The Perfect Number Theorem

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Abstract

This document presents the formal proof of the Perfect Number Theorem. The result can also be found as number 70 on the list of "top 100 mathematical theorems" [Wie]. This document was produced as result of a B.Sc. Thesis under supervision of Jaap Top and Wim H. Hesselink (University of Groningen) in 2009.

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1 Basics needed

theory *PerfectBasics*
imports *Main HOL−Computational-Algebra.Primes HOL−Algebra.Exponent*
begin

lemma *exp-is-max-div*:
  assumes *m0*: $m \neq 0$ **and** *p*: *prime p*
  shows ~ *p dvd* (*m div* (*p*^(*multiplicity p m*)))
⟨*proof*⟩

lemma *coprime-multiplicity*:
  assumes *prime* (*p*::*nat*) **and** $m > 0$
  shows *coprime p* (*m div* (*p* ^ *multiplicity p m*))
⟨*proof*⟩

theorem *simplify-sum-of-powers*: $(x - 1::nat) * (\sum i=0 .. n . x\hat{}i) = x\hat{}(n + 1) - 1$ (**is** *?l* = *?r*)
⟨*proof*⟩

end

2 Sum of divisors function

theory *Sigma*
imports *PerfectBasics HOL−Library.Infinite-Set*
begin

definition *divisors* :: *nat* ⇒ *nat set* **where**
  divisors m ≡ {*n* . *n dvd m*}

abbreviation *sigma* :: *nat* ⇒ *nat* **where**
  sigma m ≡ ∑ (*divisors*(*m*))

lemma *divisors-eq-dvd*[*iff*]: (*a* ∈ *divisors*(*n*)) ⟷ (*a dvd n*)
⟨*proof*⟩

lemma *finite-divisors* [*simp*]:
assumes *n>0* **shows** *finite* (*divisors n*)
⟨*proof*⟩

lemma *divs-of-zero-UNIV*[*simp*]: *divisors*(*0*) = *UNIV*
⟨*proof*⟩

lemma *sigma0*[*simp*]: *sigma*(*0*) = *0*
⟨*proof*⟩

lemma *sigma1*[*simp*]: *sigma*(*Suc 0*) = *1*
⟨*proof*⟩

lemma *prime-divisors*: *prime p* ⟷ *divisors p* = {*1*,*p*} ∧ *p>1*
⟨*proof*⟩

lemma *prime-imp-sigma*: *prime* (*p*::*nat*) ⟹ *sigma*(*p*) = *p*+*1*
⟨*proof*⟩

lemma *sigma-third-divisor*:
assumes *1* < *a a* < *n a dvd n*
shows *1*+*a*+*n* ≤ *sigma*(*n*)
⟨*proof*⟩

proposition *prime-iff-sigma*: *prime n* ⟷ *sigma*(*n*) = *Suc n*
⟨*proof*⟩

lemma *dvd-prime-power-iff*:
fixes *p*::*nat*
assumes *prime*: *prime p*
shows {*d*. *d dvd p*^*n*} = (λ*k*. *p*^*k*) ‘ {*0*..*n*}
⟨*proof*⟩

lemma *rewrite-sum-of-powers*:

assumes p: $(p::nat)>1$
shows $\sum$ $((\hat{}) \; p \; `\; \{0..n\}) = (\sum \; i = 0 \; .. \; n \; . \; p\hat{}i)$ (**is** $?l = ?r$)
⟨*proof*⟩

lemma *sum-of-powers-int*: $(x - 1::int) * (\sum i=0..n \; . \; x\hat{}i) = x \hat{} Suc \; n - 1$
⟨*proof*⟩

lemma *sum-of-powers-nat*: $(x - 1::nat) * (\sum i=0..n \; . \; x\hat{}i) = x \hat{} Suc \; n - 1$
(**is** $?l = ?r$)
⟨*proof*⟩

theorem *sigma-primepower*:
assumes *prime p*
shows $(p - 1) * sigma(p\hat{}e) = p\hat{}(e+1) - 1$
⟨*proof*⟩

proposition *sigma-prime-power-two*: $sigma(2\hat{}n) = 2\hat{}(n+1) - 1$
⟨*proof*⟩

lemma *prodsums-eq-sumprods*:
fixes $p :: nat$ **and** $m :: nat$
assumes *coprime p m*
shows $\sum ((\lambda k. \; p\hat{}k) \; `\; \{0..n\}) * sigma \; m = \sum \{p\hat{}k * b \mid k \; b. \; k \leq n \wedge b \; dvd \; m\}$
(**is** $?lhs = ?rhs$)
⟨*proof*⟩

lemma *div-decomp-comp*:
fixes $a::nat$
shows $coprime \; m \; n \Longrightarrow a \; dvd \; m*n \longleftrightarrow (\exists \, b \; c. \; a = b * c \wedge b \; dvd \; m \wedge c \; dvd \; n)$
⟨*proof*⟩

theorem *sigma-semimultiplicative*:
assumes *p*: *prime p* **and** *cop*: *coprime p m*
shows $sigma \; (p\hat{}n) * sigma \; m = sigma \; (p\hat{}n * m)$ (**is** $?lhs = ?rhs$)
⟨*proof*⟩

end

3 Perfect Number Theorem

theory *Perfect*
imports *Sigma*
begin

definition *perfect* :: *nat* => *bool* **where**
$perfect \; m \equiv m>0 \wedge 2*m = sigma \; m$

theorem *perfect-number-theorem*:

assumes *even*: *even m* **and** *perfect*: *perfect m*
shows $\exists\ n\ .\ m = 2\hat{}n*(2\hat{}(n+1) - 1) \wedge prime\ ((2::nat)\hat{}(n+1) - 1)$
⟨*proof*⟩

theorem *Euclid-book9-prop36*:
assumes *p*: $prime\ (2\hat{}(n+1) - (1::nat))$
shows $perfect\ (2\ \hat{}\ n * (2\ \hat{}\ (n + 1) - 1))$
⟨*proof*⟩

end

References

[Wie] Freek Wiedijk. Formalizing 100 theorems. http://www.cs.ru.nl/~freek/100/.