A simple proof that π is irrational

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Abstract

This entry provides a formalisation of Niven's famously short one-page proof that π is irrational. The proof uses only elementary algebra and analysis.

The intrinsic de Bruijn factor, i.e. the file size ratio between the gzipped Isabelle sources and a gzipped LaTeX version of the original paper's content, is roughly 4 despite the original paper's terse presentation.

Contents

1 A short proof of the irrationality of π

theory *Pi_Irrational*
imports
"HOL-Analysis.Analysis"
"Polynomial_Interpolation.Ring_Hom_Poly"
begin

1.1 Auxiliary material

lemma *fact_dvd_pochhammer:*
assumes *"m ≤ n + 1"*
shows *"fact m dvd pochhammer (int n - int m + 1) m"*
⟨*proof*⟩

lemma *factor_dvd_higher_pderiv:*
fixes *p :: "'a :: idom poly"*
assumes *"p ^ n dvd q" "i < n"*
shows *"p dvd (pderiv ^^ i) q"*
⟨*proof*⟩

lemma *fact_dvd_higher_pderiv:*
"[:fact n :: int:] dvd (pderiv ^^ n) p"
⟨*proof*⟩

lemma *higher_pderiv_eq_0_iff:*
fixes *p :: "'a::{comm_semiring_1,semiring_no_zero_divisors,semiring_char_0} poly"*
shows *"(pderiv ^^ n) p = 0 ⟷ p = 0 ∨ n > degree p"*
⟨*proof*⟩

lemma *higher_pderiv_pcompose_linear:*
shows *"(pderiv ^^ n) (pcompose p [:a, b:]) = smult (b ^ n) (pcompose ((pderiv ^^ n) p) [:a, b:])"*
⟨*proof*⟩

lemma *power_over_fact_tendsto_0:*
"(λn. (x :: real) ^ n / fact n) ⟶ 0"
⟨*proof*⟩

1.2 Main proof

locale *pi_rational =*
fixes *a b :: int*
assumes *ab: "a / b = pi"*
assumes *b: "b > 0"*
begin

context
fixes *n :: nat*

assumes *n: "n > 1"*
begin

definition *f :: "real poly"* **where**
"f = smult (1/fact n) ([:0, of_int a, -of_int b:] ^ n)"

lemma *f_mirror: "f* $\circ_p$ *[:pi, -1:] = f"*
⟨*proof*⟩

lemma *degree_f [simp]: "degree f = 2 * n"*
⟨*proof*⟩

definition *F :: "real poly"* **where**
"F = ($\sum$ *j*$\leq$*n. (-1)^j * (pderiv ^^ (2*j)) f)"*

lemma *F_mirror: "F* $\circ_p$ *[:pi, -1:] = F"*
⟨*proof*⟩

lemma *poly_F_pi: "poly F pi = poly F 0"*
⟨*proof*⟩

lemma *F_int: "poly F 0* $\in$ $\mathbb{Z}$*"*
⟨*proof*⟩

lemma *antideriv:*
"((λ*x. poly (pderiv F) x * sin x - poly F x * cos x)*
*has_field_derivative (poly f x * sin x)) (at x within A)"*
⟨*proof*⟩

lemma *bound: "pi / 2 * (a * pi) ^ n / fact n* $\geq$ *1"*
⟨*proof*⟩

end

lemma *absurd: False*
⟨*proof*⟩

end

theorem *pi_irrational: "pi* $\notin$ $\mathbb{Q}$*"*
⟨*proof*⟩

end

References

[1] I. Niven. A simple proof that π is irrational. *Bulletin of the American Mathematical Society*, 53(6):509, 1947.