

Paraconsistency

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16 August 2018

Abstract

Paraconsistency is about handling inconsistency in a coherent way. In classical and intuitionistic logic everything follows from an inconsistent theory. A paraconsistent logic avoids the explosion. Quite a few applications in computer science and engineering are discussed in the Intelligent Systems Reference Library Volume 110: Towards Paraconsistent Engineering (Springer 2016). We formalize a paraconsistent many-valued logic that we motivated and described in a special issue on logical approaches to paraconsistency (Journal of Applied Non-Classical Logics 2005). We limit ourselves to the propositional fragment of the higher-order logic. The logic is based on so-called key equalities and has a countably infinite number of truth values. We prove theorems in the logic using the definition of validity. We verify truth tables and also counterexamples for non-theorems. We prove meta-theorems about the logic and finally we investigate a case study.

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Preface

The present formalization in Isabelle essentially follows our extended abstract [1]. The Stanford Encyclopedia of Philosophy has a comprehensive overview of logical approaches to paraconsistency [2]. We have elsewhere explained the rationale for our paraconsistent many-valued logic and considered applications in multi-agent systems and natural language semantics [3, 4, 5, 6].

It is a revised and extended version of our formalization <https://github.com/logic-tools/mvl> that accompany our chapter in a book on partiality published by Cambridge Scholars Press. The GitHub link provides more information. We are grateful to the editors — Henning Christiansen, M. Dolores Jiménez López, Roussanka Loukanova and Larry Moss — for the opportunity to contribute to the book.

On Paraconsistency

Paraconsistency concerns inference systems that do not explode given a contradiction.

The Internet Encyclopedia of Philosophy has a survey article on paraconsistent logic.

The following Isabelle theory formalizes a specific paraconsistent many-valued logic.

```
theory Paraconsistency imports Main begin
```

The details about our logic are in our article in a special issue on logical approaches to paraconsistency in the Journal of Applied Non-Classical Logics (Volume 15, Number 1, 2005).

Syntax and Semantics

Syntax of Propositional Logic

Only the primed operators return indeterminate truth values.

```
type_synonym id = string
```

```
datatype fm = Pro id | Truth | Neg' fm | Con' fm fm | Eql fm fm | Eql' fm fm
```

```
abbreviation Falsity :: fm where Falsity  $\equiv$  Neg' Truth
```

```
abbreviation Dis' :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm where Dis' p q  $\equiv$  Neg' (Con' (Neg' p) (Neg' q))
```

```
abbreviation Imp :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm where Imp p q  $\equiv$  Eql p (Con' p q)
```

```
abbreviation Imp' :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm where Imp' p q  $\equiv$  Eql' p (Con' p q)
```

```
abbreviation Box :: fm  $\Rightarrow$  fm where Box p  $\equiv$  Eql p Truth
```

```
abbreviation Neg :: fm  $\Rightarrow$  fm where Neg p  $\equiv$  Box (Neg' p)
```

```
abbreviation Con :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm where Con p q  $\equiv$  Box (Con' p q)
```

```
abbreviation Dis :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm where Dis p q  $\equiv$  Box (Dis' p q)
```

```
abbreviation Cla :: fm  $\Rightarrow$  fm where Cla p  $\equiv$  Dis (Box p) (Eql p Falsity)
```

```
abbreviation Nab :: fm  $\Rightarrow$  fm where Nab p  $\equiv$  Neg (Cla p)
```

Semantics of Propositional Logic

There is a countably infinite number of indeterminate truth values.

```
datatype tv = Det bool | Indet nat
```

```
abbreviation (input) eval_neg :: tv  $\Rightarrow$  tv
```

```
where
```

```
  eval_neg x  $\equiv$ 
```

```
  (
```

```
    case x of
```

```
      Det False  $\Rightarrow$  Det True |
```

```
      Det True  $\Rightarrow$  Det False |
```

```
      Indet n  $\Rightarrow$  Indet n
```

```

)

fun eval :: (id  $\Rightarrow$  tv)  $\Rightarrow$  fm  $\Rightarrow$  tv
where
  eval i (Pro s) = i s |
  eval i Truth = Det True |
  eval i (Neg' p) = eval_neg (eval i p) |
  eval i (Con' p q) =
    (
      if eval i p = eval i q then eval i p else
      if eval i p = Det True then eval i q else
      if eval i q = Det True then eval i p else Det False
    ) |
  eval i (Eq1 p q) =
    (
      if eval i p = eval i q then Det True else Det False
    ) |
  eval i (Eq1' p q) =
    (
      if eval i p = eval i q then Det True else
      (
        case (eval i p, eval i q) of
          (Det True, _)  $\Rightarrow$  eval i q |
          (_, Det True)  $\Rightarrow$  eval i p |
          (Det False, _)  $\Rightarrow$  eval_neg (eval i q) |
          (_, Det False)  $\Rightarrow$  eval_neg (eval i p) |
          _  $\Rightarrow$  Det False
        )
      )
    )

lemma eval_equality_simplify: eval i (Eq1 p q) = Det (eval i p = eval i q)
  by simp

theorem eval_equality:
  eval i (Eq1' p q) =
    (
      if eval i p = eval i q then Det True else
      if eval i p = Det True then eval i q else
      if eval i q = Det True then eval i p else
      if eval i p = Det False then eval i (Neg' q) else
      if eval i q = Det False then eval i (Neg' p) else
      Det False
    )
  by (cases eval i p; cases eval i q) simp_all

theorem eval_negation:
  eval i (Neg' p) =
    (
      if eval i p = Det False then Det True else
      if eval i p = Det True then Det False else
      eval i p
    )
  by (cases eval i p) simp_all

corollary eval i (Cla p) = eval i (Box (Dis' p (Neg' p)))
  using eval_negation
  by simp

lemma double_negation: eval i p = eval i (Neg' (Neg' p))
  using eval_negation

```

by simp

Validity and Consistency

Validity gives the set of theorems and the logic has at least a theorem and a non-theorem.

definition valid :: fm \Rightarrow bool

where

valid p $\equiv \forall i. \text{eval } i \text{ p} = \text{Det True}$

proposition valid Truth and \neg valid Falsity

unfolding valid_def

by simp_all

Truth Tables

String Functions

The following functions support arbitrary unary and binary truth tables.

definition tv_pair_row :: tv list \Rightarrow tv \Rightarrow (tv * tv) list

where

tv_pair_row tvs tv $\equiv \text{map } (\lambda x. (tv, x)) \text{ tvs}$

definition tv_pair_table :: tv list \Rightarrow (tv * tv) list list

where

tv_pair_table tvs $\equiv \text{map } (\text{tv_pair_row tvs}) \text{ tvs}$

definition map_row :: (tv \Rightarrow tv \Rightarrow tv) \Rightarrow (tv * tv) list \Rightarrow tv list

where

map_row f tvsvs $\equiv \text{map } (\lambda(x, y). f \ x \ y) \text{ tvsvs}$

definition map_table :: (tv \Rightarrow tv \Rightarrow tv) \Rightarrow (tv * tv) list list \Rightarrow tv list list

where

map_table f tvsvss $\equiv \text{map } (\text{map_row } f) \text{ tvsvss}$

definition unary_truth_table :: fm \Rightarrow tv list \Rightarrow tv list

where

unary_truth_table p tvs \equiv
map ($\lambda x. \text{eval } ((\lambda s. \text{undefined})(''p'' := x)) \text{ p}$) tvs

definition binary_truth_table :: fm \Rightarrow tv list \Rightarrow tv list list

where

binary_truth_table p tvs \equiv
map_table ($\lambda x \ y. \text{eval } ((\lambda s. \text{undefined})(''p'' := x, ''q'' := y)) \text{ p}$) (tv_pair_table tvs)

definition digit_of_nat :: nat \Rightarrow char

where

digit_of_nat n \equiv
(if n = 1 then (CHR ''1'') else if n = 2 then (CHR ''2'') else if n = 3 then (CHR ''3'') else
if n = 4 then (CHR ''4'') else if n = 5 then (CHR ''5'') else if n = 6 then (CHR ''6'') else
if n = 7 then (CHR ''7'') else if n = 8 then (CHR ''8'') else if n = 9 then (CHR ''9'') else
(CHR ''0''))

fun string_of_nat :: nat \Rightarrow string

where

string_of_nat n =
(if n < 10 then [digit_of_nat n] else string_of_nat (n div 10) @ [digit_of_nat (n mod 10)])

```

fun string_tv :: tv  $\Rightarrow$  string
where
  string_tv (Det True) = ''*' |
  string_tv (Det False) = ''o' |
  string_tv (Indet n) = string_of_nat n

definition appends :: string list  $\Rightarrow$  string
where
  appends strs  $\equiv$  foldr append strs []

definition appends_nl :: string list  $\Rightarrow$  string
where
  appends_nl strs  $\equiv$  '' $\leftarrow$ '' @ foldr ( $\lambda$ s s'. s @ '' $\leftarrow$ '' @ s') (butlast strs) (last strs) @ '' $\leftarrow$ ''

definition string_table :: tv list list  $\Rightarrow$  string list list
where
  string_table tvss  $\equiv$  map (map string_tv) tvss

definition string_table_string :: string list list  $\Rightarrow$  string
where
  string_table_string strss  $\equiv$  appends_nl (map appends strss)

definition unary :: fm  $\Rightarrow$  tv list  $\Rightarrow$  string
where
  unary p tvs  $\equiv$  appends_nl (map string_tv (unary_truth_table p tvs))

definition binary :: fm  $\Rightarrow$  tv list  $\Rightarrow$  string
where
  binary p tvs  $\equiv$  string_table_string (string_table (binary_truth_table p tvs))

```

Main Truth Tables

The omitted Cla (for Classic) is discussed later; Nab (for Nabla) is simply the negation of it.

proposition

```

unary (Box (Pro ''p'')) [Det True, Det False, Indet 1] = ''
*
o
o
'',
  by code_simp

```

proposition

```

binary (Con' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*o12
oooo
1o1o
2oo2
'',
  by code_simp

```

proposition

```

binary (Dis' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
****
*o12
*11*
*2*2
'',
  by code_simp

```

proposition

```

unary (Neg' (Pro ''p'')) [Det True, Det False, Indet 1] = ''
o
*
1
'',
by code_simp

```

proposition

```

binary (Eq1' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*o12
o*12
11*o
22o*
'',
by code_simp

```

proposition

```

binary (Imp' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*o12
****
*1*1
*22*
'',
by code_simp

```

proposition

```

unary (Neg (Pro ''p'')) [Det True, Det False, Indet 1] = ''
o
*
o
'',
by code_simp

```

proposition

```

binary (Eq1 (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*ooo
o*oo
oo*o
ooo*
'',
by code_simp

```

proposition

```

binary (Imp (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*ooo
****
*o*o
*oo*
'',
by code_simp

```

proposition

```

unary (Nab (Pro ''p'')) [Det True, Det False, Indet 1] = ''
o
o
*
'',
by code_simp

```

proposition

```
binary (Con (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
*ooo
oooo
oooo
oooo
'',
by code_simp
```

proposition

```
binary (Dis (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
****
*ooo
*oo*
*o*o
'',
by code_simp
```

Basic Theorems

Selected Theorems and Non-Theorems

Many of the following theorems and non-theorems use assumptions and meta-variables.

```
proposition valid (Cla (Box p)) and ¬ valid (Nab (Box p))
unfolding valid_def
by simp_all
```

```
proposition valid (Cla (Cla p)) and ¬ valid (Nab (Nab p))
unfolding valid_def
by simp_all
```

```
proposition valid (Cla (Nab p)) and ¬ valid (Nab (Cla p))
unfolding valid_def
by simp_all
```

```
proposition valid (Box p) ↔ valid (Box (Box p))
unfolding valid_def
by simp
```

```
proposition valid (Neg p) ↔ valid (Neg' p)
unfolding valid_def
by simp
```

```
proposition valid (Con p q) ↔ valid (Con' p q)
unfolding valid_def
by simp
```

```
proposition valid (Dis p q) ↔ valid (Dis' p q)
unfolding valid_def
by simp
```

```
proposition valid (Eq1 p q) ↔ valid (Eq1' p q)
unfolding valid_def
using eval.simps tv.inject eval_equality eval_negation
by (metis (full_types))
```

```
proposition valid (Imp p q) ↔ valid (Imp' p q)
```

```

unfolding valid_def
using eval.simps tv.inject eval_equality eval_negation
by (metis (full_types))

proposition  $\neg$  valid (Pro ''p'')
  unfolding valid_def
  by auto

proposition  $\neg$  valid (Neg' (Pro ''p''))
proof -
  have eval ( $\lambda$ s. Det True) (Neg' (Pro ''p'')) = Det False
    by simp
  then show ?thesis
    unfolding valid_def
    using tv.inject
    by metis
qed

proposition assumes valid p shows  $\neg$  valid (Neg' p)
  using assms
  unfolding valid_def
  by simp

proposition assumes valid (Neg' p) shows  $\neg$  valid p
  using assms
  unfolding valid_def
  by force

proposition valid (Neg' (Neg' p))  $\longleftrightarrow$  valid p
  unfolding valid_def
  using double_negation
  by simp

theorem conjunction: valid (Con' p q)  $\longleftrightarrow$  valid p  $\wedge$  valid q
  unfolding valid_def
  by auto

corollary assumes valid (Con' p q) shows valid p and valid q
  using assms conjunction
  by simp_all

proposition assumes valid p and valid (Imp p q) shows valid q
  using assms eval.simps tv.inject
  unfolding valid_def
  by (metis (full_types))

proposition assumes valid p and valid (Imp' p q) shows valid q
  using assms eval.simps tv.inject eval_equality
  unfolding valid_def
  by (metis (full_types))

```

Key Equalities

The key equalities are part of the motivation for the semantic clauses.

```

proposition valid (Eq1 p (Neg' (Neg' p)))
  unfolding valid_def
  using double_negation
  by simp

```

```

proposition valid (Eq1 Truth (Neg' Falsity))
  unfolding valid_def
  by simp

proposition valid (Eq1 Falsity (Neg' Truth))
  unfolding valid_def
  by simp

proposition valid (Eq1 p (Con' p p))
  unfolding valid_def
  by simp

proposition valid (Eq1 p (Con' Truth p))
  unfolding valid_def
  by simp

proposition valid (Eq1 p (Con' p Truth))
  unfolding valid_def
  by simp

proposition valid (Eq1 Truth (Eq1' p p))
  unfolding valid_def
  by simp

proposition valid (Eq1 p (Eq1' Truth p))
  unfolding valid_def
  by simp

proposition valid (Eq1 p (Eq1' p Truth))
  unfolding valid_def
proof
  fix i
  show eval i (Eq1 p (Eq1' p Truth)) = Det True
    by (cases eval i p) simp_all
qed

proposition valid (Eq1 (Neg' p) (Eq1' Falsity p))
  unfolding valid_def
proof
  fix i
  show eval i (Eq1 (Neg' p) (Eq1' (Neg' Truth) p)) = Det True
    by (cases eval i p) simp_all
qed

proposition valid (Eq1 (Neg' p) (Eq1' p Falsity))
  unfolding valid_def
  using eval.simps eval_equality eval_negation
  by metis

```

Further Non-Theorems

Smaller Domains and Paraconsistency

Validity is relativized to a set of indeterminate truth values (called a domain).

```

definition domain :: nat set  $\Rightarrow$  tv set
where
  domain U  $\equiv$  {Det True, Det False}  $\cup$  Indet ' U

```

```

theorem universal_domain: domain {n. True} = {x. True}
proof -
  have  $\forall x. x = \text{Det True} \vee x = \text{Det False} \vee x \in \text{range Indet}$ 
    using range_eqI tv.exhaust tv.inject
  by metis
  then show ?thesis
    unfolding domain_def
    by blast
qed

definition valid_in :: nat set  $\Rightarrow$  fm  $\Rightarrow$  bool
where
  valid_in U p  $\equiv \forall i. \text{range } i \subseteq \text{domain } U \longrightarrow \text{eval } i \text{ } p = \text{Det True}$ 

abbreviation valid_boole :: fm  $\Rightarrow$  bool where valid_boole p  $\equiv$  valid_in {} p

proposition valid p  $\longleftrightarrow$  valid_in {n. True} p
  unfolding valid_def valid_in_def
  using universal_domain
  by simp

theorem valid_valid_in: assumes valid p shows valid_in U p
  using assms
  unfolding valid_in_def valid_def
  by simp

theorem transfer: assumes  $\neg$  valid_in U p shows  $\neg$  valid p
  using assms valid_valid_in
  by blast

proposition valid_in U (Neg' (Neg' p))  $\longleftrightarrow$  valid_in U p
  unfolding valid_in_def
  using double_negation
  by simp

theorem conjunction_in: valid_in U (Con' p q)  $\longleftrightarrow$  valid_in U p  $\wedge$  valid_in U q
  unfolding valid_in_def
  by auto

corollary assumes valid_in U (Con' p q) shows valid_in U p and valid_in U q
  using assms conjunction_in
  by simp_all

proposition assumes valid_in U p and valid_in U (Imp p q) shows valid_in U q
  using assms eval.simps tv.inject
  unfolding valid_in_def
  by (metis (full_types))

proposition assumes valid_in U p and valid_in U (Imp' p q) shows valid_in U q
  using assms eval.simps tv.inject eval_equality
  unfolding valid_in_def
  by (metis (full_types))

abbreviation (input) Explosion :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm
where
  Explosion p q  $\equiv$  Imp' (Con' p (Neg' p)) q

proposition valid_boole (Explosion (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def
proof (rule; rule)

```

```

fix i :: id ⇒ tv
assume range i ⊆ domain {}
then have
  i ''p'' ∈ {Det True, Det False}
  i ''q'' ∈ {Det True, Det False}
  unfolding domain_def
  by auto
then show eval i (Explosion (Pro ''p'') (Pro ''q'')) = Det True
  by (cases i ''p''; cases i ''q'') simp_all
qed

```

```

lemma explosion_counterexample: ¬ valid_in {1} (Explosion (Pro ''p'') (Pro ''q''))
proof -
  let ?i = (λs. Indet 1)(''q'' := Det False)
  have range ?i ⊆ domain {1}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Explosion (Pro ''p'') (Pro ''q'')) = Indet 1
    by simp
  moreover have Indet 1 ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

```

```

theorem explosion_not_valid: ¬ valid (Explosion (Pro ''p'') (Pro ''q''))
  using explosion_counterexample transfer
  by simp

```

```

proposition ¬ valid (Imp (Con' (Pro ''p'') (Neg' (Pro ''p''))) (Pro ''q''))
  using explosion_counterexample transfer eval.simps tv.simps
  unfolding valid_in_def
  — by smt OK

```

```

proof -
  assume *: ¬ (∀i. range i ⊆ domain U → eval i p = Det True) ⇒ ¬ valid p for U p
  assume ¬ (∀i. range i ⊆ domain {1} →
    eval i (Explosion (Pro ''p'') (Pro ''q'')) = Det True)
  then obtain i where
    **: range i ⊆ domain {1} ∧
    eval i (Explosion (Pro ''p'') (Pro ''q'')) ≠ Det True
  by blast
  then have eval i (Con' (Pro ''p'') (Neg' (Pro ''p''))) ≠
    eval i (Con' (Con' (Pro ''p'') (Neg' (Pro ''p''))) (Pro ''q''))
  by force
  then show ?thesis
    using * **
    by force
qed

```

Example: Contraposition

Contraposition is not valid.

```

abbreviation (input) Contraposition :: fm ⇒ fm ⇒ fm
where

```

```

  Contraposition p q ≡ Eql' (Imp' p q) (Imp' (Neg' q) (Neg' p))

```

```

proposition valid_boole (Contraposition (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def

```

```

proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {}
  then have
    i ''p''  $\in$  {Det True, Det False}
    i ''q''  $\in$  {Det True, Det False}
    unfolding domain_def
  by auto
  then show eval i (Contraposition (Pro ''p'') (Pro ''q'')) = Det True
  by (cases i ''p''; cases i ''q'') simp_all
qed

```

```

proposition valid_in {1} (Contraposition (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {1}
  then have
    i ''p''  $\in$  {Det True, Det False, Indet 1}
    i ''q''  $\in$  {Det True, Det False, Indet 1}
    unfolding domain_def
  by auto
  then show eval i (Contraposition (Pro ''p'') (Pro ''q'')) = Det True
  by (cases i ''p''; cases i ''q'') simp_all
qed

```

```

lemma contraposition_counterexample:  $\neg$  valid_in {1, 2} (Contraposition (Pro ''p'') (Pro ''q''))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''q'' := Indet 2)
  have range ?i  $\subseteq$  domain {1, 2}
    unfolding domain_def
  by (simp add: image_subset_iff)
  moreover have eval ?i (Contraposition (Pro ''p'') (Pro ''q'')) = Det False
  by simp
  moreover have Det False  $\neq$  Det True
  by simp
  ultimately show ?thesis
    unfolding valid_in_def
  by metis
qed

```

```

theorem contraposition_not_valid:  $\neg$  valid (Contraposition (Pro ''p'') (Pro ''q''))
  using contraposition_counterexample transfer
  by simp

```

More Than Four Truth Values Needed

Cla3 is valid for two indeterminate truth values but not for three indeterminate truth values.

```

lemma ranges: assumes range i  $\subseteq$  domain U shows eval i p  $\in$  domain U
  using assms
  unfolding domain_def
  by (induct p) auto

```

```

proposition
  unary (Cla (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  *
  *
  o
  ''

```

```

by code_simp

proposition valid_boole (Cla p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {}
  then have
    eval i p  $\in$  {Det True, Det False}
  using ranges[of i {}]
  unfolding domain_def
  by auto
  then show eval i (Cla p) = Det True
  by (cases eval i p) simp_all
qed

proposition  $\neg$  valid_in {1} (Cla (Pro ''p''))
proof -
  let ?i =  $\lambda$ s. Indet 1
  have range ?i  $\subseteq$  domain {1}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Cla (Pro ''p'')) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

abbreviation (input) Cla2 :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm
where
  Cla2 p q  $\equiv$  Dis (Dis (Cla p) (Cla q)) (Eq1 p q)

proposition
  binary (Cla2 (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  ****
  ****
  ***o
  **o*
  ''
  by code_simp

proposition valid_boole (Cla2 p q)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range: range i  $\subseteq$  domain {}
  then have
    eval i p  $\in$  {Det True, Det False}
    eval i q  $\in$  {Det True, Det False}
  using ranges[of i {}]
  unfolding domain_def
  by auto
  then show eval i (Cla2 p q) = Det True
  by (cases eval i p; cases eval i q) simp_all
qed

proposition valid_in {1} (Cla2 p q)

```

```

unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range: range i  $\subseteq$  domain {1}
  then have
    eval i p  $\in$  {Det True, Det False, Indet 1}
    eval i q  $\in$  {Det True, Det False, Indet 1}
  using ranges[of i {1}]
  unfolding domain_def
  by auto
  then show eval i (Cla2 p q) = Det True
  by (cases eval i p; cases eval i q) simp_all
qed

```

```

proposition  $\neg$  valid_in {1, 2} (Cla2 (Pro ''p'') (Pro ''q''))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''q'' := Indet 2)
  have range ?i  $\subseteq$  domain {1, 2}
  unfolding domain_def
  by (simp add: image_subset_iff)
  moreover have eval ?i (Cla2 (Pro ''p'') (Pro ''q'')) = Det False
  by simp
  moreover have Det False  $\neq$  Det True
  by simp
  ultimately show ?thesis
  unfolding valid_in_def
  by metis
qed

```

```

abbreviation (input) Cla3 :: fm  $\Rightarrow$  fm  $\Rightarrow$  fm  $\Rightarrow$  fm
where
  Cla3 p q r  $\equiv$  Dis (Dis (Cla p) (Dis (Cla q) (Cla r))) (Dis (Eql p q) (Dis (Eql p r) (Eql q r)))

```

```

proposition valid_boole (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {}
  then have
    eval i p  $\in$  {Det True, Det False}
    eval i q  $\in$  {Det True, Det False}
    eval i r  $\in$  {Det True, Det False}
  using ranges[of i {}]
  unfolding domain_def
  by auto
  then show eval i (Cla3 p q r) = Det True
  by (cases eval i p; cases eval i q; cases eval i r) simp_all
qed

```

```

proposition valid_in {1} (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {1}
  then have
    eval i p  $\in$  {Det True, Det False, Indet 1}
    eval i q  $\in$  {Det True, Det False, Indet 1}
    eval i r  $\in$  {Det True, Det False, Indet 1}
  using ranges[of i {1}]
  unfolding domain_def

```

```

    by auto
  then show eval i (Cla3 p q r) = Det True
    by (cases eval i p; cases eval i q; cases eval i r) simp_all
qed

```

```

proposition valid_in {1, 2} (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {1, 2}
  then have
    eval i p  $\in$  {Det True, Det False, Indet 1, Indet 2}
    eval i q  $\in$  {Det True, Det False, Indet 1, Indet 2}
    eval i r  $\in$  {Det True, Det False, Indet 1, Indet 2}
  using ranges[of i {1, 2}]
  unfolding domain_def
  by auto
  then show eval i (Cla3 p q r) = Det True
    by (cases eval i p; cases eval i q; cases eval i r) auto
qed

```

```

proposition  $\neg$  valid_in {1, 2, 3} (Cla3 (Pro ''p'') (Pro ''q'') (Pro ''r''))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''q'' := Indet 2, ''r'' := Indet 3)
  have range ?i  $\subseteq$  domain {1, 2, 3}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Cla3 (Pro ''p'') (Pro ''q'') (Pro ''r'')) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

```

Further Meta-Theorems

Fundamental Definitions and Lemmas

The function `props` collects the set of propositional symbols occurring in a formula.

```

fun props :: fm  $\Rightarrow$  id set
where
  props Truth = {} |
  props (Pro s) = {s} |
  props (Neg' p) = props p |
  props (Con' p q) = props p  $\cup$  props q |
  props (Eq1 p q) = props p  $\cup$  props q |
  props (Eq1' p q) = props p  $\cup$  props q

```

```

lemma relevant_props: assumes  $\forall s \in$  props p. i1 s = i2 s shows eval i1 p = eval i2 p
  using assms
  by (induct p) (simp_all, metis)

```

```

fun change_tv :: (nat  $\Rightarrow$  nat)  $\Rightarrow$  tv  $\Rightarrow$  tv
where
  change_tv f (Det b) = Det b |
  change_tv f (Indet n) = Indet (f n)

```

```

lemma change_tv_injection: assumes inj f shows inj (change_tv f)
proof -
  have change_tv f tv1 = change_tv f tv2  $\implies$  tv1 = tv2 for tv1 tv2
    using assms
  by (cases tv1; cases tv2) (simp_all add: inj_eq)
then show ?thesis
  by (simp add: injI)
qed

definition
  change_int :: (nat  $\implies$  nat)  $\implies$  (id  $\implies$  tv)  $\implies$  (id  $\implies$  tv)
where
  change_int f i  $\equiv$   $\lambda$ s. change_tv f (i s)

lemma eval_change: assumes inj f shows eval (change_int f i) p = change_tv f (eval i p)
proof (induct p)
  fix p
  assume eval (change_int f i) p = change_tv f (eval i p)
  then have eval_neg (eval (change_int f i) p) = eval_neg (change_tv f (eval i p))
    by simp
  then have eval_neg (eval (change_int f i) p) = change_tv f (eval_neg (eval i p))
    by (cases eval i p) (simp_all add: case_bool_if)
  then show eval (change_int f i) (Neg' p) = change_tv f (eval i (Neg' p))
    by simp
next
  fix p1 p2
  assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
  assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)
  show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
  proof (cases eval i p1 = eval i p2)
    assume a: eval i p1 = eval i p2
    then have yes: eval i (Con' p1 p2) = eval i p1
      by auto
    from a have change_tv f (eval i p1) = change_tv f (eval i p2)
      by auto
    then have eval (change_int f i) p1 = eval (change_int f i) p2
      using ih1 ih2
      by auto
    then have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p1
      by auto
    then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
      using yes ih1
      by auto
  next
    assume a': eval i p1  $\neq$  eval i p2
    from a' have b': eval (change_int f i) p1  $\neq$  eval (change_int f i) p2
      using assms ih1 ih2 change_tv_injection the_inv_f_f
      by metis
    show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
    proof (cases eval i p1 = Det True)
      assume a: eval i p1 = Det True
      from a a' have eval i (Con' p1 p2) = eval i p2
        by auto
      then have c: change_tv f (eval i (Con' p1 p2)) = change_tv f (eval i p2)
        by auto
      from a have b: eval (change_int f i) p1 = Det True
        using ih1
        by auto
      from b b' have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p2

```

```

    by auto
  then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
    using c ih2
    by auto
next
assume a'': eval i p1 ≠ Det True
from a'' have b'': eval (change_int f i) p1 ≠ Det True
  using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
  by metis
show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
proof (cases eval i p2 = Det True)
  assume a: eval i p2 = Det True
  from a a' a'' have eval i (Con' p1 p2) = eval i p1
    by auto
  then have c: change_tv f (eval i (Con' p1 p2)) = change_tv f (eval i p1)
    by auto
  from a have b: eval (change_int f i) p2 = Det True
    using ih2
    by auto
  from b b' b'' have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p1
    by auto
  then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
    using c ih1
    by auto
next
assume a''': eval i p2 ≠ Det True
from a' a'' a''' have eval i (Con' p1 p2) = Det False
  by auto
then have c: change_tv f (eval i (Con' p1 p2)) = Det False
  by auto
from a''' have b''': eval (change_int f i) p2 ≠ Det True
  using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
  by metis
from b' b'' b''' have eval (change_int f i) (Con' p1 p2) = Det False
  by auto
then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
  using c
  by auto
qed
qed
qed
next
fix p1 p2
assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)
have Det (eval (change_int f i) p1 = eval (change_int f i) p2) =
  Det (change_tv f (eval i p1) = change_tv f (eval i p2))
  using ih1 ih2
  by simp
also have ... = Det ((eval i p1) = (eval i p2))
  using assms change_tv_injection
  by (simp add: inj_eq)
also have ... = change_tv f (Det (eval i p1 = eval i p2))
  by simp
finally show eval (change_int f i) (Eq1 p1 p2) = change_tv f (eval i (Eq1 p1 p2))
  by simp
next
fix p1 p2
assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)

```

```

show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
proof (cases eval i p1 = eval i p2)
  assume a: eval i p1 = eval i p2
  then have yes: eval i (Eq1' p1 p2) = Det True
    by auto
  from a have change_tv f (eval i p1) = change_tv f (eval i p2)
    by auto
  then have eval (change_int f i) p1 = eval (change_int f i) p2
    using ih1 ih2
    by auto
  then have eval (change_int f i) (Eq1' p1 p2) = Det True
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using yes ih1
    by auto
next
assume a': eval i p1 ≠ eval i p2
show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
proof (cases eval i p1 = Det True)
  assume a: eval i p1 = Det True
  from a a' have yes: eval i (Eq1' p1 p2) = eval i p2
    by auto
  from a have change_tv f (eval i p1) = Det True
    by auto
  then have b: eval (change_int f i) p1 = Det True
    using ih1
    by auto
  from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
    using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
    by metis
  from b b' have eval (change_int f i) (Eq1' p1 p2) = eval (change_int f i) p2
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using ih2 yes
    by auto
next
assume a'': eval i p1 ≠ Det True
show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
proof (cases eval i p2 = Det True)
  assume a: eval i p2 = Det True
  from a a'' have yes: eval i (Eq1' p1 p2) = eval i p1
    using eval_equality[of i p1 p2]
    by auto
  from a have change_tv f (eval i p2) = Det True
    by auto
  then have b: eval (change_int f i) p2 = Det True
    using ih2
    by auto
  from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
    using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
    by metis
  from a'' have b'': eval (change_int f i) p1 ≠ Det True
    using b b'
    by auto
  from b b' b'' have eval (change_int f i) (Eq1' p1 p2) = eval (change_int f i) p1
    using eval_equality[of change_int f i p1 p2]
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using ih1 yes
    by auto

```

next

```
assume a''': eval i p2 ≠ Det True
show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
proof (cases eval i p1 = Det False)
  assume a: eval i p1 = Det False
  from a a' a'' a''' have yes: eval i (Eq1' p1 p2) = eval i (Neg' p2)
    using eval_equality[of i p1 p2]
    by auto
  from a have change_tv f (eval i p1) = Det False
    by auto
  then have b: eval (change_int f i) p1 = Det False
    using ih1
    by auto
  from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
    using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
    by metis
  from a'' have b'': eval (change_int f i) p1 ≠ Det True
    using b b'
    by auto
  from a''' have b''': eval (change_int f i) p2 ≠ Det True
    using b b' b''
    by (metis assms change_tv.simps(1) change_tv_injection inj_eq ih2)
  from b b' b'' b'''
  have eval (change_int f i) (Eq1' p1 p2) = eval (change_int f i) (Neg' p2)
    using eval_equality[of change_int f i p1 p2]
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using ih2 yes a a' a'' a''' b b' b''' eval_negation
    by metis
```

next

```
assume a''': eval i p1 ≠ Det False
show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
proof (cases eval i p2 = Det False)
  assume a: eval i p2 = Det False
  from a a' a'' a''' a'''' have yes: eval i (Eq1' p1 p2) = eval i (Neg' p1)
    using eval_equality[of i p1 p2]
    by auto
  from a have change_tv f (eval i p2) = Det False
    by auto
  then have b: eval (change_int f i) p2 = Det False
    using ih2
    by auto
  from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
    using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
    by metis
  from a'' have b'': eval (change_int f i) p1 ≠ Det True
    using change_tv.elims ih1 tv.simps(4)
    by auto
  from a''' have b''': eval (change_int f i) p2 ≠ Det True
    using b b' b''
    by (metis assms change_tv.simps(1) change_tv_injection inj_eq ih2)
  from a'''' have b''': eval (change_int f i) p1 ≠ Det False
    using b b'
    by auto
  from b b' b'' b''' b''''
  have eval (change_int f i) (Eq1' p1 p2) = eval (change_int f i) (Neg' p1)
    using eval_equality[of change_int f i p1 p2]
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using ih1 yes a a' a'' a''' a'''' b b' b'' b''' b'''' eval_negation a'' b''
```

```

    by metis
next
  assume a''''': eval i p2 ≠ Det False
  from a' a'' a''' a'''' a''''' have yes: eval i (Eq1' p1 p2) = Det False
    using eval_equality[of i p1 p2]
    by auto
  from a''''' have change_tv f (eval i p2) ≠ Det False
    using change_tv_injection inj_eq assms change_tv.simps
    by metis
  then have b: eval (change_int f i) p2 ≠ Det False
    using ih2
    by auto
  from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
    using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
    by metis
  from a'' have b'': eval (change_int f i) p1 ≠ Det True
    using change_tv.elims ih1 tv.simps(4)
    by auto
  from a''' have b''': eval (change_int f i) p2 ≠ Det True
    using b b' b''
    by (metis assms change_tv.simps(1) change_tv_injection the_inv_f_f ih2)
  from a'''' have b''''': eval (change_int f i) p1 ≠ Det False
    by (metis a'' change_tv.simps(2) ih1 string_tv.cases tv.distinct(1))
  from b b' b'' b''' b'''' have eval (change_int f i) (Eq1' p1 p2) = Det False
    using eval_equality[of change_int f i p1 p2]
    by auto
  then show eval (change_int f i) (Eq1' p1 p2) = change_tv f (eval i (Eq1' p1 p2))
    using ih1 yes a' a'' a''' a'''' a''''' b b' b'' b''' b'''' a'' b''
    by auto
  qed
  qed
  qed
  qed
  qed
qed (simp_all add: change_int_def)

```

Only a Finite Number of Truth Values Needed

Theorem `valid_in_valid` is a kind of the reverse of `valid_valid_in` (or its transfer variant).

abbreviation `is_indet :: tv ⇒ bool`

where

```
is_indet tv ≡ (case tv of Det _ ⇒ False | Indet _ ⇒ True)
```

abbreviation `get_indet :: tv ⇒ nat`

where

```
get_indet tv ≡ (case tv of Det _ ⇒ undefined | Indet n ⇒ n)
```

theorem `valid_in_valid`: assumes $\text{card } U \geq \text{card } (\text{props } p)$ and `valid_in U p` shows `valid p`

proof -

```
have finite U ⇒ card (props p) ≤ card U ⇒ valid_in U p ⇒ valid p for U p
```

proof -

```
assume assms: finite U card (props p) ≤ card U valid_in U p
```

```
show valid p
```

```
unfolding valid_def
```

```
proof
```

```
fix i
```

```
obtain f where f_p: (change_int f i) ' (props p) ⊆ (domain U) ∧ inj f
```

```
proof -
```

```
have finite U ⇒ card (props p) ≤ card U ⇒
```

```

     $\exists f. \text{change\_int } f \text{ i ' props } p \subseteq \text{domain } U \wedge \text{inj } f \text{ for } U \text{ p}$ 
proof -
  assume assms: finite U card (props p)  $\leq$  card U
  show ?thesis
  proof -
    let ?X = (get_indet ' ((i ' props p)  $\cap$  {tv. is_indet tv}))
    have d: finite (props p)
      by (induct p) auto
    then have cx: card ?X  $\leq$  card U
      using assms surj_card_le Int_lower1 card_image_le finite_Int finite_imageI le_trans
      by metis
    have f: finite ?X
      using d
      by simp
    obtain f where f_p: ( $\forall n \in ?X. f \ n \in U$ )  $\wedge$  (inj f)
    proof -
      have finite X  $\implies$  finite Y  $\implies$  card X  $\leq$  card Y  $\implies$   $\exists f. (\forall n \in X. f \ n \in Y) \wedge \text{inj } f$ 
        for X Y :: nat set
      proof -
        assume assms: finite X finite Y card X  $\leq$  card Y
        show ?thesis
        proof -
          from assms obtain Z where xyz: Z  $\subseteq$  Y  $\wedge$  card Z = card X
            by (metis card_image card_le_inj)
          then obtain f where bij_betw f X Z
            by (metis assms(1) assms(2) finite_same_card_bij infinite_super)
          then have f_p: ( $\forall n \in X. f \ n \in Y$ )  $\wedge$  inj_on f X
            using bij_betwE bij_betw_imp_inj_on xyz
            by blast
          obtain f' where f': f' = ( $\lambda n. \text{if } n \in X \text{ then } f \ n \text{ else } n + \text{Suc } (\text{Max } Y + n)$ )
            by simp
          have inj f'
            unfolding f' inj_on_def
            using assms(2) f_p le_add2 trans_le_add2 not_less_eq_eq
            by (simp, metis Max_ge add commute inj_on_eq_iff)
          moreover have ( $\forall n \in X. f' \ n \in Y$ )
            unfolding f'
            using f_p
            by auto
          ultimately show ?thesis
            by metis
        qed
      qed
    then show ( $\bigwedge f. (\forall n \in \text{get\_indet ' (i ' props p } \cap \{\text{tv. is\_indet tv}\}). f \ n \in U)$ 
       $\wedge \text{inj } f \implies \text{thesis}$ )  $\implies$  thesis
      using assms cx f
      unfolding inj_on_def
      by metis
    qed
  have (change_int f i ' (props p)  $\subseteq$  (domain U))
  proof
    fix x
    assume x  $\in$  change_int f i ' props p
    then obtain s where s_p: s  $\in$  props p  $\wedge$  change_int f i s = x
      by auto
    then have change_int f i s  $\in$  {Det True, Det False}  $\cup$  Indet ' U
    proof (cases change_int f i s  $\in$  {Det True, Det False})
      case True
      then show ?thesis
        by auto

```

```

next
  case False
  then obtain n' where change_int f i s = Indet n'
    by (cases change_int f i s) simp_all
  then have p: change_tv f (i s) = Indet n'
    by (simp add: change_int_def)
  moreover have n' ∈ U
  proof -
    obtain n'' where f n'' = n'
      using calculation change_tv.elims
      by blast
    moreover have s ∈ props p ∧ i s = (Indet n'')
      using p calculation change_tv.simps change_tv.injection the_inv_f_f f_p s_p
      by metis
    then have (Indet n'') ∈ i ' props p
      using image_iff
      by metis
    then have (Indet n'') ∈ i ' props p ∧ is_indet (Indet n'') ∧
      get_indet (Indet n'') = n''
      by auto
    then have n'' ∈ ?X
      using Int_Collect image_iff
      by metis
    ultimately show ?thesis
      using f_p
      by auto
  qed
  ultimately have change_tv f (i s) ∈ Indet ' U
    by auto
  then have change_int f i s ∈ Indet ' U
    unfolding change_int_def
    by auto
  then show ?thesis
    by auto
  qed
  then show x ∈ domain U
    unfolding domain_def
    using s_p
    by simp
  qed
  then have (change_int f i) ' (props p) ⊆ (domain U) ∧ (inj f)
    unfolding domain_def
    using f_p
    by simp
  then show ?thesis
    using f_p
    by metis
  qed
  qed
  then show (∧f. change_int f i ' props p ⊆ domain U ∧ inj f ⇒ thesis) ⇒ thesis
    using assms
    by metis
  qed
  obtain i2 where i2: i2 = (λs. if s ∈ props p then (change_int f i) s else Det True)
    by simp
  then have i2_p: ∀s ∈ props p. i2 s = (change_int f i) s
    ∀s ∈ - props p. i2 s = Det True
    by auto
  then have range i2 ⊆ (domain U)
    using i2 f_p

```

```

    unfolding domain_def
  by auto
then have eval i2 p = Det True
  using assms
  unfolding valid_in_def
  by auto
then have eval (change_int f i) p = Det True
  using relevant_props[of p i2 change_int f i] i2_p
  by auto
then have change_tv f (eval i p) = Det True
  using eval_change f_p
  by auto
then show eval i p = Det True
  by (cases eval i p) simp_all
qed
qed
then show ?thesis
  using assms subsetI sup_bot.comm_neutral image_is_empty subsetCE UnCI valid_in_def
  Un_insert_left card.empty card.infinite finite.intros(1)
  unfolding domain_def
  by metis
qed

theorem reduce: valid p  $\longleftrightarrow$  valid_in {1..card (props p)} p
  using valid_in_valid transfer
  by force

```

Case Study

Abbreviations

Entailment takes a list of assumptions.

abbreviation (input) Entail :: fm list \Rightarrow fm \Rightarrow fm

where

```
Entail l p  $\equiv$  Imp (if l = [] then Truth else fold Con' (butlast l) (last l)) p
```

theorem entailment_not_chain:

```
 $\neg$  valid (Eq1 (Entail [Pro ''p'', Pro ''q''] (Pro ''r''))
  (Box ((Imp' (Pro ''p'') (Imp' (Pro ''q'') (Pro ''r''))))))
```

proof -

```
let ?i = ( $\lambda$ s. Indet 1)(''r'' := Det False)
have eval ?i (Eq1 (Entail [Pro ''p'', Pro ''q''] (Pro ''r''))
  (Box ((Imp' (Pro ''p'') (Imp' (Pro ''q'') (Pro ''r'')))))) = Det False
  by simp
moreover have Det False  $\neq$  Det True
  by simp
ultimately show ?thesis
  unfolding valid_def
  by metis

```

qed

abbreviation (input) B0 :: fm where B0 \equiv Con' (Con' (Pro ''p'') (Pro ''q'')) (Neg' (Pro ''r''))

abbreviation (input) B1 :: fm where B1 \equiv Imp' (Con' (Pro ''p'') (Pro ''q'')) (Pro ''r'')

abbreviation (input) B2 :: fm where B2 \equiv Imp' (Pro ''r'') (Pro ''s'')

abbreviation (input) B3 :: fm where B3 \equiv Imp' (Neg' (Pro ''s'')) (Neg' (Pro ''r''))

Results

The paraconsistent logic is usable in contrast to classical logic.

```
theorem classical_logic_is_not_usable: valid_boole (Entail [B0, B1] p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {}
  then have
    i ''p''  $\in$  {Det True, Det False}
    i ''q''  $\in$  {Det True, Det False}
    i ''r''  $\in$  {Det True, Det False}
  unfolding domain_def
  by auto
  then show eval i (Entail [B0, B1] p) = Det True
    by (cases i ''p''; cases i ''q''; cases i ''r'') simp_all
qed
```

```
corollary valid_boole (Entail [B0, B1] (Pro ''r''))
  by (rule classical_logic_is_not_usable)
```

```
corollary valid_boole (Entail [B0, B1] (Neg' (Pro ''r'')))
  by (rule classical_logic_is_not_usable)
```

```
proposition  $\neg$  valid (Entail [B0, B1] (Pro ''r''))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''r'' := Det False)
  have eval ?i (Entail [B0, B1] (Pro ''r'')) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed
```

```
proposition valid_boole (Entail [B0, Box B1] p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id  $\Rightarrow$  tv
  assume range i  $\subseteq$  domain {}
  then have
    i ''p''  $\in$  {Det True, Det False}
    i ''q''  $\in$  {Det True, Det False}
    i ''r''  $\in$  {Det True, Det False}
  unfolding domain_def
  by auto
  then show eval i (Entail [B0, Box B1] p) = Det True
    by (cases i ''p''; cases i ''q''; cases i ''r'') simp_all
qed
```

```
proposition  $\neg$  valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''p'')))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''p'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''p''))) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
```

```

    unfolding valid_def
    by metis
qed

```

```

proposition  $\neg$  valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''q'')))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''q'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''q''))) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

```

```

proposition  $\neg$  valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''s'')))
proof -
  let ?i = ( $\lambda$ s. Indet 1)(''s'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''s''))) = Det False
    by simp
  moreover have Det False  $\neq$  Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

```

```

proposition valid (Entail [B0, Box B1, Box B2] (Pro ''r''))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Pro ''r'')))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Pro ''r''))
    unfolding valid_in_def
  proof (rule; rule)
    fix i :: id  $\Rightarrow$  tv
    assume range i  $\subseteq$  domain {1, 2, 3, 4}
    then have icase:
      i ''p''  $\in$  {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''q''  $\in$  {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''r''  $\in$  {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''s''  $\in$  {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
    unfolding domain_def
    by auto
  show eval i (Entail [B0, Box B1, Box B2] (Pro ''r'')) = Det True
    using icase
    by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
  qed
  ultimately show ?thesis
    using reduce
    by simp
qed

```

```

proposition valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r'')))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r''))))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r'')))
    unfolding valid_in_def
  proof (rule; rule)

```

```

fix i :: id ⇒ tv
assume range i ⊆ domain {1, 2, 3, 4}
then have icase:
  i ''p'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
  i ''q'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
  i ''r'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
  i ''s'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
  unfolding domain_def
  by auto
show eval i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r''))) = Det True
  using icase
  by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
qed
ultimately show ?thesis
  using reduce
  by simp
qed

proposition valid (Entail [B0, Box B1, Box B2] (Pro ''s''))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Pro ''s'')))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Pro ''s''))
    unfolding valid_in_def
  proof (rule; rule)
    fix i :: id ⇒ tv
    assume range i ⊆ domain {1, 2, 3, 4}
    then have icase:
      i ''p'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''q'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''r'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''s'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      unfolding domain_def
      by auto
    show eval i (Entail [B0, Box B1, Box B2] (Pro ''s'')) = Det True
      using icase
      by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
  qed
  ultimately show ?thesis
    using reduce
    by simp
qed

```

Acknowledgements

Thanks to the Isabelle developers for making a superb system and for always being willing to help.

end — Paraconsistency file

References

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