **assume** *S* ∈ *?L*
  then obtain *u* **where** *u* ∈ *verts G S* = *H.scc-of u* **by** (*auto simp*: *verts-H H.sccs-verts-conv-scc-of*)
  moreover
  then have *G.scc-of* (*tail G a*) ≠ *H.scc-of u* **using** ‹*S* ∈ *?L*› *G-scc-of-tail-not-in* **by** *auto*
  ultimately show *S* ∈ *?R*
    unfolding *G.sccs-verts-conv-scc-of*
    by (*cases u* ∈ *G.scc-of* (*tail G a*)) (*auto dest*: *scc-of-G-tail scc-of-other*)
next
  fix *S* **assume** *S* ∈ *?R*
  show *S* ∈ *?L*
  proof (*cases S* ∈ *G.sccs-verts*)
    case *True*
    with ‹*S* ∈ *?R*› **obtain** *u* **where** *u*: *u* ∈ *verts G S* = *G.scc-of u* **and** *S* ≠ *G.scc-of* (*tail G a*)
      using *H-scc-of-a-not-in* **by** (*auto simp*: *G.sccs-verts-conv-scc-of*)
    then have *G.scc-of u* ∩ *G.scc-of* (*tail G a*) = {}
    using *ends-in* **by** (*intro G.sccs-verts-disjoint*) (*auto simp*: *G.scc-of-in-sccs-verts*)
    then have *u* ∉ *G.scc-of* (*tail G a*)
      using *u* **by** (*auto dest*: *G.in-scc-of-self*)
    with *u* **show** *?thesis* **using** *scc-of-other*
      by (*auto simp*: *H.sccs-verts-conv-scc-of verts-H G.sccs-verts-conv-scc-of*)
  next
    case *False* **with** ‹*S* ∈ *?R*› *ends-in* **show** *?thesis* **by** (*auto simp*: *H.sccs-verts-conv-scc-of verts-H*)
  qed
qed

lemma *card-sccs-da*: *card H.sccs = Suc* (*card G.sccs*)
using *H-scc-of-a-not-in ends-in*
unfolding *G.card-sccs-verts*[*symmetric*] *H.card-sccs-verts*[*symmetric*] *scc-verts-da*
by (*simp add*: *card-insert-if G.finite-sccs-verts scc-of-tail-neq-head card-Suc-Diff1 G.scc-of-in-sccs-verts del*: *card-Diff-insert*)

end

sublocale *bidel-arc-not-biconnected* ⊆ *bidel-arc-same-face*
proof
note *a-in*
moreover from *a-in* **have** *head G a* ∈ *tail G* ‘ *G.face-cycle-set a*
by (*simp add*: *G.heads-face-cycle-set*[*symmetric*])
moreover have *tail G a* ∈ *tail G* ‘ *G.face-cycle-set a* **by** *simp*
ultimately obtain *p* **where** *p*: *G.trail* (*head G a*) *p* (*tail G a*) *set p* ⊆ *G.face-cycle-set a*
by (*rule G.obtain-trail-in-fcs′*)
define *p′* **where** *p′* = *G.awalk-to-apath p*
from *p* **have** *p′*: *G.apath* (*head G a*) *p′* (*tail G a*) *set p′* ⊆ *G.face-cycle-set a*
by (*auto simp*: *G.trail-def p′-def dest*: *G.apath-awalk-to-apath G.awalk-to-apath-subset*)
then have *set p′* ⊆ *arcs G*
using *a-in* **by** (*blast dest*: *G.in-face-cycle-setD*)

have ¬*set p′* ⊆ *arcs H*
proof
assume *set p′* ⊆ *arcs H*
then have *H.awalk* (*head G a*) *p′* (*tail G a*)
using *p′* **by** (*auto simp*: *G.apath-def arcs-H intro*: *H-awalkI*)
then show *False* **using** *not-reach-a* **by** (*metis H.symmetric-reachable′ H.reachable-awalk*)
qed
then have *set p′* ∩ {*a*,*a′*} ≠ {} **using** ‹*set p′* ⊆ *arcs G*› **by** (*auto simp*: *arcs-H*)
moreover
have *a* ∉ *set p′*
proof
assume *a* ∈ *set p′*
then have *head G a* ∈ *set* (*tl* (*G.awalk-verts* (*head G a*) *p′*))
using ‹*G.apath - p′ -*›
by (*cases p′*) (*auto simp*: *G.set-awalk-verts G.apath-def G.awalk-Cons-iff*, *metis imageI*)
moreover
have *head G a* ∉ *set* (*tl* (*G.awalk-verts* (*head G a*) *p′*))
using ‹*G.apath - p′ -*› **by** (*cases p′*) (*auto simp*: *G.apath-def*)
ultimately show *False* **by** *contradiction*
qed
ultimately
have *a′* ∈ *G.face-cycle-set a* **using** *p′*(*2*) **by** *auto*
then show *G.face-cycle-set a′* = *G.face-cycle-set a* **using** *G.face-cycle-set-parts* **by** *auto*

qed

locale *bidel-arc-tail-conn* = *bidel-arc* +
assumes *conn-tail*: *tail G a* $\notin$ *H.isolated-verts*

locale *bidel-arc-head-conn* = *bidel-arc* +
assumes *conn-head*: *head G a* $\notin$ *H.isolated-verts*

locale *bidel-arc-tail-isolated* = *bidel-arc* +
assumes *isolated-tail*: *tail G a* $\in$ *H.isolated-verts*

locale *bidel-arc-head-isolated* = *bidel-arc* +
assumes *isolated-head*: *head G a* $\in$ *H.isolated-verts*
begin

lemma *G-edge-succ-a′-no-loop*:
assumes *no-loop-a*: *head G a* $\neq$ *tail G a* **shows** *G-edge-succ-a′*: *edge-succ M a′* = *a′* (**is** *?t2*)
proof −
have ∗: *out-arcs G* (*tail G a′*) = {*a′*}
using *a-in isolated-head no-loop-a*
by (*auto simp*: *H.isolated-verts-def verts-H out-arcs-def arcs-H tail-H*)
obtain *edge-succ M a′* $\in$ {*a′*}
using *G.edge-succ-cyclic*[*of tail G a′*]
apply (*rule eq-on-cyclic-on-iff1* [**where** *x=a′*])
using ∗ *a-in a-neq-a′ no-loop-a* **by** *simp-all*
then show *?thesis* **by** *auto*
qed

lemma *G-face-cycle-succ-a-no-loop*:
assumes *no-loop-a*: *head G a* $\neq$ *tail G a* **shows** *G.face-cycle-succ a* = *a′*
using *assms* **by** (*auto simp*: *G.face-cycle-succ-def G-edge-succ-a′-no-loop*)

end

locale *bidel-arc-same-face-tail-conn* = *bidel-arc-same-face* + *bidel-arc-tail-conn*
begin

definition *a-neigh* :: *′b* **where**
a-neigh ≡ *SOME b. G.face-cycle-succ b* = *a*

lemma *face-cycle-succ-a-neigh*: *G.face-cycle-succ a-neigh* = *a*
proof −
have ∃ *b. G.face-cycle-succ b* = *a* **by** (*metis G.face-cycle-succ-pred*)
then show *?thesis* **unfolding** *a-neigh-def* **by** (*rule someI-ex*)

qed

lemma *a-neigh-in*: *a-neigh* ∈ *arcs G*
  using *a-in* **by** (*metis face-cycle-succ-a-neigh G.face-cycle-succ-closed*)

lemma *a-neigh-neq-a*: *a-neigh* ≠ *a*
proof
  assume *a-neigh* = *a*
  then have *G.face-cycle-set a* = {*a*} **using** *face-cycle-succ-a-neigh* **by** (*simp add*: *G.fcs-x-eq-x*)
  with *a-neq-a′ same-face G.face-cycle-set-self*[*of a′*] **show** *False* **by** *simp*
qed

lemma *a-neigh-neq-a′*: *a-neigh* ≠ *a′*
proof
  assume *A*: *a-neigh* = *a′*

  have *a-in-oa*: *a* ∈ *out-arcs G* (*tail G a*) **using** *a-in* **by** *auto*
  have *cyc*: *cyclic-on* (*edge-succ M*) (*out-arcs G* (*tail G a*))
    using *a-in* **by** (*intro G.edge-succ-cyclic*) *auto*

  from *A* **have** *G.face-cycle-succ a′* = *a* **by** (*metis face-cycle-succ-a-neigh*)
  then have *edge-succ M a* = *a* **by** (*auto simp*: *G.face-cycle-succ-def*)
  then have *card* (*out-arcs G* (*tail G a*)) = *1*
    using *cyc a-in* **by** (*auto elim*: *eq-on-cyclic-on-iff1*)
  then have *out-arcs G* (*tail G a*) = {*a*}
    using *a-in-oa* **by** (*auto simp del*: *in-out-arcs-conv dest*: *card-eq-SucD*)
  then show *False* **using** *conn-tail a-in*
    by (*auto simp*: *H.isolated-verts-def arcs-H tail-H verts-H out-arcs-def*)
qed

lemma *edge-rev-a-neigh-neq*: *edge-rev M a-neigh* ≠ *a′*
  by (*metis a-neigh-neq-a G.arev-arev*)

lemma *edge-succ-a-neq*: *edge-succ M a* ≠ *a′*
proof
  assume *edge-succ M a* = *a′*
  then have *G.face-cycle-set a′* = {*a′*}
    using *face-cycle-succ-a-neigh*
    by *auto* (*metis G.arev-arev-raw G.face-cycle-succ-def G.fcs-x-eq-x a-in comp-apply singletonD*)
  with *a-neq-a′ same-face G.face-cycle-set-self*[*of a*] **show** *False* **by** *auto*
qed

lemma *H-face-cycle-succ-a-neigh*: *H.face-cycle-succ a-neigh* = *G.face-cycle-succ a′*
  using *face-cycle-succ-a-neigh edge-succ-a-neq edge-rev-a-neigh-neq a-neigh-neq-a a-neigh-neq-a′ a-neigh-in*
  unfolding *H.face-cycle-succ-def G.face-cycle-succ-def*

by (*auto simp*: *HM-def perm-restrict-simps perm-rem-conv G.bij-edge-succ*)

lemma *H-fcs-a-neigh*: *H.face-cycle-set a-neigh* = *segment G.face-cycle-succ a′ a*
(**is** *?L* = *?R*)
proof −
{ **fix** *n* **assume** *A*: *0* < *n n* < *funpow-dist1 G.face-cycle-succ a′ a*
then have ∗: (*G.face-cycle-succ* ^^ *n*) *a′* ∈ *segment G.face-cycle-succ a′ a*
using *a-in-o* **by** (*auto simp*: *segment-altdef*)
then have (*G.face-cycle-succ* ^^ *n*) *a′* ∉ {*a*,*a′*} (*G.face-cycle-succ* ^^ *n*) *a′*
∈ *arcs G*
using *not-in-segment1*[*OF a-in-o*] *not-in-segment2*[*of a G.face-cycle-succ a′*]
by (*auto simp*: *segment-altdef a-in-o*)
} **note** *X* = *this*

{ **fix** *n* **assume** *0* < *n n* < *funpow-dist1 G.face-cycle-succ a′ a*
then have (*H.face-cycle-succ* ^^ *n*) *a-neigh* = (*G.face-cycle-succ* ^^ *n*) *a′*
proof (*induct n*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
show *?case*
proof (*cases n=0*)
case *True* **then show** *?thesis* **by** (*simp add*: *H-face-cycle-succ-a-neigh*)
next
case *False*
then have (*H.face-cycle-succ* ^^ *n*) *a-neigh* = (*G.face-cycle-succ* ^^ *n*) *a′*
using *Suc* **by** *simp*
then show *?thesis*
using *X*[*of Suc n*] *X*[*of n*] *False Suc* **by** (*simp add*: *H-fcs-eq-G-fcs*)
qed
qed
} **note** *Y* = *this*

have *fcs-a′-neq-a*: *G.face-cycle-succ a′* ≠ *a*
by (*metis* (*no-types*) *a-neigh-neq-a′ G.face-cycle-pred-succ face-cycle-succ-a-neigh*)

show *?thesis*
proof (*intro set-eqI iffI*)
fix *b* **assume** *b* ∈ *?L*

define *m* **where** *m* = *funpow-dist1 G.face-cycle-succ a′ a* − *1*

have *b-in0*: *b* ∈ *orbit H.face-cycle-succ* (*a-neigh*)
using ‹*b* ∈ *?L*› **by** (*simp add*: *H.face-cycle-set-def*[*symmetric*])

have *0* < *m*
by (*auto simp*: *m-def*) (*metis a-neigh-neq-a′ G.face-cycle-pred-succ G.face-cycle-set-def*
G.face-cycle-set-self G.face-cycle-set-succ face-cycle-succ-a-neigh fun-
pow-dist-0-eq neq0-conv

same-face)
then have *pos-dist*: $0 < funpow\text{-}dist1\ H.face\text{-}cycle\text{-}succ\ a\text{-}neigh\ b$
by (*simp add*: *m-def*)

have ∗: (*G.face-cycle-succ* ^^ *Suc m*) $a' = a$
using *a-in-o* **by** (*simp add*: *m-def funpow-dist1-prop del*: *funpow.simps*)
have (*H.face-cycle-succ* ^^ *m*) *a-neigh* = *a-neigh*
proof −
have *a* = *G.face-cycle-succ* ((*H.face-cycle-succ* ^^ *m*) *a-neigh*)
using ∗ ‹*0* < *m*› **by** (*simp add*: *Y m-def*)
then show *?thesis* **using** *face-cycle-succ-a-neigh* **by** (*metis G.face-cycle-pred-succ*)
qed
then have *funpow-dist1 H.face-cycle-succ a-neigh b* ≤ *m*
using ‹*0* < *m*› *b-in0* **by** (*intro funpow-dist1-le-self*) *simp-all*
also have ... < *funpow-dist1 G.face-cycle-succ* a' *a* **unfolding** *m-def* **by** *simp*
finally have *dist-less*: *funpow-dist1 H.face-cycle-succ a-neigh b*
< *funpow-dist1 G.face-cycle-succ* a' *a* **.**
have *b* = (*H.face-cycle-succ* ^^ *funpow-dist1 H.face-cycle-succ a-neigh b*) *a-neigh*
using *b-in0* **by** (*simp add*: *funpow-dist1-prop del*: *funpow.simps*)
also have ... = (*G.face-cycle-succ* ^^ *funpow-dist1 H.face-cycle-succ a-neigh b*) a'
using *pos-dist dist-less* **by** (*rule Y*)
also have ... ∈ *?R* **using** *pos-dist dist-less* **by** (*simp add*: *segment-altdef a-in-o del*: *funpow.simps*)
finally show *b* ∈ *?R* **.**
next
fix *b* **assume** *b* ∈ *?R*
then show *b* ∈ *?L*
using *Y*
by (*auto simp*: *segment-altdef a-in-o H.face-cycle-set-altdef Suc-le-eq*) *metis*
qed
qed

end

locale *bidel-arc-isolated-loop* =
bidel-arc-biconnected + *bidel-arc-tail-isolated*
begin

lemma *loop-a*[*simp*]: *head G a* = *tail G a*
using *isolated-tail reach-a* **by** (*auto simp*: *H.isolated-verts-def*
elim: *H.converse-reachable-cases dest*: *out-arcs-emptyD-dominates*)

end

sublocale *bidel-arc-isolated-loop* ⊆ *bidel-arc-head-isolated*
  using *isolated-tail loop-a* **by** *unfold-locales simp*

context *bidel-arc-isolated-loop* **begin**

The edges a and a' form a loop on an otherwise isolated vertex

lemma *card-isolated-verts-da*: *card H.isolated-verts* = *Suc* (*card G.isolated-verts*)
  by (*simp add*: *card-isolated-verts-da0 isolated-tail*)

lemma
  G-edge-succ-a[*simp*]: *edge-succ M a* = *a'* (**is** *?t1*) **and**
  G-edge-succ-a'[*simp*]: *edge-succ M a'* = *a* (**is** *?t2*)
proof −
  have ∗: *out-arcs G* (*tail G a*) = {*a*,*a'*}
    using *a-in isolated-tail*
    by (*auto simp*: *H.isolated-verts-def verts-H out-arcs-def arcs-H tail-H*)
  obtain *edge-succ M a'* ∈ {*a*,*a'*} *edge-succ M a'* ≠ *a'*
    using *G.edge-succ-cyclic*[*of tail G a'*]
    apply (*rule eq-on-cyclic-on-iff1* [**where** *x*=*a'*])
    using ∗ *a-in a-neq-a' loop-a* **by** *auto*
  moreover
  obtain *edge-succ M a* ∈ {*a*,*a'*} *edge-succ M a* ≠ *a*
    using *G.edge-succ-cyclic*[*of tail G a*]
    apply (*rule eq-on-cyclic-on-iff1* [**where** *x*=*a*])
    using ∗ *a-in a-neq-a' loop-a* **by** *auto*
  ultimately show *?t1 ?t2* **by** *auto*
qed

lemma
  G-face-cycle-succ-a[*simp*]: *G.face-cycle-succ a* = *a* **and**
  G-face-cycle-succ-a'[*simp*]: *G.face-cycle-succ a'* = *a'*
  by (*auto simp*: *G.face-cycle-succ-def*)

lemma
  G-face-cycle-set-a[*simp*]: *G.face-cycle-set a* = {*a*} **and**
  G-face-cycle-set-a'[*simp*]: *G.face-cycle-set a'* = {*a'*}
  unfolding *G.fcs-x-eq-x*[*symmetric*] **by** *simp-all*

end

sublocale *bidel-arc-isolated-loop* ⊆ *bidel-arc-diff-face*
  using *a-neq-a'* **by** *unfold-locales auto*

context *bidel-arc-isolated-loop* **begin**

lemma *card-face-cycle-sets-da*: *card G.face-cycle-sets* = *Suc* (*Suc* (*card H.face-cycle-sets*))
  unfolding *card-face-cycle-sets-da'* **using** *diff-face card-fcs-aa'-le*

by (*auto simp*: *card-insert-if S-def G.segment-face-cycle-x-x-eq*)

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *card-isolated-verts-da verts-H card-arcs-da card-face-cycle-sets-da card-sccs-da*)

end

locale *bidel-arc-two-isolated* =
bidel-arc-not-biconnected + *bidel-arc-tail-isolated* + *bidel-arc-head-isolated*
begin

tail G a and *head G a* form an SCC with a and a' as the only arcs.

lemma *no-loop-a*: *head G a* ≠ *tail G a*
using *not-reach-a a-in* **by** (*auto simp*: *verts-H*)

lemma *card-isolated-verts-da*: *card H.isolated-verts* = *Suc* (*Suc* (*card G.isolated-verts*))
using *no-loop-a isolated-tail isolated-head* **by** (*simp add*: *card-isolated-verts-da0 card-insert-if*)

lemma *G-edge-succ-a'*[*simp*]: *edge-succ M a'* = *a'*
using *G-edge-succ-a'-no-loop no-loop-a* **by** *simp*

lemma *G-edge-succ-a*[*simp*]: *edge-succ M a* = *a*
proof −
have ∗: *out-arcs G* (*tail G a*) = {*a*}
using *a-in isolated-tail isolated-head no-loop-a*
by (*auto simp*: *H.isolated-verts-def verts-H out-arcs-def arcs-H tail-H*)
obtain *edge-succ M a* ∈ {*a*}
using *G.edge-succ-cyclic*[*of tail G a*]
apply (*rule eq-on-cyclic-on-iff1* [**where** *x*=*a*])
using ∗ *a-in a-neq-a' no-loop-a* **by** *simp-all*
then show *?thesis* **by** *auto*
qed

lemma
G-face-cycle-succ-a[*simp*]: *G.face-cycle-succ a* = *a'* **and**
G-face-cycle-succ-a'[*simp*]: *G.face-cycle-succ a'* = *a*
by (*auto simp*: *G.face-cycle-succ-def*)

lemma
G-face-cycle-set-a[*simp*]: *G.face-cycle-set a* = {*a*,*a'*} (**is** *?t1*) **and**
G-face-cycle-set-a'[*simp*]: *G.face-cycle-set a'* = {*a*,*a'*} (**is** *?t2*)
proof −
{ fix *n* **have** (*G.face-cycle-succ* ^^ *n*) *a* ∈ {*a*,*a'*} (*G.face-cycle-succ* ^^ *n*) *a'* ∈ {*a*,*a'*}
by (*induct n*) *auto*
}

then
show *?t1 ?t2* **by** (*auto simp*: *G.face-cycle-set-altdef intro*: *exI*[**where** *x=0*] *exI*[**where** *x=1*])
qed

lemma *card-face-cycle-sets-da*: *card G.face-cycle-sets* = *Suc* (*card H.face-cycle-sets*)
unfolding *card-face-cycle-sets-da0* **using** *card-fcs-aa′-le* **by** *simp*

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *card-isolated-verts-da verts-H card-arcs-da card-face-cycle-sets-da card-sccs-da*)

end

locale *bidel-arc-tail-not-isol* = *bidel-arc-not-biconnected* +
bidel-arc-tail-conn

sublocale *bidel-arc-tail-not-isol* ⊆ *bidel-arc-same-face-tail-conn*
by *unfold-locales*

locale *bidel-arc-only-tail-not-isol* = *bidel-arc-tail-not-isol* +
bidel-arc-head-isolated

context *bidel-arc-only-tail-not-isol*
begin

lemma *card-isolated-verts-da*: *card H.isolated-verts* = *Suc* (*card G.isolated-verts*)
using *isolated-head conn-tail* **by** (*simp add*: *card-isolated-verts-da0*)

lemma *segment-a′-a-ne*: *segment G.face-cycle-succ a′ a* ≠ {}
unfolding *H-fcs-a-neigh*[*symmetric*] **by** *auto*

lemma *segment-a-a′-e*: *segment G.face-cycle-succ a a′* = {}
proof −
have *a′* = *G.face-cycle-succ a* **using** *tail-neq-head*
by (*simp add*: *G-face-cycle-succ-a-no-loop*)
then show *?thesis* **by** (*auto simp*: *segment1-empty*)
qed

lemma *card-face-cycle-sets-da*: *card H.face-cycle-sets* = *card G.face-cycle-sets*
unfolding *card-face-cycle-sets-da′* **using** *segment-a′-a-ne segment-a-a′-e card-fcs-gt-0*
by (*simp add*: *card-insert-if*)

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *card-isolated-verts-da verts-H card-arcs-da card-face-cycle-sets-da card-sccs-da*)

end

locale *bidel-arc-only-head-not-isol* = *bidel-arc-not-biconnected* +
bidel-arc-head-conn +
bidel-arc-tail-isolated
begin

interpretation *rev*: *bidel-arc G M a′*
using *a-in* **by** *unfold-locales simp*

interpretation *rev*: *bidel-arc-only-tail-not-isol G M a′*
using *a-in not-reach-a*
by *unfold-locales* (*auto simp*: *rev-H isolated-tail conn-head dest*: *H.symmetric-reachable′*)

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
using *rev.euler-genus-da* **by** (*simp add*: *rev-H rev-HM*)

end

locale *bidel-arc-two-not-isol* = *bidel-arc-tail-not-isol* +
bidel-arc-head-conn
begin

lemma *isolated-verts-da*: *H.isolated-verts* = *G.isolated-verts*
using *conn-head conn-tail* **by** (*subst isolated-da-pre*) *simp*

lemma *segment-a′-a-ne′*: *segment G.face-cycle-succ a′ a* ≠ {}
unfolding *H-fcs-a-neigh*[*symmetric*] **by** *auto*

interpretation *rev*: *bidel-arc-tail-not-isol G M a′*
using *arcs-in not-reach-a rev-H conn-head*
by *unfold-locales* (*auto dest*: *H.symmetric-reachable′*)

lemma *segment-a-a′-ne′*: *segment G.face-cycle-succ a a′* ≠ {}
using *rev.H-fcs-a-neigh*[*symmetric*] *rev-H rev-HM* **by** *auto*

lemma *card-face-cycle-sets-da*: *card H.face-cycle-sets* = *Suc* (*card G.face-cycle-sets*)
unfolding *card-face-cycle-sets-da′* **using** *segment-a′-a-ne′ segment-a-a′-ne′ card-fcs-gt-0*
by (*simp add*: *segments-neq card-insert-if*)

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *isolated-verts-da verts-H card-arcs-da card-face-cycle-sets-da card-sccs-da*)

end

locale *bidel-arc-biconnected-non-triv* = *bidel-arc-biconnected* +

bidel-arc-tail-conn

sublocale *bidel-arc-biconnected-non-triv* ⊆ *bidel-arc-head-conn*
by *unfold-locales* (*metis* (*mono-tags*) *G.in-sccs-verts-conv-reachable G.symmetric-reachable′ H.isolated-verts-in-sccs conn-tail empty-iff insert-iff reach-a reachable-daD sccs-verts-da*)

context *bidel-arc-biconnected-non-triv* **begin**

lemma *isolated-verts-da*: *H.isolated-verts* = *G.isolated-verts*
using *conn-head conn-tail* **by** (*subst isolated-da-pre*) *simp*

end

locale *bidel-arc-biconnected-same* = *bidel-arc-biconnected-non-triv* +
bidel-arc-same-face

sublocale *bidel-arc-biconnected-same* ⊆ *bidel-arc-same-face-tail-conn*
by *unfold-locales*

context *bidel-arc-biconnected-same* **begin**

interpretation *rev*: *bidel-arc-same-face-tail-conn G M a′*
using *arcs-in conn-head* **by** *unfold-locales* (*auto simp*: *same-face rev-H*)

lemma *card-face-cycle-sets-da*: *Suc* (*card H.face-cycle-sets*) ≥ (*card G.face-cycle-sets*)
unfolding *card-face-cycle-sets-da′* **using** *card-fcs-gt-0* **by** *linarith*

lemma *euler-genus-da*: *H.euler-genus* ≤ *G.euler-genus*
using *card-face-cycle-sets-da*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *isolated-verts-da verts-H card-arcs-da card-sccs-da*)

end

locale *bidel-arc-biconnected-diff* = *bidel-arc-biconnected-non-triv* +
bidel-arc-diff-face
begin

lemma *fcs-not-triv*: *G.face-cycle-set a* ≠ {*a*} ∨ *G.face-cycle-set a′* ≠ {*a′*}
proof (*rule ccontr*)
assume ¬*?thesis*
then have *G.face-cycle-succ a* = *a G.face-cycle-succ a′* = *a′*
by (*auto simp*: *G.fcs-x-eq-x*)
then have ∗: *edge-succ M a* = *a′ edge-succ M a′* = *a*
by (*auto simp*: *G.face-cycle-succ-def*)
then have (*edge-succ M* ^^ *2*) *a* = *a* **by** (*auto simp*: *eval-nat-numeral*)

{ **fix** *n*
have (*edge-succ M* ^^ *2*) *a* = *a* **by** (*auto simp*: ∗ *eval-nat-numeral*)
then have (*edge-succ M* ^^ *n*) *a* = (*edge-succ M* ^^ (*n mod 2*)) *a*
by (*auto simp*: *funpow-mod-eq*)
moreover have *n mod 2* = *0* ∨ *n mod 2* = *1* **by** *auto*
ultimately have (*edge-succ M* ^^ *n*) *a* ∈ {*a*, *a*′} **by** (*auto simp*: ∗)
}
then have *orbit* (*edge-succ M*) *a* = {*a*, *a*′}
by (*auto simp*: *orbit-altdef-permutation*[*OF G.permutation-edge-succ*] *exI*[**where** *x*=*0*] *exI*[**where** *x*=*1*] ∗)

have *out-arcs G* (*tail G a*) ⊆ {*a*,*a*′}
proof −
have *cyclic-on* (*edge-succ M*) (*out-arcs G* (*tail G a*))
using *arcs-in* **by** (*intro G.edge-succ-cyclic*) *auto*
then have *orbit* (*edge-succ M*) *a* = *out-arcs G* (*tail G a*)
using *arcs-in* **by** (*intro orbit-cyclic-eq3*) *auto*
then show *?thesis* **using** ‹*orbit* - - = {-, -}› **by** *auto*
qed
then have *out-arcs H* (*tail G a*) = {} **by** (*auto simp*: *arcs-H tail-H*)
then have *tail G a* ∈ *H.isolated-verts* **using** *arcs-in* **by** (*simp add*: *H.isolated-verts-def verts-H*)
then show *False* **using** *conn-tail* **by** *contradiction*
qed

lemma *S-ne*: *S* ≠ {}
using *fcs-not-triv* **by** (*auto simp*: *S-def G.segment-face-cycle-x-x-eq*)

lemma *card-face-cycle-sets-da*: *card G.face-cycle-sets* = *Suc* (*card H.face-cycle-sets*)
unfolding *card-face-cycle-sets-da*′ **using** *S-ne diff-face card-fcs-aa*′*-le* **by** *simp*

lemma *euler-genus-da*: *H.euler-genus* = *G.euler-genus*
unfolding *G.euler-genus-def H.euler-genus-def G.euler-char-def H.euler-char-def*
by (*simp add*: *isolated-verts-da verts-H card-arcs-da card-sccs-da card-face-cycle-sets-da*)

end

context *bidel-arc* **begin**

lemma *euler-genus-da*: *H.euler-genus* ≤ *G.euler-genus*
proof −
let *?biconnected* = *tail G a* $\rightarrow^*_H$ *head G a*
let *?isol-tail* = *tail G a* ∈ *H.isolated-verts*
let *?isol-head* = *head G a* ∈ *H.isolated-verts*
let *?same-face* = *G.face-cycle-set a*′ = *G.face-cycle-set a*
{ **assume** *?biconnected ?isol-tail*
then interpret *EG*: *bidel-arc-isolated-loop* **by** *unfold-locales*

```
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ?biconnected ¬?isol-tail ?same-face
      then interpret EG: bidel-arc-biconnected-same by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ?biconnected ¬?isol-tail ¬?same-face
      then interpret EG: bidel-arc-biconnected-diff by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ¬?biconnected ?isol-tail ?isol-head
      then interpret EG: bidel-arc-two-isolated by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ¬?biconnected ¬?isol-tail ?isol-head
      then interpret EG: bidel-arc-only-tail-not-isol by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ¬?biconnected ?isol-tail ¬?isol-head
      then interpret EG: bidel-arc-only-head-not-isol by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    moreover
    { assume ¬?biconnected ¬?isol-tail ¬?isol-head
      then interpret EG: bidel-arc-two-not-isol by unfold-locales
      have ?thesis by (simp add: EG.euler-genus-da)
    }
    ultimately show ?thesis by satx
  qed
end
```

14.3 Modifying *edge-rev*

```
definition (in pre-digraph-map) rev-swap :: 'b ⇒ 'b ⇒ 'b pre-map where
  rev-swap a b = (| edge-rev = perm-swap a b (edge-rev M), edge-succ = perm-swap
a b (edge-succ M) |)

context digraph-map begin

  lemma digraph-map-rev-swap:
    assumes arc-to-ends G a = arc-to-ends G b {a,b} ⊆ arcs G
    shows digraph-map G (rev-swap a b)
  proof
    let ?M' = rev-swap a b
```

have *tail-swap*: $\bigwedge x$. *tail G* $((a \rightleftharpoons_F b)\ x) =$ *tail G x*
using *assms* **by** (*case-tac* $x \in \{a,b\}$) (*auto simp*: *arc-to-ends-def*)
have *swap-in-arcs*: $\bigwedge x$. $(a \rightleftharpoons_F b)\ x \in$ *arcs G* $\longleftrightarrow x \in$ *arcs G*
using *assms* **by** (*case-tac* $x \in \{a,b\}$) *auto*

have *es-perm*: *edge-succ ?M′ permutes arcs G*
using *assms edge-succ-permutes* **unfolding** *permutes-conv-has-dom*
by (*auto simp*: *rev-swap-def has-dom-perm-swap*)

{
fix *x* **show** ($x \in$ *arcs G*) = (*edge-rev* (*rev-swap a b*) $x \neq x$)
using *assms*(*2*)
by (*cases* $x \in \{a,b\}$) (*auto simp*: *rev-swap-def perm-swap-def arev-dom Transposition.transpose-def split*: *if-splits*)
next
fix *x* **assume** $x \in$ *arcs G* **then show** *edge-rev ?M′* (*edge-rev ?M′ x*) = *x*
by (*auto simp*: *rev-swap-def perm-swap-comp*[*symmetric*])
next
fix *x* **assume** $x \in$ *arcs G* **then show** *tail G* (*edge-rev ?M′ x*) = *head G x*
using *assms* **by** (*case-tac* $x \in \{a,b\}$) (*auto simp*: *rev-swap-def perm-swap-def tail-swap arc-to-ends-def*)
next
show *edge-succ ?M′ permutes arcs G* **by** *fact*
next
fix *v* **assume** *A*: $v \in$ *verts G out-arcs G v* $\neq \{\}$
then obtain *c* **where** $c \in$ *out-arcs G v* **by** *blast*
have *inj* (*perm-swap a b* (*edge-succ M*)) **by** (*simp add*: *bij-is-inj bij-edge-succ*)

have *out-arcs G v* = $(a \rightleftharpoons_F b)$ ‘ *out-arcs G v*
by (*auto simp*: *tail-swap swap-swap-id swap-in-arcs intro*: *image-eqI*[**where** x=$(a \rightleftharpoons_F b)\ y$ **for** *y*])
also have $(a \rightleftharpoons_F b)$ ‘ *out-arcs G v* = $(a \rightleftharpoons_F b)$ ‘ *orbit* (*edge-succ M*) $((a \rightleftharpoons_F b)\ c)$
using *edge-succ-cyclic* **using** *A* ‹$c \in$ -›
by (*intro arg-cong*[**where** *f*=(‘) $(a \rightleftharpoons_F b)$])
(*intro orbit-cyclic-eq3*[*symmetric*], *auto simp*: *swap-in-arcs tail-swap*)
also have ... = *orbit* (*edge-succ ?M′*) *c*
by (*simp add*: *orbit-perm-swap rev-swap-def*)
finally have *oa-orb*: *out-arcs G v* = *orbit* (*edge-succ ?M′*) *c* **.**

show *cyclic-on* (*edge-succ ?M′*) (*out-arcs G v*)
unfolding *oa-orb* **using** *es-perm finite-arcs* **by** (*rule cyclic-on-orbit*)
}
qed

lemma *euler-genus-rev-swap*:
assumes *arc-to-ends G a* = *arc-to-ends G b* $\{a,b\} \subseteq$ *arcs G*
shows *pre-digraph-map.euler-genus G* (*rev-swap a b*) = *euler-genus*

proof −
let *?M′* = *rev-swap a b*

interpret *G′*: *digraph-map G ?M′* **using** *assms* **by** (*rule digraph-map-rev-swap*)

have *swap-in-arcs*: $\bigwedge x.\ (a \rightleftharpoons_F b)\ x \in arcs\ G \longleftrightarrow x \in arcs\ G$
using *assms* **by** (*case-tac* $x \in \{a,b\}$) *auto*

have *G′-fcs*: *G′.face-cycle-succ* = *perm-swap a b face-cycle-succ*
unfolding *G′.face-cycle-succ-def face-cycle-succ-def*
by (*auto simp*: *fun-eq-iff rev-swap-def perm-swap-comp*)

have $\bigwedge x.$ *G′.face-cycle-set x* = $(a \rightleftharpoons_F b)$ ‘ *face-cycle-set* $((a \rightleftharpoons_F b)\ x)$
by (*auto simp*: *face-cycle-set-def G′.face-cycle-set-def orbit-perm-swap G′-fcs imageI*)
then have *G′.face-cycle-sets* = $(\lambda S.\ (a \rightleftharpoons_F b)$ ‘ $S)$ ‘ *face-cycle-sets*
by (*auto simp*: *pre-digraph-map.face-cycle-sets-def swap-in-arcs*)
(*metis swap-swap-id image-eqI swap-in-arcs*)
then have *card G′.face-cycle-sets* = *card* $((\lambda S.\ (a \rightleftharpoons_F b)$ ‘ $S)$ ‘ *face-cycle-sets*) **by** *simp*
also have . . . = *card face-cycle-sets*
by (*rule card-image*) (*rule inj-on-f-imageI*[**where** *S*=*UNIV*], *auto*)
finally
show *pre-digraph-map.euler-genus G ?M′* = *euler-genus*
unfolding *pre-digraph-map.euler-genus-def pre-digraph-map.euler-char-def* **by** *simp*
qed

end

14.4 Conclusion

lemma *bidirected-subgraph-obtain*:
assumes *sg*: *subgraph H G arcs H* ≠ *arcs G*
assumes *fin*: *finite* (*arcs G*)
assumes *bidir*: ∃ *rev. bidirected-digraph G rev* ∃ *rev. bidirected-digraph H rev*
obtains *a a′* **where** $\{a,a'\} \subseteq$ *arcs G* − *arcs H a′* ≠ *a*
tail G a′ = *head G a head G a′*= *tail G a*
proof −
obtain *a* **where** *a*: *a* ∈ *arcs G* − *arcs H* **using** *sg* **by** *blast*

obtain *rev-G rev-H* **where** *rev*: *bidirected-digraph G rev-G bidirected-digraph H rev-H*
using *bidir* **by** *blast*
interpret *G*: *bidirected-digraph G rev-G* **by** (*rule rev*)
interpret *H*: *bidirected-digraph H rev-H* **by** (*rule rev*)

have *sg-props*: *arcs H* ⊆ *arcs G tail H* = *tail G head H* = *head G*
using *sg* **by** (*auto simp*: *subgraph-def compatible-def*)

{ **fix** *w1 w2* **assume** *A*: *tail G a = w1 head G a = w2*
have *in-arcs H w1* ∩ *out-arcs H w2* = *rev-H* ‘ (*out-arcs H w1* ∩ *in-arcs H w2*) (**is** *?Sh* = -)
unfolding *H.in-arcs-eq* **by** (*simp add*: *image-Int image-image H.inj-on-arev*)
then have *card* (*in-arcs H w1* ∩ *out-arcs H w2*) = *card* (*out-arcs H w1* ∩ *in-arcs H w2*)
by (*metis card-image H.arev-arev inj-on-inverseI*)
also have ... < *card* (*out-arcs G w1* ∩ *in-arcs G w2*) (**is** *card ?Sh1* < *card ?Sg1*)
proof (*rule psubset-card-mono*)
show *finite ?Sg1* **using** *fin* **by** (*auto simp*: *out-arcs-def*)
show *?Sh1* ⊂ *?Sg1* **using** *A a sg-props* **by** *auto*
qed
also have *?Sg1* = *rev-G* ‘ (*in-arcs G w1* ∩ *out-arcs G w2*) (**is** - = - ‘ *?Sg*)
unfolding *G.in-arcs-eq* **by** (*simp add*: *image-Int image-image G.inj-on-arev*)
also have *card* ... = *card ?Sg*
by (*metis card-image G.arev-arev inj-on-inverseI*)
finally have *card-less*: *card ?Sh* < *card ?Sg* **.**

have *S-ss*: *?Sh* ⊆ *?Sg* **using** *sg-props* **by** *auto*

have *?thesis*
proof (*cases w1 = w2*)
case *True*
have *card* (*?Sh* − {*a*}) = *card ?Sh*
using *a* **by** (*intro arg-cong*[**where** *f=card*]) *auto*
also have ... < *card ?Sg* − *1*
proof −
from *True* **have** *even* (*card ?Sg*) *even* (*card ?Sh*)
by (*auto simp*: *G.even-card-loops H.even-card-loops*)
then show *?thesis* **using** *card-less*
by *simp* (*metis Suc-pred even-Suc le-neq-implies-less lessE less-Suc-eq-le zero-less-Suc*)
qed
also have ... = *card* (*?Sg* − {*a*})
using *fin a A True* **by** (*auto simp*: *out-arcs-def card-Diff-singleton*)
finally have *card-diff-a-less*: *card* (*?Sh* − {*a*}) < *card* (*?Sg* − {*a*}) **.**
moreover
from *S-ss* **have** *?Sh* − {*a*} ⊆ *?Sg* − {*a*} **using** *S-ss* **by** *blast*
ultimately have *?Sh* − {*a*} ⊂ *?Sg* − {*a*}
by (*intro card-psubset*) *auto*
then obtain *a'* **where** *a'* ∈ (*?Sg* − {*a*})− *?Sh* **by** *blast*
then have {*a*,*a'*} ⊆ *arcs G* − *arcs H a'* ≠ *a tail G a'* = *head G a head G a'*= *tail G a*
using *A a sg-props* **by** *auto*
then show *?thesis* **..**
next
case *False*

from *card-less S-ss* **have** *?Sh* ⊂ *?Sg* **by** *auto*
then obtain a' **where** $a' \in$ *?Sg* − *?Sh* **by** *blast*
then have $\{a,a'\} \subseteq$ *arcs G* − *arcs H* $a' \neq a$ *tail G* a' = *head G a head G* a'= *tail G a*
using *A a sg-props False* **by** *auto*
then show *?thesis* **..**
qed
}
then show *?thesis* **by** *simp*
qed

lemma *subgraph-euler-genus-le*:
assumes *G*: *subgraph H G digraph-map G GM* **and** *H*: ∃ *rev. bidirected-digraph H rev*
obtains *HM* **where** *digraph-map H HM pre-digraph-map.euler-genus H HM* ≤ *pre-digraph-map.euler-genus G GM*
proof −
let *?d* = λ*G. card* (*arcs G*) + *card* (*verts G*) − *card* (*arcs H*) − *card* (*verts H*)
from *H* **obtain** *rev-H* **where** *bidirected-digraph H rev-H* **by** *blast*
then interpret *H*: *bidirected-digraph H rev-H* **.**

from *G*
have ∃ *HM. digraph-map H HM* ∧ *pre-digraph-map.euler-genus H HM* ≤ *pre-digraph-map.euler-genus G GM*
proof (*induct ?d G arbitrary*: *G GM rule*: *less-induct*)
case *less*

from *less* **interpret** *G*: *digraph-map G GM* **by** −

have *H-ss*: *arcs H* ⊆ *arcs G verts H* ⊆ *verts G* **using** ‹*subgraph H G*› **by** *auto*
then have *card-le*: *card* (*arcs H*) ≤ *card* (*arcs G*) *card* (*verts H*) ≤ *card* (*verts G*)
by (*auto intro*: *card-mono*)

have *ends*: *tail H* = *tail G head H* = *head G*
using ‹*subgraph H G*› **by** (*auto simp*: *compatible-def*)

show *?case*
proof (*cases ?d G* = *0*)
case *True*
then have *card* (*arcs H*) = *card* (*arcs G*) *card* (*verts H*) = *card* (*verts G*)
using *card-le* **by** *linarith+*
then have *arcs H* = *arcs G verts H* = *verts G*
using *H-ss* **by** (*auto simp*: *card-subset-eq*)
then have *H* = *G* **using** ‹*subgraph H G*› **by** (*auto simp*: *compatible-def*)
then have *digraph-map H GM* ∧ *pre-digraph-map.euler-genus H GM* ≤ *G.euler-genus* **by** *auto*
then show *?thesis* **..**
next

case *False*
then have *H-ne*: $(arcs\ G - arcs\ H) \neq \{\} \vee (verts\ G - verts\ H) \neq \{\}$
using *H-ss card-le* **by** *auto*

{ assume *A*: $arcs\ G - arcs\ H \neq \{\}$
then obtain $a\ a'$ **where** aa': $\{a, a'\} \subseteq arcs\ G - arcs\ H\ a' \neq a\ tail\ G\ a' = head\ G\ a\ head\ G\ a' = tail\ G\ a$
using *H-ss* ‹*subgraph H G*› **by** (*auto intro*: *bidirected-subgraph-obtain*)
let $?GM' = G.rev\text{-}swap\ (edge\text{-}rev\ GM\ a)\ a'$

interpret G': *digraph-map G* $?GM'$
using aa' **by** (*intro G.digraph-map-rev-swap*) (*auto simp*: *arc-to-ends-def*)
interpret G': *bidel-arc G* $?GM'$ *a*
using aa' **by** *unfold-locales simp*

have *edge-rev GM* $a \neq a$
using aa' **by** (*intro G.arev-neq*) *auto*
then have *er-a*: *edge-rev* $?GM'\ a = a'$
using ‹$a' \neq a$› **by** (*auto simp*: *G.rev-swap-def perm-swap-def swap-id-eq dest*: *G.arev-neq*)
then have *sg*: *subgraph H* $G'.H$
using *H-ss* aa' **by** (*intro subgraphI*) (*auto simp*: G'*.verts-H* G'*.arcs-H* G'*.tail-H* G'*.head-H ends compatible-def intro*: *H.wf-digraph* G'*.H.wf-digraph*)

have *card* $\{a,a'\} \leq$ *card* (*arcs G*)
using aa' **by** (*intro card-mono*) *auto*
then obtain *HM* **where** *HM*: *digraph-map H HM pre-digraph-map.euler-genus H HM* $\leq$ G'*.H.euler-genus*
using aa' *False* **by** *atomize-elim* (*rule less*, *auto simp*: G'*.verts-H* G'*.arcs-H card-insert-if sg er-a*)

have G'*.H.euler-genus* $\leq$ G'*.euler-genus* **by** (*rule* G'*.euler-genus-da*)
also have G'*.euler-genus* = *G.euler-genus*
using aa' **by** (*auto simp*: *G.euler-genus-rev-swap arc-to-ends-def*)
finally have *?thesis* **using** *HM* **by** *auto*
}
moreover
{ assume *A*: $arcs\ G - arcs\ H = \{\}$
then have A': $verts\ G - verts\ H \neq \{\}$ **and** *arcs-H*: *arcs H* = *arcs G* **using** *H-ss H-ne* **by** *auto*
then obtain *v* **where** *v*: $v \in verts\ G - verts\ H$ **by** *auto*
have *card-lt*: *card* (*verts H*) $<$ *card* (*verts G*)
using A' *H-ss* **by** (*intro psubset-card-mono*) *auto*

have *out-arcs G v* = *out-arcs H v* **using** *A H-ss* **by** (*auto simp*: *ends*)
then interpret *G*: *del-vert-props G GM v*
using *v* **by** *unfold-locales auto*

have *?d* (*G.del-vert v*) $<$ *?d G*

using *card-lt* **by** (*simp add*: *arcs-H G.arcs-dv G.card-verts-dv*)
moreover
have *subgraph H* (*G.del-vert v*)
using *H-ss v* **by** (*auto simp*: *subgraph-def arcs-H G.arcs-dv G.verts-del-vert H.wf-digraph*
G.H.wf-digraph compatible-def G.tail-del-vert G.head-del-vert ends)
moreover
have *bidirected-digraph* (*G.del-vert v*) (*edge-rev GM*)
using *G.arev-dom* **by** (*intro G.H.bidirected-digraphI*) (*auto simp*: *G.arcs-dv*)
ultimately
have *?thesis* **unfolding** *G.euler-genus-eq*[*symmetric*] **by** (*intro less*) *auto*
}
ultimately show *?thesis* **by** *blast*
qed
qed
then obtain *HM* **where** *digraph-map H HM pre-digraph-map.euler-genus H HM* $\leq$ *pre-digraph-map.euler-genus G GM*
by *atomize-elim*
then show *?thesis* **..**
qed

lemma (**in** *digraph-map*) *nonneg-euler-genus*: *0* $\leq$ *euler-genus*
proof −
define *H* **where** *H* = ⦇ *verts* = {}, *arcs* = {}, *tail* = *tail G*, *head* = *head G* ⦈
then have *H-simps*: *verts H* = {} *arcs H* = {} *tail H* = *tail G head H* = *head G*
by (*simp-all add*: *H-def*)

interpret *H*: *bidirected-digraph H id*
by *unfold-locales* (*auto simp*: *H-def*)
have *wf-digraph H wf-digraph G* **by** *unfold-locales*
then have *subgraph H G* **by** (*intro subgraphI*) (*auto simp*: *H-def compatible-def*)
then obtain *HM* **where** *digraph-map H HM pre-digraph-map.euler-genus H HM* $\leq$ *euler-genus*
by (*rule subgraph-euler-genus-le*) *auto*
then interpret *H*: *digraph-map H HM* **by** −

have *H.sccs* = {}
proof −
{ fix *x* **assume** ∗: *x* ∈ *H.sccs-verts*
then have *x* = {} **by** (*auto dest*: *H.sccs-verts-subsets simp*: *H-simps*)
with ∗ **have** *False* **by** (*auto simp*: *H.in-sccs-verts-conv-reachable*)
} then show *?thesis* **by** (*auto simp*: *H.sccs-verts-conv*)
qed
then have *H.euler-genus* = *0*
by (*auto simp*: *H.euler-genus-def H.euler-char-def H.isolated-verts-def H.face-cycle-sets-def H-simps*)
then show *?thesis* **using** ‹*H.euler-genus* $\leq$ -› **by** *simp*
qed

lemma *subgraph-comb-planar*:
assumes *subgraph G H comb-planar H* $\exists$ *rev. bidirected-digraph G rev* **shows** *comb-planar G*
proof −
from ‹*comb-planar H*› **obtain** *HM* **where** *digraph-map H HM* **and** *H-genus*: *pre-digraph-map.euler-genus H HM* = *0*
unfolding *comb-planar-def* **by** *metis*

obtain *GM* **where** *G*: *digraph-map G GM pre-digraph-map.euler-genus G GM* $\leq$ *pre-digraph-map.euler-genus H HM*
using *assms*(*1*) ‹*digraph-map H HM*› *assms*(*3*) **by** (*rule subgraph-euler-genus-le*)
interpret *G*: *digraph-map G GM* **by** *fact*

show *?thesis* **using** *G H-genus G.nonneg-euler-genus* **unfolding** *comb-planar-def* **by** *auto*
qed

end
theory *Kuratowski-Combinatorial*
imports
Planar-Complete
Planar-Subdivision
Planar-Subgraph
begin

theorem *comb-planar-compat*:
assumes *comb-planar G*
shows *kuratowski-planar G*
proof (*rule ccontr*)
assume ¬*?thesis*
then obtain *G0 rev-G0 K rev-K* **where** *sub*: *subgraph G0 G subdivision* (*K*, *rev-K*) (*G0*, *rev-G0*)
and *is-kur*: $K_{3,3}$ *K* $\vee$ K_5 *K*
unfolding *kuratowski-planar-def* **by** *auto*

have *comb-planar K* **using** *sub assms*
by (*metis subgraph-comb-planar subdivision-comb-planar subdivision-bidir*)
moreover
have ¬*comb-planar K* **using** *is-kur* **by** (*metis K5-not-comb-planar K33-not-comb-planar*)
ultimately
show *False* **by** *contradiction*
qed

end
theory *Simpl-Anno* **imports** *Simpl.Vcg* **begin**

definition *named-loop name* = *UNIV*

lemma *annotate-named-loop-inv*:
whileAnno b (*named-loop name*) *V c* = *whileAnno b I V c*
by (*simp add*: *whileAnno-def*)

lemma *annotate-named-loop-inv-fix*:
whileAnno b (*named-loop name*) *V c* = *whileAnnoFix b I* (λ-. *V*) (λ-. *c*)
by (*simp add*: *whileAnno-def whileAnnoFix-def*)

lemma *annotate-named-loop-var*:
whileAnno b (*named-loop name*) *V' c* = *whileAnno b I V c*
by (*simp add*: *whileAnno-def*)

lemma *annotate-named-loop-var-fix*:
whileAnno b (*named-loop name*) *V' c* = *whileAnnoFix b I* (λ-. *V*) (λ-. *c*)
by (*simp add*: *whileAnno-def whileAnnoFix-def*)

end

15 Implementation of a Non-Planarity Checker

theory *Check-Non-Planarity-Impl*
imports
Simpl.Vcg
Simpl-Anno
Graph-Theory.Graph-Theory
begin

15.1 An abstract graph datatype

type-synonym *ig-vertex* = *nat*
type-synonym *ig-edge* = *ig-vertex* × *ig-vertex*

typedef *IGraph* = {(*vs* :: *ig-vertex list*, *es* :: *ig-edge list*). *distinct vs*}
by *auto*

definition *ig-verts* :: *IGraph* ⇒ *ig-vertex list* **where**
ig-verts G ≡ *fst* (*Rep-IGraph G*)

definition *ig-arcs* :: *IGraph* ⇒ *ig-edge list* **where**
ig-arcs G ≡ *snd* (*Rep-IGraph G*)

definition *ig-verts-cnt* :: *IGraph* ⇒ *nat*
where *ig-verts-cnt G* ≡ *length* (*ig-verts G*)

definition *ig-arcs-cnt* :: *IGraph* ⇒ *nat*
where *ig-arcs-cnt G* ≡ *length* (*ig-arcs G*)

declare *ig-verts-cnt-def*[*simp*]

declare *ig-arcs-cnt-def*[*simp*]

definition *IGraph-inv* :: *IGraph* ⇒ *bool* **where**
IGraph-inv G ≡ (∀ *e* ∈ *set* (*ig-arcs G*). *fst e* ∈ *set* (*ig-verts G*) ∧ *snd e* ∈ *set* (*ig-verts G*))

definition *ig-empty* :: *IGraph* **where**
ig-empty ≡ *Abs-IGraph* ([],[])

definition *ig-add-v* :: *IGraph* ⇒ *ig-vertex* ⇒ *IGraph* **where**
ig-add-v G v = (*if v* ∈ *set* (*ig-verts G*) *then G else Abs-IGraph* (*ig-verts G* @ [*v*], *ig-arcs G*))

definition *ig-add-e* :: *IGraph* ⇒ *ig-vertex* ⇒ *ig-vertex* ⇒ *IGraph* **where**
ig-add-e G u v ≡ *Abs-IGraph* (*ig-verts G*, *ig-arcs G* @ [(*u*,*v*)])

definition *ig-in-out-arcs* :: *IGraph* ⇒ *ig-vertex* ⇒ *ig-edge list* **where**
ig-in-out-arcs G u ≡ *filter* (λ*e*. *fst e* = *u* ∨ *snd e* = *u*) (*ig-arcs G*)

definition *ig-opposite* :: *IGraph* ⇒ *ig-edge* ⇒ *ig-vertex* ⇒ *ig-vertex* **where**
ig-opposite G e u = (*if fst e* = *u then snd e else fst e*)

definition *ig-neighbors* :: *IGraph* => *ig-vertex* => *ig-vertex set* **where**
ig-neighbors G u ≡ {*v* ∈ *set* (*ig-verts G*). (*u*,*v*) ∈ *set* (*ig-arcs G*) ∨ (*v*,*u*) ∈ *set* (*ig-arcs G*)}

15.2 Code

```
procedures is-subgraph (G :: IGraph, H :: IGraph | R :: bool)
  where
    i :: nat
    v :: ig-vertex
    ends :: ig-edge
  in
    TRY
      ´i :== 0 ;;
      WHILE ´i < ig-verts-cnt ´G INV named-loop ''verts''
      DO
        ´v :== ig-verts ´G ! ´i ;;
        IF ´v ∉ set (ig-verts ´H) THEN
          RAISE ´R :== False
        FI ;;
        ´i :== ´i + 1
      OD ;;

      ´i :== 0 ;;
      WHILE ´i < ig-arcs-cnt ´G INV named-loop ''arcs''
      DO
        ´ends :== ig-arcs ´G ! ´i ;;
```

```
        IF ´ends ∉ set (ig-arcs ´H) ∧ (snd ´ends, fst ´ends) ∉ set (ig-arcs ´H)
THEN
          RAISE ´R :== False
        FI ;;
        IF fst ´ends ∉ set (ig-verts ´G) ∨ snd ´ends ∉ set (ig-verts ´G) THEN
          RAISE ´R :== False
        FI ;;
        ´i :== ´i + 1
      OD ;;
      ´R :== True
    CATCH SKIP END
```

procedures *is-loopfree* (*G* :: *IGraph* | *R* :: *bool*)

```
  where
    i :: nat
    ends :: ig-edge
    edge-map :: ig-edge ⇒ bool
  in
    TRY
      ´i :== 0 ;;
      WHILE ´i < ig-arcs-cnt ´G INV named-loop ''loop''
      DO
        ´ends :== ig-arcs ´G ! ´i ;;
        IF fst ´ends = snd ´ends THEN
          RAISE ´R :== False
        FI ;;
        ´i :== ´i + 1
      OD ;;
      ´R :== True
    CATCH SKIP END
```

procedures *select-nodes* (*G* :: *IGraph* | *R* :: *IGraph*)

```
  where
    i :: nat
    v :: ig-vertex
  in
    ´R :== ig-empty ;;

    ´i :== 0 ;;
    WHILE ´i < ig-verts-cnt ´G
    INV named-loop ''loop''
    DO
      ´v :== ig-verts ´G ! ´i ;;
      IF 2 < card (ig-neighbors ´G ´v) THEN
        ´R :== ig-add-v ´R ´v
      FI ;;
```

```
    ´i :== ´i + 1
  OD
```

procedures *find-endpoint* (*G* :: *IGraph*, *H* :: *IGraph*, *v-tail* :: *ig-vertex*, *v-next* :: *ig-vertex* | *R* :: *ig-vertex option*)

```
 where
  found :: bool
  i :: nat
  len :: nat
  io-arcs :: ig-edge list
  v0 :: ig-vertex
  v1 :: ig-vertex
  vt :: ig-vertex
 in
  TRY
    IF ´v-tail = ´v-next THEN RAISE ´R :== None FI ;;
    ´v0 :== ´v-tail ;;
    ´v1 :== ´v-next ;;
    ´len :== 1 ;;
    WHILE ´v1 ∉ set (ig-verts ´H)
    INV named-loop ''path''
    DO
      ´io-arcs :== ig-in-out-arcs ´G ´v1 ;;
      ´i :== 0 ;;
      ´found :== False ;;
      WHILE ´found = False ∧ ´i < length ´io-arcs
      INV named-loop ''arcs''
      DO
        ´vt :== ig-opposite ´G (´io-arcs ! ´i) ´v1 ;;
        IF ´vt ≠ ´v0 THEN
          ´found :== True ;;
          ´v0 :== ´v1 ;;
          ´v1 :== ´vt
        FI ;;
        ´i :== ´i + 1
      OD ;;
      ´len :== ´len + 1 ;;
      IF ¬ ´found THEN RAISE ´R :== None FI
    OD ;;
    IF ´v1 = ´v-tail THEN RAISE ´R :== None FI ;;
    ´R :== Some ´v1
  CATCH SKIP END
```

procedures *contract* (*G* :: *IGraph*, *H* :: *IGraph* | *R* :: *IGraph*)

```
 where
  i :: nat
  j :: nat
```

```
    u :: ig-vertex
    v :: ig-vertex
    vo :: ig-vertex option
    io-arcs :: ig-edge list
  in
    ´i :== 0 ;;
    WHILE ´i < ig-verts-cnt ´H
    INV named-loop ''iter-nodes''
    DO
      ´u :== ig-verts ´H ! ´i ;;
      ´io-arcs :== ig-in-out-arcs ´G ´u ;;

      ´j :== 0 ;;
      WHILE ´j < length ´io-arcs
      INV named-loop ''iter-adj''
      DO
        ´v :== ig-opposite ´G (´io-arcs ! ´j) ´u ;;
        ´vo :== CALL find-endpoint(´G, ´H, ´u, ´v) ;;
        IF ´vo ≠ None THEN
          ´H :== ig-add-e ´H ´u (the ´vo)
        FI ;;
        ´j :== ´j + 1
      OD ;;
      ´i :== ´i + 1
    OD ;;
    ´R :== ´H


procedures is-K33 (G :: IGraph | R :: bool)
  where
    i :: nat
    j :: nat
    u :: ig-vertex
    v :: ig-vertex
    blue :: ig-vertex ⇒ bool
    blue-cnt :: nat
    io-arcs :: ig-edge list
  in
    TRY
      IF ig-verts-cnt ´G ≠ 6 THEN RAISE ´R :== False FI ;;
      ´blue :== (λ-. False) ;;

      ´u :== ig-verts ´G ! 0 ;;
      ´i :== 0 ;;
      ´io-arcs :== ig-in-out-arcs ´G ´u ;;

      WHILE ´i < length ´io-arcs INV named-loop ''colorize''
      DO
        ´v :== ig-opposite ´G (´io-arcs ! ´i) ´u ;;
```

```
      ´blue :== ´blue(´v := True) ;;
      ´i :== ´i + 1
    OD ;;

    ´blue-cnt :== 0 ;;
    ´i :== 0 ;;
    WHILE ´i < ig-verts-cnt ´G INV named-loop ''component-size''
    DO
      IF ´blue (ig-verts ´G ! ´i) THEN ´blue-cnt :== ´blue-cnt + 1 FI ;;
      ´i :== ´i + 1
    OD ;;
    IF ´blue-cnt ≠ 3 THEN RAISE ´R :== False FI ;;

    ´i :== 0 ;;
    WHILE ´i < ig-verts-cnt ´G INV named-loop ''connected-outer''
    DO
      ´u :== ig-verts ´G ! ´i ;;
      ´j :== 0 ;;
      WHILE ´j < ig-verts-cnt ´G INV named-loop ''connected-inner''
      DO
        ´v :== ig-verts ´G ! ´j ;;
       IF ¬((´blue ´u = ´blue ´v) ⟷ (´u, ´v) ∉ set (ig-arcs ´G)) THEN RAISE
´R :== False FI ;;
        ´j :== ´j + 1
      OD ;;
      ´i :== ´i + 1
    OD ;;
    ´R :== True
  CATCH SKIP END
```

```
procedures is-K5 (G :: IGraph | R :: bool)
  where
    i :: nat
    j :: nat
    u :: ig-vertex
  in
    TRY
      IF ig-verts-cnt ´G ≠ 5 THEN RAISE ´R :== False FI ;;
      ´i :== 0 ;;
      WHILE ´i < 5 INV named-loop ''outer-loop''
        DO
        ´u :== ig-verts ´G ! ´i ;;
        ´j :== 0 ;;
        WHILE ´j < 5 INV named-loop ''inner-loop''
        DO
          IF ¬(´i ≠ ´j ⟷ (´u, ig-verts ´G ! ´j) ∈ set (ig-arcs ´G))
          THEN
            RAISE ´R :== False
```

```
      FI ;;
      ´j :== ´j + 1
     OD ;;
     ´i :== ´i + 1
    OD ;;
    ´R :== True
  CATCH SKIP END
```

procedures *check-kuratowski* (*G* :: *IGraph*, *K* :: *IGraph* | *R* :: *bool*)
where
H :: *IGraph*
in

```
  TRY
    ´R :== CALL is-subgraph(´K, ´G) ;;
    IF ¬´R THEN RAISE ´R :== False FI ;;
    ´R :== CALL is-loopfree(´K) ;;
    IF ¬´R THEN RAISE ´R :== False FI ;;
    ´H :== CALL select-nodes(´K) ;;
    ´H :== CALL contract(´K, ´H) ;;
    ´R :== CALL is-K5(´H) ;;
    IF ´R THEN RAISE ´R :== True FI ;;
    ´R :== CALL is-K33(´H)
  CATCH SKIP END
```

end

16 Verification of a Non-Planarity Checker

theory *Check-Non-Planarity-Verification* **imports**
Check-Non-Planarity-Impl
../Planarity/Kuratowski-Combinatorial
HOL−Library.Rewrite
HOL−Eisbach.Eisbach
begin

16.1 Graph Basics and Implementation

context *pre-digraph* **begin**

lemma *cas-nonempty-ends*:
assumes $p \neq []$ *cas u p v cas u′ p v′*
shows $u = u'$ $v = v'$
using *assms* **apply** (*metis cas-simp*)
using *assms* **by** (*metis append-Nil2 cas.simps*(*1*) *cas-append-iff cas-simp*)

lemma *awalk-nonempty-ends*:
  assumes $p \neq []$ *awalk u p v awalk u′ p v′*
  shows $u = u'$ $v = v'$
  using *assms* **by** (*auto simp*: *awalk-def intro*: *cas-nonempty-ends*)

end

lemma (**in** *pair-graph*) *verts2-awalk-distinct*:
  assumes *V*: *verts3 G* ⊆ *V V* ⊆ *pverts G u* ∈ *V*
  assumes *p*: *awalk u p v set* (*inner-verts p*) ∩ *V* = {} *progressing p*
  shows *distinct* (*inner-verts p*)
  using *p*
proof (*induct p arbitrary*: *v rule*: *rev-induct*)
  case *Nil* **then show** *?case* **by** *auto*
next
  case (*snoc e es*)
  have *distinct* (*inner-verts es*)
    apply (*rule snoc.hyps*)
      using *snoc.prems* **apply** (*auto dest*: *progressing-appendD1*)
    apply (*metis* (*opaque-lifting*, *no-types*) *disjoint-iff-not-equal in-set-inner-verts-appendI-l*)
    done
  show *?case*
  proof (*rule ccontr*)
    assume *A*: ¬*?thesis*
    then obtain *es′ e′* **where** *es* = *es′* @ [*e′*] *es* ≠ []
      by (*cases es rule*: *rev-cases*) *auto*

    have *fst e* ∈ *set* (*inner-verts es*)
      using *A* ‹*distinct* (*inner-verts es*)› ‹*es* ≠ []›
      by (*auto simp*: *inner-verts-def*)
    moreover
    have *fst e′* ≠ *fst e snd e′* = *fst e*
      using ‹*es* = *es′* @ [*e′*]› *snoc.prems*(*1*)
      by (*auto simp*: *awalk-Cons-iff dest*: *no-loops*)
    ultimately
    obtain *es″ e″* **where** *es′* = *es″* @ [*e″*]
      by (*cases es′ rule*: *rev-cases*) (*auto simp*: ‹*es* = *es′* @ [*e′*]› *inner-verts-def*)
    then have *fst e″* ≠ *fst e*
      using ‹*snd e′* = *fst e*›[*symmetric*] *snoc.prems*(*1*,*3*) **unfolding** ‹*es* = -›
     by (*simp add*: ‹*es* = -› *awalk-Cons-iff progressing-append-iff progressing-Cons*)

    have *fst e′* ∈ *set* (*inner-verts es*)
      using ‹*es* = *es′* @ [*e′*]› ‹*es′* = *es″* @ [*e″*]›
      by (*cases es″*) (*auto simp*: *inner-verts-def*)

    have *fst e* ∈ *set* (*inner-verts es′*)
      using ‹*es* = *es′* @ [*e′*]› ‹*fst e* ∈ *set* (*inner-verts es*)› ‹*fst e′* ≠ *fst e*›
      by (*cases es′*) (*auto simp*: *inner-verts-def*)
    then obtain *q e′2 e′3 r* **where** *Z*: *es′* = *q* @ [*e′2*, *e′3*] @ *r snd e′2* = *fst e fst*

```
e'3 = fst e
  proof -
    obtain e'3' where e'3' ∈ set (tl es') fst e'3' = fst e
      using ‹fst e ∈ set (inner-verts es')›
      by (cases es') (auto simp: inner-verts-def)
    then obtain q r where tl es' = q @ e'3' # r
      by (metis split-list)
    then have F2: snd (last (hd es' # q)) = fst e
      using ‹es = es' @ [e']› snoc.prems(1) ‹fst e'3' = fst e›
      apply (cases es')
       apply (case-tac [2] q rule: rev-cases)
        apply auto
      done
    then have es' = (butlast (hd es' # q)) @ [last (hd es' # q), e'3'] @ r
      using ‹tl es' = q @ e'3' # r› by (cases es') auto
    then show ?thesis using F2 ‹fst e'3' = fst e› by fact
  qed
  then have fst e'2 ≠ snd e'3
    using snoc.prems(3) unfolding ‹es = -›
    by (simp add: progressing-append-iff progressing-Cons)
  moreover
  from Z have B: fst e'2 = u ∨ fst e'2 ∈ set (inner-verts es')
    using ‹es = es' @ [e']› snoc.prems(1)
    by (cases q) (auto simp: inner-verts-def)
  then have fst e'2 ≠ fst e'
  proof
    assume fst e'2 = u
    then have fst e'2 ∉ set (inner-verts es)
      using V ‹es = es' @ [e']› snoc.prems(2)
      by (cases es') (auto simp: inner-verts-def)
    moreover
    have fst e' ∈ set (inner-verts es)
      using ‹es = es' @ [e']› ‹es' = es'' @ [e'']›
      by (cases es'') (auto simp: inner-verts-def)
    ultimately show ?thesis by auto
  next
    assume fst e'2 ∈ set (inner-verts es')
    moreover
    have fst e' ∈ set (inner-verts es)
      using ‹es = es' @ [e']› ‹es' = es'' @ [e'']›
      by (cases es'') (auto simp: inner-verts-def)
    ultimately
    show ?thesis
      using ‹distinct (inner-verts es)› unfolding ‹es = es' @ [e']›
      by (cases es') (fastforce simp: inner-verts-def)+
  qed
  moreover
  have snd e'3 ≠ fst e'
  proof (rule notI, cases)
```

```
      assume r = [] snd e'3 = fst e'
      then show False using Z ‹es = es' @ [e']› snoc.prems(3) ‹snd e' = fst e›
        by (simp add: progressing-append-iff progressing-Cons)
    next
      assume A: r ≠ [] snd e'3 = fst e'
      then obtain r0 rs where r = r0 # rs by (cases r) auto
      then have snd e'3 = fst r0
        using Z ‹es = es' @ [e']› snoc.prems(1)
        by (auto simp: awalk-Cons-iff)
      with A have fst r0 = fst e' by auto
      have ¬distinct (inner-verts es)
        by (cases q) (auto simp add: Z(1) ‹es = es' @ [e']›
          ‹r = r0 # rs› ‹fst r0 = fst e'› inner-verts-def)
      then show False using ‹distinct (inner-verts es)› by auto
    qed
    ultimately
    have card-to-fst-e: card {e'2, (snd e'3, fst e'3), e'} = 3
      by (auto simp: card-insert-if)
    moreover
    have e'3 ∈ parcs G
      using Z using snoc.prems(1) ‹es = es' @ [e']›
      by (auto intro: arcs-symmetric)
    then have (snd e'3, fst e'3) ∈ parcs G
      by (auto intro: arcs-symmetric)
    then have {e'2, (snd e'3, fst e'3), e'} ⊆ {ed ∈ parcs G. snd ed = fst e}
      using snoc.prems(1) ‹es = es' @ [e']› Z by auto
    moreover
    have fst e ∈ pverts G using snoc.prems(1) by auto
    then have card-to-fst-e-abs: card {ed ∈ parcs G. snd ed = fst e} ≤ 2
      using ‹fst e ∈ set (inner-verts es)› V snoc.prems(2)
      unfolding verts3-def in-degree-def
      by (cases es) (auto simp: inner-verts-def in-arcs-def)
    ultimately
    have {e'2, (snd e'3, fst e'3), e'} = {ed ∈ parcs G. snd ed = fst e}
      by (intro card-seteq) auto
    then show False
      using card-to-fst-e card-to-fst-e-abs by auto
  qed
qed


lemma (in wf-digraph) inner-verts-conv':
  assumes awalk u p v 2 ≤ length p shows inner-verts p = awalk-verts (head G
(hd p)) (butlast (tl p))
  using assms
  apply (cases p)
   apply (auto simp: awalk-Cons-iff; fail)
  apply (match premises in p = - # as for as ⇒ ‹cases as rule: rev-cases›)
   apply (auto simp: inner-verts-def awalk-verts-conv)
```

done

lemma *verts3-in-verts*:
assumes $x \in$ *verts3 G* **shows** $x \in$ *verts G*
using *assms* **unfolding** *verts3-def* **by** *auto*

lemma (**in** *pair-graph*) *deg2-awalk-is-iapath*:
assumes *V*: *verts3 G* $\subseteq$ *V V* $\subseteq$ *pverts G*
assumes *p*: *awalk u p v set* (*inner-verts p*) $\cap$ *V* = {} *progressing p*
assumes *in-V*: $u \in V$ $v \in V$
assumes $u \neq v$
shows *gen-iapath V u p v*
proof (*cases p*)
case *Nil* **then show** *?thesis* **using** *p*(*1*) *in-V* ‹$u \neq v$› **by** (*auto simp*: *apath-def gen-iapath-def*)
next
case (*Cons p0 ps*)
then have *ev-p*: *awalk-verts u p* = *u* # *butlast* (*tl* (*awalk-verts u p*)) @ [*v*]
using *p*(*1*) **by** (*cases p*) *auto*

have $u \notin$ *set* (*inner-verts p*) $v \notin$ *set* (*inner-verts p*)
using *p*(*2*) *in-V* **by** *auto*
with *verts2-awalk-distinct*[*OF V in-V*(*1*) *p*] **have** *distinct* (*awalk-verts u p*)
using *p*(*1*) ‹$u \neq v$› **by** (*subst ev-p*) (*auto simp*: *inner-verts-conv*[*of p u*] *verts3-def*)
then show *?thesis* **using** *p*(*1*−*2*) *in-V* ‹$u \neq v$› **by** (*auto simp*: *apath-def gen-iapath-def*)
qed

lemma (**in** *pair-graph*) *inner-verts-min-degree*:
assumes *walk-p*: *awalk u p v* **and** *progress*: *progressing p*
and *w-p*: $w \in$ *set* (*inner-verts p*)
shows *2* ≤*in-degree G w*
proof −
from *w-p* **have** *2* ≤ *length p* **using** *not-le* **by** *fastforce*
moreover
then obtain *e1 es e2* **where** *p-decomp*: *p* = *e1* # *es* @ [*e2*]
by (*metis One-nat-def Suc-1 Suc-eq-plus1 le0 list.size*(*3*) *list.size*(*4*) *neq-Nil-conv not-less-eq-eq rev-cases*)
ultimately
have *w-es*: $w \in$ *set* (*awalk-verts* (*snd e1*) *es*)
using *walk-p w-p* **by** (*auto simp*: *apath-def inner-verts-conv'*)

have *walk-es*: *awalk* (*snd e1*) *es* (*fst e2*)
using *walk-p* **by** (*auto simp*: *p-decomp awalk-simps*)
obtain *q r* **where** *es-decomp*: *es* = *q* @ *r awalk* (*snd e1*) *q w awalk w r* (*fst e2*)
using *awalk-decomp*[*OF walk-es w-es*] **by** *auto*

define *xs x y ys*

where *xs* = *butlast* (*e1* # *q*) **and** *x* = *last* (*e1* # *q*)
and *y* = *hd* (*r* @ [*e2*]) **and** *ys* = *tl* (*r* @ [*e2*])
then have *p* = *xs* @ *x* # *y* # *ys*
by (*auto simp*: *p-decomp es-decomp*)
moreover
have *awalk u* (*e1* # *q*) *w awalk w* (*r* @ [*e2*]) *v*
using *walk-p es-decomp p-decomp* **by** (*auto simp*: *awalk-Cons-iff*)
then have *inc-w*: *snd x* = *w fst y* = *w*
unfolding *x-def y-def*
apply −
apply (*auto simp*: *awalk-Cons-iff awalk-verts-conv*; *fail*)
apply (*cases r*)
apply *auto*
done
ultimately have *fst x* ≠ *snd y*
using *progress* **by** (*auto simp*: *progressing-append-iff progressing-Cons*)

have *x* ∈ *parcs G y* ∈ *parcs G*
using *walk-p* ‹*p* = *xs* @ *x* # *y* # *ys*› **by** *auto*
then have {*x*, (*snd y*, *w*)} ⊆ {*e* ∈ *parcs G*. *snd e* = *w*}
using *inc-w* **by** *auto* (*metis arcs-symmetric surjective-pairing*)
then have *card* {*x*, (*snd y*, *w*)} ≤ *in-degree G w*
unfolding *in-degree-def* **by** (*intro card-mono*) *auto*
then show *?thesis* **using** ‹*fst x* ≠ *snd y*› *inc-w*
by (*auto simp*: *card-insert-if split*: *if-split-asm*)
qed

lemma (**in** *pair-pseudo-graph*) *gen-iapath-same2E*:
assumes *verts3 G* ⊆ *V V* ⊆ *pverts G*
and *gen-iapath V u p v gen-iapath V w q x*
and *e* ∈ *set p e* ∈ *set q*
obtains *p* = *q*
using *assms same-gen-iapath-by-common-arc* **by** *metis*

definition *mk-graph′* :: *IGraph* ⇒ *ig-vertex pair-pre-digraph* **where**
mk-graph′ IG ≡ ⦇ *pverts* = *set* (*ig-verts IG*), *parcs* = *set* (*ig-arcs IG*)⦈

definition *mk-graph* :: *IGraph* ⇒ *ig-vertex pair-pre-digraph* **where**
mk-graph IG ≡ *mk-symmetric* (*mk-graph′ IG*)

lemma *verts-mkg′*: *pverts* (*mk-graph′ G*) = *set* (*ig-verts G*)
unfolding *mk-graph′-def* **by** *simp*

lemma *arcs-mkg′*: *parcs* (*mk-graph′ G*) = *set* (*ig-arcs G*)
unfolding *mk-graph′-def* **by** *simp*

lemmas *mkg′-simps* = *verts-mkg′ arcs-mkg′*

lemma *verts-mkg*: *pverts* (*mk-graph G*) = *set* (*ig-verts G*)
  unfolding *mk-graph-def* **by** (*simp add*: *mkg′-simps*)

lemma *parcs-mk-symmetric-symcl*: *parcs* (*mk-symmetric G*) = (*arcs-ends G*)^s
  by (*auto simp*: *parcs-mk-symmetric symcl-def arcs-ends-conv*)

lemma *arcs-mkg*: *parcs* (*mk-graph G*) = (*set* (*ig-arcs G*))^s
  unfolding *mk-graph-def parcs-mk-symmetric-symcl* **by** (*simp add*: *arcs-mkg′*)

lemmas *mkg-simps* = *verts-mkg arcs-mkg*

definition *iadj* :: *IGraph* ⇒ *ig-vertex* ⇒ *ig-vertex* ⇒ *bool* **where**
  iadj G u v ≡ (*u*,*v*) ∈ *set* (*ig-arcs G*) ∨ (*v*,*u*) ∈ *set* (*ig-arcs G*)

definition *loop-free G* ≡ (∀ *e* ∈ *parcs G*. *fst e* ≠ *snd e*)

lemma *ig-opposite-simps*:
  ig-opposite G (*u*,*v*) *u* = *v ig-opposite G* (*v*,*u*) *u* = *v*
unfolding *ig-opposite-def* **by** *auto*

lemma *distinct-ig-verts*:
  distinct (*ig-verts G*)
by (*cases G*) (*auto simp*: *ig-verts-def Abs-IGraph-inverse*)

lemma *set-ig-arcs-verts*:
  assumes *IGraph-inv G* (*u*,*v*) ∈ *set* (*ig-arcs G*) **shows** *u* ∈ *set* (*ig-verts G*) *v* ∈ *set* (*ig-verts G*)
  using *assms* **unfolding** *IGraph-inv-def*
  by (*auto simp*: *mkg′-simps dest*: *all-nth-imp-all-set*)

lemma *IGraph-inv-conv*:
  IGraph-inv G ⟷ *pair-fin-digraph* (*mk-graph′ G*)
proof −
  { assume ∀ *e*∈*set* (*ig-arcs G*). *fst e* ∈ *set* (*ig-verts G*) ∧ *snd e* ∈ *set* (*ig-verts G*)
    then have *pair-fin-digraph* (*mk-graph′ G*)
      by *unfold-locales* (*auto simp*: *mkg′-simps*) **}**
  moreover
  { assume *pair-fin-digraph* (*mk-graph′ G*)
    then interpret *pair-fin-digraph mk-graph′ G* **.**
    have ∀ *e*∈*set* (*ig-arcs G*). *fst e* ∈ *set* (*ig-verts G*) ∧ *snd e* ∈ *set* (*ig-verts G*)
      using *tail-in-verts head-in-verts*
      by (*fastforce simp*: *mkg′-simps in-set-conv-nth*) **}**
  ultimately
  show *?thesis* **unfolding** *IGraph-inv-def* **by** *blast*
qed

lemma *IGraph-inv-conv'*:
IGraph-inv G $\longleftrightarrow$ *pair-pseudo-graph* (*mk-graph G*)
unfolding *IGraph-inv-conv*
proof
assume *pair-fin-digraph* (*mk-graph' G*)
interpret *ppd*: *pair-fin-digraph mk-graph' G* **by** *fact*
interpret *pd*: *pair-fin-digraph mk-graph G*
unfolding *mk-graph-def* **..**
show *pair-pseudo-graph* (*mk-graph G*)
by *unfold-locales* (*auto simp*: *mk-graph-def symmetric-mk-symmetric*)
next
assume *A*: *pair-pseudo-graph* (*mk-graph G*)
interpret *ppg*: *pair-pseudo-graph mk-graph G* **by** *fact*
show *pair-fin-digraph* (*mk-graph' G*)
using *ppg.wellformed'*
by *unfold-locales* (*auto simp*: *mkg-simps mkg'-simps symcl-def*, *auto*)
qed

lemma *iadj-io-edge*:
assumes $u \in$ *set* (*ig-verts G*) $e \in$ *set* (*ig-in-out-arcs G u*)
shows *iadj G u* (*ig-opposite G e u*)
proof −
from *assms* **obtain** *v* **where** *e*: $e = (u,v) \lor e = (v,u)$ $e \in$ *set* (*ig-arcs G*)
unfolding *ig-in-out-arcs-def* **by** (*cases e*) *auto*
then have $*$: *ig-opposite G e u* $= v$ **by** *safe* (*auto simp*: *ig-opposite-def*)

show *?thesis* **using** *e* **unfolding** *iadj-def* $*$ **by** *auto*
qed

lemma *All-set-ig-verts*: ($\forall v \in$ *set* (*ig-verts G*). *P v*) $\longleftrightarrow$ ($\forall i <$ *ig-verts-cnt G*. *P* (*ig-verts G* ! *i*))
by (*metis in-set-conv-nth ig-verts-cnt-def*)

lemma *IGraph-imp-ppd-mkg'*:
assumes *IGraph-inv G* **shows** *pair-fin-digraph* (*mk-graph' G*)
using *assms* **unfolding** *IGraph-inv-conv* **by** *auto*

lemma *finite-symcl-iff*: *finite* (R^s) $\longleftrightarrow$ *finite R*
unfolding *symcl-def* **by** *blast*

lemma (**in** *pair-fin-digraph*) *pair-pseudo-graphI-mk-symmetric*:
pair-pseudo-graph (*mk-symmetric G*)
by *unfold-locales*
(*auto simp*: *parcs-mk-symmetric symmetric-mk-symmetric wellformed'*)

lemma *IGraph-imp-ppg-mkg*:
assumes *IGraph-inv G* **shows** *pair-pseudo-graph* (*mk-graph G*)
using *assms* **unfolding** *mk-graph-def*

by (*intro pair-fin-digraph.pair-pseudo-graphI-mk-symmetric IGraph-imp-ppd-mkg'*)

lemma *IGraph-lf-imp-pg-mkg*:
assumes *IGraph-inv G loop-free* (*mk-graph G*) **shows** *pair-graph* (*mk-graph G*)
proof –
interpret *ppg*: *pair-pseudo-graph mk-graph G*
using *assms*(*1*) **by** (*rule IGraph-imp-ppg-mkg*)
show *pair-graph* (*mk-graph G*)
using *assms* **by** *unfold-locales* (*auto simp*: *loop-free-def*)
qed

lemma *set-ig-arcs-imp-verts*:
assumes (*u*,*v*) ∈ *set* (*ig-arcs G*) *IGraph-inv G* **shows** *u* ∈ *set* (*ig-verts G*) *v* ∈ *set* (*ig-verts G*)
proof –
interpret *pair-pseudo-graph mk-graph G*
using *assms* **by** (*auto intro*: *IGraph-imp-ppg-mkg*)
from *assms* **have** (*u*,*v*) ∈ *parcs* (*mk-graph G*) **by** (*simp add*: *mkg-simps symcl-def*)
then have *u* ∈ *pverts* (*mk-graph G*) *v* ∈ *pverts* (*mk-graph G*) **by** (*auto dest*: *wellformed'*)
then show *u* ∈ *set* (*ig-verts G*) *v* ∈ *set* (*ig-verts G*) **by** (*auto simp*: *mkg-simps*)
qed

lemma *iadj-imp-verts*:
assumes *iadj G u v IGraph-inv G* **shows** *u* ∈ *set* (*ig-verts G*) *v* ∈ *set* (*ig-verts G*)
using *assms* **unfolding** *iadj-def* **by** (*auto dest*: *set-ig-arcs-imp-verts*)

lemma *card-ig-neighbors-indegree*:
assumes *IGraph-inv G*
shows *card* (*ig-neighbors G u*) = *in-degree* (*mk-graph G*) *u*
proof –
have *inj2*: *inj-on* (λ*e*. *ig-opposite G e u*) {*e* ∈ *parcs* (*mk-graph G*). *snd e* = *u*}
unfolding *ig-opposite-def* **by** (*rule inj-onI*) (*fastforce split*: *if-split-asm*)

have *ig-neighbors G u* = (λ*e*. *ig-opposite G e u*) ‘ {*e* ∈ *parcs* (*mk-graph G*). *snd e* = *u*}
using *assms* **unfolding** *ig-neighbors-def*
by (*auto simp*: *ig-opposite-simps symcl-def mkg-simps set-ig-arcs-verts intro*!: *rev-image-eqI*)
then have *card* (*ig-neighbors G u*) = *card* ((λ*e*. *ig-opposite G e u*) ‘ {*e* ∈ *parcs* (*mk-graph G*). *snd e* = *u*})
by *simp*
also have ... = *in-degree* (*mk-graph G*) *u*
unfolding *in-degree-def in-arcs-def with-proj-simps*
using *inj2* **by** (*rule card-image*)
finally show *?thesis* **.**
qed

lemma *iadjD*:
  assumes *iadj G u v*
  shows $\exists e \in$ *set* (*ig-in-out-arcs G u*). ($e = (u,v) \vee e = (v,u)$)
proof −
  from *assms* **obtain** *e* **where** $e \in$ *set* (*ig-arcs G*) $e = (u,v) \vee e = (v,u)$
    unfolding *iadj-def* **by** *auto*
  then show *?thesis* **unfolding** *ig-in-out-arcs-def* **by** *auto*
qed

lemma
  ig-verts-empty[*simp*]: *ig-verts ig-empty* = [] **and**
  ig-verts-add-e[*simp*]: *ig-verts* (*ig-add-e G u v*) = *ig-verts G* **and**
  ig-verts-add-v[*simp*]: *ig-verts* (*ig-add-v G v*) = *ig-verts G* @ (*if* $v \in$ *set* (*ig-verts G*) *then* [] *else* [*v*])
  unfolding *ig-verts-def ig-empty-def ig-add-e-def ig-add-v-def*
  by (*auto simp*: *Abs-IGraph-inverse distinct-ig-verts*[*simplified ig-verts-def*])

lemma
  ig-arcs-empty[*simp*]: *ig-arcs ig-empty* = [] **and**
  ig-arcs-add-e[*simp*]: *ig-arcs* (*ig-add-e G u v*) = *ig-arcs G* @ [(*u*,*v*)] **and**
  ig-arcs-add-v[*simp*]: *ig-arcs* (*ig-add-v G v*) = *ig-arcs G*
  unfolding *ig-arcs-def ig-empty-def ig-add-e-def ig-add-v-def*
  by (*auto simp*: *Abs-IGraph-inverse distinct-ig-verts*)

16.2 Total Correctness

16.2.1 Procedure *is-subgraph*

definition *is-subgraph-verts-inv* :: *IGraph* ⇒ *IGraph* ⇒ *nat* ⇒ *bool* **where**
  is-subgraph-verts-inv G H i ≡ *set* (*take i* (*ig-verts G*)) ⊆ *set* (*ig-verts H*)

definition *is-subgraph-arcs-inv* :: *IGraph* ⇒ *IGraph* ⇒ *nat* ⇒ *bool* **where**
  is-subgraph-arcs-inv G H i ≡ $\forall j < i$. *let* (*u*,*v*) = *ig-arcs G* ! *j in*
    ((*u*,*v*) ∈ *set* (*ig-arcs H*) ∨ (*v*,*u*) ∈ *set* (*ig-arcs H*))
    ∧ *u* ∈ *set* (*ig-verts G*) ∧ *v* ∈ *set* (*ig-verts G*)

lemma *is-subgraph-verts-0*: *is-subgraph-verts-inv G H 0*
  unfolding *is-subgraph-verts-inv-def* **by** *auto*

lemma *is-subgraph-verts-step*:
  assumes *is-subgraph-verts-inv G H i ig-verts G* ! *i* ∈ *set* (*ig-verts H*)
  assumes *i* < *length* (*ig-verts G*)
  shows *is-subgraph-verts-inv G H* (*Suc i*)
  using *assms* **by** (*auto simp*: *is-subgraph-verts-inv-def take-Suc-conv-app-nth*)

lemma *is-subgraph-verts-last*:
  is-subgraph-verts-inv G H (*length* (*ig-verts G*)) ⟷ *pverts* (*mk-graph G*) ⊆ *pverts* (*mk-graph H*)
  apply (*auto simp*: *is-subgraph-verts-inv-def mkg-simps*)
  done

lemma *is-subgraph-arcs-0*: *is-subgraph-arcs-inv G H 0*
  unfolding *is-subgraph-arcs-inv-def* **by** *auto*

lemma *is-subgraph-arcs-step*:
  assumes *is-subgraph-arcs-inv G H i*
    $e \in set\ (ig\text{-}arcs\ H) \lor (snd\ e,\ fst\ e) \in set\ (ig\text{-}arcs\ H)$
    $fst\ e \in set\ (ig\text{-}verts\ G)\ snd\ e \in set\ (ig\text{-}verts\ G)$
  assumes $e = ig\text{-}arcs\ G\ !\ i$
  assumes $i < length\ (ig\text{-}arcs\ G)$
  shows *is-subgraph-arcs-inv G H* (*Suc i*)
  using *assms* **by** (*auto simp*: *is-subgraph-arcs-inv-def less-Suc-eq*)

lemma *wellformed-pseudo-graph-mkg*:
  shows *pair-wf-digraph* (*mk-graph G*) = *pair-pseudo-graph*(*mk-graph G*) (**is** *?L* = *?R*)
proof
  assume *?R*
  then interpret *ppg*: *pair-pseudo-graph mk-graph G* **.**
  show *?L* **by** *unfold-locales*
next
  assume *?L*
  moreover have *symmetric* (*mk-graph G*)
    unfolding *mk-graph-def* **by** (*simp add*: *symmetric-mk-symmetric*)
  ultimately show *?R*
    unfolding *pair-wf-digraph-def*
    by *unfold-locales* (*auto simp*: *mkg-simps finite-symcl-iff*)
qed

lemma *is-subgraph-arcs-last*:
  is-subgraph-arcs-inv G H (*length* (*ig-arcs G*)) $\longleftrightarrow$ *parcs* (*mk-graph G*) $\subseteq$ *parcs* (*mk-graph H*) $\land$ *pair-pseudo-graph* (*mk-graph G*)
proof –
  have *is-subgraph-arcs-inv G H* (*length* (*ig-arcs G*))
      $= (\forall (u,v) \in set\ (ig\text{-}arcs\ G).\ ((u,v) \in set\ (ig\text{-}arcs\ H) \lor (v,u) \in set\ (ig\text{-}arcs\ H))$
        $\land\ u \in set\ (ig\text{-}verts\ G) \land v \in set\ (ig\text{-}verts\ G))$
    unfolding *is-subgraph-arcs-inv-def*
    by (*metis* (*lifting*, *no-types*) *all-nth-imp-all-set nth-mem*)
  also have ... $\longleftrightarrow$ *parcs* (*mk-graph G*) $\subseteq$ *parcs* (*mk-graph H*) $\land$ *pair-pseudo-graph* (*mk-graph G*)
    unfolding *wellformed-pseudo-graph-mkg*[*symmetric*]
    by (*auto simp*: *mkg-simps pair-wf-digraph-def symcl-def*)
  finally show *?thesis* **.**
qed

lemma *is-subgraph-verts-arcs-last*:
  assumes *is-subgraph-verts-inv G H* (*ig-verts-cnt G*)
  assumes *is-subgraph-arcs-inv G H* (*ig-arcs-cnt G*)

assumes *IGraph-inv H*
shows *subgraph* (*mk-graph G*) (*mk-graph H*) (**is** *?T1*)
pair-pseudo-graph (*mk-graph G*) (**is** *?T2*)
proof −
interpret *ppg*: *pair-pseudo-graph mk-graph G*
using *assms* **by** (*simp add*: *is-subgraph-arcs-last*)
interpret *ppgH*: *pair-pseudo-graph mk-graph H* **using** *assms* **by** (*intro IGraph-imp-ppg-mkg*)
have *wf-digraph* (*with-proj* (*mk-graph G*)) **by** *unfold-locales*
with *assms* **show** *?T1 ?T2*
by (*auto simp*: *is-subgraph-verts-last is-subgraph-arcs-last subgraph-def ppgH.wf-digraph*)
qed

lemma *is-subgraph-false*:
assumes *subgraph* (*mk-graph G*) (*mk-graph H*)
obtains $\forall\, i <$ *length* (*ig-verts G*). *ig-verts G* ! $i \in$ *set* (*ig-verts H*)
$\forall\, i <$ *length* (*ig-arcs G*). *let* (*u*,*v*) = *ig-arcs G* ! *i in*
((*u*,*v*)$\in$ *set* (*ig-arcs H*) $\vee$ (*v*,*u*) $\in$ *set* (*ig-arcs H*))
$\wedge$ *u* $\in$ *set* (*ig-verts G*) $\wedge$ *v* $\in$ *set* (*ig-verts G*)
proof
from *assms*
show $\forall\, i <$ *length* (*ig-verts G*). *ig-verts G* ! $i \in$ *set* (*ig-verts H*)
unfolding *subgraph-def* **by** (*auto simp*: *mkg-simps*)
next
from *assms* **have** *is-subgraph-arcs-inv G H* (*length* (*ig-arcs G*))
unfolding *is-subgraph-arcs-last subgraph-def wellformed-pseudo-graph-mkg*[*symmetric*]
by (*auto simp*: *wf-digraph-wp-iff*)
then show $\forall\, i <$ *length* (*ig-arcs G*). *let* (*u*,*v*) = *ig-arcs G* ! *i in*
((*u*,*v*)$\in$ *set* (*ig-arcs H*) $\vee$ (*v*,*u*) $\in$ *set* (*ig-arcs H*))
$\wedge$ *u* $\in$ *set* (*ig-verts G*) $\wedge$ *v* $\in$ *set* (*ig-verts G*)
by (*auto simp*: *is-subgraph-arcs-inv-def*)
qed

lemma (**in** *is-subgraph-impl*) *is-subgraph-spec*:
$\forall\sigma.\ \Gamma \vdash_t \{\!|\sigma.\ IGraph\text{-}inv\ \acute{}H\ |\!\}\ \acute{}R :== PROC\ is\text{-}subgraph(\acute{}G,\ \acute{}H)\ \{\!|\ \acute{}G = {}^{\sigma}G$
$\wedge\ \acute{}H = {}^{\sigma}H \wedge \acute{}R = (subgraph\ (mk\text{-}graph\ \acute{}G)\ (mk\text{-}graph\ \acute{}H) \wedge IGraph\text{-}inv\ \acute{}G)|\!\}$
apply (*vcg-step spec=none*)
apply (*rewrite*
at whileAnno - (*named-loop* ''*verts*'') - -
in for (σ)
to whileAnno -
$\{\!|\ is\text{-}subgraph\text{-}verts\text{-}inv\ \acute{}G\ \acute{}H\ \acute{}i \wedge \acute{}G = {}^{\sigma}G \wedge \acute{}H = {}^{\sigma}H \wedge \acute{}i \leq ig\text{-}verts\text{-}cnt$
$\acute{}G$
$\wedge\ IGraph\text{-}inv\ \acute{}H|\!\}$
(*MEASURE ig-verts-cnt* $\acute{}G - \acute{}i$)
-
annotate-named-loop-var)
apply (*rewrite*
at whileAnno - (*named-loop* ''*arcs*'') - -
in for (σ)

to whileAnno -
{| *is-subgraph-arcs-inv* ´*G* ´*H* ´*i* $\wedge$ ´*G* = $^\sigma$*G* $\wedge$ ´*H* = $^\sigma$*H* $\wedge$ ´*i* $\leq$ *ig-arcs-cnt* ´*G*
$\wedge$ *is-subgraph-verts-inv* ´*G* ´*H* (*length* (*ig-verts* ´*G*)) $\wedge$ *IGraph-inv* ´*H* |}
(*MEASURE ig-arcs-cnt* ´*G* − ´*i*)
-
annotate-named-loop-var)
apply *vcg*
apply (*fastforce simp*: *is-subgraph-verts-0*)
apply (*fastforce simp*: *is-subgraph-verts-step elim*: *is-subgraph-false*)
apply (*fastforce simp*: *is-subgraph-arcs-0 not-less*)
apply (*auto simp*: *is-subgraph-arcs-step elim*!: *is-subgraph-false*; *fastforce*)
apply (*fastforce simp*: *IGraph-inv-conv′ is-subgraph-verts-arcs-last*)
done

16.2.2 Procedure *is-loop-free*

definition *is-loopfree-inv G k* $\equiv$ $\forall$ *j*<*k*. *fst* (*ig-arcs G* ! *j*) $\neq$ *snd* (*ig-arcs G* ! *j*)

lemma *is-loopfree-0*:
is-loopfree-inv G 0
by (*auto simp*: *is-loopfree-inv-def*)

lemma *is-loopfree-step1*:
assumes *is-loopfree-inv G n*
assumes *fst* (*ig-arcs G* ! *n*) $\neq$ *snd* (*ig-arcs G* ! *n*)
assumes *n* < *ig-arcs-cnt G*
shows *is-loopfree-inv G* (*Suc n*)
using *assms* **unfolding** *is-loopfree-inv-def*
by (*auto intro*: *less-SucI elim*: *less-SucE*)

lemma *is-loopfree-step2*:
assumes *loop-free* (*mk-graph G*)
assumes *n* < *ig-arcs-cnt G*
shows *fst* (*ig-arcs G* ! *n*) $\neq$ *snd* (*ig-arcs G* ! *n*)
using *assms* **unfolding** *is-loopfree-inv-def loop-free-def*
by (*auto simp*: *mkg-simps symcl-def*)

lemma *is-loopfree-last*:
assumes *is-loopfree-inv G* (*ig-arcs-cnt G*)
shows *loop-free* (*mk-graph G*)
using *assms* **apply** (*auto simp*: *is-loopfree-inv-def loop-free-def mkg-simps in-set-conv-nth symcl-def*)
apply (*metis fst-eqD snd-eqD*)+
done

lemma (**in** *is-loopfree-impl*) *is-loopfree-spec*:
$\forall\sigma$. $\Gamma \vdash_t$ {|σ. *IGraph-inv* ´*G* |} ´*R* :== *PROC is-loopfree*(´*G*) {| ´*G* = $^\sigma$*G* $\wedge$ ´*R* = *loop-free* (*mk-graph* ´*G*) |}

apply (*vcg-step spec=none*)
apply (*rewrite*
at whileAnno - (*named-loop ''loop''*) - -
in for (σ)
to whileAnno -
⦃ *is-loopfree-inv ´G ´i* ∧ *´G* = $^\sigma$*G* ∧ *´i* ≤ *ig-arcs-cnt ´G* ⦄
(*MEASURE ig-arcs-cnt ´G* − *´i*)
-
annotate-named-loop-var)
apply *vcg*
apply (*fastforce simp*: *is-loopfree-0*)
apply (*fastforce intro*: *is-loopfree-step1 dest*: *is-loopfree-step2*)
apply (*fastforce simp*: *is-loopfree-last*)
done

16.2.3 Procedure *select-nodes*

definition *select-nodes-inv* :: *IGraph* ⇒ *IGraph* ⇒ *nat* ⇒ *bool* **where**
select-nodes-inv G H i ≡ *set* (*ig-verts H*) = {*v* ∈ *set* (*take i* (*ig-verts G*)). *card* (*ig-neighbors G v*) ≥ *3*} ∧ *IGraph-inv H*

lemma *select-nodes-inv-step*:
fixes *G H i*
defines *v* ≡ *ig-verts G* ! *i*
assumes *G-inv*: *IGraph-inv G*
assumes *sni-inv*: *select-nodes-inv G H i*
assumes *less*: *i* < *ig-verts-cnt G*
assumes *H'*: *H'* = (*if 3* ≤ *card* (*ig-neighbors G v*) *then ig-add-v H v else H*)
shows *select-nodes-inv G H'* (*Suc i*)
proof −
have ∗: *IGraph-inv H'* **using** *sni-inv H'*
unfolding *IGraph-inv-def select-nodes-inv-def* **by** *auto*
have *take-Suc-i*: *take* (*Suc i*) (*ig-verts G*) = *take i* (*ig-verts G*) @ [*v*]
using *less* **unfolding** *v-def* **by** (*auto simp*: *take-Suc-conv-app-nth*)
have *X*: *v* ∉ *set* (*take i* (*ig-verts G*))
using *G-inv less distinct-ig-verts* **unfolding** *v-def IGraph-inv-conv*
by (*auto simp*: *distinct-conv-nth in-set-conv-nth*)

show *?thesis*
unfolding *select-nodes-inv-def* **using** *X sni-inv*
by (*simp only*: ∗) (*auto simp*: *take-Suc-i select-nodes-inv-def H'*)
qed

definition *select-nodes-prop* :: *IGraph* ⇒ *IGraph* ⇒ *bool* **where**
select-nodes-prop G H ≡ *pverts* (*mk-graph H*) = *verts3* (*mk-graph G*)

lemma (**in** *select-nodes-impl*) *select-nodes-spec*:
∀*σ*. Γ $\vdash_t$ ⦃*σ*. *IGraph-inv ´G*⦄ *´R* :== *PROC select-nodes*(*´G*)
⦃ *select-nodes-prop* $^\sigma$*G ´R* ∧ *IGraph-inv ´R* ∧ *set* (*ig-arcs ´R*) = {}⦄

apply *vcg-step*
apply (*rewrite*
at whileAnno - (*named-loop ''loop''*) - -
in for (σ)
to whileAnno -
$\{\!|$ *select-nodes-inv* ´*G* ´*R* ´*i* $\wedge$ ´*i* $\leq$ *ig-verts-cnt* ´*G* $\wedge$ ´*G* = $^{\sigma}$*G* $\wedge$ *IGraph-inv* ´*G* $\wedge$ *set* (*ig-arcs* ´*R*) = {}$|\!\}$
(*MEASURE ig-verts-cnt* ´*G* $-$ ´*i*)
-
annotate-named-loop-var)
apply *vcg*
apply (*fastforce simp*: *select-nodes-inv-def IGraph-inv-def mkg'-simps*)
apply (*fastforce simp add*: *select-nodes-inv-step*)
apply (*fastforce simp add*: *select-nodes-inv-def select-nodes-prop-def card-ig-neighbors-indegree verts3-def mkg-simps*)
done

16.2.4 Procedure *find-endpoint*

definition *find-endpoint-path-inv* **where**
find-endpoint-path-inv G H len u v w x $\equiv$
$\exists$ *p*. *pre-digraph.awalk* (*mk-graph G*) *u p x* $\wedge$ *length p* = *len* $\wedge$
hd p = (*u*,*v*) $\wedge$ *last p* = (*w*, *x*) $\wedge$
set (*pre-digraph.inner-verts* (*mk-graph G*) *p*) $\cap$ *set* (*ig-verts H*) = {} $\wedge$
progressing p

definition *find-endpoint-arcs-inv* **where**
find-endpoint-arcs-inv G found k v0 v1 v0' v1' $\equiv$
(*found* $\longrightarrow$ ($\exists$ *i* < *k*. *v1'* = *ig-opposite G* (*ig-in-out-arcs G v1* ! *i*) *v1* $\wedge$ *v0'* = *v1* $\wedge$ *v0* $\neq$ *v1'*)) $\wedge$
($\neg$*found* $\longrightarrow$ ($\forall$ *i* < *k*. *v0* = *ig-opposite G* (*ig-in-out-arcs G v1* ! *i*) *v1*) $\wedge$ *v0* = *v0'* $\wedge$ *v1* = *v1'*)

lemma *find-endpoint-path-first*:
assumes *iadj G u v u* $\neq$ *v IGraph-inv G*
shows *find-endpoint-path-inv G H* (*Suc 0*) *u v u v*
proof $-$
interpret *ppg*: *pair-pseudo-graph mk-graph G*
using *assms* **by** (*auto intro*: *IGraph-imp-ppg-mkg*)
have (*u*,*v*) $\in$ *parcs* (*mk-graph G*)
using *assms* **by** (*auto simp*: *iadj-def mkg-simps symcl-def*)
then have *ppg.awalk u* [(*u*,*v*)] *v length* [(*u*,*v*)] = *Suc 0 hd* [(*u*,*v*)] = (*u*,*v*) *last* [(*u*,*v*)] = (*u*,*v*) *progressing* [(*u*,*v*)]
using *assms* **by** (*auto simp*: *ppg.awalk-simps iadj-imp-verts mkg-simps progressing-Cons*)
moreover
have *set* (*ppg.inner-verts* [(*u*,*v*)]) $\cap$ *set* (*ig-verts H*) = {}
by (*auto simp*: *ppg.inner-verts-def*)
ultimately

show *?thesis* **unfolding** *find-endpoint-path-inv-def* **by** *blast*
qed

lemma *find-endpoint-arcs-0*:
find-endpoint-arcs-inv G False 0 v0 v1 v0 v1
unfolding *find-endpoint-arcs-inv-def* **by** *auto*

lemma *find-endpoint-path-lastE*:
assumes *find-endpoint-path-inv G H len u v w x*
assumes *ig*: *IGraph-inv G* **and** *lf*: *loop-free* (*mk-graph G*)
assumes *snp*: *select-nodes-prop G H*
assumes *0 < len*
assumes *u*: *u* ∈ *set* (*ig-verts H*)
obtains *p* **where** *pre-digraph.awalk* (*mk-graph G*) *u* ((*u*,*v*) # *p*) *x*
and *progressing* ((*u*,*v*) # *p*)
and *set* (*pre-digraph.inner-verts* (*mk-graph G*) ((*u*,*v*) # *p*)) ∩ *set* (*ig-verts H*) = {}
and *len* ≤ *ig-verts-cnt G*
proof −
from *ig* **and** *lf* **interpret** *pair-graph mk-graph G*
by (*rule IGraph-lf-imp-pg-mkg*)
have [*simp*]: *verts3* (*mk-graph G*) = *set* (*ig-verts H*)
using *assms* **unfolding** *select-nodes-prop-def* **by** (*auto simp*: *mkg-simps*)
from *assms* **obtain** *q* **where** *q*: *awalk u q x length q* = *len hd q* = (*u*,*v*)
and *iv*: *set* (*inner-verts q*) ∩ *verts3* (*mk-graph G*) = {}
and *prg*: *progressing q*
unfolding *find-endpoint-path-inv-def* **by** *auto*
moreover then obtain *q0 qs* **where** *q* = *q0* # *qs* **using** ‹*0* < *len*› **by** (*cases q*) *auto*
moreover
have *len* ≤ *ig-verts-cnt G*
proof −
have *ev-q*: *awalk-verts u q* = *u* # *inner-verts q* @ [*x*]
unfolding *inner-verts-conv*[*of q u*] **using** *q* ‹*q* = *q0* # *qs*› **by** *auto*
then have *len-ev*: *length* (*awalk-verts u q*) = *2* + *length* (*inner-verts q*)
by *auto*

have *set-av*: *set* (*awalk-verts u q*) ⊆ *pverts* (*mk-graph G*)
using *q*(*1*) **by** *auto*

from *snp u* **have** *u* ∈ *verts3* (*mk-graph G*) **by** *simp*
moreover
with - - **have** *distinct* (*inner-verts q*)
using *q*(*1*) *iv prg* **by** (*rule verts2-awalk-distinct*) (*auto simp*: *verts3-def*)
ultimately
have *distinct* (*u* # *inner-verts q*) **using** *iv* **by** *auto*
moreover
have *set* (*u* # *inner-verts q*) ⊆ *pverts* (*mk-graph G*)
using *ev-q set-av* **by** *auto*

ultimately
have *length* (*u* # *inner-verts q*) ≤ *card* (*pverts* (*mk-graph G*))
  by (*metis card-mono distinct-card finite-set verts-mkg*)
then have *length* (*awalk-verts u q*) ≤ *1* + *card* (*pverts* (*mk-graph G*))
  by (*simp add*: *len-ev*)
then have *length q* ≤ *card* (*pverts* (*mk-graph G*))
  by (*auto simp*: *length-awalk-verts*)
also have ... ≤ *ig-verts-cnt G* **by** (*auto simp*: *mkg-simps card-length*)
finally show *?thesis* **by** (*simp add*: *q*)
qed
ultimately show *?thesis* **by** (*intro that*) *auto*
qed

lemma *find-endpoint-path-last1*:
assumes *find-endpoint-path-inv G H len u v w x*
assumes *ig*: *IGraph-inv G* **and** *lf*: *loop-free* (*mk-graph G*)
assumes *snp*: *select-nodes-prop G H*
assumes *0* < *len*
assumes *mem*: *u* ∈ *set* (*ig-verts H*) *x* ∈ *set* (*ig-verts H*) *u* ≠ *x*
shows ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*,*v*) # *p*) *x*
proof −
from *ig* **and** *lf* **interpret** *pair-graph mk-graph G*
  by (*rule IGraph-lf-imp-pg-mkg*)
have [*simp*]: *verts3* (*mk-graph G*) = *set* (*ig-verts H*)
  ⋀*x*. *x* ∈ *set* (*ig-verts H*) ⟹ *x* ∈ *pverts* (*mk-graph G*)
 using *assms* **unfolding** *select-nodes-prop-def* **by** (*auto simp*: *mkg-simps verts3-def*)
show *?thesis*
  apply (*rule find-endpoint-path-lastE*[*OF assms*(*1*−*5*) *mem*(*1*)])
  by (*drule deg2-awalk-is-iapath*[*rotated 2*]) (*auto simp*: *mem*)
qed

lemma *find-endpoint-path-last2D*:
assumes *path*: *find-endpoint-path-inv G H len u v w u*
assumes *ig*: *IGraph-inv G* **and** *lf*: *loop-free* (*mk-graph G*)
assumes *snp*: *select-nodes-prop G H*
assumes *0* < *len*
assumes *mem*: *u* ∈ *set* (*ig-verts H*)
assumes *iapath*: *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*,*v*) # *p*) *x*
shows *False*
proof −
from *ig* **and** *lf* **interpret** *pair-graph mk-graph G*
  by (*rule IGraph-lf-imp-pg-mkg*)
have [*simp*]: *verts3* (*mk-graph G*) = *set* (*ig-verts H*)
  using *assms* **unfolding** *select-nodes-prop-def* **by** (*auto simp*: *mkg-simps*)
have *V*: *verts3* (*mk-graph G*) ⊆ *verts3* (*mk-graph G*) *verts3* (*mk-graph G*) ⊆ *pverts* (*mk-graph G*)
  using *verts3-in-verts*[**where** *G*=*mk-graph G*] **by** *auto*

obtain *q* **where** *walk-q*: *awalk u* ((*u*, *v*) # *q*) *u* **and**

progress-q: *progressing* ((*u*, *v*) # *q*) **and**
iv-q: *set* (*inner-verts* ((*u*, *v*) # *q*)) ∩ *verts3* (*mk-graph G*) = {}
by (*rule find-endpoint-path-lastE*[*OF path ig lf snp* ‹*0* < *len*› *mem*]) *auto*

from *iapath* **have** *walk-p*: *awalk u* ((*u*,*v*) # *p*) *x* **and**
iv-p: *set* (*inner-verts* ((*u*, *v*) # *p*)) ∩ *verts3* (*mk-graph G*) = {} **and**
uv-verts3: *u* ∈ *verts3* (*mk-graph G*) *x* ∈ *verts3* (*mk-graph G*)
unfolding *gen-iapath-def apath-def* **by** *auto*
from *iapath* **have** *progress-p*: *progressing* ((*u*,*v*) # *p*)
unfolding *gen-iapath-def* **by** (*auto intro*: *apath-imp-progressing*)

from *V walk-q walk-p progress-q progress-p iv-q iv-p*
have (*u*,*v*) # *q* = (*u*,*v*) # *p*
apply (*rule same-awalk-by-common-arc*[**where** *e*=(*u*,*v*)])
using *uv-verts3*
apply *auto*
done
then show *False*
by (*metis iapath apath-nonempty-ends gen-iapath-def awalk-nonempty-ends*(*2*) *walk-p walk-q*)
qed

lemma *find-endpoint-arcs-last*:
assumes *arcs*: *find-endpoint-arcs-inv G False* (*length* (*ig-in-out-arcs G v1*)) *v0 v1 v0a v1a*
assumes *path*: *find-endpoint-path-inv G H len v-tail v-next v0 v1*
assumes *ig*: *IGraph-inv G* **and** *lf*: *loop-free* (*mk-graph G*)
assumes *snp*: *select-nodes-prop G H*
assumes *mem*: *v-tail* ∈ *set* (*ig-verts H*)
assumes *0* < *len*
shows ¬ *pre-digraph.iapath* (*mk-graph G*) *v-tail* ((*v-tail*, *v-next*) # *p*) *x*
proof
let ¬*?A* = *?thesis*
assume *?A*

interpret *pair-graph mk-graph G* **using** *ig lf* **by** (*rule IGraph-lf-imp-pg-mkg*)

have *v3G-eq*: *verts3* (*mk-graph G*) = *set* (*ig-verts H*)
using *assms* **unfolding** *select-nodes-prop-def* **by** (*auto simp*: *mkg-simps*)

If no extending edge was found (as implied by *find-endpoint-arcs-inv G False* (*length* (*ig-in-out-arcs G v1*)) *v0 v1 v0a v1a*), the last vertex of the walk computed (as implied by *find-endpoint-path-inv G H len v-tail v-next v0 v1*) is of degree 1. Hence we consider all vertices except the degree-2 nodes.

define *V* **where** *V* = {*v* ∈ *pverts* (*mk-graph G*). *in-degree* (*mk-graph G*) *v* ≠ *2*}

have *V*: *verts3* (*mk-graph G*) ⊆ *V V* ⊆ *pverts* (*mk-graph G*)
unfolding *verts3-def V-def* **by** *auto*

from ‹*?A*› **have** *walk-p*: *awalk v-tail* ((*v-tail*, *v-next*) # *p*) *x* **and**
progress-p: *progressing* ((*v-tail*, *v-next*) # *p*)
by (*auto simp*: *gen-iapath-def apath-def intro*: *apath-imp-progressing*)
have *iapath-V-p*: *gen-iapath V v-tail* ((*v-tail*, *v-next*) # *p*) *x*
proof −
{ fix *u* **assume** *A*: $u \in$ *set* (*inner-verts* ((*v-tail*, *v-next*) # *p*))
then have $u \in$ *pverts* (*mk-graph G*) **using** ‹*?A*›
by (*auto 2 4 simp*: *set-inner-verts gen-iapath-def apath-Cons-iff dest*: *awalkI-apath*)
with *A* ‹*?A*› *inner-verts-min-degree*[*OF walk-p progress-p A*] **have** $u \notin V$
unfolding *gen-iapath-def verts3-def V-def* **by** *auto* **}**
with ‹*?A*› *V* **show** *?thesis* **by** (*auto simp*: *gen-iapath-def*)
qed

have *arcs-p*: (*v-tail*, *v-next*) $\in$ *set* ((*v-tail*, *v-next*) # *p*)
unfolding *gen-iapath-def apath-def* **by** *auto*

have *id-x*: $2 <$ *in-degree* (*mk-graph G*) *x*
using ‹*?A*› **unfolding** *gen-iapath-def verts3-def* **by** *auto*

from *arcs* **have** *edge-no-pr*: $\bigwedge e.\ e \in$ *set* (*ig-in-out-arcs G v1*) $\Longrightarrow$
v0 = *ig-opposite G e v1* **and***v0* = *v0a v1* = *v1a*
by (*auto simp*: *find-endpoint-arcs-inv-def in-set-conv-nth*)

have {$e \in$ *parcs* (*mk-graph G*). *snd e* = *v1*} $\subseteq$ {(*v0*,*v1*)} (**is** *?L* $\subseteq$ *?R*)
proof
fix *e* **assume** $e \in$ *?L*
then have *fst e* $\neq$ *snd e* **by** (*auto dest*: *no-loops*)
moreover
from ‹$e \in$ *?L*› **have** $e \in$ *set* (*ig-in-out-arcs G v1*) $\vee$ (*snd e*, *fst e*) $\in$ *set* (*ig-in-out-arcs G v1*)
by (*auto simp*: *mkg-simps ig-in-out-arcs-def symcl-def*)
then have *v0* = *ig-opposite G e v1* $\vee$ *v0* = *ig-opposite G* (*snd e*, *fst e*) *v1*
by (*auto intro*: *edge-no-pr*)
ultimately show $e \in$ *?R* **using** ‹$e \in$ *?L*› **by** (*auto simp*: *ig-opposite-def*)
qed
then have *id-v1*: *in-degree* (*mk-graph G*) *v1* $\leq$ *card* {(*v0*,*v1*)}
unfolding *in-degree-def in-arcs-def* **by** (*intro card-mono*) *auto*

from *path* **obtain** *q* **where** *walk-q*: *awalk v-tail q v1* **and**
q-props: *length q* = *len hd q* = (*v-tail*, *v-next*) **and**
iv-q': *set* (*inner-verts q*) $\cap$ *verts3* (*mk-graph G*) = {} **and**
progress-q: *progressing q*
by (*auto simp*: *find-endpoint-path-inv-def v3G-eq*)
then have *v1* $\in$ *pverts* (*mk-graph G*)
by (*metis awalk-last-in-verts*)
then have *v1* $\in$ *V* **using** *id-v1* **unfolding** *V-def* **by** *auto*

{ **fix** *u* **assume** *A*: $u \in set$ (*inner-verts q*)
then have $u \in set$ (*pawalk-verts v-tail q*) **using** *walk-q*
by (*auto simp*: *inner-verts-conv*[**where** *u=v-tail*] *awalk-def dest*: *in-set-butlastD list-set-tl*)
then have $u \in$ *pverts* (*mk-graph G*) **using** *walk-q* **by** *auto*
with *A iv-q′ inner-verts-min-degree*[*OF walk-q progress-q A*] **have** $u \notin V$
unfolding *verts3-def V-def* **by** *auto* }
then have *iv-q*: *set* (*inner-verts q*) $\cap$ *V* = {} **by** *auto*

have *arcs-q*: (*v-tail*, *v-next*) $\in$ *set q*
using *q-props* ‹*0* < *len*› **by** (*cases q*) *auto*

have *neq*: *v-tail* $\neq$ *v1*
using *find-endpoint-path-last2D*[*OF - ig lf snp* ‹*0* < *len*› ‹*v-tail* $\in$ -› ‹*?A*›] *path* **by** *auto*

have *in-V*: *v-tail* $\in$ *V* **using** *iapath-V-p* **unfolding** *gen-iapath-def* **by** *auto*
have *iapath-V-q*: *gen-iapath V v-tail q v1*
using *V walk-q iv-q progress-q in-V* ‹*v1* $\in$ *V*› *neq* **by** (*rule deg2-awalk-is-iapath*)

have ((*v-tail*, *v-next*) # *p*) = *q*
using *V iapath-V-p iapath-V-q arcs-p arcs-q*
by (*rule same-gen-iapath-by-common-arc*)
then have *v1* = *x* **using** *walk-p walk-q* **by** *auto*
then show *False* **using** *id-v1 id-x* **by** *auto*
qed

lemma *find-endpoint-arcs-step1E*:
assumes *find-endpoint-arcs-inv G False k v0 v1 v0′ v1′*
assumes *ig-opposite G* (*ig-in-out-arcs G v1* ! *k*) *v1′* $\neq$ *v0′*
obtains *v0* = *v0′ v1* = *v1′ find-endpoint-arcs-inv G True* (*Suc k*) *v0 v1 v1* (*ig-opposite G* (*ig-in-out-arcs G v1* ! *k*) *v1*)
using *assms* **unfolding** *find-endpoint-arcs-inv-def*
by (*auto intro*: *less-SucI elim*: *less-SucE*)

lemma *find-endpoint-arcs-step2E*:
assumes *find-endpoint-arcs-inv G False k v0 v1 v0′ v1′*
assumes *ig-opposite G* (*ig-in-out-arcs G v1* ! *k*) *v1′* = *v0′*
obtains *v0* = *v0′ v1* = *v1′ find-endpoint-arcs-inv G False* (*Suc k*) *v0 v1 v0 v1*
using *assms* **unfolding** *find-endpoint-arcs-inv-def*
by (*auto intro*: *less-SucI elim*: *less-SucE*)

lemma *find-endpoint-path-step*:
assumes *path*: *find-endpoint-path-inv G H len u v w x* **and** *0* < *len*
assumes *arcs*: *find-endpoint-arcs-inv G True k w x w′ x′*
$k \leq$ *length* (*ig-in-out-arcs G x*)
assumes *ig*: *IGraph-inv G*
assumes *not-end*: $x \notin$ *set* (*ig-verts H*)
shows *find-endpoint-path-inv G H* (*Suc len*) *u v w′ x′*

proof −
  interpret *pg*: *pair-pseudo-graph mk-graph G*
    using *ig* **by** (*auto intro*: *IGraph-imp-ppg-mkg*)
  from *path* **obtain** *p* **where** *awalk*: *pg.awalk u p x* **and**
      p: *length p* = *len hd p* = (*u*, *v*) *last p* = (*w*, *x*) **and**
      iv: *set* (*pg.inner-verts p*) ∩ *set* (*ig-verts H*) = {} **and**
      progress: *progressing p*
    by (*auto simp*: *find-endpoint-path-inv-def*)

  define *p'* **where** *p'* = *p* @ [(*x*,*x'*)]

  from *path* **have** *x* ∈ *set* (*ig-verts G*)
    by (*metis awalk pg.awalk-last-in-verts verts-mkg*)

  with *arcs* **have** *iadj G x x' x* = *w' w* ≠ *x'*
    using ‹*x* ∈ *set* (*ig-verts G*)› **unfolding** *find-endpoint-arcs-inv-def*
    by (*auto intro*: *iadj-io-edge*)
  then have (*x*,*x'*) ∈ *parcs* (*mk-graph G*) *x'* ∈ *set* (*ig-verts G*)
     using *ig* **unfolding** *iadj-def* **by** (*auto simp*: *mkg-simps set-ig-arcs-imp-verts symcl-def*)
  then have *pg.awalk u p' x'*
    unfolding *p'-def* **using** *awalk* **by** (*auto simp*: *pg.awalk-simps mkg-simps*)
  moreover
  have *length p'* = *Suc len hd p'* = (*u*,*v*) *last p'* = (*w'*,*x'*)
    using ‹*x* = *w'*› ‹*0* < *len*› *p* **by** (*auto simp*: *p'-def*)
  moreover
  have *set* (*pg.inner-verts p'*) ∩ *set* (*ig-verts H*) = {}
   using *iv not-end p* ‹*0* < *len*› **unfolding** *p'-def* **by** (*auto simp*: *pg.inner-verts-def*)
  moreover
  { fix *ys y z zs* **have** *p'* ≠ *ys* @ [(*y*,*z*), (*z*,*y*)] @ *zs*
    proof
      let ¬*?A* = *?thesis*
      assume *?A*
      from *progress* **have** ⋀*zs*. *p* ≠ *ys* @ (*y*,*z*) # (*z*,*y*) # *zs*
        by (*auto simp*: *progressing-append-iff progressing-Cons*)
      with ‹*?A*› **have** *zs* = [] **unfolding** *p'-def* **by** (*cases zs rule*: *rev-cases*) *auto*
       then show *False* **using** ‹*?A*› **using** ‹*w* ≠ *x'*› ‹*last p* = (*w*,*x*)› **unfolding** *p'-def* **by** *auto*
    qed }
  then have *progressing p'* **by** (*auto simp*: *progressing-def*)
  ultimately show *?thesis* **unfolding** *find-endpoint-path-inv-def* **by** *blast*
qed

lemma *no-loop-path*:
  assumes *u* = *v* **and** *ig*: *IGraph-inv G*
  shows ¬ (∃ *p w*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *v*) # *p*) *w*)
proof −
  interpret *ppg*: *pair-pseudo-graph mk-graph G*
    using *ig* **by** (*rule IGraph-imp-ppg-mkg*)

from ‹*u* = *v*› **show** *?thesis*
by (*auto simp*: *ppg.gen-iapath-def ppg.apath-Cons-iff*)
(*metis hd-in-set ppg.awalk-verts-non-Nil ppg.awhd-of-awalk pre-digraph.awalkI-apath*)
qed

lemma (**in** *find-endpoint-impl*) *find-endpoint-spec*:
$\forall \sigma.\ \Gamma \vdash_t \{\!|\sigma.$ *select-nodes-prop* ´*G* ´*H* ∧ *loop-free* (*mk-graph* ´*G*) ∧ ´*v-tail* ∈ *set* (*ig-verts* ´*H*) ∧ *iadj* ´*G* ´*v-tail* ´*v-next* ∧ *IGraph-inv* ´*G*$|\!\}$
´*R* :== *PROC find-endpoint*(´*G*, ´*H*, ´*v-tail*, ´*v-next*)
$\{\!|$*case* ´*R of None* ⇒ ¬(∃ *p w*. *pre-digraph.iapath* (*mk-graph* $^\sigma$*G*) $^\sigma$*v-tail* (($^\sigma$*v-tail*, $^\sigma$*v-next*) # *p*) *w*)
| *Some w* ⇒ (∃ *p*. *pre-digraph.iapath* (*mk-graph* $^\sigma$*G*) $^\sigma$*v-tail* (($^\sigma$*v-tail*, $^\sigma$*v-next*) # *p*) *w*) $|\!\}$
apply *vcg-step*
apply (*rewrite*
at whileAnno - (*named-loop* ''*path*'') - -
in for (σ)
to whileAnno -
$\{\!|$ *find-endpoint-path-inv* ´*G* ´*H* ´*len* ´*v-tail* ´*v-next* ´*v0* ´*v1*
∧ ´*v-tail* = $^\sigma$*v-tail* ∧ ´*v-next* = $^\sigma$*v-next* ∧ ´*G* = $^\sigma$*G* ∧ ´*H* = $^\sigma$*H*
∧ *0* < ´ *len*
∧ ´*v-tail* ∈ *set* (*ig-verts* ´*H*) ∧ *select-nodes-prop* ´*G* ´*H* ∧ *IGraph-inv* ´*G* ∧ *loop-free* (*mk-graph* ´*G*) $|\!\}$
(*MEASURE Suc* (*ig-verts-cnt* ´*G*) − ´*len*)
-
annotate-named-loop-var)
apply (*rewrite*
at whileAnno - (*named-loop* ''*arcs*'') - -
in for (σ)
to whileAnnoFix -
(λ(*v0*,*v1*,*len*). $\{\!|$ *find-endpoint-arcs-inv* ´*G* ´*found* ´*i v0 v1* ´*v0* ´*v1*
∧ ´*i* ≤ *length* (*ig-in-out-arcs* ´*G v1*) ∧ ´*io-arcs* = *ig-in-out-arcs* ´*G v1*
∧ ´*v-tail* = $^\sigma$*v-tail* ∧ ´*v-next* = $^\sigma$*v-next* ∧ ´*G* = $^\sigma$*G* ∧ ´*H* = $^\sigma$ *H*
∧ ´*len* = *len*
∧ ´*v-tail* ∈ *set* (*ig-verts* ´*H*) ∧ *select-nodes-prop* ´*G* ´*H* ∧ *IGraph-inv* ´*G* $|\!\}$)
(λ-. (*MEASURE length* ´*io-arcs* − ´*i*))
-
annotate-named-loop-var-fix)
apply *vcg*
apply (*fastforce simp*: *find-endpoint-path-first no-loop-path*)
apply (*match* **premises in** *find-endpoint-path-inv* - - - - - *v0 v1* **for** *v0 v1*
⇒ ‹*rule exI*[*where x*=*v0*], *rule exI*[*where x*=*v1*]›)
apply (*fastforce simp*: *find-endpoint-arcs-last find-endpoint-arcs-0 find-endpoint-path-step elim*: *find-endpoint-path-lastE*)
apply (*fastforce elim*: *find-endpoint-arcs-step1E find-endpoint-arcs-step2E*)
apply (*fastforce dest*: *find-endpoint-path-last1 find-endpoint-path-last2D*)
done

16.2.5 Procedure *contract*

definition *contract-iter-nodes-inv* **where**
contract-iter-nodes-inv G H k ≡
set (*ig-arcs H*) = (⋃ *i*<*k*. {(*u*,*v*). *u* = (*ig-verts H* ! *i*) ∧ (∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*)})

definition *contract-iter-adj-inv* :: *IGraph* ⇒ *IGraph* ⇒ *IGraph* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
contract-iter-adj-inv G H0 H u l ≡ (*set* (*ig-arcs H*) − ({*u*} × *UNIV*) = *set* (*ig-arcs H0*)) ∧
ig-verts H = *ig-verts H0* ∧
(∀ *v*. (*u*,*v*) ∈ *set* (*ig-arcs H*) ⟷
((∃ *j*<*l*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*)))

lemma *contract-iter-adj-invE*:
assumes *contract-iter-adj-inv G H0 H u l*
obtains *set* (*ig-arcs H*) − ({*u*} × *UNIV*) = *set* (*ig-arcs H0*) *ig-verts H* = *ig-verts H0*
⋀*v*. (*u*,*v*) ∈ *set* (*ig-arcs H*) ⟷ ((∃ *j*<*l*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*))
using *assms* **unfolding** *contract-iter-adj-inv-def* **by** *auto*

lemma *contract-iter-adj-inv-def'*:
contract-iter-adj-inv G H0 H u l ⟷ (
set (*ig-arcs H*) − ({*u*} × *UNIV*) = *set* (*ig-arcs H0*)) ∧ *ig-verts H* = *ig-verts H0* ∧
(∀ *v*. ((∃ *j*<*l*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*) ⟶ (*u*,*v*) ∈ *set* (*ig-arcs H*)) ∧
((*u*,*v*) ∈ *set* (*ig-arcs H*) ⟶ ((∃ *j*<*l*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*))))
unfolding *contract-iter-adj-inv-def* **by** *metis*

lemma *select-nodes-prop-add-e*[*simp*]:
select-nodes-prop G (*ig-add-e H u v*) = *select-nodes-prop G H*
by (*simp add*: *select-nodes-prop-def mkg-simps*)

lemma *contract-iter-adj-inv-step1*:
assumes *pair-pseudo-graph* (*mk-graph G*)
assumes *ciai*: *contract-iter-adj-inv G H0 H u l*
assumes *iapath*: *pre-digraph.iapath* (*mk-graph G*) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *l*) *u*) # *p*) *w*
shows *contract-iter-adj-inv G H0* (*ig-add-e H u w*) *u* (*Suc l*)
proof −
interpret *pair-pseudo-graph mk-graph G* **by** *fact*
{ **fix** *v j* **assume** ∗: *j* < *Suc l* ∃ *p*. *iapath u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*
then have (*u*, *v*) ∈ *set* (*ig-arcs* (*ig-add-e H u w*))
proof (*cases j* < *l*)

```
      case True with ∗ ciai show ?thesis
        by (auto simp: contract-iter-adj-inv-def)[]
    next
      case False with ∗ have j = l by arith
      with ∗(2) obtain q where ∗∗: iapath u ((u, ig-opposite G (ig-in-out-arcs G
u ! l) u) # q) v
        by metis
      with iapath have p = q
        using verts3-in-verts[where G=mk-graph G]
        by (auto elim: gen-iapath-same2E[rotated 2])
      with ∗∗ iapath have v = w
      by (auto simp: pre-digraph.gen-iapath-def pre-digraph.apath-def elim: pre-digraph.awalk-nonempty-ends[ro
      then show ?thesis by simp
    qed }
  moreover
  { fix v assume ∗: (u,v) ∈ set (ig-arcs (ig-add-e H u w))
    have (∃ j<Suc l. ∃ p. gen-iapath (verts3 (mk-graph G)) u ((u, ig-opposite G
(ig-in-out-arcs G u ! j) u) # p) v)
    proof cases
      assume v = w then show ?thesis using iapath by auto
    next
      assume v ≠ w then show ?thesis using ciai ∗
        unfolding contract-iter-adj-inv-def by (auto intro: less-SucI)
    qed }
  moreover
  have set (ig-arcs (ig-add-e H u w)) − ({u} × UNIV) = set (ig-arcs H0)
      ig-verts (ig-add-e H u w) = ig-verts H0
    using ciai unfolding contract-iter-adj-inv-def by auto
  ultimately
  show ?thesis unfolding contract-iter-adj-inv-def by metis
qed

lemma contract-iter-adj-inv-step2:
  assumes ciai: contract-iter-adj-inv G H0 H u l
  assumes iapath: ⋀p w. ¬pre-digraph.iapath (mk-graph G) u ((u, ig-opposite G
(ig-in-out-arcs G u ! l) u) # p) w
  shows contract-iter-adj-inv G H0 H u (Suc l)
proof −
  { fix v j assume ∗: j < Suc l ∃ p. pre-digraph.iapath (mk-graph G) u ((u,
ig-opposite G (ig-in-out-arcs G u ! j) u) # p) v
    then have (u, v) ∈ set (ig-arcs H)
    proof (cases j < l)
      case True with ∗ ciai show ?thesis
        by (auto simp: contract-iter-adj-inv-def)
    next
      case False with ∗ have j = l by auto
      with ∗ show ?thesis using iapath by metis
    qed }
  moreover
```

{ **fix** *v* **assume** ∗: (*u*,*v*) ∈ *set* (*ig-arcs H*)
then have (∃ *j*<*Suc l*. ∃ *p*. *pre-digraph.gen-iapath* (*mk-graph G*) (*verts3* (*mk-graph G*)) *u* ((*u*, *ig-opposite G* (*ig-in-out-arcs G u* ! *j*) *u*) # *p*) *v*)
using *ciai* **unfolding** *contract-iter-adj-inv-def* **by** (*auto intro*: *less-SucI*) }
moreover
have *set* (*ig-arcs H*) − ({*u*} × *UNIV*) = *set* (*ig-arcs H0*) *ig-verts H* = *ig-verts H0*
using *ciai* **unfolding** *contract-iter-adj-inv-def* **by** (*auto simp*:)
ultimately
show *?thesis* **unfolding** *contract-iter-adj-inv-def* **by** *metis*
qed

definition *contract-iter-adj-prop* **where**
contract-iter-adj-prop G H0 H u ≡ *ig-verts H* = *ig-verts H0*
∧ *set* (*ig-arcs H*) = *set* (*ig-arcs H0*) ∪ ({*u*} × {*v*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*})

lemma *contract-iter-adj-propI*:
assumes *nodes*: *contract-iter-nodes-inv G H i*
assumes *ciai*: *contract-iter-adj-inv G H H′ u* (*length* (*ig-in-out-arcs G u*))
assumes *u*: *u* = *ig-verts H* ! *i*
shows *contract-iter-adj-prop G H H′ u*
proof −
have *ig-verts H′* = *ig-verts H*
using *ciai* **unfolding** *contract-iter-adj-inv-def* **by** *auto*
moreover
have *set* (*ig-arcs H′*) ⊆ *set* (*ig-arcs H*) ∪({*u*} × {*v*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*})
using *ciai* **unfolding** *contract-iter-adj-inv-def* **by** *auto*
moreover
{ **fix** *v p* **assume** *path*: *pre-digraph.iapath* (*mk-graph G*) *u p v*
then obtain *e es* **where** *p* = *e* # *es* **by** (*cases p*) (*auto simp*: *pre-digraph.gen-iapath-def*)
then have *e* ∈ *parcs* (*mk-graph G*) **using** *path*
by (*auto simp*: *pre-digraph.gen-iapath-def pre-digraph.apath-def pre-digraph.awalk-def*)
moreover
then obtain *w* **where** *e* = (*u*,*w*) **using** ‹*p* = *e* # *es*› *path*
by (*cases e*) (*auto simp*: *pre-digraph.gen-iapath-def pre-digraph.apath-def pre-digraph.awalk-def pre-digraph.cas.simps*)
ultimately
have (*u*,*w*) ∈ *set* (*ig-arcs G*) ∨ (*w*,*u*) ∈ *set* (*ig-arcs G*)
unfolding *mk-graph-def* **by** (*auto simp*: *parcs-mk-symmetric mkg′-simps*)
then obtain *e′* **where** *H1*: *e′* = (*u*,*w*) ∨ *e′* = (*w*,*u*) **and** *e′* ∈ *set* (*ig-arcs G*)
by *auto*
then have *e′* ∈ *set* (*ig-in-out-arcs G u*)
unfolding *ig-in-out-arcs-def* **by** *auto*
then obtain *k* **where** *H2*: *ig-in-out-arcs G u* ! *k* = *e′ k* < *length* (*ig-in-out-arcs G u*)

by (*auto simp*: *in-set-conv-nth*)
have *opp-e'*: *ig-opposite G e' u = w* **using** *H1* **unfolding** *ig-opposite-def* **by** *auto*
have (*u*,*v*) ∈ *set* (*ig-arcs H'*)
using *ciai* **unfolding** *contract-iter-adj-inv-def'*
apply *safe*
apply (*erule allE*[**where** *x*=*v*])
apply *safe*
apply (*erule notE*)
apply (*rule exI*[**where** *x*=*k*])
apply (*simp add*: *H2 opp-e'*)
using *path* ‹*e* = (*u*,*w*)› ‹*p* = *e* # *es*› **by** *auto* **}**
then have *set* (*ig-arcs H*) ∪({*u*} × {*v*. ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*}) ⊆ *set* (*ig-arcs H'*)
using *ciai* **unfolding** *contract-iter-adj-inv-def* **by** *auto*
ultimately
show *?thesis* **unfolding** *contract-iter-adj-prop-def* **by** *blast*
qed

lemma *contract-iter-nodes-inv-step*:
assumes *nodes*: *contract-iter-nodes-inv G H i*
assumes *adj*: *contract-iter-adj-inv G H H'* (*ig-verts H* ! *i*) (*length* (*ig-in-out-arcs G* (*ig-verts H* ! *i*)))
assumes *snp*: *select-nodes-prop G H*
shows *contract-iter-nodes-inv G H'* (*Suc i*)
proof −
have *ciap*: *contract-iter-adj-prop G H H'* (*ig-verts H* ! *i*)
using *nodes adj* **by** (*rule contract-iter-adj-propI*) *simp*
then have *ie-H'*: *set* (*ig-arcs H'*) = *set* (*ig-arcs H*) ∪ {(*u*,*v*). *u* = *ig-verts H'* ! *i* ∧ (∃ *p*. *pre-digraph.gen-iapath* (*mk-graph G*) (*verts3* (*mk-graph G*)) *u p v*)}
and [*simp*]: *ig-verts H'* = *ig-verts H*
unfolding *contract-iter-adj-prop-def* **by** *auto*
have *ie-H*: *set* (*ig-arcs H*) = (⋃ *j*<*i*. {(*u*, *v*). *u* = *ig-verts H'* ! *j* ∧ (∃ *p*. *pre-digraph.gen-iapath* (*mk-graph G*) (*verts3* (*mk-graph G*)) *u p v*)})
using *nodes* **unfolding** *contract-iter-nodes-inv-def* **by** *simp*

have ∗: ⋀*S k*. (⋃ *i* < *Suc k*. *S i*) = (⋃ *i* < *k*. *S i*) ∪ *S k*
by (*metis UN-insert lessThan-Suc sup-commute*)

show *?thesis* **by** (*simp only*: *contract-iter-nodes-inv-def ie-H ie-H'* ∗)
qed

lemma *contract-iter-nodes-0*:
assumes *set* (*ig-arcs H*) = {} **shows** *contract-iter-nodes-inv G H 0*
using *assms* **unfolding** *contract-iter-nodes-inv-def* **by** *simp*

lemma *contract-iter-adj-0*:
assumes *nodes*: *contract-iter-nodes-inv G H i*
assumes *i*: *i* < *ig-verts-cnt H*

shows *contract-iter-adj-inv G H H* (*ig-verts H* ! *i*) *0*
using *assms distinct-ig-verts*
unfolding *contract-iter-adj-inv-def contract-iter-nodes-inv-def*
by (*auto simp*: *distinct-conv-nth*)

lemma *snp-vertexes*:
assumes *select-nodes-prop G H u* ∈ *set* (*ig-verts H*) **shows** *u* ∈ *set* (*ig-verts G*)
using *assms* **unfolding** *select-nodes-prop-def* **by** (*auto simp*: *verts3-def mkg-simps*)

lemma *igraph-ig-add-eI*:
assumes *IGraph-inv G*
assumes *u* ∈ *set* (*ig-verts G*) *v* ∈ *set* (*ig-verts G*)
shows *IGraph-inv* (*ig-add-e G u v*)
using *assms* **unfolding** *IGraph-inv-def* **by** *auto*

lemma *snp-iapath-ends-in*:
assumes *select-nodes-prop G H*
assumes *pre-digraph.iapath* (*mk-graph G*) *u p v*
shows *u* ∈ *set* (*ig-verts H*) *v* ∈ *set* (*ig-verts H*)
using *assms* **unfolding** *pre-digraph.gen-iapath-def select-nodes-prop-def verts3-def*
by (*auto simp*: *mkg-simps*)

lemma *contract-iter-nodes-last*:
assumes *nodes*: *contract-iter-nodes-inv G H* (*ig-verts-cnt H*)
assumes *snp*: *select-nodes-prop G H*
assumes *igraph*: *IGraph-inv G*
shows *mk-graph′ H* = *contr-graph* (*mk-graph G*) (**is** *?t1*)
and *symmetric* (*mk-graph′ H*) (**is** *?t2*)
proof −
interpret *ppg-mkG*: *pair-pseudo-graph mk-graph G*
using *igraph* **by** (*rule IGraph-imp-ppg-mkg*)
{ fix *u v p* **assume** *pre-digraph.iapath* (*mk-graph G*) *u p v*
then have ∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *v p u*
using *ppg-mkG.gen-iapath-rev-path*[**where** *u*=*u* **and** *v*=*v*, *symmetric*] **by** *auto*
}
then have *ie-sym*: ⋀*u v*. (∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*) ⟷ (∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *v p u*)
by *auto*

from *nodes* **have** *set* (*ig-arcs H*) = {(*u*, *v*). *u* ∈ *set* (*ig-verts H*) ∧ (∃ *p*. *pre-digraph.gen-iapath* (*mk-graph G*) (*verts3* (*mk-graph G*)) *u p v*)}
unfolding *contract-iter-nodes-inv-def* **by** (*auto simp*: *in-set-conv-nth*)
then have ∗: *set* (*ig-arcs H*) = {(*u*,*v*). (∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*)}
using *snp* **by** (*auto simp*: *snp-iapath-ends-in*(*1*))
then have ∗∗: *set* (*ig-arcs H*) = (λ(*a*,*b*). (*b*,*a*)) ‘ {(*u*,*v*). (∃ *p*. *pre-digraph.iapath* (*mk-graph G*) *u p v*)}
using *ie-sym* **by** *fastforce*

```
  have sym: symmetric (mk-graph' H)
    unfolding symmetric-conv by (auto simp: mkg'-simps ∗ ie-sym)

  have pverts (mk-graph' H) = verts3 (mk-graph G)
   using snp unfolding select-nodes-prop-def by (simp add: mkg-simps mkg'-simps)
  moreover
  have parcs (mk-graph' H) = {(u,v). (∃ p. ppg-mkG.iapath u p v)}
    using ∗  by (auto simp: mkg-simps mkg'-simps)
  ultimately show ?t1 ?t2
    using snp sym unfolding gen-contr-graph-def select-nodes-prop-def by auto
qed

lemma (in contract-impl) contract-spec:
  ∀ σ. Γ ⊢t ⦃σ. select-nodes-prop ´G ´H ∧ IGraph-inv ´G ∧ loop-free (mk-graph
´G) ∧ IGraph-inv ´H ∧ set (ig-arcs ´H) = {}⦄
    ´R :== PROC contract(´G, ´H)
  ⦃´G = σ G ∧ mk-graph' ´R = contr-graph (mk-graph ´G) ∧ symmetric (mk-graph'
´R) ∧ IGraph-inv ´R⦄
  apply vcg-step
  apply (rewrite
    at whileAnno - (named-loop ''iter-nodes'') - -
    in for (σ)
    to whileAnno -
      ⦃contract-iter-nodes-inv ´G ´H ´i
          ∧ select-nodes-prop ´G ´H ∧ ´i ≤ ig-verts-cnt ´H ∧ IGraph-inv ´G ∧
loop-free (mk-graph ´G)
        ∧ IGraph-inv ´H ∧ ´G = σ G⦄
      (MEASURE ig-verts-cnt ´H − ´i)
      -
    annotate-named-loop-var)
  apply (rewrite
    at whileAnno - (named-loop ''iter-adj'') - -
    in for (σ)
    to whileAnnoFix -
      (λ(H, u, i). ⦃contract-iter-adj-inv ´G H ´H u ´j
        ∧ select-nodes-prop ´G ´H ∧ ´u = u ∧ ´j ≤ length (ig-in-out-arcs ´G ´u)
∧ ´io-arcs = ig-in-out-arcs ´G ´u
         ∧ u ∈ set (ig-verts ´H)  ∧  IGraph-inv ´G  ∧ loop-free (mk-graph ´G) ∧
IGraph-inv ´H ∧ ´G = σ G ∧ ´i = i⦄)
      (λ-. (MEASURE length ´io-arcs − ´j))
      -
    annotate-named-loop-var-fix)
  apply vcg
     apply (fastforce simp: contract-iter-nodes-0)
     apply (match premises in select-nodes-prop - H for H ⇒ ‹rule exI[where
x=H]›)
    apply (fastforce simp: contract-iter-adj-0 contract-iter-nodes-inv-step elim: con-
tract-iter-adj-invE)
   apply (fastforce simp: contract-iter-adj-inv-step2 contract-iter-adj-inv-step1
```

IGraph-imp-ppg-mkg igraph-ig-add-eI snp-iapath-ends-in iadj-io-edge snp-vertexes)
apply (*fastforce simp*: *not-less intro*: *contract-iter-nodes-last*)
done

16.2.6 Procedure *is-K33*

definition *is-K33-colorize-inv* :: *IGraph* $\Rightarrow$ *ig-vertex* $\Rightarrow$ *nat* $\Rightarrow$ (*ig-vertex* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
is-K33-colorize-inv G u k blue $\equiv$ $\forall$ *v* $\in$ *set* (*ig-verts G*). *blue v* $\longleftrightarrow$
($\exists$ *i* < *k*. *v* = *ig-opposite G* (*ig-in-out-arcs G u* ! *i*) *u*)

definition *is-K33-component-size-inv* :: *IGraph* $\Rightarrow$ *nat* $\Rightarrow$ (*ig-vertex* $\Rightarrow$ *bool*) $\Rightarrow$ *nat* $\Rightarrow$ *bool* **where**
is-K33-component-size-inv G k blue cnt $\equiv$ *cnt* = *card* {*i*. *i* < *k* $\wedge$ *blue* (*ig-verts G* ! *i*)}

definition *is-K33-outer-inv* :: *IGraph* $\Rightarrow$ *nat* $\Rightarrow$ (*ig-vertex* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
is-K33-outer-inv G k blue $\equiv$ $\forall$ *i* < *k*. $\forall$ *v* $\in$ *set* (*ig-verts G*).
blue (*ig-verts G* ! *i*) = *blue v* $\longleftrightarrow$ (*ig-verts G* ! *i*, *v*) $\notin$ *set* (*ig-arcs G*)

definition *is-K33-inner-inv* :: *IGraph* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ (*ig-vertex* $\Rightarrow$ *bool*) $\Rightarrow$ *bool* **where**
is-K33-inner-inv G k l blue $\equiv$ $\forall$ *j* < *l*.
blue (*ig-verts G* ! *k*) = *blue* (*ig-verts G* ! *j*) $\longleftrightarrow$ (*ig-verts G* ! *k*, *ig-verts G* ! *j*) $\notin$ *set* (*ig-arcs G*)

lemma *is-K33-colorize-0*: *is-K33-colorize-inv G u 0* (λ-. *False*)
unfolding *is-K33-colorize-inv-def* **by** *auto*

lemma *is-K33-component-size-0*: *is-K33-component-size-inv G 0 blue 0*
unfolding *is-K33-component-size-inv-def* **by** *auto*

lemma *is-K33-outer-0*: *is-K33-outer-inv G 0 blue*
unfolding *is-K33-outer-inv-def* **by** *auto*

lemma *is-K33-inner-0*: *is-K33-inner-inv G k 0 blue*
unfolding *is-K33-inner-inv-def* **by** *auto*

lemma *is-K33-colorize-last*:
assumes *u* $\in$ *set* (*ig-verts G*)
shows *is-K33-colorize-inv G u* (*length* (*ig-in-out-arcs G u*)) *blue*
= ($\forall$ *v* $\in$ *set* (*ig-verts G*). *blue v* $\longleftrightarrow$ *iadj G u v*) (**is** *?L* = *?R*)
proof −
{ **fix** *v*
have ($\exists$ *i*<*length* (*ig-in-out-arcs G u*). *v* = *ig-opposite G* (*ig-in-out-arcs G u* ! *i*) *u*)
$\longleftrightarrow$ ($\exists$ *e* $\in$ *set* (*ig-in-out-arcs G u*). *v* = *ig-opposite G e u*) (**is** *?A* $\longleftrightarrow$ -)
by *auto* (*auto simp*: *in-set-conv-nth*)
also have ... $\longleftrightarrow$ *iadj G u v*

    using *assms* **by** (*force simp*: *iadj-io-edge ig-opposite-simps dest*: *iadjD*)
  finally have *?A* $\longleftrightarrow$ *iadj G u v* **. }**
 then show *?thesis* **unfolding** *is-K33-colorize-inv-def* **by** *auto*
qed

lemma *is-K33-component-size-last*:
 assumes k = *ig-verts-cnt G*
 shows *is-K33-component-size-inv G k blue cnt* $\longleftrightarrow$ *card* {$u \in$ *set* (*ig-verts G*). *blue u*} = *cnt*
proof −
 have $*$: {$u \in$ *set* (*ig-verts G*). *blue u*} = (λn. *ig-verts G* ! n) ' {i. $i <$ *ig-verts-cnt G* $\wedge$ *blue* (*ig-verts G* ! i)}
  by (*auto simp*: *in-set-conv-nth*)
 have *inj-on* (λn. *ig-verts G* ! n) {i. $i <$ *ig-verts-cnt G* $\wedge$ *blue* (*ig-verts G* ! i)}
  using *distinct-ig-verts* **by** (*auto simp*: *nth-eq-iff-index-eq intro*: *inj-onI*)
 with *assms* **show** *?thesis*
  unfolding $*$ *is-K33-component-size-inv-def*
  by (*auto intro*: *card-image*)
qed

lemma *is-K33-outer-last*:
 is-K33-outer-inv G (*ig-verts-cnt G*) *blue* $\longleftrightarrow$ ($\forall u \in$ *set* (*ig-verts G*). $\forall v \in$ *set* (*ig-verts G*).
  blue u = *blue v* $\longleftrightarrow$ (*u*,*v*) $\notin$ *set* (*ig-arcs G*))
 unfolding *is-K33-outer-inv-def* **by** (*simp add*: *All-set-ig-verts*)

lemma *is-K33-inner-last*:
 is-K33-inner-inv G k (*ig-verts-cnt G*) *blue* $\longleftrightarrow$ ($\forall v \in$ *set* (*ig-verts G*).
  blue (*ig-verts G* ! k) = *blue v* $\longleftrightarrow$ (*ig-verts G* ! k, v) $\notin$ *set* (*ig-arcs G*))
 unfolding *is-K33-inner-inv-def* **by** (*simp add*: *All-set-ig-verts*)

lemma *is-K33-colorize-step*:
 fixes *G u i blue*
 assumes *colorize*: *is-K33-colorize-inv G u k blue*
 shows *is-K33-colorize-inv G u* (*Suc k*) (*blue* (*ig-opposite G* (*ig-in-out-arcs G u* ! k) u := *True*))
 using *assms* **by** (*auto simp*: *is-K33-colorize-inv-def elim*: *less-SucE intro*: *less-SucI*)

lemma *is-K33-component-size-step1*:
 assumes *comp*:*is-K33-component-size-inv G k blue blue-cnt*
 assumes *blue*: *blue* (*ig-verts G* ! k)
 shows *is-K33-component-size-inv G* (*Suc k*) *blue* (*Suc blue-cnt*)
proof −
 have {i. $i <$ *Suc k* $\wedge$ *blue* (*ig-verts G* ! i)}
   = *insert k* {i. $i < k \wedge$ *blue* (*ig-verts G* ! i)}
  using *blue* **by** *auto*
 with *comp* **show** *?thesis*
  unfolding *is-K33-component-size-inv-def* **by** *auto*
qed

```
lemma is-K33-component-size-step2:
  assumes comp:is-K33-component-size-inv G k blue blue-cnt
  assumes blue: ¬blue (ig-verts G ! k)
  shows is-K33-component-size-inv G (Suc k) blue blue-cnt
proof –
  have {i. i < Suc k ∧ blue (ig-verts G ! i)} = {i. i < k ∧ blue (ig-verts G ! i)}
    using blue by (auto elim: less-SucE)
  with comp show ?thesis
    unfolding is-K33-component-size-inv-def by auto
qed

lemma is-K33-outer-step:
  assumes is-K33-outer-inv G i blue
  assumes is-K33-inner-inv G i (ig-verts-cnt G) blue
  shows is-K33-outer-inv G (Suc i) blue
  using assms unfolding is-K33-outer-inv-def is-K33-inner-last
  by (auto intro: less-SucI elim: less-SucE)

lemma is-K33-inner-step:
  assumes is-K33-inner-inv G i j blue
  assumes (blue (ig-verts G ! i) = blue (ig-verts G ! j)) ⟷ (ig-verts G ! i, ig-verts G ! j) ∉ set (ig-arcs G)
  shows is-K33-inner-inv G i (Suc j) blue
  using assms by (auto simp: is-K33-inner-inv-def elim: less-SucE)

lemma K33-mkg'I:
  fixes G col cnt
  defines u ≡ ig-verts G ! 0
  assumes ig: IGraph-inv G
  assumes iv-cnt: ig-verts-cnt G = 6 and c1-cnt: cnt = 3
  assumes colorize: is-K33-colorize-inv G u (length (ig-in-out-arcs G u)) blue
  assumes comp: is-K33-component-size-inv G (ig-verts-cnt G) blue cnt
  assumes outer: is-K33-outer-inv G (ig-verts-cnt G) blue
  shows K_3,3 (mk-graph' G)
proof –
  have u ∈ set (ig-verts G) unfolding u-def using iv-cnt by auto
  then have (∀ v∈set (ig-verts G). blue v ⟷ iadj G u v)
    using colorize by (rule is-K33-colorize-last[THEN iffD1])

  define U V where U = {u ∈ set (ig-verts G). ¬blue u} and V = {v ∈ set (ig-verts G). blue v}
  then have UV-set: U ⊆ set (ig-verts G) V ⊆ set (ig-verts G) U ∪ V = set (ig-verts G) U ∩ V = {}
    and fin-UV: finite U finite V by auto

  have card-verts: card (set (ig-verts G)) = 6
    using iv-cnt distinct-ig-verts by (simp add: distinct-card)
```

from *ig comp c1-cnt* **have** *card V = 3* **by** (*simp add*: *is-K33-component-size-last V-def*)
moreover have *card* (*U* ∪ *V*) = *6* **using** *UV-set distinct-ig-verts iv-cnt*
by (*auto simp*: *distinct-card*)
ultimately have *card U = 3*
by (*simp add*: *card-Un-disjoint*[*OF fin-UV UV-set*(*4*)])
note *cards* = ‹*card V = 3*› ‹*card U = 3*› *card-verts*

from *is-K33-outer-last*[*THEN iffD1*, *OF outer*]
have (∀ *u*∈*U*. ∀ *v*∈*V*. (*u*, *v*) ∈ *set* (*ig-arcs G*) ∧ (*v*, *u*) ∈ *set* (*ig-arcs G*))
∧ (∀ *u*∈*U*. ∀ *u'*∈*U*. (*u*, *u'*) ∉ *set* (*ig-arcs G*))
∧ (∀ *v*∈*V*. ∀ *v'*∈*V*. (*v*, *v'*) ∉ *set* (*ig-arcs G*))
unfolding *U-def V-def* **by** *auto*
then have *U* × *V* ⊆ *set* (*ig-arcs G*) *V* × *U* ⊆ *set* (*ig-arcs G*)
U × *U* ∩ *set* (*ig-arcs G*) = {} *V* × *V* ∩ *set* (*ig-arcs G*) = {}
by *auto*
moreover have *set* (*ig-arcs G*) ⊆ (*U* ∪ *V*) × (*U* ∪ *V*)
unfolding ‹*U* ∪ *V* = -› **by** (*auto simp*: *ig set-ig-arcs-verts*)
ultimately
have *conn*: *set* (*ig-arcs G*) = *U* × *V* ∪ *V* × *U*
by *blast*

interpret *ppg-mkg'*: *pair-fin-digraph mk-graph' G*
using *ig* **by** (*auto intro*: *IGraph-imp-ppd-mkg'*)

show *?thesis*
unfolding *complete-bipartite-digraph-pair-def mkg'-simps*
using *cards UV-set conn* **by** *simp metis*
qed

lemma *K33-mkg'E*:
assumes *K33*: $K_{3,3}$ (*mk-graph' G*)
assumes *ig*: *IGraph-inv G*
assumes *colorize*: *is-K33-colorize-inv G u* (*length* (*ig-in-out-arcs G u*)) *blue*
and *u*: *u* ∈ *set* (*ig-verts G*)
obtains *is-K33-component-size-inv G* (*ig-verts-cnt G*) *blue 3*
is-K33-outer-inv G (*ig-verts-cnt G*) *blue*
proof −
from *K33* **obtain** *U V* **where**
verts-G: *set* (*ig-verts G*) = *U* ∪ *V* **and**
arcs-G: *set* (*ig-arcs G*) = *U* × *V* ∪ *V* × *U* **and**
disj-UV: *U* ∩ *V* = {} **and**
card: *card U = 3 card V = 3*
unfolding *complete-bipartite-digraph-pair-def mkg'-simps* **by** *auto*

from *colorize u* **have** *col-adj*: ⋀*v*. *v* ∈ *set* (*ig-verts G*) ⟹ *blue v* ⟷ *iadj G u v*
using *is-K33-colorize-last* **by** *auto*

```
  have iadj-conv: ⋀u v. iadj G u v ⟷ (u,v) ∈ U × V ∪ V × U
    unfolding iadj-def arcs-G by auto

  { assume u ∈ U
    then have V = {v ∈ set (ig-verts G). blue v}
      using disj-UV by (auto simp: iadj-conv verts-G col-adj)
    then have is-K33-component-size-inv G (ig-verts-cnt G) blue 3
      using ig card by (simp add: is-K33-component-size-last)
    moreover
    have ⋀v. v ∈ U ∪ V ⟹ blue v ⟷ v ∈ V
      using ‹u ∈ U› disj-UV by (auto simp: verts-G col-adj iadj-conv)
    then have is-K33-outer-inv G (ig-verts-cnt G) blue
      using ‹U ∩ V = {}› by (subst is-K33-outer-last) (auto simp: arcs-G verts-G
)
    ultimately have ?thesis by (rule that) }
  moreover
  { assume u ∈ V
    then have U = {v ∈ set (ig-verts G). blue v}
      using disj-UV by (auto simp: iadj-conv verts-G col-adj)
    then have is-K33-component-size-inv G (ig-verts-cnt G) blue 3
      using ig card by (simp add: is-K33-component-size-last)
    moreover
    have ⋀v. v ∈ U ∪ V ⟹ blue v ⟷ v ∈ U
      using ‹u ∈ V› disj-UV by (auto simp: verts-G col-adj iadj-conv)
    then have is-K33-outer-inv G (ig-verts-cnt G) blue
      using ‹U ∩ V = {}› by (subst is-K33-outer-last) (auto simp: arcs-G verts-G
)
    ultimately have ?thesis by (rule that) }
  ultimately show ?thesis using verts-G u by blast
qed

lemma K33-card:
  assumes K_{3,3} (mk-graph' G) shows ig-verts-cnt G = 6
proof -
  from assms have card (verts (mk-graph' G)) = 6
    unfolding complete-bipartite-digraph-pair-def by (auto simp: card-Un-disjoint)
  then show ?thesis
    using distinct-ig-verts by (auto simp: mkg'-simps distinct-card)
qed

abbreviation (input) is-K33-colorize-inv-last :: IGraph ⇒ (ig-vertex ⇒ bool) ⇒
bool where
  is-K33-colorize-inv-last G blue ≡ is-K33-colorize-inv G (ig-verts G ! 0) (length
(ig-in-out-arcs G (ig-verts G ! 0))) blue

abbreviation (input) is-K33-component-size-inv-last :: IGraph ⇒ (ig-vertex ⇒
bool) ⇒ bool where
 is-K33-component-size-inv-last G blue ≡ is-K33-component-size-inv G (ig-verts-cnt
G) blue 3
```

lemma *is-K33-outerD*:
  assumes *is-K33-outer-inv G* (*ig-verts-cnt G*) *blue*
  assumes i < *ig-verts-cnt G j* < *ig-verts-cnt G*
  shows (*blue* (*ig-verts G* ! *i*) = *blue* (*ig-verts G* ! *j*)) ⟷ (*ig-verts G* ! *i*, *ig-verts G* ! *j*) ∉ *set* (*ig-arcs G*)
  using *assms* **unfolding** *is-K33-outer-last* **by** *auto*

lemma (**in** *is-K33-impl*) *is-K33-spec*:
  ∀*σ*. Γ $\vdash_t$ ⦃*σ*. *IGraph-inv* ´*G* ∧ *symmetric* (*mk-graph'* ´*G*)⦄
    ´*R* :== *PROC is-K33*(´*G*)
  ⦃ ´*G* = $^\sigma$*G* ∧ ´*R* = $K_{3,3}$(*mk-graph'* ´*G*) ⦄
  apply *vcg-step*
  apply (*rewrite*
    at whileAnno - (*named-loop* ''*colorize*'') - -
    in for (*σ*)
    to whileAnno -
      ⦃ *is-K33-colorize-inv* ´*G* ´*u* ´*i* ´*blue* ∧ ´*i* ≤ *length* ´*io-arcs*
        ∧ ´*io-arcs* = *ig-in-out-arcs* ´*G* ´*u* ∧ ´*u* = *ig-verts* ´*G* ! *0* ∧ ´*G* = $^\sigma$*G* ∧ *IGraph-inv* ´*G*
        ∧ ´*u* = *ig-verts* ´*G* ! *0* ∧ *ig-verts-cnt* ´*G* = *6*⦄
      (*MEASURE length* ´*io-arcs* − ´*i*)
      -
    annotate-named-loop-var)
  apply (*rewrite*
    at whileAnno - (*named-loop* ''*component-size*'') - -
    in for (*σ*)
    to whileAnnoFix -
      (*λblue*. ⦃ *is-K33-component-size-inv* ´*G* ´*i* ´*blue* ´*blue-cnt*
      ∧ ´*i* ≤ *ig-verts-cnt* ´*G* ∧ ´*blue* = *blue* ∧ ´*G* = $^\sigma$*G* ∧ *IGraph-inv* ´*G*
      ∧ *ig-verts-cnt* ´*G* = *6* ∧ *is-K33-colorize-inv-last* ´*G* ´*blue* ⦄)
      (*λ*-. (*MEASURE ig-verts-cnt* ´*G* − ´*i*))
      -
    annotate-named-loop-var-fix)
  apply (*rewrite*
    at whileAnno - (*named-loop* ''*connected-outer*'') - -
    in for (*σ*)
    to whileAnnoFix -
      (*λblue*. ⦃ *is-K33-outer-inv* ´*G* ´*i* ´*blue* ∧ ´*i* ≤ *ig-verts-cnt* ´*G*
        ∧ ´*blue* = *blue* ∧ ´*G* = $^\sigma$*G* ∧ *IGraph-inv* ´*G*
      ∧ *ig-verts-cnt* ´*G* = *6* ∧ *is-K33-colorize-inv-last* ´*G* ´*blue* ∧ *is-K33-component-size-inv-last* ´*G* ´*blue* ⦄)
      (*λ*-. (*MEASURE ig-verts-cnt* ´*G* − ´*i*))
      -
    annotate-named-loop-var-fix)
  apply (*rewrite*
    at whileAnno - (*named-loop* ''*connected-inner*'') - -
    in for (*σ*)
    to whileAnnoFix -

(λ(*i*,*blue*). ⦃ *is-K33-inner-inv ´G ´i ´j ´blue* ∧ *´j* ≤ *ig-verts-cnt ´G*
∧ *´i* = *i* ∧ *´i* < *ig-verts-cnt ´G* ∧ *´blue* = *blue* ∧ *´G* = $^{\sigma}$ *G* ∧ *IGraph-inv*
´G ∧ *´u* = *ig-verts ´G ! ´i*
∧ *ig-verts-cnt ´G* = *6* ∧ *is-K33-colorize-inv-last ´G ´blue* ∧ *is-K33-component-size-inv-last*
´G ´blue ⦄)
(λ-. (*MEASURE ig-verts-cnt ´G* − *´j*))
-
annotate-named-loop-var-fix)
apply *vcg*
apply (*fastforce simp*: *is-K33-colorize-0 is-K33-component-size-0 is-K33-outer-0*
is-K33-component-size-last
elim: *K33-mkg′E dest*: *K33-card intro*: *K33-mkg′I*)
apply (*fastforce simp add*: *is-K33-colorize-step*)
apply (*fastforce simp*: *is-K33-colorize-0 is-K33-component-size-0 is-K33-outer-0*
is-K33-component-size-last
elim: *K33-mkg′E intro*: *K33-mkg′I*)
apply (*fastforce simp*: *is-K33-component-size-step1 is-K33-component-size-step2*)
apply (*fastforce simp*: *is-K33-inner-0 is-K33-outer-step*)
apply (*simp only*: *simp-thms*)
apply (*intro conjI allI impI notI*)
apply (*fastforce elim*: *K33-mkg′E dest*: *is-K33-outerD*)
apply (*fastforce elim*: *K33-mkg′E dest*: *is-K33-outerD*)
apply (*simp add*: *is-K33-inner-step*; *fail*)
apply *linarith*
done

16.2.7 Procedure *is-K5*

definition
is-K5-outer-inv G k ≡ ∀ *i*<*k*. ∀ *v* ∈ *set* (*ig-verts G*). *ig-verts G ! i* ≠ *v*
⟷ (*ig-verts G ! i*, *v*) ∈ *set* (*ig-arcs G*)

definition
is-K5-inner-inv G k l ≡ ∀ *j* < *l*. *ig-verts G ! k* ≠ *ig-verts G ! j*
⟷ (*ig-verts G ! k*, *ig-verts G ! j*) ∈ *set* (*ig-arcs G*)

lemma *K5-card*:
assumes K_5 (*mk-graph′ G*) **shows** *ig-verts-cnt G* = *5*
using *assms distinct-ig-verts* **unfolding** *complete-digraph-pair-def*
by (*auto simp add*: *mkg′-simps distinct-card*)

lemma *is-K5-inner-0*: *is-K5-inner-inv G k 0*
unfolding *is-K5-inner-inv-def* **by** *auto*

lemma *is-K5-inner-last*:
assumes *l* = *ig-verts-cnt G*
shows *is-K5-inner-inv G k l* ⟷ (∀ *v* ∈ *set* (*ig-verts G*). *ig-verts G ! k* ≠ *v*
⟷ (*ig-verts G ! k*, *v*) ∈ *set* (*ig-arcs G*))
proof −

have $\bigwedge v.\ v \in set\ (ig\text{-}verts\ G) \Longrightarrow \exists j{<}ig\text{-}verts\text{-}cnt\ G.\ ig\text{-}verts\ G\ !\ j = v$
by (*auto simp*: *in-set-conv-nth*)
then show *?thesis* **using** *assms* **unfolding** *is-K5-inner-inv-def*
by *auto metis*
qed

lemma *is-K5-outer-step*:
assumes *is-K5-outer-inv G k*
assumes *is-K5-inner-inv G k* (*ig-verts-cnt G*)
shows *is-K5-outer-inv G* (*Suc k*)
using *assms* **unfolding** *is-K5-outer-inv-def*
by (*auto simp*: *is-K5-inner-last elim*: *less-SucE*)

lemma *is-K5-outer-last*:
assumes *is-K5-outer-inv G* (*ig-verts-cnt G*)
assumes *IGraph-inv G ig-verts-cnt G* = *5 symmetric* (*mk-graph′ G*)
shows K_5 (*mk-graph′ G*)
proof −
interpret *ppg-mkg′*: *pair-fin-digraph mk-graph′ G*
using *assms*(*2*) **by** (*auto intro*: *IGraph-imp-ppd-mkg′*)
have $\bigwedge u\ v.\ (u, v) \in set\ (ig\text{-}arcs\ G) \Longrightarrow u \neq v$
using *assms*(*1*,*2*) **unfolding** *is-K5-outer-inv-def ig-verts-cnt-def*
by (*metis in-set-conv-nth set-ig-arcs-verts*(*2*))
then interpret *ppg-mkg′*: *pair-graph* (*mk-graph′ G*)
using *assms*(*4*) **by** *unfold-locales* (*auto simp*: *mkg′-simps arc-to-ends-def*)
have $\bigwedge a\ b.\ a \in pverts\ (mk\text{-}graph'\ G) \Longrightarrow$
$b \in pverts\ (mk\text{-}graph'\ G) \Longrightarrow a \neq b \Longrightarrow (a, b) \in parcs\ (mk\text{-}graph'\ G)$
using *assms*(*1*) **unfolding** *is-K5-outer-inv-def mkg′-simps*
by (*metis in-set-conv-nth ig-verts-cnt-def*)
moreover
have *card* (*pverts* (*mk-graph′ G*)) = *5*
using ‹*ig-verts-cnt G* = *5*› *distinct-ig-verts* **by** (*auto simp*: *mkg′-simps dis-tinct-card*)
ultimately
show *?thesis*
unfolding *complete-digraph-pair-def*
by (*auto dest*: *ppg-mkg′.in-arcsD1 ppg-mkg′.in-arcsD2 ppg-mkg′.no-loops′*)
qed

lemma *is-K5-inner-step*:
assumes *is-K5-inner-inv G k l*
assumes $k < ig\text{-}verts\text{-}cnt\ G$
assumes $k \neq l \longleftrightarrow (ig\text{-}verts\ G\ !\ k,\ ig\text{-}verts\ G\ !\ l) \in set\ (ig\text{-}arcs\ G)$
shows *is-K5-inner-inv G k* (*Suc l*)
using *assms distinct-ig-verts* **unfolding** *is-K5-inner-inv-def*
apply (*auto elim*: *less-SucE*)
by (*metis* (*opaque-lifting*, *no-types*) *Suc-lessD less-SucE less-trans-Suc linorder-neqE-nat nth-eq-iff-index-eq*)

lemma *iK5E*:
assumes K_5 (*mk-graph′ G*)
obtains $\textit{ig-verts-cnt}\ G = 5\ [\![i < \textit{ig-verts-cnt}\ G;\ j < \textit{ig-verts-cnt}\ G]\!] \Longrightarrow i \neq j$
$\longleftrightarrow (\textit{ig-verts}\ G\ !\ i,\ \textit{ig-verts}\ G\ !\ j) \in \textit{set}\ (\textit{ig-arcs}\ G)$
proof
show $\textit{ig-verts-cnt}\ G = 5$
$i < \textit{ig-verts-cnt}\ G \Longrightarrow j < \textit{ig-verts-cnt}\ G \Longrightarrow$
$(i \neq j) = ((\textit{ig-verts}\ G\ !\ i,\ \textit{ig-verts}\ G\ !\ j) \in \textit{set}\ (\textit{ig-arcs}\ G))$
using *assms distinct-ig-verts*
by (*auto simp*: *complete-digraph-pair-def mkg′-simps distinct-card nth-eq-iff-index-eq*)
qed

lemma (**in** *is-K5-impl*) *is-K5-spec*:
$\forall \sigma.\ \Gamma \vdash_t \{\!|\sigma.\ \textit{IGraph-inv}\ \acute{}G \wedge \textit{symmetric}\ (\textit{mk-graph}'\ \acute{}G)|\!\}$
$\acute{}R :== \textit{PROC is-K5}(\acute{}G)$
$\{\!|\ \acute{}G = {}^{\sigma}G \wedge \acute{}R = K_5(\textit{mk-graph}'\ \acute{}G)\ |\!\}$
apply *vcg-step*
apply (*rewrite*
at whileAnno - (*named-loop ''outer-loop''*) - -
in for (σ)
to whileAnno -
$\{\!|\ \textit{is-K5-outer-inv}\ \acute{}G\ \acute{}i \wedge \acute{}i \leq 5 \wedge \textit{IGraph-inv}\ \acute{}G \wedge \textit{symmetric}\ (\textit{mk-graph}'$
$\acute{}G) \wedge \acute{}G = {}^{\sigma}G \wedge \textit{ig-verts-cnt}\ \acute{}G = 5\ |\!\}$
$(\textit{MEASURE}\ 5 - \acute{}i)$
-
annotate-named-loop-var)
apply (*rewrite*
at whileAnno - (*named-loop ''inner-loop''*) - -
in for (σ)
to whileAnnoFix -
$(\lambda i.\ \{\!|\ \textit{is-K5-inner-inv}\ \acute{}G\ \acute{}i\ \acute{}j$
$\wedge\ \acute{}j \leq 5 \wedge \acute{}i < 5 \wedge \textit{IGraph-inv}\ \acute{}G \wedge \textit{symmetric}\ (\textit{mk-graph}'\ \acute{}G) \wedge \acute{}G =$
${}^{\sigma}G \wedge \acute{}i = i$
$\wedge\ \textit{ig-verts-cnt}\ \acute{}G = 5 \wedge \acute{}u = \textit{ig-verts}\ \acute{}G\ !\ \acute{}i|\!\})$
$(\lambda\text{-}.\ (\textit{MEASURE}\ 5 - \acute{}j))$
-
annotate-named-loop-var-fix)
apply *vcg*
apply (*fastforce simp*: *is-K5-outer-inv-def intro*: *K5-card*)
apply (*fastforce simp add*: *is-K5-inner-0 is-K5-outer-step*)
apply (*fastforce simp*: *is-K5-inner-step elim*: *iK5E*)
apply (*fastforce simp*: *is-K5-outer-last*)
done

16.2.8 Soundness of the Checker

lemma *planar-theorem*:
assumes *pair-pseudo-graph G pair-pseudo-graph K*

and *subgraph K G*
and $K_{3,3}$ (*contr-graph K*) ∨ K_5 (*contr-graph K*)
shows ¬*kuratowski-planar G*
using *assms*
by (*auto dest*: *pair-pseudo-graph.kuratowski-contr*)

definition *witness* :: *'a pair-pre-digraph* ⇒ *'a pair-pre-digraph* ⇒ *bool* **where**
witness G K ≡ *loop-free K* ∧ *pair-pseudo-graph K* ∧ *subgraph K G*
∧ ($K_{3,3}$ (*contr-graph K*) ∨ K_5 (*contr-graph K*))

lemma *witness* (*mk-graph G*) (*mk-graph K*) ⟷ *pair-pre-digraph.certify* (*mk-graph G*) (*mk-graph K*) ∧ *loop-free* (*mk-graph K*)
by (*auto simp*: *witness-def pair-pre-digraph.certify-def Let-def wf-digraph-wp-iff wellformed-pseudo-graph-mkg*)

lemma *pwd-imp-ppg-mkg*:
assumes *pair-wf-digraph* (*mk-graph G*)
shows *pair-pseudo-graph* (*mk-graph G*)
proof −
interpret *pair-wf-digraph mk-graph G* **by** *fact*
show *?thesis*
apply *unfold-locales*
apply (*auto simp*: *mkg-simps finite-symcl-iff*)
apply (*auto simp*: *mk-graph-def symmetric-mk-symmetric*)
done
qed

theorem (**in** *check-kuratowski-impl*) *check-kuratowski-spec*:
∀σ. Γ $\vdash_t$ ⦃σ. *pair-wf-digraph* (*mk-graph ´G*)⦄
´R :== *PROC check-kuratowski*(*´G*, *´K*)
⦃ *´G* = $^{\sigma}$*G* ∧ *´K* = $^{\sigma}$*K* ∧ *´R* ⟷ *witness* (*mk-graph ´G*) (*mk-graph ´K*)⦄
by *vcg* (*auto simp*: *witness-def IGraph-inv-conv' pwd-imp-ppg-mkg*)

lemma *check-kuratowski-correct*:
assumes *pair-pseudo-graph G*
assumes *witness G K*
shows ¬*kuratowski-planar G*
using *assms*
by (*intro planar-theorem*[**where** *K*=*K*]) (*auto simp*: *witness-def*)

lemma *check-kuratowski-correct-comb*:
assumes *pair-pseudo-graph G*
assumes *witness G K*
shows ¬*comb-planar G*
using *assms* **by** (*metis check-kuratowski-correct comb-planar-compat*)

lemma *check-kuratowski-complete*:
assumes *pair-pseudo-graph G pair-pseudo-graph K loop-free K*

assumes *subgraph K G*
assumes *subdivision-pair H K* $K_{3,3}$ *H* $\vee$ K_5 *H*
shows *witness G K*
using *assms* **by** (*auto simp*: *witness-def intro*: *K33-contractedI K5-contractedI*)

end
theory *AutoCorres-Misc* **imports**
../l4v/lib/OptionMonadWP
begin

17 Auxilliary Lemmas for Autocorres

17.1 Option monad

definition *owhile-inv* :: $('a \Rightarrow 's \Rightarrow bool) \Rightarrow ('a \Rightarrow ('s,'a)\ lookup) \Rightarrow 'a \Rightarrow ('a \Rightarrow 's \Rightarrow bool) \Rightarrow 'a\ rel \Rightarrow ('s,'a)\ lookup$ **where**
$owhile\text{-}inv\ c\ b\ a\ I\ R \equiv owhile\ c\ b\ a$

lemma *owhile-unfold*: $owhile\ C\ B\ r\ s = ocondition\ (C\ r)\ (B\ r\ |{>}{>}\ owhile\ C\ B)\ (oreturn\ r)\ s$
by (*auto simp*: *ocondition-def obind-def oreturn-def owhile-def option-while-simps split*: *option.split*)

lemma *ovalidNF-owhile*:
assumes $\bigwedge s.\ P\ r\ s \Longrightarrow I\ r\ s$
and $\bigwedge r\ s.\ ovalidNF\ (\lambda s'.\ I\ r\ s' \wedge C\ r\ s' \wedge s' = s)\ (B\ r)\ (\lambda r'\ s'.\ I\ r'\ s' \wedge (r', r) \in R)$
and *wf R*
and $\bigwedge r\ s.\ I\ r\ s \Longrightarrow \neg\ C\ r\ s \Longrightarrow Q\ r\ s$
shows $ovalidNF\ (P\ r)\ (OptionMonad.owhile\ C\ B\ r)\ Q$
unfolding *ovalidNF-def*
proof (*intro allI impI*)
fix *s* **assume** *P r s*
then have *I r s* **by** *fact*
moreover note ‹*wf R*›
moreover have $\bigwedge r\ r'.\ I\ r\ s \Longrightarrow C\ r\ s \Longrightarrow B\ r\ s = Some\ r' \Longrightarrow (r', r) \in R$
using *assms*(*2*) **unfolding** *ovalidNF-def* **by** *fastforce*
moreover have $\bigwedge r\ r'.\ I\ r\ s \Longrightarrow C\ r\ s \Longrightarrow B\ r\ s = Some\ r' \Longrightarrow I\ r'\ s$
using *assms*(*2*) **unfolding** *ovalidNF-def* **by** *blast*
moreover have $\bigwedge r.\ I\ r\ s \Longrightarrow C\ r\ s \Longrightarrow B\ r\ s = None \Longrightarrow$
$None \neq None \wedge (\forall r'.\ None = Some\ r' \longrightarrow Q\ r'\ s)$
using *assms*(*2*) **unfolding** *ovalidNF-def* **by** *blast*
moreover have $\bigwedge r.\ I\ r\ s \Longrightarrow \neg\ C\ r\ s \Longrightarrow Some\ r \neq None \wedge (\forall r'.\ Some\ r = Some\ r' \longrightarrow Q\ r'\ s)$
using *assms*(*4*) **unfolding** *ovalidNF-def* **by** *blast*
ultimately
show $owhile\ C\ B\ r\ s \neq None \wedge (\forall r'.\ owhile\ C\ B\ r\ s = Some\ r' \longrightarrow Q\ r'\ s)$
by (*rule owhile-rule*[**where** *I*=*I*])
qed

lemma *ovalidNF-owhile-inv*[*wp*]:
assumes $\bigwedge r\ s.\ ovalidNF\ (\lambda s'.\ I\ r\ s' \wedge\ C\ r\ s' \wedge\ s' = s)\ (B\ r)\ (\lambda r'\ s'.\ I\ r'\ s' \wedge (r', r) \in R)$
and *wf R*
and $\bigwedge r\ s.\ I\ r\ s \Longrightarrow \neg\ C\ r\ s \Longrightarrow Q\ r\ s$
shows *ovalidNF* (*I r*) (*owhile-inv C B r I R*) *Q*
unfolding *owhile-inv-def* **using** - *assms* **by** (*rule ovalidNF-owhile*)

end
theory *Setup-AutoCorres*
imports
Case-Labeling.Case-Labeling
HOL−Eisbach.Eisbach
AutoCorres-Misc
begin

18 AutoCorres setup for VCG labelling

Theorem collections for the VCG

ML-file ‹*../../Case-Labeling/util.ML*›

```
ML ‹
  fun vcg-tac nt-rules nt-comb ctxt =
    let
      val rules = Named-Theorems.get ctxt nt-rules
      val comb = Named-Theorems.get ctxt nt-comb
     in REPEAT-ALL-NEW-FWD ( resolve-tac ctxt rules ORELSE' (resolve-tac
ctxt comb THEN' resolve-tac ctxt rules)) end
›
```

named-theorems *vcg-l*
named-theorems *vcg-l-comb*
named-theorems *vcg-elim*
named-theorems *vcg-simp*

method-setup *vcg-l* = ‹
Scan.succeed (*fn ctxt* => *SIMPLE-METHOD* (*FIRSTGOAL* (*vcg-tac* @{*named-theorems vcg-l*} @{*named-theorems vcg-l-comb*} *ctxt*)))
›

method *vcg-l′* = (*vcg-l*; (*elim vcg-elim*)?; (*unfold vcg-simp*)?)

method *vcg-casify* = (*rule Initial-Label*, *vcg-l′*, *casify*)

18.1 Labeled VCG theorems for branching

definition *BRANCH P* $\equiv$ *P*

named-theorems *branch-l*
named-theorems *branch-l-comb*

context begin
  interpretation *Labeling-Syntax* **.**

  lemma *DC-if* [*branch-l*]:
    fixes *ct* **defines** $ct' \equiv \lambda pos\ name.\ (name,\ pos,[])\ \#\ ct$
    assumes $a \Longrightarrow C\langle Suc\ inp, ct'\ inp\ ''then'',\ outp'{:}\ b\rangle$
    assumes $\neg a \Longrightarrow C\langle Suc\ outp', ct'\ outp'\ ''else'',\ outp{:}\ c\rangle$
    shows $C\langle inp, ct, outp{:}\ BRANCH\ (\mathit{if}\ a\ \mathit{then}\ b\ \mathit{else}\ c)\rangle$
    using *assms*(*2*−) **unfolding** *LABEL-simps BRANCH-def* **by** *auto*

  lemma *DC-final*:
    assumes $V\langle(''g'', inp, []),\ ct{:}\ a\rangle$
    shows $C\langle inp, ct, Suc\ inp{:}\ a\rangle$
    using *assms* **unfolding** *LABEL-simps BRANCH-def* **by** *auto*

end

method-setup *branch-l* = ‹
Scan.succeed (*fn ctxt* => *SIMPLE-METHOD* (*FIRSTGOAL* (*vcg-tac* @{*named-theorems branch-l*} @{*named-theorems branch-l-comb*} *ctxt*)))
›

method *branch-casify* = ((*rule Initial-Label*, *branch-l*; (*rule DC-final*)?), *casify*)

18.2 Labelled VCG theorems for the option monad

definition
  lpred-conj :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ *bool*) (**infixr** ‹*land*› *35*)
where
  lpred-conj P Q $\equiv \lambda x.\ P\ x \wedge Q\ x$

context begin
  interpretation *Labeling-Syntax* **.**

  lemma *ovalidNF-obind-K-bind* [*vcg-l*]:
    assumes *CTXT* (*Suc OC1*) *CT OC* (*ovalidNF R g Q*)
      and *CTXT IC CT OC1* (*ovalidNF P f* (λ-. *R*))
    shows *CTXT IC CT OC* (*ovalidNF P* (*f* |>> *K-bind g*) *Q*)
    using *assms* **unfolding** *LABEL-simps* **by** *wp*

  lemma *L-ovalidNF-obind-oreturn* [*vcg-l*]:
    assumes *CTXT IC CT OC* (*ovalidNF P* (*g x*) *Q*)
    shows *CTXT IC CT OC* (*ovalidNF P* (*oreturn x* |>> *g*) *Q*)

using *assms* **by** (*simp add*: *LABEL-simps*)

lemma *L-ovalidNF-obind*[*vcg-l*]:
assumes ⋀*r*. *CTXT* (*Suc OC1*) (("*bind*", *Suc OC1*, [*VAR r*]) # *CT*) *OC*
(*ovalidNF* (*R r*) (*g r*) *Q*)
and *CTXT IC CT OC1* (*ovalidNF P f R*)
shows *CTXT IC CT OC* (*ovalidNF P* (*f* |>> (λ*r*. *g r*)) *Q*)
using *assms* **unfolding** *LABEL-simps* **by** *wp*

lemma *ovalidNF-K-bind*[*vcg-l*]:
assumes *CTXT IC CT OC* (*ovalidNF P f Q*)
shows *CTXT IC CT OC* (*ovalidNF P* (*K-bind f x*) *Q*)
using *assms* **by** *simp*

lemma *L-ovalidNF-prod-case*[*vcg-l*]:
assumes ⋀*x y*. *SPLIT v* (*x*,*y*) ⟹ *CTXT IC CT OC* (*ovalidNF* (*P x y*) (*B x y*) *Q*)
shows *CTXT IC CT OC* (*ovalidNF* (*case v of* (*x*, *y*) ⇒ *P x y*) (*case v of* (*x*, *y*) ⇒ *B x y*) *Q*)
using *assms* **unfolding** *LABEL-simps* **by** (*auto simp*: *ovalidNF-def*)

lemma *L-ovalidNF-oreturn-NF*[*vcg-l*]:
shows *CTXT IC CT IC* (*ovalidNF* (*P x*) (*oreturn x*) *P*)
unfolding *LABEL-simps* **by** *wp*

lemma *L-ovalidNF-owhile-inv*[*vcg-l*]:
fixes *CT IC*
defines *CT'* ≡ λ*r*. ("*while*", *IC*, [*VAR r*]) # *CT*
assumes ⋀*r s*. *CTXT IC* (("*invariant*", *IC*, [*VAR s*]) # *CT' r*) *OC*
(*ovalidNF*
(*BIND* "*loop-inv*" *IC* (*I r*) *land*
BIND "*loop-cond*" *IC* (*C r*) *land*
BIND "*loop-var*" *IC* (λ*s'*. *s'* = *s*))
(*B r*)
(λ*r'*. *BIND* "*inv*" *IC* (*I r'*) *land BIND* "*var*" *IC* (λ-. (*r'*, *r*) ∈ *R*)))
and ⋀*r*. *VC* ("*wf*", *OC*, []) (*CT' r*) (*wf R*)
and ⋀*r s*. *I r s* ⟹ ¬ *C r s* ⟹
VC ("*postcondition*", *Suc OC*, [*VAR s*]) (*CT' r*) (*Q r s*)
shows *CTXT IC CT* (*Suc OC*) (*ovalidNF* (*I r*) (*owhile-inv C B r I R*) *Q*)
using *assms* **unfolding** *LABEL-simps lpred-conj-def* **by** *wp auto*

lemma *L-ovalidNF-wp-comb2*[*vcg-l-comb*]:
assumes *CTXT IC CT OC* (*ovalidNF P f Q*)
and ⋀*s*. *P' s* ⟹ *VC* ("*weaken*", *IC*, [*VAR s*]) *CT* (*P s*)
shows *CTXT IC CT OC* (*ovalidNF P' f Q*)
using *assms* **unfolding** *LABEL-simps* **by** (*rule ovalidNF-wp-comb2*)

lemma *L-condition-NF-wp*[*vcg-l*]:
fixes *CT IC*

defines $CT' \equiv (''if'', IC, []) \# CT$
assumes $CTXT\ IC\ ((''then'', IC, []) \# CT')\ OC1\ (ovalidNF\ L\ l\ Q)$
and $CTXT\ (Suc\ OC1)\ ((''else'', Suc\ OC1, []) \# CT')\ OC\ (ovalidNF\ R\ r\ Q)$
shows $CTXT\ IC\ CT\ OC\ (ovalidNF\ (\lambda s.\ BRANCH\ (if\ C\ s\ then\ L\ s\ else\ R\ s))\ (ocondition\ C\ l\ r)\ Q)$
using *assms* **unfolding** *LABEL-simps BRANCH-def* **by** *wp*

lemma *L-ogets-NF-wp*[*vcg-l*]: $CTXT\ IC\ CT\ IC\ (ovalidNF\ (\lambda s.\ P\ (f\ s)\ s)\ (ogets\ f)\ P)$
unfolding *LABEL-simps* **by** *wp*

lemma *elim-land*[*vcg-elim*]:
assumes $(P\ land\ Q)\ s$ **obtains** $P\ s\ Q\ s$
using *assms* **by** (*auto simp*: *lpred-conj-def*)

lemma *simp-bind*[*vcg-simp*]: $BIND\ ct\ n\ P\ s \longleftrightarrow BIND\ ct\ n\ (P\ s)$
by (*auto simp*: *LABEL-simps*)

lemma *simp-land*[*vcg-simp*]: $(P\ land\ Q)\ s \longleftrightarrow P\ s \wedge Q\ s$
by (*auto simp*: *lpred-conj-def*)
end

end

19 Verififcation of a Planarity Checker

theory *Check-Planarity-Verification*
imports
../Planarity/Graph-Genus
Setup-AutoCorres
HOL−Library.Rewrite
begin

19.1 Implementation Types

type-synonym $IVert = nat$
type-synonym $IEdge = IVert \times IVert$
type-synonym $IGraph = IVert\ list \times IEdge\ list$

abbreviation (*input*) *ig-edges* :: $IGraph \Rightarrow IEdge\ list$ **where**
$ig\text{-}edges\ G \equiv snd\ G$

abbreviation (*input*) *ig-verts* :: $IGraph \Rightarrow IVert\ list$ **where**
$ig\text{-}verts\ G \equiv fst\ G$

definition *ig-tail* :: $IGraph \Rightarrow nat \Rightarrow IVert$ **where**
$ig\text{-}tail\ IG\ a = fst\ (ig\text{-}edges\ IG\ !\ a)$

definition *ig-head* :: $IGraph \Rightarrow nat \Rightarrow IVert$ **where**

ig-head IG a = *snd* (*ig-edges IG* ! *a*)

type-synonym *IMap* = (*nat* ⇒ *nat*) × (*nat* ⇒ *nat*) × (*nat* ⇒ *nat*)

definition *im-rev* :: *IMap* ⇒ (*nat* ⇒ *nat*) **where**
im-rev iM = *fst iM*

definition *im-succ* :: *IMap* ⇒ (*nat* ⇒ *nat*) **where**
im-succ iM = *fst* (*snd iM*)

definition *im-pred* :: *IMap* ⇒ (*nat* ⇒ *nat*) **where**
im-pred iM = *snd* (*snd iM*)

definition *mk-graph* :: *IGraph* ⇒ (*IVert*, *nat*) *pre-digraph* **where**
mk-graph IG ≡ (|
verts = *set* (*ig-verts IG*),
arcs = {*0*..< *length* (*ig-edges IG*)},
tail = *ig-tail IG*,
head = *ig-head IG*
|)

lemma *mkg-simps*:
verts (*mk-graph IG*) = *set* (*ig-verts IG*)
tail (*mk-graph IG*) = *ig-tail IG*
head (*mk-graph IG*) = *ig-head IG*
by (*auto simp*: *mk-graph-def*)

lemma *arcs-mkg*: *arcs* (*mk-graph IG*) = {*0*..< *length* (*ig-edges IG*)}
by (*auto simp*: *mk-graph-def*)

lemma *arc-to-ends-mkg*: *arc-to-ends* (*mk-graph IG*) *a* = *ig-edges IG* ! *a*
by (*auto simp*: *arc-to-ends-def mkg-simps ig-tail-def ig-head-def*)

definition *mk-map* :: (-, *nat*) *pre-digraph* ⇒ *IMap* ⇒ *nat pre-map* **where**
mk-map G iM ≡ (|
edge-rev = *perm-restrict* (*im-rev iM*) (*arcs G*),
edge-succ = *perm-restrict* (*im-succ iM*) (*arcs G*)
|)

lemma *mkm-simps*:
edge-rev (*mk-map G iM*) = *perm-restrict* (*im-rev iM*) (*arcs G*)
edge-succ (*mk-map G iM*) = *perm-restrict* (*im-succ iM*) (*arcs G*)
by (*auto simp*: *mk-map-def*)

lemma *es-eq-im*: *a* ∈ *arcs* (*mk-graph iG*) ⟹ *edge-succ* (*mk-map* (*mk-graph iG*) *iM*) *a* = *im-succ iM a*
by (*auto simp*: *mkm-simps arcs-mkg perm-restrict-simps*)

19.2 Implementation

```
definition is-map iG iM ≡
  DO ecnt ← oreturn (length (snd iG));
     vcnt ← oreturn (length (fst iG));
     (i, revOk) ← owhile
      (λ(i, ok) s. i < ecnt ∧ ok)
      (λ(i, ok).
        DO
          j ← oreturn (im-rev iM i);
          revIn ← oreturn (j < length (ig-edges iG));
          revNeq ← oreturn (j ≠ i);
          revRevs ← oreturn (ig-edges iG ! j = prod.swap (ig-edges iG ! i));
          invol ← oreturn (im-rev iM j = i);
          oreturn (i + 1, revIn ∧ revNeq ∧ revRevs ∧ invol)
        OD)
      (0, True);
     (i, succPerm) ← owhile
      (λ(i, ok) s. i < ecnt ∧ ok)
      (λ(i, ok).
        DO
          j ← oreturn (im-succ iM i);
          succIn ← oreturn (j < length (ig-edges iG));
          succEnd ← oreturn (ig-tail iG i = ig-tail iG j);
          isPerm ← oreturn (im-pred iM j = i);
          oreturn (i + 1, succIn ∧ succEnd ∧ isPerm)
        OD)
      (0, True);
     (i, succOrbits, V, A) ← owhile
      (λ(i, ok, V, A) s. i < ecnt ∧ succPerm ∧ ok)
      (λ(i, ok, V, A).
        DO
          (x,V,A) ← ocondition (λ-. ig-tail iG i ∈ V)
            (oreturn (i ∈ A, V, A))
            (DO
              (A',j) ← owhile
                (λ(A',j) s. j ∉ A')
                (λ(A',j). DO
                    A' ← oreturn (insert j A');
                    j ← oreturn (im-succ iM j);
                    oreturn (A', j)
                  OD)
                ({},i);
              V ← oreturn (insert (ig-tail iG j) V);
              oreturn (True,V,A ∪ A')
            OD);
          oreturn (i + 1, x, V, A)
        OD)
      (0, True, {}, {});
     oreturn (revOk ∧ succPerm ∧ succOrbits)
```

```
  OD

definition isolated-nodes :: IGraph ⇒ - ⇒ nat option where
isolated-nodes iG ≡
  DO ecnt ← oreturn (length (snd iG));
     vcnt ← oreturn (length (fst iG));
     (i, nz) ←
     owhile
      (λ(i, nz) a. i < vcnt)
      (λ(i, nz).
         DO v ← oreturn (fst iG ! i);
            j ← oreturn 0;
              ret ← ocondition (λs. j < ecnt) (oreturn (ig-tail iG j ≠ v)) (oreturn
False);
            ret ← ocondition (λs. ret) (oreturn (ig-head iG j ≠ v)) (oreturn ret);
            (j, -) ←
            owhile
             (λ(j, cond) a. cond)
             (λ(j, cond).
                 DO j ← oreturn (j + 1);
                       cond ← ocondition (λs. j < ecnt) (oreturn (ig-tail iG j ≠ v))
(oreturn False);
                 cond ← ocondition (λs. cond) (oreturn (ig-head iG j ≠ v)) (oreturn
cond);
                    oreturn (j, cond)
                 OD)
             (j, ret);
            nz ← oreturn (if j = ecnt then nz + 1 else nz);
            oreturn (i + 1, nz)
         OD)
      (0, 0);
     oreturn nz
  OD

definition face-cycles :: IGraph ⇒ nat pre-map ⇒ - ⇒ nat option where
face-cycles iG iM ≡
  DO ecnt ← oreturn (length (snd iG));
     (edge-info, c, i) ←
     owhile
      (λ(edge-info, c, i) s. i < ecnt)
      (λ(edge-info, c, i).
         DO (edge-info, c) ←
            ocondition (λs. i ∉ edge-info)
             (DO j ← oreturn i;
                 edge-info ← oreturn (insert j edge-info);
                 ret' ← oreturn (pre-digraph-map.face-cycle-succ iM j);
                 (edge-info, j) ←
                 owhile
```

```
                    (λ(edge-info, j) s. i ≠ j)
                    (λ(edge-info, j).
                         oreturn (insert j edge-info, pre-digraph-map.face-cycle-succ iM
j))
                    (edge-info, ret');
                   oreturn (edge-info, c + 1)
                 OD)
                (oreturn (edge-info, c));
               oreturn (edge-info, c, i + 1)
             OD)
      ({}, 0, 0);
     oreturn c
  OD
```

definition *euler-genus iG iM c* ≡
```
   DO n ← oreturn (length (ig-edges iG));
      m ← oreturn (length (ig-verts iG));
      nz ← isolated-nodes iG;
      fc ← face-cycles iG iM;
      oreturn ((int n div 2 + 2 * int c − int m − int nz − int fc) div 2)
   OD
```

definition *certify iG iM c* ≡
```
  DO
     map ← is-map iG iM;
     ocondition (λ-. map)
       (DO
          gen ← euler-genus iG (mk-map (mk-graph iG) iM) c;
          oreturn (gen = 0)
       OD)
       (oreturn False)
  OD
```

19.3 Verification

context begin
  interpretation *Labeling-Syntax* **.**
  lemma *trivial-label*: $P \Longrightarrow CTXT\ IC\ CT\ OC\ P$
    unfolding *LABEL-simps* **.**
end

lemma *ovalidNF-wp*:
  assumes *ovalidNF P c* ($\lambda r\ s.\ r = x$)
  shows *ovalidNF* ($\lambda s.\ Q\ x\ s \wedge P\ s$) *c Q*
  using *assms* **unfolding** *ovalidNF-def* **by** *auto*

19.3.1 *is-map*

definition *is-map-rev-ok-inv iG iM k ok* ≡ *ok* ⟷ (∀ *i* < *k*.
  im-rev iM i < *length* (*ig-edges iG*)

$\wedge$ *ig-edges iG* ! *im-rev iM i* = *prod.swap* (*ig-edges iG* ! *i*)
$\wedge$ *im-rev iM i* $\neq$ *i*
$\wedge$ *im-rev iM* (*im-rev iM i*) = *i*)

definition *is-map-succ-perm-inv iG iM k ok* $\equiv$ *ok* $\longleftrightarrow$ ($\forall$ *i* < *k*.
im-succ iM i < *length* (*ig-edges iG*)
$\wedge$ *ig-tail iG* (*im-succ iM i*) = *ig-tail iG i*
$\wedge$ *im-pred iM* (*im-succ iM i*) = *i*)

definition *is-map-succ-orbits-inv iG iM k ok V A* $\equiv$
A = ($\bigcup$ *i*<(*if ok then k else k* − *1*). *orbit* (*im-succ iM*) *i*) $\wedge$
V = {*ig-tail iG i* | *i*. *i*<(*if ok then k else k* − *1*)} $\wedge$
ok = ($\forall$ *i* < *k*. $\forall$ *j* < *k*. *ig-tail iG i* = *ig-tail iG j* $\longrightarrow$ *j* $\in$ *orbit* (*im-succ iM*) *i*)

definition *is-map-succ-orbits-inner-inv iG iM i j A'* $\equiv$
A' = (*if i* = *j* $\wedge$ *i* $\notin$ *A' then* {} *else* {*i*} $\cup$ *segment* (*im-succ iM*) *i j*)
$\wedge$ *j* $\in$ *orbit* (*im-succ iM*) *i*

definition *is-map-final iG k ok* $\equiv$ (*ok* $\longrightarrow$ *k* = *length* (*ig-edges iG*)) $\wedge$ *k* $\leq$ *length* (*ig-edges iG*)

lemma *bij-betwI-finite-dom*:
assumes *finite A f* $\in$ *A* $\to$ *A* $\bigwedge$*a*. *a* $\in$ *A* $\Longrightarrow$ *g* (*f a*) = *a*
shows *bij-betw f A A*
proof −
have *inj-on f A* **by** (*metis assms*(*3*) *inj-onI*)
moreover
then have *f* ‘ *A* = *A* **by** (*metis Pi-iff assms*(*1*−*2*) *endo-inj-surj image-subsetI*)
ultimately show *?thesis* **unfolding** *bij-betw-def* **by** *simp*
qed

lemma *permutesI-finite-dom*:
assumes *finite A*
assumes *f* $\in$ *A* $\to$ *A*
assumes $\bigwedge$*a*. *a* $\notin$ *A* $\Longrightarrow$ *f a* = *a*
assumes $\bigwedge$*a*. *a* $\in$ *A* $\Longrightarrow$ *g* (*f a*) = *a*
shows *f permutes A*
using *assms* **by** (*intro bij-imp-permutes bij-betwI-finite-dom*)

lemma *orbit-ss*:
assumes *f* $\in$ *A* $\to$ *A a* $\in$ *A*
shows *orbit f a* $\subseteq$ *A*
proof −
{ fix *x* **assume** *x* $\in$ *orbit f a* **then have** *x* $\in$ *A* **using** *assms* **by** *induct auto* **}**
then show *?thesis* **by** *blast*

qed

lemma *segment-eq-orbit*:
  assumes $y \notin$ *orbit f x* **shows** *segment f x y* = *orbit f x*
proof (*intro set-eqI iffI*)
  fix *z* **assume** $z \in$ *segment f x y* **then show** $z \in$ *orbit f x* **by** (*rule segmentD-orbit*)
next
  fix *z* **assume** $z \in$ *orbit f x* **then show** $z \in$ *segment f x y*
    using *assms* **by** *induct* (*auto intro*: *segment.intros orbit-eqI elim*: *orbit.cases*)
qed

lemma *funpow-in-funcset*:
  assumes $x \in A$ $f \in A \to A$ **shows** $(f \frown\frown n)\ x \in A$
  using *assms* **by** (*induct n*) *auto*

lemma *funpow-eq-funcset*:
  assumes $x \in A$ $f \in A \to A$ $\bigwedge y.\ y \in A \Longrightarrow f\ y = g\ y$
  shows $(f \frown\frown n)\ x = (g \frown\frown n)\ x$
  using *assms* **by** (*induct n*) (*auto*, *metis funpow-in-funcset*)

lemma *funpow-dist1-eq-funcset*:
  assumes $y \in$ *orbit f x* $x \in A$ $f \in A \to A$ $\bigwedge y.\ y \in A \Longrightarrow f\ y = g\ y$
  shows *funpow-dist1 f x y* = *funpow-dist1 g x y*
proof −
  have $y = (f \frown\frown$ *funpow-dist1 f x y*$)\ x$ **by** (*metis assms*(*1*) *funpow-dist1-prop*)
  also have ... $= (g \frown\frown$ *funpow-dist1 f x y*$)\ x$ **by** (*metis assms*(*2*−) *funpow-eq-funcset*)
  finally have ∗: $y = (g \frown\frown$ *funpow-dist1 f x y*$)\ x$ **.**
  then have $(g \frown\frown$ *funpow-dist1 g x y*$)\ x = y$ **by** (*metis funpow-dist1-prop1 zero-less-Suc*)
  with ∗ **have** *gf*: *funpow-dist1 g x y* ≤ *funpow-dist1 f x y*
    by (*metis funpow-dist1-least not-le zero-less-Suc*)

  have $(f \frown\frown$ *funpow-dist1 g x y*$)\ x = y$
  using ‹$(g \frown\frown$ *funpow-dist1 g x y*$)\ x = y$› **by** (*metis assms*(*2*−) *funpow-eq-funcset*)
  then have *fg*: *funpow-dist1 f x y* ≤ *funpow-dist1 g x y*
    using ‹$y = (f \frown\frown -)\ x$› **by** (*metis funpow-dist1-least not-le zero-less-Suc*)

  from *gf fg* **show** *?thesis* **by** *simp*
qed

lemma *segment-cong0*:
  assumes $x \in A$ $f \in A \to A$ $\bigwedge y.\ y \in A \Longrightarrow f\ y = g\ y$ **shows** *segment f x y* = *segment g x y*
proof (*cases* $y \in$ *orbit f x*)
  case *True*
  moreover
  from *assms* **have** *orbit f x* = *orbit g x* **by** (*rule orbit-cong0*)
  moreover
  have $(f \frown\frown n)\ x = (g \frown\frown n)\ x \land (f \frown\frown n)\ x \in A$ **for** *n*

```
    by (induct n rule: nat.induct) (insert assms, auto)
  ultimately show ?thesis
   using True by (auto simp: segment-altdef funpow-dist1-eq-funcset[OF - assms])
next
  case False
  moreover from assms have orbit f x = orbit g x by (rule orbit-cong0)
  ultimately show ?thesis by (simp add: segment-eq-orbit)
qed

lemma rev-ok-final:
  assumes wf-iG: wf-digraph (mk-graph iG)
  assumes rev: is-map-rev-ok-inv iG iM rev-i rev-ok is-map-final iG rev-i rev-ok
 shows rev-ok ⟷ bidirected-digraph (mk-graph iG) (edge-rev (mk-map (mk-graph
iG) iM)) (is ?L ⟷ ?R)
proof
  assume rev-ok
  interpret wf-digraph mk-graph iG by (rule wf-iG)
  have rev-inv-sep:
      ⋀i. i < length (ig-edges iG) ⟹ im-rev iM i < length (ig-edges iG)
        ⋀i. i < length (ig-edges iG) ⟹ ig-edges iG ! im-rev iM i = prod.swap
(ig-edges iG ! i)
      ⋀i. i < length (ig-edges iG) ⟹ im-rev iM i ≠ i
      ⋀i. i < length (ig-edges iG) ⟹ im-rev iM (im-rev iM i) = i
    using rev ‹rev-ok› by (auto simp: is-map-rev-ok-inv-def is-map-final-def)
  moreover
  { fix i assume i < length (ig-edges iG)
    then have ig-tail iG (im-rev iM i) = ig-head iG i
         using rev-inv-sep(2) by (cases ig-edges iG ! i) (auto simp: ig-head-def
ig-tail-def)
  }
  ultimately show ?R
   using wf by unfold-locales (auto simp: mkg-simps arcs-mkg mkm-simps perm-restrict-def)
next
  assume ?R
  let ?rev = perm-restrict (im-rev iM) (arcs (mk-graph iG))
 interpret bidirected-digraph mk-graph iG perm-restrict (im-rev iM) (arcs (mk-graph
iG))
    using ‹?R› by (simp add: mkm-simps mkg-simps)
  have ⋀a. a ∈ arcs (mk-graph iG) ⟹ ?rev a ∈ arcs (mk-graph iG)
       ⋀a. a ∈ arcs (mk-graph iG) ⟹
        arc-to-ends (mk-graph iG) (?rev a) = prod.swap (arc-to-ends (mk-graph iG)
a)
       ⋀a. a ∈ arcs (mk-graph iG) ⟹ ?rev a ≠ a
       ⋀a. a ∈ arcs (mk-graph iG) ⟹ ?rev (?rev a) = a
    by (auto simp: arev-dom)
  then show rev-ok
    using rev unfolding is-map-rev-ok-inv-def is-map-final-def
    by (simp add: perm-restrict-simps arcs-mkg arc-to-ends-mkg)
qed
```

locale *is-map-postcondition0* =
fixes *iG iM rev-ok succ-i succ-ok*
assumes *succ-perm*: *is-map-succ-perm-inv iG iM succ-i succ-ok is-map-final iG succ-i succ-ok*
begin

lemma *succ-ok-tail-eq*:
succ-ok $\Longrightarrow$ *i* < *length* (*ig-edges iG*) $\Longrightarrow$ *ig-tail iG* (*im-succ iM i*) = *ig-tail iG i*
using *succ-perm* **unfolding** *is-map-succ-perm-inv-def is-map-final-def* **by** *auto*

lemma *succ-ok-imp-pred*:
succ-ok $\Longrightarrow$ *i* < *length* (*ig-edges iG*) $\Longrightarrow$ *im-pred iM* (*im-succ iM i*) = *i*
using *succ-perm* **unfolding** *is-map-succ-perm-inv-def is-map-final-def* **by** *auto*

lemma *succ-ok-imp-permutes*:
assumes *succ-ok*
shows *edge-succ* (*mk-map* (*mk-graph iG*) *iM*) *permutes arcs* (*mk-graph iG*)
proof −
from *assms* **have** ∀ *a* ∈ *arcs* (*mk-graph iG*). *edge-succ* (*mk-map* (*mk-graph iG*) *iM*) *a* ∈ *arcs* (*mk-graph iG*)
using *succ-perm* **unfolding** *is-map-succ-perm-inv-def is-map-final-def*
by (*auto simp*: *mkg-simps mkm-simps arcs-mkg perm-restrict-def*)
with *succ-ok-imp-pred*[*OF assms*] **show** *?thesis*
by − (*rule permutesI-finite-dom*[**where** *g*=*im-pred iM*], *auto simp*: *perm-restrict-simps mkm-simps arcs-mkg*)
qed

lemma *es-A2A*: *succ-ok* $\Longrightarrow$ *edge-succ* (*mk-map* (*mk-graph iG*) *iM*) ∈ *arcs* (*mk-graph iG*) → *arcs* (*mk-graph iG*)
using *succ-ok-imp-permutes* **by** (*auto dest*: *permutes-in-image*)

lemma *im-succ-le-length*: *succ-ok* $\Longrightarrow$ *i* < *length* (*ig-edges iG*) $\Longrightarrow$ *im-succ iM i* < *length* (*ig-edges iG*)
using *is-map-final-def is-map-succ-perm-inv-def succ-perm*(*1*) *succ-perm*(*2*) **by** *auto*

lemma *orbit-es-eq-im*:
succ-ok $\Longrightarrow$ *a* ∈ *arcs* (*mk-graph iG*) $\Longrightarrow$ *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *a* = *orbit* (*im-succ iM*) *a*
using - *es-A2A es-eq-im* **by** (*rule orbit-cong0*)

lemma *segment-es-eq-im*:
succ-ok $\Longrightarrow$ *a* ∈ *arcs* (*mk-graph iG*) $\Longrightarrow$ *segment* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *a b* = *segment* (*im-succ iM*) *a b*
using - *es-A2A es-eq-im* **by** (*rule segment-cong0*)

lemma *in-orbit-im-succE*:
assumes $j \in$ *orbit* (*im-succ iM*) *i succ-ok* $i <$ *length* (*ig-edges iG*)
obtains *ig-tail iG j* = *ig-tail iG i* $j <$ *length* (*ig-edges iG*)
using *assms es-A2A* **by** *induct* (*force simp add*: *succ-ok-tail-eq es-eq-im arcs-mkg*)+

lemma *self-in-orbit-im-succ*:
assumes *succ-ok* $i <$ *length* (*ig-edges iG*) **shows** $i \in$ *orbit* (*im-succ iM*) *i*
proof −
have $i \in$ *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *i*
using *assms succ-ok-imp-permutes*
by (*intro permutation-self-in-orbit*) (*auto simp*: *permutation-permutes arcs-mkg*)
with *assms* **show** *?thesis* **by** (*simp add*:*orbit-es-eq-im arcs-mkg*)
qed

end

locale *is-map-postcondition* = *is-map-postcondition0* +
fixes *so-i so-ok V A*
assumes *rev*: *rev-ok* $\longleftrightarrow$ *bidirected-digraph* (*mk-graph iG*) (*edge-rev* (*mk-map* (*mk-graph iG*) *iM*))
assumes *succ-orbits*: *is-map-succ-orbits-inv iG iM so-i so-ok V A succ-ok* $\longrightarrow$ *is-map-final iG so-i so-ok*
begin

lemma *ok-imp-digraph*:
assumes *rev-ok succ-ok so-ok*
shows *digraph-map* (*mk-graph iG*) (*mk-map* (*mk-graph iG*) *iM*)
proof −
interpret *bidirected-digraph mk-graph iG edge-rev* (*mk-map* (*mk-graph iG*) *iM*)
using ‹*rev-ok*› **by** (*simp add*: *rev*)

from ‹*succ-ok*› **have** *perm*: *edge-succ* (*mk-map* (*mk-graph iG*) *iM*) *permutes arcs* (*mk-graph iG*)
by (*simp add*: *succ-ok-imp-permutes*)
from ‹*succ-ok*› **have** *ig-tail*: $\bigwedge a.\ a \in$ *arcs* (*mk-graph iG*) $\Longrightarrow$ *ig-tail iG* (*im-succ iM a*) = *ig-tail iG a*
by (*simp-all add*: *succ-ok-tail-eq arcs-mkg*)

{ fix *v* **assume** $v \in$ *verts* (*mk-graph iG*) *out-arcs* (*mk-graph iG*) $v \neq \{\}$
then obtain *a* **where** *a*: $a \in$ *arcs* (*mk-graph iG*) *tail* (*mk-graph iG*) $a = v$ **by** *auto metis*
then have *out-arcs* (*mk-graph iG*) $v = \{b \in$ *arcs* (*mk-graph iG*). *ig-tail iG a* = *ig-tail iG b*$\}$
by (*auto simp*: *mkg-simps*)
also have $\ldots \subseteq$ *orbit* (*im-succ iM*) *a*
proof −
have ($\forall$ *i*<*length* (*snd iG*). $\forall$ *j*<*length* (*snd iG*).
ig-tail iG i = *ig-tail iG j* $\longrightarrow j \in$ *orbit* (*im-succ iM*) *i*)
using ‹*succ-ok* ›‹*so-ok*› *succ-orbits* **unfolding** *is-map-succ-orbits-inv-def*

is-map-final-def **by** *metis*
with *a* **show** *?thesis* **by** (*auto simp*: *arcs-mkg*)
qed
finally have *out-arcs* (*mk-graph iG*) *v* $\subseteq$ *orbit* (*im-succ iM*) *a* **.**
moreover
have *orbit* (*im-succ iM*) *a* $\subseteq$ *out-arcs* (*mk-graph iG*) *v*
proof –
{ fix *x* **assume** *x* $\in$ *orbit* (*im-succ iM*) *a* **then have** *tail* (*mk-graph iG*) *x* = *v*
using *a ig-tail*
apply *induct*
apply (*auto simp*: *mkg-simps intro*: *orbit.intros*)
by (*metis* ‹*succ-ok*› *contra-subsetD orbit-es-eq-im permutes-orbit-subset perm*)
} moreover
have *orbit* (*im-succ iM*) *a* $\subseteq$ *arcs* (*mk-graph iG*)
using - *a*(*1*) **apply** (*rule orbit-ss*)
using *assms arcs-mkg is-map-final-def is-map-succ-perm-inv-def succ-perm*(*1*) *succ-perm*(*2*) **by** *auto*
ultimately
show *?thesis* **by** *auto*
qed
ultimately
have *out-arcs* (*mk-graph iG*) *v* = *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *a*
using ‹*succ-ok*› *a* **by** (*auto simp*: *orbit-es-eq-im*)
then
have *cyclic-on* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) (*out-arcs* (*mk-graph iG*) *v*)
unfolding *cyclic-on-def* **using** *a* **by** *force*
}
with *perm* **show** *?thesis*
using ‹*rev-ok*› **by** *unfold-locales* (*auto simp*: *mkg-simps arcs-mkg*)
qed

lemma *digraph-imp-ok*:
assumes *dm*: *digraph-map* (*mk-graph iG*) (*mk-map* (*mk-graph iG*) *iM*)
assumes *pred*: $\bigwedge i.\ i < $ *length* (*ig-edges iG*) $\Longrightarrow$ *im-pred iM* (*im-succ iM i*) = *i*
obtains *rev-ok succ-ok so-ok*
proof
interpret *dm*: *digraph-map mk-graph iG mk-map* (*mk-graph iG*) *iM* **by** (*fact dm*)

show *rev-ok* **unfolding** *rev* **by** *unfold-locales*

show *succ-ok*
proof –
{ fix *i* **assume** *i* $\in$ *arcs* (*mk-graph iG*)
then have

edge-succ (*mk-map* (*mk-graph iG*) *iM*) *i* ∈ *arcs* (*mk-graph iG*)
tail (*mk-graph iG*) (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*) *i*) = *tail* (*mk-graph iG*) *i*
by *auto*
then have
im-succ iM i < *length* (*snd iG*)
ig-tail iG (*im-succ iM i*) = *ig-tail iG i*
unfolding *es-eq-im*[*OF* ‹*i* ∈ *arcs* -›] **by** (*auto simp*: *arcs-mkg mkg-simps*)
}
then have (∀ *i*<*length* (*ig-edges iG*).
im-succ iM i < *length* (*snd iG*) ∧
ig-tail iG (*im-succ iM i*) = *ig-tail iG i* ∧ *im-pred iM* (*im-succ iM i*) = *i*)
using *pred* **by** (*auto simp*: *arcs-mkg es-eq-im*)
with *succ-perm* **show** *?thesis*
unfolding *is-map-succ-perm-inv-def is-map-final-def* **by** *simp*
qed

show *so-ok*
proof −
{ fix *i j* **assume** *i* < *length* (*ig-edges iG*) *j* < *length* (*ig-edges iG*) *ig-tail iG i* = *ig-tail iG j*
then have *A*: *i* ∈ *arcs* (*mk-graph iG*) *j* ∈ *arcs* (*mk-graph iG*) *tail* (*mk-graph iG*) *i* = *tail* (*mk-graph iG*) *j*
by (*auto simp*: *mkg-simps arcs-mkg*)
then have *cyclic-on* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) (*out-arcs* (*mk-graph iG*) (*tail* (*mk-graph iG*) *i*))
by (*auto intro*!: *dm.edge-succ-cyclic*)
then have *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *i* = *out-arcs* (*mk-graph iG*) (*ig-tail iG i*)
by (*simp add*: ‹*i* ∈ *arcs* (*mk-graph iG*)› *mkg-simps orbit-cyclic-eq3*)
then have *j* ∈ *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *i* **using** *A* **by** (*simp add*: *mkg-simps*)
also have *orbit* (*edge-succ* (*mk-map* (*mk-graph iG*) *iM*)) *i* = *orbit* (*im-succ iM*) *i*
using ‹*i* ∈ *arcs* -›
by (*rule orbit-cong0*) (*fastforce*, *simp add*: *es-eq-im*)
finally have *j* ∈ *orbit* (*im-succ iM*) *i* **.**
}
then show *?thesis*
using *succ-orbits* **unfolding** *is-map-succ-orbits-inv-def is-map-final-def*
by *safe* (*simp-all only*: ‹*succ-ok*› *simp-thms*)
qed
qed

end

lemma *all-less-Suc-eq*: (∀ *x* < *Suc n*. *P x*) ⟷ (∀ *x* < *n*. *P x*) ∧ *P n*
by (*auto elim*: *less-SucE*)

```
lemma in-orbit-imp-in-segment:
  assumes y ∈ orbit f x x ≠ y bij f shows y ∈ segment f x (f y)
  using assms
proof induct
  case base then show ?case by (auto intro: segment.intros simp: bij-iff)
next
  case (step y)
  show ?case
  proof (cases x = y)
    case True then show ?thesis using step by (auto intro: segment.intros simp:
bij-iff)
  next
    case False
    with step have f y ≠ f (f y) by (metis bij-is-inj inv-f-f not-in-segment2)
    then show ?thesis using step False
      by (auto intro: segment.intros segment-step-2 bij-is-inj)
  qed
qed


lemma ovalidNF-is-map:
  ovalidNF (λs. distinct (ig-verts iG) ∧ wf-digraph (mk-graph iG))
    (is-map iG iM)
  (λr s. r ⟷ digraph-map (mk-graph iG) (mk-map (mk-graph iG) iM) ∧ (∀ i <
length (ig-edges iG). im-pred iM (im-succ iM i) = i))

  unfolding is-map-def
  apply (rewrite
    in oreturn (length (ig-edges iG)) |>> (λecnt. ¤)
    to owhile-inv - - -
      (λ(i, ok) s. is-map-rev-ok-inv iG iM i ok
        ∧ i ≤ ecnt ∧ wf-digraph (mk-graph iG))
      (measure (λ(i, ok). ecnt − i))
     owhile-inv-def[symmetric] )
  apply (rewrite
    in owhile-inv - - - - - |>> (λ(rev-i, rev-ok). ¤)
    in oreturn (length (ig-edges iG)) |>> (λecnt. ¤)
    to owhile-inv - - -
      (λ(i, ok) s. is-map-succ-perm-inv iG iM i ok
        ∧ rev-ok = bidirected-digraph (mk-graph iG) (edge-rev (mk-map (mk-graph
iG) iM))
        ∧ i ≤ ecnt ∧ wf-digraph (mk-graph iG))
      (measure (λ(i, ok). ecnt − i))
     owhile-inv-def[symmetric] )
  apply (rewrite
    in owhile-inv - - - - - |>> (λ(succ-i, succ-ok). ¤)
    in owhile-inv - - - - - |>> (λ(rev-i, rev-ok). ¤)
    in oreturn (length (ig-edges iG)) |>> (λecnt. ¤)
    to owhile-inv - - -
```

(λ(i, ok, V, A) s. is-map-succ-orbits-inv iG iM i ok V A
∧ rev-ok = bidirected-digraph (mk-graph iG) (edge-rev (mk-map (mk-graph iG) iM))
∧ is-map-succ-perm-inv iG iM succ-i succ-ok ∧ is-map-final iG succ-i succ-ok
∧ i ≤ ecnt ∧ wf-digraph (mk-graph iG))
(measure (λ(i, ok, V, A). ecnt − i))
owhile-inv-def[symmetric])
apply (rewrite
in owhile-inv - (λ(i, ok, V, A). ⌑) - - -
in owhile-inv - - - - - |>> (λ(succ-i, succ-ok). ⌑)
in owhile-inv - - - - - |>> (λ(rev-i, rev-ok). ⌑)
in oreturn (length (ig-edges iG)) |>> (λecnt. ⌑)
to owhile-inv - - -
(λ(A′, j) s. is-map-succ-orbits-inner-inv iG iM i j A′
∧ ig-tail iG i ∉ V ∧ succ-ok ∧ ok ∧ is-map-succ-orbits-inv iG iM i ok V A
∧ rev-ok = bidirected-digraph (mk-graph iG) (edge-rev (mk-map (mk-graph iG) iM))
∧ is-map-succ-perm-inv iG iM succ-i succ-ok ∧ is-map-final iG succ-i succ-ok
∧ i < ecnt ∧ wf-digraph (mk-graph iG))
(measure (λ(A′,j). length (ig-edges iG) − card A′))
owhile-inv-def[symmetric])
proof vcg-casify
let ?es = edge-succ (mk-map (mk-graph iG) iM)

{ **case** weaken **then show** ?case **by** (auto simp: is-map-rev-ok-inv-def)
}
{ **case** (while i ok)
{ **case** invariant
case weaken **then show** ?case **by** (auto simp: is-map-rev-ok-inv-def elim: less-SucE)
}
{ **case** wf **show** ?case **by** auto
}
{ **case** postcondition
then have ok ⟷ bidirected-digraph (mk-graph iG) (edge-rev (mk-map (mk-graph iG) iM))
by (intro rev-ok-final) (auto simp: is-map-final-def)
with postcondition **show** ?case **by** (auto simp: is-map-succ-perm-inv-def)
}
}
case (bind - rev-ok)
{ **case** (while i ok)
{ **case** invariant **case** weaken
then show ?case **by** (auto simp: is-map-succ-perm-inv-def elim: less-SucE)
}
{ **case** wf **show** ?case **by** auto
}
{ **case** postcondition
then show ?case **by** (auto simp: is-map-final-def is-map-succ-orbits-inv-def)

```
    }
  }
  case (bind succ-i succ-ok)
  { case (while i ok V A)
    { case invariant
      { case weaken
        then interpret pc0: is-map-postcondition0 iG iM rev-ok succ-i succ-ok
          by unfold-locales auto
        from weaken.loop-cond have i < length (ig-edges iG) succ-ok ok by auto
        with weaken.loop-inv have
            V: V = {ig-tail iG k |k. k < i} and
            A: A = (⋃ k<i. orbit (im-succ iM) k)
          by (simp-all add: is-map-succ-orbits-inv-def)
        show ?case
        proof branch-casify
          case then case g
         have V': V = {ig-tail iG ia |ia. ia < (if i ∈ A then Suc i else Suc i − 1)}
            using g ‹V = -› by (auto elim: less-SucE)

          have is-map-succ-orbits-inv iG iM (Suc i) (i ∈ A) V A
          proof (cases i ∈ A)
            case True
            obtain j where j: j < i i ∈ orbit (im-succ iM) j
              using True ‹A = -› by auto
            have i-in-less-i: ∃x∈{..<i}. i ∈ orbit (im-succ iM) x
              using True ‹A = -› by auto
             have A': A = (⋃ i<if True then Suc i else Suc i − 1. orbit (im-succ
iM) i)
               using True unfolding ‹A = -› by (auto 4 3 intro: orbit-trans elim:
less-SucE)

             have X: ∀k<i. ∀l<i. ig-tail iG k = ig-tail iG l ⟶ l ∈ orbit (im-succ
iM) k
              using weaken unfolding is-map-succ-orbits-inv-def by metis
            moreover
            { fix j assume j: j < i ig-tail iG j = ig-tail iG i
             from i-in-less-i obtain k where k: k < i i ∈ orbit (im-succ iM) k by
auto
              then have ig-tail iG k = ig-tail iG i
                using ‹succ-ok› ‹i < -› by (auto elim: pc0.in-orbit-im-succE)
              then have k ∈ orbit (im-succ iM) j
                using j ‹ig-tail iG k = -› k X by auto
            then have i ∈ orbit (im-succ iM) j using k by (auto intro: orbit-trans)
            }
            ultimately
              have ∀k<Suc i. ∀l<Suc i. ig-tail iG k = ig-tail iG l ⟶ l ∈ orbit
(im-succ iM) k
              unfolding all-less-Suc-eq using ‹i < -› ‹succ-ok›
              by (auto intro: orbit-swap pc0.self-in-orbit-im-succ)
```

```
      with True show ?thesis
        by (simp only: A' V' simp-thms is-map-succ-orbits-inv-def)
    next
      case False

      from V g obtain j where j: j < i ig-tail iG j = ig-tail iG i by auto
      with False show ?thesis
        by (auto 0 3 simp: is-map-succ-orbits-inv-def V' A intro: exI[where
x=j] exI[where x=i])
    qed
    then show ?case using weaken by auto
  next
    case else case g
    have is-map-succ-orbits-inner-inv iG iM i i {}
      unfolding is-map-succ-orbits-inner-inv-def
      using ‹succ-ok› ‹i < -› by (auto simp: pc0.self-in-orbit-im-succ)
    with g weaken show ?case by blast
  qed
}
{ case if case else case (while A' i')
  { case invariant case weaken
    then interpret pc0: is-map-postcondition0 iG iM rev-ok succ-i succ-ok
      by unfold-locales auto
    have succ-ok i < length (ig-edges iG) i' ∈ orbit (im-succ iM) i
      using weaken by (auto simp: is-map-succ-orbits-inner-inv-def)
    have i' < length (ig-edges iG)
      using ‹i' ∈ -› ‹succ-ok› ‹i < -› by (rule pc0.in-orbit-im-succE)

    { assume i' ∈ orbit (im-succ iM) i i ≠ i'
      then have i' ∈ orbit (?es) i
      by (subst pc0.orbit-es-eq-im) (auto simp add: ‹succ-ok› ‹i < -› arcs-mkg)
      then have i' ∈ segment (?es) i (?es i')
        using ‹i ≠ i'› pc0.succ-ok-imp-permutes ‹succ-ok ›
       by (intro in-orbit-imp-in-segment) (auto simp: permutes-conv-has-dom)
      then have i' ∈ segment (im-succ iM) i (im-succ iM i')
        by (subst pc0.segment-es-eq-im[symmetric] es-eq-im[symmetric];
            auto simp add: ‹succ-ok› ‹i < -› ‹i' < -› arcs-mkg)+
    } note X = this

    { fix x assume x ∈ segment (im-succ iM) i i' i ≠ i'
      then have x ∈ segment (?es) i i'
        by (subst pc0.segment-es-eq-im) (auto simp add: ‹succ-ok› ‹i < -› ‹i'
< -› arcs-mkg)
      then have x ∈ segment (?es) i (?es i')
        using ‹i ≠ i'› pc0.succ-ok-imp-permutes ‹succ-ok ›
        by (auto simp: permutes-conv-has-dom bij-is-inj intro: segment-step-2)
      then have x ∈ segment (im-succ iM) i (im-succ iM i')
        by (subst pc0.segment-es-eq-im[symmetric] es-eq-im[symmetric];
            auto simp add: ‹succ-ok› ‹i < -› ‹i' < -› arcs-mkg)+
```

} **note** *Y* = *this*

have *Z*: *is-map-succ-orbits-inner-inv iG iM i* (*im-succ iM i'*) (*insert i' A'*)
  using *weaken* **unfolding** *is-map-succ-orbits-inner-inv-def*
  by (*auto dest*: *segment-step-2D X Y simp*: *orbit.intros segment1-empty split*: *if-splits*)

have $A' \subseteq$ *orbit* (*im-succ iM*) *i*
  using *weaken* **unfolding** *is-map-succ-orbits-inner-inv-def*
by (*auto simp*: *pc0.self-in-orbit-im-succ dest*: *segmentD-orbit split*: *if-splits*)
also have $\ldots \subseteq$ *arcs* (*mk-graph iG*)
  by (*rule orbit-ss*) (*auto simp*: *arcs-mkg pc0.im-succ-le-length* ‹*succ-ok*› ‹$i < -$›)
finally have *card* $A' <$ *card* (*arcs* (*mk-graph iG*)) *finite* A'
  using ‹$i' \notin A'$› ‹$i' < -$›
  by − (*intro psubset-card-mono*, *auto simp*: *arcs-mkg intro*: *finite-subset*)
then have *card* $A' <$ *length* (*ig-edges iG*) **by** (*simp add*: *arcs-mkg*)
show *?case*
  using *weaken Z* ‹*card* $A' <$ *length* -› **by** (*auto simp*: *card-insert-if* ‹*finite* A'›)
}
{ **case** *wf* **show** *?case* **by** *simp*
}
{ **case** *postcondition*
then interpret *pc0*: *is-map-postcondition0 iG iM rev-ok succ-i succ-ok*
  by *unfold-locales auto*
from *postcondition* **have** *ok succ-ok* $i <$ *length* (*ig-edges iG*) **by** *simp-all*

from *postcondition*
have $i' \in A'$
  $A' = ($*if* $i = i' \wedge i \notin A'$ *then* $\{\}$ *else* $\{i\} \cup$ *segment* (*im-succ iM*) $i\ i')$
  $i' \in$ *orbit* (*im-succ iM*) *i*
  ig-tail iG i $\notin V$
  by (*simp-all add*: *is-map-succ-orbits-inner-inv-def*)
moreover
then have $i = i'$ **by** (*simp split*: *if-splits add*: *not-in-segment2*)
ultimately
have $A' = \{i\} \cup$ *segment* (*im-succ iM*) *i i* **by** *simp*
also have *segment* (*im-succ iM*) *i i* = *segment ?es i i*
  by (*auto simp*: *pc0.segment-es-eq-im* ‹*succ-ok*› ‹$i < -$› *arcs-mkg*)
also have $\ldots =$ *orbit ?es i* $- \{i\}$
  using *pc0.succ-ok-imp-permutes* ‹*succ-ok*›
  by (*auto simp*: *permutation-permutes arcs-mkg intro*!: *segment-x-x-eq*)
also have $\ldots =$ *orbit* (*im-succ iM*) *i* $- \{i\}$
  by (*auto simp*: *pc0.orbit-es-eq-im* ‹*succ-ok*› ‹$i < -$› *arcs-mkg*)
finally
have A': $A' =$ *orbit* (*im-succ iM*) *i*
  using ‹$i < -$› ‹*succ-ok*› **by** (*auto simp*: *pc0.self-in-orbit-im-succ*)

from *postcondition*
have $A = (\bigcup k<i.\ orbit\ (im\text{-}succ\ iM)\ k)$
unfolding *is-map-succ-orbits-inner-inv-def* **by** (*simp add*: *is-map-succ-orbits-inv-def*)
have $A \cup A' = (\bigcup k<Suc\ i.\ orbit\ (im\text{-}succ\ iM)\ k)$
unfolding A' ‹A = -› **by** (*auto 2 3 elim*: *less-SucE*)

from *postcondition* **have** $V = \{ig\text{-}tail\ iG\ ia\ |ia.\ ia < i\}$
by (*auto simp*: ‹*ok*› *is-map-succ-orbits-inv-def*)
then have V': $insert\ (ig\text{-}tail\ iG\ i')\ V = \{ig\text{-}tail\ iG\ ia\ |ia.\ ia < Suc\ i\}$
by (*auto simp add*: ‹$i = i'$› *elim*: *less-SucE*)

have $*$: $\bigwedge k.\ k < i \Longrightarrow ig\text{-}tail\ iG\ k \neq ig\text{-}tail\ iG\ i$
using ‹V = -› ‹$ig\text{-}tail\ iG\ i \notin V$› **by** *auto*

from *postcondition* **have** $(\forall k<i.\ \forall l<i.\ ig\text{-}tail\ iG\ k = ig\text{-}tail\ iG\ l \longrightarrow l \in orbit\ (im\text{-}succ\ iM)\ k)$
by (*simp add*: *is-map-succ-orbits-inv-def* ‹*ok*›)
then have X: $(\forall k<Suc\ i.\ \forall l<Suc\ i.\ ig\text{-}tail\ iG\ k = ig\text{-}tail\ iG\ l \longrightarrow l \in orbit\ (im\text{-}succ\ iM)\ k)$
by (*auto simp add*: *all-less-Suc-eq pc0.self-in-orbit-im-succ* ‹*succ-ok*› ‹i < -› *dest*: $*$)

have $is\text{-}map\text{-}succ\text{-}orbits\text{-}inv\ iG\ iM\ (i + 1)\ True\ (insert\ (ig\text{-}tail\ iG\ i')\ V)\ (A \cup A')$
unfolding *is-map-succ-orbits-inv-def* **by** (*simp add*: ‹$A \cup A'$ = -› V' X)
then show *?case*
using *postcondition* ‹i < -› **by** *auto*
}
}
}
{ case *wf* **show** *?case* **by** *auto*
}
{ case *postcondition*
interpret *pc*: *is-map-postcondition iG iM rev-ok succ-i succ-ok i ok V A*
using *postcondition* **by** *unfold-locales* (*auto simp*: *is-map-final-def*)

show *?case* (**is** *?L* = *?R*)
by (*auto simp add*: *pc.ok-imp-digraph dest*: *pc.succ-ok-imp-pred elim*: *pc.digraph-imp-ok*)
}
}
qed

declare *ovalidNF-is-map*[*THEN ovalidNF-wp*, *THEN trivial-label*, *vcg-l*]

19.3.2 *isolated-nodes*

definition *inv-isolated-nodes s iG vcnt ecnt* ≡
vcnt = *length* (*ig-verts iG*)

$\wedge$ *ecnt* = *length* (*ig-edges iG*)
$\wedge$ *distinct* (*ig-verts iG*)
$\wedge$ *sym-digraph* (*mk-graph iG*)

definition *inv-isolated-nodes-outer iG i nz* $\equiv$
nz = *card* (*pre-digraph.isolated-verts* (*mk-graph iG*) $\cap$ *set* (*take i* (*ig-verts iG*)))

definition *inv-isolated-nodes-inner iG v j* $\equiv$
$\forall$ *k* < *j*. *v* $\neq$ *ig-tail iG k* $\wedge$ *v* $\neq$ *ig-head iG k*

lemma (**in** *sym-digraph*) *in-arcs-empty-iff*:
in-arcs G v = {} $\longleftrightarrow$ *out-arcs G v* = {}
by (*auto simp*: *out-arcs-def in-arcs-def*)
(*metis graph-symmetric in-arcs-imp-in-arcs-ends reachableE*)+

lemma *take-nth-distinct*:
$[\![$*distinct xs*; *n* < *length xs*; *xs* ! *n* $\in$ *set* (*take n xs*)$]\!] \Longrightarrow$ *False*
by (*fastforce simp*: *distinct-conv-nth in-set-conv-nth*)

lemma *ovalidNF-isolated-nodes*:
ovalidNF (λ*s*. *distinct* (*ig-verts iG*) $\wedge$ *sym-digraph* (*mk-graph iG*))
(*isolated-nodes iG*)
(λ*r s*. *r* = (*card* (*pre-digraph.isolated-verts* (*mk-graph iG*))))
unfolding *isolated-nodes-def*
apply (*rewrite*
in *oreturn* (*length* (*ig-verts iG*)) |>> (λ*vcnt*. ¤)
in *oreturn* (*length* (*ig-edges iG*)) |>> (λ*ecnt*. ¤)
to owhile-inv - - -
(λ(*i*, *nz*) *s*. *inv-isolated-nodes s iG vcnt ecnt*
$\wedge$ *inv-isolated-nodes-outer iG i nz*
$\wedge$ *i* $\leq$ *vcnt*)
(*measure* (λ(*i*, *nz*). *vcnt* $-$ *i*))
owhile-inv-def[*symmetric*])
apply (*rewrite*
in *oreturn* (*fst iG* ! *i*) |>> (λ*v*. ¤)
in *owhile-inv* - (λ(*i*, *nz*). ¤)
in *oreturn* (*length* (*ig-verts iG*)) |>> (λ*vcnt*. ¤)
in *oreturn* (*length* (*ig-edges iG*)) |>> (λ*ecnt*. ¤)
to owhile-inv - - -
(λ(*j*, *ret*) *s*. *inv-isolated-nodes s iG vcnt ecnt*
$\wedge$ *inv-isolated-nodes-inner iG v j*
$\wedge$ *inv-isolated-nodes-outer iG i nz*
$\wedge$ *v* = *ig-verts iG* ! *i*
$\wedge$ *ret* = (*j* < *ecnt* $\wedge$ *ig-tail iG j* $\neq$ *v* $\wedge$ *ig-head iG j* $\neq$ *v*)
$\wedge$ *i* < *vcnt*
$\wedge$ *j* $\leq$ *ecnt*)
(*measure* (λ(*j*, *ret*). *ecnt* $-$ *j*))
owhile-inv-def[*symmetric*])

proof *vcg-casify*
  case (*weaken s*)
  then show *?case*
    by (*auto simp*: *inv-isolated-nodes-def inv-isolated-nodes-outer-def*)
next
  case (*while i nz*)
  { case *invariant*
    { case (*weaken s'*)
    then show *?case* **unfolding** *BRANCH-def* **by** (*auto simp*: *inv-isolated-nodes-inner-def*)
    next
      case *bind*
      case *bind*
      case (*while j cond*)
      { case *invariant*
        { case *weaken*
          show *?case*
          proof *branch-casify*
            case *else* **case** *else* **case** *g*
            with *weaken* **have** *length* (*ig-edges iG*) = *j* + *1* **by** *linarith*
            with *weaken* **show** *?case*
              by (*auto simp*: *inv-isolated-nodes-inner-def elim*: *less-SucE*)
        qed (*insert weaken*, *auto simp*: *inv-isolated-nodes-inner-def elim*: *less-SucE*)
        }
      next
        case *wf* **show** *?case* **by** *auto*
      next
        case *postcondition*
        interpret *G*: *sym-digraph mk-graph iG* **using** *postcondition* **by** (*simp add*: *inv-isolated-nodes-def*)

        have *?var* **using** *postcondition* **by** *auto*

        let *?v* = *ig-verts iG* ! *i*

        { assume *A*: *j* = *length* (*snd iG*)
          have *?v* ∈ *pre-digraph.isolated-verts* (*mk-graph iG*)
              using *A postcondition* **by** (*auto simp*: *pre-digraph.isolated-verts-def mkg-simps inv-isolated-nodes-inner-def arcs-mkg*)

          have *distinct* (*ig-verts iG*) *?v* = *ig-verts iG* ! *i i* < *length* (*ig-verts iG*)
            using *postcondition* **by** (*auto simp*: *inv-isolated-nodes-def*)
          then have *?v* ∉ *set* (*take i* (*ig-verts iG*))
            by (*metis take-nth-distinct*)

          have *Suc* (*card* (*pre-digraph.isolated-verts* (*mk-graph iG*) ∩ *set* (*take i* (*fst iG*))))
            = *card* (*insert ?v* (*pre-digraph.isolated-verts* (*mk-graph iG*) ∩ *set* (*take i* (*fst iG*)))) (**is** - = *card ?S*)
            using ‹*?v* ∉ -› **by** *simp*

also have *?S* = *pre-digraph.isolated-verts* (*mk-graph iG*) ∩ *set* (*take* (*Suc i*) (*fst iG*))
using ‹*?v* ∈ -› ‹*i* < -› ‹*?v* = -› **by** (*auto simp*: *take-Suc-conv-app-nth*)
finally
have *inv-isolated-nodes-outer iG* (*Suc i*) (*Suc nz*)
using *postcondition* **by** (*auto simp*: *inv-isolated-nodes-outer-def*)
}
moreover
{ assume *A*: *j* ≠ *length* (*snd iG*)

then have ∗: *j* ∈ (*out-arcs* (*mk-graph iG*) *?v* ∪ *in-arcs* (*mk-graph iG*) *?v*)
using *postcondition* **by** (*auto simp*: *arcs-mkg mkg-simps ig-tail-def ig-head-def*)
then have *out-arcs* (*mk-graph iG*) *?v* ≠ {}
by (*auto simp del*: *in-in-arcs-conv in-out-arcs-conv*)
(*auto simp*: *G.in-arcs-empty-iff*[*symmetric*])
then have *?v* ∉ *pre-digraph.isolated-verts* (*mk-graph iG*)
by (*auto simp*: *pre-digraph.isolated-verts-def*)
then have *inv-isolated-nodes-outer iG* (*Suc i*) *nz*
using *postcondition* **by** (*auto simp*: *inv-isolated-nodes-outer-def take-Suc-conv-app-nth*)
}
ultimately
have *?inv* **using** *postcondition* **by** *auto*
from ‹*?var*› ‹*?inv*› **show** *?case* **by** *blast*
}
}
next
case *wf* **show** *?case* **by** *auto*
next
case *postcondition*
have *pre-digraph.isolated-verts* (*mk-graph iG*) ∩ *set* (*fst iG*) = *pre-digraph.isolated-verts* (*mk-graph iG*)
by (*auto simp*: *pre-digraph.isolated-verts-def mkg-simps*)
with *postcondition* **show** *?case*
by (*auto simp*: *inv-isolated-nodes-def inv-isolated-nodes-outer-def*)
}
qed

declare *ovalidNF-isolated-nodes*[*THEN ovalidNF-wp*, *THEN trivial-label*, *vcg-l*]

19.3.3 *face-cycles*

definition *inv-face-cycles s iG iM ecnt* ≡
ecnt = *length* (*ig-edges iG*)
∧ *digraph-map* (*mk-graph iG*) *iM*

definition *fcs-upto* :: *nat pre-map* ⇒ *nat* ⇒ *nat set set* **where**
fcs-upto iM i ≡ {*pre-digraph-map.face-cycle-set iM k* | *k*. *k* < *i*}

definition *inv-face-cycles-outer s iG iM i c edge-info* $\equiv$
  let fcs = *fcs-upto iM i in*
  c = *card fcs*
  $\wedge$ ($\forall$ *k* < *length* (*ig-edges iG*). *k* $\in$ *edge-info* $\longleftrightarrow$ *k* $\in \bigcup$*fcs*)

definition *inv-face-cycles-inner s iG iM i j c edge-info* $\equiv$
  j $\in$ *pre-digraph-map.face-cycle-set iM i*
  $\wedge$ *c* = *card* (*fcs-upto iM i*)
  $\wedge$ *i* $\notin \bigcup$(*fcs-upto iM i*)
  $\wedge$ ($\forall$ *k* < *length* (*ig-edges iG*). *k* $\in$ *edge-info* $\longleftrightarrow$
    (*k* $\in \bigcup$(*fcs-upto iM i*)
   $\vee$ ($\exists$ *l* < *funpow-dist1* (*pre-digraph-map.face-cycle-succ iM*) *i j*. (*pre-digraph-map.face-cycle-succ iM* ^^ *l*) *i* = *k*)))

lemma *finite-fcs-upto*: *finite* (*fcs-upto iM i*)
  by (*auto simp*: *fcs-upto-def*)

lemma *card-orbit-eq-funpow-dist1*:
  assumes *x* $\in$ *orbit f x* **shows** *card* (*orbit f x*) = *funpow-dist1 f x x*
proof −
  have *card* (*orbit f x*) = *card* ((λ*n*. (*f* ^^ *n*) *x*) ‘ {*0*..<*funpow-dist1 f x x*})
    using *assms* **by** (*simp only*: *orbit-conv-funpow-dist1*[*symmetric*])
  also have ... = *card* {*0*..<*funpow-dist1 f x x*}
    using *assms* **by** (*intro card-image inj-on-funpow-dist1*)
  finally show *?thesis* **by** *simp*
qed

lemma *funpow-dist1-le*:
  assumes *y* $\in$ *orbit f x x* $\in$ *orbit f x*
  shows *funpow-dist1 f x y* $\leq$ *funpow-dist1 f x x*
    using *assms* **by** (*intro funpow-dist1-le-self funpow-dist1-prop*) *simp-all*

lemma *funpow-dist1-le-card*:
  assumes *y* $\in$ *orbit f x x* $\in$ *orbit f x*
  shows *funpow-dist1 f x y* $\leq$ *card* (*orbit f x*)
  using *funpow-dist1-le*[*OF assms*] **using** *assms*
  by (*simp add*: *card-orbit-eq-funpow-dist1*)

lemma (**in** *digraph-map*) *funpow-dist1-le-card-fcs*:
  assumes *b* $\in$ *face-cycle-set a*
  shows *funpow-dist1 face-cycle-succ a b* $\leq$ *card* (*face-cycle-set a*)
  by (*metis assms face-cycle-set-def face-cycle-set-self funpow-dist1-le-card*)

lemma *funpow-dist1-f-eq*:
  assumes *b* $\in$ *orbit f a a* $\in$ *orbit f a a* $\neq$ *b*
  shows *funpow-dist1 f a* (*f b*) = *Suc* (*funpow-dist1 f a b*)
proof −
  have *f-inj*: *inj-on* (λ*n*. (*f* ^^ *n*) *a*) {*0*..<*funpow-dist1 f a a*}

by (*rule inj-on-funpow-dist1*) (*rule assms*)
have *funpow-dist1 f a b* $\leq$ *funpow-dist1 f a a*
using *assms* **by** (*intro funpow-dist1-le*)
moreover
have *funpow-dist1 f a b* $\neq$ *funpow-dist1 f a a*
by (*metis assms funpow-dist1-prop*)
ultimately
have *f-less*: *funpow-dist1 f a b* $<$ *funpow-dist1 f a a* **by** *simp*

have *f-Suc-eq*: (*f* ^^ *Suc* (*funpow-dist1 f a b*)) *a* = *f b*
using *assms* **by** (*metis funpow.simps*(*2*) *o-apply funpow-dist1-prop*)
show *?thesis*
proof (*cases f b* = *a*)
case *True*
then show *?thesis*
by (*metis Suc-lessI f-Suc-eq f-less assms*(*2*) *funpow.simps*(*1*) *funpow-neq-less-funpow-dist1 id-apply old.nat.distinct*(*1*) *zero-less-Suc*)
next
case *False*
then have ∗: *Suc* (*funpow-dist1 f a b*) $<$ *funpow-dist1 f a a*
using *f-Suc-eq* **by** (*metis assms*(*2*) *f-less funpow-dist1-prop le-less-Suc-eq less-Suc-eq-le not-less-eq*)
from *f-inj* **have** ∗∗: $\bigwedge n.\ n <$ *funpow-dist1 f a a* $\Longrightarrow n \neq$ *Suc* (*funpow-dist1 f a b*) $\Longrightarrow$ (*f* ^^ *n*) *a* $\neq$ *f b*
using *f-Suc-eq* **by** (*auto dest*!: *inj-onD*) (*metis* ∗ *assms*(*2*) *f-Suc-eq funpow-neq-less-funpow-dist1*)
show *?thesis*
proof (*rule ccontr*)
assume *A*: ¬*?thesis*
have (*f* ^^ (*funpow-dist1 f a* (*f b*))) *a* = *f b*
using *assms* **by** (*intro funpow-dist1-prop*) (*simp add*: *orbit.intros*)
with *A* ∗∗ **have** *funpow-dist1 f a a* $\leq$ (*funpow-dist1 f a* (*f b*))
by (*metis less-Suc-eq-le not-less-eq*)
then have *Suc* (*funpow-dist1 f a b*) $<$ (*funpow-dist1 f a* (*f b*)) **using** ∗ **by** *linarith*
then have (*f* ^^ *Suc* (*funpow-dist1 f a b*)) *a* $\neq$ *f b*
by (*intro funpow-dist1-least*) *simp-all*
then show *False* **using** *f-Suc-eq* **by** *simp*
qed
qed
qed

lemma (**in** −) *funpow-dist1-less-f*:
assumes *b* ∈ *orbit f a a* ∈ *orbit f a a* $\neq$ *b*
shows *funpow-dist1 f a b* $<$ *funpow-dist1 f a* (*f b*)
using *assms* **by** (*simp add*: *funpow-dist1-f-eq*)

lemma *ovalidNF-face-cycles*:
ovalidNF (λ*s. digraph-map* (*mk-graph iG*) *iM*)

(*face-cycles iG iM*)
(*λr s. r = card* (*pre-digraph-map.face-cycle-sets* (*mk-graph iG*) *iM*))

unfolding *face-cycles-def*
apply (*rewrite*
in *oreturn* (*length* (*ig-edges iG*)) |>> (*λecnt.* ¤)
to owhile-inv - - -
(λ(*edge-info*, *c*, *i*) *s. inv-face-cycles s iG iM ecnt*
∧ *inv-face-cycles-outer s iG iM i c edge-info*
∧ *i* ≤ *ecnt*)
(*measure* (λ(*edge-info*, *c*, *i*). *ecnt* − *i*))
owhile-inv-def[*symmetric*]
)
apply (*rewrite*
in *owhile-inv* - (λ(-, *c*, *i*). ¤)
in *oreturn* (*length* (*ig-edges iG*)) |>> (*λecnt.* ¤)
to owhile-inv - - -
(λ(*edge-info*, *j*) *s. inv-face-cycles s iG iM ecnt*
∧ *inv-face-cycles-inner s iG iM i j c edge-info*
∧ *i* < *ecnt*)
(*measure* (λ(*edge-info*, *j*). *card* (*pre-digraph-map.face-cycle-set iM i*) − *funpow-dist1* (*pre-digraph-map.face-cycle-succ iM*) *i j*))
owhile-inv-def[*symmetric*]
)
proof *vcg-casify*
{ case (*weaken s*)
then show *?case* **by** (*auto simp add*: *inv-face-cycles-def inv-face-cycles-outer-def fcs-upto-def*)
}
{ case (*while edge-info c i*)
{ case (*postcondition s*)
moreover have *fcs-upto iM* (*length* (*ig-edges iG*))
= *pre-digraph-map.face-cycle-sets* (*mk-graph iG*) *iM*
by (*auto simp*: *pre-digraph-map.face-cycle-sets-def arcs-mkg fcs-upto-def*)
ultimately show *?case* **by** (*auto simp*: *inv-face-cycles-outer-def Let-def*)
}
{ case (*invariant s*)
{ case (*weaken s′*)
interpret *G*: *digraph-map mk-graph iG iM*
using *weaken* **by** (*auto simp*: *inv-face-cycles-def*)
show *?case*
proof *branch-casify*
case *else* **case** *g*
then have *G.face-cycle-set i* ∈ {*G.face-cycle-set k* |*k. k* < *i*}
using *weaken* **by** (*auto simp*: *inv-face-cycles-outer-def fcs-upto-def dest*: *G.face-cycle-eq*)
then have {*G.face-cycle-set k* |*k. k* < *Suc i*} = {*G.face-cycle-set k* |*k. k* < *i*}
by (*auto elim*: *less-SucE*)

then have *inv-face-cycles-outer s′ iG iM* $(i + 1)$ *c edge-info*
using *weaken* **unfolding** *inv-face-cycles-outer-def* **by** (*auto simp*: *fcs-upto-def*)
then have *?inv* **using** *weaken* **by** *auto*
then show *?case* **using** *weaken* **by** *auto*
next
case *then* **case** *g*
have *fd1-triv*: $\bigwedge f\, x.$ *funpow-dist1 f x* $(f\ x) = 1$
by (*simp add*: *funpow-dist-0*)
have *fcs-in*: *G.face-cycle-succ i* $\in$ *G.face-cycle-set i*
by (*simp add*: *G.face-cycle-succ-inI*)

have *i-not-in-fcs*: $i \notin \bigcup$(*fcs-upto iM i*)
using *g weaken*
by (*auto simp*: *inv-face-cycles-outer-def fcs-upto-def*)

from *weaken* **show** *?case*
unfolding *inv-face-cycles-inner-def inv-face-cycles-outer-def*
using *i-not-in-fcs* **by** (*auto simp*: *fd1-triv fcs-in fcs-upto-def*)
qed
}
{ case *if* **case** *then*
{ case (*while edge-info j*)
{ case (*postcondition s′*)

interpret *G*: *digraph-map mk-graph iG iM*
using *postcondition* **by** (*auto simp*: *inv-face-cycles-def*)

have *?var* **using** *postcondition* **by** *auto*

have *fu-Suc*: *fcs-upto iM* (*Suc j*) = *fcs-upto iM j* $\cup$ {*G.face-cycle-set j*}
by (*auto simp*: *fcs-upto-def elim*: *less-SucE*)
moreover
have *G.face-cycle-set j* $\notin$ *fcs-upto iM j c* = *card* (*fcs-upto iM j*)
using *postcondition* **by** (*auto simp*: *inv-face-cycles-inner-def*)
ultimately
have *Suc c* = *card* (*fcs-upto iM* (*Suc j*)) **by** (*simp add*: *finite-fcs-upto*)

have $*$: $\forall\, k <$ *length* (*snd iG*). $k \in$ *edge-info* $\longleftrightarrow$ ($\exists\, x \in$*fcs-upto iM* (*Suc j*). $k \in x$)
proof −
have $*$: $j \in$ *orbit G.face-cycle-succ j*
by (*simp add*: *G.face-cycle-set-def*[*symmetric*])
have $\bigwedge k.$ ($\exists\, l <$*funpow-dist1 G.face-cycle-succ j j*. (*G.face-cycle-succ* ^^ *l*) $j = k$) $\longleftrightarrow$ ($k \in$ *G.face-cycle-set j*)
by (*auto simp*: *G.face-cycle-set-def orbit-conv-funpow-dist1*[*OF* $*$])
moreover
from *postcondition* **have** *inv-face-cycles-inner s′ iG iM j j c edge-info*
by *auto*

ultimately
show *?thesis* **unfolding** *inv-face-cycles-inner-def fu-Suc* **by** *auto*
qed

have *?inv* **using** *postcondition* ∗
by (*auto simp*: *inv-face-cycles-outer-def* ‹*Suc c* = -›)
with ‹*?var*› **show** *?case* **by** *blast*
}
{ case (*invariant* s')
{ case (*weaken* s'')
interpret *G*: *digraph-map mk-graph iG iM*
using *weaken* **by** (*auto simp*: *inv-face-cycles-def*)
have $j \in$ *G.face-cycle-set i*
using *weaken* **by** (*auto simp*: *inv-face-cycles-inner-def*)
then have $j \in$ *arcs* (*mk-graph iG*)
by (*metis G.face-cycle-set-def G.funpow-face-cycle-succ-no-arc G.in-face-cycle-setD*
funpow-dist1-prop weaken.loop-cond)

have *A*: $j \in$ *pre-digraph-map.face-cycle-set iM i*
using *weaken* **by** (*auto simp*: *inv-face-cycles-inner-def*)
then have A': (*G.face-cycle-succ* ^^ *funpow-dist1 G.face-cycle-succ i j*) *i* = *j*
by (*intro funpow-dist1-prop*) (*simp add*: *G.face-cycle-set-def*[*symmetric*])

{ fix *k*
have ∗: $\bigwedge i\ n\ f\ x\ .\ i < n \Longrightarrow \exists j{<}n.\ (f$ ^^ $j)\ x = (f$ ^^ $i)\ x$ **by** *auto*

have (∃ *l*<*funpow-dist1 G.face-cycle-succ i* (*G.face-cycle-succ j*). (*G.face-cycle-succ* ^^ *l*) *i* = *k*)
⟷ (∃ *l*<*Suc* (*funpow-dist1 G.face-cycle-succ i j*). (*G.face-cycle-succ* ^^ *l*) *i* = *k*) (**is** *?L* ⟷ -)
using *A* ‹$i \neq j$›
by (*subst funpow-dist1-f-eq*) (*simp-all add*: *G.face-cycle-set-def*[*symmetric*])
also have . . . ⟷ (∃ *l*<*funpow-dist1 G.face-cycle-succ i j*. (*G.face-cycle-succ* ^^ *l*) *i* = *k*) ∨ *k* = *j* (**is** - ⟷ *?R*)
using A' **by** (*fastforce elim*: *less-SucE*
intro: ∗ *exI*[**where** *x*=(*funpow-dist1 G.face-cycle-succ i j*)])
finally have *?L* ⟷ *?R* **.**
} note *B* = *this*

have *?inv*
using *weaken* **unfolding** *inv-face-cycles-inner-def B*
by (*auto simp*: *G.face-cycle-succ-inI*)

have *X*: *funpow-dist1 G.face-cycle-succ i j* < *card* (*G.face-cycle-set i*)
proof −
have *funpow-dist1 G.face-cycle-succ i j* ≤ *funpow-dist1 G.face-cycle-succ i i*

using - - *A* **unfolding** *G.face-cycle-set-def*
apply (*rule funpow-dist1-le-self*)
apply (*rule funpow-dist1-prop*)
unfolding *G.face-cycle-set-def*[*symmetric*]
by *simp-all*
moreover have *funpow-dist1 G.face-cycle-succ i j* $\neq$ *funpow-dist1 G.face-cycle-succ i i*
by (*metis A G.face-cycle-set-def G.face-cycle-set-self funpow-dist1-prop weaken.loop-cond*)
ultimately
have *funpow-dist1 G.face-cycle-succ i j* < *funpow-dist1 G.face-cycle-succ i i*
by *simp*
also have ... $\leq$ *card* (*G.face-cycle-set i*)
by (*rule G.funpow-dist1-le-card-fcs*) *simp*
finally show *?thesis* **.**
qed

have *?var*
apply *simp*
using - *X* **apply** (*rule diff-less-mono2*)
apply (*rule funpow-dist1-less-f*)
using ‹*i* $\neq$ *j*› *A* **by** (*auto simp*: *G.face-cycle-set-def*[*symmetric*])
with ‹*?inv*› **show** *?case* **by** *blast*
}
}
}
}
}
}
qed *auto*
declare *ovalidNF-face-cycles*[*THEN ovalidNF-wp*, *THEN trivial-label*, *vcg-l*]

lemma *ovalidNF-euler-genus*:
ovalidNF (λ*s. distinct* (*ig-verts iG*) $\wedge$ *digraph-map* (*mk-graph iG*) *iM* $\wedge$ *c* = *card* (*pre-digraph.sccs* (*mk-graph iG*)))
(*euler-genus iG iM c*)
(λ*r s. r* = *pre-digraph-map.euler-genus* (*mk-graph iG*) *iM*)

unfolding *euler-genus-def*
proof *vcg-casify*
case *weaken*
have *distinct* (*ig-verts iG*) **using** *weaken* **by** *simp*
interpret *G*: *digraph-map mk-graph iG iM* **using** *weaken* **by** *simp*
have *len-card*:
length (*ig-verts iG*) = *card* (*verts* (*mk-graph iG*))
length (*ig-edges iG*) = *card* (*arcs* (*mk-graph iG*))
using ‹*distinct* -› **by** (*auto simp*: *mkg-simps arcs-mkg distinct-card*)
show *?case*

using *weaken* **by** (*auto simp*: *G.euler-genus-def G.euler-char-def len-card*)
qed

declare *ovalidNF-euler-genus*[*THEN ovalidNF-wp*, *THEN trivial-label*, *vcg-l*]

lemma *ovalidNF-certify*:
ovalidNF (λ*s. distinct* (*ig-verts iG*) $\wedge$ *fin-digraph* (*mk-graph iG*) $\wedge$ *c* = *card* (*pre-digraph.sccs* (*mk-graph iG*)))
(*certify iG iM c*)
(λ*r s. r* $\longleftrightarrow$ *pre-digraph-map.euler-genus* (*mk-graph iG*) (*mk-map* (*mk-graph iG*) *iM*) = *0*
$\wedge$ *digraph-map* (*mk-graph iG*) (*mk-map* (*mk-graph iG*) *iM*)
$\wedge$ ($\forall$ *i* < *length* (*ig-edges iG*). *im-pred iM* (*im-succ iM i*) = *i*))

unfolding *certify-def*
proof *vcg-casify*
case *weaken*
then interpret *fin-digraph mk-graph iG* **by** *auto*
from *weaken* **show** *?case* **by** (*auto simp*: *BRANCH-def intro*: *wf-digraph*)
qed

end
theory *Planarity-Certificates*
imports
Planarity/Kuratowski-Combinatorial
Verification/Check-Non-Planarity-Verification
Verification/Check-Planarity-Verification
begin

end

References

[1] L. Noschinski. *Formalizing Graph Theory and Planarity Certificates.* PhD thesis, Technische Universität München, München, Nov. 2015.