Prime Number Theorem with Remainder Term

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Abstract

We have formalized the proof of the Prime Number Theorem with remainder term. This is the first formalized version of PNT with an explicit error term.

There are many useful results in this AFP entry.

First, the main result, prime number theorem with remainder:

$$\pi(x) = \mathrm{Li}(x) + O\left(x \exp\left(-\sqrt{\log x}/3653\right)\right)$$

Second, the zero-free region of the Riemann zeta function:

$$\zeta(\beta + i\gamma) \neq 0 \text{ when } \beta \geq 1 - \frac{1}{952320} \left(\log(|\gamma| + 2)\right)^{-1}$$

Moreover, we proved a revised version of Perron's formula, together with the zero-free region we can prove the main result.

Contents

theory *PNT_Notation*
imports
Prime_Number_Theorem.Prime_Counting_Functions
begin

definition *PNT_const_*C_1 ≡ *1* / *952320* :: *real*

abbreviation *nat_powr*
(**infixr** ‹*nat′_powr*› *80*)
where
n nat_powr x ≡ (*of_nat n*) *powr x*

bundle *pnt_syntax*
begin
notation *PNT_const_*C_1 (‹C_1›)
notation *norm* (‹‖_‖›)
notation *Suc* (‹_₊› [*101*] *100*)
end

end
theory *PNT_Remainder_Library*
imports
PNT_Notation
begin
unbundle *pnt_syntax*

1 Auxiliary library for prime number theorem

1.1 Zeta function

lemma *pre_zeta_1_bound*:
assumes *0* < *Re s*
shows ‖*pre_zeta 1 s*‖ ≤ ‖*s*‖ / *Re s*
⟨*proof*⟩

lemma *zeta_pole_eq*:
assumes *s* ≠ *1*
shows *zeta s* = *pre_zeta 1 s* + *1* / (*s* − *1*)
⟨*proof*⟩

definition *zeta′* **where** *zeta′ s* ≡ *pre_zeta 1 s* ∗ (*s* − *1*) + *1*

lemma *zeta′_analytic*:
zeta′ analytic_on UNIV
⟨*proof*⟩

lemma *zeta′_analytic_on* [*analytic_intros*]:
zeta′ analytic_on A ⟨*proof*⟩

lemma *zeta′_holomorphic_on* [*holomorphic_intros*]:
zeta′ holomorphic_on A ⟨*proof*⟩

lemma *zeta_eq_zeta′*:
zeta s = *zeta′ s* / (*s* − *1*)

⟨*proof*⟩

lemma *zeta′_1* [*simp*]: *zeta′ 1 = 1* ⟨*proof*⟩

lemma *zeta_eq_zero_iff_zeta′*:
shows $s \neq 1 \Longrightarrow$ *zeta′ s = 0* $\longleftrightarrow$ *zeta s = 0*
⟨*proof*⟩

lemma *zeta′_eq_zero_iff*:
shows *zeta′ s = 0* $\longleftrightarrow$ *zeta s = 0* $\wedge$ $s \neq 1$
⟨*proof*⟩

lemma *zeta_eq_zero_iff*:
shows *zeta s = 0* $\longleftrightarrow$ *zeta′ s = 0* $\vee$ *s = 1*
⟨*proof*⟩

1.2 Logarithm derivatives

definition *logderiv f x* $\equiv$ *deriv f x / f x*
definition *log_differentiable*
(**infixr** ‹(*log′_differentiable*)› *50*)
where
f log_differentiable x $\equiv$ (*f field_differentiable* (*at x*)) $\wedge$ *f x* $\neq$ *0*

lemma *logderiv_prod′*:
fixes *f* :: *′n* ⇒ *′f* ⇒ *′f* :: *real_normed_field*
assumes *fin*: *finite I*
and *lder*: $\bigwedge$*i. i* ∈ *I* $\Longrightarrow$ *f i log_differentiable a*
shows *logderiv* (λ*x*. ∏ *i*∈*I*. *f i x*) *a* = (∑ *i*∈*I*. *logderiv* (*f i*) *a*) (**is** *?P*)
and (λ*x*. ∏ *i*∈*I*. *f i x*) *log_differentiable a* (**is** *?Q*)
⟨*proof*⟩

lemma *logderiv_prod*:
fixes *f* :: *′n* ⇒ *′f* ⇒ *′f* :: *real_normed_field*
assumes *lder*: $\bigwedge$*i. i* ∈ *I* $\Longrightarrow$ *f i log_differentiable a*
shows *logderiv* (λ*x*. ∏ *i*∈*I*. *f i x*) *a* = (∑ *i*∈*I*. *logderiv* (*f i*) *a*) (**is** *?P*)
and (λ*x*. ∏ *i*∈*I*. *f i x*) *log_differentiable a* (**is** *?Q*)
⟨*proof*⟩

lemma *logderiv_mult*:
assumes *f log_differentiable a*
and *g log_differentiable a*
shows *logderiv* (λ*z*. *f z* ∗ *g z*) *a* = *logderiv f a* + *logderiv g a* (**is** *?P*)
and (λ*z*. *f z* ∗ *g z*) *log_differentiable a* (**is** *?Q*)
⟨*proof*⟩

lemma *logderiv_cong_ev*:
assumes $\forall_F$ *x in nhds x*. *f x = g x*
and *x = y*
shows *logderiv f x = logderiv g y*
⟨*proof*⟩

lemma *logderiv_linear*:
assumes *z* ≠ *a*
shows *logderiv* (λ*w*. *w* − *a*) *z* = *1* / (*z* − *a*)
and (λ*w*. *w* − *z*) *log_differentiable a*

⟨*proof*⟩

lemma *deriv_shift*:
  assumes *f field_differentiable at* $(a + x)$
  shows *deriv* $(\lambda t.\ f\ (a + t))\ x = $ *deriv* $f\ (a + x)$
⟨*proof*⟩

lemma *logderiv_shift*:
  assumes *f field_differentiable at* $(a + x)$
  shows *logderiv* $(\lambda t.\ f\ (a + t))\ x = $ *logderiv* $f\ (a + x)$
  ⟨*proof*⟩

lemma *logderiv_inverse*:
  assumes $x \neq 0$
  shows *logderiv* $(\lambda x.\ 1\ /\ x)\ x = -\ 1\ /\ x$
⟨*proof*⟩

lemma *logderiv_zeta_eq_zeta'*:
  assumes $s \neq 1$ *zeta* $s \neq 0$
  shows *logderiv zeta s* = *logderiv zeta' s* $-\ 1\ /\ (s - 1)$
⟨*proof*⟩

lemma *analytic_logderiv* [*analytic_intros*]:
  assumes *f analytic_on A* $\bigwedge z.\ z \in A \Longrightarrow f\ z \neq 0$
  shows $(\lambda s.$ *logderiv f s*$)$ *analytic_on A*
  ⟨*proof*⟩

1.3 Lemmas of integration and integrability

lemma *powr_has_integral*:
  fixes *a b w* :: *real*
  assumes *Hab*: $a \leq b$ **and** *Hw*: $w > 0 \wedge w \neq 1$
  shows $((\lambda x.\ w$ *powr* $x)$ *has_integral w powr b / ln w − w powr a / ln w*$)$ $\{a..b\}$
⟨*proof*⟩

lemma *powr_integrable*:
  fixes *a b w* :: *real*
  assumes *Hab*: $a \leq b$ **and** *Hw*: $w > 0 \wedge w \neq 1$
  shows $(\lambda x.\ w$ *powr* $x)$ *integrable_on* $\{a..b\}$
⟨*proof*⟩

lemma *powr_integral_bound_gt_1*:
  fixes *a b w* :: *real*
  assumes *Hab*: $a \leq b$ **and** *Hw*: $w > 1$
  shows *integral* $\{a..b\}$ $(\lambda x.\ w$ *powr* $x) \leq w$ *powr* $b\ /\ |$*ln w*$|$
⟨*proof*⟩

lemma *powr_integral_bound_lt_1*:
  fixes *a b w* :: *real*
  assumes *Hab*: $a \leq b$ **and** *Hw*: $0 < w \wedge w < 1$
  shows *integral* $\{a..b\}$ $(\lambda x.\ w$ *powr* $x) \leq w$ *powr* $a\ /\ |$*ln w*$|$
⟨*proof*⟩

lemma *set_integrableI_bounded*:
  fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
  shows $A \in$ *sets M*

$\Longrightarrow (\lambda x.\ indicator\ A\ x *_R f\ x) \in borel\_measurable\ M$
$\Longrightarrow emeasure\ M\ A < \infty$
$\Longrightarrow (AE\ x\ in\ M.\ x \in A \longrightarrow norm\ (f\ x) \le B)$
$\Longrightarrow set\_integrable\ M\ A\ f$
⟨*proof*⟩

lemma *integrable_cut'*:
fixes $a\ b\ c :: real$ **and** $f :: real \Rightarrow real$
assumes $a \le b\ b \le c$
and *Hf*: $\bigwedge x.\ a \le x \Longrightarrow f\ integrable\_on\ \{a..x\}$
shows $f\ integrable\_on\ \{b..c\}$
⟨*proof*⟩

lemma *integration_by_part'*:
fixes $a\ b :: real$
and $f\ g :: real \Rightarrow 'a :: \{real\_normed\_field, banach\}$
and $f'\ g' :: real \Rightarrow 'a$
assumes $a \le b$
and $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f\ has\_vector\_derivative\ f'\ x)\ (at\ x)$
and $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has\_vector\_derivative\ g'\ x)\ (at\ x)$
and *int*: $(\lambda x.\ f\ x * g'\ x)\ integrable\_on\ \{a..b\}$
shows $((\lambda x.\ f'\ x * g\ x)\ has\_integral$
$f\ b * g\ b - f\ a * g\ a - integral\{a..b\}\ (\lambda x.\ f\ x * g'\ x))\ \{a..b\}$
⟨*proof*⟩

lemma *integral_bigo*:
fixes $a :: real$ **and** $f\ g :: real \Rightarrow real$
assumes *f_bound*: $f \in O(g)$
and *Hf*: $\bigwedge x.\ a \le x \Longrightarrow f\ integrable\_on\ \{a..x\}$
and *Hf'*: $\bigwedge x.\ a \le x \Longrightarrow (\lambda x.\ |f\ x|)\ integrable\_on\ \{a..x\}$
and *Hg'*: $\bigwedge x.\ a \le x \Longrightarrow (\lambda x.\ |g\ x|)\ integrable\_on\ \{a..x\}$
shows $(\lambda x.\ integral\{a..x\}\ f) \in O(\lambda x.\ 1 + integral\{a..x\}\ (\lambda x.\ |g\ x|))$
⟨*proof*⟩

lemma *integral_linepath_same_Re*:
assumes *Ha*: $Re\ a = Re\ b$
and *Hb*: $Im\ a < Im\ b$
and *Hf*: $(f\ has\_contour\_integral\ x)\ (linepath\ a\ b)$
shows $((\lambda t.\ f\ (Complex\ (Re\ a)\ t) * \mathrm{i})\ has\_integral\ x)\ \{Im\ a..Im\ b\}$
⟨*proof*⟩

1.4 Lemmas on asymptotics

lemma *eventually_at_top_linorderI'*:
fixes $c :: 'a :: \{no\_top, linorder\}$
assumes *h*: $\bigwedge x.\ c < x \Longrightarrow P\ x$
shows $eventually\ P\ at\_top$
⟨*proof*⟩

lemma *eventually_le_imp_bigo*:
assumes $\forall_F\ x\ in\ F.\ \|f\ x\| \le g\ x$
shows $f \in O[F](g)$
⟨*proof*⟩

lemma *eventually_le_imp_bigo'*:
assumes $\forall_F\ x\ in\ F.\ \|f\ x\| \le g\ x$

shows $(\lambda x.\ \|f\ x\|) \in O[F](g)$
$\langle proof\rangle$

lemma *le_imp_bigo*:
assumes $\bigwedge x.\ \|f\ x\| \leq g\ x$
shows $f \in O[F](g)$
$\langle proof\rangle$

lemma *le_imp_bigo'*:
assumes $\bigwedge x.\ \|f\ x\| \leq g\ x$
shows $(\lambda x.\ \|f\ x\|) \in O[F](g)$
$\langle proof\rangle$

lemma *exp_bigo*:
fixes $f\ g :: real \Rightarrow real$
assumes $\forall_F\ x\ in\ at\_top.\ f\ x \leq g\ x$
shows $(\lambda x.\ exp\ (f\ x)) \in O(\lambda x.\ exp\ (g\ x))$
$\langle proof\rangle$

lemma *ev_le_imp_exp_bigo*:
fixes $f\ g :: real \Rightarrow real$
assumes *hf*: $\forall_F\ x\ in\ at\_top.\ 0 < f\ x$
and *hg*: $\forall_F\ x\ in\ at\_top.\ 0 < g\ x$
and *le*: $\forall_F\ x\ in\ at\_top.\ ln\ (f\ x) \leq ln\ (g\ x)$
shows $f \in O(g)$
$\langle proof\rangle$

lemma *smallo_ln_diverge_1*:
fixes $f :: real \Rightarrow real$
assumes *f_ln*: $f \in o(ln)$
shows $LIM\ x\ at\_top.\ x * exp\ (-\ f\ x) :> at\_top$
$\langle proof\rangle$

lemma *ln_ln_asymp_pos*: $\forall_F\ x :: real\ in\ at\_top.\ 0 < ln\ (ln\ x)$ $\langle proof\rangle$
lemma *ln_asymp_pos*: $\forall_F\ x :: real\ in\ at\_top.\ 0 < ln\ x$ $\langle proof\rangle$
lemma *x_asymp_pos*: $\forall_F\ x :: real\ in\ at\_top.\ 0 < x$ $\langle proof\rangle$

1.5 Lemmas of *floor*, *ceil* and *nat_powr*

lemma *nat_le_self*: $0 \leq x \Longrightarrow nat\ (int\ x) \leq x$ $\langle proof\rangle$
lemma *floor_le*: $\bigwedge x :: real.\ \lfloor x\rfloor \leq x$ $\langle proof\rangle$
lemma *ceil_ge*: $\bigwedge x :: real.\ x \leq \lceil x\rceil$ $\langle proof\rangle$

lemma *nat_lt_real_iff*:
$(n :: nat) < (a :: real) = (n < nat\ \lceil a\rceil)$
$\langle proof\rangle$

lemma *nat_le_real_iff*:
$(n :: nat) \leq (a :: real) = (n < nat\ (\lfloor a\rfloor + 1))$
$\langle proof\rangle$

lemma *of_real_nat_power*: $n\ nat\_powr\ (of\_real\ x :: complex) = of\_real\ (n\ nat\_powr\ x)$ **for** $n\ x$
$\langle proof\rangle$

lemma *norm_nat_power*: $\|n\ nat\_powr\ (s :: complex)\| = n\ powr\ (Re\ s)$
$\langle proof\rangle$

1.6 Elementary estimation of *exp* and *ln*

lemma *ln_when_ge_3*:
  1 < *ln x* **if** *3* ≤ *x* **for** *x* :: *real*
⟨*proof*⟩

lemma *exp_lemma_1*:
  fixes *x* :: *real*
  assumes *1* ≤ *x*
  shows *1* + *exp x* ≤ *exp* (*2* ∗ *x*)
⟨*proof*⟩

lemma *ln_bound_1*:
  fixes *t* :: *real*
  assumes *Ht*: *0* ≤ *t*
  shows *ln* (*14* + *4* ∗ *t*) ≤ *4* ∗ *ln* (*t* + *2*)
⟨*proof*⟩

1.7 Miscellaneous lemmas

abbreviation *fds_zeta_complex* :: *complex fds* ≡ *fds_zeta*

lemma *powr_mono_lt_1_cancel*:
  fixes *x a b* :: *real*
  assumes *Hx*: *0* < *x* ∧ *x* < *1*
  shows (*x powr a* ≤ *x powr b*) = (*b* ≤ *a*)
  ⟨*proof*⟩

abbreviation *mangoldt_real* :: _ ⇒ *real* ≡ *mangoldt*
abbreviation *mangoldt_complex* :: _ ⇒ *complex* ≡ *mangoldt*

lemma *norm_fds_mangoldt_complex*:
  ⋀*n*. ‖*fds_nth* (*fds mangoldt_complex*) *n*‖ = *mangoldt_real n* ⟨*proof*⟩

lemma *suminf_norm_bound*:
  fixes *f* :: *nat* ⇒ *'a* :: *banach*
  assumes *summable g*
    and ⋀*n*. ‖*f n*‖ ≤ *g n*
  shows ‖*suminf f*‖ ≤ (∑*n*. *g n*)
⟨*proof*⟩

lemma C_1*_gt_zero*: *0* < C_1 ⟨*proof*⟩

unbundle *no pnt_syntax*

end
theory *Relation_of_PNTs*
imports
  PNT_Remainder_Library
begin
unbundle *pnt_syntax*
unbundle *prime_counting_syntax*

2 Implication relation of many forms of prime number theorem

definition *rem_est* :: *real* ⇒ *real* ⇒ *real* ⇒ _ **where**
$rem\_est\ c\ m\ n \equiv O(\lambda\ x.\ x * exp\ (-c * ln\ x\ powr\ m * ln\ (ln\ x)\ powr\ n))$

definition *Li* :: *real* ⇒ *real* **where** $Li\ x \equiv integral\ \{2..x\}\ (\lambda x.\ 1\ /\ ln\ x)$
definition *PNT_1* **where** $PNT\_1\ c\ m\ n \equiv ((\lambda x.\ \pi\ x - Li\ x) \in rem\_est\ c\ m\ n)$
definition *PNT_2* **where** $PNT\_2\ c\ m\ n \equiv ((\lambda x.\ \vartheta\ x - x) \in rem\_est\ c\ m\ n)$
definition *PNT_3* **where** $PNT\_3\ c\ m\ n \equiv ((\lambda x.\ \psi\ x - x) \in rem\_est\ c\ m\ n)$

lemma *rem_est_compare_powr*:
fixes *c m n* :: *real*
assumes *h*: $0 < m\ m < 1$
shows $(\lambda x.\ x\ powr\ (2\ /\ 3)) \in rem\_est\ c\ m\ n$
⟨*proof*⟩

lemma *PNT_3_imp_PNT_2*:
fixes *c m n* :: *real*
assumes *h*: $0 < m\ m < 1$ **and** *PNT_3 c m n*
shows *PNT_2 c m n*
⟨*proof*⟩

definition r_1 **where** $r_1\ x \equiv \pi\ x - Li\ x$ **for** *x*
definition r_2 **where** $r_2\ x \equiv \vartheta\ x - x$ **for** *x*

lemma *pi_represent_by_theta*:
fixes *x* :: *real*
assumes $2 \le x$
shows $\pi\ x = \vartheta\ x\ /\ (ln\ x) + integral\ \{2..x\}\ (\lambda t.\ \vartheta\ t\ /\ (t * (ln\ t)^2))$
⟨*proof*⟩

lemma *Li_integrate_by_part*:
fixes *x* :: *real*
assumes $2 \le x$
shows
$(\lambda x.\ 1\ /\ (ln\ x)^2)$ *integrable_on* $\{2..x\}$
$Li\ x = x\ /\ (ln\ x) - 2\ /\ (ln\ 2) + integral\ \{2..x\}\ (\lambda t.\ 1\ /\ (ln\ t)^2)$
⟨*proof*⟩

lemma *ϑ_integrable*:
fixes *x* :: *real*
assumes $2 \le x$
shows $(\lambda t.\ \vartheta\ t\ /\ (t * (ln\ t)^2))$ *integrable_on* $\{2..x\}$
⟨*proof*⟩

lemma r_1*_represent_by_*r_2:
fixes *x* :: *real*
assumes *Hx*: $2 \le x$
shows $(\lambda t.\ r_2\ t\ /\ (t * (ln\ t)^2))$ *integrable_on* $\{2..x\}$ (**is** *?P*)
$r_1\ x = r_2\ x\ /\ (ln\ x) + 2\ /\ ln\ 2 + integral\ \{2..x\}\ (\lambda t.\ r_2\ t\ /\ (t * (ln\ t)^2))$ (**is** *?Q*)
⟨*proof*⟩

lemma *exp_integral_asymp*:
fixes *f f′* :: *real* ⇒ *real*

assumes *cf*: *continuous_on* $\{a..\}$ *f*
  and *der*: $\bigwedge x.\ a < x \Longrightarrow$ *DERIV f x :> f' x*
  and *td*: $((\lambda x.\ x * f'\ x) \longrightarrow 0)$ *at_top*
  and *f_ln*: $f \in o(ln)$
shows $(\lambda x.\ integral\ \{a..x\}\ (\lambda t.\ exp\ (-f\ t))) \sim[at\_top]\ (\lambda x.\ x * exp(-f\ x))$
⟨*proof*⟩

lemma *x_mul_exp_larger_than_const*:
  fixes *c* :: *real* **and** *g* :: *real* ⇒ *real*
  assumes *g_ln*: $g \in o(ln)$
  shows $(\lambda x.\ c) \in O(\lambda x.\ x * exp(-g\ x))$
⟨*proof*⟩

lemma *integral_bigo_exp'*:
  fixes *a* :: *real* **and** *f g g'* :: *real* ⇒ *real*
  assumes *f_bound*: $f \in O(\lambda x.\ exp(-g\ x))$
  and *Hf*: $\bigwedge x.\ a \le x \Longrightarrow$ *f integrable_on* $\{a..x\}$
  and *Hf'*: $\bigwedge x.\ a \le x \Longrightarrow (\lambda x.\ |f\ x|)$ *integrable_on* $\{a..x\}$
  and *Hg*: *continuous_on* $\{a..\}$ *g*
  and *der*: $\bigwedge x.\ a < x \Longrightarrow$ *DERIV g x :> g' x*
  and *td*: $((\lambda x.\ x * g'\ x) \longrightarrow 0)$ *at_top*
  and *g_ln*: $g \in o(ln)$
  shows $(\lambda x.\ integral\{a..x\}\ f) \in O(\lambda x.\ x * exp(-g\ x))$
⟨*proof*⟩

lemma *integral_bigo_exp*:
  fixes *a b* :: *real* **and** *f g g'* :: *real* ⇒ *real*
  assumes *le*: $a \le b$
  and *f_bound*: $f \in O(\lambda x.\ exp(-g\ x))$
  and *Hf*: $\bigwedge x.\ a \le x \Longrightarrow$ *f integrable_on* $\{a..x\}$
  and *Hf'*: $\bigwedge x.\ b \le x \Longrightarrow (\lambda x.\ |f\ x|)$ *integrable_on* $\{b..x\}$
  and *Hg*: *continuous_on* $\{b..\}$ *g*
  and *der*: $\bigwedge x.\ b < x \Longrightarrow$ *DERIV g x :> g' x*
  and *td*: $((\lambda x.\ x * g'\ x) \longrightarrow 0)$ *at_top*
  and *g_ln*:$g \in o(ln)$
  shows $(\lambda x.\ integral\ \{a..x\}\ f) \in O(\lambda x.\ x * exp(-g\ x))$
⟨*proof*⟩

lemma *integrate_*r_2*_estimate*:
  fixes *c m n* :: *real*
  assumes *hm*: $0 < m\ m < 1$
  and *h*: $r_2 \in$ *rem_est c m n*
  shows $(\lambda x.\ integral\ \{2..x\}\ (\lambda t.\ r_2\ t\ /\ (t * (ln\ t)^2))) \in$ *rem_est c m n*
⟨*proof*⟩

lemma r_2*_div_ln_estimate*:
  fixes *c m n* :: *real*
  assumes *hm*: $0 < m\ m < 1$
  and *h*: $r_2 \in$ *rem_est c m n*
  shows $(\lambda x.\ r_2\ x\ /\ (ln\ x) + 2\ /\ ln\ 2) \in$ *rem_est c m n*
⟨*proof*⟩

lemma *PNT_2_imp_PNT_1*:
  fixes *l* :: *real*
  assumes *h*: $0 < m\ m < 1$ **and** *PNT_2 c m n*

shows *PNT_1 c m n*
⟨*proof*⟩

theorem *PNT_3_imp_PNT_1*:
fixes *l* :: *real*
assumes *h* : *0 < m m < 1* **and** *PNT_3 c m n*
shows *PNT_1 c m n*
⟨*proof*⟩

hide_const (**open**) r_1 r_2
unbundle *no prime_counting_syntax* **and** *no pnt_syntax*

end
theory *PNT_Complex_Analysis_Lemmas*
imports
PNT_Remainder_Library
begin
unbundle *pnt_syntax*

3 Some basic theorems in complex analysis

3.1 Introduction rules for holomorphic functions and analytic functions

lemma *holomorphic_on_shift* [*holomorphic_intros*]:
assumes *f holomorphic_on* ((λ*z. s* + *z*) ‘ *A*)
shows (λ*z. f* (*s* + *z*)) *holomorphic_on A*
⟨*proof*⟩

lemma *holomorphic_logderiv* [*holomorphic_intros*]:
assumes *f holomorphic_on A open A* ⋀*z. z* ∈ *A* ⟹ *f z* ≠ *0*
shows (λ*s. logderiv f s*) *holomorphic_on A*
⟨*proof*⟩

lemma *holomorphic_glue_to_analytic*:
assumes *o*: *open S open T*
and *hf*: *f holomorphic_on S*
and *hg*: *g holomorphic_on T*
and *hI*: ⋀*z. z* ∈ *S* ⟹ *z* ∈ *T* ⟹ *f z* = *g z*
and *hU*: *U* ⊆ *S* ∪ *T*
obtains *h*
where *h analytic_on U*
⋀*z. z* ∈ *S* ⟹ *h z* = *f z*
⋀*z. z* ∈ *T* ⟹ *h z* = *g z*
⟨*proof*⟩

lemma *analytic_on_powr_right* [*analytic_intros*]:
assumes *f analytic_on s*
shows (λ*z. w powr f z*) *analytic_on s*
⟨*proof*⟩

3.2 Factorization of analytic function on compact region

definition *not_zero_on* (**infixr** ‹*not′_zero′_on*› *46*)
where *f not_zero_on S* ≡ ∃ *z* ∈ *S*. *f z* ≠ *0*

lemma *not_zero_on_obtain*:
  assumes f *not_zero_on* S **and** $S \subseteq T$
  obtains t **where** $f\ t \neq 0$ **and** $t \in T$
⟨*proof*⟩

lemma *analytic_on_holomorphic_connected*:
  assumes *hf*: f *analytic_on* S
    and *con*: *connected* A
    and *ne*: $\xi \in A$ **and** *AS*: $A \subseteq S$
  obtains $T\ T'$ **where**
    f *holomorphic_on* T f *holomorphic_on* T'
    open T *open* T' $A \subseteq T$ $S \subseteq T'$ *connected* T
⟨*proof*⟩

lemma *analytic_factor_zero*:
  assumes *hf*: f *analytic_on* S
    and *KS*: $K \subseteq S$ **and** *con*: *connected* K
    and ξK: $\xi \in K$ **and** ξz: $f\ \xi = 0$
    and *nz*: f *not_zero_on* K
  obtains $g\ r\ n$
    where $0 < n$ $0 < r$
      g *analytic_on* S g *not_zero_on* K
      $\bigwedge z.\ z \in S \Longrightarrow f\ z = (z - \xi)\hat{}n * g\ z$
      $\bigwedge z.\ z \in \mathit{ball}\ \xi\ r \Longrightarrow g\ z \neq 0$
⟨*proof*⟩

lemma *analytic_compact_finite_zeros*:
  assumes *af*: f *analytic_on* S
    and *KS*: $K \subseteq S$
    and *con*: *connected* K
    and *cm*: *compact* K
    and *nz*: f *not_zero_on* K
  shows *finite* $\{z \in K.\ f\ z = 0\}$
⟨*proof*⟩

3.2.1 Auxiliary propositions for theorem *analytic_factorization*

definition *analytic_factor_p'* **where**
‹*analytic_factor_p'* $f\ S\ K \equiv$
  $\exists\, g\ n.\ \exists\, \alpha :: nat \Rightarrow complex.$
      g *analytic_on* S
    $\wedge\ (\forall\, z \in K.\ g\ z \neq 0)$
    $\wedge\ (\forall\, z \in S.\ f\ z = g\ z * (\prod k{<}n.\ z - \alpha\ k))$
    $\wedge\ \alpha$ ‘ $\{..{<}n\} \subseteq K$›

definition *analytic_factor_p* **where**
‹*analytic_factor_p* $F \equiv$
  $\forall\, f\ S\ K.$ f *analytic_on* S
    $\longrightarrow K \subseteq S$
    $\longrightarrow$ *connected* K
    $\longrightarrow$ *compact* K
    $\longrightarrow$ f *not_zero_on* K
    $\longrightarrow \{z \in K.\ f\ z = 0\} = F$
    $\longrightarrow$ *analytic_factor_p'* $f\ S\ K$›

lemma *analytic_factorization_E*:

shows *analytic_factor_p* {}
⟨*proof*⟩

lemma *analytic_factorization_I*:
assumes *ind*: *analytic_factor_p F*
and *ξni*: $\xi \notin F$
shows *analytic_factor_p* (*insert ξ F*)
⟨*proof*⟩

A nontrivial analytic function on connected compact region can be factorized as a everywhere-non-zero function and linear terms $z - s_0$ for all zeros s_0. Note that the connected assumption of K may be removed, but we remain it just for simplicity of proof.

theorem *analytic_factorization*:
assumes *af*: *f analytic_on S*
and *KS*: $K \subseteq S$
and *con*: *connected K*
and *compact K*
and *f not_zero_on K*
obtains *g n* **and** α :: *nat* ⇒ *complex* **where**
g analytic_on S
$\bigwedge z.\ z \in K \Longrightarrow g\ z \neq 0$
$\bigwedge z.\ z \in S \Longrightarrow f\ z = g\ z * (\prod k<n.\ (z - \alpha\ k))$
α ‘ {..<n} $\subseteq K$
⟨*proof*⟩

3.3 Schwarz theorem in complex analysis

lemma *Schwarz_Lemma1*:
fixes *f* :: *complex* ⇒ *complex*
and ξ :: *complex*
assumes *f holomorphic_on ball 0 1*
and $f\ 0 = 0$
and $\bigwedge z.\ \|z\| < 1 \Longrightarrow \|f\ z\| \le 1$
and $\|\xi\| < 1$
shows $\|f\ \xi\| \le \|\xi\|$
⟨*proof*⟩

theorem *Schwarz_Lemma2*:
fixes *f* :: *complex* ⇒ *complex*
and ξ :: *complex*
assumes *holf*: *f holomorphic_on ball 0 R*
and *hR*: $0 < R$ **and** *nz*: $f\ 0 = 0$
and *bn*: $\bigwedge z.\ \|z\| < R \Longrightarrow \|f\ z\| \le 1$
and *ξR*: $\|\xi\| < R$
shows $\|f\ \xi\| \le \|\xi\|\ /\ R$
⟨*proof*⟩

3.4 Borel-Carathedory theorem

Borel-Carathedory theorem, from book *Theorem 5.5*, *The Theory of Functions*, *E. C. Titchmarsh*

lemma *Borel_Caratheodory1*:
assumes *hr*: $0 < R\ 0 < r\ r < R$
and *f0*: $f\ 0 = 0$
and *hf*: $\bigwedge z.\ \|z\| < R \Longrightarrow Re\ (f\ z) \le A$
and *holf*: *f holomorphic_on* (*ball 0 R*)

and *zr*: $\|z\| \le r$
shows $\|f\ z\| \le 2{*}r/(R{-}r) * A$
⟨*proof*⟩

lemma *Borel_Caratheodory2*:
assumes *hr*: $0 < R$ $0 < r$ $r < R$
and *hf*: $\bigwedge z.\ \|z\| < R \Longrightarrow Re\ (f\ z - f\ 0) \le A$
and *holf*: *f holomorphic_on* (*ball 0 R*)
and *zr*: $\|z\| \le r$
shows $\|f\ z - f\ 0\| \le 2{*}r/(R{-}r) * A$
⟨*proof*⟩

theorem *Borel_Caratheodory3*:
assumes *hr*: $0 < R$ $0 < r$ $r < R$
and *hf*: $\bigwedge w.\ w \in ball\ s\ R \Longrightarrow Re\ (f\ w - f\ s) \le A$
and *holf*: *f holomorphic_on* (*ball s R*)
and *zr*: $z \in ball\ s\ r$
shows $\|f\ z - f\ s\| \le 2{*}r/(R{-}r) * A$
⟨*proof*⟩

3.5 Lemma 3.9

These lemmas is referred to the following material: Theorem 3.9, *The Theory of the Riemann Zeta−Function*, *E. C. Titchmarsh*, *D. R. Heath−Brown*.

lemma *lemma_3_9_beta1*:
fixes $f\ M\ r\ s_0$
assumes *zl*: $0 < r$ $0 \le M$
and *hf*: *f holomorphic_on ball 0 r*
and *ne*: $\bigwedge z.\ z \in ball\ 0\ r \Longrightarrow f\ z \ne 0$
and *bn*: $\bigwedge z.\ z \in ball\ 0\ r \Longrightarrow \|f\ z\ /\ f\ 0\| \le exp\ M$
shows $\|logderiv\ f\ 0\| \le 4 * M\ /\ r$
and $\forall s{\in}cball\ 0\ (r\ /\ 4).\ \|logderiv\ f\ s\| \le 8 * M\ /\ r$
⟨*proof*⟩

lemma *lemma_3_9_beta1'*:
fixes $f\ M\ r\ s_0$
assumes *zl*: $0 < r$ $0 \le M$
and *hf*: *f holomorphic_on ball s r*
and *ne*: $\bigwedge z.\ z \in ball\ s\ r \Longrightarrow f\ z \ne 0$
and *bn*: $\bigwedge z.\ z \in ball\ s\ r \Longrightarrow \|f\ z\ /\ f\ s\| \le exp\ M$
and *hs*: $z \in cball\ s\ (r\ /\ 4)$
shows $\|logderiv\ f\ z\| \le 8 * M\ /\ r$
⟨*proof*⟩

lemma *lemma_3_9_beta2*:
fixes *f M r*
assumes *zl*: $0 < r$ $0 \le M$
and *af*: *f analytic_on cball 0 r*
and *f0*: $f\ 0 \ne 0$
and *rz*: $\bigwedge z.\ z \in cball\ 0\ r \Longrightarrow Re\ z > 0 \Longrightarrow f\ z \ne 0$
and *bn*: $\bigwedge z.\ z \in cball\ 0\ r \Longrightarrow \|f\ z\ /\ f\ 0\| \le exp\ M$
and *hg*: $\Gamma \subseteq \{z \in cball\ 0\ (r\ /\ 2).\ f\ z = 0\}$
shows $-\ Re\ (logderiv\ f\ 0) \le 8 * M\ /\ r + Re\ (\sum z{\in}\Gamma.\ 1\ /\ z)$
⟨*proof*⟩

theorem *lemma_3_9_beta3*:
fixes *f M r* **and** *s* :: *complex*
assumes *zl*: $0 < r$ $0 \le M$
and *af*: *f analytic_on cball s r*
and *f0*: $f\ s \neq 0$
and *rz*: $\bigwedge z.\ z \in cball\ s\ r \Longrightarrow Re\ z > Re\ s \Longrightarrow f\ z \neq 0$
and *bn*: $\bigwedge z.\ z \in cball\ s\ r \Longrightarrow \|f\ z\ /\ f\ s\| \le exp\ M$
and *hg*: $\Gamma \subseteq \{z \in cball\ s\ (r\ /\ 2).\ f\ z = 0\}$
shows $-\ Re\ (logderiv\ f\ s) \le 8 * M\ /\ r + Re\ (\sum z{\in}\Gamma.\ 1\ /\ (z - s))$
⟨*proof*⟩

unbundle *no pnt_syntax*

end
theory *Zeta_Zerofree*
imports
PNT_Complex_Analysis_Lemmas
begin
unbundle *pnt_syntax*

4 Zero-free region of zeta function

lemma *cos_inequality_1*:
fixes *x* :: *real*
shows $3 + 4 * cos\ x + cos\ (2 * x) \ge 0$
⟨*proof*⟩

lemma *multiplicative_fds_zeta*:
completely_multiplicative_function (*fds_nth fds_zeta_complex*)
⟨*proof*⟩

lemma *fds_mangoldt_eq*:
fds mangoldt_complex = −(*fds_deriv fds_zeta* / *fds_zeta*)
⟨*proof*⟩

lemma *abs_conv_abscissa_log_deriv*:
abs_conv_abscissa (*fds_deriv fds_zeta_complex* / *fds_zeta*) ≤ *1*
⟨*proof*⟩

lemma *abs_conv_abscissa_mangoldt*:
abs_conv_abscissa (*fds mangoldt_complex*) ≤ *1*
⟨*proof*⟩

lemma
assumes *s*: $Re\ s > 1$
shows *eval_fds_mangoldt*: *eval_fds* (*fds mangoldt*) *s* = − *deriv zeta s* / *zeta s*
and *abs_conv_mangoldt*: *fds_abs_converges* (*fds mangoldt*) *s*
⟨*proof*⟩

lemma *sums_mangoldt*:
fixes *s* :: *complex*
assumes *s*: $Re\ s > 1$
shows (($\lambda n.$ *mangoldt n* / *n nat_powr s*) *has_sum* − *deriv zeta s* / *zeta s*) {*1*..}
⟨*proof*⟩

lemma *sums_Re_logderiv_zeta*:
fixes σ t :: *real*
assumes *s*: $\sigma > 1$
shows ((λn. *mangoldt_real n* $*$ *n nat_powr* $(-\sigma)$ $*$ *cos* $(t * \ln n)$)
has_sum Re ($-$ *deriv zeta* (*Complex* σ *t*) / *zeta* (*Complex* σ *t*))) $\{1..\}$
⟨*proof*⟩

lemma *logderiv_zeta_ineq*:
fixes σ t :: *real*
assumes *s*: $\sigma > 1$
shows $3 *$ *Re* (*logderiv zeta* (*Complex* σ *0*)) $+ 4 *$ *Re* (*logderiv zeta* (*Complex* σ *t*))
$+$ *Re* (*logderiv zeta* (*Complex* σ $(2*t)$)) ≤ 0 (**is** $?x \leq 0$)
⟨*proof*⟩

lemma *sums_zeta_real*:
fixes *r* :: *real*
assumes $1 < r$
shows ($\sum n$. (n_+) *powr* $-r$) $=$ *Re* (*zeta r*)
⟨*proof*⟩

lemma *inverse_zeta_bound'*:
assumes $1 <$ *Re s*
shows $\|$*inverse* (*zeta s*)$\| \leq$ *Re* (*zeta* (*Re s*))
⟨*proof*⟩

lemma *zeta_bound'*:
assumes $1 <$ *Re s*
shows $\|$*zeta s*$\| \leq$ *Re* (*zeta* (*Re s*))
⟨*proof*⟩

lemma *zeta_bound_trivial'*:
assumes $1 / 2 \leq$ *Re s* $\wedge$ *Re s* ≤ 2
and $|$*Im s*$| \geq 1 / 11$
shows $\|$*zeta s*$\| \leq 12 + 2 * |$*Im s*$|$
⟨*proof*⟩

lemma *zeta_bound_gt_1*:
assumes $1 <$ *Re s*
shows $\|$*zeta s*$\| \leq$ *Re s* / (*Re s* $- 1$)
⟨*proof*⟩

lemma *zeta_bound_trivial*:
assumes $1 / 2 \leq$ *Re s* **and** $|$*Im s*$| \geq 1 / 11$
shows $\|$*zeta s*$\| \leq 12 + 2 * |$*Im s*$|$
⟨*proof*⟩

lemma *zeta_nonzero_small_imag'*:
assumes $|$*Im s*$| \leq 13 / 22$ **and** *Re s* $\geq 1 / 2$ **and** *Re s* < 1
shows *zeta s* $\neq 0$
⟨*proof*⟩

lemma *zeta_nonzero_small_imag*:
assumes $|$*Im s*$| \leq 13 / 22$ **and** *Re s* > 0 **and** $s \neq 1$
shows *zeta s* $\neq 0$
⟨*proof*⟩

lemma *inverse_zeta_bound*:
  assumes $1 < Re\ s$
  shows $\|inverse\ (zeta\ s)\| \le Re\ s\ /\ (Re\ s - 1)$
⟨*proof*⟩

lemma *deriv_zeta_bound*:
  fixes s :: *complex*
  assumes *Hr*: $0 < r$ **and** *Hs*: $s \neq 1$
    and *hB*: $\bigwedge w.\ \|s - w\| = r \Longrightarrow \|pre\_zeta\ 1\ w\| \le B$
  shows $\|deriv\ zeta\ s\| \le B\ /\ r + 1\ /\ \|s - 1\|^2$
⟨*proof*⟩

lemma *zeta_lower_bound*:
  assumes $0 < Re\ s\ s \neq 1$
  shows $1\ /\ \|s - 1\| - \|s\|\ /\ Re\ s \le \|zeta\ s\|$
⟨*proof*⟩

lemma *logderiv_zeta_bound*:
  fixes σ :: *real*
  assumes $1 < \sigma\ \sigma \le 23\ /\ 20$
  shows $\|logderiv\ zeta\ \sigma\| \le 5\ /\ 4 * (1\ /\ (\sigma - 1))$
⟨*proof*⟩

lemma *Re_logderiv_zeta_bound*:
  fixes σ :: *real*
  assumes $1 < \sigma\ \sigma \le 23\ /\ 20$
  shows $Re\ (logderiv\ zeta\ \sigma) \ge -\ 5\ /\ 4 * (1\ /\ (\sigma - 1))$
⟨*proof*⟩

locale *zeta_bound_param* =
  fixes $\vartheta\ \varphi$ :: $real \Rightarrow real$
  assumes *zeta_bn′*: $\bigwedge z.\ 1 - \vartheta\ (Im\ z) \le Re\ z \Longrightarrow Im\ z \ge 1\ /\ 11 \Longrightarrow \|zeta\ z\| \le exp\ (\varphi\ (Im\ z))$
    and ϑ*_pos*: $\bigwedge t.\ 0 < \vartheta\ t \wedge \vartheta\ t \le 1\ /\ 2$
    and φ*_pos*: $\bigwedge t.\ 1 \le \varphi\ t$
    and *inv_*ϑ: $\bigwedge t.\ \varphi\ t\ /\ \vartheta\ t \le 1\ /\ 960 * exp\ (\varphi\ t)$
    and *mo*ϑ: *antimono* ϑ **and** *mo*φ: *mono* φ
begin
  definition $region \equiv \{z.\ 1 - \vartheta\ (Im\ z) \le Re\ z \wedge Im\ z \ge 1\ /\ 11\}$
  lemma *zeta_bn*: $\bigwedge z.\ z \in region \Longrightarrow \|zeta\ z\| \le exp\ (\varphi\ (Im\ z))$
    ⟨*proof*⟩
  lemma ϑ*_pos′*: $\bigwedge t.\ 0 < \vartheta\ t \wedge \vartheta\ t \le 1$
    ⟨*proof*⟩
  lemma φ*_pos′*: $\bigwedge t.\ 0 < \varphi\ t$ ⟨*proof*⟩
end

locale *zeta_bound_param_1* = *zeta_bound_param* +
  fixes γ :: *real*
  assumes γ*_cnd*: $\gamma \ge 13\ /\ 22$
begin
  definition r **where** $r \equiv \vartheta\ (2 * \gamma + 1)$
end

locale *zeta_bound_param_2* = *zeta_bound_param_1* +
  fixes $\sigma\ \delta$ :: *real*

assumes σ_cnd: $\sigma \geq 1 + exp\ (-\ \varphi(2 * \gamma + 1))$
and δ_cnd: $\delta = \gamma \vee \delta = 2 * \gamma$
begin
definition s **where** $s \equiv Complex\ \sigma\ \delta$
end

context *zeta_bound_param_2* **begin**
declare *dist_complex_def* [*simp*] *norm_minus_commute* [*simp*]
declare *legacy_Complex_simps* [*simp*]

lemma *cball_lm*:
assumes $z \in cball\ s\ r$
shows $r \leq 1$ $|Re\ z - \sigma| \leq r$ $|Im\ z - \delta| \leq r$
$1\ /\ 11 \leq Im\ z$ $Im\ z \leq 2 * \gamma + r$
⟨*proof*⟩

lemma *cball_in_region*:
shows $cball\ s\ r \subseteq region$
⟨*proof*⟩

lemma *Re_s_gt_1*:
shows $1 < Re\ s$
⟨*proof*⟩

lemma *zeta_analytic_on_region*:
shows *zeta analytic_on region*
⟨*proof*⟩

lemma *zeta_div_bound*:
assumes $z \in cball\ s\ r$
shows $\|zeta\ z\ /\ zeta\ s\| \leq exp\ (3 * \varphi\ (2 * \gamma + 1))$
⟨*proof*⟩

lemma *logderiv_zeta_bound*:
shows $Re\ (logderiv\ zeta\ s) \geq -\ 24 * \varphi\ (2 * \gamma + 1)\ /\ r$
and $\bigwedge\beta.\ \sigma - r\ /\ 2 \leq \beta \Longrightarrow zeta\ (Complex\ \beta\ \delta) = 0 \Longrightarrow$
$Re\ (logderiv\ zeta\ s) \geq -\ 24 * \varphi\ (2 * \gamma + 1)\ /\ r + 1\ /\ (\sigma - \beta)$
⟨*proof*⟩
end

context *zeta_bound_param_1* **begin**
lemma *zeta_nonzero_region'*:
assumes $1 + 1\ /\ 960 * (r\ /\ \varphi\ (2 * \gamma + 1)) - r\ /\ 2 \leq \beta$
and $zeta\ (Complex\ \beta\ \gamma) = 0$
shows $1 - \beta \geq 1\ /\ 29760 * (r\ /\ \varphi\ (2 * \gamma + 1))$
⟨*proof*⟩

lemma *zeta_nonzero_region*:
assumes $zeta\ (Complex\ \beta\ \gamma) = 0$
shows $1 - \beta \geq 1\ /\ 29760 * (r\ /\ \varphi\ (2 * \gamma + 1))$
⟨*proof*⟩
end

context *zeta_bound_param* **begin**
theorem *zeta_nonzero_region*:

assumes *zeta* (*Complex* β γ) = *0* **and** *Complex* β $\gamma \neq 1$
shows $1 - \beta \geq 1 \ / \ 29760 * (\vartheta\ (2 * |\gamma| + 1) \ / \ \varphi\ (2 * |\gamma| + 1))$
⟨*proof*⟩
end

lemma *zeta_bound_param_nonneg*:
fixes ϑ φ :: *real* ⇒ *real*
assumes *zeta_bn'*: $\bigwedge z.\ 1 - \vartheta\ (Im\ z) \leq Re\ z \Longrightarrow Im\ z \geq 1\ /\ 11 \Longrightarrow \|zeta\ z\| \leq exp\ (\varphi\ (Im\ z))$
and *ϑ_pos*: $\bigwedge t.\ 0 \leq t \Longrightarrow 0 < \vartheta\ t \wedge \vartheta\ t \leq 1\ /\ 2$
and *φ_pos*: $\bigwedge t.\ 0 \leq t \Longrightarrow 1 \leq \varphi\ t$
and *inv_ϑ*: $\bigwedge t.\ 0 \leq t \Longrightarrow \varphi\ t\ /\ \vartheta\ t \leq 1\ /\ 960 * exp\ (\varphi\ t)$
and *moϑ*: $\bigwedge x\ y.\ 0 \leq x \Longrightarrow x \leq y \Longrightarrow \vartheta\ y \leq \vartheta\ x$
and *moφ*: $\bigwedge x\ y.\ 0 \leq x \Longrightarrow x \leq y \Longrightarrow \varphi\ x \leq \varphi\ y$
shows *zeta_bound_param* (λ*t*. ϑ (*max 0 t*)) (λ*t*. φ (*max 0 t*))
⟨*proof*⟩

interpretation *classical_zeta_bound*:
zeta_bound_param λ*t*. *1* / *2* λ*t*. *4* ∗ *ln* (*12* + *2* ∗ *max 0 t*)
⟨*proof*⟩

theorem *zeta_nonzero_region*:
assumes *zeta* (*Complex* β γ) = *0* **and** *Complex* β $\gamma \neq 1$
shows $1 - \beta \geq C_1\ /\ ln\ (|\gamma| + 2)$
⟨*proof*⟩

unbundle *no pnt_syntax*

end
theory *PNT_Subsummable*
imports
PNT_Remainder_Library
begin
unbundle *pnt_syntax*

definition *has_subsum* **where** *has_subsum f S x* ≡ (λ*n*. *if n* ∈ *S then f n else 0*) *sums x*
definition *subsum* **where** *subsum f S* ≡ ∑*n*. *if n* ∈ *S then f n else 0*
definition *subsummable* (**infix** ‹*subsummable*› *50*)
where *f subsummable S* ≡ *summable* (λ*n*. *if n* ∈ *S then f n else 0*)

syntax *_subsum* :: *pttrn* ⇒ *nat set* ⇒ *'a* ⇒ *'a*
(‹(*2*∑ '_ ∈ (_)./ _)› [*0*, *0*, *10*] *10*)
syntax_consts *_subsum* == *subsum*
translations
∑ ' *x*∈*S*. *t* => *CONST subsum* (λ*x*. *t*) *S*

syntax *_subsum_prop* :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a*
(‹(*2*∑ '_ | (_)./ _)› [*0*, *0*, *10*] *10*)
syntax_consts *_subsum_prop* == *subsum*
translations
∑ ' *x*|*P*. *t* => *CONST subsum* (λ*x*. *t*) {*x*. *P*}

syntax *_subsum_ge* :: *pttrn* ⇒ *nat* ⇒ *'a* ⇒ *'a*
(‹(*2*∑ '_ ≥ _./ _)› [*0*, *0*, *10*] *10*)
syntax_consts *_subsum_ge* == *subsum*
translations

$\sum$ ' $x \geq n.\ t$ => *CONST subsum* ($\lambda x.\ t$) {n..}

lemma *has_subsum_finite*:
finite F ⟹ *has_subsum f F* (*sum f F*)
⟨*proof*⟩

lemma *has_subsum_If_finite_set*:
assumes *finite F*
shows *has_subsum* (λn. *if* $n \in F$ *then f n else 0*) *A* (*sum f* ($F \cap A$))
⟨*proof*⟩

lemma *has_subsum_If_finite*:
assumes *finite* {$n \in A$. *p n*}
shows *has_subsum* (λn. *if p n then f n else 0*) *A* (*sum f* {$n \in A$. *p n*})
⟨*proof*⟩

lemma *has_subsum_univ*:
f sums v ⟹ *has_subsum f UNIV v*
⟨*proof*⟩

lemma *subsumI*:
fixes *f* :: *nat* ⇒ *'a* :: {*t2_space*, *comm_monoid_add*}
shows *has_subsum f A x* ⟹ *x* = *subsum f A*
⟨*proof*⟩

lemma *has_subsum_summable*:
has_subsum f A x ⟹ *f subsummable A*
⟨*proof*⟩

lemma *subsummable_sums*:
fixes *f* :: *nat* ⇒ *'a* :: {*comm_monoid_add*, *t2_space*}
shows *f subsummable S* ⟹ *has_subsum f S* (*subsum f S*)
⟨*proof*⟩

lemma *has_subsum_diff_finite*:
fixes *S* :: *'a* :: {*topological_ab_group_add*, *t2_space*}
assumes *finite F has_subsum f A S F* ⊆ *A*
shows *has_subsum f* (*A* − *F*) (*S* − *sum f F*)
⟨*proof*⟩

lemma *subsum_split*:
fixes *f* :: *nat* ⇒ *'a* :: {*topological_ab_group_add*, *t2_space*}
assumes *f subsummable A finite F F* ⊆ *A*
shows *subsum f A* = *sum f F* + *subsum f* (*A* − *F*)
⟨*proof*⟩

lemma *has_subsum_zero* [*simp*]: *has_subsum* (λn. *0*) *A 0* ⟨*proof*⟩
lemma *zero_subsummable* [*simp*]: (λn. *0*) *subsummable A* ⟨*proof*⟩
lemma *zero_subsum* [*simp*]: ($\sum$ '$n \in A$. *0* :: *'a* :: {*comm_monoid_add*, *t2_space*}) = *0* ⟨*proof*⟩

lemma *has_subsum_minus*:
fixes *f* :: *nat* ⇒ *'a* :: *real_normed_vector*
assumes *has_subsum f A a has_subsum g A b*
shows *has_subsum* (λn. *f n* − *g n*) *A* (*a* − *b*)
⟨*proof*⟩

lemma *subsum_minus*:
  assumes *f subsummable A g subsummable A*
  shows $subsum\ f\ A - subsum\ g\ A = (\sum\text{`}n{\in}A.\ f\ n - g\ n :: {'}a :: real\_normed\_vector)$
  ⟨*proof*⟩

lemma *subsummable_minus*:
  assumes *f subsummable A g subsummable A*
  shows $(\lambda n.\ f\ n - g\ n :: {'}a :: real\_normed\_vector)\ subsummable\ A$
  ⟨*proof*⟩

lemma *has_subsum_uminus*:
  assumes *has_subsum f A a*
  shows $has\_subsum\ (\lambda n. - f\ n :: {'}a :: real\_normed\_vector)\ A\ (- a)$
⟨*proof*⟩

lemma *subsum_uminus*:
  $f\ subsummable\ A \Longrightarrow - subsum\ f\ A = (\sum\text{`}n{\in}A. - f\ n :: {'}a :: real\_normed\_vector)$
  ⟨*proof*⟩

lemma *subsummable_uminus*:
  $f\ subsummable\ A \Longrightarrow (\lambda n. - f\ n :: {'}a :: real\_normed\_vector)\ subsummable\ A$
  ⟨*proof*⟩

lemma *has_subsum_add*:
  fixes $f :: nat \Rightarrow {'}a :: real\_normed\_vector$
  assumes *has_subsum f A a has_subsum g A b*
  shows $has\_subsum\ (\lambda n.\ f\ n + g\ n)\ A\ (a + b)$
⟨*proof*⟩

lemma *subsum_add*:
  assumes *f subsummable A g subsummable A*
  shows $subsum\ f\ A + subsum\ g\ A = (\sum\text{`}n{\in}A.\ f\ n + g\ n :: {'}a :: real\_normed\_vector)$
  ⟨*proof*⟩

lemma *subsummable_add*:
  assumes *f subsummable A g subsummable A*
  shows $(\lambda n.\ f\ n + g\ n :: {'}a :: real\_normed\_vector)\ subsummable\ A$
  ⟨*proof*⟩

lemma *subsum_cong*:
  $(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow subsum\ f\ A = subsum\ g\ A$
  ⟨*proof*⟩

lemma *subsummable_cong*:
  fixes $f :: nat \Rightarrow {'}a :: real\_normed\_vector$
  shows $(\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x) \Longrightarrow (f\ subsummable\ A) = (g\ subsummable\ A)$
  ⟨*proof*⟩

lemma *subsum_norm_bound*:
  fixes $f :: nat \Rightarrow {'}a :: banach$
  assumes $g\ subsummable\ A\ \bigwedge n.\ n \in A \Longrightarrow \|f\ n\| \le g\ n$
  shows $\|subsum\ f\ A\| \le subsum\ g\ A$
  ⟨*proof*⟩

lemma *eval_fds_subsum*:
fixes $f :: 'a :: \{nat\_power, banach, real\_normed\_field\}$ *fds*
assumes *fds_converges f s*
shows *has_subsum* $(\lambda n.\ fds\_nth\ f\ n\ /\ nat\_power\ n\ s)\ \{1..\}\ (eval\_fds\ f\ s)$
⟨*proof*⟩

lemma *fds_abs_subsummable*:
fixes $f :: 'a :: \{nat\_power, banach, real\_normed\_field\}$ *fds*
assumes *fds_abs_converges f s*
shows $(\lambda n.\ \|fds\_nth\ f\ n\ /\ nat\_power\ n\ s\|)$ *subsummable* $\{1..\}$
⟨*proof*⟩

lemma *subsum_mult2*:
fixes $f :: nat \Rightarrow 'a :: real\_normed\_algebra$
shows *f subsummable A* $\Longrightarrow (\sum`x{\in}A.\ f\ x * c) = subsum\ f\ A * c$
⟨*proof*⟩

lemma *subsummable_mult2*:
fixes $f :: nat \Rightarrow 'a :: real\_normed\_algebra$
assumes *f subsummable A*
shows $(\lambda x.\ f\ x * c)$ *subsummable A*
⟨*proof*⟩

lemma *subsum_ge_limit*:
$lim\ (\lambda N.\ \sum n = m..N.\ f\ n) = (\sum`n \geq m.\ f\ n)$
⟨*proof*⟩

lemma *has_subsum_ge_limit*:
fixes $f :: nat \Rightarrow 'a :: \{t2\_space, comm\_monoid\_add, topological\_space\}$
assumes $((\lambda N.\ \sum n = m..N.\ f\ n) \longrightarrow l)\ at\_top$
shows *has_subsum f* $\{m..\}$ *l*
⟨*proof*⟩

lemma *eval_fds_complex*:
fixes $f ::$ *complex fds*
assumes *fds_converges f s*
shows *has_subsum* $(\lambda n.\ fds\_nth\ f\ n\ /\ n\ nat\_powr\ s)\ \{1..\}\ (eval\_fds\ f\ s)$
⟨*proof*⟩

lemma *eval_fds_complex_subsum*:
fixes $f ::$ *complex fds*
assumes *fds_converges f s*
shows $eval\_fds\ f\ s = (\sum`n \geq 1.\ fds\_nth\ f\ n\ /\ n\ nat\_powr\ s)$
$(\lambda n.\ fds\_nth\ f\ n\ /\ n\ nat\_powr\ s)$ *subsummable* $\{1..\}$
⟨*proof*⟩

lemma *has_sum_imp_has_subsum*:
fixes $x :: 'a :: \{comm\_monoid\_add, t2\_space\}$
assumes $(f\ has\_sum\ x)\ A$
shows *has_subsum f A x*
⟨*proof*⟩

unbundle *no pnt_syntax*

end

theory *Perron_Formula*
imports
PNT_Remainder_Library
PNT_Subsummable
begin
unbundle *pnt_syntax*

5 Perron's formula

This version of Perron's theorem is referenced to: *Perron′s Formula and the Prime Number Theorem for Automorphic L−Functions, Jianya Liu, Y. Ye*

A contour integral estimation lemma that will be used both in proof of Perron's formula and the prime number theorem.

```
lemma perron_aux_3':
  fixes f :: complex ⇒ complex and a b B T :: real
  assumes Ha: 0 < a and Hb: 0 < b and hT: 0 < T
    and Hf: ⋀t. t ∈ {−T..T} ⟹ ‖f (Complex b t)‖ ≤ B
    and Hf': (λs. f s ∗ a powr s / s) contour_integrable_on (linepath (Complex b (−T)) (Complex b T))
  shows ‖1 / (2 ∗ pi ∗ i) ∗ contour_integral (linepath (Complex b (−T)) (Complex b T)) (λs. f s ∗ a powr
s / s)‖
      ≤ B ∗ a powr b ∗ ln (1 + T / b)
⟨proof⟩

locale perron_locale =
  fixes b B H T x :: real and f :: complex fds
  assumes Hb: 0 < b and hT: b ≤ T
    and Hb': abs_conv_abscissa f < b
    and hH: 1 < H and hH': b + 1 ≤ H and Hx: 0 < x
    and hB: (∑`n ≥ 1. ‖fds_nth f n‖ / n nat_powr b) ≤ B
begin
definition r where r a ≡
  if a ≠ 1 then min (1 / (2 ∗ T ∗ |ln a|)) (2 + ln (T / b))
  else (2 + ln (T / b))
definition path where path ≡ linepath (Complex b (−T)) (Complex b T)
definition img_path where img_path ≡ path_image path
definition σa where σa ≡ abs_conv_abscissa f
definition region where region = {n :: nat. x − x / H ≤ n ∧ n ≤ x + x / H}
definition F where F (a :: real) ≡
  1 / (2 ∗ pi ∗ i) ∗ contour_integral path (λs. a powr s / s) − (if 1 ≤ a then 1 else 0)
definition F' where F' (n :: nat) ≡ F (x / n)

lemma hT': 0 < T ⟨proof⟩
lemma cond: 0 < b b ≤ T 0 < T ⟨proof⟩

lemma perron_integrable:
  assumes (0 :: real) < a
  shows (λs. a powr s / s) contour_integrable_on (linepath (Complex b (−T)) (Complex b T))
⟨proof⟩

lemma perron_aux_1':
  fixes U :: real
  assumes hU: 0 < U and Ha: 1 < a
  shows ‖F a‖ ≤ 1 / pi ∗ a powr b / (T ∗ |ln a|) + a powr −U ∗ T / (pi ∗ U)
⟨proof⟩
```

lemma *perron_aux_1*:
  assumes *Ha*: *1 < a*
  shows ‖*F a*‖ ≤ *1 / pi ∗ a powr b / (T ∗ |ln a|)* (**is** _ ≤ *?x*)
⟨*proof*⟩

lemma *perron_aux_2′*:
  fixes *U* :: *real*
  assumes *hU*: *0 < U b < U* **and** *Ha*: *0 < a ∧ a < 1*
  shows ‖*F a*‖ ≤ *1 / pi ∗ a powr b / (T ∗ |ln a|) + a powr U ∗ T / (pi ∗ U)*
⟨*proof*⟩

lemma *perron_aux_2*:
  assumes *Ha*: *0 < a ∧ a < 1*
  shows ‖*F a*‖ ≤ *1 / pi ∗ a powr b / (T ∗ |ln a|)* (**is** _ ≤ *?x*)
⟨*proof*⟩

lemma *perron_aux_3*:
  assumes *Ha*: *0 < a*
  shows ‖*1 / (2 ∗ pi ∗* i*) ∗ contour_integral path (λs. a powr s / s)*‖ ≤ *a powr b ∗ ln (1 + T / b)*
⟨*proof*⟩

lemma *perron_aux′*:
  assumes *Ha*: *0 < a*
  shows ‖*F a*‖ ≤ *a powr b ∗ r a*
⟨*proof*⟩

lemma *r_bound*:
  assumes *Hn*: *1 ≤ n*
  shows *r (x / n) ≤ H / T + (if n ∈ region then 2 + ln (T / b) else 0)*
⟨*proof*⟩

lemma *perron_aux*:
  assumes *Hn*: *0 < n*
  shows ‖*F′ n*‖ ≤ *1 / n nat_powr b ∗ (x powr b ∗ H / T)*
    + (if n ∈ region then 3 ∗ (2 + ln (T / b)) else 0) (**is** *?P ≤ ?Q*)
⟨*proof*⟩

definition *a* **where** *a n ≡ fds_nth f n*

lemma *finite_region*: *finite region*
  ⟨*proof*⟩

lemma *zero_notin_region*: *0 ∉ region*
  ⟨*proof*⟩

lemma *path_image_conv*:
  assumes *s ∈ img_path*
  shows *conv_abscissa f < s · 1*
⟨*proof*⟩

lemma *converge_on_path*:
  assumes *s ∈ img_path*
  shows *fds_converges f s*
  ⟨*proof*⟩

lemma *summable_on_path*:
  assumes $s \in img\_path$
  shows $(\lambda n.\ a\ n\ /\ n\ nat\_powr\ s)$ *subsummable* $\{1..\}$
  ⟨*proof*⟩

lemma *zero_notin_path*:
  shows $0 \notin closed\_segment\ (Complex\ b\ (-\ T))\ (Complex\ b\ T)$
  ⟨*proof*⟩

lemma *perron_bound*:
  $\|\sum`n \geq 1.\ a\ n * F'\ n\| \leq x\ powr\ b * H * B\ /\ T$
    $+\ 3 * (2 + ln\ (T\ /\ b)) * (\sum n{\in}region.\ \|a\ n\|)$
⟨*proof*⟩

lemma *perron*:
  $(\lambda s.\ eval\_fds\ f\ s * x\ powr\ s\ /\ s)$ *contour_integrable_on path*
  $\|sum\_upto\ a\ x - 1\ /\ (2 * pi * \mathrm{i}) * contour\_integral\ path\ (\lambda s.\ eval\_fds\ f\ s * x\ powr\ s\ /\ s)\|$
    $\leq x\ powr\ b * H * B\ /\ T + 3 * (2 + ln\ (T\ /\ b)) * (\sum n{\in}region.\ \|a\ n\|)$
⟨*proof*⟩
end

theorem *perron_formula*:
  fixes $b\ B\ H\ T\ x :: real$ **and** $f :: complex\ fds$
  assumes *Hb*: $0 < b$ **and** *hT*: $b \leq T$
    and *Hb'*: $abs\_conv\_abscissa\ f < b$
    and *hH*: $1 < H$ **and** *hH'*: $b + 1 \leq H$ **and** *Hx*: $0 < x$
    and *hB*: $(\sum`n \geq 1.\ \|fds\_nth\ f\ n\|\ /\ n\ nat\_powr\ b) \leq B$
  shows $(\lambda s.\ eval\_fds\ f\ s * x\ powr\ s\ /\ s)$ *contour_integrable_on* $(linepath\ (Complex\ b\ (-T))\ (Complex\ b$
$T))$
      $\|sum\_upto\ (fds\_nth\ f)\ x - 1\ /\ (2 * pi * \mathrm{i}) *$
        $contour\_integral\ (linepath\ (Complex\ b\ (-T))\ (Complex\ b\ T))\ (\lambda s.\ eval\_fds\ f\ s * x\ powr\ s\ /\ s)\|$
        $\leq x\ powr\ b * H * B\ /\ T + 3 * (2 + ln\ (T\ /\ b)) * (\sum n\ |\ x - x\ /\ H \leq n \wedge n \leq x + x\ /\ H.$
$\|fds\_nth\ f\ n\|)$
⟨*proof*⟩

theorem *perron_asymp*:
  fixes $b\ x :: real$
  assumes *b*: $b > 0$ $ereal\ b > abs\_conv\_abscissa\ f$
  assumes *x*: $0 < x$ $x \notin \mathbb{N}$
  defines $L \equiv (\lambda T.\ linepath\ (Complex\ b\ (-T))\ (Complex\ b\ T))$
  shows $((\lambda T.\ contour\_integral\ (L\ T)\ (\lambda s.\ eval\_fds\ f\ s * of\_real\ x\ powr\ s\ /\ s))$
        $\longrightarrow 2 * pi * \mathrm{i} * sum\_upto\ (\lambda n.\ fds\_nth\ f\ n)\ x)\ at\_top$
⟨*proof*⟩

unbundle *no pnt_syntax*

end
theory *PNT_with_Remainder*
imports
  Relation_of_PNTs
  Zeta_Zerofree
  Perron_Formula
begin
unbundle *pnt_syntax*

6 Estimation of the order of $\frac{\zeta'(s)}{\zeta(s)}$

notation *primes_psi* (‹ψ›)

lemma *zeta_div_bound'*:
  assumes $1 + exp\ (- 4 * ln\ (14 + 4 * t)) \le \sigma$
    and $13\ /\ 22 \le t$
    and $z \in cball\ (Complex\ \sigma\ t)\ (1\ /\ 2)$
  shows $\|zeta\ z\ /\ zeta\ (Complex\ \sigma\ t)\| \le exp\ (12 * ln\ (14 + 4 * t))$
⟨*proof*⟩

lemma *zeta_div_bound*:
  assumes $1 + exp\ (- 4 * ln\ (14 + 4 * |t|)) \le \sigma$
    and $13\ /\ 22 \le |t|$
    and $z \in cball\ (Complex\ \sigma\ t)\ (1\ /\ 2)$
  shows $\|zeta\ z\ /\ zeta\ (Complex\ \sigma\ t)\| \le exp\ (12 * ln\ (14 + 4 * |t|))$
⟨*proof*⟩

definition C_2 **where** $C_2 \equiv 319979520 :: real$

lemma *C₂_gt_zero*: $0 < C_2$ ⟨*proof*⟩

lemma *logderiv_zeta_order_estimate'*:
$\forall_F\ t\ in\ (abs\ going\_to\ at\_top).$
  $\forall\sigma.\ 1 - 1\ /\ 7 * C_1\ /\ ln\ (|t| + 3) \le \sigma$
  $\longrightarrow \|logderiv\ zeta\ (Complex\ \sigma\ t)\| \le C_2 * (ln\ (|t| + 3))^2$
⟨*proof*⟩

definition C_3 **where**
$C_3 \equiv SOME\ T.\ 0 < T \land$
  $(\forall t.\ T \le |t| \longrightarrow$
    $(\forall\sigma.\ 1 - 1\ /\ 7 * C_1\ /\ ln\ (|t| + 3) \le \sigma$
     $\longrightarrow \|logderiv\ zeta\ (Complex\ \sigma\ t)\| \le C_2 * (ln\ (|t| + 3))^2))$

lemma *C₃_prop*:
  $0 < C_3 \land$
  $(\forall t.\ C_3 \le |t| \longrightarrow$
    $(\forall\sigma.\ 1 - 1\ /\ 7 * C_1\ /\ ln\ (|t| + 3) \le \sigma$
    $\longrightarrow \|logderiv\ zeta\ (Complex\ \sigma\ t)\| \le C_2 * (ln\ (|t| + 3))^2))$
⟨*proof*⟩

lemma *C₃_gt_zero*: $0 < C_3$ ⟨*proof*⟩

lemma *logderiv_zeta_order_estimate*:
  assumes $1 - 1\ /\ 7 * C_1\ /\ ln\ (|t| + 3) \le \sigma\ \ C_3 \le |t|$
  shows $\|logderiv\ zeta\ (Complex\ \sigma\ t)\| \le C_2 * (ln\ (|t| + 3))^2$
  ⟨*proof*⟩

definition *zeta_zerofree_region*
  where $zeta\_zerofree\_region \equiv \{s.\ s \ne 1 \land 1 - C_1\ /\ ln\ (|Im\ s| + 2) < Re\ s\}$
definition *logderiv_zeta_region*
  where $logderiv\_zeta\_region \equiv \{s.\ C_3 \le |Im\ s| \land 1 - 1\ /\ 7 * C_1\ /\ ln\ (|Im\ s| + 3) \le Re\ s\}$
definition *zeta_strip_region*
  where $zeta\_strip\_region\ \sigma\ T \equiv \{s.\ s \ne 1 \land \sigma \le Re\ s \land |Im\ s| \le T\}$
definition *zeta_strip_region'*

where $zeta\_strip\_region'\ \sigma\ T \equiv \{s.\ s \neq 1 \wedge \sigma \leq Re\ s \wedge C_3 \leq |Im\ s| \wedge |Im\ s| \leq T\}$

lemma *strip_in_zerofree_region*:
assumes $1 - C_1\ /\ ln\ (T + 2) < \sigma$
shows $zeta\_strip\_region\ \sigma\ T \subseteq zeta\_zerofree\_region$
⟨*proof*⟩

lemma *strip_in_logderiv_zeta_region*:
assumes $1 - 1\ /\ 7 * C_1\ /\ ln\ (T + 3) \leq \sigma$
shows $zeta\_strip\_region'\ \sigma\ T \subseteq logderiv\_zeta\_region$
⟨*proof*⟩

lemma *strip_condition_imp*:
assumes $0 \leq T\ 1 - 1\ /\ 7 * C_1\ /\ ln\ (T + 3) \leq \sigma$
shows $1 - C_1\ /\ ln\ (T + 2) < \sigma$
⟨*proof*⟩

lemma *zeta_zerofree_region*:
assumes $s \in zeta\_zerofree\_region$
shows $zeta\ s \neq 0$
⟨*proof*⟩

lemma *logderiv_zeta_region_estimate*:
assumes $s \in logderiv\_zeta\_region$
shows $\|logderiv\ zeta\ s\| \leq C_2 * (ln\ (|Im\ s| + 3))^2$
⟨*proof*⟩

definition C_4 :: *real* **where** $C_4 \equiv 1\ /\ 6666241$

lemma C_4*_prop*:
$\forall_F\ x\ in\ at\_top.\ C_4\ /\ ln\ x \leq C_1\ /\ (7 * ln\ (x + 3))$
⟨*proof*⟩

lemma C_4*_gt_zero*: $0 < C_4$ ⟨*proof*⟩

definition C_5*_prop* **where**
$C_5\_prop\ C_5 \equiv$
$0 < C_5 \wedge (\forall_F\ x\ in\ at\_top.\ (\forall t.\ |t| \leq x$
$\longrightarrow \|logderiv\ zeta\ (Complex\ (1 - C_4\ /\ ln\ x)\ t)\| \leq C_5 * (ln\ x)^2))$

lemma *logderiv_zeta_bound_vertical'*:
$\exists C_5.\ C_5\_prop\ C_5$
⟨*proof*⟩

definition C_5 **where** $C_5 \equiv SOME\ C_5.\ C_5\_prop\ C_5$

lemma
C_5*_gt_zero*: $0 < C_5$ (**is** *?prop_1*) **and**
logderiv_zeta_bound_vertical:
$\forall_F\ x\ in\ at\_top.\ \forall t.\ |t| \leq x$
$\longrightarrow \|logderiv\ zeta\ (Complex\ (1 - C_4\ /\ ln\ x)\ t)\| \leq C_5 * (ln\ x)^2$ (**is** *?prop_2*)
⟨*proof*⟩

7 Deducing prime number theorem using Perron's formula

locale *prime_number_theorem* =
  fixes $c\ \varepsilon :: real$
  assumes *Hc*: $0 < c$ **and** *Hc'*: $c * c < 2 * C_4$ **and** *Hε*: $0 < \varepsilon\ 2 * \varepsilon < c$
begin
notation *primes_psi* (‹ψ›)
definition H **where** $H\ x \equiv exp\ (c\ /\ 2 * (ln\ x)\ powr\ (1\ /\ 2))$ **for** $x :: real$
definition T **where** $T\ x \equiv exp\ (c * (ln\ x)\ powr\ (1\ /\ 2))$ **for** $x :: real$
definition a **where** $a\ x \equiv 1 - C_4\ /\ (c * (ln\ x)\ powr\ (1\ /\ 2))$ **for** $x :: real$
definition b **where** $b\ x \equiv 1 + 1\ /\ (ln\ x)$ **for** $x :: real$
definition B **where** $B\ x \equiv 5\ /\ 4 * ln\ x$ **for** $x :: real$
definition f **where** $f\ x\ s \equiv x\ powr\ s\ /\ s * logderiv\ zeta\ s$ **for** $x :: real$ **and** $s :: complex$
definition R **where** $R\ x \equiv$
  $x\ powr\ (b\ x) * H\ x * B\ x\ /\ T\ x + 3 * (2 + ln\ (T\ x\ /\ b\ x))$
  $* (\sum n \mid x - x\ /\ H\ x \le n \wedge n \le x + x\ /\ H\ x.\ \|fds\_nth\ (fds\ mangoldt\_complex)\ n\|)$ **for** $x :: real$
definition Rc' **where** $Rc' \equiv O(\lambda x.\ x * exp\ (-\ (c\ /\ 2 - \varepsilon) * ln\ x\ powr\ (1\ /\ 2)))$
definition Rc **where** $Rc \equiv O(\lambda x.\ x * exp\ (-\ (c\ /\ 2 - 2 * \varepsilon) * ln\ x\ powr\ (1\ /\ 2)))$
definition z_1 **where** $z_1\ x \equiv Complex\ (a\ x)\ (-\ T\ x)$ **for** x
definition z_2 **where** $z_2\ x \equiv Complex\ (b\ x)\ (-\ T\ x)$ **for** x
definition z_3 **where** $z_3\ x \equiv Complex\ (b\ x)\ (T\ x)$ **for** x
definition z_4 **where** $z_4\ x \equiv Complex\ (a\ x)\ (T\ x)$ **for** x
definition *rect* **where** $rect\ x \equiv cbox\ (z_1\ x)\ (z_3\ x)$ **for** x
definition *rect'* **where** $rect'\ x \equiv rect\ x - \{1\}$ **for** x
definition P_t **where** $P_t\ x\ t \equiv linepath\ (Complex\ (a\ x)\ t)\ (Complex\ (b\ x)\ t)$ **for** $x\ t$
definition P_1 **where** $P_1\ x \equiv linepath\ (z_1\ x)\ (z_4\ x)$ **for** x
definition P_2 **where** $P_2\ x \equiv linepath\ (z_2\ x)\ (z_3\ x)$ **for** x
definition P_3 **where** $P_3\ x \equiv P_t\ x\ (-\ T\ x)$ **for** x
definition P_4 **where** $P_4\ x \equiv P_t\ x\ (T\ x)$ **for** x
definition P_r **where** $P_r\ x \equiv rectpath\ (z_1\ x)\ (z_3\ x)$ **for** x

lemma *Rc_eq_rem_est*:
  $Rc = rem\_est\ (c\ /\ 2 - 2 * \varepsilon)\ (1\ /\ 2)\ 0$
⟨*proof*⟩

lemma *residue_f*:
  $residue\ (f\ x)\ 1 = -\ x$
⟨*proof*⟩

lemma *rect_in_strip*:
  $rect\ x - \{1\} \subseteq zeta\_strip\_region\ (a\ x)\ (T\ x)$
  ⟨*proof*⟩

lemma *rect_in_strip'*:
  $\{s \in rect\ x.\ C_3 \le |Im\ s|\} \subseteq zeta\_strip\_region'\ (a\ x)\ (T\ x)$
  ⟨*proof*⟩

lemma
  rect'_in_zerofree: $\forall_F\ x\ in\ at\_top.\ rect'\ x \subseteq zeta\_zerofree\_region$ **and**
  rect_in_logderiv_zeta: $\forall_F\ x\ in\ at\_top.\ \{s \in rect\ x.\ C_3 \le |Im\ s|\} \subseteq logderiv\_zeta\_region$
⟨*proof*⟩

lemma *zeta_nonzero_in_rect*:
  $\forall_F\ x\ in\ at\_top.\ \forall s.\ s \in rect'\ x \longrightarrow zeta\ s \ne 0$
  ⟨*proof*⟩

lemma *zero_notin_rect*: $\forall_F$ *x in at_top*. $0 \notin$ *rect′ x*
⟨*proof*⟩

lemma *f_analytic*:
$\forall_F$ *x in at_top*. *f x analytic_on rect′ x*
⟨*proof*⟩

lemma *path_image_in_rect_1*:
assumes $0 \leq T\ x \wedge a\ x \leq b\ x$
shows *path_image* $(P_1\ x) \subseteq$ *rect x* $\wedge$ *path_image* $(P_2\ x) \subseteq$ *rect x*
⟨*proof*⟩

lemma *path_image_in_rect_2*:
assumes $0 \leq T\ x \wedge a\ x \leq b\ x \wedge t \in \{-T\ x..T\ x\}$
shows *path_image* $(P_t\ x\ t) \subseteq$ *rect x*
⟨*proof*⟩

definition *path_in_rect′* **where**
path_in_rect′ x $\equiv$
path_image $(P_1\ x) \subseteq$ *rect′ x* $\wedge$ *path_image* $(P_2\ x) \subseteq$ *rect′ x* $\wedge$
path_image $(P_3\ x) \subseteq$ *rect′ x* $\wedge$ *path_image* $(P_4\ x) \subseteq$ *rect′ x*

lemma *path_image_in_rect′*:
assumes $0 < T\ x \wedge a\ x < 1 \wedge 1 < b\ x$
shows *path_in_rect′ x*
⟨*proof*⟩

lemma *asymp_1*:
$\forall_F$ *x in at_top*. $0 < T\ x \wedge a\ x < 1 \wedge 1 < b\ x$
⟨*proof*⟩

lemma *f_continuous_on*:
$\forall_F$ *x in at_top*. $\forall\ A \subseteq$ *rect′ x*. *continuous_on A* $(f\ x)$
⟨*proof*⟩

lemma *contour_integrability*:
$\forall_F$ *x in at_top*.
f x contour_integrable_on $P_1\ x \wedge$ *f x contour_integrable_on* $P_2\ x \wedge$
f x contour_integrable_on $P_3\ x \wedge$ *f x contour_integrable_on* $P_4\ x$
⟨*proof*⟩

lemma *contour_integral_rectpath′*:
assumes *f x analytic_on* (*rect′ x*) $0 < T\ x \wedge a\ x < 1 \wedge 1 < b\ x$
shows *contour_integral* $(P_r\ x)$ $(f\ x) = -\ 2 * pi * \mathrm{i} * x$
⟨*proof*⟩

lemma *contour_integral_rectpath*:
$\forall_F$ *x in at_top*. *contour_integral* $(P_r\ x)$ $(f\ x) = -\ 2 * pi * \mathrm{i} * x$
⟨*proof*⟩

lemma *valid_paths*:
valid_path $(P_1\ x)$ *valid_path* $(P_2\ x)$ *valid_path* $(P_3\ x)$ *valid_path* $(P_4\ x)$
⟨*proof*⟩

lemma *integral_rectpath_split*:

assumes $f\ x$ *contour_integrable_on* $P_1\ x \land f\ x$ *contour_integrable_on* $P_2\ x \land$
$f\ x$ *contour_integrable_on* $P_3\ x \land f\ x$ *contour_integrable_on* $P_4\ x$
shows *contour_integral* $(P_3\ x)\ (f\ x)$ + *contour_integral* $(P_2\ x)\ (f\ x)$
− *contour_integral* $(P_4\ x)\ (f\ x)$ − *contour_integral* $(P_1\ x)\ (f\ x)$ = *contour_integral* $(P_r\ x)\ (f\ x)$
⟨*proof*⟩

lemma P_2*_eq*:
$\forall_F$ *x in at_top. contour_integral* $(P_2\ x)\ (f\ x) + 2 * pi * \mathrm{i} * x$
= *contour_integral* $(P_1\ x)\ (f\ x)$ − *contour_integral* $(P_3\ x)\ (f\ x)$ + *contour_integral* $(P_4\ x)\ (f\ x)$
⟨*proof*⟩

lemma *estimation_*P_1:
$(\lambda x.\ \|$*contour_integral* $(P_1\ x)\ (f\ x)\|) \in Rc$
⟨*proof*⟩

lemma *estimation_*$P_t{}'$:
assumes *h*:
$1 < x \land max\ 1\ C_3 \leq T\ x\ a\ x < 1 \land 1 < b\ x$
$\{s \in rect\ x.\ C_3 \leq |Im\ s|\} \subseteq$ *logderiv_zeta_region*
$f\ x$ *contour_integrable_on* $P_3\ x \land f\ x$ *contour_integrable_on* $P_4\ x$
and *Ht*: $|t| = T\ x$
shows $\|$*contour_integral* $(P_t\ x\ t)\ (f\ x)\| \leq C_2 * exp\ 1 * x\ /\ T\ x * (ln\ (T\ x + 3))^2 * (b\ x - a\ x)$
⟨*proof*⟩

lemma *estimation_*P_t:
$(\lambda x.\ \|$*contour_integral* $(P_3\ x)\ (f\ x)\|) \in Rc \land$
$(\lambda x.\ \|$*contour_integral* $(P_4\ x)\ (f\ x)\|) \in Rc$
⟨*proof*⟩

lemma *Re_path_*P_2:
$\bigwedge z.\ z \in$ *path_image* $(P_2\ x) \Longrightarrow Re\ z = b\ x$
⟨*proof*⟩

lemma *estimation_*P_2:
$(\lambda x.\ \|1\ /\ (2 * pi * \mathrm{i}) *$ *contour_integral* $(P_2\ x)\ (f\ x) + x\|) \in Rc$
⟨*proof*⟩

lemma *estimation_R*:
$R \in Rc$
⟨*proof*⟩

lemma *perron_psi*:
$\forall_F$ *x in at_top.* $\|\psi\ x + 1\ /\ (2 * pi * \mathrm{i}) *$ *contour_integral* $(P_2\ x)\ (f\ x)\| \leq R\ x$
⟨*proof*⟩

lemma *estimation_perron_psi*:
$(\lambda x.\ \|\psi\ x + 1\ /\ (2 * pi * \mathrm{i}) *$ *contour_integral* $(P_2\ x)\ (f\ x)\|) \in Rc$
⟨*proof*⟩

theorem *prime_number_theorem*:
PNT_3 $(c\ /\ 2 - 2 * \varepsilon)\ (1\ /\ 2)\ 0$
⟨*proof*⟩

no_notation *primes_psi* (‹ψ›)
end

unbundle *prime_counting_syntax*

theorem *prime_number_theorem*:
shows $(\lambda x.\ \pi\ x - Li\ x) \in O(\lambda x.\ x * exp\ (- 1\ /\ 3653 * (ln\ x)\ powr\ (1\ /\ 2)))$
⟨*proof*⟩

hide_const (**open**) $C_3\ C_4\ C_5$
unbundle *no prime_counting_syntax* **and** *no pnt_syntax*

end