A Sound Type System for Physical Quantities, Units, and Measurements

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Abstract

The present Isabelle theory builds a formal model for both the *International System of Quantities* (ISQ) and the *International System of Units* (SI), which are both fundamental for physics and engineering [2]. Both the ISQ and the SI are deeply integrated into Isabelle's type system. Quantities are parameterised by *dimension types*, which correspond to base vectors, and thus only quantities of the same dimension can be equated. Since the underlying "algebra of quantities" from [2] induces congruences on quantity and SI types, specific tactic support is developed to capture these. Our construction is validated by a test-set of known equivalences between both quantities and SI units. Moreover, the presented theory can be used for type-safe conversions between the SI system and others, like the British Imperial System (BIS).

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Chapter 1

ISQ and SI: An Introduction

Modern Physics is based on the concept of quantifiable properties of physical phenomena such as mass, length, time, current, etc. These phenomena, called *quantities*, are linked via an *algebra of quantities* to derived concepts such as speed, force, and energy. The latter allows for a *dimensional analysis* of physical equations, which had already been the backbone of Newtonian Physics. In parallel, physicians developed their own research field called "metrology" defined as a scientific study of the *measurement* of physical quantities.

The relevant international standard for quantities and measurements is distributed by the *Bureau International des Poids et des Mesures* (BIPM), which also provides the *Vocabulaire International de Mètrologie* (VIM) [2]. The VIM actually defines two systems: the *International System of Quantities* (ISQ) and the *International System of Units* (SI, abbreviated from the French Système international (dunités)). The latter is also documented in the *SI Brochure* [3], a standard that is updated periodically, most recently in 2019. Finally, the VIM defines concrete reference measurement procedures as well as a terminology for measurement errors.

Conceived as a refinement of the ISQ, the SI comprises a coherent system of units of measurement built on seven base units, which are the metre, kilogram, second, ampere, kelvin, mole, candela, and a set of twenty prefixes to the unit names and unit symbols, such as milli- and kilo-, that may be used when specifying multiples and fractions of the units. The system also specifies names for 22 derived units, such as lumen and watt, for other common physical quantities. While there is still nowadays a wealth of different measuring systems such as the *British Imperial System* (BIS) and the *United States Customary System* (USC), the SI is more or less the de-facto reference behind all these systems.

The present Isabelle theory builds a formal model for both the ISQ and the SI, together with a deep integration into Isabelle's type system [5]. Quan-

tities and units are represented in a way that they have a *quantity type* as well as a *unit type* based on its base vectors and their magnitudes. Since the algebra of quantities induces congruences on quantity and SI types, specific tactic support has been developed to capture these. Our construction is validated by a test-set of known equivalences between both quantities and SI units. Moreover, the presented theory can be used for type-safe conversions between the SI system and others, like the British Imperial System (BIS).

In the following we describe the overall theory architecture in more detail. Our ISQ model provides the following fundamental concepts:

1. *dimensions* represented by a type *(int, 'd::enum) dimvec* , i.e. a *'d*-indexed vector space of integers representing the exponents of the dimension vector. *'d* is constrained to be a dimension type later.

2. *quantities* represented by type *('a, 'd::enum) Quantity*, which are constructed as a vector space and a magnitude type *'a*.

3. quantity calculus consisting of *quantity equations* allowing to infer that $LT^{-1}T^{-1}M = MLT^{-2} = F$ (the left-hand-side equals mass times acceleration which is equal to force).

4. a kind of equivalence relation $\cong_Q$ on quantities, permitting to relate quantities of different dimension types.

5. *base quantities* for *length*, *mass*, *time*, *electric current*, *temperature*, *amount of substance*, and *luminous intensity*, serving as concrete instance of the vector instances, and for syntax a set of the symbols L, M, T, I, Θ, N, J corresponding to the above mentioned base vectors.

6. *(Abstract) Measurement Systems* represented by type *('a, 'd::enum, 's::unit_system) Measurement_System*, which are a refinement of quantities. The refinement is modelled by a polymorphic record extensions; as a consequence, Measurement Systems inherit the algebraic properties of quantities.

7. *derived dimensions* such as *volume* L^3 or energy $ML^2\,T^{-2}$ corresponding to *derived quantities*.

Then, through a fresh type-constructor *SI*, the abstract measurement systems are instantiated to the SI system — the *British Imperial System* (BIS) is constructed analogously. Technically, *SI* is a tag-type that represents the fact that the magnitude of a quantity is actually a quantifiable entity in the sense of the SI system. In other words, this means that the magnitude 1 in quantity *1[L]* actually refers to one metre intended to be measured according to the SI standard. At this point, it becomes impossible, for example, to add to one foot, in the sense of the BIS, to one metre in the SI without creating a type-inconsistency.

The theory of the SI is created by specialising the *Measurement_System*-type with the SI-tag-type and adding new infrastructure. The SI theory provides the following fundamental concepts:

1. measuring units and types corresponding to the ISQ base quantities sich as *metre*, *kilogram*, *second*, *ampere*, *kelvin*, *mole* and *candela* (together with procedures how to measure a metre, for example, which are defined in accompanying standards);

2. a standardised set of symbols for units such as m, kg, s, A, K, mol, and cd;

3. a standardised set of symbols of SI prefixes for multiples of SI units, such as $giga$ $(= 10^9)$, $kilo$ $(= 10^3)$, $milli$ $(= 10^-3)$, etc.; and a set of

4. *unit equations* and conversion equations such as $J = kg\,m^2/s^2$ or $1km/h = 1/3.6\,m/s$.

As a result, it is possible to express "4500.0 kilogram times metre per second squared" which has the type $\mathbb{R}\ [M \cdot L \cdot T^{-3} \cdot SI]$. This type means that the magnitude 4500 of the dimension $M \cdot L \cdot T^{-3}$ is a quantity intended to be measured in the SI-system, which means that it actually represents a force measured in Newtons. In the example, the *magnitude* type of the measurement unit is the real numbers ($\mathbb{R}$). In general, however, magnitude types can be arbitrary types from the HOL library, so for example integer numbers (*int*), integer numbers representable by 32 bits (*int32*), IEEE-754 floating-point numbers (*float*), or, a vector in the three-dimensional space $\mathbb{R}^3$. Thus, our type-system allows to capture both conceptual entities in physics as well as implementation issues in concrete physical calculations on a computer.

As mentioned before, it is a main objective of this work to support the quantity calculus of ISQ and the resulting equations on derived SI entities (cf. [3]), both from a type checking as well as a proof-checking perspective. Our design objectives are not easily reconciled, however, and so some substantial theory engineering is required. On the one hand, we want a deep integration of dimensions and units into the Isabelle type system. On the other, we need to do normal-form calculations on types, so that, for example, the units m and $ms^{-1}s$ can be equated.

Isabelle's type system follows the Curry-style paradigm, which rules out the possibility of direct calculations on type-terms (in contrast to Coq-like systems). However, our semantic interpretation of ISQ and SI allows for the foundation of the heterogeneous equivalence relation $\cong_Q$ in semantic terms. This means that we can relate quantities with syntactically different dimension types, yet with same dimension semantics. This paves the way

for derived rules that do computations of terms, which represent type computations indirectly. This principle is the basis for the tactic support, which allows for the dimensional type checking of key definitions of the SI system. Some examples are given below.

theorem *metre-definition*:
$1 *_Q$ *metre* $\cong_Q$ (**c** / (*299792458* $*_Q$ **1**))·*second*
by *si-calc*

theorem *kilogram-definition*:
$1 *_Q$ *kilogram* $\cong_Q$ (**h** / (*6.62607015* · *1*/(*10^34*) $*_Q$ **1**))·*metre*$^{-\mathbf{2}}$·*second*
by *si-calc*

These equations are both adapted from the SI Brochure, and give the concrete definitions for the metre and kilogram in terms of the physical constants **c** (speed of light) and **h** (Planck constant). They are both proved using the tactic *si-calc*.

This work has drawn inspiration from some previous formalisations of the ISQ and SI, notably Hayes and Mahoney's formalisation in Z [4] and Aragon's algebraic structure for physical quantities [1]. To the best of our knowledge, our mechanisation represents the most comprehensive account of ISQ and SI in a theory prover.

Chapter 2

Preliminaries

2.1 Integer Powers

theory *Power-int*
imports *HOL.Real*
begin

The standard HOL power operator is only for natural powers. This operator allows integers.

definition *intpow* :: *'a*::{*linordered-field*} ⇒ *int* ⇒ *'a* (**infixr** $\hat{}_Z$ *80*) **where**
intpow x n = (*if* (*n* < *0*) *then inverse* (*x* ^ *nat* (−*n*)) *else* (*x* ^ *nat n*))

lemma *intpow-zero* [*simp*]: *x* $\hat{}_Z$ *0* = *1*
⟨*proof*⟩

lemma *intpow-spos* [*simp*]: *x* > *0* ⟹ *x* $\hat{}_Z$ *n* > *0*
⟨*proof*⟩

lemma *intpow-one* [*simp*]: *x* $\hat{}_Z$ *1* = *x*
⟨*proof*⟩

lemma *one-intpow* [*simp*]: *1* $\hat{}_Z$ *n* = *1*
⟨*proof*⟩

lemma *intpow-plus*: *x* > *0* ⟹ *x* $\hat{}_Z$ (*m* + *n*) = *x* $\hat{}_Z$ *m* ∗ *x* $\hat{}_Z$ *n*
⟨*proof*⟩

lemma *intpow-mult-combine*: *x* > *0* ⟹ *x* $\hat{}_Z$ *m* ∗ (*x* $\hat{}_Z$ *n* ∗ *y*) = *x* $\hat{}_Z$ (*m* + *n*) ∗ *y*
⟨*proof*⟩

lemma *intpow-pos* [*simp*]: *n* ≥ *0* ⟹ *x* $\hat{}_Z$ *n* = *x* ^ *nat n*
⟨*proof*⟩

lemma *intpow-uminus*: *x* $\hat{}_Z$ −*n* = *inverse* (*x* $\hat{}_Z$ *n*)

⟨*proof*⟩

lemma *intpow-uminus-nat*: $n \geq 0 \Longrightarrow x \mathbin{\hat{}_Z} -n = \mathit{inverse}\ (x \mathbin{\hat{}} \mathit{nat}\ n)$
⟨*proof*⟩

lemma *intpow-inverse*: $\mathit{inverse}\ a \mathbin{\hat{}_Z} n = \mathit{inverse}\ (a \mathbin{\hat{}_Z} n)$
⟨*proof*⟩

lemma *intpow-mult-distrib*: $(x * y) \mathbin{\hat{}_Z} m = x \mathbin{\hat{}_Z} m * y \mathbin{\hat{}_Z} m$
⟨*proof*⟩

end

2.2 Enumeration Extras

theory *Enum-extra*
imports *HOL−Library.Code-Cardinality*
begin

2.2.1 First Index Function

The following function extracts the index of the first occurrence of an element in a list, assuming it is indeed an element.

fun *first-ind* :: $'a\ \mathit{list} \Rightarrow 'a \Rightarrow \mathit{nat} \Rightarrow \mathit{nat}$ **where**
$\mathit{first\text{-}ind}\ []\ y\ i = \mathit{undefined}$ |
$\mathit{first\text{-}ind}\ (x \# xs)\ y\ i = (\mathit{if}\ (x = y)\ \mathit{then}\ i\ \mathit{else}\ \mathit{first\text{-}ind}\ xs\ y\ (\mathit{Suc}\ i))$

lemma *first-ind-length*:
$x \in \mathit{set}(xs) \Longrightarrow \mathit{first\text{-}ind}\ xs\ x\ i < \mathit{length}(xs) + i$
⟨*proof*⟩

lemma *nth-first-ind*:
$[\![\ \mathit{distinct}\ xs;\ x \in \mathit{set}(xs)\ ]\!] \Longrightarrow xs\ !\ (\mathit{first\text{-}ind}\ xs\ x\ i - i) = x$
⟨*proof*⟩

lemma *first-ind-nth*:
$[\![\ \mathit{distinct}\ xs;\ i < \mathit{length}\ xs\ ]\!] \Longrightarrow \mathit{first\text{-}ind}\ xs\ (xs\ !\ i)\ j = i + j$
⟨*proof*⟩

2.2.2 Enumeration Indices

syntax
-ENUM :: $\mathit{type} \Rightarrow \mathit{logic}$ (*ENUM'*(*-'*))

translations
ENUM('*a*) => *CONST Enum.enum* :: ('*a*::*enum*) *list*

Extract a unique natural number associated with an enumerated value by using its index in the characteristic list *enum-class.enum*.

definition *enum-ind* :: *'a*::*enum* ⇒ *nat* **where**
enum-ind (*x* :: *'a*::*enum*) = *first-ind ENUM*(*'a*) *x 0*

lemma *length-enum-CARD*: *length ENUM*(*'a*) = *CARD*(*'a*)
⟨*proof*⟩

lemma *CARD-length-enum*: *CARD*(*'a*) = *length ENUM*(*'a*)
⟨*proof*⟩

lemma *enum-ind-less-CARD* [*simp*]: *enum-ind* (*x* :: *'a*::*enum*) < *CARD*(*'a*)
⟨*proof*⟩

lemma *enum-nth-ind* [*simp*]: *Enum.enum* ! (*enum-ind x*) = *x*
⟨*proof*⟩

lemma *enum-distinct-conv-nth*:
assumes *i* < *CARD*(*'a*) *j* < *CARD*(*'a*) *ENUM*(*'a*) ! *i* = *ENUM*(*'a*) ! *j*
shows *i* = *j*
⟨*proof*⟩

lemma *enum-ind-nth* [*simp*]:
assumes *i* < *CARD*(*'a*::*enum*)
shows *enum-ind* (*ENUM*(*'a*) ! *i*) = *i*
⟨*proof*⟩

lemma *enum-ind-spec*:
enum-ind (*x* :: *'a*::*enum*) = (*THE i*. *i* < *CARD*(*'a*) ∧ *Enum.enum* ! *i* = *x*)
⟨*proof*⟩

lemma *enum-ind-inj*: *inj* (*enum-ind* :: *'a*::*enum* ⇒ *nat*)
⟨*proof*⟩

lemma *enum-ind-neq* [*simp*]: *x* ≠ *y* ⟹ *enum-ind x* ≠ *enum-ind y*
⟨*proof*⟩

end

2.3 Multiplication Groups

theory *Groups-mult*
imports *Main*
begin

The HOL standard library only has groups based on addition. Here, we build one based on multiplication.

notation *times* (**infixl** · *70*)

class *group-mult* = *inverse* + *monoid-mult* +
assumes *left-inverse*: *inverse a* · *a* = *1*

assumes *multi-inverse-conv-div* [*simp*]: $a \cdot (inverse\ b) = a\ /\ b$
begin

lemma *div-conv-mult-inverse*: $a\ /\ b = a \cdot (inverse\ b)$
⟨*proof*⟩

sublocale *mult*: *group times 1 inverse*
⟨*proof*⟩

lemma *diff-self* [*simp*]: $a\ /\ a = 1$
⟨*proof*⟩

lemma *mult-distrib-inverse* [*simp*]: $(a * b)\ /\ b = a$
⟨*proof*⟩

end

class *ab-group-mult* = *comm-monoid-mult* + *group-mult*
begin

lemma *mult-distrib-inverse'* [*simp*]: $(a * b)\ /\ a = b$
⟨*proof*⟩

lemma *inverse-distrib*: $inverse\ (a * b) = (inverse\ a) * (inverse\ b)$
⟨*proof*⟩

lemma *inverse-divide* [*simp*]: $inverse\ (a\ /\ b) = b\ /\ a$
⟨*proof*⟩

end

abbreviation (*input*) *npower* :: *'a*::{*power*,*inverse*} ⇒ *nat* ⇒ *'a* ((-^^-) [*1000*,*999*] *999*)
where $npower\ x\ n \equiv inverse\ (x \hat{} n)$

end

Chapter 3

International System of Quantities

3.1 Quantity Dimensions

theory *ISQ-Dimensions*
imports *Groups-mult Power-int Enum-extra*
HOL.Transcendental
HOL−Eisbach.Eisbach
begin

3.1.1 Preliminaries

class *unitary* = *finite* +
assumes *unitary-unit-pres*: *card* (*UNIV*::′*a set*) = *1*
begin

definition *unit* = (*undefined*::′*a*)

lemma *UNIV-unitary*: *UNIV* = {*a*::′*a*}
⟨*proof*⟩

lemma *eq-unit*: (*a*::′*a*) = *b*
⟨*proof*⟩

end

lemma *unitary-intro*: (*UNIV*::′*s set*) = {*a*} ⟹ *OFCLASS*(′*s*, *unitary-class*)
⟨*proof*⟩

named-theorems *si-def* **and** *si-eq*

instantiation *unit* :: *comm-monoid-add*
begin
definition *zero-unit* = ()

definition *plus-unit* (*x*::*unit*) (*y*::*unit*) = ()
instance ⟨*proof*⟩
end

instantiation *unit* :: *comm-monoid-mult*
begin
definition *one-unit* = ()
definition *times-unit* (*x*::*unit*) (*y*::*unit*) = ()
instance ⟨*proof*⟩
end

instantiation *unit* :: *inverse*
begin
definition *inverse-unit* (*x*::*unit*) = ()
definition *divide-unit* (*x*::*unit*) (*y*::*unit*) = ()
instance ⟨*proof*⟩
end

instance *unit* :: *ab-group-mult*
⟨*proof*⟩

3.1.2 Dimension Vectors

Quantity dimensions are used to distinguish quantities of different kinds. Only quantities of the same kind can be compared and combined: it is a mistake to add a length to a mass, for example. Dimensions are often expressed in terms of seven base quantities, which can be combined to form derived quantities. Consequently, a dimension associates with each of the base quantities an integer that denotes the power to which it is raised. We use a special vector type to represent dimensions, and then specialise this to the seven major dimensions.

typedef ('*n*, '*d*) *dimvec* = *UNIV* :: ('*d*::*enum* ⇒ '*n*) *set*
morphisms *dim-nth dim-lambda* ⟨*proof*⟩

declare *dim-lambda-inject* [*simplified*, *simp*]
declare *dim-nth-inverse* [*simp*]
declare *dim-lambda-inverse* [*simplified*, *simp*]

instantiation *dimvec* :: (*zero*, *enum*) *one*
begin
definition *one-dimvec* :: ('*a*, '*b*) *dimvec* **where** *one-dimvec* = *dim-lambda* (λ *i*. *0*)
instance ⟨*proof*⟩
end

instantiation *dimvec* :: (*plus*, *enum*) *times*
begin
definition *times-dimvec* :: ('*a*, '*b*) *dimvec* ⇒ ('*a*, '*b*) *dimvec* ⇒ ('*a*, '*b*) *dimvec*
where

times-dimvec x y = dim-lambda (λ i. dim-nth x i + dim-nth y i)
instance ⟨*proof*⟩
end

instance *dimvec* :: (*comm-monoid-add*, *enum*) *comm-monoid-mult*
⟨*proof*⟩

We also define the inverse and division operations, and an abelian group, which will allow us to perform dimensional analysis.

instantiation *dimvec* :: ({*plus*,*uminus*}, *enum*) *inverse*
begin
definition *inverse-dimvec* :: ('*a*, '*b*) *dimvec* ⇒ ('*a*, '*b*) *dimvec* **where**
inverse-dimvec x = dim-lambda (λ i. − dim-nth x i)

definition *divide-dimvec* :: ('*a*, '*b*) *dimvec* ⇒ ('*a*, '*b*) *dimvec* ⇒ ('*a*, '*b*) *dimvec*
where
[*code-unfold*]: *divide-dimvec x y = x ∗ (inverse y)*

instance ⟨*proof*⟩
end

instance *dimvec* :: (*ab-group-add*, *enum*) *ab-group-mult*
⟨*proof*⟩

3.1.3 Code Generation

Dimension vectors can be represented using lists, which enables code generation and thus efficient proof.

definition *mk-dimvec* :: '*n list* ⇒ ('*n*::*ring-1*, '*d*::*enum*) *dimvec*
where *mk-dimvec ds = (if (length ds = CARD('d)) then dim-lambda (λ d. ds ! enum-ind d) else 1)*

code-datatype *mk-dimvec*

lemma *mk-dimvec-inj*: *inj-on* (*mk-dimvec* :: '*n list* ⇒ ('*n*::*ring-1*, '*d*::*enum*) *dimvec*) {*xs*. *length xs = CARD('d)*}
⟨*proof*⟩

lemma *mk-dimvec-eq-iff* [*simp*]:
assumes *length x = CARD('d) length y = CARD('d)*
shows ((*mk-dimvec x* :: ('*n*::*ring-1*, '*d*::*enum*) *dimvec*) = *mk-dimvec y*) ⟷ (*x = y*)
⟨*proof*⟩

lemma *one-mk-dimvec* [*code*, *si-def*]: (*1*::('*n*::*ring-1*, '*a*::*enum*) *dimvec*) = *mk-dimvec* (*replicate CARD('a) 0*)
⟨*proof*⟩

lemma *times-mk-dimvec* [*code*, *si-def*]:
(*mk-dimvec xs* ∗ *mk-dimvec ys* :: (*'n*::*ring-1*, *'a*::*enum*) *dimvec*) =
(*if* (*length xs* = *CARD*(*'a*) ∧ *length ys* = *CARD*(*'a*))
then mk-dimvec (*map* (λ (*x*, *y*). *x* + *y*) (*zip xs ys*))
else if length xs = *CARD*(*'a*) *then mk-dimvec xs else mk-dimvec ys*)
⟨*proof*⟩

lemma *power-mk-dimvec* [*si-def*]:
(*power* (*mk-dimvec xs*) *n* :: (*'n*::*ring-1*, *'a*::*enum*) *dimvec*) =
(*if* (*length xs* = *CARD*(*'a*)) *then mk-dimvec* (*map* ((∗) (*of-nat n*)) *xs*) *else mk-dimvec xs*)
⟨*proof*⟩

lemma *inverse-mk-dimvec* [*code*, *si-def*]:
(*inverse* (*mk-dimvec xs*) :: (*'n*::*ring-1*, *'a*::*enum*) *dimvec*) =
(*if* (*length xs* = *CARD*(*'a*)) *then mk-dimvec* (*map uminus xs*) *else 1*)
⟨*proof*⟩

lemma *divide-mk-dimvec* [*code*, *si-def*]:
(*mk-dimvec xs* / *mk-dimvec ys* :: (*'n*::*ring-1*, *'a*::*enum*) *dimvec*) =
(*if* (*length xs* = *CARD*(*'a*) ∧ *length ys* = *CARD*(*'a*))
then mk-dimvec (*map* (λ (*x*, *y*). *x* − *y*) (*zip xs ys*))
else if length ys = *CARD*(*'a*) *then mk-dimvec* (*map uminus ys*) *else mk-dimvec xs*)
⟨*proof*⟩

A base dimension is a dimension where precisely one component has power 1: it is the dimension of a base quantity. Here we define the seven base dimensions.

definition *mk-BaseDim* :: *'d*::*enum* ⇒ (*int*, *'d*) *dimvec* **where**
mk-BaseDim d = *dim-lambda* (λ *i*. *if* (*i* = *d*) *then 1 else 0*)

lemma *mk-BaseDim-neq* [*simp*]: *x* ≠ *y* ⟹ *mk-BaseDim x* ≠ *mk-BaseDim y*
⟨*proof*⟩

lemma *mk-BaseDim-code* [*code*]: *mk-BaseDim* (*d*::*'d*::*enum*) = *mk-dimvec* (*list-update* (*replicate CARD*(*'d*) *0*) (*enum-ind d*) *1*)
⟨*proof*⟩

definition *is-BaseDim* :: (*int*, *'d*::*enum*) *dimvec* ⇒ *bool*
where *is-BaseDim x* ≡ (∃ *i*. *x* = *dim-lambda* ((λ *x*. *0*)(*i* := *1*)))

lemma *is-BaseDim-mk* [*simp*]: *is-BaseDim* (*mk-BaseDim x*)
⟨*proof*⟩

3.1.4 Dimension Semantic Domain

We next specialise dimension vectors to the usual seven place vector.

datatype *sdim* = *Length* | *Mass* | *Time* | *Current* | *Temperature* | *Amount* | *Intensity*

lemma *sdim-UNIV*: (*UNIV* :: *sdim set*) = {*Length*, *Mass*, *Time*, *Current*, *Temperature*, *Amount*, *Intensity*}
⟨*proof*⟩

lemma *CARD-sdim* [*simp*]: *CARD*(*sdim*) = 7
⟨*proof*⟩

instantiation *sdim* :: *enum*
begin
definition *enum-sdim* = [*Length*, *Mass*, *Time*, *Current*, *Temperature*, *Amount*, *Intensity*]
definition *enum-all-sdim P* ⟷ *P Length* ∧ *P Mass* ∧ *P Time* ∧ *P Current* ∧ *P Temperature* ∧ *P Amount* ∧ *P Intensity*
definition *enum-ex-sdim P* ⟷ *P Length* ∨ *P Mass* ∨ *P Time* ∨ *P Current* ∨ *P Temperature* ∨ *P Amount* ∨ *P Intensity*
instance
⟨*proof*⟩
end

instantiation *sdim* :: *card-UNIV*
begin
definition *finite-UNIV* = *Phantom*(*sdim*) *True*
definition *card-UNIV* = *Phantom*(*sdim*) 7
instance ⟨*proof*⟩
end

lemma *sdim-enum* [*simp*]:
enum-ind Length = 0 *enum-ind Mass* = 1 *enum-ind Time* = 2 *enum-ind Current* = 3
enum-ind Temperature = 4 *enum-ind Amount* = 5 *enum-ind Intensity* = 6
⟨*proof*⟩

type-synonym *Dimension* = (*int*, *sdim*) *dimvec*

abbreviation *LengthBD* (**L**) **where L** ≡ *mk-BaseDim Length*
abbreviation *MassBD* (**M**) **where M** ≡ *mk-BaseDim Mass*
abbreviation *TimeBD* (**T**) **where T** ≡ *mk-BaseDim Time*
abbreviation *CurrentBD* (**I**) **where I** ≡ *mk-BaseDim Current*
abbreviation *TemperatureBD* (**Θ**) **where Θ** ≡ *mk-BaseDim Temperature*
abbreviation *AmountBD* (**N**) **where N** ≡ *mk-BaseDim Amount*
abbreviation *IntensityBD* (**J**) **where J** ≡ *mk-BaseDim Intensity*

abbreviation *BaseDimensions* ≡ {**L**, **M**, **T**, **I**, **Θ**, **N**, **J**}

lemma *BD-mk-dimvec* [*si-def*]:
L = *mk-dimvec* [1, 0, 0, 0, 0, 0, 0]

M = *mk-dimvec* [*0*, *1*, *0*, *0*, *0*, *0*, *0*]
T = *mk-dimvec* [*0*, *0*, *1*, *0*, *0*, *0*, *0*]
I = *mk-dimvec* [*0*, *0*, *0*, *1*, *0*, *0*, *0*]
Θ = *mk-dimvec* [*0*, *0*, *0*, *0*, *1*, *0*, *0*]
N = *mk-dimvec* [*0*, *0*, *0*, *0*, *0*, *1*, *0*]
J = *mk-dimvec* [*0*, *0*, *0*, *0*, *0*, *0*, *1*]
⟨*proof*⟩

The following lemma confirms that there are indeed seven unique base dimensions.

lemma *seven-BaseDimensions*: *card BaseDimensions* = *7*
⟨*proof*⟩

We can use the base dimensions and algebra to form dimension expressions. Some examples are shown below.

term $\mathbf{L}\cdot\mathbf{M}\cdot\mathbf{T}^{-2}$
term $\mathbf{M}\cdot\mathbf{L}^{-3}$

value $\mathbf{L}\cdot\mathbf{M}\cdot\mathbf{T}^{-2}$

lemma $\mathbf{L}\cdot\mathbf{M}\cdot\mathbf{T}^{-2}$ = *mk-dimvec* [*1*, *1*, − *2*, *0*, *0*, *0*, *0*]
⟨*proof*⟩

3.1.5 Dimension Type Expressions

Classification

We provide a syntax for dimension type expressions, which allows representation of dimensions as types in Isabelle. This will allow us to represent quantities that are parametrised by a particular dimension type. We first must characterise the subclass of types that represent a dimension.

The mechanism in Isabelle to characterize a certain subclass of Isabelle type expressions are *type classes*. The following type class is used to link particular Isabelle types to an instance of the type *Dimension*. It requires that any such type has the cardinality *1*::′*a*, since a dimension type is used only to mark a quantity.

class *dim-type* = *unitary* +
fixes *dim-ty-sem* :: ′*a itself* ⇒ *Dimension*

syntax
-*QD* :: *type* ⇒ *logic* (*QD*′(-′))

translations
QD(′*a*) == *CONST dim-ty-sem TYPE*(′*a*)

The notation *QD*(′*a*) allows to obtain the dimension of a dimension type ′*a*.

The subset of basic dimension types can be characterized by the following type class:

class *basedim-type* = *dim-type* +
assumes *is-BaseDim*: *is-BaseDim QD('a)*

Base Dimension Type Expressions

The definition of the basic dimension type constructors is straightforward via a one-elementary set, *unit set*. The latter is adequate since we need just an abstract syntax for type expressions, so just one value for the `dimension`-type symbols. We define types for each of the seven base dimensions, and also for dimensionless quantities.

typedef *Length* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Length*
typedef *Mass* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Mass*
typedef *Time* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Time*
typedef *Current* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Current*
typedef *Temperature* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Temperature*
typedef *Amount* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Amount*
typedef *Intensity* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-Intensity*
typedef *NoDimension* = *UNIV* :: *unit set* ⟨*proof*⟩ **setup-lifting** *type-definition-NoDimension*

type-synonym *M* = *Mass*
type-synonym *L* = *Length*
type-synonym *T* = *Time*
type-synonym *I* = *Current*
type-synonym Θ = *Temperature*
type-synonym *N* = *Amount*
type-synonym *J* = *Intensity*
type-notation *NoDimension* (**1**)

translations
(*type*) *M* <= (*type*) *Mass*
(*type*) *L* <= (*type*) *Length*
(*type*) *T* <= (*type*) *Time*
(*type*) *I* <= (*type*) *Current*
(*type*) Θ <= (*type*) *Temperature*
(*type*) *N* <= (*type*) *Amount*
(*type*) *J* <= (*type*) *Intensity*

Next, we embed the base dimensions into the dimension type expressions by instantiating the class *basedim-type* with each of the base dimension types.

instantiation *Length* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Length* (-::*Length itself*) = **L**
instance ⟨*proof*⟩
end

instantiation *Mass* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Mass* (-::*Mass itself*) = **M**
instance ⟨*proof*⟩
end

instantiation *Time* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Time* (-::*Time itself*) = **T**
instance ⟨*proof*⟩
end

instantiation *Current* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Current* (-::*Current itself*) = **I**
instance ⟨*proof*⟩
end

instantiation *Temperature* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Temperature* (-::*Temperature itself*) = **Θ**
instance ⟨*proof*⟩
end

instantiation *Amount* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Amount* (-::*Amount itself*) = **N**
instance ⟨*proof*⟩
end

instantiation *Intensity* :: *basedim-type*
begin
definition [*si-eq*]: *dim-ty-sem-Intensity* (-::*Intensity itself*) = **J**
instance ⟨*proof*⟩
end

instantiation *NoDimension* :: *dim-type*
begin
definition [*si-eq*]: *dim-ty-sem-NoDimension* (-::*NoDimension itself*) = (*1*::*Dimension*)
instance ⟨*proof*⟩
end

lemma *base-dimension-types* [*simp*]:
is-BaseDim QD(*Length*) *is-BaseDim QD*(*Mass*) *is-BaseDim QD*(*Time*) *is-BaseDim QD*(*Current*)
is-BaseDim QD(*Temperature*) *is-BaseDim QD*(*Amount*) *is-BaseDim QD*(*Intensity*)

⟨*proof*⟩

Dimension Type Constructors: Inner Product and Inverse

Dimension type expressions can be constructed by multiplication and division of the base dimension types above. Consequently, we need to define multiplication and inverse operators at the type level as well. On the class of dimension types (in which we have already inserted the base dimension types), the definitions of the type constructors for inner product and inverse is straightforward.

typedef ('a::*dim-type*, 'b::*dim-type*) *DimTimes* (**infixl** · *69*) = *UNIV* :: *unit set* ⟨*proof*⟩
setup-lifting *type-definition-DimTimes*

The type 'a · 'b is parameterised by two types, 'a and 'b that must both be elements of the *dim-type* class. As with the base dimensions, it is a unitary type as its purpose is to represent dimension type expressions. We instantiate *dim-type* with this type, where the semantics of a product dimension expression is the product of the underlying dimensions. This means that multiplication of two dimension types yields a dimension type.

instantiation *DimTimes* :: (*dim-type*, *dim-type*) *dim-type*
begin
  definition *dim-ty-sem-DimTimes* :: ('a · 'b) *itself* ⇒ *Dimension* **where**
  [*si-eq*]: *dim-ty-sem-DimTimes x* = *QD*('a) ∗ *QD*('b)
  instance ⟨*proof*⟩
end

Similarly, we define inversion of dimension types and prove that dimension types are closed under this.

typedef 'a *DimInv* ((-$^{-1}$) [*999*] *999*) = *UNIV* :: *unit set* ⟨*proof*⟩
setup-lifting *type-definition-DimInv*
instantiation *DimInv* :: (*dim-type*) *dim-type*
begin
  definition *dim-ty-sem-DimInv* :: ('a^{-1}) *itself* ⇒ *Dimension* **where**
  [*si-eq*]: *dim-ty-sem-DimInv x* = *inverse QD*('a)
  instance ⟨*proof*⟩
end

Dimension Type Syntax

A division is expressed, as usual, by multiplication with an inverted dimension.

type-synonym ('a, 'b) *DimDiv* = 'a · ('b^{-1}) (**infixl** '/ *69*)

A number of further type synonyms allow for more compact notation:

type-synonym 'a *DimSquare* = 'a · 'a ((-)^2 [*999*] *999*)
type-synonym 'a *DimCube* = 'a · 'a · 'a ((-)^3 [*999*] *999*)
type-synonym 'a *DimQuart* = 'a · 'a · 'a · 'a ((-)^4 [*999*] *999*)

type-synonym *'a DimInvSquare* = $('a^2)^{-1}$ $((\text{-})^{-2}$ [*999*] *999*)
type-synonym *'a DimInvCube* = $('a^3)^{-1}$ $((\text{-})^{-3}$ [*999*] *999*)
type-synonym *'a DimInvQuart* = $('a^4)^{-1}$ $((\text{-})^{-4}$ [*999*] *999*)

translations (*type*) $'a^{-2}$ <= (*type*) $('a^2)^{-1}$
translations (*type*) $'a^{-3}$ <= (*type*) $('a^3)^{-1}$
translations (*type*) $'a^{-4}$ <= (*type*) $('a^4)^{-1}$

⟨*ML*⟩

Derived Dimension Types

type-synonym *Area* = L^2
type-synonym *Volume* = L^3
type-synonym *Acceleration* = $L{\cdot}T^{-1}$
type-synonym *Frequency* = T^{-1}
type-synonym *Energy* = $L^2{\cdot}M{\cdot}T^{-2}$
type-synonym *Power* = $L^2{\cdot}M{\cdot}T^{-3}$
type-synonym *Force* = $L{\cdot}M{\cdot}T^{-2}$
type-synonym *Pressure* = $L^{-1}{\cdot}M{\cdot}T^{-2}$
type-synonym *Charge* = $I{\cdot}T$
type-synonym *PotentialDifference* = $L^2{\cdot}M{\cdot}T^{-3}{\cdot}I^{-1}$
type-synonym *Capacitance* = $L^{-2}{\cdot}M^{-1}{\cdot}T^4{\cdot}I^2$

3.1.6 ML Functions

We define ML functions for converting a dimension to an integer vector, and vice-versa. These are useful for normalising dimension types.

⟨*ML*⟩

end

3.2 Quantities

theory *ISQ-Quantities*
imports *ISQ-Dimensions*
begin

3.2.1 Quantity Semantic Domain

Here, we give a semantic domain for particular values of physical quantities. A quantity is usually expressed as a number and a measurement unit, and the goal is to support this. First, though, we give a more general semantic domain where a quantity has a magnitude and a dimension.

record (*'a*, *'d*::*enum*) *Quantity* =
mag :: *'a* — Magnitude of the quantity.

dim :: (*int*, *'d*) *dimvec* — Dimension of the quantity – denotes the kind of quantity.

The quantity type is parametric as we permit the magnitude to be represented using any kind of numeric type, such as *int*, *rat*, or *real*, though we usually minimally expect a field.

lemma *Quantity-eq-intro*:
assumes *mag x = mag y dim x = dim y more x = more y*
shows *x = y*
⟨*proof*⟩

We can define several arithmetic operators on quantities. Multiplication takes multiplies both the magnitudes and the dimensions.

instantiation *Quantity-ext* :: (*times*, *enum*, *times*) *times*
begin
definition *times-Quantity-ext* ::
('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme*
where [*si-def*]: *times-Quantity-ext x y* = (| *mag* = *mag x* · *mag y*, *dim* = *dim x* · *dim y*,
... = *more x* · *more y* |)
instance ⟨*proof*⟩
end

lemma *mag-times* [*simp*]: *mag* (*x* · *y*) = *mag x* · *mag y* ⟨*proof*⟩
lemma *dim-times* [*simp*]: *dim* (*x* · *y*) = *dim x* · *dim y* ⟨*proof*⟩
lemma *more-times* [*simp*]: *more* (*x* · *y*) = *more x* · *more y* ⟨*proof*⟩

The zero and one quantities are both dimensionless quantities with magnitude of *0*::*'a* and *1*::*'a*, respectively.

instantiation *Quantity-ext* :: (*zero*, *enum*, *zero*) *zero*
begin
definition *zero-Quantity-ext* = (| *mag* = *0*, *dim* = *1*, ... = *0* |)
instance ⟨*proof*⟩
end

lemma *mag-zero* [*simp*]: *mag 0* = *0* ⟨*proof*⟩
lemma *dim-zero* [*simp*]: *dim 0* = *1* ⟨*proof*⟩
lemma *more-zero* [*simp*]: *more 0* = *0* ⟨*proof*⟩

instantiation *Quantity-ext* :: (*one*, *enum*, *one*) *one*
begin
definition [*si-def*]: *one-Quantity-ext* = (| *mag* = *1*, *dim* = *1*, ... = *1* |)
instance ⟨*proof*⟩
end

lemma *mag-one* [*simp*]: *mag 1* = *1* ⟨*proof*⟩
lemma *dim-one* [*simp*]: *dim 1* = *1* ⟨*proof*⟩

lemma *more-one* [*simp*]: *more 1 = 1* ⟨*proof*⟩

Quantity inversion inverts both the magnitude and the dimension. Similarly, division of one quantity by another, divides both the magnitudes and the dimensions.

instantiation *Quantity-ext* :: (*inverse*, *enum*, *inverse*) *inverse*
begin
definition *inverse-Quantity-ext* :: ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme* **where**
[*si-def*]: *inverse-Quantity-ext x* = (‖ *mag* = *inverse* (*mag x*), *dim* = *inverse* (*dim x*), ... = *inverse* (*more x*) ‖)
definition *divide-Quantity-ext* :: ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme* **where**
[*si-def*]: *divide-Quantity-ext x y* = (‖ *mag* = *mag x* / *mag y*, *dim* = *dim x* / *dim y*, ... = *more x* / *more y* ‖)
instance ⟨*proof*⟩
end

lemma *mag-inverse* [*simp*]: *mag* (*inverse x*) = *inverse* (*mag x*)
⟨*proof*⟩

lemma *dim-inverse* [*simp*]: *dim* (*inverse x*) = *inverse* (*dim x*)
⟨*proof*⟩

lemma *more-inverse* [*simp*]: *more* (*inverse x*) = *inverse* (*more x*)
⟨*proof*⟩

lemma *mag-divide* [*simp*]: *mag* (*x* / *y*) = *mag x* / *mag y*
⟨*proof*⟩

lemma *dim-divide* [*simp*]: *dim* (*x* / *y*) = *dim x* / *dim y*
⟨*proof*⟩

lemma *more-divide* [*simp*]: *more* (*x* / *y*) = *more x* / *more y*
⟨*proof*⟩

As for dimensions, quantities form a commutative monoid and an abelian group.

instance *Quantity-ext* :: (*comm-monoid-mult*, *enum*, *comm-monoid-mult*) *comm-monoid-mult*
⟨*proof*⟩

instance *Quantity-ext* :: (*ab-group-mult*, *enum*, *ab-group-mult*) *ab-group-mult*
⟨*proof*⟩

We can also define a partial order on quantities.

instantiation *Quantity-ext* :: (*ord*, *enum*, *ord*) *ord*
begin
definition *less-eq-Quantity-ext* :: ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ ('*a*, '*b*, '*c*) *Quantity-scheme* ⇒ *bool*

where *less-eq-Quantity-ext x y* = (*mag x* ≤ *mag y* ∧ *dim x* = *dim y* ∧ *more x* ≤ *more y*)
definition *less-Quantity-ext* :: (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ *bool*
where *less-Quantity-ext x y* = (*x* ≤ *y* ∧ ¬ *y* ≤ *x*)

instance ⟨*proof*⟩

end

instance *Quantity-ext* :: (*order*, *enum*, *order*) *order*
⟨*proof*⟩

We can define plus and minus as well, but these are partial operators as they are defined only when the quantities have the same dimension.

instantiation *Quantity-ext* :: (*plus*, *enum*, *plus*) *plus*
begin
definition *plus-Quantity-ext* :: (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme*
where [*si-def*]:
dim x = *dim y* ⟹
plus-Quantity-ext x y = (| *mag* = *mag x* + *mag y*, *dim* = *dim x*, ... = *more x* + *more y* |)
instance ⟨*proof*⟩
end

instantiation *Quantity-ext* :: (*uminus*, *enum*, *uminus*) *uminus*
begin
definition *uminus-Quantity-ext* :: (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme* **where**
[*si-def*]: *uminus-Quantity-ext x* = (| *mag* = − *mag x* , *dim* = *dim x*, ... = − *more x* |)
instance ⟨*proof*⟩
end

instantiation *Quantity-ext* :: (*minus*, *enum*, *minus*) *minus*
begin
definition *minus-Quantity-ext* :: (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme* ⇒ (*'a*, *'b*, *'c*) *Quantity-scheme* **where**
[*si-def*]:
dim x = *dim y* ⟹
minus-Quantity-ext x y = (| *mag* = *mag x* − *mag y*, *dim* = *dim x*, ... = *more x* − *more y* |)
instance ⟨*proof*⟩
end

3.2.2 Measurement Systems

class *unit-system* = *unitary*

lemma *unit-system-intro*: $(UNIV::'s\ set) = \{a\} \Longrightarrow OFCLASS('s, unit\text{-}system\text{-}class)$
⟨*proof*⟩

record $('a, 'd::enum, 's::unit\text{-}system)$ *Measurement-System* = $('a, 'd::enum)$ *Quantity* +
unit-sys :: $'s$ — The system of units being employed

definition *mmore* = *Record.iso-tuple-snd Measurement-System-ext-Tuple-Iso*

lemma *mmore* [*simp*]: *mmore* (| *unit-sys* = x, ... = y |) = y
⟨*proof*⟩

lemma *mmore-ext* [*simp*]: (|*unit-sys* = *unit*, ... = *mmore a*|) = a
⟨*proof*⟩

lemma *Measurement-System-eq-intro*:
assumes *mag* x = *mag* y *dim* x = *dim* y *more* x = *more* y
shows $x = y$
⟨*proof*⟩

instantiation *Measurement-System-ext* :: (*unit-system*, *zero*) *zero*
begin
definition *zero-Measurement-System-ext* :: $('a, 'b)$ *Measurement-System-ext*
where [*si-def*]: *zero-Measurement-System-ext* = (| *unit-sys* = *unit*, ... = *0* |)
instance ⟨*proof*⟩
end

instantiation *Measurement-System-ext* :: (*unit-system*, *one*) *one*
begin
definition *one-Measurement-System-ext* :: $('a, 'b)$ *Measurement-System-ext*
where [*si-def*]: *one-Measurement-System-ext* = (| *unit-sys* = *unit*, ... = *1* |)
instance ⟨*proof*⟩
end

instantiation *Measurement-System-ext* :: (*unit-system*, *times*) *times*
begin
definition *times-Measurement-System-ext* ::
$('a, 'b)$ *Measurement-System-ext* ⇒ $('a, 'b)$ *Measurement-System-ext* ⇒ $('a, 'b)$ *Measurement-System-ext*
where [*si-def*]: *times-Measurement-System-ext* x y = (| *unit-sys* = *unit*, ... = *mmore* x · *mmore* y |)
instance ⟨*proof*⟩
end

instantiation *Measurement-System-ext* :: (*unit-system*, *inverse*) *inverse*
begin
definition *inverse-Measurement-System-ext* :: $('a, 'b)$ *Measurement-System-ext* ⇒ $('a, 'b)$ *Measurement-System-ext* **where**

[*si-def*]: *inverse-Measurement-System-ext x* = (| *unit-sys* = *unit*, ... = *inverse* (*mmore x*) |)
definition *divide-Measurement-System-ext* ::
('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext*
where [*si-def*]: *divide-Measurement-System-ext x y* = (| *unit-sys* = *unit*, ... = *mmore x* / *mmore y* |)
instance ⟨*proof*⟩
end

instance *Measurement-System-ext* :: (*unit-system*, *comm-monoid-mult*) *comm-monoid-mult*
⟨*proof*⟩

instance *Measurement-System-ext* :: (*unit-system*, *ab-group-mult*) *ab-group-mult*
⟨*proof*⟩

instantiation *Measurement-System-ext* :: (*unit-system*, *ord*) *ord*
begin
definition *less-eq-Measurement-System-ext* :: ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* ⇒ *bool*
where *less-eq-Measurement-System-ext x y* = (*mmore x* ≤ *mmore y*)
definition *less-Measurement-System-ext* :: ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* ⇒ *bool*
where *less-Measurement-System-ext x y* = (*x* ≤ *y* ∧ ¬ *y* ≤ *x*)
instance ⟨*proof*⟩

end

instance *Measurement-System-ext* :: (*unit-system*, *order*) *order*
⟨*proof*⟩

instantiation *Measurement-System-ext* :: (*unit-system*, *plus*) *plus*
begin
definition *plus-Measurement-System-ext* ::
('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext*
where [*si-def*]:
plus-Measurement-System-ext x y = (| *unit-sys* = *unit*, ... = *mmore x* + *mmore y* |)
instance ⟨*proof*⟩
end

instantiation *Measurement-System-ext* :: (*unit-system*, *uminus*) *uminus*
begin
definition *uminus-Measurement-System-ext* :: ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* **where**
[*si-def*]: *uminus-Measurement-System-ext x* = (| *unit-sys* = *unit*, ... = − *mmore x* |)
instance ⟨*proof*⟩

end

instantiation *Measurement-System-ext* :: (*unit-system*, *minus*) *minus*
begin
definition *minus-Measurement-System-ext* ::
('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* ⇒ ('*a*, '*b*) *Measurement-System-ext* **where**
[*si-def*]:
minus-Measurement-System-ext x y = (| *unit-sys* = *unit*, ... = *mmore x* − *mmore y* |)
instance ⟨*proof*⟩
end

3.2.3 Dimension Typed Quantities

We can now define the type of quantities with parametrised dimension types.

typedef (**overloaded**) ('*n*, '*d*::*dim-type*, '*s*::*unit-system*) *QuantT* (-[-, -] [*999*,*0*,*0*] *999*)
= {*x* :: ('*n*, *sdim*, '*s*) *Measurement-System*. *dim x* = *QD*('*d*)}
morphisms *fromQ toQ* ⟨*proof*⟩

setup-lifting *type-definition-QuantT*

A dimension typed quantity is parameterised by two types: '*a*, the numeric type for the magntitude, and '*d* for the dimension expression, which is an element of *dim-type*. The type '*n*['*d*, '*s*] is to ('*n*, '*d*, '*s*) *Measurement-System* as dimension types are to *Dimension*. Specifically, an element of '*n*['*d*, '*s*] is a quantity whose dimension is '*d*.
Intuitively, the formula *x* can be read as "*x* is a quantity of '*d*", for example it might be a quantity of length, or a quantity of mass.

Since quantities can have dimension type expressions that are distinct, but denote the same dimension, it is necessary to define the following function for coercion between two dimension expressions. This requires that the underlying dimensions are the same.

definition *coerceQuantT* :: 'd_2 *itself* ⇒ '*a*['d_1::*dim-type*, '*s*::*unit-system*] ⇒ '*a*['d_2::*dim-type*, '*s*] **where**
[*si-def*]: *QD*('d_1) = *QD*('d_2) ⟹ *coerceQuantT t x* = (*toQ* (*fromQ x*))

syntax
-*QCOERCE* :: *type* ⇒ *logic* ⇒ *logic* (*QCOERCE*[-])

translations
QCOERCE['*t*] == *CONST coerceQuantT TYPE*('*t*)

3.2.4 Predicates on Typed Quantities

The standard HOL order ($\leq$) and equality ($=$) have the homogeneous type $'a \Rightarrow 'a \Rightarrow bool$ and so they cannot compare values of different types. Consequently, we define a heterogeneous order and equivalence on typed quantities.

lift-definition *qless-eq* :: $'n$::*order*[$'a$::*dim-type*, $'s$::*unit-system*] $\Rightarrow$ $'n$[$'b$::*dim-type*, $'s$] $\Rightarrow$ *bool* (**infix** $\lesssim_Q$ *50*)
is ($\leq$) ⟨*proof*⟩

lift-definition *qequiv* :: $'n$[$'a$::*dim-type*, $'s$::*unit-system*] $\Rightarrow$ $'n$[$'b$::*dim-type*, $'s$] $\Rightarrow$ *bool* (**infix** $\cong_Q$ *50*)
is ($=$) ⟨*proof*⟩

These are both fundamentally the same as the usual order and equality relations, but they permit potentially different dimension types, $'a$ and $'b$. Two typed quantities are comparable only when the two dimension types have the same semantic dimension.

lemma *qequiv-refl* [*simp*]: $a \cong_Q a$
⟨*proof*⟩

lemma *qequiv-sym*: $a \cong_Q b \Longrightarrow b \cong_Q a$
⟨*proof*⟩

lemma *qequiv-trans*: $[\![a \cong_Q b; b \cong_Q c]\!] \Longrightarrow a \cong_Q c$
⟨*proof*⟩

theorem *qeq-iff-same-dim*:
fixes x y :: $'a$[$'d$::*dim-type*, $'s$::*unit-system*]
shows $x \cong_Q y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *coerceQuant-eq-iff*:
fixes x :: $'a$[$'d_1$::*dim-type*, $'s$::*unit-system*]
assumes $QD('d_1) = QD('d_2$::*dim-type*$)$
shows (*coerceQuantT TYPE*($'d_2$) x) $\cong_Q x$
⟨*proof*⟩

lemma *coerceQuant-eq-iff2*:
fixes x :: $'a$[$'d_1$::*dim-type*, $'s$::*unit-system*]
assumes $QD('d_1) = QD('d_2$::*dim-type*$)$ **and** $y = ($*coerceQuantT TYPE*($'d_2$) $x)$
shows $x \cong_Q y$
⟨*proof*⟩

lemma *updown-eq-iff*:
fixes x :: $'a$[$'d_1$::*dim-type*, $'s$::*unit-system*] **fixes** y :: $'a$[$'d_2$::*dim-type*, $'s$]
assumes $QD('d_1) = QD('d_2$::*dim-type*$)$ **and** $y = (toQ\ (fromQ\ x))$
shows $x \cong_Q y$

⟨*proof*⟩

This is more general that $y = x \Longrightarrow x \cong_Q y$, since x and y may have different type.

lemma *qeq*:
fixes x :: $'a['d_1$::*dim-type*, $'s$::*unit-system*] **fixes** y :: $'a['d_2$::*dim-type*, $'s]$
assumes $x \cong_Q y$
shows $QD('d_1) = QD('d_2)$
⟨*proof*⟩

3.2.5 Operators on Typed Quantities

We define several operators on typed quantities. These variously compose the dimension types as well. Multiplication composes the two dimension types. Inverse constructs and inverted dimension type. Division is defined in terms of multiplication and inverse.

lift-definition
qtimes :: ($'n$::*comm-ring-1*)[$'a$::*dim-type*, $'s$::*unit-system*] ⇒ $'n['b$::*dim-type*, $'s]$ ⇒ $'n['a \cdot 'b, 's]$ (**infixl** · *69*)
is (∗) ⟨*proof*⟩

lift-definition
qinverse :: ($'n$::*field*)[$'a$::*dim-type*, $'s$::*unit-system*] ⇒ $'n['a^{-1}, 's]$ ($(\text{-}^{\mathbf{-1}})$ [*999*] *999*)

is *inverse* ⟨*proof*⟩

abbreviation (*input*)
qdivide :: ($'n$::*field*)[$'a$::*dim-type*, $'s$::*unit-system*] ⇒ $'n['b$::*dim-type*, $'s]$ ⇒ $'n['a/'b, 's]$ (**infixl** *'/ 70*) **where**
qdivide x y ≡ $x \cdot y^{\mathbf{-1}}$

We also provide some helpful notations for expressing heterogeneous powers.

abbreviation *qsq* $((\text{-})^{\mathbf{2}}$ [*999*] *999*) **where** $u^{\mathbf{2}} \equiv u \cdot u$
abbreviation *qcube* $((\text{-})^{\mathbf{3}}$ [*999*] *999*) **where** $u^{\mathbf{3}} \equiv u \cdot u \cdot u$
abbreviation *qquart* $((\text{-})^{\mathbf{4}}$ [*999*] *999*) **where** $u^{\mathbf{4}} \equiv u \cdot u \cdot u \cdot u$

abbreviation *qneq-sq* $((\text{-})^{\mathbf{-2}}$ [*999*] *999*) **where** $u^{\mathbf{-2}} \equiv (u^{\mathbf{2}})^{\mathbf{-1}}$
abbreviation *qneq-cube* $((\text{-})^{\mathbf{-3}}$ [*999*] *999*) **where** $u^{\mathbf{-3}} \equiv (u^{\mathbf{3}})^{\mathbf{-1}}$
abbreviation *qneq-quart* $((\text{-})^{\mathbf{-4}}$ [*999*] *999*) **where** $u^{\mathbf{-4}} \equiv (u^{\mathbf{3}})^{\mathbf{-1}}$

Analogous to the $(*_R)$ operator for vectors, we define the following scalar multiplication that scales an existing quantity by a numeric value. This operator is especially important for the representation of quantity values, which consist of a numeric value and a unit.

lift-definition *scaleQ* :: $'a$ ⇒ $'a$::*comm-ring-1*[$'d$::*dim-type*, $'s$::*unit-system*] ⇒ $'a['d, 's]$ (**infixr** $*_Q$ *63*)
is $\lambda\ r\ x.$ (| *mag* = $r * mag\ x$, *dim* = $QD('d)$, *unit-sys* = *unit* |) ⟨*proof*⟩

Finally, we instantiate the arithmetic types classes where possible. We do not instantiate *times* because this results in a nonsensical homogeneous product on quantities.

instantiation *QuantT* :: (*zero*, *dim-type*, *unit-system*) *zero*
begin
lift-definition *zero-QuantT* :: ('*a*, '*b*, '*c*) *QuantT* **is** ⦇ *mag* = *0*, *dim* = *QD*('*b*), *unit-sys* = *unit* ⦈
⟨*proof*⟩
instance ⟨*proof*⟩
end

instantiation *QuantT* :: (*one*, *dim-type*, *unit-system*) *one*
begin
lift-definition *one-QuantT* :: ('*a*, '*b*, '*c*) *QuantT* **is** ⦇ *mag* = *1*, *dim* = *QD*('*b*), *unit-sys* = *unit* ⦈
⟨*proof*⟩
instance ⟨*proof*⟩
end

The following specialised one element has both magnitude and dimension 1: it is a dimensionless quantity.

abbreviation *qone* :: '*n*::*one*[**1**, '*s*::*unit-system*] (**1**) **where** *qone* ≡ *1*

Unlike for semantic quantities, the plus operator on typed quantities is total, since the type system ensures that the dimensions (and the dimension types) must be the same.

instantiation *QuantT* :: (*plus*, *dim-type*, *unit-system*) *plus*
begin
lift-definition *plus-QuantT* :: '*a*['*b*, '*c*] ⇒ '*a*['*b*, '*c*] ⇒ '*a*['*b*, '*c*]
is λ *x y*. ⦇ *mag* = *mag x* + *mag y*, *dim* = *QD*('*b*), *unit-sys* = *unit* ⦈
⟨*proof*⟩
instance ⟨*proof*⟩
end

We can also show that typed quantities are commutative *additive* monoids. Indeed, addition is a much easier operator to deal with in typed quantities, unlike product.

instance *QuantT* :: (*semigroup-add*,*dim-type*,*unit-system*) *semigroup-add*
⟨*proof*⟩

instance *QuantT* :: (*ab-semigroup-add*,*dim-type*,*unit-system*) *ab-semigroup-add*
⟨*proof*⟩

instance *QuantT* :: (*monoid-add*,*dim-type*,*unit-system*) *monoid-add*
⟨*proof*⟩

instance *QuantT* :: (*comm-monoid-add*,*dim-type*,*unit-system*) *comm-monoid-add*

⟨*proof*⟩

instantiation *QuantT* :: (*uminus*,*dim-type*,*unit-system*) *uminus*
begin
lift-definition *uminus-QuantT* :: *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*]
is λ *x*. (| *mag* = − *mag x*, *dim* = *dim x*, *unit-sys* = *unit* |) ⟨*proof*⟩
instance ⟨*proof*⟩
end

instantiation *QuantT* :: (*minus*,*dim-type*,*unit-system*) *minus*
begin
lift-definition *minus-QuantT* :: *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*]
is λ *x y*. (| *mag* = *mag x* − *mag y*, *dim* = *dim x*, *unit-sys* = *unit* |) ⟨*proof*⟩

instance ⟨*proof*⟩
end

instance *QuantT* :: (*numeral*,*dim-type*,*unit-system*) *numeral* ⟨*proof*⟩

Moreover, types quantities also form an additive group.

instance *QuantT* :: (*ab-group-add*,*dim-type*,*unit-system*) *ab-group-add*
⟨*proof*⟩

Typed quantities helpfully can be both partially and a linearly ordered.

instantiation *QuantT* :: (*order*,*dim-type*,*unit-system*) *order*
begin
lift-definition *less-eq-QuantT* :: *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*] ⇒ *bool* **is** λ *x y*. *mag x* ≤ *mag y* ⟨*proof*⟩
lift-definition *less-QuantT* :: *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*] ⇒ *bool* **is** λ *x y*. *mag x* < *mag y* ⟨*proof*⟩
instance ⟨*proof*⟩
end

instance *QuantT* :: (*linorder*,*dim-type*,*unit-system*) *linorder*
⟨*proof*⟩

instantiation *QuantT* :: (*scaleR*,*dim-type*,*unit-system*) *scaleR*
begin
lift-definition *scaleR-QuantT* :: *real* ⇒ *'a*['*b*,'*c*] ⇒ *'a*['*b*,'*c*]
is λ *n q*. (| *mag* = *n* ∗$_R$ *mag q*, *dim* = *dim q*, *unit-sys* = *unit* |) ⟨*proof*⟩
instance ⟨*proof*⟩
end

instance *QuantT* :: (*real-vector*,*dim-type*,*unit-system*) *real-vector*
⟨*proof*⟩

instantiation *QuantT* :: (*norm*,*dim-type*,*unit-system*) *norm*
begin
lift-definition *norm-QuantT* :: *'a*['*b*,'*c*] ⇒ *real*

is $\lambda\ x.\ norm\ (mag\ x)$ ⟨*proof*⟩
instance ⟨*proof*⟩
end

instantiation *QuantT* :: (*sgn-div-norm*,*dim-type*,*unit-system*) *sgn-div-norm*
begin
definition *sgn-QuantT* :: $'a['b,'c] \Rightarrow 'a['b,'c]$ **where**
sgn-QuantT $x = x\ /_R\ norm\ x$
instance ⟨*proof*⟩
end

instantiation *QuantT* :: (*dist-norm*,*dim-type*,*unit-system*) *dist-norm*
begin
definition *dist-QuantT* :: $'a['b,'c] \Rightarrow 'a['b,'c] \Rightarrow real$ **where**
dist-QuantT $x\ y = norm\ (x - y)$
instance
⟨*proof*⟩
end

instantiation *QuantT* :: ({*uniformity-dist*,*dist-norm*},*dim-type*,*unit-system*) *uniformity-dist*
begin
definition *uniformity-QuantT* :: $('a['b,'c] \times 'a['b,'c])\ filter$ **where**
uniformity-QuantT = $(INF\ e{\in}\{0 <..\}.\ principal\ \{(x,\ y).\ dist\ x\ y < e\})$
instance
⟨*proof*⟩
end

instantiation *QuantT* :: ({*dist-norm*,*open-uniformity*,*uniformity-dist*},*dim-type*,*unit-system*)
open-uniformity
begin

definition *open-QuantT* :: $('a['b,'c])\ set \Rightarrow bool$ **where**
open-QuantT $U = (\forall\, x{\in}U.\ eventually\ (\lambda(x',\ y).\ x' = x \longrightarrow y \in U)\ uniformity)$
instance ⟨*proof*⟩
end

Quantities form a real normed vector space.

instance *QuantT* :: (*real-normed-vector*,*dim-type*,*unit-system*) *real-normed-vector*
⟨*proof*⟩

end

3.3 Proof Support for Quantities

theory *ISQ-Proof*
imports *ISQ-Quantities*
begin

named-theorems *si-transfer*

definition *magQ* :: *'a*['*u*::*dim-type*, '*s*::*unit-system*] ⇒ *'a* ([-]$_Q$) **where**
[*si-def*]: *magQ x* = *mag* (*fromQ x*)

definition *dimQ* :: *'a*['*u*::*dim-type*, '*s*::*unit-system*] ⇒ *Dimension* **where**
[*si-def*]: *dimQ x* = *dim* (*fromQ x*)

lemma *quant-eq-iff-mag-eq* [*si-eq*]:
$x = y \longleftrightarrow [x]_Q = [y]_Q$
⟨*proof*⟩

lemma *quant-eqI* [*si-transfer*]:
$[x]_Q = [y]_Q \Longrightarrow x = y$
⟨*proof*⟩

lemma *quant-equiv-iff* [*si-eq*]:
fixes *x* :: *'a*['u_1::*dim-type*, '*s*::*unit-system*] **and** *y* :: *'a*['u_2::*dim-type*, '*s*::*unit-system*]
shows $x \cong_Q y \longleftrightarrow [x]_Q = [y]_Q \wedge QD('u_1) = QD('u_2)$
⟨*proof*⟩

lemma *quant-equivI* [*si-transfer*]:
fixes *x* :: *'a*['u_1::*dim-type*, '*s*::*unit-system*] **and** *y* :: *'a*['u_2::*dim-type*, '*s*::*unit-system*]
assumes $QD('u_1) = QD('u_2)$ $QD('u_1) = QD('u_2) \Longrightarrow [x]_Q = [y]_Q$
shows $x \cong_Q y$
⟨*proof*⟩

lemma *quant-le-iff-magn-le* [*si-eq*]:
$x \leq y \longleftrightarrow [x]_Q \leq [y]_Q$
⟨*proof*⟩

lemma *quant-leI* [*si-transfer*]:
$[x]_Q \leq [y]_Q \Longrightarrow x \leq y$
⟨*proof*⟩

lemma *quant-less-iff-magn-less* [*si-eq*]:
$x < y \longleftrightarrow [x]_Q < [y]_Q$
⟨*proof*⟩

lemma *quant-lessI* [*si-transfer*]:
$[x]_Q < [y]_Q \Longrightarrow x < y$
⟨*proof*⟩

lemma *magQ-zero* [*si-eq*]: $[0]_Q = 0$
⟨*proof*⟩

lemma *magQ-one* [*si-eq*]: $[1]_Q = 1$
⟨*proof*⟩

lemma *magQ-plus* [*si-eq*]: $[\![x + y]\!]_Q = [\![x]\!]_Q + [\![y]\!]_Q$
⟨*proof*⟩

lemma *magQ-minus* [*si-eq*]: $[\![x - y]\!]_Q = [\![x]\!]_Q - [\![y]\!]_Q$
⟨*proof*⟩

lemma *magQ-uminus* [*si-eq*]: $[\![- x]\!]_Q = - [\![x]\!]_Q$
⟨*proof*⟩

lemma *magQ-scaleQ* [*si-eq*]: $[\![x *_Q y]\!]_Q = x * [\![y]\!]_Q$
⟨*proof*⟩

lemma *magQ-qtimes* [*si-eq*]: $[\![x \cdot y]\!]_Q = [\![x]\!]_Q \cdot [\![y]\!]_Q$
⟨*proof*⟩

lemma *magQ-qinverse* [*si-eq*]: $[\![x^{-1}]\!]_Q = inverse\ [\![x]\!]_Q$
⟨*proof*⟩

lemma *magQ-qdivivide* [*si-eq*]: $[\![(x::('a::field)[-,-])\ /\ y]\!]_Q = [\![x]\!]_Q\ /\ [\![y]\!]_Q$
⟨*proof*⟩

lemma *magQ-numeral* [*si-eq*]: $[\![numeral\ n]\!]_Q = numeral\ n$
⟨*proof*⟩

lemma *magQ-coerce* [*si-eq*]:
fixes $q :: 'a['d_1::dim\text{-}type, 's::unit\text{-}system]$ **and** $t :: 'd_2::dim\text{-}type\ itself$
assumes $QD('d_1) = QD('d_2)$
shows $[\![coerceQuantT\ t\ q]\!]_Q = [\![q]\!]_Q$
⟨*proof*⟩

lemma *dimQ* [*simp*]: $dimQ(x :: 'a['d::dim\text{-}type, 's::unit\text{-}system]) = QD('d)$
⟨*proof*⟩

The following tactic breaks an SI conjecture down to numeric and unit properties

method *si-simp* **uses** *add* =
(*rule-tac si-transfer*; *simp add*: *add si-eq field-simps*)

The next tactic additionally compiles the semantics of the underlying units

method *si-calc* **uses** *add* =
(*si-simp add*: *add*; *simp add*: *si-def add*)

lemma $QD(N \cdot \Theta \cdot N) = QD(\Theta \cdot N^2)$ ⟨*proof*⟩

end

3.4 Algebraic Laws

theory *ISQ-Algebra*
imports *ISQ-Proof*
begin

3.4.1 Quantity Scale

lemma *scaleQ-add-right*: $a *_Q x + y = (a *_Q x) + (a *_Q y)$
⟨*proof*⟩

lemma *scaleQ-add-left*: $a + b *_Q x = (a *_Q x) + (b *_Q x)$
⟨*proof*⟩

lemma *scaleQ-scaleQ* [*simp*]: $a *_Q b *_Q x = a \cdot b *_Q x$
⟨*proof*⟩

lemma *scaleQ-one* [*simp*]: $1 *_Q x = x$
⟨*proof*⟩

lemma *scaleQ-zero* [*simp*]: $0 *_Q x = 0$
⟨*proof*⟩

lemma *scaleQ-inv*: $-a *_Q x = a *_Q -x$
⟨*proof*⟩

lemma *scaleQ-as-qprod*: $a *_Q x \cong_Q (a *_Q \mathbf{1}) \cdot x$
⟨*proof*⟩

lemma *mult-scaleQ-left* [*simp*]: $(a *_Q x) \cdot y = a *_Q x \cdot y$
⟨*proof*⟩

lemma *mult-scaleQ-right* [*simp*]: $x \cdot (a *_Q y) = a *_Q x \cdot y$
⟨*proof*⟩

3.4.2 Field Laws

lemma *qtimes-commute*: $x \cdot y \cong_Q y \cdot x$
⟨*proof*⟩

lemma *qtimes-assoc*: $(x \cdot y) \cdot z \cong_Q x \cdot (y \cdot z)$
⟨*proof*⟩

lemma *qtimes-left-unit*: $\mathbf{1} \cdot x \cong_Q x$
⟨*proof*⟩

lemma *qtimes-right-unit*: $x \cdot \mathbf{1} \cong_Q x$
⟨*proof*⟩

The following weak congruences will allow for replacing equivalences in con-

texts built from product and inverse.

lemma *qtimes-weak-cong-left*:
assumes $x \cong_Q y$
shows $x \cdot z \cong_Q y \cdot z$
⟨*proof*⟩

lemma *qtimes-weak-cong-right*:
assumes $x \cong_Q y$
shows $z \cdot x \cong_Q z \cdot y$
⟨*proof*⟩

lemma *qinverse-weak-cong*:
assumes $x \cong_Q y$
shows $x^{-1} \cong_Q y^{-1}$
⟨*proof*⟩

lemma *scaleQ-cong*:
assumes $y \cong_Q z$
shows $x *_Q y \cong_Q x *_Q z$
⟨*proof*⟩

lemma *qinverse-qinverse*: $x^{-1-1} \cong_Q x$
⟨*proof*⟩

lemma *qinverse-nonzero-iff-nonzero*: $x^{-1} = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *qinverse-qtimes*: $(x \cdot y)^{-1} \cong_Q x^{-1} \cdot y^{-1}$
⟨*proof*⟩

lemma *qinverse-qdivide*: $(x \,/\, y)^{-1} \cong_Q y \,/\, x$
⟨*proof*⟩

lemma *qtimes-cancel*: $x \neq 0 \Longrightarrow x \,/\, x \cong_Q \mathbf{1}$
⟨*proof*⟩

end

3.5 Units

theory *ISQ-Units*
imports *ISQ-Proof*
begin

Parallel to the base quantities, there are base units. In the implementation of the SI unit system, we fix these to be precisely those quantities that have a base dimension and a magnitude of $1::'a$. Consequently, a base unit corresponds to a unit in the algebraic sense.

lift-definition *is-base-unit* :: *'a*::*one*[*'d*::*dim-type*, *'s*::*unit-system*] ⇒ *bool*
is λ *x*. *mag x* = *1* ∧ *is-BaseDim* (*dim x*) ⟨*proof*⟩

definition *mk-base-unit* :: *'u itself* ⇒ *'s itself* ⇒ (*'a*::*one*)[*'u*::*basedim-type*, *'s*::*unit-system*]

where *mk-base-unit t s* = *1*

syntax *-mk-base-unit* :: *type* ⇒ *type* ⇒ *logic* (*BUNIT'*(-, -'))
translations *BUNIT*('*a*, '*s*) == *CONST mk-base-unit TYPE*('*a*) *TYPE*('*s*)

lemma *mk-base-unit*: *is-base-unit* (*mk-base-unit a s*)
⟨*proof*⟩

lemma *magQ-mk* [*si-eq*]: ⟦*BUNIT*('*u*::*basedim-type*, '*s*::*unit-system*)⟧$_Q$ = *1*
⟨*proof*⟩

end

3.6 Conversion Between Unit Systems

theory *ISQ-Conversion*
imports *ISQ-Units*
begin

3.6.1 Conversion Schemas

A conversion schema provides factors for each of the base units for converting between two systems of units. We currently only support conversion between systems that can meaningfully characterise a subset of the seven SI dimensions.

record *ConvSchema* =
cLengthF :: *rat*
cMassF :: *rat*
cTimeF :: *rat*
cCurrentF :: *rat*
cTemperatureF :: *rat*
cAmountF :: *rat*
cIntensityF :: *rat*

We require that all the factors of greater than zero.

typedef (*'s*$_1$::*unit-system*, *'s*$_2$::*unit-system*) *Conversion* ((-/ ⇒$_U$ -) [*1*, *0*] *0*) =
{*c* :: *ConvSchema*. *cLengthF c* > *0* ∧ *cMassF c* > *0* ∧ *cTimeF c* > *0* ∧ *cCurrentF c* > *0*
∧ *cTemperatureF c* > *0* ∧ *cAmountF c* > *0* ∧ *cIntensityF c* > *0*}
⟨*proof*⟩

setup-lifting *type-definition-Conversion*

lift-definition *LengthF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cLengthF* ⟨*proof*⟩
lift-definition *MassF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cMassF* ⟨*proof*⟩
lift-definition *TimeF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cTimeF* ⟨*proof*⟩
lift-definition *CurrentF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cCurrentF* ⟨*proof*⟩
lift-definition *TemperatureF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cTemperatureF* ⟨*proof*⟩
lift-definition *AmountF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cAmountF* ⟨*proof*⟩
lift-definition *IntensityF* :: $('s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ *rat* **is** *cIntensityF* ⟨*proof*⟩

lemma *Conversion-props* [*simp*]: *LengthF c > 0 MassF c > 0 TimeF c > 0 CurrentF c > 0*
TemperatureF c > 0 AmountF c > 0 IntensityF c > 0
⟨*proof*⟩

3.6.2 Conversion Algebra

lift-definition *convid* :: $'s$::*unit-system* $\Rightarrow_U$ $'s$ (id_C)
is
(| *cLengthF = 1*
, *cMassF = 1*
, *cTimeF = 1*
, *cCurrentF = 1*
, *cTemperatureF = 1*
, *cAmountF = 1*
, *cIntensityF = 1* |) ⟨*proof*⟩

lift-definition *convcomp* ::
($'s_2 \Rightarrow_U$ $'s_3$::*unit-system*) $\Rightarrow$ ($'s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ ($'s_1 \Rightarrow_U$ $'s_3$) (**infixl** $\circ_C$ *55*) **is**
λ c_1 c_2. (| *cLengthF = cLengthF* c_1 $*$ *cLengthF* c_2, *cMassF = cMassF* c_1 $*$ *cMassF* c_2
, *cTimeF = cTimeF* c_1 $*$ *cTimeF* c_2, *cCurrentF = cCurrentF* c_1 $*$ *cCurrentF* c_2
, *cTemperatureF = cTemperatureF* c_1 $*$ *cTemperatureF* c_2
, *cAmountF = cAmountF* c_1 $*$ *cAmountF* c_2, *cIntensityF = cIntensityF* c_1 $*$ *cIntensityF* c_2 |)
⟨*proof*⟩

lift-definition *convinv* :: ($'s_1$::*unit-system* $\Rightarrow_U$ $'s_2$::*unit-system*) $\Rightarrow$ ($'s_2 \Rightarrow_U$ $'s_1$) (inv_C) **is**
λ *c*. (| *cLengthF = inverse* (*cLengthF c*), *cMassF = inverse* (*cMassF c*), *cTimeF = inverse* (*cTimeF c*)
, *cCurrentF = inverse* (*cCurrentF c*), *cTemperatureF = inverse* (*cTemperatureF*

c)
, *cAmountF* = *inverse* (*cAmountF c*), *cIntensityF* = *inverse* (*cIntensityF c*)
⟧ ⟨*proof*⟩

lemma *convinv-inverse* [*simp*]: *convinv* (*convinv c*) = *c*
⟨*proof*⟩

lemma *convcomp-inv* [*simp*]: *c* ∘$_C$ *inv*$_C$ *c* = *id*$_C$
⟨*proof*⟩

lemma *inv-convcomp* [*simp*]: *inv*$_C$ *c* ∘$_C$ *c* = *id*$_C$
⟨*proof*⟩

lemma *Conversion-invs* [*simp*]: *LengthF* (*inv*$_C$ *x*) = *inverse* (*LengthF x*) *MassF* (*inv*$_C$ *x*) = *inverse* (*MassF x*)
TimeF (*inv*$_C$ *x*) = *inverse* (*TimeF x*) *CurrentF* (*inv*$_C$ *x*) = *inverse* (*CurrentF x*)
TemperatureF (*inv*$_C$ *x*) = *inverse* (*TemperatureF x*) *AmountF* (*inv*$_C$ *x*) = *inverse* (*AmountF x*)
IntensityF (*inv*$_C$ *x*) = *inverse* (*IntensityF x*)
⟨*proof*⟩

lemma *Conversion-comps* [*simp*]: *LengthF* (*c*$_1$ ∘$_C$ *c*$_2$) = *LengthF c*$_1$ ∗ *LengthF c*$_2$
MassF (*c*$_1$ ∘$_C$ *c*$_2$) = *MassF c*$_1$ ∗ *MassF c*$_2$
TimeF (*c*$_1$ ∘$_C$ *c*$_2$) = *TimeF c*$_1$ ∗ *TimeF c*$_2$
CurrentF (*c*$_1$ ∘$_C$ *c*$_2$) = *CurrentF c*$_1$ ∗ *CurrentF c*$_2$
TemperatureF (*c*$_1$ ∘$_C$ *c*$_2$) = *TemperatureF c*$_1$ ∗ *TemperatureF c*$_2$
AmountF (*c*$_1$ ∘$_C$ *c*$_2$) = *AmountF c*$_1$ ∗ *AmountF c*$_2$
IntensityF (*c*$_1$ ∘$_C$ *c*$_2$) = *IntensityF c*$_1$ ∗ *IntensityF c*$_2$
⟨*proof*⟩

3.6.3 Conversion Functions

definition *dconvfactor* :: ('*s*$_1$::*unit-system* ⇒$_U$ '*s*$_2$::*unit-system*) ⇒ *Dimension* ⇒ *rat* **where**
dconvfactor c d =
LengthF c ^$_Z$ *dim-nth d Length*
∗ *MassF c* ^$_Z$ *dim-nth d Mass*
∗ *TimeF c* ^$_Z$ *dim-nth d Time*
∗ *CurrentF c* ^$_Z$ *dim-nth d Current*
∗ *TemperatureF c* ^$_Z$ *dim-nth d Temperature*
∗ *AmountF c* ^$_Z$ *dim-nth d Amount*
∗ *IntensityF c* ^$_Z$ *dim-nth d Intensity*

lemma *dconvfactor-pos* [*simp*]: *dconvfactor c d* > *0*
⟨*proof*⟩

lemma *dconvfactor-nz* [*simp*]: *dconvfactor c d* ≠ *0*
⟨*proof*⟩

lemma *dconvfactor-convinv*: *dconvfactor* (*convinv c*) *d* = *inverse* (*dconvfactor c d*)
⟨*proof*⟩

lemma *dconvfactor-id* [*simp*]: *dconvfactor* id_C *d* = *1*
⟨*proof*⟩

lemma *dconvfactor-compose*:
dconvfactor (c_1 $\circ_C$ c_2) *d* = *dconvfactor* c_1 *d* * *dconvfactor* c_2 *d*
⟨*proof*⟩

lemma *dconvfactor-inverse*:
dconvfactor c (*inverse d*) = *inverse* (*dconvfactor c d*)
⟨*proof*⟩

lemma *dconvfactor-times*:
dconvfactor c (*x* · *y*) = *dconvfactor c x* · *dconvfactor c y*
⟨*proof*⟩

lift-definition *qconv* :: ('s_1, 's_2) *Conversion* ⇒ ('*a*::*field-char-0*)['*d*::*dim-type*, 's_1::*unit-system*] ⇒ '*a*['*d*, 's_2::*unit-system*]
is λ *c q*. (| *mag* = *of-rat* (*dconvfactor c* (*dim q*)) * *mag q*, *dim* = *dim q*, *unit-sys* = *unit* |) ⟨*proof*⟩

lemma *magQ-qconv*: ⟦*qconv c q*⟧$_Q$ = *of-rat* (*dconvfactor c* (*dimQ q*)) * ⟦*q*⟧$_Q$
⟨*proof*⟩

lemma *qconv-id* [*simp*]: *qconv* id_C *x* = *x*
⟨*proof*⟩

lemma *qconv-comp*: *qconv* (c_1 $\circ_C$ c_2) *x* = *qconv* c_1 (*qconv* c_2 *x*)
⟨*proof*⟩

lemma *qconv-convinv* [*simp*]: *qconv* (*convinv c*) (*qconv c x*) = *x*
⟨*proof*⟩

lemma *qconv-scaleQ* [*simp*]: *qconv c* (*d* *$_Q$ *x*) = *d* *$_Q$ *qconv c x*
⟨*proof*⟩

lemma *qconv-plus* [*simp*]: *qconv c* (*x* + *y*) = *qconv c x* + *qconv c y*
⟨*proof*⟩

lemma *qconv-minus* [*simp*]: *qconv c* (*x* − *y*) = *qconv c x* − *qconv c y*
⟨*proof*⟩

lemma *qconv-qmult* [*simp*]: *qconv c* (*x* ⋅ *y*) = *qconv c x* ⋅ *qconv c y*
⟨*proof*⟩

lemma *qconv-qinverse* [*simp*]: *qconv c* $(x^{-\mathbf{1}}) = (qconv\ c\ x)^{-\mathbf{1}}$
⟨*proof*⟩

lemma *qconv-Length* [*simp*]: *qconv c BUNIT*(*L*, -) = *LengthF c* $*_Q$ *BUNIT*(*L*, -)
⟨*proof*⟩

lemma *qconv-Mass* [*simp*]: *qconv c BUNIT*(*M*, -) = *MassF c* $*_Q$ *BUNIT*(*M*, -)
⟨*proof*⟩

lemma *qconv-Time* [*simp*]: *qconv c BUNIT*(*T*, -) = *TimeF c* $*_Q$ *BUNIT*(*T*, -)
⟨*proof*⟩

lemma *qconv-Current* [*simp*]: *qconv c BUNIT*(*I*, -) = *CurrentF c* $*_Q$ *BUNIT*(*I*, -)
⟨*proof*⟩

lemma *qconv-Temperature* [*simp*]: *qconv c BUNIT*(Θ, -) = *TemperatureF c* $*_Q$ *BUNIT*(Θ, -)
⟨*proof*⟩

lemma *qconv-Amount* [*simp*]: *qconv c BUNIT*(*N*, -) = *AmountF c* $*_Q$ *BUNIT*(*N*, -)
⟨*proof*⟩

lemma *qconv-Intensity* [*simp*]: *qconv c BUNIT*(*J*, -) = *IntensityF c* $*_Q$ *BUNIT*(*J*, -)
⟨*proof*⟩

end

3.7 Meta-Theory for ISQ

theory *ISQ*
imports *ISQ-Dimensions ISQ-Quantities ISQ-Proof ISQ-Algebra ISQ-Units ISQ-Conversion*
begin end

Chapter 4

International System of Units

4.1 SI Units Semantics

theory *SI-Units*
imports *ISQ*
begin

An SI unit is simply a particular kind of quantity with an SI tag.

typedef $SI = UNIV :: unit\ set\ \langle proof \rangle$

instance $SI :: unit\text{-}system$
$\langle proof \rangle$

abbreviation $SI \equiv unit :: SI$

type-synonym $('n, 'd)\ SIUnitT = ('n, 'd, SI)\ QuantT$ (-[-] [*999*,*0*] *999*)

We now define the seven base units. Effectively, these definitions axiomatise given names for the $1::'a$ elements of the base quantities.

abbreviation $metre \equiv BUNIT(L, SI)$
abbreviation $kilogram \equiv BUNIT(M, SI)$
abbreviation $ampere \equiv BUNIT(I, SI)$
abbreviation $kelvin \equiv BUNIT(\Theta, SI)$
abbreviation $mole \equiv BUNIT(N, SI)$
abbreviation $candela \equiv BUNIT(J, SI)$

The second is commonly used in unit systems other than SI. Consequently, we define it polymorphically, and require that the system type instantiate a type class to use it.

class $time\text{-}second = unit\text{-}system$

instance $SI :: time\text{-}second\ \langle proof \rangle$

abbreviation *second* $\equiv$ *BUNIT*(*T*, *'a*::*time-second*)

Note that as a consequence of our construction, the term *metre* is a SI Unit constant of SI-type *'a*[*L*, *SI*], so a unit of dimension *L* with the magnitude of type *'a*. A magnitude instantiation can be, e.g., an integer, a rational number, a real number, or a vector of type $real^3$. Note than when considering vectors, dimensions refer to the *norm* of the vector, not to its components.

lemma *BaseUnits*:
is-base-unit metre is-base-unit second is-base-unit kilogram is-base-unit ampere
is-base-unit kelvin is-base-unit mole is-base-unit candela
⟨*proof*⟩

The effect of the above encoding is that we can use the SI base units as synonyms for their corresponding dimensions at the type level.

type-synonym *'a metre* = *'a*[*Length*, *SI*]
type-synonym *'a second* = *'a*[*Time*, *SI*]
type-synonym *'a kilogram* = *'a*[*Mass*, *SI*]
type-synonym *'a ampere* = *'a*[*Current*, *SI*]
type-synonym *'a kelvin* = *'a*[*Temperature*, *SI*]
type-synonym *'a mole* = *'a*[*Amount*, *SI*]
type-synonym *'a candela* = *'a*[*Intensity*, *SI*]

We can therefore construct a quantity such as *5*, which unambiguously identifies that the unit of 5 is metres using the type system. This works because each base unit it the one element.

4.1.1 Example Unit Equations

lemma (*metre* $\cdot$ $second^{-\mathbf{1}}$) $\cdot$ *second* $\cong_Q$ *metre*
⟨*proof*⟩

4.1.2 Metrification

class *metrifiable* = *unit-system* +
fixes *convschema* :: *'a itself* $\Rightarrow$ (*'a*, *SI*) *Conversion* ($schema_C$)

instantiation *SI* :: *metrifiable*
begin
lift-definition *convschema-SI* :: *SI itself* $\Rightarrow$ (*SI*, *SI*) *Conversion*
is λ *s*.
⦇ *cLengthF* = *1*
, *cMassF* = *1*
, *cTimeF* = *1*
, *cCurrentF* = *1*
, *cTemperatureF* = *1*
, *cAmountF* = *1*
, *cIntensityF* = *1* ⦈ ⟨*proof*⟩

instance ⟨*proof*⟩
end

abbreviation *metrify* :: ('*a*::*field-char-0*)['*d*::*dim-type*, '*s*::*metrifiable*] ⇒ '*a*['*d*::*dim-type*, *SI*] **where**
metrify ≡ *qconv* (*convschema* (*TYPE*('*s*)))

Conversion via SI units

abbreviation *qmconv* ::
's_1 *itself* ⇒ 's_2 *itself*
⇒ ('*a*::*field-char-0*)['*d*::*dim-type*, 's_1::*metrifiable*]
⇒ '*a*['*d*::*dim-type*, 's_2::*metrifiable*] **where**
qmconv s_1 s_2 *x* ≡ *qconv* (*inv*$_C$ (*schema*$_C$ s_2) ∘$_C$ *schema*$_C$ s_1) *x*

syntax
-*qmconv* :: *type* ⇒ *type* ⇒ *logic* (*QMC*'(- → -'))

translations
QMC('s_1 → 's_2) == *CONST qmconv TYPE*('s_1) *TYPE*('s_2)

lemma *qmconv-self*: *QMC*('*s*::*metrifiable* → '*s*) = *id*
⟨*proof*⟩

end

4.2 Centimetre-Gram-Second System

theory *CGS*
imports *SI-Units*
begin

4.2.1 Preliminaries

typedef *CGS* = *UNIV* :: *unit set* ⟨*proof*⟩
instance *CGS* :: *unit-system*
⟨*proof*⟩
instance *CGS* :: *time-second* ⟨*proof*⟩
abbreviation *CGS* ≡ *unit* :: *CGS*

4.2.2 Base Units

abbreviation *centimetre* ≡ *BUNIT*(*L*, *CGS*)
abbreviation *gram* ≡ *BUNIT*(*M*, *CGS*)

4.2.3 Conversion to SI

instantiation *CGS* :: *metrifiable*
begin

lift-definition *convschema-CGS* :: *CGS itself* ⇒ (*CGS*, *SI*) *Conversion* **is**
λ *x*. (| *cLengthF* = *0.01*, *cMassF* = *0.001*, *cTimeF* = *1*
, *cCurrentF* = *1*, *cTemperatureF* = *1*, *cAmountF* = *1*, *cIntensityF* = *1* |)
⟨*proof*⟩

instance ⟨*proof*⟩
end

lemma *CGS-SI-simps* [*simp*]: *LengthF* (*convschema* (*a*::*CGS itself*)) = *0.01 MassF* (*convschema a*) = *0.001*
TimeF (*convschema a*) = *1 CurrentF* (*convschema a*) = *1 TemperatureF* (*convschema a*) = *1*
⟨*proof*⟩

4.2.4 Conversion Examples

lemma *metrify* ((*100*::*rat*) $*_Q$ *centimetre*) = *1* $*_Q$ *metre*
⟨*proof*⟩

end

4.3 Physical Constants

theory *SI-Constants*
imports *SI-Units*
begin

4.3.1 Core Derived Units

abbreviation (*input*) *hertz* ≡ $second^{-1}$

abbreviation *radian* ≡ *metre* · $metre^{-1}$

abbreviation *steradian* ≡ $metre^2$ · $metre^{-2}$

abbreviation *joule* ≡ *kilogram* · $metre^2$ · $second^{-2}$

type-synonym *'a joule* = *'a*[$M \cdot L^2 \cdot T^{-2}$, *SI*]

abbreviation *watt* ≡ *kilogram* · $metre^2$ · $second^{-3}$

type-synonym *'a watt* = *'a*[$M \cdot L^2 \cdot T^{-3}$, *SI*]

abbreviation *coulomb* ≡ *ampere* · *second*

type-synonym *'a coulomb* = *'a*[$I \cdot T$, *SI*]

abbreviation *lumen* ≡ *candela* · *steradian*

type-synonym *'a lumen* = *'a*[$J \cdot (L^2 \cdot L^{-2})$, *SI*]

4.3.2 Constants

The most general types we support must form a field into which the natural numbers can be injected.

default-sort *field-char-0*

Hyperfine transition frequency of frequency of Cs

abbreviation *caesium-frequency*:: *'a*[T^{-1},*SI*] (Δv_{Cs}) **where**
caesium-frequency ≡ *9192631770* $*_Q$ *hertz*

Speed of light in vacuum

abbreviation *speed-of-light* :: *'a*[$L \cdot T^{-1}$,*SI*] (**c**) **where**
speed-of-light ≡ *299792458* $*_Q$ (*metre*•*second*$^{-\mathbf{1}}$)

Planck constant

abbreviation *Planck* :: *'a*[$M \cdot L^2 \cdot T^{-2} \cdot T$,*SI*] (**h**) **where**
Planck ≡ (*6.62607015* · *1*/(*10^34*)) $*_Q$ (*joule*•*second*)

Elementary charge

abbreviation *elementary-charge* :: *'a*[$I \cdot T$,*SI*] (**e**) **where**
elementary-charge ≡ (*1.602176634* · *1*/(*10^19*)) $*_Q$ *coulomb*

The Boltzmann constant

abbreviation *Boltzmann* :: *'a*[$M \cdot L^2 \cdot T^{-2} \cdot \Theta^{-1}$,*SI*] (**k**) **where**
Boltzmann ≡ (*1.380649*·*1*/(*10^23*)) $*_Q$ (*joule* / *kelvin*)

The Avogadro number

abbreviation *Avogadro* :: *'a*[N^{-1},*SI*] (N_A) **where**
Avogadro ≡ *6.02214076*·(*10^23*) $*_Q$ (*mole*$^{-\mathbf{1}}$)

abbreviation *max-luminous-frequency* :: *'a*[T^{-1},*SI*] **where**
max-luminous-frequency ≡ (*540*·*10^12*) $*_Q$ *hertz*

The luminous efficacy of monochromatic radiation of frequency *max-luminous-frequency*.

abbreviation *luminous-efficacy* :: *'a*[$J \cdot (L^2 \cdot L^{-2}) \cdot (M \cdot L^2 \cdot T^{-3})^{-1}$,*SI*] ($K_{cd}$)
where
luminous-efficacy ≡ *683* $*_Q$ (*lumen*/*watt*)

4.3.3 Checking Foundational Equations of the SI System

theorem *second-definition*:
1 $*_Q$ *second* $\cong_Q$ (*9192631770* $*_Q$ **1**) / Δv_{Cs}
⟨*proof*⟩

theorem *metre-definition*:

1 $*_Q$ *metre* $\cong_Q$ (**c** / (*299792458* $*_Q$ **1**))·*second*
1 $*_Q$ *metre* $\cong_Q$ (*9192631770* / *299792458*) $*_Q$ (**c** / Δv_{Cs})
⟨*proof*⟩

theorem *kilogram-definition*:
((*1* $*_Q$ *kilogram*)::*'a kilogram*) $\cong_Q$ (**h** / (*6.62607015* · *1*/(*10^34*) $*_Q$ **1**))·*metre*$^{-\mathbf{2}}$·*second*

⟨*proof*⟩

abbreviation *approx-ice-point* ≡ *273.15* $*_Q$ *kelvin*

default-sort *type*

end

4.4 SI Prefixes

theory *SI-Prefix*
imports *SI-Constants*
begin

4.4.1 Definitions

Prefixes are simply numbers that can be composed with units using the scalar multiplication operator ($*_Q$).

default-sort *ring-char-0*

definition *deca* :: *'a* **where** [*si-eq*]: *deca* = *10^1*

definition *hecto* :: *'a* **where** [*si-eq*]: *hecto* = *10^2*

definition *kilo* :: *'a* **where** [*si-eq*]: *kilo* = *10^3*

definition *mega* :: *'a* **where** [*si-eq*]: *mega* = *10^6*

definition *giga* :: *'a* **where** [*si-eq*]: *giga* = *10^9*

definition *tera* :: *'a* **where** [*si-eq*]: *tera* = *10^12*

definition *peta* :: *'a* **where** [*si-eq*]: *peta* = *10^15*

definition *exa* :: *'a* **where** [*si-eq*]: *exa* = *10^18*

definition *zetta* :: *'a* **where** [*si-eq*]: *zetta* = *10^21*

definition *yotta* :: *'a* **where** [*si-eq*]: *yotta* = *10^24*

default-sort *field-char-0*

definition *deci* :: *'a* **where** [*si-eq*]: *deci = 1/10^1*

definition *centi* :: *'a* **where** [*si-eq*]: *centi = 1/10^2*

definition *milli* :: *'a* **where** [*si-eq*]: *milli = 1/10^3*

definition *micro* :: *'a* **where** [*si-eq*]: *micro = 1/10^6*

definition *nano* :: *'a* **where** [*si-eq*]: *nano = 1/10^9*

definition *pico* :: *'a* **where** [*si-eq*]: *pico = 1/10^12*

definition *femto* :: *'a* **where** [*si-eq*]: *femto = 1/10^15*

definition *atto* :: *'a* **where** [*si-eq*]: *atto = 1/10^18*

definition *zepto* :: *'a* **where** [*si-eq*]: *zepto = 1/10^21*

definition *yocto* :: *'a* **where** [*si-eq*]: *yocto = 1/10^24*

4.4.2 Examples

lemma $2.3 *_Q (centi *_Q metre)^{\mathbf{3}} = 2.3 \cdot 1/10\hat{}6 *_Q metre^{\mathbf{3}}$
⟨*proof*⟩

lemma $1 *_Q (centi *_Q metre)^{-\mathbf{1}} = 100 *_Q metre^{-\mathbf{1}}$
⟨*proof*⟩

4.4.3 Binary Prefixes

Although not in general applicable to physical quantities, we include these prefixes for completeness.

default-sort *ring-char-0*

definition *kibi* :: *'a* **where** [*si-eq*]: *kibi = 2^10*

definition *mebi* :: *'a* **where** [*si-eq*]: *mebi = 2^20*

definition *gibi* :: *'a* **where** [*si-eq*]: *gibi = 2^30*

definition *tebi* :: *'a* **where** [*si-eq*]: *tebi = 2^40*

definition *pebi* :: *'a* **where** [*si-eq*]: *pebi = 2^50*

definition *exbi* :: *'a* **where** [*si-eq*]: *exbi = 2^60*

definition *zebi* :: *'a* **where** [*si-eq*]: *zebi = 2^70*

definition *yobi* :: *'a* **where** [*si-eq*]: *yobi = 2^80*

default-sort *type*

end

4.5 Derived SI-Units

theory *SI-Derived*
imports *SI-Prefix*
begin

4.5.1 Definitions

abbreviation $newton \equiv kilogram \cdot metre \cdot second^{-2}$

type-synonym $'a\ newton = 'a[M \cdot L \cdot T^{-2}, SI]$

abbreviation $pascal \equiv kilogram \cdot metre^{-1} \cdot second^{-2}$

type-synonym $'a\ pascal = 'a[M \cdot L^{-1} \cdot T^{-2}, SI]$

abbreviation $volt \equiv kilogram \cdot metre^{2} \cdot second^{-3} \cdot ampere^{-1}$

type-synonym $'a\ volt = 'a[M \cdot L^2 \cdot T^{-3} \cdot I^{-1}, SI]$

abbreviation $farad \equiv kilogram^{-1} \cdot metre^{-2} \cdot second^{4} \cdot ampere^{2}$

type-synonym $'a\ farad = 'a[M^{-1} \cdot L^{-2} \cdot T^4 \cdot I^2, SI]$

abbreviation $ohm \equiv kilogram \cdot metre^{2} \cdot second^{-3} \cdot ampere^{-2}$

type-synonym $'a\ ohm = 'a[M \cdot L^2 \cdot T^{-3} \cdot I^{-2}, SI]$

abbreviation $siemens \equiv kilogram^{-1} \cdot metre^{-2} \cdot second^{3} \cdot ampere^{2}$

abbreviation $weber \equiv kilogram \cdot metre^{2} \cdot second^{-2} \cdot ampere^{-1}$

abbreviation $tesla \equiv kilogram \cdot second^{-2} \cdot ampere^{-1}$

abbreviation $henry \equiv kilogram \cdot metre^{2} \cdot second^{-2} \cdot ampere^{-2}$

abbreviation $lux \equiv candela \cdot steradian \cdot metre^{-2}$

abbreviation (*input*) $becquerel \equiv second^{-1}$

abbreviation $gray \equiv metre^{2} \cdot second^{-2}$

abbreviation *sievert* ≡ *metre*$^{\mathbf{2}}$ · *second*$^{-\mathbf{2}}$

abbreviation *katal* ≡ *mole* · *second*$^{-\mathbf{1}}$

definition *degrees-celcius* :: *'a*::*field-char-0* ⇒ *'a*[Θ] (*-XXXC* [*999*] *999*)
where [*si-eq*]: *degrees-celcius x* = (*x* ∗$_Q$ *kelvin*) + *approx-ice-point*

definition [*si-eq*]: *gram* = *milli* ∗$_Q$ *kilogram*

4.5.2 Equivalences

lemma *joule-alt-def*: *joule* ≅$_Q$ *newton* · *metre*
⟨*proof*⟩

lemma *watt-alt-def*: *watt* ≅$_Q$ *joule* / *second*
⟨*proof*⟩

lemma *volt-alt-def*: *volt* = *watt* / *ampere*
⟨*proof*⟩

lemma *farad-alt-def*: *farad* ≅$_Q$ *coulomb* / *volt*
⟨*proof*⟩

lemma *ohm-alt-def*: *ohm* ≅$_Q$ *volt* / *ampere*
⟨*proof*⟩

lemma *siemens-alt-def*: *siemens* ≅$_Q$ *ampere* / *volt*
⟨*proof*⟩

lemma *weber-alt-def*: *weber* ≅$_Q$ *volt* · *second*
⟨*proof*⟩

lemma *tesla-alt-def*: *tesla* ≅$_Q$ *weber* / *metre*$^{\mathbf{2}}$
⟨*proof*⟩

lemma *henry-alt-def*: *henry* ≅$_Q$ *weber* / *ampere*
⟨*proof*⟩

lemma *lux-alt-def*: *lux* = *lumen* / *metre*$^{\mathbf{2}}$
⟨*proof*⟩

lemma *gray-alt-def*: *gray* ≅$_Q$ *joule* / *kilogram*
⟨*proof*⟩

lemma *sievert-alt-def*: *sievert* ≅$_Q$ *joule* / *kilogram*
⟨*proof*⟩

4.5.3 Properties

lemma *kilogram*: *kilo* ∗$_Q$ *gram* = *kilogram*

⟨*proof*⟩

lemma *celcius-to-kelvin*: $TXXXC = (T *_Q kelvin) + (273.15 *_Q kelvin)$
⟨*proof*⟩

end

4.6 Non-SI Units Accepted for SI use

theory *SI-Accepted*
imports *SI-Derived*
begin

definition [*si-def*, *si-eq*]: $minute = 60 *_Q second$

definition [*si-def*, *si-eq*]: $hour = 60 *_Q minute$

definition [*si-def*, *si-eq*]: $day = 24 *_Q hour$

definition [*si-def*, *si-eq*]: $astronomical\text{-}unit = 149597870700 *_Q metre$

definition $degree :: 'a::real\text{-}field[L/L]$ **where**
[*si-def*, *si-eq*]: $degree = (2 \cdot (of\text{-}real\ pi)\ /\ 180) *_Q radian$

abbreviation $degrees\ (\text{-}XXX\ [999]\ 999)$ **where** $nXXX \equiv n *_Q degree$

definition [*si-def*, *si-eq*]: $litre = 1/1000 *_Q metre^3$

definition [*si-def*, *si-eq*]: $tonne = 10\hat{}3 *_Q kilogram$

definition [*si-def*, *si-eq*]: $dalton = 1.66053906660 * (1\ /\ 10\hat{}27) *_Q kilogram$

4.6.1 Example Unit Equations

lemma $1 *_Q hour = 3600 *_Q second$
⟨*proof*⟩

lemma $watt \cdot hour \cong_Q 3600 *_Q joule$ ⟨*proof*⟩

lemma $25 *_Q metre\ /\ second = 90 *_Q (kilo *_Q metre)\ /\ hour$
⟨*proof*⟩

end

4.7 Imperial Units via SI Units

theory *SI-Imperial*
imports *SI-Accepted*

begin

4.7.1 Units of Length

default-sort *field-char-0*

The units of length are defined in terms of the international yard, as standardised in 1959.

definition *yard* :: $'a[L]$ **where**
[*si-eq*]: *yard* = *0.9144* $*_Q$ *metre*

definition *foot* :: $'a[L]$ **where**
[*si-eq*]: *foot* = *1/3* $*_Q$ *yard*

lemma *foot-alt-def*: *foot* = *0.3048* $*_Q$ *metre*
⟨*proof*⟩

definition *inch* :: $'a[L]$ **where**
[*si-eq*]: *inch* = (*1* / *36*) $*_Q$ *yard*

lemma *inch-alt-def*: *inch* = *25.4* $*_Q$ *milli* $*_Q$ *metre*
⟨*proof*⟩

definition *mile* :: $'a[L]$ **where**
[*si-eq*]: *mile* = *1760* $*_Q$ *yard*

lemma *mile-alt-def*: *mile* = *1609.344* $*_Q$ *metre*
⟨*proof*⟩

definition *nautical-mile* :: $'a[L]$ **where**
[*si-eq*]: *nautical-mile* = *1852* $*_Q$ *metre*

4.7.2 Units of Mass

The units of mass are defined in terms of the international yard, as standardised in 1959.

definition *pound* :: $'a[M]$ **where**
[*si-eq*]: *pound* = *0.45359237* $*_Q$ *kilogram*

definition *ounce* :: $'a[M]$ **where**
[*si-eq*]: *ounce* = *1/16* $*_Q$ *pound*

definition *stone* :: $'a[M]$ **where**
[*si-eq*]: *stone* = *14* $*_Q$ *pound*

4.7.3 Other Units

definition *knot* :: $'a[L \cdot T^{-1}]$ **where**

[*si-eq*]: *knot* = *1* $*_Q$ (*nautical-mile* / *hour*)

definition *pint* :: *'a*[*Volume*] **where**
[*si-eq*]: *pint* = *0.56826125* $*_Q$ *litre*

definition *gallon* :: *'a*[*Volume*] **where**
[*si-eq*]: *gallon* = *8* $*_Q$ *pint*

definition *degrees-farenheit* :: *'a* ⇒ *'a*[Θ] (*-XXXF* [*999*] *999*)
where [*si-eq*]: *degrees-farenheit x* = (*x* + *459.67*)·*5*/*9* $*_Q$ *kelvin*

default-sort *type*

4.7.4 Unit Equations

lemma *miles-to-feet*: *mile* = *5280* $*_Q$ *foot*
⟨*proof*⟩

lemma *mph-to-kmh*: *1* $*_Q$ (*mile* / *hour*) = *1.609344* $*_Q$ ((*kilo* $*_Q$ *metre*) / *hour*)
⟨*proof*⟩

lemma *farenheit-to-celcius*: *TXXXF* = ((*T* − *32*) · *5*/*9*)*XXXC*
⟨*proof*⟩

end

4.8 Meta-Theory for SI Units

theory *SI*
imports *SI-Units SI-Constants SI-Prefix SI-Derived SI-Accepted SI-Imperial*
begin end

4.9 Astronomical Constants

theory *SI-Astronomical*
imports *SI HOL−Decision-Procs.Approximation*
begin

We create a number of astronomical constants and prove relationships between some of them. For this, we use the approximation method that can compute bounds on transcendental functions.

definition *julian-year* :: *'a*::*field*[*T*] **where**
[*si-eq*]: *julian-year* = *365.25* $*_Q$ *day*

definition *light-year* :: *'a*::*field-char-0*[*L*] **where**
light-year = *QCOERCE*[*L*] (**c** · *julian-year*)

We need to apply a coercion in the definition of light year to convert the

dimension type from $L \cdot T^{-1} \cdot T$ to L. The correctness of this coercion is confirmed by the following equivalence theorem.

lemma *light-year*: $light\text{-}year \cong_Q \mathbf{c} \cdot julian\text{-}year$
⟨*proof*⟩

lemma *light-year-eq* [*si-eq*]: $[\![light\text{-}year]\!]_Q = [\![\mathbf{c} \cdot julian\text{-}year]\!]_Q$
⟨*proof*⟩

HOL can characterise *pi* exactly and so we also give an exact value for the parsec.

definition *parsec* :: *real*[*L*] **where**
[*si-eq*]: $parsec = 648000 \;/\; pi *_Q astronomical\text{-}unit$

We calculate some conservative bounds on the parsec: it is somewhere between 3.26 and 3.27 light-years.

lemma *parsec-lb*: $3.26 *_Q light\text{-}year < parsec$
⟨*proof*⟩

lemma *parsec-ub*: $parsec < 3.27 *_Q light\text{-}year$
⟨*proof*⟩

The full beauty of the approach is perhaps revealed here, with the type of a classical three-dimensional gravitation field:

type-synonym $gravitation\text{-}field = real^3[L] \Rightarrow (real^3[L \cdot T^{-2}])$

end

4.10 Parsing and Pretty Printing of SI Units

theory *SI-Pretty*
imports *SI*
begin

4.10.1 Syntactic SI Units

The following syntactic representation can apply at both the type and value level.

nonterminal *si*

syntax
-si-metre :: *si* (*m*)
-si-kilogram :: *si* (*kg*)
-si-second :: *si* (*s*)
-si-ampere :: *si* (*A*)
-si-kelvin :: *si* (*K*)
-si-mole :: *si* (*mol*)

-si-candela :: *si* (*cd*)

-si-square :: *si* ⇒ *si* ((-)^2 [*999*] *999*)
-si-cube :: *si* ⇒ *si* ((-)^3 [*999*] *999*)
-si-quart :: *si* ⇒ *si* ((-)^4 [*999*] *999*)

-si-inverse :: *si* ⇒ *si* ((-$^{-1}$) [*999*] *999*)
-si-invsquare :: *si* ⇒ *si* ((-)$^{-2}$ [*999*] *999*)
-si-invcube :: *si* ⇒ *si* ((-)$^{-3}$ [*999*] *999*)
-si-invquart :: *si* ⇒ *si* ((-)$^{-4}$ [*999*] *999*)

-si-times :: *si* ⇒ *si* ⇒ *si* (**infixl** · *70*)

4.10.2 Type Notation

Pretty notation for SI units at the type level.

no-type-notation *SIUnitT* (-[-] [*999*,*0*] *999*)

syntax
-si-unit :: *type* ⇒ *si* ⇒ *type* (-[-] [*999*,*0*] *999*)
-si-print :: *type* ⇒ *si* (*SIPRINT*$'$(-$'$))

translations
(*type*) $'a$[*SIPRINT*($'d$)] == (*type*) $'a$[$'d$, *SI*]
(*si*) *SIPRINT*($'d$)^2 == (*si*) *SIPRINT*($'d^2$)
(*si*) *SIPRINT*($'d$)^3 == (*si*) *SIPRINT*($'d^3$)
(*si*) *SIPRINT*($'d$)^4 == (*si*) *SIPRINT*($'d^4$)
(*si*) *SIPRINT*($'d$)$^{-1}$ == (*si*) *SIPRINT*($'d^{-1}$)
(*si*) *SIPRINT*($'d$)$^{-2}$ == (*si*) *SIPRINT*($'d^{-2}$)
(*si*) *SIPRINT*($'d$)$^{-3}$ == (*si*) *SIPRINT*($'d^{-3}$)
(*si*) *SIPRINT*($'d$)$^{-4}$ == (*si*) *SIPRINT*($'d^{-4}$)
(*si*) *SIPRINT*($'d_1$) · *SIPRINT*($'d_2$) == (*si*) *SIPRINT*($'d_1$ · $'d_2$)
(*si*) *m* == (*si*) *SIPRINT*(*L*)
(*si*) *kg* == (*si*) *SIPRINT*(*M*)
(*si*) *s* == (*si*) *SIPRINT*(*T*)
(*si*) *A* == (*si*) *SIPRINT*(*I*)
(*si*) *K* == (*si*) *SIPRINT*(Θ)
(*si*) *mol* == (*si*) *SIPRINT*(*N*)
(*si*) *cd* == (*si*) *SIPRINT*(*J*)

-si-invsquare x <= *-si-inverse* (*-si-square x*)
-si-invcube x <= *-si-inverse* (*-si-cube x*)
-si-invquart x <= *-si-inverse* (*-si-quart x*)

-si-invsquare x <= *-si-square* (*-si-inverse x*)
-si-invcube x <= *-si-cube* (*-si-inverse x*)
-si-invquart x <= *-si-quart* (*-si-inverse x*)

typ *real*[$m \cdot s^{-2}$]

typ $real[m{\cdot}s^{-2}{\cdot}A^2]$
term $5 *_Q joule$

4.10.3 Value Notations

Pretty notation for SI units at the type level. Currently, it is not possible to support prefixes, as this would require a more sophisticated cartouche parser.

definition $SIQ\ n\ u = n *_Q u$

syntax
$\text{-}si\text{-}term$ :: $si \Rightarrow logic\ (SI'(\text{-}'))$
$\text{-}siq\text{-}term$ :: $logic \Rightarrow si \Rightarrow logic\ (SI[\text{-}, \text{-}])$
$\text{-}siq\text{-}print$ :: $logic \Rightarrow si$

translations
$\text{-}siq\text{-}term\ n\ u => CONST\ SIQ\ n\ (\text{-}si\text{-}term\ u)$
$\text{-}siq\text{-}term\ n\ (\text{-}siq\text{-}print\ u) <= CONST\ SIQ\ n\ u$
$\text{-}si\text{-}term\ (\text{-}si\text{-}times\ x\ y) == (\text{-}si\text{-}term\ x) \cdot (\text{-}si\text{-}term\ y)$
$\text{-}si\text{-}term\ (\text{-}si\text{-}inverse\ x) == (\text{-}si\text{-}term\ x)^{-\mathbf{1}}$
$\text{-}si\text{-}term\ (\text{-}si\text{-}square\ x) == (\text{-}si\text{-}term\ x)^{\mathbf{2}}$
$\text{-}si\text{-}term\ (\text{-}si\text{-}cube\ x) == (\text{-}si\text{-}term\ x)^{\mathbf{2}}$
$SI(m) => CONST\ metre$
$SI(kg) => CONST\ kilogram$
$SI(s) => CONST\ second$
$SI(A) => CONST\ ampere$
$SI(K) => CONST\ kelvin$
$SI(mol) => CONST\ mole$
$SI(cd) => CONST\ candela$

$\text{-}si\text{-}inverse\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{-\mathbf{1}})$
$\text{-}si\text{-}invsquare\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{-\mathbf{2}})$
$\text{-}si\text{-}invcube\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{-\mathbf{3}})$
$\text{-}si\text{-}invquart\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{-\mathbf{4}})$

$\text{-}si\text{-}square\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{\mathbf{2}})$
$\text{-}si\text{-}cube\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{\mathbf{3}})$
$\text{-}si\text{-}quart\ (\text{-}siq\text{-}print\ x) <= \text{-}siq\text{-}print\ (x^{\mathbf{4}})$
$\text{-}si\text{-}times\ (\text{-}siq\text{-}print\ x)\ (\text{-}siq\text{-}print\ y) <= \text{-}siq\text{-}print\ (x \cdot y)$

$\text{-}si\text{-}metre <= \text{-}siq\text{-}print\ (CONST\ metre)$
$\text{-}si\text{-}kilogram <= \text{-}siq\text{-}print\ (CONST\ kilogram)$
$\text{-}si\text{-}second <= \text{-}siq\text{-}print\ (CONST\ second)$
$\text{-}si\text{-}ampere <= \text{-}siq\text{-}print\ (CONST\ ampere)$
$\text{-}si\text{-}kelvin <= \text{-}siq\text{-}print\ (CONST\ kelvin)$
$\text{-}si\text{-}mole <= \text{-}siq\text{-}print\ (CONST\ mole)$
$\text{-}si\text{-}candela <= \text{-}siq\text{-}print\ (CONST\ candela)$

term $SI[5,\ m^2]$

term $SI[22,\ m{\cdot}s^{-1}]$

end

4.11 British Imperial System (1824/1897)

theory *BIS*
imports *ISQ SI-Units CGS*
begin

The values in the British Imperial System (BIS) are derived from the UK Weights and Measures Act 1824.

4.11.1 Preliminaries

typedef $BIS = UNIV :: unit\ set\ \langle proof\rangle$
instance $BIS :: unit\text{-}system$
$\langle proof\rangle$
instance $BIS :: time\text{-}second\ \langle proof\rangle$
abbreviation $BIS \equiv unit :: BIS$

4.11.2 Base Units

abbreviation $yard \equiv BUNIT(L,\ BIS)$
abbreviation $pound \equiv BUNIT(M,\ BIS)$
abbreviation $rankine \equiv BUNIT(\Theta,\ BIS)$

We chose Rankine rather than Farenheit as this is more compatible with the SI system and avoids the need for having an offset in conversion functions.

4.11.3 Derived Units

definition [*si-eq*]: $foot = 1/3 *_Q yard$

definition [*si-eq*]: $inch = 1/12 *_Q foot$

definition [*si-eq*]: $furlong = 220 *_Q yard$

definition [*si-eq*]: $mile = 1760 *_Q yard$

definition [*si-eq*]: $acre = 4840 *_Q yard^{\mathbf{3}}$

definition [*si-eq*]: $ounce = 1/12 *_Q pound$

definition [*si-eq*]: $gallon = 277.421 *_Q inch^{\mathbf{3}}$

definition [*si-eq*]: $quart = 1/4 *_Q gallon$

definition [*si-eq*]: *pint* = *1/8* $*_Q$ *gallon*

definition [*si-eq*]: *peck* = *2* $*_Q$ *gallon*

definition [*si-eq*]: *bushel* = *8* $*_Q$ *gallon*

definition [*si-eq*]: *minute* = *60* $*_Q$ *second*

definition [*si-eq*]: *hour* = *60* $*_Q$ *minute*

4.11.4 Conversion to SI

instantiation *BIS* :: *metrifiable*
begin

lift-definition *convschema-BIS* :: *BIS itself* ⇒ (*BIS*, *SI*) *Conversion* **is**
λ *x*. ⦇ *cLengthF* = *0.9143993*, *cMassF* = *0.453592338*, *cTimeF* = *1*
, *cCurrentF* = *1*, *cTemperatureF* = *5/9*, *cAmountF* = *1*, *cIntensityF* = *1* ⦈
⟨*proof*⟩

instance ⟨*proof*⟩
end

lemma *BIS-SI-simps* [*simp*]: *LengthF* (*convschema* (*a*::*BIS itself*)) = *0.9143993*
MassF (*convschema a*) = *0.453592338*
TimeF (*convschema a*) = *1*
CurrentF (*convschema a*) = *1*
TemperatureF (*convschema a*) = *5/9*
⟨*proof*⟩

4.11.5 Conversion Examples

lemma *metrify* (*foot* :: *rat*[*L*, *BIS*]) = *0.9143993 / 3* $*_Q$ *metre*
⟨*proof*⟩

lemma *metrify* ((*70*::*rat*) $*_Q$ *mile / hour*) = (*704087461 / 22500000*) $*_Q$ (*metre / second*)
⟨*proof*⟩

lemma *QMC*(*CGS* → *BIS*) ((*1*::*rat*) $*_Q$ *centimetre*) = *100000 / 9143993* $*_Q$ *yard*
⟨*proof*⟩

end

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