Paraconsistency

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Abstract

Paraconsistency is about handling inconsistency in a coherent way. In classical and intuitionistic logic everything follows from an inconsistent theory. A paraconsistent logic avoids the explosion. Quite a few applications in computer science and engineering are discussed in the Intelligent Systems Reference Library Volume 110: Towards Paraconsistent Engineering (Springer 2016). We formalize a paraconsistent many-valued logic that we motivated and described in a special issue on logical approaches to paraconsistency (Journal of Applied Non-Classical Logics 2005). We limit ourselves to the propositional fragment of the higher-order logic. The logic is based on so-called key equalities and has a countably infinite number of truth values. We prove theorems in the logic using the definition of validity. We verify truth tables and also counterexamples for non-theorems. We prove meta-theorems about the logic and finally we investigate a case study.

Contents

Preface

The present formalization in Isabelle essentially follows our extended abstract [1]. The Stanford Encyclopedia of Philosophy has a comprehensive overview of logical approaches to paraconsistency [2]. We have elsewhere explained the rationale for our paraconsistent many-valued logic and considered applications in multi-agent systems and natural language semantics [3, 4, 5, 6].

It is a revised and extended version of our formalization https://github.com/logic-tools/mvl that accompany our chapter in a book on partiality published by Cambridge Scholars Press. The GitHub link provides more information. We are grateful to the editors — Henning Christiansen, M. Dolores Jiménez López, Roussanka Loukanova and Larry Moss — for the opportunity to contribute to the book.

On Paraconsistency

Paraconsistency concerns inference systems that do not explode given a contradiction.

The Internet Encyclopedia of Philosophy has a survey article on paraconsistent logic.

The following Isabelle theory formalizes a specific paraconsistent many-valued logic.

```
theory Paraconsistency imports Main begin
```

The details about our logic are in our article in a special issue on logical approaches to paraconsistency in the Journal of Applied Non-Classical Logics (Volume 15, Number 1, 2005).

Syntax and Semantics

Syntax of Propositional Logic

Only the primed operators return indeterminate truth values.

```
type_synonym id = string

datatype fm = Pro id | Truth | Neg' fm | Con' fm fm | Eql fm fm | Eql' fm fm

abbreviation Falsity :: fm where Falsity ≡ Neg' Truth

abbreviation Dis' :: fm ⇒ fm ⇒ fm where Dis' p q ≡ Neg' (Con' (Neg' p) (Neg' q))

abbreviation Imp :: fm ⇒ fm ⇒ fm where Imp p q ≡ Eql p (Con' p q)

abbreviation Imp' :: fm ⇒ fm ⇒ fm where Imp' p q ≡ Eql' p (Con' p q)

abbreviation Box :: fm ⇒ fm where Box p ≡ Eql p Truth

abbreviation Neg :: fm ⇒ fm where Neg p ≡ Box (Neg' p)

abbreviation Con :: fm ⇒ fm ⇒ fm where Con p q ≡ Box (Con' p q)

abbreviation Dis :: fm ⇒ fm ⇒ fm where Dis p q ≡ Box (Dis' p q)

abbreviation Cla :: fm ⇒ fm where Cla p ≡ Dis (Box p) (Eql p Falsity)

abbreviation Nab :: fm ⇒ fm where Nab p ≡ Neg (Cla p)
```

Semantics of Propositional Logic

There is a countably infinite number of indeterminate truth values.

```
datatype tv = Det bool | Indet nat

abbreviation (input) eval_neg :: tv ⇒ tv
where
  eval_neg x ≡
    (
      case x of
        Det False ⇒ Det True |
        Det True ⇒ Det False |
        Indet n ⇒ Indet n
```

```
    )

fun eval :: (id ⇒ tv) ⇒ fm ⇒ tv
where
  eval i (Pro s) = i s |
  eval i Truth = Det True |
  eval i (Neg' p) = eval_neg (eval i p) |
  eval i (Con' p q) =
    (
      if eval i p = eval i q then eval i p else
      if eval i p = Det True then eval i q else
      if eval i q = Det True then eval i p else Det False
    ) |
  eval i (Eql p q) =
    (
      if eval i p = eval i q then Det True else Det False
    ) |
  eval i (Eql' p q) =
    (
      if eval i p = eval i q then Det True else
        (
          case (eval i p, eval i q) of
            (Det True, _) ⇒ eval i q |
            (_, Det True) ⇒ eval i p |
            (Det False, _) ⇒ eval_neg (eval i q) |
            (_, Det False) ⇒ eval_neg (eval i p) |
            _ ⇒ Det False
        )
    )

lemma eval_equality_simplify: eval i (Eql p q) = Det (eval i p = eval i q)
  by simp

theorem eval_equality:
  eval i (Eql' p q) =
    (
      if eval i p = eval i q then Det True else
      if eval i p = Det True then eval i q else
      if eval i q = Det True then eval i p else
      if eval i p = Det False then eval i (Neg' q) else
      if eval i q = Det False then eval i (Neg' p) else
      Det False
    )
  by (cases eval i p; cases eval i q) simp_all

theorem eval_negation:
  eval i (Neg' p) =
    (
      if eval i p = Det False then Det True else
      if eval i p = Det True then Det False else
      eval i p
    )
  by (cases eval i p) simp_all

corollary eval i (Cla p) = eval i (Box (Dis' p (Neg' p)))
  using eval_negation
  by simp

lemma double_negation: eval i p = eval i (Neg' (Neg' p))
  using eval_negation
```

```
  by simp
```

Validity and Consistency

Validity gives the set of theorems and the logic has at least a theorem and a non-theorem.

```
definition valid :: fm ⇒ bool
where
  valid p ≡ ∀i. eval i p = Det True

proposition valid Truth and ¬ valid Falsity
  unfolding valid_def
  by simp_all
```

Truth Tables

String Functions

The following functions support arbitrary unary and binary truth tables.

```
definition tv_pair_row :: tv list ⇒ tv ⇒ (tv * tv) list
where
  tv_pair_row tvs tv ≡ map (λx. (tv, x)) tvs

definition tv_pair_table :: tv list ⇒ (tv * tv) list list
where
  tv_pair_table tvs ≡ map (tv_pair_row tvs) tvs

definition map_row :: (tv ⇒ tv ⇒ tv) ⇒ (tv * tv) list ⇒ tv list
where
  map_row f tvtvs ≡ map (λ(x, y). f x y) tvtvs

definition map_table :: (tv ⇒ tv ⇒ tv) ⇒ (tv * tv) list list ⇒ tv list list
where
  map_table f tvtvss ≡ map (map_row f) tvtvss

definition unary_truth_table :: fm ⇒ tv list ⇒ tv list
where
  unary_truth_table p tvs ≡
      map (λx. eval ((λs. undefined)(''p'' := x)) p) tvs

definition binary_truth_table :: fm ⇒ tv list ⇒ tv list list
where
  binary_truth_table p tvs ≡
      map_table (λx y. eval ((λs. undefined)(''p'' := x, ''q'' := y)) p) (tv_pair_table tvs)

definition digit_of_nat :: nat ⇒ char
where
  digit_of_nat n ≡
   (if n = 1 then (CHR ''1'') else if n = 2 then (CHR ''2'') else if n = 3 then (CHR ''3'') else
    if n = 4 then (CHR ''4'') else if n = 5 then (CHR ''5'') else if n = 6 then (CHR ''6'') else
    if n = 7 then (CHR ''7'') else if n = 8 then (CHR ''8'') else if n = 9 then (CHR ''9'') else
      (CHR ''0''))

fun string_of_nat :: nat ⇒ string
where
  string_of_nat n =
      (if n < 10 then [digit_of_nat n] else string_of_nat (n div 10) @ [digit_of_nat (n mod 10)])
```

```
fun string_tv :: tv ⇒ string
where
  string_tv (Det True) = ''*'' |
  string_tv (Det False) = ''o'' |
  string_tv (Indet n) = string_of_nat n

definition appends :: string list ⇒ string
where
  appends strs ≡ foldr append strs []

definition appends_nl :: string list ⇒ string
where
  appends_nl strs ≡ ''↩  '' @ foldr (λs s'. s @ ''↩  '' @ s') (butlast strs) (last strs) @ ''↩''

definition string_table :: tv list list ⇒ string list list
where
  string_table tvss ≡ map (map string_tv) tvss

definition string_table_string :: string list list ⇒ string
where
  string_table_string strss ≡ appends_nl (map appends strss)

definition unary :: fm ⇒ tv list ⇒ string
where
  unary p tvs ≡ appends_nl (map string_tv (unary_truth_table p tvs))

definition binary :: fm ⇒ tv list ⇒ string
where
  binary p tvs ≡ string_table_string (string_table (binary_truth_table p tvs))
```

Main Truth Tables

The omitted Cla (for Classic) is discussed later; Nab (for Nabla) is simply the negation of it.

```
proposition
  unary (Box (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  *
  o
  o
''
  by code_simp

proposition
  binary (Con' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *o12
  oooo
  1o1o
  2oo2
''
  by code_simp

proposition
  binary (Dis' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  ****
  *o12
  *11*
  *2*2
''
  by code_simp
```

```
proposition
  unary (Neg' (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  o
  *
  1
''
  by code_simp

proposition
  binary (Eql' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *o12
  o*12
  11*o
  22o*
''
  by code_simp

proposition
  binary (Imp' (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *o12
  ****
  *1*1
  *22*
''
  by code_simp

proposition
  unary (Neg (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  o
  *
  o
''
  by code_simp

proposition
  binary (Eql (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *ooo
  o*oo
  oo*o
  ooo*
''
  by code_simp

proposition
  binary (Imp (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *ooo
  ****
  *o*o
  *oo*
''
  by code_simp

proposition
  unary (Nab (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  o
  o
  *
''
  by code_simp
```

```
proposition
  binary (Con (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  *ooo
  oooo
  oooo
  oooo
''
  by code_simp
```

```
proposition
  binary (Dis (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  ****
  *ooo
  *oo*
  *o*o
''
  by code_simp
```

Basic Theorems

Selected Theorems and Non-Theorems

Many of the following theorems and non-theorems use assumptions and meta-variables.

```
proposition valid (Cla (Box p)) and ¬ valid (Nab (Box p))
  unfolding valid_def
  by simp_all
```

```
proposition valid (Cla (Cla p)) and ¬ valid (Nab (Nab p))
  unfolding valid_def
  by simp_all
```

```
proposition valid (Cla (Nab p)) and ¬ valid (Nab (Cla p))
  unfolding valid_def
  by simp_all
```

```
proposition valid (Box p) ⟷ valid (Box (Box p))
  unfolding valid_def
  by simp
```

```
proposition valid (Neg p) ⟷ valid (Neg' p)
  unfolding valid_def
  by simp
```

```
proposition valid (Con p q) ⟷ valid (Con' p q)
  unfolding valid_def
  by simp
```

```
proposition valid (Dis p q) ⟷ valid (Dis' p q)
  unfolding valid_def
  by simp
```

```
proposition valid (Eql p q) ⟷ valid (Eql' p q)
  unfolding valid_def
  using eval.simps tv.inject eval_equality eval_negation
  by (metis (full_types))
```

```
proposition valid (Imp p q) ⟷ valid (Imp' p q)
```

```
  unfolding valid_def
  using eval.simps tv.inject eval_equality eval_negation
  by (metis (full_types))

proposition ¬ valid (Pro ''p'')
  unfolding valid_def
  by auto

proposition ¬ valid (Neg' (Pro ''p''))
proof -
  have eval (λs. Det True) (Neg' (Pro ''p'')) = Det False
    by simp
  then show ?thesis
    unfolding valid_def
    using tv.inject
    by metis
qed

proposition assumes valid p shows ¬ valid (Neg' p)
  using assms
  unfolding valid_def
  by simp

proposition assumes valid (Neg' p) shows ¬ valid p
  using assms
  unfolding valid_def
  by force

proposition valid (Neg' (Neg' p)) ⟷ valid p
  unfolding valid_def
  using double_negation
  by simp

theorem conjunction: valid (Con' p q) ⟷ valid p ∧ valid q
  unfolding valid_def
  by auto

corollary assumes valid (Con' p q) shows valid p and valid q
  using assms conjunction
  by simp_all

proposition assumes valid p and valid (Imp p q) shows valid q
  using assms eval.simps tv.inject
  unfolding valid_def
  by (metis (full_types))

proposition assumes valid p and valid (Imp' p q) shows valid q
  using assms eval.simps tv.inject eval_equality
  unfolding valid_def
  by (metis (full_types))
```

Key Equalities

The key equalities are part of the motivation for the semantic clauses.

```
proposition valid (Eql p (Neg' (Neg' p)))
  unfolding valid_def
  using double_negation
  by simp
```

```
proposition valid (Eql Truth (Neg' Falsity))
  unfolding valid_def
  by simp

proposition valid (Eql Falsity (Neg' Truth))
  unfolding valid_def
  by simp

proposition valid (Eql p (Con' p p))
  unfolding valid_def
  by simp

proposition valid (Eql p (Con' Truth p))
  unfolding valid_def
  by simp

proposition valid (Eql p (Con' p Truth))
  unfolding valid_def
  by simp

proposition valid (Eql Truth (Eql' p p))
  unfolding valid_def
  by simp

proposition valid (Eql p (Eql' Truth p))
  unfolding valid_def
  by simp

proposition valid (Eql p (Eql' p Truth))
  unfolding valid_def
proof
  fix i
  show eval i (Eql p (Eql' p Truth)) = Det True
    by (cases eval i p) simp_all
qed

proposition valid (Eql (Neg' p) (Eql' Falsity p))
  unfolding valid_def
proof
  fix i
  show eval i (Eql (Neg' p) (Eql' (Neg' Truth) p)) = Det True
    by (cases eval i p) simp_all
qed

proposition valid (Eql (Neg' p) (Eql' p Falsity))
  unfolding valid_def
  using eval.simps eval_equality eval_negation
  by metis
```

Further Non-Theorems

Smaller Domains and Paraconsistency

Validity is relativized to a set of indeterminate truth values (called a domain).

```
definition domain :: nat set ⇒ tv set
where
  domain U ≡ {Det True, Det False} ∪ Indet ‘ U
```

```
theorem universal_domain: domain {n. True} = {x. True}
proof -
  have ∀x. x = Det True ∨ x = Det False ∨ x ∈ range Indet
    using range_eqI tv.exhaust tv.inject
    by metis
  then show ?thesis
    unfolding domain_def
    by blast
qed

definition valid_in :: nat set ⇒ fm ⇒ bool
where
  valid_in U p ≡ ∀i. range i ⊆ domain U ⟶ eval i p = Det True

abbreviation valid_boole :: fm ⇒ bool where valid_boole p ≡ valid_in {} p

proposition valid p ⟷ valid_in {n. True} p
  unfolding valid_def valid_in_def
  using universal_domain
  by simp

theorem valid_valid_in: assumes valid p shows valid_in U p
  using assms
  unfolding valid_in_def valid_def
  by simp

theorem transfer: assumes ¬ valid_in U p shows ¬ valid p
  using assms valid_valid_in
  by blast

proposition valid_in U (Neg' (Neg' p)) ⟷ valid_in U p
  unfolding valid_in_def
  using double_negation
  by simp

theorem conjunction_in: valid_in U (Con' p q) ⟷ valid_in U p ∧ valid_in U q
  unfolding valid_in_def
  by auto

corollary assumes valid_in U (Con' p q) shows valid_in U p and valid_in U q
  using assms conjunction_in
  by simp_all

proposition assumes valid_in U p and valid_in U (Imp p q) shows valid_in U q
  using assms eval.simps tv.inject
  unfolding valid_in_def
  by (metis (full_types))

proposition assumes valid_in U p and valid_in U (Imp' p q) shows valid_in U q
  using assms eval.simps tv.inject eval_equality
  unfolding valid_in_def
  by (metis (full_types))

abbreviation (input) Explosion :: fm ⇒ fm ⇒ fm
where
  Explosion p q ≡ Imp' (Con' p (Neg' p)) q

proposition valid_boole (Explosion (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def
proof (rule; rule)
```

```
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      i ''p'' ∈ {Det True, Det False}
      i ''q'' ∈ {Det True, Det False}
    unfolding domain_def
    by auto
  then show eval i (Explosion (Pro ''p'') (Pro ''q'')) = Det True
    by (cases i ''p''; cases i ''q'') simp_all
qed

lemma explosion_counterexample: ¬ valid_in {1} (Explosion (Pro ''p'') (Pro ''q''))
proof -
  let ?i = (λs. Indet 1)(''q'' := Det False)
  have range ?i ⊆ domain {1}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Explosion (Pro ''p'') (Pro ''q'')) = Indet 1
    by simp
  moreover have Indet 1 ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

theorem explosion_not_valid: ¬ valid (Explosion (Pro ''p'') (Pro ''q''))
  using explosion_counterexample transfer
  by simp

proposition ¬ valid (Imp (Con' (Pro ''p'') (Neg' (Pro ''p''))) (Pro ''q''))
  using explosion_counterexample transfer eval.simps tv.simps
  unfolding valid_in_def
  — by smt OK
proof -
  assume *: ¬ (∀i. range i ⊆ domain U ⟶ eval i p = Det True) ⟹ ¬ valid p for U p
  assume ¬ (∀i. range i ⊆ domain {1} ⟶
      eval i (Explosion (Pro ''p'') (Pro ''q'')) = Det True)
  then obtain i where
    **: range i ⊆ domain {1} ∧
        eval i (Explosion (Pro ''p'') (Pro ''q'')) ≠ Det True
    by blast
  then have eval i (Con' (Pro ''p'') (Neg' (Pro ''p''))) ≠
      eval i (Con' (Con' (Pro ''p'') (Neg' (Pro ''p''))) (Pro ''q''))
    by force
  then show ?thesis
    using * **
    by force
qed
```

Example: Contraposition

Contraposition is not valid.

```
abbreviation (input) Contraposition :: fm ⇒ fm ⇒ fm
where
  Contraposition p q ≡ Eql' (Imp' p q) (Imp' (Neg' q) (Neg' p))

proposition valid_boole (Contraposition (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def
```

```
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      i ''p'' ∈ {Det True, Det False}
      i ''q'' ∈ {Det True, Det False}
    unfolding domain_def
    by auto
  then show eval i (Contraposition (Pro ''p'') (Pro ''q'')) = Det True
    by (cases i ''p''; cases i ''q'') simp_all
qed

proposition valid_in {1} (Contraposition (Pro ''p'') (Pro ''q''))
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {1}
  then have
      i ''p'' ∈ {Det True, Det False, Indet 1}
      i ''q'' ∈ {Det True, Det False, Indet 1}
    unfolding domain_def
    by auto
  then show eval i (Contraposition (Pro ''p'') (Pro ''q'')) = Det True
    by (cases i ''p''; cases i ''q'') simp_all
qed

lemma contraposition_counterexample: ¬ valid_in {1, 2} (Contraposition (Pro ''p'') (Pro ''q''))
proof -
  let ?i = (λs. Indet 1)(''q'' := Indet 2)
  have range ?i ⊆ domain {1, 2}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Contraposition (Pro ''p'') (Pro ''q'')) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

theorem contraposition_not_valid: ¬ valid (Contraposition (Pro ''p'') (Pro ''q''))
  using contraposition_counterexample transfer
  by simp
```

More Than Four Truth Values Needed

Cla3 is valid for two indeterminate truth values but not for three indeterminate truth values.

```
lemma ranges: assumes range i ⊆ domain U shows eval i p ∈ domain U
  using assms
  unfolding domain_def
  by (induct p) auto

proposition
  unary (Cla (Pro ''p'')) [Det True, Det False, Indet 1] = ''
  *
  *
  o
''
```

```
  by code_simp

proposition valid_boole (Cla p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      eval i p ∈ {Det True, Det False}
    using ranges[of i {}]
    unfolding domain_def
    by auto
  then show eval i (Cla p) = Det True
    by (cases eval i p) simp_all
qed

proposition ¬ valid_in {1} (Cla (Pro ''p''))
proof -
  let ?i = λs. Indet 1
  have range ?i ⊆ domain {1}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Cla (Pro ''p'')) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

abbreviation (input) Cla2 :: fm ⇒ fm ⇒ fm
where
  Cla2 p q ≡ Dis (Dis (Cla p) (Cla q)) (Eql p q)

proposition
  binary (Cla2 (Pro ''p'') (Pro ''q'')) [Det True, Det False, Indet 1, Indet 2] = ''
  ****
  ****
  ***o
  **o*
''
  by code_simp

proposition valid_boole (Cla2 p q)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range: range i ⊆ domain {}
  then have
      eval i p ∈ {Det True, Det False}
      eval i q ∈ {Det True, Det False}
    using ranges[of i {}]
    unfolding domain_def
    by auto
  then show eval i (Cla2 p q) = Det True
    by (cases eval i p; cases eval i q) simp_all
qed

proposition valid_in {1} (Cla2 p q)
```

```
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range: range i ⊆ domain {1}
  then have
      eval i p ∈ {Det True, Det False, Indet 1}
      eval i q ∈ {Det True, Det False, Indet 1}
    using ranges[of i {1}]
    unfolding domain_def
    by auto
  then show eval i (Cla2 p q) = Det True
    by (cases eval i p; cases eval i q) simp_all
qed

proposition ¬ valid_in {1, 2} (Cla2 (Pro ''p'') (Pro ''q''))
proof -
  let ?i = (λs. Indet 1)(''q'' := Indet 2)
  have range ?i ⊆ domain {1, 2}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Cla2 (Pro ''p'') (Pro ''q'')) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed

abbreviation (input) Cla3 :: fm ⇒ fm ⇒ fm ⇒ fm
where
  Cla3 p q r ≡ Dis (Dis (Cla p) (Dis (Cla q) (Cla r))) (Dis (Eql p q) (Dis (Eql p r) (Eql q r)))

proposition valid_boole (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      eval i p ∈ {Det True, Det False}
      eval i q ∈ {Det True, Det False}
      eval i r ∈ {Det True, Det False}
    using ranges[of i {}]
    unfolding domain_def
    by auto
  then show eval i (Cla3 p q r) = Det True
    by (cases eval i p; cases eval i q; cases eval i r) simp_all
qed

proposition valid_in {1} (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {1}
  then have
      eval i p ∈ {Det True, Det False, Indet 1}
      eval i q ∈ {Det True, Det False, Indet 1}
      eval i r ∈ {Det True, Det False, Indet 1}
    using ranges[of i {1}]
    unfolding domain_def
```

```
    by auto
  then show eval i (Cla3 p q r) = Det True
    by (cases eval i p; cases eval i q; cases eval i r) simp_all
qed

proposition valid_in {1, 2} (Cla3 p q r)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {1, 2}
  then have
      eval i p ∈ {Det True, Det False, Indet 1, Indet 2}
      eval i q ∈ {Det True, Det False, Indet 1, Indet 2}
      eval i r ∈ {Det True, Det False, Indet 1, Indet 2}
    using ranges[of i {1, 2}]
    unfolding domain_def
    by auto
  then show eval i (Cla3 p q r) = Det True
    by (cases eval i p; cases eval i q; cases eval i r) auto
qed

proposition ¬ valid_in {1, 2, 3} (Cla3 (Pro ''p'') (Pro ''q'') (Pro ''r''))
proof -
  let ?i = (λs. Indet 1)(''q'' := Indet 2, ''r'' := Indet 3)
  have range ?i ⊆ domain {1, 2, 3}
    unfolding domain_def
    by (simp add: image_subset_iff)
  moreover have eval ?i (Cla3 (Pro ''p'') (Pro ''q'') (Pro ''r'')) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_in_def
    by metis
qed
```

Further Meta-Theorems

Fundamental Definitions and Lemmas

The function props collects the set of propositional symbols occurring in a formula.

```
fun props :: fm ⇒ id set
where
  props Truth = {} |
  props (Pro s) = {s} |
  props (Neg' p) = props p |
  props (Con' p q) = props p ∪ props q |
  props (Eql p q) = props p ∪ props q |
  props (Eql' p q) = props p ∪ props q

lemma relevant_props: assumes ∀s ∈ props p. i1 s = i2 s shows eval i1 p = eval i2 p
  using assms
  by (induct p) (simp_all, metis)

fun change_tv :: (nat ⇒ nat) ⇒ tv ⇒ tv
where
  change_tv f (Det b) = Det b |
  change_tv f (Indet n) = Indet (f n)
```

```
lemma change_tv_injection: assumes inj f shows inj (change_tv f)
proof -
  have change_tv f tv1 = change_tv f tv2 ⟹ tv1 = tv2 for tv1 tv2
    using assms
    by (cases tv1; cases tv2) (simp_all add: inj_eq)
  then show ?thesis
    by (simp add: injI)
qed

definition
  change_int :: (nat ⇒ nat) ⇒ (id ⇒ tv) ⇒ (id ⇒ tv)
where
  change_int f i ≡ λs. change_tv f (i s)

lemma eval_change: assumes inj f shows eval (change_int f i) p = change_tv f (eval i p)
proof (induct p)
  fix p
  assume eval (change_int f i) p = change_tv f (eval i p)
  then have eval_neg (eval (change_int f i) p) = eval_neg (change_tv f (eval i p))
    by simp
  then have eval_neg (eval (change_int f i) p) = change_tv f (eval_neg (eval i p))
    by (cases eval i p) (simp_all add: case_bool_if)
  then show eval (change_int f i) (Neg' p) = change_tv f (eval i (Neg' p))
    by simp
next
  fix p1 p2
  assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
  assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)
  show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
  proof (cases eval i p1 = eval i p2)
    assume a: eval i p1 = eval i p2
    then have yes: eval i (Con' p1 p2) = eval i p1
      by auto
    from a have change_tv f (eval i p1) = change_tv f (eval i p2)
      by auto
    then have eval (change_int f i) p1 = eval (change_int f i) p2
      using ih1 ih2
      by auto
    then have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p1
      by auto
    then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
      using yes ih1
      by auto
  next
    assume a': eval i p1 ≠ eval i p2
    from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
      using assms ih1 ih2 change_tv_injection the_inv_f_f
      by metis
    show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
    proof (cases eval i p1 = Det True)
      assume a: eval i p1 = Det True
      from a a' have eval i (Con' p1 p2) = eval i p2
        by auto
      then have c: change_tv f (eval i (Con' p1 p2)) = change_tv f (eval i p2)
        by auto
      from a have b: eval (change_int f i) p1 = Det True
        using ih1
        by auto
      from b b' have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p2
```

```
        by auto
      then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
        using c ih2
        by auto
    next
      assume a'': eval i p1 ≠ Det True
      from a'' have b'': eval (change_int f i) p1 ≠ Det True
        using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
        by metis
      show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
      proof (cases eval i p2 = Det True)
        assume a: eval i p2 = Det True
        from a a' a'' have eval i (Con' p1 p2) = eval i p1
          by auto
        then have c: change_tv f (eval i (Con' p1 p2)) = change_tv f (eval i p1)
          by auto
        from a have b: eval (change_int f i) p2 = Det True
          using ih2
          by auto
        from b b' b'' have eval (change_int f i) (Con' p1 p2) = eval (change_int f i) p1
          by auto
        then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
          using c ih1
          by auto
      next
        assume a''': eval i p2 ≠ Det True
        from a' a'' a''' have eval i (Con' p1 p2) = Det False
          by auto
        then have c: change_tv f (eval i (Con' p1 p2)) = Det False
          by auto
        from a''' have b''': eval (change_int f i) p2 ≠ Det True
          using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
          by metis
        from b' b'' b''' have eval (change_int f i) (Con' p1 p2) = Det False
          by auto
        then show eval (change_int f i) (Con' p1 p2) = change_tv f (eval i (Con' p1 p2))
          using c
          by auto
      qed
    qed
  qed
next
  fix p1 p2
  assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
  assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)
  have Det (eval (change_int f i) p1 = eval (change_int f i) p2) =
      Det (change_tv f (eval i p1) = change_tv f (eval i p2))
    using ih1 ih2
    by simp
  also have ... = Det ((eval i p1) = (eval i p2))
    using assms change_tv_injection
    by (simp add: inj_eq)
  also have ... = change_tv f (Det (eval i p1 = eval i p2))
    by simp
  finally show eval (change_int f i) (Eql p1 p2) = change_tv f (eval i (Eql p1 p2))
    by simp
next
  fix p1 p2
  assume ih1: eval (change_int f i) p1 = change_tv f (eval i p1)
  assume ih2: eval (change_int f i) p2 = change_tv f (eval i p2)
```

```
show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
proof (cases eval i p1 = eval i p2)
  assume a: eval i p1 = eval i p2
  then have yes: eval i (Eql' p1 p2) = Det True
    by auto
  from a have change_tv f (eval i p1) = change_tv f (eval i p2)
    by auto
  then have eval (change_int f i) p1 = eval (change_int f i) p2
    using ih1 ih2
    by auto
  then have eval (change_int f i) (Eql' p1 p2) = Det True
    by auto
  then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
    using yes ih1
    by auto
next
  assume a': eval i p1 ≠ eval i p2
  show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
  proof (cases eval i p1 = Det True)
    assume a: eval i p1 = Det True
    from a a' have yes: eval i (Eql' p1 p2) = eval i p2
      by auto
    from a have change_tv f (eval i p1) = Det True
      by auto
    then have b: eval (change_int f i) p1 = Det True
      using ih1
      by auto
    from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
      using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
      by metis
    from b b' have eval (change_int f i) (Eql' p1 p2) = eval (change_int f i) p2
      by auto
    then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
      using ih2 yes
      by auto
  next
    assume a'': eval i p1 ≠ Det True
    show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
    proof (cases eval i p2 = Det True)
      assume a: eval i p2 = Det True
      from a a' a'' have yes: eval i (Eql' p1 p2) = eval i p1
        using eval_equality[of i p1 p2]
        by auto
      from a have change_tv f (eval i p2) = Det True
        by auto
      then have b: eval (change_int f i) p2 = Det True
        using ih2
        by auto
      from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
        using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
        by metis
      from a'' have b'': eval (change_int f i) p1 ≠ Det True
        using b b'
        by auto
      from b b' b'' have eval (change_int f i) (Eql' p1 p2) = eval (change_int f i) p1
        using eval_equality[of change_int f i p1 p2]
        by auto
      then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
        using ih1 yes
        by auto
```

```
next
  assume a''': eval i p2 ≠ Det True
  show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
  proof (cases eval i p1 = Det False)
    assume a: eval i p1 = Det False
    from a a' a'' a''' have yes: eval i (Eql' p1 p2) = eval i (Neg' p2)
      using eval_equality[of i p1 p2]
      by auto
    from a have change_tv f (eval i p1) = Det False
      by auto
    then have b: eval (change_int f i) p1 = Det False
      using ih1
      by auto
    from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
      using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
      by metis
    from a'' have b'': eval (change_int f i) p1 ≠ Det True
      using b b'
      by auto
    from a''' have b''': eval (change_int f i) p2 ≠ Det True
      using b b' b''
      by (metis assms change_tv.simps(1) change_tv_injection inj_eq ih2)
    from b b' b'' b'''
    have eval (change_int f i) (Eql' p1 p2) = eval (change_int f i) (Neg' p2)
      using eval_equality[of change_int f i p1 p2]
      by auto
    then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
      using ih2 yes a a' a''' b b' b''' eval_negation
      by metis
  next
    assume a'''': eval i p1 ≠ Det False
    show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
    proof (cases eval i p2 = Det False)
      assume a: eval i p2 = Det False
      from a a' a'' a''' a'''' have yes: eval i (Eql' p1 p2) = eval i (Neg' p1)
        using eval_equality[of i p1 p2]
        by auto
      from a have change_tv f (eval i p2) = Det False
        by auto
      then have b: eval (change_int f i) p2 = Det False
        using ih2
        by auto
      from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
        using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
        by metis
      from a'' have b'': eval (change_int f i) p1 ≠ Det True
        using change_tv.elims ih1 tv.simps(4)
        by auto
      from a''' have b''': eval (change_int f i) p2 ≠ Det True
        using b b' b''
        by (metis assms change_tv.simps(1) change_tv_injection inj_eq ih2)
      from a'''' have b'''': eval (change_int f i) p1 ≠ Det False
        using b b'
        by auto
      from b b' b'' b''' b''''
      have eval (change_int f i) (Eql' p1 p2) = eval (change_int f i) (Neg' p1)
        using eval_equality[of change_int f i p1 p2]
        by auto
      then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
        using ih1 yes a a' a''' a'''' b b' b''' b'''' eval_negation a'' b''
```

```
          by metis
      next
        assume a''''': eval i p2 ≠ Det False
        from a' a'' a''' a'''' a''''' have yes: eval i (Eql' p1 p2) = Det False
          using eval_equality[of i p1 p2]
          by auto
        from a''''' have change_tv f (eval i p2) ≠ Det False
          using change_tv_injection inj_eq assms change_tv.simps
          by metis
        then have b: eval (change_int f i) p2 ≠ Det False
          using ih2
          by auto
        from a' have b': eval (change_int f i) p1 ≠ eval (change_int f i) p2
          using assms ih1 ih2 change_tv_injection the_inv_f_f change_tv.simps
          by metis
        from a'' have b'': eval (change_int f i) p1 ≠ Det True
          using change_tv.elims ih1 tv.simps(4)
          by auto
        from a''' have b''': eval (change_int f i) p2 ≠ Det True
          using b b' b''
          by (metis assms change_tv.simps(1) change_tv_injection the_inv_f_f ih2)
        from a'''' have b'''': eval (change_int f i) p1 ≠ Det False
          by (metis a'' change_tv.simps(2) ih1 string_tv.cases tv.distinct(1))
        from b b' b'' b''' b'''' have eval (change_int f i) (Eql' p1 p2) = Det False
          using eval_equality[of change_int f i p1 p2]
          by auto
        then show eval (change_int f i) (Eql' p1 p2) = change_tv f (eval i (Eql' p1 p2))
          using ih1 yes a' a''' a'''' b b' b''' b'''' a'' b''
          by auto
      qed
    qed
  qed
 qed
qed
qed (simp_all add: change_int_def)
```

Only a Finite Number of Truth Values Needed

Theorem valid_in_valid is a kind of the reverse of valid_valid_in (or its transfer variant).

```
abbreviation is_indet :: tv ⇒ bool
where
  is_indet tv ≡ (case tv of Det _ ⇒ False | Indet _ ⇒ True)

abbreviation get_indet :: tv ⇒ nat
where
  get_indet tv ≡ (case tv of Det _ ⇒ undefined | Indet n ⇒ n)

theorem valid_in_valid: assumes card U ≥ card (props p) and valid_in U p shows valid p
proof -
  have finite U ⟹ card (props p) ≤ card U ⟹ valid_in U p ⟹ valid p for U p
  proof -
    assume assms: finite U card (props p) ≤ card U valid_in U p
    show valid p
      unfolding valid_def
    proof
      fix i
      obtain f where f_p: (change_int f i) ` (props p) ⊆ (domain U) ∧ inj f
      proof -
        have finite U ⟹ card (props p) ≤ card U ⟹
```

```
    ∃f. change_int f i ‘ props p ⊆ domain U ∧ inj f for U p
proof -
  assume assms: finite U card (props p) ≤ card U
  show ?thesis
  proof -
    let ?X = (get_indet ‘ ((i ‘ props p) ∩ {tv. is_indet tv}))
    have d: finite (props p)
      by (induct p) auto
    then have cx: card ?X ≤ card U
      using assms surj_card_le Int_lower1 card_image_le finite_Int finite_imageI le_trans
      by metis
    have f: finite ?X
      using d
      by simp
    obtain f where f_p: (∀n ∈ ?X. f n ∈ U) ∧ (inj f)
    proof -
      have finite X ⟹ finite Y ⟹ card X ≤ card Y ⟹ ∃f. (∀n ∈ X. f n ∈ Y) ∧ inj f
          for X Y :: nat set
      proof -
        assume assms: finite X finite Y card X ≤ card Y
        show ?thesis
        proof -
          from assms obtain Z where xyz: Z ⊆ Y ∧ card Z = card X
            by (metis card_image card_le_inj)
          then obtain f where bij_betw f X Z
            by (metis assms(1) assms(2) finite_same_card_bij infinite_super)
          then have f_p: (∀n ∈ X. f n ∈ Y) ∧ inj_on f X
            using bij_betwE bij_betw_imp_inj_on xyz
            by blast
          obtain f' where f': f' = (λn. if n ∈ X then f n else n + Suc (Max Y + n))
            by simp
          have inj f'
            unfolding f' inj_on_def
            using assms(2) f_p le_add2 trans_le_add2 not_less_eq_eq
            by (simp, metis Max_ge add.commute inj_on_eq_iff)
          moreover have (∀n ∈ X. f' n ∈ Y)
            unfolding f'
            using f_p
            by auto
          ultimately show ?thesis
            by metis
        qed
      qed
      then show (⋀f. (∀n ∈ get_indet ‘ (i ‘ props p ∩ {tv. is_indet tv}). f n ∈ U)
          ∧ inj f ⟹ thesis) ⟹ thesis
        using assms cx f
        unfolding inj_on_def
        by metis
    qed
    have (change_int f i) ‘ (props p) ⊆ (domain U)
    proof
      fix x
      assume x ∈ change_int f i ‘ props p
      then obtain s where s_p: s ∈ props p ∧ change_int f i s = x
        by auto
      then have change_int f i s ∈ {Det True, Det False} ∪ Indet ‘ U
      proof (cases change_int f i s ∈ {Det True, Det False})
        case True
        then show ?thesis
          by auto
```

```
          next
            case False
            then obtain n' where change_int f i s = Indet n'
              by (cases change_int f i s) simp_all
            then have p: change_tv f (i s) = Indet n'
              by (simp add: change_int_def)
            moreover have n' ∈ U
            proof -
              obtain n'' where f n'' = n'
                using calculation change_tv.elims
                by blast
              moreover have s ∈ props p ∧ i s = (Indet n'')
                using p calculation change_tv.simps change_tv_injection the_inv_f_f f_p s_p
                by metis
              then have (Indet n'') ∈ i ‘ props p
                using image_iff
                by metis
              then have (Indet n'') ∈ i ‘ props p ∧ is_indet (Indet n'') ∧
                  get_indet (Indet n'') = n''
                by auto
              then have n'' ∈ ?X
                using Int_Collect image_iff
                by metis
              ultimately show ?thesis
                using f_p
                by auto
            qed
            ultimately have change_tv f (i s) ∈ Indet ‘ U
              by auto
            then have change_int f i s ∈ Indet ‘ U
              unfolding change_int_def
              by auto
            then show ?thesis
              by auto
          qed
          then show x ∈ domain U
            unfolding domain_def
            using s_p
            by simp
        qed
        then have (change_int f i) ‘ (props p) ⊆ (domain U) ∧ (inj f)
          unfolding domain_def
          using f_p
          by simp
        then show ?thesis
          using f_p
          by metis
      qed
    qed
    then show (⋀f. change_int f i ‘ props p ⊆ domain U ∧ inj f ⟹ thesis) ⟹ thesis
      using assms
      by metis
  qed
  obtain i2 where i2: i2 = (λs. if s ∈ props p then (change_int f i) s else Det True)
    by simp
  then have i2_p: ∀s ∈ props p. i2 s = (change_int f i) s
        ∀s ∈ - props p. i2 s = Det True
    by auto
  then have range i2 ⊆ (domain U)
    using i2 f_p
```

```
        unfolding domain_def
        by auto
      then have eval i2 p = Det True
        using assms
        unfolding valid_in_def
        by auto
      then have eval (change_int f i) p = Det True
        using relevant_props[of p i2 change_int f i] i2_p
        by auto
      then have change_tv f (eval i p) = Det True
        using eval_change f_p
        by auto
      then show eval i p = Det True
        by (cases eval i p) simp_all
    qed
  qed
  then show ?thesis
    using assms subsetI sup_bot.comm_neutral image_is_empty subsetCE UnCI valid_in_def
        Un_insert_left card.empty card.infinite finite.intros(1)
    unfolding domain_def
    by metis
qed

theorem reduce: valid p ⟷ valid_in {1..card (props p)} p
  using valid_in_valid transfer
  by force
```

Case Study

Abbreviations

Entailment takes a list of assumptions.

```
abbreviation (input) Entail :: fm list ⇒ fm ⇒ fm
where
  Entail l p ≡ Imp (if l = [] then Truth else fold Con’ (butlast l) (last l)) p

theorem entailment_not_chain:
  ¬ valid (Eql (Entail [Pro ’’p’’, Pro ’’q’’] (Pro ’’r’’))
      (Box ((Imp’ (Pro ’’p’’) (Imp’ (Pro ’’q’’) (Pro ’’r’’))))))
proof -
  let ?i = (λs. Indet 1)(’’r’’ := Det False)
  have eval ?i (Eql (Entail [Pro ’’p’’, Pro ’’q’’] (Pro ’’r’’))
      (Box ((Imp’ (Pro ’’p’’) (Imp’ (Pro ’’q’’) (Pro ’’r’’)))))) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

abbreviation (input) B0 :: fm where B0 ≡ Con’ (Con’ (Pro ’’p’’) (Pro ’’q’’)) (Neg’ (Pro ’’r’’))

abbreviation (input) B1 :: fm where B1 ≡ Imp’ (Con’ (Pro ’’p’’) (Pro ’’q’’)) (Pro ’’r’’)

abbreviation (input) B2 :: fm where B2 ≡ Imp’ (Pro ’’r’’) (Pro ’’s’’)

abbreviation (input) B3 :: fm where B3 ≡ Imp’ (Neg’ (Pro ’’s’’)) (Neg’ (Pro ’’r’’))
```

Results

The paraconsistent logic is usable in contrast to classical logic.

```
theorem classical_logic_is_not_usable: valid_boole (Entail [B0, B1] p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      i ''p'' ∈ {Det True, Det False}
      i ''q'' ∈ {Det True, Det False}
      i ''r'' ∈ {Det True, Det False}
    unfolding domain_def
    by auto
  then show eval i (Entail [B0, B1] p) = Det True
    by (cases i ''p''; cases i ''q''; cases i ''r'') simp_all
qed

corollary valid_boole (Entail [B0, B1] (Pro ''r''))
  by (rule classical_logic_is_not_usable)

corollary valid_boole (Entail [B0, B1] (Neg' (Pro ''r'')))
  by (rule classical_logic_is_not_usable)

proposition ¬ valid (Entail [B0, B1] (Pro ''r''))
proof -
  let ?i = (λs. Indet 1)(''r'' := Det False)
  have eval ?i (Entail [B0, B1] (Pro ''r'')) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

proposition valid_boole (Entail [B0, Box B1] p)
  unfolding valid_in_def
proof (rule; rule)
  fix i :: id ⇒ tv
  assume range i ⊆ domain {}
  then have
      i ''p'' ∈ {Det True, Det False}
      i ''q'' ∈ {Det True, Det False}
      i ''r'' ∈ {Det True, Det False}
    unfolding domain_def
    by auto
  then show eval i (Entail [B0, Box B1] p) = Det True
    by (cases i ''p''; cases i ''q''; cases i ''r'') simp_all
qed

proposition ¬ valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''p'')))
proof -
  let ?i = (λs. Indet 1)(''p'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''p''))) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
```

```
    unfolding valid_def
    by metis
qed

proposition ¬ valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''q'')))
proof -
  let ?i = (λs. Indet 1)(''q'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''q''))) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

proposition ¬ valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''s'')))
proof -
  let ?i = (λs. Indet 1)(''s'' := Det True)
  have eval ?i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''s''))) = Det False
    by simp
  moreover have Det False ≠ Det True
    by simp
  ultimately show ?thesis
    unfolding valid_def
    by metis
qed

proposition valid (Entail [B0, Box B1, Box B2] (Pro ''r''))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Pro ''r'')))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Pro ''r''))
    unfolding valid_in_def
  proof (rule; rule)
    fix i :: id ⇒ tv
    assume range i ⊆ domain {1, 2, 3, 4}
    then have icase:
      i ''p'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''q'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''r'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''s'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      unfolding domain_def
      by auto
    show eval i (Entail [B0, Box B1, Box B2] (Pro ''r'')) = Det True
      using icase
      by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
  qed
  ultimately show ?thesis
    using reduce
    by simp
qed

proposition valid (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r'')))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r''))))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r'')))
    unfolding valid_in_def
  proof (rule; rule)
```

```
    fix i :: id ⇒ tv
    assume range i ⊆ domain {1, 2, 3, 4}
    then have icase:
      i ''p'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''q'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''r'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''s'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      unfolding domain_def
      by auto
    show eval i (Entail [B0, Box B1, Box B2] (Neg' (Pro ''r''))) = Det True
      using icase
      by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
  qed
  ultimately show ?thesis
    using reduce
    by simp
qed

proposition valid (Entail [B0, Box B1, Box B2] (Pro ''s''))
proof -
  have {1..card (props (Entail [B0, Box B1, Box B2] (Pro ''s'')))} = {1, 2, 3, 4}
    by code_simp
  moreover have valid_in {1, 2, 3, 4} (Entail [B0, Box B1, Box B2] (Pro ''s''))
    unfolding valid_in_def
  proof (rule; rule)
    fix i :: id ⇒ tv
    assume range i ⊆ domain {1, 2, 3, 4}
    then have icase:
      i ''p'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''q'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''r'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      i ''s'' ∈ {Det True, Det False, Indet 1, Indet 2, Indet 3, Indet 4}
      unfolding domain_def
      by auto
    show eval i (Entail [B0, Box B1, Box B2] (Pro ''s'')) = Det True
      using icase
      by (cases i ''p''; cases i ''q''; cases i ''r''; cases i ''s'') simp_all
  qed
  ultimately show ?thesis
    using reduce
    by simp
qed
```

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end — Paraconsistency file

References

[1] A. S. Jensen and J. Villadsen. *Paraconsistent Computational Logic.* In P. Blackburn, K. F. Jørgensen, N. Jones, and E. Palmgren, editors, 8th Scandinavian Logic Symposium: Abstracts, pages 59–61, Roskilde University, 2012.

[2] G. Priest, K. Tanaka and Z. Weber. *Paraconsistent Logic.* In E. N. Zalta et al., editors, Stanford Encyclopedia of Philosophy, Online Entry http://plato.stanford.edu/entries/logic-paraconsistent/ Spring Edition, 2015.

[3] J. Villadsen. *Supra-logic: Using Transfinite Type Theory with Type Variables for Paraconsistency.* Logical Approaches to Paraconsistency, Journal of Applied Non-Classical Logics, 15(1):45–58, 2005.

[4] J. Villadsen. *Infinite-Valued Propositional Type Theory for Semantics.* In J.-Y. Béziau and A. Costa-Leite, editors, Dimensions of Logical Concepts, pages 277–297, Unicamp Coleç. CLE 54, 2009.

[5] J. Villadsen. *Nabla: A Linguistic System Based on Type Theory.* Foundations of Communication and Cognition (New Series), LIT Verlag, 2010.

[6] J. Villadsen. *Multi-dimensional Type Theory: Rules, Categories and Combinators for Syntax and Semantics.* In P. Blache, H. Christiansen, V. Dahl, D. Duchier, and J. Villadsen, editors, Constraints and Language, pages 167–189, Cambridge Scholars Press, 2014.