# PAC Checker

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**Abstract**

Generating and checking proof certificates is important to increase the trust in automated reasoning tools. In recent years formal verification using computer algebra became more important and is heavily used in automated circuit verification. An existing proof format which covers algebraic reasoning and allows efficient proof checking is the practical algebraic calculus. In this development, we present the verified checker Pastèque that is obtained by synthesis via the Refinement Framework.

This is the formalization going with our FMCAD'20 tool presentation [1].

## Contents

**theory** *PAC-More-Poly*
**imports** *HOL−Library.Poly-Mapping HOL−Algebra.Polynomials Polynomials.MPoly-Type-Class HOL−Algebra.Module HOL−Library.Countable-Set*
**begin**

# 1 Overview

One solution to check circuit of multipliers is to use algebraic method, like producing proofs on polynomials. We are here interested in checking PAC proofs on the Boolean ring. The idea is the following: each variable represents an input or the output of a gate and we want to prove the bitwise multiplication of the input bits yields the output, namely the bitwise representation of the multiplication of the input (modulo $2^n$ where $n$ is the number of bits of the circuit).

Algebraic proof systems typically reason over polynomials in a ring $\mathbb{K}[X]$, where the variables $X$ represent Boolean values. The aim of an algebraic proof is to derive whether a polynomial $f$ can be derived from a given set of polynomials $G = \{g_1, \ldots, g_l\} \subseteq \mathbb{K}[X]$ together with the Boolean value constraints $B(X) = \{x_i^2 - x_i \mid x_i \in X\}$. In algebraic terms this means to show that the polynomial $f \in \langle G \cup B(X)\rangle$.

In our setting we set $\mathbb{K} = \mathbb{Z}$ and we treat the Boolean value constraints implicitly, i.e., we consider proofs in the ring $\mathbb{Z}[X]/\langle B(X)\rangle$ to admit shorter proofs

The checker takes as input 3 files:

1. an input file containing all polynomials that are initially present;
2. a target (or specification) polynomial ;

3. a "proof" file to check that contains the proof in PAC format that shows that the specification is in the ideal generated by the polynomials present initially.

Each step of the proof is either an addition of two polynomials previously derived, a multiplication from a previously derived polynomial and an arbitrary polynomial, and the deletion a derived polynomial.

One restriction on the proofs compared to generic PAC proofs is that $x^2 = x$ in the Boolean ring we are considering.

The checker can produce two outputs: valid (meaning that each derived polynomial in the proof has been correctly derived and the specification polynomial was also derived at some point [either in the proof or as input]) or invalid (without proven information what went wrong).

The development is organised as follows:

- `PAC_Specification.thy` (this file) contains the specification as described above on ideals without any peculiarities on the PAC proof format
- `PAC_Checker_Specification.thy` specialises to the PAC format and enters the nondeterminism monad to prepare the subsequent refinements.
- `PAC_Checker.thy` contains the refined version where polynomials are represented as lists.
- `PAC_Checker_Synthesis.thy` contains the efficient implementation with imperative data structure like a hash set.
- `PAC_Checker_MLton.thy` contains the code generation and the command to compile the file with the ML compiler MLton.

Here is an example of a proof and an input file (taken from the appendix of our FMCAD paper [1], available at http://fmv.jku.at/pacheck_pasteque):

```
<res.input>       <res.proof>
 1 x*y;            3  = fz, -z+1;
 2 y*z-y-z+1;      4  *  3,  y-1, -fz*y+fz-y*z+y+z-1;
                   5  +  2,    4, -fz*y+fz;
                   2  d;
                   4  d;
<res.target>       6  *  1,   fz, fz*x*y;
 -x*z+x;           1  d;
                   7  *  5,    x, -fz*x*y+fz*x;
                   8  +  6,    7, fz*x;
                   9  *  3,    x, -fz*x-x*z+x;
                  10  +  8,    9, -x*z+x;
```

Each line starts with a number that is used to index the (conclusion) polynomial. In the proof, there are four kind of steps:

1. `3 = fz, -z+1;` is an extension that introduces a new variable (in this case `fz`);

2. `4 * 3, y-1, -fz*y+fz-y*z+y+z-1;` is a multiplication of the existing polynomial with index 3 by the arbitrary polynomial `y-1` and generates the new polynomial `-fz*y+fz-y*z+y+z-1` with index 4;

3. `5 + 2, 4, -fz*y+fz;` is an addition of the existing polynomials with index 2 and 4 and generates the new polynomial `-fz*y+fz` with index 5;

4. `1 d;` deletes the polynomial with index 1 and it cannot be reused in subsequent steps.

Remark that unlike DRAT checker, we do forward checking and check every derived polynomial. The target polynomial can also be part of the input file.

# 2 Libraries

## 2.1 More Polynomials

Here are more theorems on polynomials. Most of these facts are extremely trivial and should probably be generalised and moved to the Isabelle distribution.

**lemma** *Const$_0$-add*:
‹*Const$_0$* (*a* + *b*) = *Const$_0$* *a* + *Const$_0$* *b*›
⟨*proof*⟩

**lemma** *Const-mult*:
‹*Const* (*a* ∗ *b*) = *Const a* ∗ *Const b*›
⟨*proof*⟩

**lemma** *Const$_0$-mult*:
‹*Const$_0$* (*a* ∗ *b*) = *Const$_0$* *a* ∗ *Const$_0$* *b*›
⟨*proof*⟩

**lemma** *Const0*[*simp*]:
‹*Const 0* = *0*›
⟨*proof*⟩

**lemma** (**in** −) *Const-uminus*[*simp*]:
‹*Const* (−*n*) = − *Const n*›
⟨*proof*⟩

**lemma** [*simp*]: ‹*Const$_0$ 0* = *0*›
‹*MPoly 0* = *0*›
⟨*proof*⟩

**lemma** *Const-add*:
‹*Const* (*a* + *b*) = *Const a* + *Const b*›
⟨*proof*⟩

**instance** *mpoly* :: (*comm-semiring-1*) *comm-semiring-1*
⟨*proof*⟩

**lemma** *degree-uminus*[*simp*]:
‹*degree* (−*A*) *x′* = *degree A x′*›
⟨*proof*⟩

**lemma** *degree-sum-notin*:
‹$x' \notin vars\ B \Longrightarrow degree\ (A + B)\ x' = degree\ A\ x'$›
⟨*proof*⟩

**lemma** *degree-notin-vars*:
‹$x \notin (vars\ B) \Longrightarrow degree\ (B :: 'a :: \{monoid\text{-}add\}\ mpoly)\ x = 0$›
⟨*proof*⟩

**lemma** *not-in-vars-coeff0*:
‹$x \notin vars\ p \Longrightarrow MPoly\text{-}Type.coeff\ p\ (monomial\ (Suc\ 0)\ x) = 0$›
⟨*proof*⟩

**lemma** *keys-add'*:
$p \in keys\ (f + g) \Longrightarrow p \in keys\ f \cup keys\ g$
⟨*proof*⟩

**lemma** *keys-mapping-sum-add*:
‹$finite\ A \Longrightarrow keys\ (mapping\text{-}of\ (\sum v \in A.\ f\ v)) \subseteq \bigcup (keys\ `\ mapping\text{-}of\ `\ f\ `\ UNIV)$›
⟨*proof*⟩

**lemma** *vars-sum-vars-union*:
**fixes** $f$ :: ‹$int\ mpoly \Rightarrow int\ mpoly$›
**assumes** ‹$finite\ \{v.\ f\ v \neq 0\}$›
**shows** ‹$vars\ (\sum v \mid f\ v \neq 0.\ f\ v * v) \subseteq \bigcup (vars\ `\ \{v.\ f\ v \neq 0\}) \cup \bigcup (vars\ `\ f\ `\ \{v.\ f\ v \neq 0\})$›
(**is** ‹$?A \subseteq ?B$›)
⟨*proof*⟩

**lemma** *vars-in-right-only*:
$x \in vars\ q \Longrightarrow x \notin vars\ p \Longrightarrow x \in vars\ (p+q)$
⟨*proof*⟩

**lemma** [*simp*]:
‹$vars\ 0 = \{\}$›
⟨*proof*⟩

**lemma** *vars-Un-nointer*:
‹$keys\ (mapping\text{-}of\ p) \cap\ keys\ (mapping\text{-}of\ q) = \{\} \Longrightarrow vars\ (p + q) = vars\ p \cup vars\ q$›
⟨*proof*⟩

**lemmas** [*simp*] = *zero-mpoly.rep-eq*

**lemma** *polynomial-sum-monoms*:
**fixes** $p$ :: ‹$'a :: \{comm\text{-}monoid\text{-}add,cancel\text{-}comm\text{-}monoid\text{-}add\}\ mpoly$›
**shows**
‹$p = (\sum x{\in}keys\ (mapping\text{-}of\ p).\ MPoly\text{-}Type.monom\ x\ (MPoly\text{-}Type.coeff\ p\ x))$›
‹$keys\ (mapping\text{-}of\ p) \subseteq I \Longrightarrow finite\ I \Longrightarrow p = (\sum x{\in}I.\ MPoly\text{-}Type.monom\ x\ (MPoly\text{-}Type.coeff\ p\ x))$›
⟨*proof*⟩

**lemma** *vars-mult-monom*:
**fixes** $p$ :: ‹$int\ mpoly$›
**shows** ‹$vars\ (p * (monom\ (monomial\ (Suc\ 0)\ x')\ 1)) = (if\ p = 0\ then\ \{\}\ else\ insert\ x'\ (vars\ p))$›

⟨*proof*⟩

**term** ‹(*x'*, *u*, *lookup u x'*, *A*)›
**lemma** *in-mapping-mult-single*:
‹*x* ∈ (λ*x*. *lookup x x'*) ‘ *keys* (*A* ∗ (*Var*$_0$ *x'* :: (*nat* ⇒$_0$ *nat*) ⇒$_0$ *'b* :: {*monoid-mult*,*zero-neq-one*,*semiring-0*}))
⟷
*x* > *0* ∧ *x* − *1* ∈ (λ*x*. *lookup x x'*) ‘ *keys* (*A*)›
⟨*proof*⟩

**lemma** *Max-Suc-Suc-Max*:
‹*finite A* ⟹ *A* ≠ {} ⟹ *Max* (*insert 0* (*Suc* ‘ *A*)) =
*Suc* (*Max* (*insert 0 A*))›
⟨*proof*⟩

**lemma** [*simp*]:
‹*keys* (*Var*$_0$ *x'* :: (*'a* ⇒$_0$ *nat*) ⇒$_0$ *'b* :: {*zero-neq-one*}) = {*Poly-Mapping.single x' 1*}›
⟨*proof*⟩

**lemma** *degree-mult-Var*:
‹*degree* (*A* ∗ *Var x'*) *x'* = (*if A* = *0 then 0 else Suc* (*degree A x'*))› **for** *A* :: ‹*int mpoly*›
⟨*proof*⟩

**lemma** *degree-mult-Var'*:
‹*degree* (*Var x'* ∗ *A*) *x'* = (*if A* = *0 then 0 else Suc* (*degree A x'*))› **for** *A* :: ‹*int mpoly*›
⟨*proof*⟩

**lemma** *degree-times-le*:
‹*degree* (*A* ∗ *B*) *x* ≤ *degree A x* + *degree B x*›
⟨*proof*⟩

**lemma** *monomial-inj*:
*monomial c s* = *monomial* (*d*::*'b*::*zero-neq-one*) *t* ⟷ (*c* = *0* ∧ *d* = *0*) ∨ (*c* = *d* ∧ *s* = *t*)
⟨*proof*⟩

**lemma** *MPoly-monomial-power'*:
‹*MPoly* (*monomial 1 x'*) ^(*n*+*1*) = *MPoly* (*monomial* (*1*) (((λ*x*. *x* + *x'*) ^^ *n*) *x'*))›
⟨*proof*⟩

**lemma** *MPoly-monomial-power*:
‹*n* > *0* ⟹ *MPoly* (*monomial 1 x'*) ^(*n*) = *MPoly* (*monomial* (*1*) (((λ*x*. *x* + *x'*) ^^ (*n* − *1*)) *x'*))›
⟨*proof*⟩

**lemma** *vars-uminus*[*simp*]:
‹*vars* (−*p*) = *vars p*›
⟨*proof*⟩

**lemma** *coeff-uminus*[*simp*]:
‹*MPoly-Type.coeff* (−*p*) *x* = −*MPoly-Type.coeff p x*›
⟨*proof*⟩

**definition** *decrease-key*::*'a* ⇒ (*'a* ⇒$_0$ *'b*::{*monoid-add*, *minus*,*one*}) ⇒ (*'a* ⇒$_0$ *'b*) **where**
*decrease-key k0 f* = *Abs-poly-mapping* (λ*k*. *if k* = *k0* ∧ *lookup f k* ≠ *0 then lookup f k* − *1 else lookup*

*f k*)

**lemma** *remove-key-lookup*:
*lookup* (*decrease-key k0 f*) *k* = (*if k* = *k0* ∧ *lookup f k* ≠ *0 then lookup f k* − *1 else lookup f k*)
⟨*proof*⟩

**lemma** *polynomial-split-on-var*:
**fixes** *p* :: ‹'*a* :: {*comm-monoid-add*,*cancel-comm-monoid-add*,*semiring-0*,*comm-semiring-1*} *mpoly*›
**obtains** *q r* **where**
‹*p* = *monom* (*monomial* (*Suc 0*) *x'*) *1* ∗ *q* + *r*› **and**
‹*x'* ∉ *vars r*›
⟨*proof*⟩

**lemma** *polynomial-split-on-var2*:
**fixes** *p* :: ‹*int mpoly*›
**assumes** ‹*x'* ∉ *vars s*›
**obtains** *q r* **where**
‹*p* = (*monom* (*monomial* (*Suc 0*) *x'*) *1* − *s*) ∗ *q* + *r*› **and**
‹*x'* ∉ *vars r*›
⟨*proof*⟩

**lemma** *finit-whenI*[*intro*]:
‹*finite* {*x*. (*0* :: *nat*) < (*y x*)} ⟹ *finite* {*x*. *0* < (*y x when x* ≠ *x'*)}›
⟨*proof*⟩

**lemma** *polynomial-split-on-var-diff-sq2*:
**fixes** *p* :: ‹*int mpoly*›
**obtains** *q r s* **where**
‹*p* = *monom* (*monomial* (*Suc 0*) *x'*) *1* ∗ *q* + *r* + *s* ∗ (*monom* (*monomial* (*Suc 0*) *x'*) *1*^*2* − *monom* (*monomial* (*Suc 0*) *x'*) *1*)› **and**
‹*x'* ∉ *vars r*› **and**
‹*x'* ∉ *vars q*›
⟨*proof*⟩

**lemma** *polynomial-decomp-alien-var*:
**fixes** *q A b* :: ‹*int mpoly*›
**assumes**
*q*: ‹*q* = *A* ∗ (*monom* (*monomial* (*Suc 0*) *x'*) *1*) + *b*› **and**
*x*: ‹*x'* ∉ *vars q*› ‹*x'* ∉ *vars b*›
**shows**
‹*A* = *0*› **and**
‹*q* = *b*›
⟨*proof*⟩

**lemma** *polynomial-decomp-alien-var2*:
**fixes** *q A b* :: ‹*int mpoly*›
**assumes**
*q*: ‹*q* = *A* ∗ (*monom* (*monomial* (*Suc 0*) *x'*) *1* + *p*) + *b*› **and**
*x*: ‹*x'* ∉ *vars q*› ‹*x'* ∉ *vars b*› ‹*x'* ∉ *vars p*›
**shows**
‹*A* = *0*› **and**
‹*q* = *b*›
⟨*proof*⟩

**lemma** *vars-unE*: ‹$x \in vars\ (a * b) \Longrightarrow (x \in vars\ a \Longrightarrow thesis) \Longrightarrow (x \in vars\ b \Longrightarrow thesis) \Longrightarrow thesis$›
⟨*proof*⟩

**lemma** *in-keys-minusI1*:
**assumes** $t \in keys\ p$ **and** $t \notin keys\ q$
**shows** $t \in keys\ (p - q)$
⟨*proof*⟩

**lemma** *in-keys-minusI2*:
**fixes** $t :: ‹'a›$ **and** $q :: ‹'a \Rightarrow_0 'b :: \{cancel\text{-}comm\text{-}monoid\text{-}add, group\text{-}add\}›$
**assumes** $t \in keys\ q$ **and** $t \notin keys\ p$
**shows** $t \in keys\ (p - q)$
⟨*proof*⟩

**lemma** *in-vars-addE*:
‹$x \in vars\ (p + q) \Longrightarrow (x \in vars\ p \Longrightarrow thesis) \Longrightarrow (x \in vars\ q \Longrightarrow thesis) \Longrightarrow thesis$›
⟨*proof*⟩

**lemma** *lookup-monomial-If*:
‹$lookup\ (monomial\ v\ k) = (\lambda k'.\ if\ k = k'\ then\ v\ else\ 0)$›
⟨*proof*⟩

**lemma** *vars-mult-Var*:
‹$vars\ (Var\ x * p) = (if\ p = 0\ then\ \{\}\ else\ insert\ x\ (vars\ p))$› **for** $p :: ‹int\ mpoly›$
⟨*proof*⟩

**lemma** *keys-mult-monomial*:
‹$keys\ (monomial\ (n :: int)\ k * mapping\text{-}of\ a) = (if\ n = 0\ then\ \{\}\ else\ ((+)\ k)\ `\ keys\ (mapping\text{-}of\ a))$›
⟨*proof*⟩

**lemma** *vars-mult-Const*:
‹$vars\ (Const\ n * a) = (if\ n = 0\ then\ \{\}\ else\ vars\ a)$› **for** $a :: ‹int\ mpoly›$
⟨*proof*⟩

**lemma** *coeff-minus*: $coeff\ p\ m - coeff\ q\ m = coeff\ (p - q)\ m$
⟨*proof*⟩

**lemma** *Const-1-eq-1*: ‹$Const\ (1 :: int) = (1 :: int\ mpoly)$›
⟨*proof*⟩

**lemma** [*simp*]:
‹$vars\ (1 :: int\ mpoly) = \{\}$›
⟨*proof*⟩

## 2.2 More Ideals

**lemma**
**fixes** $A :: ‹((‘x \Rightarrow_0 nat) \Rightarrow_0 ‘a :: comm\text{-}ring\text{-}1)\ set›$
**assumes** ‹$p \in ideal\ A$›
**shows** ‹$p * q \in ideal\ A$›
⟨*proof*⟩

The following theorem is very close to *More-Modules.ideal* $(insert\ ?a\ ?S) = \{x.\ \exists k.\ x - k *$

*?a ∈ More-Modules.ideal ?S*}, except that it is more useful if we need to take an element of *More-Modules.ideal* (*insert a S*).

**lemma** *ideal-insert'*:
‹*More-Modules.ideal* (*insert a S*) = {*y*. ∃ *x k*. *y* = *x* + *k* ∗ *a* ∧ *x* ∈ *More-Modules.ideal S*}›
⟨*proof*⟩

**lemma** *ideal-mult-right-in*:
‹*a* ∈ *ideal A* ⟹ *a* ∗ *b* ∈ *More-Modules.ideal A*›
⟨*proof*⟩

**lemma** *ideal-mult-right-in2*:
‹*a* ∈ *ideal A* ⟹ *b* ∗ *a* ∈ *More-Modules.ideal A*›
⟨*proof*⟩

**lemma** [*simp*]: ‹*vars* (*Var x* :: *'a* :: {*zero-neq-one*} *mpoly*) = {*x*}›
⟨*proof*⟩

**lemma** *vars-minus-Var-subset*:
‹*vars* (*p'* − *Var x* :: *'a* :: {*ab-group-add*,*one*,*zero-neq-one*} *mpoly*) ⊆ *V* ⟹ *vars p'* ⊆ *insert x V*›
⟨*proof*⟩

**lemma** *vars-add-Var-subset*:
‹*vars* (*p'* + *Var x* :: *'a* :: {*ab-group-add*,*one*,*zero-neq-one*} *mpoly*) ⊆ *V* ⟹ *vars p'* ⊆ *insert x V*›
⟨*proof*⟩

**lemma** *coeff-monomila-in-varsD*:
‹*coeff p* (*monomial* (*Suc 0*) *x*) ≠ *0* ⟹ *x* ∈ *vars* (*p* :: *int mpoly*)›
⟨*proof*⟩

**lemma** *coeff-MPoly-monomial*[*simp*]:
‹(*MPoly-Type.coeff* (*MPoly* (*monomial a m*)) *m*) = *a*›
⟨*proof*⟩

**end**

**theory** *Finite-Map-Multiset*
**imports**
*HOL−Library.Finite-Map*
*Nested-Multisets-Ordinals.Duplicate-Free-Multiset*
**begin**

**notation** *image-mset* (**infixr** ‹'#› *90*)

# 3 Finite maps and multisets

## 3.1 Finite sets and multisets

**abbreviation** *mset-fset* :: ‹*'a fset* ⇒ *'a multiset*› **where**
‹*mset-fset N* ≡ *mset-set* (*fset N*)›

**definition** *fset-mset* :: ‹*'a multiset* ⇒ *'a fset*› **where**
‹*fset-mset N* ≡ *Abs-fset* (*set-mset N*)›

**lemma** *fset-mset-mset-fset*: ‹*fset-mset* (*mset-fset N*) = *N*›

⟨*proof*⟩

**lemma** *mset-fset-fset-mset*[*simp*]:
‹*mset-fset* (*fset-mset N*) = *remdups-mset N*›
⟨*proof*⟩

**lemma** *in-mset-fset-fmember*[*simp*]: ‹*x* ∈# *mset-fset N* ⟷ *x* |∈| *N*›
⟨*proof*⟩

**lemma** *in-fset-mset-mset*[*simp*]: ‹*x* |∈| *fset-mset N* ⟷ *x* ∈# *N*›
⟨*proof*⟩

## 3.2 Finite map and multisets

Roughly the same as *ran* and *dom*, but with duplication in the content (unlike their finite sets counterpart) while still working on finite domains (unlike a function mapping). Remark that *dom-m* (the keys) does not contain duplicates, but we keep for symmetry (and for easier use of multiset operators as in the definition of *ran-m*).

**definition** *dom-m* **where**
‹*dom-m N* = *mset-fset* (*fmdom N*)›

**definition** *ran-m* **where**
‹*ran-m N* = *the* ‘# *fmlookup N* ‘# *dom-m N*›

**lemma** *dom-m-fmdrop*[*simp*]: ‹*dom-m* (*fmdrop C N*) = *remove1-mset C* (*dom-m N*)›
⟨*proof*⟩

**lemma** *dom-m-fmdrop-All*: ‹*dom-m* (*fmdrop C N*) = *removeAll-mset C* (*dom-m N*)›
⟨*proof*⟩

**lemma** *dom-m-fmupd*[*simp*]: ‹*dom-m* (*fmupd k C N*) = *add-mset k* (*remove1-mset k* (*dom-m N*))›
⟨*proof*⟩

**lemma** *distinct-mset-dom*: ‹*distinct-mset* (*dom-m N*)›
⟨*proof*⟩

**lemma** *in-dom-m-lookup-iff*: ‹*C* ∈# *dom-m N'* ⟷ *fmlookup N' C* ≠ *None*›
⟨*proof*⟩

**lemma** *in-dom-in-ran-m*[*simp*]: ‹*i* ∈# *dom-m N* ⟹ *the* (*fmlookup N i*) ∈# *ran-m N*›
⟨*proof*⟩

**lemma** *fmupd-same*[*simp*]:
‹*x1* ∈# *dom-m x1aa* ⟹ *fmupd x1* (*the* (*fmlookup x1aa x1*)) *x1aa* = *x1aa*›
⟨*proof*⟩

**lemma** *ran-m-fmempty*[*simp*]: ‹*ran-m fmempty* = {#}› **and**
*dom-m-fmempty*[*simp*]: ‹*dom-m fmempty* = {#}›
⟨*proof*⟩

**lemma** *fmrestrict-set-fmupd*:
‹*a* ∈ *xs* ⟹ *fmrestrict-set xs* (*fmupd a C N*) = *fmupd a C* (*fmrestrict-set xs N*)›
‹*a* ∉ *xs* ⟹ *fmrestrict-set xs* (*fmupd a C N*) = *fmrestrict-set xs N*›
⟨*proof*⟩

**lemma** *fset-fmdom-fmrestrict-set*:
‹*fset* (*fmdom* (*fmrestrict-set xs N*)) = *fset* (*fmdom N*) ∩ *xs*›
⟨*proof*⟩

**lemma** *dom-m-fmrestrict-set*: ‹*dom-m* (*fmrestrict-set* (*set xs*) *N*) = *mset xs* ∩# *dom-m N*›
⟨*proof*⟩

**lemma** *dom-m-fmrestrict-set'*: ‹*dom-m* (*fmrestrict-set xs N*) = *mset-set* (*xs* ∩ *set-mset* (*dom-m N*))›
⟨*proof*⟩

**lemma** *indom-mI*: ‹*fmlookup m x* = *Some y* ⟹ *x* ∈# *dom-m m*›
⟨*proof*⟩

**lemma** *fmupd-fmdrop-id*:
**assumes** ‹*k* |∈| *fmdom N'*›
**shows** ‹*fmupd k* (*the* (*fmlookup N' k*)) (*fmdrop k N'*) = *N'*›
⟨*proof*⟩

**lemma** *fm-member-split*: ‹*k* |∈| *fmdom N'* ⟹ ∃ *N''* *v*. *N'* = *fmupd k v N''* ∧ *the* (*fmlookup N' k*) = *v*
∧
*k* |∉| *fmdom N''*›
⟨*proof*⟩

**lemma** ‹*fmdrop k* (*fmupd k va N''*) = *fmdrop k N''*›
⟨*proof*⟩

**lemma** *fmap-ext-fmdom*:
‹(*fmdom N* = *fmdom N'*) ⟹ (⋀ *x*. *x* |∈| *fmdom N* ⟹ *fmlookup N x* = *fmlookup N' x*) ⟹
*N* = *N'*›
⟨*proof*⟩

**lemma** *fmrestrict-set-insert-in*:
‹*xa* ∈ *fset* (*fmdom N*) ⟹
*fmrestrict-set* (*insert xa l1*) *N* = *fmupd xa* (*the* (*fmlookup N xa*)) (*fmrestrict-set l1 N*)›
⟨*proof*⟩

**lemma** *fmrestrict-set-insert-notin*:
‹*xa* ∉ *fset* (*fmdom N*) ⟹
*fmrestrict-set* (*insert xa l1*) *N* = *fmrestrict-set l1 N*›
⟨*proof*⟩

**lemma** *fmrestrict-set-insert-in-dom-m*[*simp*]:
‹*xa* ∈# *dom-m N* ⟹
*fmrestrict-set* (*insert xa l1*) *N* = *fmupd xa* (*the* (*fmlookup N xa*)) (*fmrestrict-set l1 N*)›
⟨*proof*⟩

**lemma** *fmrestrict-set-insert-notin-dom-m*[*simp*]:
‹*xa* ∉# *dom-m N* ⟹
*fmrestrict-set* (*insert xa l1*) *N* = *fmrestrict-set l1 N*›
⟨*proof*⟩

**lemma** *fmlookup-restrict-set-id*: ‹*fset* (*fmdom N*) ⊆ *A* ⟹ *fmrestrict-set A N* = *N*›
⟨*proof*⟩

**lemma** *fmlookup-restrict-set-id'*: ‹*set-mset* (*dom-m N*) ⊆ *A* ⟹ *fmrestrict-set A N* = *N*›

⟨*proof*⟩

**lemma** *ran-m-mapsto-upd*:
**assumes**
*NC*: ‹*C* ∈# *dom-m N*›
**shows** ‹*ran-m* (*fmupd C C′ N*) = *add-mset C′* (*remove1-mset* (*the* (*fmlookup N C*)) (*ran-m N*))›
⟨*proof*⟩

**lemma** *ran-m-mapsto-upd-notin*:
**assumes** *NC*: ‹*C* ∉# *dom-m N*›
**shows** ‹*ran-m* (*fmupd C C′ N*) = *add-mset C′* (*ran-m N*)›
⟨*proof*⟩

**lemma** *image-mset-If-eq-notin*:
‹*C* ∉# *A* ⟹ {#*f* (*if x* = *C then a x else b x*). *x* ∈# *A*#} = {# *f*(*b x*). *x* ∈# *A* #}›
⟨*proof*⟩

**lemma** *filter-mset-cong2*:
(⋀*x*. *x* ∈# *M* ⟹ *f x* = *g x*) ⟹ *M* = *N* ⟹ *filter-mset f M* = *filter-mset g N*
⟨*proof*⟩

**lemma** *ran-m-fmdrop*:
‹*C* ∈# *dom-m N* ⟹ *ran-m* (*fmdrop C N*) = *remove1-mset* (*the* (*fmlookup N C*)) (*ran-m N*)›
⟨*proof*⟩

**lemma** *ran-m-fmdrop-notin*:
‹*C* ∉# *dom-m N* ⟹ *ran-m* (*fmdrop C N*) = *ran-m N*›
⟨*proof*⟩

**lemma** *ran-m-fmdrop-If*:
‹*ran-m* (*fmdrop C N*) = (*if C* ∈# *dom-m N then remove1-mset* (*the* (*fmlookup N C*)) (*ran-m N*) *else ran-m N*)›
⟨*proof*⟩

**lemma** *dom-m-empty-iff*[*iff*]:
‹*dom-m NU* = {#} ⟷ *NU* = *fmempty*›
⟨*proof*⟩

**end**

**theory** *WB-Sort*
**imports**
*Refine-Imperative-HOL.IICF*
*HOL−Library.Rewrite*
*Nested-Multisets-Ordinals.Duplicate-Free-Multiset*
**begin**

This a complete copy-paste of the IsaFoL version because sharing is too hard.

Every element between *lo* and *hi* can be chosen as pivot element.

**definition** *choose-pivot* :: ‹('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ '*a list* ⇒ *nat* ⇒ *nat* ⇒ *nat nres*› **where**
‹*choose-pivot* - - - *lo hi* = *SPEC*(λ*k*. *k* ≥ *lo* ∧ *k* ≤ *hi*)›

The element at index *p* partitions the subarray *lo*..*hi*. This means that every element

**definition** *isPartition-wrt* :: ‹('*b* ⇒ '*b* ⇒ *bool*) ⇒ '*b list* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool*› **where**

‹*isPartition-wrt R xs lo hi p* ≡ (∀ *i*. *i* ≥ *lo* ∧ *i* < *p* ⟶ *R* (*xs*!*i*) (*xs*!*p*)) ∧ (∀ *j*. *j* > *p* ∧ *j* ≤ *hi* ⟶ *R* (*xs*!*p*) (*xs*!*j*))›

**lemma** *isPartition-wrtI*:
‹(⋀ *i*. ⟦*i* ≥ *lo*; *i* < *p*⟧ ⟹ *R* (*xs*!*i*) (*xs*!*p*)) ⟹ (⋀ *j*. ⟦*j* > *p*; *j* ≤ *hi*⟧ ⟹ *R* (*xs*!*p*) (*xs*!*j*)) ⟹ *isPartition-wrt R xs lo hi p*›
⟨*proof*⟩

**definition** *isPartition* :: ‹'*a* :: *order list* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool*› **where**
‹*isPartition xs lo hi p* ≡ *isPartition-wrt* (≤) *xs lo hi p*›

**abbreviation** *isPartition-map* :: ‹('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ '*a list* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool*› **where**
‹*isPartition-map R h xs i j k* ≡ *isPartition-wrt* (λ*a b*. *R* (*h a*) (*h b*)) *xs i j k*›

**lemma** *isPartition-map-def'*:
‹*lo* ≤ *p* ⟹ *p* ≤ *hi* ⟹ *hi* < *length xs* ⟹ *isPartition-map R h xs lo hi p* = *isPartition-wrt R* (*map h xs*) *lo hi p*›
⟨*proof*⟩

Example: 6 is the pivot element (with index 4); 7::'*a* is equal to the *length xs* − *1*.

**lemma** ‹*isPartition* [*0*,*5*,*3*,*4*,*6*,*9*,*8*,*10*::*nat*] *0* 7 *4*›
⟨*proof*⟩

**definition** *sublist* :: ‹'*a list* ⇒ *nat* ⇒ *nat* ⇒ '*a list*› **where**
‹*sublist xs i j* ≡ *take* (*Suc j* − *i*) (*drop i xs*)›

**lemma** *take-Suc0*:
*l*≠[] ⟹ *take* (*Suc 0*) *l* = [*l*!*0*]
*0* < *length l* ⟹ *take* (*Suc 0*) *l* = [*l*!*0*]
*Suc n* ≤ *length l* ⟹ *take* (*Suc 0*) *l* = [*l*!*0*]
⟨*proof*⟩

**lemma** *sublist-single*: ‹*i* < *length xs* ⟹ *sublist xs i i* = [*xs*!*i*]›
⟨*proof*⟩

**lemma** *insert-eq*: ‹*insert a b* = *b* ∪ {*a*}›
⟨*proof*⟩

**lemma** *sublist-nth*: ‹⟦*lo* ≤ *hi*; *hi* < *length xs*; *k*+*lo* ≤ *hi*⟧ ⟹ (*sublist xs lo hi*)!*k* = *xs*!(*lo*+*k*)›
⟨*proof*⟩

**lemma** *sublist-length*: ‹⟦*i* ≤ *j*; *j* < *length xs*⟧ ⟹ *length* (*sublist xs i j*) = *1* + *j* − *i*›
⟨*proof*⟩

**lemma** *sublist-not-empty*: ‹⟦*i* ≤ *j*; *j* < *length xs*; *xs* ≠ []⟧ ⟹ *sublist xs i j* ≠ []›
⟨*proof*⟩

**lemma** *sublist-app*: ‹⟦*i1* ≤ *i2*; *i2* ≤ *i3*⟧ ⟹ *sublist xs i1 i2* @ *sublist xs* (*Suc i2*) *i3* = *sublist xs i1 i3*›
⟨*proof*⟩

**definition** *sorted-sublist-wrt* :: ‹('b ⇒ 'b ⇒ bool) ⇒ 'b list ⇒ nat ⇒ nat ⇒ bool› **where**
‹*sorted-sublist-wrt R xs lo hi* = *sorted-wrt R* (*sublist xs lo hi*)›

**definition** *sorted-sublist* :: ‹'a :: *linorder list* ⇒ *nat* ⇒ *nat* ⇒ *bool*› **where**
‹*sorted-sublist xs lo hi* = *sorted-sublist-wrt* (≤) *xs lo hi*›

**abbreviation** *sorted-sublist-map* :: ‹('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ 'a list ⇒ nat ⇒ nat ⇒ bool›
**where**
‹*sorted-sublist-map R h xs lo hi* ≡ *sorted-sublist-wrt* (λ*a b*. *R* (*h a*) (*h b*)) *xs lo hi*›

**lemma** *sorted-sublist-map-def'*:
‹*lo* < *length xs* ⟹ *sorted-sublist-map R h xs lo hi* ≡ *sorted-sublist-wrt R* (*map h xs*) *lo hi*›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-refl*: ‹*i* < *length xs* ⟹ *sorted-sublist-wrt R xs i i*›
⟨*proof*⟩

**lemma** *sorted-sublist-refl*: ‹*i* < *length xs* ⟹ *sorted-sublist xs i i*›
⟨*proof*⟩

**lemma** *sublist-map*: ‹*sublist* (*map f xs*) *i j* = *map f* (*sublist xs i j*)›
⟨*proof*⟩

**lemma** *take-set*: ‹*j* ≤ *length xs* ⟹ *x* ∈ *set* (*take j xs*) ≡ (∃ *k*. *k* < *j* ∧ *xs*!*k* = *x*)›
⟨*proof*⟩

**lemma** *drop-set*: ‹*j* ≤ *length xs* ⟹ *x* ∈ *set* (*drop j xs*) ≡ (∃ *k*. *j*≤*k*∧*k*<*length xs* ∧ *xs*!*k*=*x*)›
⟨*proof*⟩

**lemma** *sublist-el*: ‹*i* ≤ *j* ⟹ *j* < *length xs* ⟹ *x* ∈ *set* (*sublist xs i j*) ≡ (∃ *k*. *k* < *Suc j*−*i* ∧ *xs*!(*i*+*k*)=*x*)›
⟨*proof*⟩

**lemma** *sublist-el'*: ‹*i* ≤ *j* ⟹ *j* < *length xs* ⟹ *x* ∈ *set* (*sublist xs i j*) ≡ (∃ *k*. *i*≤*k*∧*k*≤*j* ∧ *xs*!*k*=*x*)›
⟨*proof*⟩

**lemma** *sublist-lt*: ‹*hi* < *lo* ⟹ *sublist xs lo hi* = []›
⟨*proof*⟩

**lemma** *nat-le-eq-or-lt*: ‹(*a* :: *nat*) ≤ *b* = (*a* = *b* ∨ *a* < *b*)›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-le*: ‹*hi* ≤ *lo* ⟹ *hi* < *length xs* ⟹ *sorted-sublist-wrt R xs lo hi*›
⟨*proof*⟩

Elements in a sorted sublists are actually sorted

**lemma** *sorted-sublist-wrt-nth-le*:
**assumes** ‹*sorted-sublist-wrt R xs lo hi*› **and** ‹*lo* ≤ *hi*› **and** ‹*hi* < *length xs*› **and**
‹*lo* ≤ *i*› **and** ‹*i* < *j*› **and** ‹*j* ≤ *hi*›
**shows** ‹*R* (*xs*!*i*) (*xs*!*j*)›
⟨*proof*⟩

We can make the assumption $i < j$ weaker if we have a reflexivie relation.

**lemma** *sorted-sublist-wrt-nth-le'*:
**assumes** *ref*: ‹$\bigwedge x.\ R\ x\ x$›
**and** ‹*sorted-sublist-wrt R xs lo hi*› **and** ‹$lo \leq hi$› **and** ‹$hi < length\ xs$›
**and** ‹$lo \leq i$› **and** ‹$i \leq j$› **and** ‹$j \leq hi$›
**shows** ‹$R\ (xs!i)\ (xs!j)$›
⟨*proof*⟩

**lemma** *sorted-sublist-le*: ‹$hi \leq lo \Longrightarrow hi < length\ xs \Longrightarrow$ *sorted-sublist xs lo hi*›
⟨*proof*⟩

**lemma** *sorted-sublist-map-le*: ‹$hi \leq lo \Longrightarrow hi < length\ xs \Longrightarrow$ *sorted-sublist-map R h xs lo hi*›
⟨*proof*⟩

**lemma** *sublist-cons*: ‹$lo < hi \Longrightarrow hi < length\ xs \Longrightarrow sublist\ xs\ lo\ hi = xs!lo\ \#\ sublist\ xs\ (Suc\ lo)\ hi$›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-cons'*:
‹*sorted-sublist-wrt R xs* $(lo+1)$ *hi* $\Longrightarrow lo \leq hi \Longrightarrow hi < length\ xs \Longrightarrow (\forall j.\ lo<j \wedge j \leq hi \longrightarrow R\ (xs!lo)\ (xs!j)) \Longrightarrow$ *sorted-sublist-wrt R xs lo hi*›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-cons*:
**assumes** *trans*: ‹$(\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z]\!] \Longrightarrow R\ x\ z)$› **and**
‹*sorted-sublist-wrt R xs* $(lo+1)$ *hi*› **and**
‹$lo \leq hi$› **and** ‹$hi < length\ xs$ › **and** ‹$R\ (xs!lo)\ (xs!(lo+1))$›
**shows** ‹*sorted-sublist-wrt R xs lo hi*›
⟨*proof*⟩

**lemma** *sorted-sublist-map-cons*:
‹$(\bigwedge x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)) \Longrightarrow$
*sorted-sublist-map R h xs* $(lo+1)$ *hi* $\Longrightarrow lo \leq hi \Longrightarrow hi < length\ xs \Longrightarrow R\ (h\ (xs!lo))\ (h\ (xs!(lo+1))) \Longrightarrow$ *sorted-sublist-map R h xs lo hi*›
⟨*proof*⟩

**lemma** *sublist-snoc*: ‹$lo < hi \Longrightarrow hi < length\ xs \Longrightarrow sublist\ xs\ lo\ hi = sublist\ xs\ lo\ (hi-1)\ @\ [xs!hi]$›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-snoc'*:
‹*sorted-sublist-wrt R xs lo* $(hi-1) \Longrightarrow lo \leq hi \Longrightarrow hi < length\ xs \Longrightarrow (\forall j.\ lo \leq j \wedge j < hi \longrightarrow R\ (xs!j)\ (xs!hi)) \Longrightarrow$ *sorted-sublist-wrt R xs lo hi*›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-snoc*:
**assumes** *trans*: ‹$(\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z]\!] \Longrightarrow R\ x\ z)$› **and**
‹*sorted-sublist-wrt R xs lo* $(hi-1)$› **and**
‹$lo \leq hi$› **and** ‹$hi < length\ xs$› **and** ‹$(R\ (xs!(hi-1))\ (xs!hi))$›
**shows** ‹*sorted-sublist-wrt R xs lo hi*›

⟨*proof*⟩

**lemma** *sublist-split*: ‹$lo \leq hi \Longrightarrow lo < p \Longrightarrow p < hi \Longrightarrow hi < length\ xs \Longrightarrow sublist\ xs\ lo\ p\ @\ sublist\ xs\ (p+1)\ hi = sublist\ xs\ lo\ hi$›
⟨*proof*⟩

**lemma** *sublist-split-part*: ‹$lo \leq hi \Longrightarrow lo < p \Longrightarrow p < hi \Longrightarrow hi < length\ xs \Longrightarrow sublist\ xs\ lo\ (p-1)\ @\ xs!p\ \#\ sublist\ xs\ (p+1)\ hi = sublist\ xs\ lo\ hi$›
⟨*proof*⟩

A property for partitions (we always assume that $R$ is transitive.

**lemma** *isPartition-wrt-trans*:
‹$(\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z]\!] \Longrightarrow R\ x\ z) \Longrightarrow$
$isPartition\text{-}wrt\ R\ xs\ lo\ hi\ p \Longrightarrow$
$(\forall i\ j.\ lo \leq i \wedge i < p \wedge p < j \wedge j \leq hi \longrightarrow R\ (xs!i)\ (xs!j))$›
⟨*proof*⟩

**lemma** *isPartition-map-trans*:
‹$(\bigwedge x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)) \Longrightarrow$
$hi < length\ xs \Longrightarrow$
$isPartition\text{-}map\ R\ h\ xs\ lo\ hi\ p \Longrightarrow$
$(\forall i\ j.\ lo \leq i \wedge i < p \wedge p < j \wedge j \leq hi \longrightarrow R\ (h\ (xs!i))\ (h\ (xs!j)))$›
⟨*proof*⟩

**lemma** *merge-sorted-wrt-partitions-between'*:
‹$lo \leq hi \Longrightarrow lo < p \Longrightarrow p < hi \Longrightarrow hi < length\ xs \Longrightarrow$
$isPartition\text{-}wrt\ R\ xs\ lo\ hi\ p \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ (p-1) \Longrightarrow sorted\text{-}sublist\text{-}wrt\ R\ xs\ (p+1)\ hi \Longrightarrow$
$(\forall i\ j.\ lo \leq i \wedge i < p \wedge p < j \wedge j \leq hi \longrightarrow R\ (xs!i)\ (xs!j)) \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ hi$›
⟨*proof*⟩

**lemma** *merge-sorted-wrt-partitions-between*:
‹$(\bigwedge x\ y\ z.\ [\![R\ x\ y;\ R\ y\ z]\!] \Longrightarrow R\ x\ z) \Longrightarrow$
$isPartition\text{-}wrt\ R\ xs\ lo\ hi\ p \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ (p-1) \Longrightarrow sorted\text{-}sublist\text{-}wrt\ R\ xs\ (p+1)\ hi \Longrightarrow$
$lo \leq hi \Longrightarrow hi < length\ xs \Longrightarrow lo < p \Longrightarrow p < hi \Longrightarrow hi < length\ xs \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ hi$›
⟨*proof*⟩

The main theorem to merge sorted lists

**lemma** *merge-sorted-wrt-partitions*:
‹$isPartition\text{-}wrt\ R\ xs\ lo\ hi\ p \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ (p - Suc\ 0) \Longrightarrow sorted\text{-}sublist\text{-}wrt\ R\ xs\ (Suc\ p)\ hi \Longrightarrow$
$lo \leq hi \Longrightarrow lo \leq p \Longrightarrow p \leq hi \Longrightarrow hi < length\ xs \Longrightarrow$
$(\forall i\ j.\ lo \leq i \wedge i < p \wedge p < j \wedge j \leq hi \longrightarrow R\ (xs!i)\ (xs!j)) \Longrightarrow$
$sorted\text{-}sublist\text{-}wrt\ R\ xs\ lo\ hi$›
⟨*proof*⟩

**theorem** *merge-sorted-map-partitions*:
‹$(\bigwedge x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)) \Longrightarrow$
$isPartition\text{-}map\ R\ h\ xs\ lo\ hi\ p \Longrightarrow$
$sorted\text{-}sublist\text{-}map\ R\ h\ xs\ lo\ (p - Suc\ 0) \Longrightarrow sorted\text{-}sublist\text{-}map\ R\ h\ xs\ (Suc\ p)\ hi \Longrightarrow$
$lo \leq hi \Longrightarrow lo \leq p \Longrightarrow p \leq hi \Longrightarrow hi < length\ xs \Longrightarrow$

*sorted-sublist-map R h xs lo hi*›
⟨*proof*⟩

**lemma** *partition-wrt-extend*:
‹*isPartition-wrt R xs lo′ hi′ p* ⟹
*hi* < *length xs* ⟹
*lo* ≤ *lo′* ⟹ *lo′* ≤ *hi* ⟹ *hi′* ≤ *hi* ⟹
*lo′* ≤ *p* ⟹ *p* ≤ *hi′* ⟹
(⋀ *i*. *lo*≤*i* ⟹ *i* <*lo′* ⟹ *R* (*xs*!*i*) (*xs*!*p*)) ⟹
(⋀ *j*. *hi′*<*j* ⟹ *j*≤*hi* ⟹ *R* (*xs*!*p*) (*xs*!*j*)) ⟹
*isPartition-wrt R xs lo hi p*›
⟨*proof*⟩

**lemma** *partition-map-extend*:
‹*isPartition-map R h xs lo′ hi′ p* ⟹
*hi* < *length xs* ⟹
*lo* ≤ *lo′* ⟹ *lo′* ≤ *hi* ⟹ *hi′* ≤ *hi* ⟹
*lo′* ≤ *p* ⟹ *p* ≤ *hi′* ⟹
(⋀ *i*. *lo*≤*i* ⟹ *i* <*lo′* ⟹ *R* (*h* (*xs*!*i*)) (*h* (*xs*!*p*))) ⟹
(⋀ *j*. *hi′*<*j* ⟹ *j*≤*hi* ⟹ *R* (*h* (*xs*!*p*)) (*h* (*xs*!*j*))) ⟹
*isPartition-map R h xs lo hi p*›
⟨*proof*⟩

**lemma** *isPartition-empty*:
‹(⋀ *j*. ⟦*lo* < *j*; *j* ≤ *hi*⟧ ⟹ *R* (*xs* ! *lo*) (*xs* ! *j*)) ⟹
*isPartition-wrt R xs lo hi lo*›
⟨*proof*⟩

**lemma** *take-ext*:
‹(∀ *i*<*k*. *xs′*!*i*=*xs*!*i*) ⟹
*k* < *length xs* ⟹ *k* < *length xs′* ⟹
*take k xs′* = *take k xs*›
⟨*proof*⟩

**lemma** *drop-ext′*:
‹(∀ *i*. *i*≥*k* ∧ *i*<*length xs* ⟶ *xs′*!*i*=*xs*!*i*) ⟹
*0*<*k* ⟹ *xs*≠[] ⟹ — These corner cases will be dealt with in the next lemma
*length xs′*=*length xs* ⟹
*drop k xs′* = *drop k xs*›
⟨*proof*⟩

**lemma** *drop-ext*:
‹(∀ *i*. *i*≥*k* ∧ *i*<*length xs* ⟶ *xs′*!*i*=*xs*!*i*) ⟹
*length xs′*=*length xs* ⟹
*drop k xs′* = *drop k xs*›
⟨*proof*⟩

**lemma** *sublist-ext′*:
‹(∀ *i*. *lo*≤*i*∧*i*≤*hi* ⟶ *xs′*!*i*=*xs*!*i*) ⟹
*length xs′* = *length xs* ⟹

*lo* ≤ *hi* ⟹ *Suc hi* < *length xs* ⟹
*sublist xs' lo hi* = *sublist xs lo hi*›
⟨*proof*⟩

**lemma** *lt-Suc*: ‹(*a* < *b*) = (*Suc a* = *b* ∨ *Suc a* < *b*)›
⟨*proof*⟩

**lemma** *sublist-until-end-eq-drop*: ‹*Suc hi* = *length xs* ⟹ *sublist xs lo hi* = *drop lo xs*›
⟨*proof*⟩

**lemma** *sublist-ext*:
‹(∀ *i*. *lo*≤*i*∧*i*≤*hi* ⟶ *xs'*!*i*=*xs*!*i*) ⟹
*length xs'* = *length xs* ⟹
*lo* ≤ *hi* ⟹ *hi* < *length xs* ⟹
*sublist xs' lo hi* = *sublist xs lo hi*›
⟨*proof*⟩

**lemma** *sorted-wrt-lower-sublist-still-sorted*:
**assumes** ‹*sorted-sublist-wrt R xs lo* (*lo'* − *Suc 0*)› **and**
‹*lo* ≤ *lo'*› **and** ‹*lo'* < *length xs*› **and**
‹(∀ *i*. *lo*≤*i*∧*i*<*lo'* ⟶ *xs'*!*i*=*xs*!*i*)› **and** ‹*length xs'* = *length xs*›
**shows** ‹*sorted-sublist-wrt R xs' lo* (*lo'* − *Suc 0*)›
⟨*proof*⟩

**lemma** *sorted-map-lower-sublist-still-sorted*:
**assumes** ‹*sorted-sublist-map R h xs lo* (*lo'* − *Suc 0*)› **and**
‹*lo* ≤ *lo'*› **and** ‹*lo'* < *length xs*› **and**
‹(∀ *i*. *lo*≤*i*∧*i*<*lo'* ⟶ *xs'*!*i*=*xs*!*i*)› **and** ‹*length xs'* = *length xs*›
**shows** ‹*sorted-sublist-map R h xs' lo* (*lo'* − *Suc 0*)›
⟨*proof*⟩

**lemma** *sorted-wrt-upper-sublist-still-sorted*:
**assumes** ‹*sorted-sublist-wrt R xs* (*hi'*+*1*) *hi*› **and**
‹*lo* ≤ *lo'*› **and** ‹*hi* < *length xs*› **and**
‹∀ *j*. *hi'*<*j*∧*j*≤*hi* ⟶ *xs'*!*j*=*xs*!*j*› **and** ‹*length xs'* = *length xs*›
**shows** ‹*sorted-sublist-wrt R xs'* (*hi'*+*1*) *hi*›
⟨*proof*⟩

**lemma** *sorted-map-upper-sublist-still-sorted*:
**assumes** ‹*sorted-sublist-map R h xs* (*hi'*+*1*) *hi*› **and**
‹*lo* ≤ *lo'*› **and** ‹*hi* < *length xs*› **and**
‹∀ *j*. *hi'*<*j*∧*j*≤*hi* ⟶ *xs'*!*j*=*xs*!*j*› **and** ‹*length xs'* = *length xs*›
**shows** ‹*sorted-sublist-map R h xs'* (*hi'*+*1*) *hi*›
⟨*proof*⟩

The specification of the partition function

**definition** *partition-spec* :: ‹('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ '*a list* ⇒ *nat* ⇒ *nat* ⇒ '*a list* ⇒ *nat* ⇒ *bool*› **where**
‹*partition-spec R h xs lo hi xs' p* ≡
*mset xs'* = *mset xs* ∧ — The list is a permutation
*isPartition-map R h xs' lo hi p* ∧ — We have a valid partition on the resulting list
*lo* ≤ *p* ∧ *p* ≤ *hi* ∧ — The partition index is in bounds
(∀ *i*. *i*<*lo* ⟶ *xs'*!*i*=*xs*!*i*) ∧ (∀ *i*. *hi*<*i*∧*i*<*length xs'* ⟶ *xs'*!*i*=*xs*!*i*)› — Everything else is unchanged.

**lemma** *in-set-take-conv-nth*:
‹$x \in set\ (take\ n\ xs) \longleftrightarrow (\exists\, m{<}min\ n\ (length\ xs).\ xs\ !\ m = x)$›
⟨*proof*⟩

**lemma** *mset-drop-upto*: ‹$mset\ (drop\ a\ N) = \{\#N!i.\ i \in\# mset\text{-}set\ \{a..{<}length\ N\}\#\}$›
⟨*proof*⟩

**lemma** *mathias*:
**assumes**
*Perm*: ‹$mset\ xs' = mset\ xs$›
**and** *I*: ‹$lo{\le}i$› ‹$i{\le}hi$› ‹$xs'!i{=}x$›
**and** *Bounds*: ‹$hi < length\ xs$›
**and** *Fix*: ‹$\bigwedge i.\ i{<}lo \Longrightarrow xs'!i = xs!i$› ‹$\bigwedge j.\ [\![hi{<}j;\ j{<}length\ xs]\!] \Longrightarrow xs'!j = xs!j$›
**shows** ‹$\exists\, j.\ lo{\le}j \wedge j{\le}hi \wedge xs!j = x$›
⟨*proof*⟩

If we fix the left and right rest of two permutated lists, then the sublists are also permutations.

But we only need that the sets are equal.

**lemma** *mset-sublist-incl*:
**assumes** *Perm*: ‹$mset\ xs' = mset\ xs$›
**and** *Fix*: ‹$\bigwedge i.\ i{<}lo \Longrightarrow xs'!i = xs!i$› ‹$\bigwedge j.\ [\![hi{<}j;\ j{<}length\ xs]\!] \Longrightarrow xs'!j = xs!j$›
**and** *bounds*: ‹$lo \le hi$› ‹$hi < length\ xs$›
**shows** ‹$set\ (sublist\ xs'\ lo\ hi) \subseteq set\ (sublist\ xs\ lo\ hi)$›
⟨*proof*⟩

**lemma** *mset-sublist-eq*:
**assumes** ‹$mset\ xs' = mset\ xs$›
**and** ‹$\bigwedge i.\ i{<}lo \Longrightarrow xs'!i = xs!i$›
**and** ‹$\bigwedge j.\ [\![hi{<}j;\ j{<}length\ xs]\!] \Longrightarrow xs'!j = xs!j$›
**and** *bounds*: ‹$lo \le hi$› ‹$hi < length\ xs$›
**shows** ‹$set\ (sublist\ xs'\ lo\ hi) = set\ (sublist\ xs\ lo\ hi)$›
⟨*proof*⟩

Our abstract recursive quicksort procedure. We abstract over a partition procedure.

**definition** *quicksort* :: ‹$('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow nat \times nat \times 'a\ list \Rightarrow 'a\ list\ nres$› **where**
‹*quicksort R h* = $(\lambda(lo,hi,xs0).\ do\ \{$
*RECT* $(\lambda f\ (lo,hi,xs).\ do\ \{$
*ASSERT*$(lo \le hi \wedge hi < length\ xs \wedge mset\ xs = mset\ xs0)$; — Premise for a partition function
$(xs,\ p) \leftarrow$ *SPEC*$(uncurry\ (partition\text{-}spec\ R\ h\ xs\ lo\ hi))$; — Abstract partition function
*ASSERT*$(mset\ xs = mset\ xs0)$;
$xs \leftarrow (if\ p{-}1{\le}lo\ then\ RETURN\ xs\ else\ f\ (lo,\ p{-}1,\ xs))$;
*ASSERT*$(mset\ xs = mset\ xs0)$;
*if* $hi{\le}p{+}1$ *then RETURN xs else* $f\ (p{+}1,\ hi,\ xs)$
$\})\ (lo,hi,xs0)$
$\}$)›

As premise for quicksor, we only need that the indices are ok.

**definition** *quicksort-pre* :: ‹$('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ list \Rightarrow\ nat \Rightarrow nat \Rightarrow 'a\ list \Rightarrow bool$›
**where**
‹*quicksort-pre R h xs0 lo hi xs* $\equiv lo \le hi \wedge hi < length\ xs \wedge mset\ xs = mset\ xs0$›

**definition** *quicksort-post* :: ‹$('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow nat \Rightarrow nat \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool$›
**where**

‹*quicksort-post R h lo hi xs xs'* ≡
*mset xs'* = *mset xs* ∧
*sorted-sublist-map R h xs' lo hi* ∧
(∀ *i*. *i*<*lo* ⟶ *xs'*!*i* = *xs*!*i*) ∧
(∀ *j*. *hi*<*j*∧*j*<*length xs* ⟶ *xs'*!*j* = *xs*!*j*)›

Convert Pure to HOL

**lemma** *quicksort-postI*:
‹⟦*mset xs'* = *mset xs*; *sorted-sublist-map R h xs' lo hi*; (⋀ *i*. ⟦*i*<*lo*⟧ ⟹ *xs'*!*i* = *xs*!*i*); (⋀ *j*. ⟦*hi*<*j*; *j*<*length xs*⟧ ⟹ *xs'*!*j* = *xs*!*j*)⟧ ⟹ *quicksort-post R h lo hi xs xs'*›
⟨*proof*⟩

The first case for the correctness proof of (abstract) quicksort: We assume that we called the partition function, and we have *p* − *1* ≤ *lo* and *hi* ≤ *p* + *1*.

**lemma** *quicksort-correct-case1*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *x* ≠ *y* ⟹ *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹*p*−*1* ≤ *lo*› ‹*hi* ≤ *p*+*1*›
**shows** ‹*quicksort-post R h lo hi xs xs'*›
⟨*proof*⟩

In the second case, we have to show that the precondition still holds for (p+1, hi, x') after the partition.

**lemma** *quicksort-correct-case2*:
**assumes**
*pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹¬ *hi* ≤ *p* + *1*›
**shows** ‹*quicksort-pre R h xs0* (*Suc p*) *hi xs'*›
⟨*proof*⟩

**lemma** *quicksort-post-set*:
**assumes** ‹*quicksort-post R h lo hi xs xs'*›
**and** *bounds*: ‹*lo* ≤ *hi*› ‹*hi* < *length xs*›
**shows** ‹*set* (*sublist xs' lo hi*) = *set* (*sublist xs lo hi*)›
⟨*proof*⟩

In the third case, we have run quicksort recursively on (p+1, hi, xs') after the partition, with hi<=p+1 and p-1<=lo.

**lemma** *quicksort-correct-case3*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *x* ≠ *y* ⟹ *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹*p* − *Suc 0* ≤ *lo*› ‹¬ *hi* ≤ *Suc p*›
**and** *IH1'*: ‹*quicksort-post R h* (*Suc p*) *hi xs' xs''*›
**shows** ‹*quicksort-post R h lo hi xs xs''*›
⟨*proof*⟩

In the 4th case, we have to show that the premise holds for (*lo*, *p* − *1*, *xs'*), in case ¬ *p* − *1* ≤ *lo*

Analogous to case 2.

**lemma** *quicksort-correct-case4*:
**assumes**
*pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹¬ *p − Suc 0 ≤ lo* ›
**shows** ‹*quicksort-pre R h xs0 lo* (*p−Suc 0*) *xs'*›
⟨*proof*⟩

In the 5th case, we have run quicksort recursively on (lo, p-1, xs').

**lemma** *quicksort-correct-case5*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *x* ≠ *y* ⟹ *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹¬ *p − Suc 0 ≤ lo*› ‹*hi ≤ Suc p*›
**and** *IH1'*: ‹*quicksort-post R h lo* (*p − Suc 0*) *xs' xs''*›
**shows** ‹*quicksort-post R h lo hi xs xs''*›
⟨*proof*⟩

In the 6th case, we have run quicksort recursively on (lo, p-1, xs'). We show the precondition on the second call on (p+1, hi, xs")

**lemma** *quicksort-correct-case6*:
**assumes**
*pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹¬ *p − Suc 0 ≤ lo*› ‹¬ *hi ≤ Suc p*›
**and** *IH1*: ‹*quicksort-post R h lo* (*p − Suc 0*) *xs' xs''*›
**shows** ‹*quicksort-pre R h xs0* (*Suc p*) *hi xs''*›
⟨*proof*⟩

In the 7th (and last) case, we have run quicksort recursively on (lo, p-1, xs'). We show the postcondition on the second call on (p+1, hi, xs")

**lemma** *quicksort-correct-case7*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *x* ≠ *y* ⟹ *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *pre*: ‹*quicksort-pre R h xs0 lo hi xs*›
**and** *part*: ‹*partition-spec R h xs lo hi xs' p*›
**and** *ifs*: ‹¬ *p − Suc 0 ≤ lo*› ‹¬ *hi ≤ Suc p*›
**and** *IH1'*: ‹*quicksort-post R h lo* (*p − Suc 0*) *xs' xs''*›
**and** *IH2'*: ‹*quicksort-post R h* (*Suc p*) *hi xs'' xs'''*›
**shows** ‹*quicksort-post R h lo hi xs xs'''*›
⟨*proof*⟩

We can now show the correctness of the abstract quicksort procedure, using the refinement framework and the above case lemmas.

**lemma** *quicksort-correct*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *x* ≠ *y* ⟹ *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *Pre*: ‹*lo0 ≤ hi0*› ‹*hi0 < length xs0*›
**shows** ‹*quicksort R h* (*lo0*,*hi0*,*xs0*) ≤ ⇓ *Id* (*SPEC*(λ*xs*. *quicksort-post R h lo0 hi0 xs0 xs*))›
⟨*proof*⟩

**definition** *partition-main-inv* :: ‹('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ nat ⇒ nat ⇒ 'a list ⇒ (nat×nat×'a list) ⇒ bool› **where**
‹*partition-main-inv R h lo hi xs0 p* ≡
*case p of* (*i*,*j*,*xs*) ⇒
*j* < *length xs* ∧ *j* ≤ *hi* ∧ *i* < *length xs* ∧ *lo* ≤ *i* ∧ *i* ≤ *j* ∧ *mset xs* = *mset xs0* ∧
(∀ *k*. *k* ≥ *lo* ∧ *k* < *i* ⟶ *R* (*h* (*xs*!*k*)) (*h* (*xs*!*hi*))) ∧ — All elements from *lo* to *i* − *1* are smaller than the pivot
(∀ *k*. *k* ≥ *i* ∧ *k* < *j* ⟶ *R* (*h* (*xs*!*hi*)) (*h* (*xs*!*k*))) ∧ — All elements from *i* to *j* − *1* are greater than the pivot
(∀ *k*. *k* < *lo* ⟶ *xs*!*k* = *xs0*!*k*) ∧ — Everything below *lo* is unchanged
(∀ *k*. *k* ≥ *j* ∧ *k* < *length xs* ⟶ *xs*!*k* = *xs0*!*k*) — All elements from *j* are unchanged (including everyting above *hi*)
›

The main part of the partition function. The pivot is assumed to be the last element. This is exactly the "Lomuto partition scheme" partition function from Wikipedia.

**definition** *partition-main* :: ‹('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ nat ⇒ nat ⇒ 'a list ⇒ ('a list × nat) *nres*› **where**
‹*partition-main R h lo hi xs0* = *do* {
*ASSERT*(*hi* < *length xs0*);
*pivot* ← *RETURN* (*h* (*xs0* ! *hi*));
(*i*,*j*,*xs*) ← *WHILE*$_T^{partition\text{-}main\text{-}inv\ R\ h\ lo\ hi\ xs0}$ — We loop from *j* = *lo* to *j* = *hi* − *1*.
(λ(*i*,*j*,*xs*). *j* < *hi*)
(λ(*i*,*j*,*xs*). *do* {
*ASSERT*(*i* < *length xs* ∧ *j* < *length xs*);
*if R* (*h* (*xs*!*j*)) *pivot*
*then RETURN* (*i*+*1*, *j*+*1*, *swap xs i j*)
*else RETURN* (*i*, *j*+*1*, *xs*)
})
(*lo*, *lo*, *xs0*); — i and j are both initialized to lo
*ASSERT*(*i* < *length xs* ∧ *j* = *hi* ∧ *lo* ≤ *i* ∧ *hi* < *length xs* ∧ *mset xs* = *mset xs0*);
*RETURN* (*swap xs i hi*, *i*)
}›

**lemma** *partition-main-correct*:
**assumes** *bounds*: ‹*hi* < *length xs*› ‹*lo* ≤ *hi*› **and**
*trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**shows** ‹*partition-main R h lo hi xs* ≤ *SPEC*(λ(*xs'*, *p*). *mset xs* = *mset xs'* ∧
*lo* ≤ *p* ∧ *p* ≤ *hi* ∧ *isPartition-map R h xs' lo hi p* ∧ (∀ *i*. *i*<*lo* ⟶ *xs'*!*i*=*xs*!*i*) ∧ (∀ *i*. *hi*<*i*∧*i*<*length xs'* ⟶ *xs'*!*i*=*xs*!*i*))›
⟨*proof*⟩

**definition** *partition-between* :: ‹('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ nat ⇒ nat ⇒ 'a list ⇒ ('a list × nat) *nres*› **where**
‹*partition-between R h lo hi xs0* = *do* {
*ASSERT*(*hi* < *length xs0* ∧ *lo* ≤ *hi*);
*k* ← *choose-pivot R h xs0 lo hi*; — choice of pivot

*ASSERT*(*k* < *length xs0*);
*xs* ← *RETURN* (*swap xs0 k hi*); — move the pivot to the last position, before we start the actual loop
*ASSERT*(*length xs* = *length xs0*);
*partition-main R h lo hi xs*
}›

**lemma** *partition-between-correct*:
**assumes** ‹*hi* < *length xs*› **and** ‹*lo* ≤ *hi*› **and**
‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** ‹⋀*x y*. *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**shows** ‹*partition-between R h lo hi xs* ≤ *SPEC*(*uncurry* (*partition-spec R h xs lo hi*))›
⟨*proof*⟩

We use the median of the first, the middle, and the last element.

**definition** *choose-pivot3* **where**
‹*choose-pivot3 R h xs lo* (*hi*::*nat*) = *do* {
*ASSERT*(*lo* < *length xs*);
*ASSERT*(*hi* < *length xs*);
*let k'* = (*hi* − *lo*) *div 2*;
*let k* = *lo* + *k'*;
*ASSERT*(*k* < *length xs*);
*let start* = *h* (*xs* ! *lo*);
*let mid* = *h* (*xs* ! *k*);
*let end* = *h* (*xs* ! *hi*);
*if* (*R start mid* ∧ *R mid end*) ∨ (*R end mid* ∧ *R mid start*) *then RETURN k*
*else if* (*R start end* ∧ *R end mid*) ∨ (*R mid end* ∧ *R end start*) *then RETURN hi*
*else RETURN lo*
}›

— We only have to show that this procedure yields a valid index between *lo* and *hi*.
**lemma** *choose-pivot3-choose-pivot*:
**assumes** ‹*lo* < *length xs*› ‹*hi* < *length xs*› ‹*hi* ≥ *lo*›
**shows** ‹*choose-pivot3 R h xs lo hi* ≤ ⇓ *Id* (*choose-pivot R h xs lo hi*)›
⟨*proof*⟩

The refined partion function: We use the above pivot function and fold instead of non-deterministic iteration.

**definition** *partition-between-ref*
:: ‹('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *nat* ⇒ *nat* ⇒ '*a list* ⇒ ('*a list* × *nat*) *nres*›
**where**
‹*partition-between-ref R h lo hi xs0* = *do* {
*ASSERT*(*hi* < *length xs0* ∧ *hi* < *length xs0* ∧ *lo* ≤ *hi*);
*k* ← *choose-pivot3 R h xs0 lo hi*; — choice of pivot
*ASSERT*(*k* < *length xs0*);
*xs* ← *RETURN* (*swap xs0 k hi*); — move the pivot to the last position, before we start the actual loop
*ASSERT*(*length xs* = *length xs0*);
*partition-main R h lo hi xs*
}›

**lemma** *partition-main-ref'*:
‹*partition-main R h lo hi xs*
≤ ⇓ ((λ *a b c d*. *Id*) *a b c d*) (*partition-main R h lo hi xs*)›

⟨*proof*⟩

**lemma** *Down-id-eq*:
‹⇓*Id x* = *x*›
⟨*proof*⟩

**lemma** *partition-between-ref-partition-between*:
‹*partition-between-ref R h lo hi xs* ≤ (*partition-between R h lo hi xs*)›
⟨*proof*⟩

Technical lemma for sepref

**lemma** *partition-between-ref-partition-between′*:
‹(*uncurry2* (*partition-between-ref R h*), *uncurry2* (*partition-between R h*)) ∈
(*nat-rel* ×$_r$ *nat-rel*) ×$_r$ ⟨*Id*⟩*list-rel* →$_f$ ⟨⟨*Id*⟩*list-rel* ×$_r$ *nat-rel*⟩*nres-rel*›
⟨*proof*⟩

Example instantiation for pivot

**definition** *choose-pivot3-impl* **where**
‹*choose-pivot3-impl* = *choose-pivot3* (≤) *id*›

**lemma** *partition-between-ref-correct*:
**assumes** *trans*: ‹⋀ *x y z*. ⟦*R* (*h x*) (*h y*); *R* (*h y*) (*h z*)⟧ ⟹ *R* (*h x*) (*h z*)› **and** *lin*: ‹⋀*x y*. *R* (*h x*) (*h y*) ∨ *R* (*h y*) (*h x*)›
**and** *bounds*: ‹*hi* < *length xs*› ‹*lo* ≤ *hi*›
**shows** ‹*partition-between-ref R h lo hi xs* ≤ *SPEC* (*uncurry* (*partition-spec R h xs lo hi*))›
⟨*proof*⟩

Refined quicksort algorithm: We use the refined partition function.

**definition** *quicksort-ref* :: ‹- ⇒ - ⇒ *nat* × *nat* × *′a list* ⇒ *′a list nres*› **where**
‹*quicksort-ref R h* = (λ(*lo*,*hi*,*xs0*).
*do* {
*RECT* (λ*f* (*lo*,*hi*,*xs*). *do* {
*ASSERT*(*lo* ≤ *hi* ∧ *hi* < *length xs0* ∧ *mset xs* = *mset xs0*);
(*xs*, *p*) ← *partition-between-ref R h lo hi xs*; — This is the refined partition function. Note that we need the premises (trans,lin,bounds) here.
*ASSERT*(*mset xs* = *mset xs0* ∧ *p* ≥ *lo* ∧ *p* < *length xs0*);
*xs* ← (*if p*−*1*≤*lo then RETURN xs else f* (*lo*, *p*−*1*, *xs*));
*ASSERT*(*mset xs* = *mset xs0*);
*if hi*≤*p*+*1 then RETURN xs else f* (*p*+*1*, *hi*, *xs*)
}) (*lo*,*hi*,*xs0*)
})›

**lemma** *fref-to-Down-curry2*:
‹(*uncurry2 f*, *uncurry2 g*) ∈ [*P*]$_f$ *A* → ⟨*B*⟩*nres-rel* ⟹
(⋀*x x′ y y′ z z′*. *P* ((*x′*, *y′*), *z′*) ⟹ (((*x*, *y*), *z*), ((*x′*, *y′*), *z′*)) ∈ *A*⟹
*f x y z* ≤ ⇓ *B* (*g x′ y′ z′*))›
⟨*proof*⟩

**lemma** *fref-to-Down-curry*:
‹(*f*, *g*) ∈ [*P*]$_f$ *A* → ⟨*B*⟩*nres-rel* ⟹

$(\bigwedge x\ x'\ .\ P\ x' \Longrightarrow (x,\ x') \in A \Longrightarrow$
$f\ x\ \leq \Downarrow B\ (g\ x'))$›
⟨*proof*⟩

**lemma** *quicksort-ref-quicksort*:
**assumes** *bounds*: ‹$hi < length\ xs$› ‹$lo \leq hi$› **and**
*trans*: ‹$\bigwedge\ x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)$› **and** *lin*: ‹$\bigwedge x\ y.\ R\ (h\ x)\ (h\ y) \vee R\ (h\ y)\ (h\ x)$›
**shows** ‹$quicksort\text{-}ref\ R\ h\ x0 \leq \Downarrow Id\ (quicksort\ R\ h\ x0)$›
⟨*proof*⟩
**definition** *full-quicksort* **where**
‹$full\text{-}quicksort\ R\ h\ xs \equiv if\ xs = []\ then\ RETURN\ xs\ else\ quicksort\ R\ h\ (0,\ length\ xs - 1,\ xs)$›

**definition** *full-quicksort-ref* **where**
‹$full\text{-}quicksort\text{-}ref\ R\ h\ xs \equiv$
$if\ List.null\ xs\ then\ RETURN\ xs$
$else\ quicksort\text{-}ref\ R\ h\ (0,\ length\ xs - 1,\ xs)$›

**definition** *full-quicksort-impl* :: ‹$nat\ list \Rightarrow nat\ list\ nres$› **where**
‹$full\text{-}quicksort\text{-}impl\ xs = full\text{-}quicksort\text{-}ref\ (\leq)\ id\ xs$›

**lemma** *full-quicksort-ref-full-quicksort*:
**assumes** *trans*: ‹$\bigwedge\ x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)$› **and** *lin*: ‹$\bigwedge x\ y.\ R\ (h\ x)\ (h\ y) \vee R\ (h\ y)\ (h\ x)$›
**shows** ‹$(full\text{-}quicksort\text{-}ref\ R\ h,\ full\text{-}quicksort\ R\ h) \in$
$\langle Id\rangle list\text{-}rel \rightarrow_f \langle\ \langle Id\rangle list\text{-}rel\rangle nres\text{-}rel$›
⟨*proof*⟩

**lemma** *sublist-entire*:
‹$sublist\ xs\ 0\ (length\ xs - 1) = xs$›
⟨*proof*⟩

**lemma** *sorted-sublist-wrt-entire*:
**assumes** ‹$sorted\text{-}sublist\text{-}wrt\ R\ xs\ 0\ (length\ xs - 1)$›
**shows** ‹$sorted\text{-}wrt\ R\ xs$›
⟨*proof*⟩

**lemma** *sorted-sublist-map-entire*:
**assumes** ‹$sorted\text{-}sublist\text{-}map\ R\ h\ xs\ 0\ (length\ xs - 1)$›
**shows** ‹$sorted\text{-}wrt\ (\lambda\ x\ y.\ R\ (h\ x)\ (h\ y))\ xs$›
⟨*proof*⟩

Final correctness lemma

**theorem** *full-quicksort-correct-sorted*:
**assumes**
*trans*: ‹$\bigwedge x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)$› **and** *lin*: ‹$\bigwedge x\ y.\ x \neq y \Longrightarrow R\ (h\ x)\ (h\ y) \vee R\ (h\ y)\ (h\ x)$›
**shows** ‹$full\text{-}quicksort\ R\ h\ xs \leq \Downarrow Id\ (SPEC(\lambda xs'.\ mset\ xs' = mset\ xs \wedge sorted\text{-}wrt\ (\lambda\ x\ y.\ R\ (h\ x)\ (h\ y))\ xs'))$›
⟨*proof*⟩

**lemma** *full-quicksort-correct*:
**assumes**
*trans*: ‹$\bigwedge x\ y\ z.\ [\![R\ (h\ x)\ (h\ y);\ R\ (h\ y)\ (h\ z)]\!] \Longrightarrow R\ (h\ x)\ (h\ z)$› **and**
*lin*: ‹$\bigwedge x\ y.\ R\ (h\ x)\ (h\ y) \vee R\ (h\ y)\ (h\ x)$›
**shows** ‹*full-quicksort R h xs* $\leq \Downarrow$ *Id* (*SPEC*($\lambda xs'.$ *mset* $xs' =$ *mset xs*))›
⟨*proof*⟩

**end**

**theory** *More-Loops*
**imports**
*Refine-Monadic.Refine-While*
*Refine-Monadic.Refine-Foreach*
*HOL−Library.Rewrite*
**begin**

## 3.3 More Theorem about Loops

Most theorem below have a counterpart in the Refinement Framework that is weaker (by missing assertions for example that are critical for code generation).

**lemma** *Down-id-eq*:
‹$\Downarrow$*Id x* = *x*›
⟨*proof*⟩

**lemma** *while-upt-while-direct1*:
$b \geq a \Longrightarrow$
*do* {
(-,$\sigma$) ← *WHILE*$_T$ (*FOREACH-cond c*) ($\lambda x.$ *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*})
([*a*..<*b*],$\sigma$);
*RETURN* $\sigma$
} ≤ *do* {
(-,$\sigma$) ← *WHILE*$_T$ ($\lambda(i,\ x).\ i < b \wedge c\ x$) ($\lambda(i,\ x).$ *do* {*ASSERT* ($i < b$); $\sigma'$←*f i x*; *RETURN* ($i$+1,$\sigma'$)}) (*a*,$\sigma$);
*RETURN* $\sigma$
}
⟨*proof*⟩

**lemma** *while-upt-while-direct2*:
$b \geq a \Longrightarrow$
*do* {
(-,$\sigma$) ← *WHILE*$_T$ (*FOREACH-cond c*) ($\lambda x.$ *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*})
([*a*..<*b*],$\sigma$);
*RETURN* $\sigma$
} ≥ *do* {
(-,$\sigma$) ← *WHILE*$_T$ ($\lambda(i,\ x).\ i < b \wedge c\ x$) ($\lambda(i,\ x).$ *do* {*ASSERT* ($i < b$); $\sigma'$←*f i x*; *RETURN* ($i$+1,$\sigma'$)}) (*a*,$\sigma$);
*RETURN* $\sigma$
}
⟨*proof*⟩

**lemma** *while-upt-while-direct*:
$b \geq a \Longrightarrow$
*do* {

(-,σ) ← $WHILE_T$ (*FOREACH-cond c*) (λx. *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*})
([*a*..<*b*],σ);
*RETURN* σ
} = *do* {
(-,σ) ← $WHILE_T$ (λ(*i*, *x*). *i* < *b* ∧ *c x*) (λ(*i*, *x*). *do* {*ASSERT* (*i* < *b*); σ'←*f i x*; *RETURN* (*i*+*1*,σ')}) (*a*,σ);
*RETURN* σ
}
⟨*proof*⟩

**lemma** *while-nfoldli*:
*do* {
(-,σ) ← $WHILE_T$ (*FOREACH-cond c*) (λx. *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*}) (*l*,σ);
*RETURN* σ
} ≤ *nfoldli l c f* σ
⟨*proof*⟩
**lemma** *nfoldli-while*: *nfoldli l c f* σ
≤
($WHILE_T{}^I$
(*FOREACH-cond c*) (λx. *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*}) (*l*, σ)
≫=
(λ(-, σ). *RETURN* σ))
⟨*proof*⟩

**lemma** *while-eq-nfoldli*: *do* {
(-,σ) ← $WHILE_T$ (*FOREACH-cond c*) (λx. *do* {*ASSERT* (*FOREACH-cond c x*); *FOREACH-body f x*}) (*l*,σ);
*RETURN* σ
} = *nfoldli l c f* σ
⟨*proof*⟩

**end**

**theory** *PAC-Specification*
**imports** *PAC-More-Poly*
**begin**

# 4 Specification of the PAC checker

## 4.1 Ideals

**type-synonym** *int-poly* = ‹*int mpoly*›
**definition** *polynomial-bool* :: ‹*int-poly set*› **where**
‹*polynomial-bool* = (λc. *Var c* ^ 2 − *Var c*) ‘ *UNIV*›

**definition** *pac-ideal* **where**
‹*pac-ideal A* ≡ *ideal* (*A* ∪ *polynomial-bool*)›

**lemma** *X2-X-in-pac-ideal*:
‹*Var c* ^ 2 − *Var c* ∈ *pac-ideal A*›
⟨*proof*⟩

**lemma** *pac-idealI1*[*intro*]:

‹$p \in A \Longrightarrow p \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *pac-idealI2*[*intro*]:
‹$p \in$ *ideal* $A \Longrightarrow p \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *pac-idealI3*[*intro*]:
‹$p \in$ *ideal* $A \Longrightarrow p*q \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *pac-ideal-Xsq2-iff*:
‹*Var* $c \hat{} 2 \in$ *pac-ideal* $A \longleftrightarrow$ *Var* $c \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *diff-in-polynomial-bool-pac-idealI*:
**assumes** *a1*: $p \in$ *pac-ideal* $A$
**assumes** *a2*: $p - p' \in$ *More-Modules.ideal polynomial-bool*
**shows** ‹$p' \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *diff-in-polynomial-bool-pac-idealI2*:
**assumes** *a1*: $p \in A$
**assumes** *a2*: $p - p' \in$ *More-Modules.ideal polynomial-bool*
**shows** ‹$p' \in$ *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *pac-ideal-alt-def*:
‹*pac-ideal* $A =$ *ideal* $(A \cup$ *ideal polynomial-bool*$)$›
⟨*proof*⟩

The equality on ideals is restricted to polynomials whose variable appear in the set of ideals. The function restrict sets:

**definition** *restricted-ideal-to* **where**
‹*restricted-ideal-to* $B\ A = \{p \in A.$ *vars* $p\ \subseteq B\}$›

**abbreviation** *restricted-ideal-to*$_I$ **where**
‹*restricted-ideal-to*$_I$ $B\ A \equiv$ *restricted-ideal-to* $B$ (*pac-ideal* (*set-mset* $A$))›

**abbreviation** *restricted-ideal-to*$_V$ **where**
‹*restricted-ideal-to*$_V$ $B \equiv$ *restricted-ideal-to* $(\bigcup$(*vars* ‘ *set-mset* $B$))›

**abbreviation** *restricted-ideal-to*$_{V\,I}$ **where**
‹*restricted-ideal-to*$_{V\,I}$ $B\ A \equiv$ *restricted-ideal-to* $(\bigcup$(*vars* ‘ *set-mset* $B$)) (*pac-ideal* (*set-mset* $A$))›

**lemma** *restricted-idealI*:
‹$p \in$ *pac-ideal* (*set-mset* $A$) $\Longrightarrow$ *vars* $p \subseteq C \Longrightarrow p \in$ *restricted-ideal-to*$_I$ $C\ A$›
⟨*proof*⟩

**lemma** *pac-ideal-insert-already-in*:
‹$pq \in$ *pac-ideal* (*set-mset* $A$) $\Longrightarrow$ *pac-ideal* (*insert* $pq$ (*set-mset* $A$)) $=$ *pac-ideal* (*set-mset* $A$)›
⟨*proof*⟩

**lemma** *pac-ideal-add*:

‹$p \in\# A \Longrightarrow q \in\# A \Longrightarrow p + q \in$ *pac-ideal* (*set-mset A*)›
⟨*proof*⟩
**lemma** *pac-ideal-mult*:
‹$p \in\# A \Longrightarrow p * q \in$ *pac-ideal* (*set-mset A*)›
⟨*proof*⟩

**lemma** *pac-ideal-mono*:
‹$A \subseteq B \Longrightarrow$ *pac-ideal A* $\subseteq$ *pac-ideal B*›
⟨*proof*⟩

## 4.2 PAC Format

The PAC format contains three kind of steps:

- `add` that adds up two polynomials that are known.
- `mult` that multiply a known polynomial with another one.
- `del` that removes a polynomial that cannot be reused anymore.

To model the simplification that happens, we add the $p - p' \in$ *polynomial-bool* stating that $p$ and $p'$ are equivalent.

**type-synonym** *pac-st* = ‹(*nat set* × *int-poly multiset*)›

**inductive** *PAC-Format* :: ‹*pac-st* ⇒ *pac-st* ⇒ *bool*› **where**
*add*:
‹*PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}$, *add-mset p' A*)›
**if**
‹$p \in\# A$› ‹$q \in\# A$›
‹$p+q - p' \in$ *ideal polynomial-bool*›
‹*vars* $p' \subseteq \mathcal{V}$› |
*mult*:
‹*PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}$, *add-mset p' A*)›
**if**
‹$p \in\# A$›
‹$p*q - p' \in$ *ideal polynomial-bool*›
‹*vars* $p' \subseteq \mathcal{V}$›
‹*vars* $q \subseteq \mathcal{V}$› |
*del*:
‹$p \in\# A \Longrightarrow$ *PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}$, *A* − {#*p*#})› |
*extend-pos*:
‹*PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V} \cup \{x' \in$ *vars* $(-$ *Var x* + *p'*$)$. $x' \notin \mathcal{V}\}$, *add-mset* $(-$ *Var x* + *p'*$)$ *A*)›
**if**
‹$(p')^2 - p' \in$ *ideal polynomial-bool*›
‹*vars* $p' \subseteq \mathcal{V}$›
‹$x \notin \mathcal{V}$›

In the PAC format above, we have a technical condition on the normalisation: *vars* $p' \subseteq$ *vars* $(p + q)$ is here to ensure that we don't normalise *0* to $(Var\ x)^2 - Var\ x$ for a new variable $x$. This is completely obvious for the normalisation process we have in mind when we write the specification, but we must add it explicitly because we are too general.

**lemmas** *PAC-Format-induct-split* =
*PAC-Format.induct*[*split-format*(*complete*), *of V A V' A'* **for** *V A V' A'*]

**lemma** *PAC-Format-induct*[*consumes 1*, *case-names add mult del ext*]:
**assumes**
‹*PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}'$, *A'*)› **and**
*cases*:
‹$\bigwedge p\ q\ p'\ A\ \mathcal{V}.\ p \in\# A \Longrightarrow q \in\# A \Longrightarrow p+q-p' \in$ *ideal polynomial-bool* $\Longrightarrow$ *vars* $p' \subseteq \mathcal{V} \Longrightarrow P\ \mathcal{V}\ A\ \mathcal{V}$ (*add-mset* $p'$ *A*)›
‹$\bigwedge p\ q\ p'\ A\ \mathcal{V}.\ p \in\# A \Longrightarrow p*q-p' \in$ *ideal polynomial-bool* $\Longrightarrow$ *vars* $p' \subseteq \mathcal{V} \Longrightarrow$ *vars* $q \subseteq \mathcal{V} \Longrightarrow$ $P\ \mathcal{V}\ A\ \mathcal{V}$ (*add-mset* $p'$ *A*)›
‹$\bigwedge p\ A\ \mathcal{V}.\ p \in\# A \Longrightarrow P\ \mathcal{V}\ A\ \mathcal{V}\ (A - \{\#p\#\})$›
‹$\bigwedge p'\ x\ r.$
$(p')\hat{}2 - (p') \in$ *ideal polynomial-bool* $\Longrightarrow$ *vars* $p' \subseteq \mathcal{V} \Longrightarrow$
$x \notin \mathcal{V} \Longrightarrow P\ \mathcal{V}\ A\ (\mathcal{V} \cup \{x' \in$ *vars* $(p' - Var\ x).\ x' \notin \mathcal{V}\})$ (*add-mset* $(p' - Var\ x)\ A$)›
**shows**
‹$P\ \mathcal{V}\ A\ \mathcal{V}'\ A'$›
⟨*proof*⟩

The theorem below (based on the proof ideal by Manuel Kauers) is the correctness theorem of extensions. Remark that the assumption *vars* $q \subseteq \mathcal{V}$ is only used to show that $x' \notin$ *vars* $q$.

**lemma** *extensions-are-safe*:
**assumes** ‹$x' \in$ *vars* $p$› **and**
$x'$: ‹$x' \notin \mathcal{V}$› **and**
‹$\bigcup$ (*vars* ‘ *set-mset A*) $\subseteq \mathcal{V}$› **and**
*p-x-coeff*: ‹*coeff p* (*monomial* (*Suc 0*) $x'$) = *1*› **and**
*vars-q*: ‹*vars q* $\subseteq \mathcal{V}$› **and**
*q*: ‹$q \in$ *More-Modules.ideal* (*insert p* (*set-mset A* $\cup$ *polynomial-bool*))› **and**
*leading*: ‹$x' \notin$ *vars* $(p - Var\ x')$› **and**
*diff*: ‹$(Var\ x' - p)^2 - (Var\ x' - p) \in$ *More-Modules.ideal polynomial-bool*›
**shows**
‹$q \in$ *More-Modules.ideal* (*set-mset A* $\cup$ *polynomial-bool*)›
⟨*proof*⟩

**lemma** *extensions-are-safe-uminus*:
**assumes** ‹$x' \in$ *vars* $p$› **and**
$x'$: ‹$x' \notin \mathcal{V}$› **and**
‹$\bigcup$ (*vars* ‘ *set-mset A*) $\subseteq \mathcal{V}$› **and**
*p-x-coeff*: ‹*coeff p* (*monomial* (*Suc 0*) $x'$) = $-1$› **and**
*vars-q*: ‹*vars q* $\subseteq \mathcal{V}$› **and**
*q*: ‹$q \in$ *More-Modules.ideal* (*insert p* (*set-mset A* $\cup$ *polynomial-bool*))› **and**
*leading*: ‹$x' \notin$ *vars* $(p + Var\ x')$› **and**
*diff*: ‹$(Var\ x' + p)\hat{}2 - (Var\ x' + p) \in$ *More-Modules.ideal polynomial-bool*›
**shows**
‹$q \in$ *More-Modules.ideal* (*set-mset A* $\cup$ *polynomial-bool*)›
⟨*proof*⟩

This is the correctness theorem of a PAC step: no polynomials are added to the ideal.

**lemma** *vars-subst-in-left-only*:
‹$x \notin$ *vars* $p \Longrightarrow x \in$ *vars* $(p - Var\ x)$› **for** $p$ :: ‹*int mpoly*›
⟨*proof*⟩

**lemma** *vars-subst-in-left-only-diff-iff*:
**fixes** $p$ :: ‹*int mpoly*›
**assumes** ‹$x \notin$ *vars* $p$›
**shows** ‹*vars* $(p - Var\ x) =$ *insert* $x$ (*vars* $p$)›
⟨*proof*⟩

**lemma** *vars-subst-in-left-only-iff*:
‹$x \notin vars\ p \Longrightarrow vars\ (p + Var\ x) = insert\ x\ (vars\ p)$› **for** $p$ :: ‹*int mpoly*›
⟨*proof*⟩

**lemma** *coeff-add-right-notin*:
‹$x \notin vars\ p \Longrightarrow MPoly\text{-}Type.coeff\ (Var\ x - p)\ (monomial\ (Suc\ 0)\ x) = 1$›
⟨*proof*⟩

**lemma** *coeff-add-left-notin*:
‹$x \notin vars\ p \Longrightarrow MPoly\text{-}Type.coeff\ (p - Var\ x)\ (monomial\ (Suc\ 0)\ x) = -1$› **for** $p$ :: ‹*int mpoly*›
⟨*proof*⟩

**lemma** *ideal-insert-polynomial-bool-swap*: ‹$r - s \in ideal\ polynomial\text{-}bool \Longrightarrow$
*More-Modules.ideal* (*insert r* (*A* ∪ *polynomial-bool*)) = *More-Modules.ideal* (*insert s* (*A* ∪ *polynomial-bool*))›
⟨*proof*⟩

**lemma** *PAC-Format-subset-ideal*:
‹*PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}'$, *B*) $\Longrightarrow \bigcup(vars\ `\ set\text{-}mset\ A) \subseteq \mathcal{V} \Longrightarrow$
*restricted-ideal-to*$_I$ $\mathcal{V}$ *B* ⊆ *restricted-ideal-to*$_I$ $\mathcal{V}$ *A* ∧ $\mathcal{V} \subseteq \mathcal{V}' \wedge \bigcup(vars\ `\ set\text{-}mset\ B) \subseteq \mathcal{V}'$›
⟨*proof*⟩

In general, if deletions are disallowed, then the stronger $B = pac\text{-}ideal\ A$ holds.

**lemma** *restricted-ideal-to-restricted-ideal-to*$_I$*D*:
‹*restricted-ideal-to* $\mathcal{V}$ (*set-mset A*) ⊆ *restricted-ideal-to*$_I$ $\mathcal{V}$ *A*›
⟨*proof*⟩

**lemma** *rtranclp-PAC-Format-subset-ideal*:
‹*rtranclp PAC-Format* ($\mathcal{V}$, *A*) ($\mathcal{V}'$, *B*) $\Longrightarrow \bigcup(vars\ `\ set\text{-}mset\ A) \subseteq \mathcal{V} \Longrightarrow$
*restricted-ideal-to*$_I$ $\mathcal{V}$ *B* ⊆ *restricted-ideal-to*$_I$ $\mathcal{V}$ *A* ∧ $\mathcal{V} \subseteq \mathcal{V}' \wedge \bigcup(vars\ `\ set\text{-}mset\ B) \subseteq \mathcal{V}'$›
⟨*proof*⟩

**end**

**theory** *PAC-Map-Rel*
**imports**
*Refine-Imperative-HOL.IICF Finite-Map-Multiset*
**begin**

# 5 Hash-Map for finite mappings

This function declares hash-maps for (*'a*, *'b*) *fmap*, that are nicer to use especially here where everything is finite.

**definition** *fmap-rel* **where**
[*to-relAPP*]:
*fmap-rel K V* ≡ {(*m1*, *m2*).
(∀ *i j*. *i* |∈| *fmdom m2* ⟶ (*j*, *i*) ∈ *K* ⟶ (*the* (*fmlookup m1 j*), *the* (*fmlookup m2 i*)) ∈ *V*) ∧
*fset* (*fmdom m1*) ⊆ *Domain K* ∧ *fset* (*fmdom m2*) ⊆ *Range K* ∧
(∀ *i j*. (*i*, *j*) ∈ *K* ⟶ *j* |∈| *fmdom m2* ⟷ *i* |∈| *fmdom m1*)}

**lemma** *fmap-rel-alt-def*:

‹⟨K, V⟩fmap-rel ≡
{(m1, m2).
(∀ i j. i ∈# dom-m m2 ⟶
(j, i) ∈ K ⟶ (the (fmlookup m1 j), the (fmlookup m2 i)) ∈ V) ∧
fset (fmdom m1) ⊆ Domain K ∧
fset (fmdom m2) ⊆ Range K ∧
(∀ i j. (i, j) ∈ K ⟶ (j ∈# dom-m m2) = (i ∈# dom-m m1))}
›
⟨proof⟩

**lemma** *fmdom-empty-fmempty-iff*[*simp*]: ‹fmdom m = {||} ⟷ m = fmempty›
⟨proof⟩

**lemma** *fmap-rel-empty1-simp*[*simp*]:
(fmempty,m)∈⟨K,V⟩fmap-rel ⟷ m=fmempty
⟨proof⟩

**lemma** *fmap-rel-empty2-simp*[*simp*]:
(m,fmempty)∈⟨K,V⟩fmap-rel ⟷ m=fmempty
⟨proof⟩

**sepref-decl-intf** ('k,'v) *f-map* **is** ('k, 'v) *fmap*

**lemma** [*synth-rules*]: ⟦*INTF-OF-REL K TYPE*('k); *INTF-OF-REL V TYPE*('v)⟧
⟹ *INTF-OF-REL* (⟨K,V⟩fmap-rel) *TYPE*(('k,'v) *f-map*) ⟨proof⟩

## 5.1 Operations

**sepref-decl-op** *fmap-empty*: *fmempty* :: ⟨K,V⟩fmap-rel ⟨proof⟩

**sepref-decl-op** *fmap-is-empty*: (=) *fmempty* :: ⟨K,V⟩fmap-rel → *bool-rel*
⟨proof⟩

**lemma** *fmap-rel-fmupd-fmap-rel*:
‹(A, B) ∈ ⟨K, R⟩fmap-rel ⟹ (p, p′) ∈ K ⟹ (q, q′) ∈ R ⟹
(fmupd p q A, fmupd p′ q′ B) ∈ ⟨K, R⟩fmap-rel›
**if** *single-valued K single-valued* ($K^{-1}$)
⟨proof⟩

**sepref-decl-op** *fmap-update*: *fmupd* :: K → V → ⟨K,V⟩fmap-rel → ⟨K,V⟩fmap-rel
**where** *single-valued K single-valued* ($K^{-1}$)
⟨proof⟩

**lemma** *remove1-mset-eq-add-mset-iff*:
‹remove1-mset a A = add-mset a A′ ⟷ A = add-mset a (add-mset a A′)›
⟨proof⟩

**lemma** *fmap-rel-fmdrop-fmap-rel*:
‹(fmdrop p A, fmdrop p′ B) ∈ ⟨K, R⟩fmap-rel›
**if** *single*: *single-valued K single-valued* ($K^{-1}$) **and**
*H0*: ‹(A, B) ∈ ⟨K, R⟩fmap-rel› ‹(p, p′) ∈ K›
⟨proof⟩

**sepref-decl-op** *fmap-delete*: *fmdrop* :: K → ⟨K,V⟩fmap-rel → ⟨K,V⟩fmap-rel

**where** *single-valued K single-valued* $(K^{-1})$
⟨*proof*⟩

**lemma** *fmap-rel-nat-the-fmlookup*[*intro*]:
‹$(A,\ B) \in \langle S,\ R\rangle$*fmap-rel* $\Longrightarrow$ $(p,\ p') \in S \Longrightarrow p' \in\#$ *dom-m B* $\Longrightarrow$
(*the* (*fmlookup A p*), *the* (*fmlookup B p'*)) $\in R$›
⟨*proof*⟩

**lemma** *fmap-rel-in-dom-iff*:
‹$(aa,\ a'a) \in \langle K,\ V\rangle$*fmap-rel* $\Longrightarrow$
$(a,\ a') \in K \Longrightarrow$
$a' \in\#$ *dom-m* $a'a \longleftrightarrow$
$a \in\#$ *dom-m aa*›
⟨*proof*⟩

**lemma** *fmap-rel-fmlookup-rel*:
‹$(a,\ a') \in K \Longrightarrow (aa,\ a'a) \in \langle K,\ V\rangle$*fmap-rel* $\Longrightarrow$
(*fmlookup aa a*, *fmlookup* $a'a$ $a'$) $\in \langle V\rangle$*option-rel*›
⟨*proof*⟩

**sepref-decl-op** *fmap-lookup*: *fmlookup* :: $\langle K,V\rangle$*fmap-rel* $\to$ $K$ $\to$ $\langle V\rangle$*option-rel*
⟨*proof*⟩

**lemma** *in-fdom-alt*: $k\in\#$*dom-m m* $\longleftrightarrow \neg$*is-None* (*fmlookup m k*)
⟨*proof*⟩

**sepref-decl-op** *fmap-contains-key*: $\lambda k\ m.\ k\in\#$*dom-m m* :: $K \to \langle K,V\rangle$*fmap-rel* $\to$ *bool-rel*
⟨*proof*⟩

## 5.2 Patterns

**lemma** *pat-fmap-empty*[*pat-rules*]: *fmempty* $\equiv$ *op-fmap-empty* ⟨*proof*⟩

**lemma** *pat-map-is-empty*[*pat-rules*]:
(=) $m$*fmempty* $\equiv$ *op-fmap-is-empty*$m
(=) $fmempty$m \equiv$ *op-fmap-is-empty*$m
(=) $(dom-m$m)${#} \equiv$ *op-fmap-is-empty*$m
(=) ${#}$(dom-m$m) \equiv$ *op-fmap-is-empty*$m
⟨*proof*⟩

**lemma** *op-map-contains-key*[*pat-rules*]:
(∈#) $ $k$ $ (*dom-m*$m) $\equiv$ *op-fmap-contains-key*$'k$'m
⟨*proof*⟩

## 5.3 Mapping to Normal Hashmaps

**abbreviation** *map-of-fmap* :: ‹('k ⇒ 'v option) ⇒ ('k, 'v) fmap› **where**
‹*map-of-fmap h* ≡ *Abs-fmap h*›

**definition** *map-fmap-rel* **where**
‹*map-fmap-rel* = *br map-of-fmap* (λ*a*. *finite* (*dom a*))›

**lemma** *fmdrop-set-None*:
‹(*op-map-delete*, *fmdrop*) ∈ *Id* → *map-fmap-rel* → *map-fmap-rel*›
⟨*proof*⟩

**lemma** *map-upd-fmupd*:
‹(*op-map-update*, *fmupd*) ∈ *Id* → *Id* → *map-fmap-rel* → *map-fmap-rel*›
⟨*proof*⟩

Technically *op-map-lookup* has the arguments in the wrong direction.

**definition** *fmlookup'* **where**
[*simp*]: ‹*fmlookup'* *A* *k* = *fmlookup* *k* *A*›

**lemma** [*def-pat-rules*]:
‹((∈#)$*k*$(*dom-m*$*A*)) ≡ *Not*$(*is-None*$(*fmlookup'*$*k*$*A*))›
⟨*proof*⟩

**lemma** *op-map-lookup-fmlookup*:
‹(*op-map-lookup*, *fmlookup'*) ∈ *Id* → *map-fmap-rel* → ⟨*Id*⟩*option-rel*›
⟨*proof*⟩

**abbreviation** *hm-fmap-assn* **where**
‹*hm-fmap-assn* *K* *V* ≡ *hr-comp* (*hm.assn* *K* *V*) *map-fmap-rel*›

**lemmas** *fmap-delete-hnr* [*sepref-fr-rules*] =
*hm.delete-hnr*[*FCOMP* *fmdrop-set-None*]

**lemmas** *fmap-update-hnr* [*sepref-fr-rules*] =
*hm.update-hnr*[*FCOMP* *map-upd-fmupd*]

**lemmas** *fmap-lookup-hnr* [*sepref-fr-rules*] =
*hm.lookup-hnr*[*FCOMP* *op-map-lookup-fmlookup*]

**lemma** *fmempty-empty*:
‹(*uncurry0* (*RETURN* *op-map-empty*), *uncurry0* (*RETURN* *fmempty*)) ∈ *unit-rel* →$_f$ ⟨*map-fmap-rel*⟩*nres-rel*›
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] =
*hm.empty-hnr*[*FCOMP* *fmempty-empty*, *unfolded* *op-fmap-empty-def*[*symmetric*]]

**abbreviation** *iam-fmap-assn* **where**
‹*iam-fmap-assn* *K* *V* ≡ *hr-comp* (*iam.assn* *K* *V*) *map-fmap-rel*›

**lemmas** *iam-fmap-delete-hnr* [*sepref-fr-rules*] =
*iam.delete-hnr*[*FCOMP* *fmdrop-set-None*]

**lemmas** *iam-ffmap-update-hnr* [*sepref-fr-rules*] =
*iam.update-hnr*[*FCOMP* *map-upd-fmupd*]

**lemmas** *iam-ffmap-lookup-hnr* [*sepref-fr-rules*] =
*iam.lookup-hnr*[*FCOMP* *op-map-lookup-fmlookup*]

**definition** *op-iam-fmap-empty* **where**
‹*op-iam-fmap-empty* = *fmempty*›

**lemma** *iam-fmempty-empty*:
‹(*uncurry0* (*RETURN op-map-empty*), *uncurry0* (*RETURN op-iam-fmap-empty*)) ∈ *unit-rel* →$_f$ ⟨*map-fmap-rel*⟩*nres-rel*›
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] =
*iam.empty-hnr*[*FCOMP fmempty-empty*, *unfolded op-iam-fmap-empty-def*[*symmetric*]]

**definition** *upper-bound-on-dom* **where**
‹*upper-bound-on-dom A* = *SPEC*(λ*n*. ∀ *i* ∈#(*dom-m A*). *i* < *n*)›

**lemma** [*sepref-fr-rules*]:
‹((*Array.len*), *upper-bound-on-dom*) ∈ (*iam-fmap-assn nat-assn V*)$^k$ →$_a$ *nat-assn*›
⟨*proof*⟩

**lemma** *fmap-rel-nat-rel-dom-m*[*simp*]:
‹(*A*, *B*) ∈ ⟨*nat-rel*, *R*⟩*fmap-rel* ⟹ *dom-m A* = *dom-m B*›
⟨*proof*⟩

**lemma** *ref-two-step'*:
‹*A* ≤ *B* ⟹ ⇓ *R A* ≤ ⇓ *R B*›
⟨*proof*⟩

**end**

**theory** *PAC-Checker-Specification*
**imports** *PAC-Specification*
*Refine-Imperative-HOL.IICF*
*Finite-Map-Multiset*
**begin**

# 6 Checker Algorithm

In this level of refinement, we define the first level of the implementation of the checker, both with the specification as on ideals and the first version of the loop.

## 6.1 Specification

**datatype** *status* =
*is-failed*: *FAILED* |
*is-success*: *SUCCESS* |
*is-found*: *FOUND*

**lemma** *is-success-alt-def*:
‹*is-success a* ⟷ *a* = *SUCCESS*›
⟨*proof*⟩

**datatype** ('*a*, '*b*, '*lbls*) *pac-step* =
*Add* (*pac-src1*: '*lbls*) (*pac-src2*: '*lbls*) (*new-id*: '*lbls*) (*pac-res*: '*a*) |
*Mult* (*pac-src1*: '*lbls*) (*pac-mult*: '*a*) (*new-id*: '*lbls*) (*pac-res*: '*a*) |
*Extension* (*new-id*: '*lbls*) (*new-var*: '*b*) (*pac-res*: '*a*) |
*Del* (*pac-src1*: '*lbls*)

**type-synonym** *pac-state* = ‹(*nat set* × *int-poly multiset*)›

**definition** *PAC-checker-specification*
:: ‹*int-poly* ⇒ *int-poly multiset* ⇒ (*status* × *nat set* × *int-poly multiset*) *nres*›
**where**
‹*PAC-checker-specification spec A* = *SPEC*(λ(*b*, *V*, *B*).
(¬*is-failed b* ⟶ *restricted-ideal-to*$_I$ (⋃(*vars* ‘ *set-mset A*) ∪ *vars spec*) *B* ⊆ *restricted-ideal-to*$_I$ (⋃(*vars* ‘ *set-mset A*) ∪ *vars spec*) *A*) ∧
(*is-found b* ⟶ *spec* ∈ *pac-ideal* (*set-mset A*)))›

**definition** *PAC-checker-specification-spec*
:: ‹*int-poly* ⇒ *pac-state* ⇒ (*status* × *pac-state*) ⇒ *bool*›
**where**
‹*PAC-checker-specification-spec spec* = (λ(*V*, *A*) (*b*, *B*). (¬*is-failed b* ⟶ ⋃(*vars* ‘ *set-mset A*) ⊆ *V*) ∧
(*is-success b* ⟶ *PAC-Format*** (*V*, *A*) *B*) ∧
(*is-found b* ⟶ *PAC-Format*** (*V*, *A*) *B* ∧ *spec* ∈ *pac-ideal* (*set-mset A*)))›

**abbreviation** *PAC-checker-specification2*
:: ‹*int-poly* ⇒ (*nat set* × *int-poly multiset*) ⇒ (*status* × (*nat set* × *int-poly multiset*)) *nres*›
**where**
‹*PAC-checker-specification2 spec A* ≡ *SPEC*(*PAC-checker-specification-spec spec A*)›

**definition** *PAC-checker-specification-step-spec*
:: ‹*pac-state* ⇒ *int-poly* ⇒ *pac-state* ⇒ (*status* × *pac-state*) ⇒ *bool*›
**where**
‹*PAC-checker-specification-step-spec* = (λ(*V*$_0$, *A*$_0$) *spec* (*V*, *A*) (*b*, *B*).
(*is-success b* ⟶
⋃(*vars* ‘ *set-mset A*$_0$) ⊆ *V*$_0$ ∧
⋃(*vars* ‘ *set-mset A*) ⊆ *V* ∧ *PAC-Format*** (*V*$_0$, *A*$_0$) (*V*, *A*) ∧ *PAC-Format*** (*V*, *A*) *B*) ∧
(*is-found b* ⟶
⋃(*vars* ‘ *set-mset A*$_0$) ⊆ *V*$_0$ ∧
⋃(*vars* ‘ *set-mset A*) ⊆ *V* ∧ *PAC-Format*** (*V*$_0$, *A*$_0$) (*V*, *A*) ∧ *PAC-Format*** (*V*, *A*) *B* ∧
*spec* ∈ *pac-ideal* (*set-mset A*$_0$)))›

**abbreviation** *PAC-checker-specification-step2*
:: ‹*pac-state* ⇒ *int-poly* ⇒ *pac-state* ⇒ (*status* × *pac-state*) *nres*›
**where**
‹*PAC-checker-specification-step2 A*$_0$ *spec A* ≡ *SPEC*(*PAC-checker-specification-step-spec A*$_0$ *spec A*)›

**definition** *normalize-poly-spec* :: ‹-› **where**
‹*normalize-poly-spec p* = *SPEC* (λ*r*. *p* − *r* ∈ *ideal polynomial-bool* ∧ *vars r* ⊆ *vars p*)›

**lemma** *normalize-poly-spec-alt-def*:
‹*normalize-poly-spec p* = *SPEC* (λ*r*. *r* − *p* ∈ *ideal polynomial-bool* ∧ *vars r* ⊆ *vars p*)›
⟨*proof*⟩

**definition** *mult-poly-spec* :: ‹*int mpoly* ⇒ *int mpoly* ⇒ *int mpoly nres*› **where**
‹*mult-poly-spec p q* = *SPEC* (λ*r*. *p* ∗ *q* − *r* ∈ *ideal polynomial-bool*)›

**definition** *check-add* :: ‹(*nat*, *int mpoly*) *fmap* ⇒ *nat set* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *int mpoly* ⇒ *bool nres*› **where**
‹*check-add A V p q i r* =
*SPEC*(λ*b*. *b* ⟶ *p* ∈# *dom-m A* ∧ *q* ∈# *dom-m A* ∧ *i* ∉# *dom-m A* ∧ *vars r* ⊆ *V* ∧

*the (fmlookup A p) + the (fmlookup A q) − r ∈ ideal polynomial-bool)*›

**definition** *check-mult* :: ‹(*nat*, *int mpoly*) *fmap* ⇒ *nat set* ⇒ *nat* ⇒ *int mpoly* ⇒ *nat* ⇒ *int mpoly* ⇒ *bool nres*› **where**
‹*check-mult A* $\mathcal{V}$ *p q i r* =
*SPEC*(λ*b*. *b* ⟶ *p* ∈# *dom-m A* ∧ *i* ∉# *dom-m A* ∧ *vars q* ⊆ $\mathcal{V}$ ∧ *vars r* ⊆ $\mathcal{V}$ ∧
*the* (*fmlookup A p*) ∗ *q* − *r* ∈ *ideal polynomial-bool*)›

**definition** *check-extension* :: ‹(*nat*, *int mpoly*) *fmap* ⇒ *nat set* ⇒ *nat* ⇒ *nat* ⇒ *int mpoly* ⇒ (*bool*) *nres*› **where**
‹*check-extension A* $\mathcal{V}$ *i v p* =
*SPEC*(λ*b*. *b* ⟶ (*i* ∉# *dom-m A* ∧
(*v* ∉ $\mathcal{V}$ ∧
(*p*+*Var v*)$^2$ − (*p*+*Var v*) ∈ *ideal polynomial-bool* ∧
*vars* (*p*+*Var v*) ⊆ $\mathcal{V}$)))›

**fun** *merge-status* **where**
‹*merge-status* (*FAILED*) - = *FAILED*› |
‹*merge-status* - (*FAILED*) = *FAILED*› |
‹*merge-status FOUND* - = *FOUND*› |
‹*merge-status* - *FOUND* = *FOUND*› |
‹*merge-status* - - = *SUCCESS*›

**type-synonym** *fpac-step* = ‹*nat set* × (*nat*, *int-poly*) *fmap*›

**definition** *check-del* :: ‹(*nat*, *int mpoly*) *fmap* ⇒ *nat* ⇒ *bool nres*› **where**
‹*check-del A p* =
*SPEC*(λ*b*. *b* ⟶ *True*)›

## 6.2 Algorithm

**definition** *PAC-checker-step*
:: ‹*int-poly* ⇒ (*status* × *fpac-step*) ⇒ (*int-poly*, *nat*, *nat*) *pac-step* ⇒
(*status* × *fpac-step*) *nres*›
**where**
‹*PAC-checker-step* = (λ*spec* (*stat*, ($\mathcal{V}$, *A*)) *st*. *case st of*
*Add* - - - - ⇒
*do* {
*r* ← *normalize-poly-spec* (*pac-res st*);
*eq* ← *check-add A* $\mathcal{V}$ (*pac-src1 st*) (*pac-src2 st*) (*new-id st*) *r*;
*st′* ← *SPEC*(λ*st′*. (¬*is-failed st′* ∧ *is-found st′* ⟶ *r* − *spec* ∈ *ideal polynomial-bool*));
*if eq*
*then RETURN* (*merge-status stat st′*,
$\mathcal{V}$, *fmupd* (*new-id st*) *r A*)
*else RETURN* (*FAILED*, ($\mathcal{V}$, *A*))
}
| *Del* - ⇒
*do* {
*eq* ← *check-del A* (*pac-src1 st*);
*if eq*
*then RETURN* (*stat*, ($\mathcal{V}$, *fmdrop* (*pac-src1 st*) *A*))
*else RETURN* (*FAILED*, ($\mathcal{V}$, *A*))
}
| *Mult* - - - - ⇒
*do* {
*r* ← *normalize-poly-spec* (*pac-res st*);

```
        q ← normalize-poly-spec (pac-mult st);
        eq ← check-mult A 𝒱 (pac-src1 st) q (new-id st) r;
        st' ← SPEC(λst'. (¬is-failed st' ∧ is-found st' ⟶ r − spec ∈ ideal polynomial-bool));
        if eq
        then RETURN (merge-status stat st',
          𝒱, fmupd (new-id st) r A)
        else RETURN (FAILED, (𝒱, A))
    }
    | Extension - - - ⇒
        do {
          r ← normalize-poly-spec (pac-res st − Var (new-var st));
        (eq) ← check-extension A 𝒱 (new-id st) (new-var st) r;
        if eq
        then do {
         RETURN (stat,
          insert (new-var st) 𝒱, fmupd (new-id st) (r) A)}
        else RETURN (FAILED, (𝒱, A))
    }
)›
```

**definition** *polys-rel* :: ‹((*nat*, *int mpoly*)*fmap* × -) *set*› **where**
‹*polys-rel* = {(*A*, *B*). *B* = (*ran-m A*)}›

**definition** *polys-rel-full* :: ‹((*nat set* × (*nat*, *int mpoly*)*fmap*) × -) *set*› **where**
‹*polys-rel-full* = {((𝒱, *A*), (𝒱' , *B*)). (*A*, *B*) ∈ *polys-rel* ∧ 𝒱 = 𝒱'}›

**lemma** *polys-rel-update-remove*:
‹*x13* ∉# *dom-m A* ⟹ *x11* ∈# *dom-m A* ⟹ *x12* ∈# *dom-m A* ⟹ *x11* ≠ *x12* ⟹ (*A*,*B*) ∈ *polys-rel*
⟹
(*fmupd x13 r* (*fmdrop x11* (*fmdrop x12 A*)),
*add-mset r B* − {#*the* (*fmlookup A x11*), *the* (*fmlookup A x12*)#})
∈ *polys-rel*›
‹*x13* ∉# *dom-m A* ⟹ *x11* ∈# *dom-m A* ⟹ (*A*,*B*) ∈ *polys-rel* ⟹
(*fmupd x13 r* (*fmdrop x11 A*),*add-mset r B* − {#*the* (*fmlookup A x11*)#})
∈ *polys-rel*›
‹*x13* ∉# *dom-m A* ⟹ (*A*,*B*) ∈ *polys-rel* ⟹
(*fmupd x13 r A*, *add-mset r B*) ∈ *polys-rel*›
‹*x13* ∈# *dom-m A* ⟹ (*A*,*B*) ∈ *polys-rel* ⟹
(*fmdrop x13 A*, *remove1-mset* (*the* (*fmlookup A x13*)) *B*) ∈ *polys-rel*›
⟨*proof*⟩

**lemma** *polys-rel-in-dom-inD*:
‹(*A*, *B*) ∈ *polys-rel* ⟹
*x12* ∈# *dom-m A* ⟹
*the* (*fmlookup A x12*) ∈# *B*›
⟨*proof*⟩

**lemma** *PAC-Format-add-and-remove*:
‹*r* − *x14* ∈ *More-Modules.ideal polynomial-bool* ⟹
(*A*, *B*) ∈ *polys-rel* ⟹
*x12* ∈# *dom-m A* ⟹
*x13* ∉# *dom-m A* ⟹
*vars r* ⊆ 𝒱 ⟹
2 ∗ *the* (*fmlookup A x12*) − *r* ∈ *More-Modules.ideal polynomial-bool* ⟹
*PAC-Format*** (𝒱, *B*) (𝒱, *remove1-mset* (*the* (*fmlookup A x12*)) (*add-mset r B*))›

‹$r - x14 \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
$(A, B) \in$ *polys-rel* $\Longrightarrow$
*the* (*fmlookup A x11*) + *the* (*fmlookup A x12*) $- r \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*x11* $\in\#$ *dom-m A* $\Longrightarrow$
*x12* $\in\#$ *dom-m A* $\Longrightarrow$
*vars r* $\subseteq \mathcal{V} \Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$ $(\mathcal{V},$ *add-mset r B*)›
‹$r - x14 \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
$(A, B) \in$ *polys-rel* $\Longrightarrow$
*x11* $\in\#$ *dom-m A* $\Longrightarrow$
*x12* $\in\#$ *dom-m A* $\Longrightarrow$
*the* (*fmlookup A x11*) + *the* (*fmlookup A x12*) $- r \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*vars r* $\subseteq \mathcal{V} \Longrightarrow$
*x11* $\neq$ *x12* $\Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$
$(\mathcal{V},$ *add-mset r B* $-$ {#*the* (*fmlookup A x11*), *the* (*fmlookup A x12*)#})›
‹$(A, B) \in$ *polys-rel* $\Longrightarrow$
$r - x34 \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*x11* $\in\#$ *dom-m A* $\Longrightarrow$
*the* (*fmlookup A x11*) $*$ *x32* $- r \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*vars x32* $\subseteq \mathcal{V} \Longrightarrow$
*vars r* $\subseteq \mathcal{V} \Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$ $(\mathcal{V},$ *add-mset r B*)›
‹$(A, B) \in$ *polys-rel* $\Longrightarrow$
$r - x34 \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*x11* $\in\#$ *dom-m A* $\Longrightarrow$
*the* (*fmlookup A x11*) $*$ *x32* $- r \in$ *More-Modules.ideal polynomial-bool* $\Longrightarrow$
*vars x32* $\subseteq \mathcal{V} \Longrightarrow$
*vars r* $\subseteq \mathcal{V} \Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$ $(\mathcal{V},$ *remove1-mset* (*the* (*fmlookup A x11*)) (*add-mset r B*))›
‹$(A, B) \in$ *polys-rel* $\Longrightarrow$
*x12* $\in\#$ *dom-m A* $\Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$ $(\mathcal{V},$ *remove1-mset* (*the* (*fmlookup A x12*)) *B*)›
‹$(A, B) \in$ *polys-rel* $\Longrightarrow$
$(p' +$ *Var x*$)^2 - (p' +$ *Var x*$) \in$ *ideal polynomial-bool* $\Longrightarrow$
$x \notin \mathcal{V} \Longrightarrow$
$x \notin$ *vars*$(p' +$ *Var x*$) \Longrightarrow$
*vars*$(p' +$ *Var x*$) \subseteq \mathcal{V} \Longrightarrow$
*PAC-Format*** $(\mathcal{V}, B)$
(*insert x* $\mathcal{V}$, *add-mset p' B*)›
⟨*proof*⟩

**abbreviation** *status-rel* :: ‹(*status* × *status*) *set*› **where**
‹*status-rel* ≡ *Id*›

**lemma** *is-merge-status*[*simp*]:
‹*is-failed* (*merge-status a st'*) ⟷ *is-failed a* ∨ *is-failed st'*›
‹*is-found* (*merge-status a st'*) ⟷ ¬*is-failed a* ∧ ¬*is-failed st'* ∧ (*is-found a* ∨ *is-found st'*)›
‹*is-success* (*merge-status a st'*) ⟷ (*is-success a* ∧ *is-success st'*)›
⟨*proof*⟩

**lemma** *status-rel-merge-status*:
‹(*merge-status a b*, *SUCCESS*) ∉ *status-rel* ⟷
(*a* = *FAILED*) ∨ (*b* = *FAILED*) ∨

$a = FOUND \vee (b = FOUND)$›
⟨*proof*⟩

**lemma** *Ex-status-iff*:
‹$(\exists\, a.\ P\ a) \longleftrightarrow P\ SUCCESS \vee P\ FOUND \vee (P\ (FAILED))$›
⟨*proof*⟩

**lemma** *is-failed-alt-def*:
‹*is-failed st'* $\longleftrightarrow \neg$*is-success st'* $\wedge \neg$*is-found st'*›
⟨*proof*⟩

**lemma** *merge-status-eq-iff*[*simp*]:
‹*merge-status a SUCCESS* = *SUCCESS* $\longleftrightarrow$ *a* = *SUCCESS*›
‹*merge-status a SUCCESS* = *FOUND* $\longleftrightarrow$ *a* = *FOUND*›
‹*merge-status SUCCESS a* = *SUCCESS* $\longleftrightarrow$ *a* = *SUCCESS*›
‹*merge-status SUCCESS a* = *FOUND* $\longleftrightarrow$ *a* = *FOUND*›
‹*merge-status SUCCESS a* = *FAILED* $\longleftrightarrow$ *a* = *FAILED*›
‹*merge-status a SUCCESS* = *FAILED* $\longleftrightarrow$ *a* = *FAILED*›
‹*merge-status FOUND a* = *FAILED* $\longleftrightarrow$ *a* = *FAILED*›
‹*merge-status a FOUND* = *FAILED* $\longleftrightarrow$ *a* = *FAILED*›
‹*merge-status a FOUND* = *SUCCESS* $\longleftrightarrow$ *False*›
‹*merge-status a b* = *FOUND* $\longleftrightarrow$ (*a* = *FOUND* $\vee$ *b* = *FOUND*) $\wedge$ (*a* $\neq$ *FAILED* $\wedge$ *b* $\neq$ *FAILED*)›
⟨*proof*⟩

**lemma** *fmdrop-irrelevant*: ‹*x11* $\notin\#$ *dom-m A* $\Longrightarrow$ *fmdrop x11 A* = *A*›
⟨*proof*⟩

**lemma** *PAC-checker-step-PAC-checker-specification2*:
**fixes** *a* :: ‹*status*›
**assumes** *AB*: ‹((*V*, *A*),(*V*$_B$, *B*)) ∈ *polys-rel-full*› **and**
‹¬*is-failed a*› **and**
[*simp*,*intro*]: ‹*a* = *FOUND* $\Longrightarrow$ *spec* ∈ *pac-ideal* (*set-mset A*$_0$)› **and**
*A*$_0$*B*: ‹*PAC-Format*** (*V*$_0$, *A*$_0$) (*V*, *B*)› **and**
*spec*$_0$: ‹*vars spec* ⊆ *V*$_0$› **and**
*vars-A*$_0$: ‹$\bigcup$ (*vars* ‘ *set-mset A*$_0$) ⊆ *V*$_0$›
**shows** ‹*PAC-checker-step spec* (*a*, (*V*, *A*)) *st* ≤ ⇓ (*status-rel* ×$_r$ *polys-rel-full*) (*PAC-checker-specification-step2* (*V*$_0$, *A*$_0$) *spec* (*V*, *B*))›
⟨*proof*⟩

**definition** *PAC-checker*
:: ‹*int-poly* ⇒ *fpac-step* ⇒ *status* ⇒ (*int-poly*, *nat*, *nat*) *pac-step list* ⇒
(*status* × *fpac-step*) *nres*›
**where**
‹*PAC-checker spec A b st* = *do* {
(*S*, -) ← *WHILE*$_T$
(λ((*b* :: *status*, *A* :: *fpac-step*), *st*). ¬*is-failed b* ∧ *st* ≠ [])
(λ((*bA*), *st*). *do* {
*ASSERT*(*st* ≠ []);
*S* ← *PAC-checker-step spec* (*bA*) (*hd st*);
*RETURN* (*S*, *tl st*)
})
((*b*, *A*), *st*);
*RETURN S*
}›

**lemma** *PAC-checker-specification-spec-trans*:
‹*PAC-checker-specification-spec spec A* (*st*, *x2*) ⟹
*PAC-checker-specification-step-spec A spec x2* (*st'*, *x1a*) ⟹
*PAC-checker-specification-spec spec A* (*st'*, *x1a*)›
⟨*proof*⟩

**lemma** *RES-SPEC-eq*:
‹*RES* Φ = *SPEC*(λ*P*. *P* ∈ Φ)›
⟨*proof*⟩

**lemma** *is-failed-is-success-completeD*:
‹¬ *is-failed x* ⟹ ¬*is-success x* ⟹ *is-found x*›
⟨*proof*⟩

**lemma** *PAC-checker-PAC-checker-specification2*:
‹(*A*, *B*) ∈ *polys-rel-full* ⟹
¬*is-failed a* ⟹
(*a* = *FOUND* ⟹ *spec* ∈ *pac-ideal* (*set-mset* (*snd B*))) ⟹
⋃(*vars* ‘ *set-mset* (*ran-m* (*snd A*))) ⊆ *fst B* ⟹
*vars spec* ⊆ *fst B* ⟹
*PAC-checker spec A a st* ≤ ⇓ (*status-rel* ×$_r$ *polys-rel-full*) (*PAC-checker-specification2 spec B*)›
⟨*proof*⟩

**definition** *remap-polys-polynomial-bool* :: ‹*int mpoly* ⇒ *nat set* ⇒ (*nat*, *int-poly*) *fmap* ⇒ (*status* × *fpac-step*) *nres*› **where**
‹*remap-polys-polynomial-bool spec* = (λ𝒱 *A*.
*SPEC*(λ(*st*, 𝒱', *A'*). (¬*is-failed st* ⟶
*dom-m A* = *dom-m A'* ∧
(∀ *i* ∈# *dom-m A*. *the* (*fmlookup A i*) − *the* (*fmlookup A' i*) ∈ *ideal polynomial-bool*) ∧
⋃(*vars* ‘ *set-mset* (*ran-m A*)) ⊆ 𝒱' ∧
⋃(*vars* ‘ *set-mset* (*ran-m A'*)) ⊆ 𝒱') ∧
(*st* = *FOUND* ⟶ *spec* ∈# *ran-m A'*)))›

**definition** *remap-polys-change-all* :: ‹*int mpoly* ⇒ *nat set* ⇒ (*nat*, *int-poly*) *fmap* ⇒ (*status* × *fpac-step*) *nres*› **where**
‹*remap-polys-change-all spec* = (λ𝒱 *A*. *SPEC* (λ(*st*, 𝒱', *A'*).
(¬*is-failed st* ⟶
*pac-ideal* (*set-mset* (*ran-m A*)) = *pac-ideal* (*set-mset* (*ran-m A'*)) ∧
⋃(*vars* ‘ *set-mset* (*ran-m A*)) ⊆ 𝒱' ∧
⋃(*vars* ‘ *set-mset* (*ran-m A'*)) ⊆ 𝒱') ∧
(*st* = *FOUND* ⟶ *spec* ∈# *ran-m A'*)))›

**lemma** *fmap-eq-dom-iff*:
‹*A* = *A'* ⟷ *dom-m A* = *dom-m A'* ∧ (∀ *i* ∈# *dom-m A*. *the* (*fmlookup A i*) = *the* (*fmlookup A' i*))›
⟨*proof*⟩

**lemma** *ideal-remap-incl*:
‹*finite A'* ⟹ (∀ *a'*∈*A'*. ∃ *a*∈*A*. *a*−*a'* ∈ *B*) ⟹ *ideal* (*A'* ∪ *B*) ⊆ *ideal* (*A* ∪ *B*)›
⟨*proof*⟩

**lemma** *pac-ideal-remap-eq*:
‹*dom-m b* = *dom-m ba* ⟹
∀ *i*∈#*dom-m ba*.

*the (fmlookup b i) − the (fmlookup ba i)*
*∈ More-Modules.ideal polynomial-bool ⟹*
*pac-ideal ((λx. the (fmlookup b x)) ‘ set-mset (dom-m ba)) = pac-ideal ((λx. the (fmlookup ba x)) ‘ set-mset (dom-m ba))*›
⟨*proof*⟩

**lemma** *remap-polys-polynomial-bool-remap-polys-change-all*:
‹*remap-polys-polynomial-bool spec 𝒱 A ≤ remap-polys-change-all spec 𝒱 A*›
⟨*proof*⟩

**definition** *remap-polys* :: ‹*int mpoly ⇒ nat set ⇒ (nat, int-poly) fmap ⇒ (status × fpac-step) nres*›
**where**
‹*remap-polys spec = (λ𝒱 A. do{*
*dom ← SPEC(λdom. set-mset (dom-m A) ⊆ dom ∧ finite dom);*

*failed ← SPEC(λ-::bool. True);*
*if failed*
*then do {*
*RETURN (FAILED, 𝒱, fmempty)*
*}*
*else do {*
*(b, N) ← FOREACH dom*
*(λi (b, 𝒱, A′).*
*if i ∈# dom-m A*
*then do {*
*p ← SPEC(λp. the (fmlookup A i) − p ∈ ideal polynomial-bool ∧ vars p ⊆ vars (the (fmlookup A i)));*
*eq ← SPEC(λeq. eq ⟶ p = spec);*
*𝒱 ← SPEC(λ𝒱′. 𝒱 ∪ vars (the (fmlookup A i)) ⊆ 𝒱′);*
*RETURN(b ∨ eq, 𝒱, fmupd i p A′)*
*} else RETURN (b, 𝒱, A′))*
*(False, 𝒱, fmempty);*
*RETURN (if b then FOUND else SUCCESS, N)*
*}*
*})*›

**lemma** *remap-polys-spec*:
‹*remap-polys spec 𝒱 A ≤ remap-polys-polynomial-bool spec 𝒱 A*›
⟨*proof*⟩

## 6.3 Full Checker

**definition** *full-checker*
:: ‹*int-poly ⇒ (nat, int-poly) fmap ⇒ (int-poly, nat,nat) pac-step list ⇒ (status × -) nres*›
**where**
‹*full-checker spec0 A pac = do {*
*spec ← normalize-poly-spec spec0;*
*(st, 𝒱, A) ← remap-polys-change-all spec {} A;*
*if is-failed st then*
*RETURN (st, 𝒱, A)*
*else do {*
*𝒱 ← SPEC(λ𝒱′. 𝒱 ∪ vars spec0 ⊆ 𝒱′);*
*PAC-checker spec (𝒱, A) st pac*
*}*
}›

**lemma** *restricted-ideal-to-mono*:
‹*restricted-ideal-to*$_I$ $\mathcal{V}$ $I$ ⊆ *restricted-ideal-to*$_I$ $\mathcal{V}'$ $J$ ⟹
$\mathcal{U} \subseteq \mathcal{V}$ ⟹
*restricted-ideal-to*$_I$ $\mathcal{U}$ $I$ ⊆ *restricted-ideal-to*$_I$ $\mathcal{U}$ $J$›
⟨*proof*⟩

**lemma** *pac-ideal-idemp*: ‹*pac-ideal* (*pac-ideal* $A$) = *pac-ideal* $A$›
⟨*proof*⟩

**lemma** *full-checker-spec*:
**assumes** ‹($A$, $A'$) ∈ *polys-rel*›
**shows**
‹*full-checker spec* $A$ *pac* ≤ ⇓{(($st$, $G$), ($st'$, $G'$)). ($st$, $st'$) ∈ *status-rel* ∧
($st$ ≠ *FAILED* ⟶ ($G$, $G'$) ∈ *polys-rel-full*)}
(*PAC-checker-specification spec* ($A'$))›
⟨*proof*⟩

**lemma** *full-checker-spec'*:
**shows**
‹(*uncurry2 full-checker*, *uncurry2* (λ*spec* $A$ -. *PAC-checker-specification spec* $A$)) ∈
(*Id* ×$_r$ *polys-rel*) ×$_r$ *Id* →$_f$ ⟨{(($st$, $G$), ($st'$, $G'$)). ($st$, $st'$) ∈ *status-rel* ∧
($st$ ≠ *FAILED* ⟶ ($G$, $G'$) ∈ *polys-rel-full*)}⟩*nres-rel*›
⟨*proof*⟩

**end**
**theory** *PAC-Polynomials*
**imports** *PAC-Specification Finite-Map-Multiset*
**begin**

# 7 Polynomials of strings

Isabelle's definition of polynomials only work with variables of type *nat*. Therefore, we introduce a version that uses strings by using an injective function that converts a string to a natural number. It exists because strings are countable. Remark that the whole development is independent of the function.

## 7.1 Polynomials and Variables

**lemma** *poly-embed-EX*:
‹∃φ. *bij* (φ :: *string* ⇒ *nat*)›
⟨*proof*⟩

Using a multiset instead of a list has some advantage from an abstract point of view. First, we can have monomials that appear several times and the coefficient can also be zero. Basically, we can represent un-normalised polynomials, which is very useful to talk about intermediate states in our program.

**type-synonym** *term-poly* = ‹*string multiset*›
**type-synonym** *mset-polynomial* =
‹(*term-poly* ∗ *int*) *multiset*›

**definition** *normalized-poly* :: ‹*mset-polynomial* ⇒ *bool*› **where**

‹*normalized-poly p* ⟷
*distinct-mset* (*fst* ‘# *p*) ∧
*0* ∉# *snd* ‘# *p*›

**lemma** *normalized-poly-simps*[*simp*]:
‹*normalized-poly* {#}›
‹*normalized-poly* (*add-mset t p*) ⟷ *snd t* ≠ *0* ∧
*fst t* ∉# *fst* ‘# *p* ∧ *normalized-poly p*›
⟨*proof*⟩

**lemma** *normalized-poly-mono*:
‹*normalized-poly B* ⟹ *A* ⊆# *B* ⟹ *normalized-poly A*›
⟨*proof*⟩

**definition** *mult-poly-by-monom* :: ‹*term-poly* ∗ *int* ⇒ *mset-polynomial* ⇒ *mset-polynomial*› **where**
‹*mult-poly-by-monom* = (λ*ys q*. *image-mset* (λ*xs*. (*fst xs* + *fst ys*, *snd ys* ∗ *snd xs*)) *q*)›

**definition** *mult-poly-raw* :: ‹*mset-polynomial* ⇒ *mset-polynomial* ⇒ *mset-polynomial*› **where**
‹*mult-poly-raw p q* =
(*sum-mset* ((λ*y*. *mult-poly-by-monom y q*) ‘# *p*))›

**definition** *remove-powers* :: ‹*mset-polynomial* ⇒ *mset-polynomial*› **where**
‹*remove-powers xs* = *image-mset* (*apfst remdups-mset*) *xs*›

**definition** *all-vars-mset* :: ‹*mset-polynomial* ⇒ *string multiset*› **where**
‹*all-vars-mset p* = ∑# (*fst* ‘# *p*)›

**abbreviation** *all-vars* :: ‹*mset-polynomial* ⇒ *string set*› **where**
‹*all-vars p* ≡ *set-mset* (*all-vars-mset p*)›

**definition** *add-to-coefficient* :: ‹- ⇒ *mset-polynomial* ⇒ *mset-polynomial*› **where**
‹*add-to-coefficient* = (λ(*a*, *n*) *b*. {#(*a*′, -) ∈# *b*. *a*′ ≠ *a*#} +
(*if n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}) = *0 then* {#}
*else* {#(*a*, *n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}))#}))›

**definition** *normalize-poly* :: ‹*mset-polynomial* ⇒ *mset-polynomial*› **where**
‹*normalize-poly p* = *fold-mset add-to-coefficient* {#} *p*›

**lemma** *add-to-coefficient-simps*:
‹*n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}) ≠ *0* ⟹
*add-to-coefficient* (*a*, *n*) *b* = {#(*a*′, -) ∈# *b*. *a*′ ≠ *a*#} +
{#(*a*, *n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}))#}›
‹*n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}) = *0* ⟹
*add-to-coefficient* (*a*, *n*) *b* = {#(*a*′, -) ∈# *b*. *a*′ ≠ *a*#}› **and**
*add-to-coefficient-simps-If*:
‹*add-to-coefficient* (*a*, *n*) *b* = {#(*a*′, -) ∈# *b*. *a*′ ≠ *a*#} +
(*if n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}) = *0 then* {#}
*else* {#(*a*, *n* + *sum-mset* (*snd* ‘# {#(*a*′, -) ∈# *b*. *a*′ = *a*#}))#})›
⟨*proof*⟩

**interpretation** *comp-fun-commute* ‹*add-to-coefficient*›
⟨*proof*⟩

**lemma** *normalized-poly-normalize-poly*[*simp*]:
‹*normalized-poly* (*normalize-poly p*)›
⟨*proof*⟩

## 7.2 Addition

**inductive** *add-poly-p* :: ‹*mset-polynomial* × *mset-polynomial* × *mset-polynomial* ⇒ *mset-polynomial* × *mset-polynomial* × *mset-polynomial* ⇒ *bool*› **where**
*add-new-coeff-r*:
‹*add-poly-p* (*p*, *add-mset x q*, *r*) (*p*, *q*, *add-mset x r*)› |
*add-new-coeff-l*:
‹*add-poly-p* (*add-mset x p*, *q*, *r*) (*p*, *q*, *add-mset x r*)› |
*add-same-coeff-l*:
‹*add-poly-p* (*add-mset* (*x*, *n*) *p*, *q*, *add-mset* (*x*, *m*) *r*) (*p*, *q*, *add-mset* (*x*, *n* + *m*) *r*)› |
*add-same-coeff-r*:
‹*add-poly-p* (*p*, *add-mset* (*x*, *n*) *q*, *add-mset* (*x*, *m*) *r*) (*p*, *q*, *add-mset* (*x*, *n* + *m*) *r*)› |
*rem-0-coeff*:
‹*add-poly-p* (*p*, *q*, *add-mset* (*x*, *0*) *r*) (*p*, *q*, *r*)›

**inductive-cases** *add-poly-pE*: ‹*add-poly-p S T*›

**lemmas** *add-poly-p-induct* =
*add-poly-p.induct*[*split-format*(*complete*)]

**lemma** *add-poly-p-empty-l*:
‹*add-poly-p*** (*p*, *q*, *r*) ({#}, *q*, *p* + *r*)›
⟨*proof*⟩

**lemma** *add-poly-p-empty-r*:
‹*add-poly-p*** (*p*, *q*, *r*) (*p*, {#}, *q* + *r*)›
⟨*proof*⟩

**lemma** *add-poly-p-sym*:
‹*add-poly-p* (*p*, *q*, *r*) (*p*′, *q*′, *r*′) ⟷ *add-poly-p* (*q*, *p*, *r*) (*q*′, *p*′, *r*′)›
⟨*proof*⟩

**lemma** *wf-if-measure-in-wf*:
‹*wf R* ⟹ (⋀*a b*. (*a*, *b*) ∈ *S* ⟹ (ν *a*, ν *b*)∈*R*) ⟹ *wf S*›
⟨*proof*⟩

**lemma** *lexn-n*:
‹*n* > *0* ⟹ (*x* # *xs*, *y* # *ys*) ∈ *lexn r n* ⟷
(*length xs* = *n*−*1* ∧ *length ys* = *n*−*1*) ∧ ((*x*, *y*) ∈ *r* ∨ (*x* = *y* ∧ (*xs*, *ys*) ∈ *lexn r* (*n* − *1*)))›
⟨*proof*⟩

**lemma** *wf-add-poly-p*:
‹*wf* {(*x*, *y*). *add-poly-p y x*}›
⟨*proof*⟩

**lemma** *mult-poly-by-monom-simps*[*simp*]:
‹*mult-poly-by-monom t* {#} = {#}›
‹*mult-poly-by-monom t* (*ps* + *qs*) = *mult-poly-by-monom t ps* + *mult-poly-by-monom t qs*›
‹*mult-poly-by-monom a* (*add-mset p ps*) = *add-mset* (*fst a* + *fst p*, *snd a* ∗ *snd p*) (*mult-poly-by-monom a ps*)›
⟨*proof*⟩

**inductive** *mult-poly-p* :: ‹*mset-polynomial* ⇒ *mset-polynomial* × *mset-polynomial* ⇒ *mset-polynomial* × *mset-polynomial* ⇒ *bool*›
  **for** *q* :: *mset-polynomial* **where**
*mult-step*:
  ‹*mult-poly-p q* (*add-mset* (*xs*, *n*) *p*, *r*) (*p*, (λ(*ys*, *m*). (*remdups-mset* (*xs* + *ys*), *n* ∗ *m*)) `# *q* + *r*)›

**lemmas** *mult-poly-p-induct* = *mult-poly-p.induct*[*split-format*(*complete*)]

## 7.3 Normalisation

**inductive** *normalize-poly-p* :: ‹*mset-polynomial* ⇒ *mset-polynomial* ⇒ *bool*›**where**
*rem-0-coeff*[*simp*, *intro*]:
  ‹*normalize-poly-p p q* ⟹ *normalize-poly-p* (*add-mset* (*xs*, *0*) *p*) *q*› |
*merge-dup-coeff*[*simp*, *intro*]:
  ‹*normalize-poly-p p q* ⟹ *normalize-poly-p* (*add-mset* (*xs*, *m*) (*add-mset* (*xs*, *n*) *p*)) (*add-mset* (*xs*, *m* + *n*) *q*)› |
*same*[*simp*, *intro*]:
  ‹*normalize-poly-p p p*› |
*keep-coeff*[*simp*, *intro*]:
  ‹*normalize-poly-p p q* ⟹ *normalize-poly-p* (*add-mset x p*) (*add-mset x q*)›

## 7.4 Correctness

This locales maps string polynomials to real polynomials.

**locale** *poly-embed* =
  **fixes** φ :: ‹*string* ⇒ *nat*›
  **assumes** φ-*inj*: ‹*inj* φ›
**begin**

**definition** *poly-of-vars* :: *term-poly* ⇒ ('*a* :: {*comm-semiring-1*}) *mpoly* **where**
  ‹*poly-of-vars xs* = *fold-mset* (λ*a b*. *Var* (φ *a*) ∗ *b*) (*1* :: '*a mpoly*) *xs*›

**lemma** *poly-of-vars-simps*[*simp*]:
  **shows**
  ‹*poly-of-vars* (*add-mset x xs*) = *Var* (φ *x*) ∗ (*poly-of-vars xs* :: ('*a* :: {*comm-semiring-1*}) *mpoly*)› (**is** *?A*) **and**
  ‹*poly-of-vars* (*xs* + *ys*) = *poly-of-vars xs* ∗ (*poly-of-vars ys* :: ('*a* :: {*comm-semiring-1*}) *mpoly*)› (**is** *?B*)
⟨*proof*⟩

**definition** *mononom-of-vars* **where**
  ‹*mononom-of-vars* ≡ (λ(*xs*, *n*). (+) (*Const n* ∗ *poly-of-vars xs*))›

**interpretation** *comp-fun-commute* ‹*mononom-of-vars*›
  ⟨*proof*⟩

**lemma** [*simp*]:
  ‹*poly-of-vars* {#} = *1*›
  ⟨*proof*⟩

**lemma** *mononom-of-vars-add*[*simp*]:
  ‹*NO-MATCH 0 b* ⟹ *mononom-of-vars xs b* = *Const* (*snd xs*) ∗ *poly-of-vars* (*fst xs*) + *b*›

⟨*proof*⟩

**definition** *polynomial-of-mset* :: ‹*mset-polynomial* ⇒ -› **where**
‹*polynomial-of-mset p* = *sum-mset* (*mononom-of-vars* ‘# *p*) *0*›

**lemma** *polynomial-of-mset-append*[*simp*]:
‹*polynomial-of-mset* (*xs* + *ys*) = *polynomial-of-mset xs* + *polynomial-of-mset ys*›
⟨*proof*⟩

**lemma** *polynomial-of-mset-Cons*[*simp*]:
‹*polynomial-of-mset* (*add-mset x ys*) = *Const* (*snd x*) ∗ *poly-of-vars* (*fst x*) + *polynomial-of-mset ys*›
⟨*proof*⟩

**lemma** *polynomial-of-mset-empty*[*simp*]:
‹*polynomial-of-mset* {#} = *0*›
⟨*proof*⟩

**lemma** *polynomial-of-mset-mult-poly-by-monom*[*simp*]:
‹*polynomial-of-mset* (*mult-poly-by-monom x ys*) =
(*Const* (*snd x*) ∗ *poly-of-vars* (*fst x*) ∗ *polynomial-of-mset ys*)›
⟨*proof*⟩

**lemma** *polynomial-of-mset-mult-poly-raw*[*simp*]:
‹*polynomial-of-mset* (*mult-poly-raw xs ys*) = *polynomial-of-mset xs* ∗ *polynomial-of-mset ys*›
⟨*proof*⟩

**lemma** *polynomial-of-mset-uminus*:
‹*polynomial-of-mset* {#*case x of* (*a*, *b*) ⇒ (*a*, − *b*). *x* ∈# *za*#} =
− *polynomial-of-mset za*›
⟨*proof*⟩

**lemma** *X2-X-polynomial-bool-mult-in*:
‹*Var* (*x1*) ∗ (*Var* (*x1*) ∗ *p*) − *Var* (*x1*) ∗ *p* ∈ *More-Modules.ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *polynomial-of-list-remove-powers-polynomial-bool*:
‹(*polynomial-of-mset xs*) − *polynomial-of-mset* (*remove-powers xs*) ∈ *ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *add-poly-p-polynomial-of-mset*:
‹*add-poly-p* (*p*, *q*, *r*) (*p′*, *q′*, *r′*) ⟹
*polynomial-of-mset r* + (*polynomial-of-mset p* + *polynomial-of-mset q*) =
*polynomial-of-mset r′* + (*polynomial-of-mset p′* + *polynomial-of-mset q′*)›
⟨*proof*⟩

**lemma** *rtranclp-add-poly-p-polynomial-of-mset*:
‹*add-poly-p*** (*p*, *q*, *r*) (*p′*, *q′*, *r′*) ⟹
*polynomial-of-mset r* + (*polynomial-of-mset p* + *polynomial-of-mset q*) =
*polynomial-of-mset r′* + (*polynomial-of-mset p′* + *polynomial-of-mset q′*)›
⟨*proof*⟩

**lemma** *rtranclp-add-poly-p-polynomial-of-mset-full*:

‹*add-poly-p*** (*p*, *q*, {#}) ({#}, {#}, *r'*) ⟹
*polynomial-of-mset r'* = (*polynomial-of-mset p* + *polynomial-of-mset q*)›
⟨*proof*⟩

**lemma** *poly-of-vars-remdups-mset*:
‹*poly-of-vars* (*remdups-mset* (*xs*)) − (*poly-of-vars xs*)
∈ *More-Modules.ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *polynomial-of-mset-mult-map*:
‹*polynomial-of-mset*
{#*case x of* (*ys*, *n*) ⇒ (*remdups-mset* (*ys* + *xs*), *n* ∗ *m*). *x* ∈# *q*#} −
*Const m* ∗ (*poly-of-vars xs* ∗ *polynomial-of-mset q*)
∈ *More-Modules.ideal polynomial-bool*›
(**is** ‹*?P q* ∈ -›)
⟨*proof*⟩

**lemma** *mult-poly-p-mult-ideal*:
‹*mult-poly-p q* (*p*, *r*) (*p'*, *r'*) ⟹
(*polynomial-of-mset p'* ∗ *polynomial-of-mset q* + *polynomial-of-mset r'*) − (*polynomial-of-mset p* ∗ *polynomial-of-mset q* + *polynomial-of-mset r*)
∈ *ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *rtranclp-mult-poly-p-mult-ideal*:
‹(*mult-poly-p q*)** (*p*, *r*) (*p'*, *r'*) ⟹
(*polynomial-of-mset p'* ∗ *polynomial-of-mset q* + *polynomial-of-mset r'*) − (*polynomial-of-mset p* ∗ *polynomial-of-mset q* + *polynomial-of-mset r*)
∈ *ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *rtranclp-mult-poly-p-mult-ideal-final*:
‹(*mult-poly-p q*)** (*p*, {#}) ({#}, *r*) ⟹
(*polynomial-of-mset r*) − (*polynomial-of-mset p* ∗ *polynomial-of-mset q*)
∈ *ideal polynomial-bool*›
⟨*proof*⟩

**lemma** *normalize-poly-p-poly-of-mset*:
‹*normalize-poly-p p q* ⟹ *polynomial-of-mset p* = *polynomial-of-mset q*›
⟨*proof*⟩

**lemma** *rtranclp-normalize-poly-p-poly-of-mset*:
‹*normalize-poly-p*** *p q* ⟹ *polynomial-of-mset p* = *polynomial-of-mset q*›
⟨*proof*⟩

**end**

It would be nice to have the property in the other direction too, but this requires a deep dive into the definitions of polynomials.

**locale** *poly-embed-bij* = *poly-embed* +
**fixes** *V N*
**assumes** *φ-bij*: ‹*bij-betw φ V N*›
**begin**

**definition** $\varphi'$ :: ‹*nat* ⇒ *string*› **where**
‹$\varphi'$ = *the-inv-into V* $\varphi$›

**lemma** $\varphi'$-$\varphi$[*simp*]:
‹$x \in V \Longrightarrow \varphi'\ (\varphi\ x) = x$›
⟨*proof*⟩

**lemma** $\varphi$-$\varphi'$[*simp*]:
‹$x \in N \Longrightarrow \varphi\ (\varphi'\ x) = x$›
⟨*proof*⟩

**end**

**end**

**theory** *PAC-Polynomials-Term*
**imports** *PAC-Polynomials*
*Refine-Imperative-HOL.IICF*
**begin**

# 8 Terms

We define some helper functions.

## 8.1 Ordering

**lemma** *fref-to-Down-curry-left*:
**fixes** *f*:: ‹'*a* ⇒ '*b* ⇒ '*c nres*› **and**
*A*::‹(('*a* × '*b*) × '*d*) *set*›
**shows**
‹(*uncurry f*, *g*) ∈ $[P]_f$ *A* → ⟨*B*⟩*nres-rel* ⟹
(⋀*a b x'*. *P x'* ⟹ ((*a*, *b*), *x'*) ∈ *A* ⟹ *f a b* ≤ ⇓ *B* (*g x'*))›
⟨*proof*⟩

**lemma** *fref-to-Down-curry-right*:
**fixes** *g* :: ‹'*a* ⇒ '*b* ⇒ '*c nres*› **and** *f* :: ‹'*d* ⇒ - *nres*› **and**
*A*::‹('*d* × ('*a* × '*b*)) *set*›
**shows**
‹(*f*, *uncurry g*) ∈ $[P]_f$ *A* → ⟨*B*⟩*nres-rel* ⟹
(⋀*a b x'*. *P* (*a*,*b*) ⟹ (*x'*, (*a*, *b*)) ∈ *A* ⟹ *f x'* ≤ ⇓ *B* (*g a b*))›
⟨*proof*⟩

**type-synonym** *term-poly-list* = ‹*string list*›
**type-synonym** *llist-polynomial* = ‹(*term-poly-list* × *int*) *list*›

We instantiate the characters with typeclass linorder to be able to talk abourt sorted and so on.

**definition** *less-eq-char* :: ‹*char* ⇒ *char* ⇒ *bool*› **where**
‹*less-eq-char c d* = (((*of-char c*) :: *nat*) ≤ *of-char d*)›

**definition** *less-char* :: ‹*char* ⇒ *char* ⇒ *bool*› **where**
‹*less-char c d* = (((*of-char c*) :: *nat*) < *of-char d*)›

**global-interpretation** *char*: *linorder less-eq-char less-char*

⟨*proof*⟩

**abbreviation** *less-than-char* :: ‹(*char* × *char*) *set*› **where**
‹*less-than-char* ≡ *p2rel less-char*›

**lemma** *less-than-char-def*:
‹(*x*,*y*) ∈ *less-than-char* ⟷ *less-char x y*›
⟨*proof*⟩

**lemma** *trans-less-than-char*[*simp*]:
‹*trans less-than-char*› **and**
*irrefl-less-than-char*:
‹*irrefl less-than-char*› **and**
*antisym-less-than-char*:
‹*antisym less-than-char*›
⟨*proof*⟩

## 8.2 Polynomials

**definition** *var-order-rel* :: ‹(*string* × *string*) *set*› **where**
‹*var-order-rel* ≡ *lexord less-than-char*›

**abbreviation** *var-order* :: ‹*string* ⇒ *string* ⇒ *bool*› **where**
‹*var-order* ≡ *rel2p var-order-rel*›

**abbreviation** *term-order-rel* :: ‹(*term-poly-list* × *term-poly-list*) *set*› **where**
‹*term-order-rel* ≡ *lexord var-order-rel*›

**abbreviation** *term-order* :: ‹*term-poly-list* ⇒ *term-poly-list* ⇒ *bool*› **where**
‹*term-order* ≡ *rel2p term-order-rel*›

**definition** *term-poly-list-rel* :: ‹(*term-poly-list* × *term-poly*) *set*› **where**
‹*term-poly-list-rel* = {(*xs*, *ys*).
*ys* = *mset xs* ∧
*distinct xs* ∧
*sorted-wrt* (*rel2p var-order-rel*) *xs*}›

**definition** *unsorted-term-poly-list-rel* :: ‹(*term-poly-list* × *term-poly*) *set*› **where**
‹*unsorted-term-poly-list-rel* = {(*xs*, *ys*).
*ys* = *mset xs* ∧ *distinct xs*}›

**definition** *poly-list-rel* :: ‹- ⇒ ((*'a* × *int*) *list* × *mset-polynomial*) *set*› **where**
‹*poly-list-rel R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel* ∧
*0* ∉# *snd* ‘# *ys*}›

**definition** *sorted-poly-list-rel-wrt* :: ‹(*'a* ⇒ *'a* ⇒ *bool*)
⇒ (*'a* × *string multiset*) *set* ⇒ ((*'a* × *int*) *list* × *mset-polynomial*) *set*› **where**
‹*sorted-poly-list-rel-wrt S R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel* ∧
*sorted-wrt S* (*map fst xs*) ∧
*distinct* (*map fst xs*) ∧
*0* ∉# *snd* ‘# *ys*}›

**abbreviation** *sorted-poly-list-rel* **where**
‹*sorted-poly-list-rel R* ≡ *sorted-poly-list-rel-wrt R term-poly-list-rel*›

**abbreviation** *sorted-poly-rel* **where**
‹*sorted-poly-rel* ≡ *sorted-poly-list-rel term-order*›

**definition** *sorted-repeat-poly-list-rel-wrt* :: ‹('*a* ⇒ '*a* ⇒ *bool*)
⇒ ('*a* × *string multiset*) *set* ⇒ (('*a* × *int*) *list* × *mset-polynomial*) *set*› **where**
‹*sorted-repeat-poly-list-rel-wrt S R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel* ∧
*sorted-wrt S* (*map fst xs*) ∧
*0* ∉# *snd* '# *ys*}›

**abbreviation** *sorted-repeat-poly-list-rel* **where**
‹*sorted-repeat-poly-list-rel R* ≡ *sorted-repeat-poly-list-rel-wrt R term-poly-list-rel*›

**abbreviation** *sorted-repeat-poly-rel* **where**
‹*sorted-repeat-poly-rel* ≡ *sorted-repeat-poly-list-rel* (*rel2p* (*Id* ∪ *lexord var-order-rel*))›

**abbreviation** *unsorted-poly-rel* **where**
‹*unsorted-poly-rel* ≡ *poly-list-rel term-poly-list-rel*›

**lemma** *sorted-poly-list-rel-empty-l*[*simp*]:
‹([], *s'*) ∈ *sorted-poly-list-rel-wrt S T* ⟷ *s'* = {#}›
⟨*proof*⟩

**definition** *fully-unsorted-poly-list-rel* :: ‹- ⇒ (('*a* × *int*) *list* × *mset-polynomial*) *set*› **where**
‹*fully-unsorted-poly-list-rel R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel*}›

**abbreviation** *fully-unsorted-poly-rel* **where**
‹*fully-unsorted-poly-rel* ≡ *fully-unsorted-poly-list-rel unsorted-term-poly-list-rel*›

**lemma** *fully-unsorted-poly-list-rel-empty-iff*[*simp*]:
‹(*p*, {#}) ∈ *fully-unsorted-poly-list-rel R* ⟷ *p* = []›
‹([], *p'*) ∈ *fully-unsorted-poly-list-rel R* ⟷ *p'* = {#}›
⟨*proof*⟩

**definition** *poly-list-rel-with0* :: ‹- ⇒ (('*a* × *int*) *list* × *mset-polynomial*) *set*› **where**
‹*poly-list-rel-with0 R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel*}›

**abbreviation** *unsorted-poly-rel-with0* **where**
‹*unsorted-poly-rel-with0* ≡ *fully-unsorted-poly-list-rel term-poly-list-rel*›

**lemma** *poly-list-rel-with0-empty-iff*[*simp*]:
‹(*p*, {#}) ∈ *poly-list-rel-with0 R* ⟷ *p* = []›
‹([], *p'*) ∈ *poly-list-rel-with0 R* ⟷ *p'* = {#}›
⟨*proof*⟩

**definition** *sorted-repeat-poly-list-rel-with0-wrt* :: ‹('*a* ⇒ '*a* ⇒ *bool*)
⇒ ('*a* × *string multiset*) *set* ⇒ (('*a* × *int*) *list* × *mset-polynomial*) *set*› **where**

‹*sorted-repeat-poly-list-rel-with0-wrt S R* = {(*xs*, *ys*).
(*xs*, *ys*) ∈ ⟨*R* ×$_r$ *int-rel*⟩*list-rel O list-mset-rel* ∧
*sorted-wrt S* (*map fst xs*)}›

**abbreviation** *sorted-repeat-poly-list-rel-with0* **where**
‹*sorted-repeat-poly-list-rel-with0 R* ≡ *sorted-repeat-poly-list-rel-with0-wrt R term-poly-list-rel*›

**abbreviation** *sorted-repeat-poly-rel-with0* **where**
‹*sorted-repeat-poly-rel-with0* ≡ *sorted-repeat-poly-list-rel-with0* (*rel2p* (*Id* ∪ *lexord var-order-rel*))›

**lemma** *term-poly-list-relD*:
‹(*xs*, *ys*) ∈ *term-poly-list-rel* ⟹ *distinct xs*›
‹(*xs*, *ys*) ∈ *term-poly-list-rel* ⟹ *ys* = *mset xs*›
‹(*xs*, *ys*) ∈ *term-poly-list-rel* ⟹ *sorted-wrt* (*rel2p var-order-rel*) *xs*›
‹(*xs*, *ys*) ∈ *term-poly-list-rel* ⟹ *sorted-wrt* (*rel2p* (*Id* ∪ *var-order-rel*)) *xs*›
⟨*proof*⟩

**end**
**theory** *PAC-Polynomials-Operations*
**imports** *PAC-Polynomials-Term PAC-Checker-Specification*
**begin**

## 8.3 Addition

In this section, we refine the polynomials to list. These lists will be used in our checker to represent the polynomials and execute operations.

There is one *key* difference between the list representation and the usual representation: in the former, coefficients can be zero and monomials can appear several times. This makes it easier to reason on intermediate representation where this has not yet been sanitized.

**fun** *add-poly-l'* :: ‹*llist-polynomial* × *llist-polynomial* ⇒ *llist-polynomial*› **where**
‹*add-poly-l'* (*p*, []) = *p*› |
‹*add-poly-l'* ([], *q*) = *q*› |
‹*add-poly-l'* ((*xs*, *n*) # *p*, (*ys*, *m*) # *q*) =
(*if xs* = *ys then if n* + *m* = *0 then add-poly-l'* (*p*, *q*) *else*
*let pq* = *add-poly-l'* (*p*, *q*) *in*
((*xs*, *n* + *m*) # *pq*)
*else if* (*xs*, *ys*) ∈ *term-order-rel*
*then*
*let pq* = *add-poly-l'* (*p*, (*ys*, *m*) # *q*) *in*
((*xs*, *n*) # *pq*)
*else*
*let pq* = *add-poly-l'* ((*xs*, *n*) # *p*, *q*) *in*
((*ys*, *m*) # *pq*)
)›

**definition** *add-poly-l* :: ‹*llist-polynomial* × *llist-polynomial* ⇒ *llist-polynomial nres*› **where**
‹*add-poly-l* = *REC*$_T$
(λ*add-poly-l* (*p*, *q*).
*case* (*p*,*q*) *of*
(*p*, []) ⇒ *RETURN p*
| ([], *q*) ⇒ *RETURN q*
| ((*xs*, *n*) # *p*, (*ys*, *m*) # *q*) ⇒
(*if xs* = *ys then if n* + *m* = *0 then add-poly-l* (*p*, *q*) *else*
*do* {

```
        pq ← add-poly-l (p, q);
        RETURN ((xs, n + m) # pq)
    }
  else if (xs, ys) ∈ term-order-rel
    then do {
        pq ← add-poly-l (p, (ys, m) # q);
        RETURN ((xs, n) # pq)
  }
  else do {
        pq ← add-poly-l ((xs, n) # p, q);
        RETURN ((ys, m) # pq)
  }))›
```

**definition** *nonzero-coeffs* **where**
‹*nonzero-coeffs a* ⟷ *0* ∉# *snd* ‘# *a*›

**lemma** *nonzero-coeffs-simps*[*simp*]:
‹*nonzero-coeffs* {#}›
‹*nonzero-coeffs* (*add-mset* (*xs*, *n*) *a*) ⟷ *nonzero-coeffs a* ∧ *n* ≠ *0*›
⟨*proof*⟩

**lemma** *nonzero-coeffsD*:
‹*nonzero-coeffs a* ⟹ (*x*, *n*) ∈# *a* ⟹ *n* ≠ *0*›
⟨*proof*⟩

**lemma** *sorted-poly-list-rel-ConsD*:
‹((*ys*, *n*) # *p*, *a*) ∈ *sorted-poly-list-rel S* ⟹ (*p*, *remove1-mset* (*mset ys*, *n*) *a*) ∈ *sorted-poly-list-rel S*
∧
(*mset ys*, *n*) ∈# *a* ∧ (∀ *x* ∈ *set p*. *S ys* (*fst x*)) ∧ *sorted-wrt* (*rel2p var-order-rel*) *ys* ∧
*distinct ys* ∧ *ys* ∉ *set* (*map fst p*) ∧ *n* ≠ *0* ∧ *nonzero-coeffs a*›
⟨*proof*⟩

**lemma** *sorted-poly-list-rel-Cons-iff*:
‹((*ys*, *n*) # *p*, *a*) ∈ *sorted-poly-list-rel S* ⟷ (*p*, *remove1-mset* (*mset ys*, *n*) *a*) ∈ *sorted-poly-list-rel S*
∧
(*mset ys*, *n*) ∈# *a* ∧ (∀ *x* ∈ *set p*. *S ys* (*fst x*)) ∧ *sorted-wrt* (*rel2p var-order-rel*) *ys* ∧
*distinct ys* ∧ *ys* ∉ *set* (*map fst p*) ∧ *n* ≠ *0* ∧ *nonzero-coeffs a*›
⟨*proof*⟩

**lemma** *sorted-repeat-poly-list-rel-ConsD*:
‹((*ys*, *n*) # *p*, *a*) ∈ *sorted-repeat-poly-list-rel S* ⟹ (*p*, *remove1-mset* (*mset ys*, *n*) *a*) ∈ *sorted-repeat-poly-list-rel*
*S* ∧
(*mset ys*, *n*) ∈# *a* ∧ (∀ *x* ∈ *set p*. *S ys* (*fst x*)) ∧ *sorted-wrt* (*rel2p var-order-rel*) *ys* ∧
*distinct ys* ∧ *n* ≠ *0* ∧ *nonzero-coeffs a*›
⟨*proof*⟩

**lemma** *sorted-repeat-poly-list-rel-Cons-iff*:
‹((*ys*, *n*) # *p*, *a*) ∈ *sorted-repeat-poly-list-rel S* ⟷ (*p*, *remove1-mset* (*mset ys*, *n*) *a*) ∈ *sorted-repeat-poly-list-rel*
*S* ∧
(*mset ys*, *n*) ∈# *a* ∧ (∀ *x* ∈ *set p*. *S ys* (*fst x*)) ∧ *sorted-wrt* (*rel2p var-order-rel*) *ys* ∧
*distinct ys* ∧ *n* ≠ *0* ∧ *nonzero-coeffs a*›
⟨*proof*⟩

**lemma** *add-poly-p-add-mset-sum-0*:
‹$n + m = 0 \Longrightarrow$*add-poly-p*$^{**}$ ($A$, $Aa$, {#}) ({#}, {#}, $r$) $\Longrightarrow$
*add-poly-p*$^{**}$
(*add-mset* (*mset ys*, $n$) $A$, *add-mset* (*mset ys*, $m$) $Aa$, {#})
({#}, {#}, $r$)›
⟨*proof*⟩

**lemma** *monoms-add-poly-l'D*:
‹($aa$, $ba$) ∈ *set* (*add-poly-l'* $x$) $\Longrightarrow$ $aa \in$ *fst* ‘ *set* (*fst* $x$) ∨ $aa \in$ *fst* ‘ *set* (*snd* $x$)›
⟨*proof*⟩

**lemma** *add-poly-p-add-to-result*:
‹*add-poly-p*$^{**}$ ($A$, $B$, $r$) ($A'$, $B'$, $r'$) $\Longrightarrow$
*add-poly-p*$^{**}$
($A$, $B$, $p + r$) ($A'$, $B'$, $p + r'$)›
⟨*proof*⟩

**lemma** *add-poly-p-add-mset-comb*:
‹*add-poly-p*$^{**}$ ($A$, $Aa$, {#}) ({#}, {#}, $r$) $\Longrightarrow$
*add-poly-p*$^{**}$
(*add-mset* ($xs$, $n$) $A$, $Aa$, {#})
({#}, {#}, *add-mset* ($xs$, $n$) $r$)›
⟨*proof*⟩

**lemma** *add-poly-p-add-mset-comb2*:
‹*add-poly-p*$^{**}$ ($A$, $Aa$, {#}) ({#}, {#}, $r$) $\Longrightarrow$
*add-poly-p*$^{**}$
(*add-mset* ($ys$, $n$) $A$, *add-mset* ($ys$, $m$) $Aa$, {#})
({#}, {#}, *add-mset* ($ys$, $n + m$) $r$)›
⟨*proof*⟩

**lemma** *add-poly-p-add-mset-comb3*:
‹*add-poly-p*$^{**}$ ($A$, $Aa$, {#}) ({#}, {#}, $r$) $\Longrightarrow$
*add-poly-p*$^{**}$
($A$, *add-mset* ($ys$, $m$) $Aa$, {#})
({#}, {#}, *add-mset* ($ys$, $m$) $r$)›
⟨*proof*⟩

**lemma** *total-on-lexord*:
‹*Relation.total-on UNIV R* $\Longrightarrow$ *Relation.total-on UNIV* (*lexord R*)›
⟨*proof*⟩

**lemma** *antisym-lexord*:
‹*antisym R* $\Longrightarrow$ *irrefl R* $\Longrightarrow$ *antisym* (*lexord R*)›
⟨*proof*⟩

**lemma** *less-than-char-linear*:
‹($a$, $b$) ∈ *less-than-char* ∨
$a = b$ ∨ ($b$, $a$) ∈ *less-than-char*›
⟨*proof*⟩

**lemma** *total-on-lexord-less-than-char-linear*:
‹$xs \neq ys \Longrightarrow$ ($xs$, $ys$) ∉ *lexord* (*lexord less-than-char*) ⟷

$(ys, xs) \in lexord\ (lexord\ less\text{-}than\text{-}char)\rangle$
⟨*proof*⟩

**lemma** *sorted-poly-list-rel-nonzeroD*:
‹$(p, r) \in sorted\text{-}poly\text{-}list\text{-}rel\ term\text{-}order \Longrightarrow$
$nonzero\text{-}coeffs\ (r)\rangle$
‹$(p, r) \in sorted\text{-}poly\text{-}list\text{-}rel\ (rel2p\ (lexord\ (lexord\ less\text{-}than\text{-}char))) \Longrightarrow$
$nonzero\text{-}coeffs\ (r)\rangle$
⟨*proof*⟩

**lemma** *add-poly-l'-add-poly-p*:
**assumes** ‹$(pq, pq') \in sorted\text{-}poly\text{-}rel \times_r sorted\text{-}poly\text{-}rel\rangle$
**shows** ‹$\exists r.\ (add\text{-}poly\text{-}l'\ pq, r) \in sorted\text{-}poly\text{-}rel \wedge$
$add\text{-}poly\text{-}p^{**}\ (fst\ pq', snd\ pq', \{\#\})\ (\{\#\}, \{\#\}, r)\rangle$
⟨*proof*⟩

**lemma** *add-poly-l-add-poly*:
‹$add\text{-}poly\text{-}l\ x = RETURN\ (add\text{-}poly\text{-}l'\ x)\rangle$
⟨*proof*⟩

**lemma** *add-poly-l-spec*:
‹$(add\text{-}poly\text{-}l, uncurry\ (\lambda p\ q.\ SPEC(\lambda r.\ add\text{-}poly\text{-}p^{**}\ (p, q, \{\#\})\ (\{\#\}, \{\#\}, r)))) \in$
$sorted\text{-}poly\text{-}rel \times_r sorted\text{-}poly\text{-}rel \to_f \langle sorted\text{-}poly\text{-}rel\rangle nres\text{-}rel\rangle$
⟨*proof*⟩

**definition** *sort-poly-spec* :: ‹*llist-polynomial* ⇒ *llist-polynomial nres*› **where**
‹$sort\text{-}poly\text{-}spec\ p =$
$SPEC(\lambda p'.\ mset\ p = mset\ p' \wedge sorted\text{-}wrt\ (rel2p\ (Id \cup term\text{-}order\text{-}rel))\ (map\ fst\ p'))\rangle$

**lemma** *sort-poly-spec-id*:
**assumes** ‹$(p, p') \in unsorted\text{-}poly\text{-}rel\rangle$
**shows** ‹$sort\text{-}poly\text{-}spec\ p \le \Downarrow\ (sorted\text{-}repeat\text{-}poly\text{-}rel)\ (RETURN\ p')\rangle$
⟨*proof*⟩

## 8.4 Multiplication

**fun** *mult-monoms* :: ‹*term-poly-list* ⇒ *term-poly-list* ⇒ *term-poly-list*› **where**
‹$mult\text{-}monoms\ p\ [] = p\rangle$ |
‹$mult\text{-}monoms\ []\ p = p\rangle$ |
‹$mult\text{-}monoms\ (x\ \#\ p)\ (y\ \#\ q) =$
$(if\ x = y\ then\ x\ \#\ mult\text{-}monoms\ p\ q$
$else\ if\ (x, y) \in var\text{-}order\text{-}rel\ then\ x\ \#\ mult\text{-}monoms\ p\ (y\ \#\ q)$
$else\ y\ \#\ mult\text{-}monoms\ (x\ \#\ p)\ q)\rangle$

**lemma** *term-poly-list-rel-empty-iff*[*simp*]:
‹$([], q') \in term\text{-}poly\text{-}list\text{-}rel \longleftrightarrow q' = \{\#\}\rangle$
⟨*proof*⟩

**lemma** *mset-eqD-set-mset*: ‹$mset\ xs = A \Longrightarrow set\ xs = set\text{-}mset\ A\rangle$
⟨*proof*⟩

**lemma** *term-poly-list-rel-Cons-iff*:
‹$(y\ \#\ p, p') \in term\text{-}poly\text{-}list\text{-}rel \longleftrightarrow$
$(p, remove1\text{-}mset\ y\ p') \in term\text{-}poly\text{-}list\text{-}rel \wedge$

$y \in\# p' \wedge y \notin set\ p \wedge y \notin\# remove1\text{-}mset\ y\ p' \wedge$
$(\forall x \in\# mset\ p.\ (y,\ x) \in var\text{-}order\text{-}rel)$›
⟨*proof*⟩

**lemma** *var-order-rel-antisym*[*simp*]:
‹$(y,\ y) \notin var\text{-}order\text{-}rel$›
⟨*proof*⟩

**lemma** *term-poly-list-rel-remdups-mset*:
‹$(p,\ p') \in term\text{-}poly\text{-}list\text{-}rel \Longrightarrow$
$(p,\ remdups\text{-}mset\ p') \in term\text{-}poly\text{-}list\text{-}rel$›
⟨*proof*⟩

**lemma** *var-notin-notin-mult-monomsD*:
‹$y \in set\ (mult\text{-}monoms\ p\ q) \Longrightarrow y \in set\ p \vee y \in set\ q$›
⟨*proof*⟩

**lemma** *term-poly-list-rel-set-mset*:
‹$(p,\ q) \in term\text{-}poly\text{-}list\text{-}rel \Longrightarrow set\ p = set\text{-}mset\ q$›
⟨*proof*⟩

**lemma** *mult-monoms-spec*:
‹$(mult\text{-}monoms, (\lambda a\ b.\ remdups\text{-}mset\ (a + b))) \in term\text{-}poly\text{-}list\text{-}rel \to term\text{-}poly\text{-}list\text{-}rel \to term\text{-}poly\text{-}list\text{-}rel$›
⟨*proof*⟩

**definition** *mult-monomials* :: ‹*term-poly-list* × *int* ⇒ *term-poly-list* × *int* ⇒ *term-poly-list* × *int*›
**where**
‹$mult\text{-}monomials = (\lambda(x,\ a)\ (y,\ b).\ (mult\text{-}monoms\ x\ y,\ a * b))$›

**definition** *mult-poly-raw* :: ‹*llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial*› **where**
‹$mult\text{-}poly\text{-}raw\ p\ q = foldl\ (\lambda b\ x.\ map\ (mult\text{-}monomials\ x)\ q\ @\ b)\ []\ p$›

**fun** *map-append* **where**
‹$map\text{-}append\ f\ b\ [] = b$› |
‹$map\text{-}append\ f\ b\ (x \#\ xs) = f\ x \#\ map\text{-}append\ f\ b\ xs$›

**lemma** *map-append-alt-def*:
‹$map\text{-}append\ f\ b\ xs = map\ f\ xs\ @\ b$›
⟨*proof*⟩

**lemma** *foldl-append-empty*:
‹$NO\text{-}MATCH\ []\ xs \Longrightarrow foldl\ (\lambda b\ x.\ f\ x\ @\ b)\ xs\ p = foldl\ (\lambda b\ x.\ f\ x\ @\ b)\ []\ p\ @\ xs$›
⟨*proof*⟩

**lemma** *poly-list-rel-empty-iff*[*simp*]:
‹$([],\ r) \in poly\text{-}list\text{-}rel\ R \longleftrightarrow r = \{\#\}$›
⟨*proof*⟩

**lemma** *mult-poly-raw-simp*[*simp*]:
‹$mult\text{-}poly\text{-}raw\ []\ q = []$›
‹$mult\text{-}poly\text{-}raw\ (x \#\ p)\ q = mult\text{-}poly\text{-}raw\ p\ q\ @\ map\ (mult\text{-}monomials\ x)\ q$›
⟨*proof*⟩

**lemma** *sorted-poly-list-relD*:
‹$(q,\ q') \in$ *sorted-poly-list-rel* $R \Longrightarrow q' = (\lambda(a,\ b).\ (mset\ a,\ b))$ ‘$\#$ *mset* $q$›
⟨*proof*⟩

**lemma** *list-all2-in-set-ExD*:
‹*list-all2* $R\ p\ q \Longrightarrow x \in set\ p \Longrightarrow \exists\, y \in set\ q.\ R\ x\ y$›
⟨*proof*⟩

**inductive-cases** *mult-poly-p-elim*: ‹*mult-poly-p* $q\ (A,\ r)\ (B,\ r')$›

**lemma** *mult-poly-p-add-mset-same*:
‹$(mult\text{-}poly\text{-}p\ q')^{**}\ (A,\ r)\ (B,\ r') \Longrightarrow (mult\text{-}poly\text{-}p\ q')^{**}\ (add\text{-}mset\ x\ A,\ r)\ (add\text{-}mset\ x\ B,\ r')$›
⟨*proof*⟩

**lemma** *mult-poly-raw-mult-poly-p*:
**assumes** ‹$(p,\ p') \in$ *sorted-poly-rel*› **and** ‹$(q,\ q') \in$ *sorted-poly-rel*›
**shows** ‹$\exists\, r.\ (mult\text{-}poly\text{-}raw\ p\ q,\ r) \in unsorted\text{-}poly\text{-}rel \wedge (mult\text{-}poly\text{-}p\ q')^{**}\ (p',\ \{\#\})\ (\{\#\},\ r)$›
⟨*proof*⟩

**fun** *merge-coeffs* :: ‹*llist-polynomial* ⇒ *llist-polynomial*› **where**
‹*merge-coeffs* [] = []› |
‹*merge-coeffs* $[(xs,\ n)]$ = $[(xs,\ n)]$› |
‹*merge-coeffs* $((xs,\ n)\ \#\ (ys,\ m)\ \#\ p) =$
(*if* $xs = ys$
*then if* $n + m \neq 0$ *then merge-coeffs* $((xs,\ n + m)\ \#\ p)$ *else merge-coeffs* $p$
*else* $(xs,\ n)\ \#$ *merge-coeffs* $((ys,\ m)\ \#\ p)$)›

**abbreviation** (**in** −)*mononoms* :: ‹*llist-polynomial* ⇒ *term-poly-list set*› **where**
‹*mononoms* $p \equiv fst$ ‘*set* $p$›

**lemma** *fst-normalize-polynomial-subset*:
‹*mononoms* (*merge-coeffs* $p$) $\subseteq$ *mononoms* $p$›
⟨*proof*⟩

**lemma** *fst-normalize-polynomial-subsetD*:
‹$(a,\ b) \in$ *set* (*merge-coeffs* $p$) $\Longrightarrow a \in$ *mononoms* $p$›
⟨*proof*⟩

**lemma** *distinct-merge-coeffs*:
**assumes** ‹*sorted-wrt* $R$ (*map fst xs*)› **and** ‹*transp* $R$› ‹*antisymp* $R$›
**shows** ‹*distinct* (*map fst* (*merge-coeffs xs*))›
⟨*proof*⟩

**lemma** *in-set-merge-coeffsD*:
‹$(a,\ b) \in$ *set* (*merge-coeffs* $p$) $\Longrightarrow \exists\, b.\ (a,\ b) \in$ *set* $p$›
⟨*proof*⟩

**lemma** *rtranclp-normalize-poly-add-mset*:
‹*normalize-poly-p*$^{**}$ $A\ r \Longrightarrow$ *normalize-poly-p*$^{**}$ (*add-mset* $x\ A$) (*add-mset* $x\ r$)›
⟨*proof*⟩

**lemma** *nonzero-coeffs-diff*:

‹*nonzero-coeffs A* ⟹ *nonzero-coeffs* (*A* − *B*)›
⟨*proof*⟩

**lemma** *merge-coeffs-is-normalize-poly-p*:
‹(*xs*, *ys*) ∈ *sorted-repeat-poly-rel* ⟹ ∃ *r*. (*merge-coeffs xs*, *r*) ∈ *sorted-poly-rel* ∧ *normalize-poly-p*** *ys r*›
⟨*proof*⟩

## 8.5 Normalisation

**definition** *normalize-poly* **where**
‹*normalize-poly p* = *do* {
  *p* ← *sort-poly-spec p*;
  *RETURN* (*merge-coeffs p*)
}›
**definition** *sort-coeff* :: ‹*string list* ⇒ *string list nres*› **where**
‹*sort-coeff ys* = *SPEC*(λ*xs*. *mset xs* = *mset ys* ∧ *sorted-wrt* (*rel2p* (*Id* ∪ *var-order-rel*)) *xs*)›

**lemma** *distinct-var-order-Id-var-order*:
‹*distinct a* ⟹ *sorted-wrt* (*rel2p* (*Id* ∪ *var-order-rel*)) *a* ⟹
  *sorted-wrt var-order a*›
⟨*proof*⟩

**definition** *sort-all-coeffs* :: ‹*llist-polynomial* ⇒ *llist-polynomial nres*› **where**
‹*sort-all-coeffs xs* = *monadic-nfoldli xs* (λ-. *RETURN True*) (λ(*a*, *n*) *b*. *do* {*a* ← *sort-coeff a*; *RETURN* ((*a*, *n*) # *b*)}) []›

**lemma** *sort-all-coeffs-gen*:
**assumes** ‹(∀ *xs* ∈ *mononoms xs'*. *sorted-wrt* (*rel2p* (*var-order-rel*)) *xs*)› **and**
  ‹∀ *x* ∈ *mononoms* (*xs* @ *xs'*). *distinct x*›
**shows** ‹*monadic-nfoldli xs* (λ-. *RETURN True*) (λ(*a*, *n*) *b*. *do* {*a* ← *sort-coeff a*; *RETURN* ((*a*, *n*) # *b*)}) *xs'* ≤
  ⇓*Id* (*SPEC*(λ*ys*. *map* (λ(*a*,*b*). (*mset a*, *b*)) (*rev xs* @ *xs'*) = *map* (λ(*a*,*b*). (*mset a*, *b*)) (*ys*) ∧
  (∀ *xs* ∈ *mononoms ys*. *sorted-wrt* (*rel2p* (*var-order-rel*)) *xs*)))›
⟨*proof*⟩

**definition** *shuffle-coefficients* **where**
‹*shuffle-coefficients xs* = (*SPEC*(λ*ys*. *map* (λ(*a*,*b*). (*mset a*, *b*)) (*rev xs*) = *map* (λ(*a*,*b*). (*mset a*, *b*)) *ys* ∧
  (∀ *xs* ∈ *mononoms ys*. *sorted-wrt* (*rel2p* (*var-order-rel*)) *xs*)))›

**lemma** *sort-all-coeffs*:
‹∀ *x* ∈ *mononoms xs*. *distinct x* ⟹
*sort-all-coeffs xs* ≤ ⇓ *Id* (*shuffle-coefficients xs*)›
⟨*proof*⟩

**lemma** *unsorted-term-poly-list-rel-mset*:
‹(*ys*, *aa*) ∈ *unsorted-term-poly-list-rel* ⟹ *mset ys* = *aa*›
⟨*proof*⟩

**lemma** *RETURN-map-alt-def*:
‹*RETURN o* (*map f*) =
  *REC*$_T$ (λ*g xs*.
    *case xs of*
      [] ⇒ *RETURN* []

$|\ x\ \#\ xs \Rightarrow do\ \{xs \leftarrow g\ xs;\ RETURN\ (f\ x\ \#\ xs)\}) \rangle$
⟨*proof*⟩

**lemma** *fully-unsorted-poly-rel-Cons-iff*:
‹((*ys*, *n*) # *p*, *a*) ∈ *fully-unsorted-poly-rel* ⟷
(*p*, *remove1-mset* (*mset ys*, *n*) *a*) ∈ *fully-unsorted-poly-rel* ∧
(*mset ys*, *n*) ∈# *a* ∧ *distinct ys*›
⟨*proof*⟩

**lemma** *map-mset-unsorted-term-poly-list-rel*:
‹(⋀*a*. *a* ∈ *mononoms s* ⟹ *distinct a*) ⟹ ∀ *x* ∈ *mononoms s*. *distinct x* ⟹
(∀ *xs* ∈ *mononoms s*. *sorted-wrt* (*rel2p* (*Id* ∪ *var-order-rel*)) *xs*) ⟹
(*s*, *map* (λ(*a*, *y*). (*mset a*, *y*)) *s*)
∈ ⟨*term-poly-list-rel* ×$_r$ *int-rel*⟩*list-rel*›
⟨*proof*⟩

**lemma** *list-rel-unsorted-term-poly-list-relD*:
‹(*p*, *y*) ∈ ⟨*unsorted-term-poly-list-rel* ×$_r$ *int-rel*⟩*list-rel* ⟹
*mset y* = (λ(*a*, *y*). (*mset a*, *y*)) ‘# *mset p* ∧ (∀ *x* ∈ *mononoms p*. *distinct x*)›
⟨*proof*⟩

**lemma** *shuffle-terms-distinct-iff*:
**assumes** ‹*map* (λ(*a*, *y*). (*mset a*, *y*)) *p* = *map* (λ(*a*, *y*). (*mset a*, *y*)) *s*›
**shows** ‹(∀ *x*∈*set p*. *distinct* (*fst x*)) ⟷ (∀ *x*∈*set s*. *distinct* (*fst x*))›
⟨*proof*⟩

**lemma**
‹(*p*, *y*) ∈ ⟨*unsorted-term-poly-list-rel* ×$_r$ *int-rel*⟩*list-rel* ⟹
(*a*, *b*) ∈ *set p* ⟹ *distinct a*›
⟨*proof*⟩

**lemma** *sort-all-coeffs-unsorted-poly-rel-with0*:
**assumes** ‹(*p*, *p*′) ∈ *fully-unsorted-poly-rel*›
**shows** ‹*sort-all-coeffs p* ≤ ⇓ (*unsorted-poly-rel-with0*) (*RETURN p*′)›
⟨*proof*⟩

**lemma** *sort-poly-spec-id*′:
**assumes** ‹(*p*, *p*′) ∈ *unsorted-poly-rel-with0*›
**shows** ‹*sort-poly-spec p* ≤ ⇓ (*sorted-repeat-poly-rel-with0*) (*RETURN p*′)›
⟨*proof*⟩

**fun** *merge-coeffs0* :: ‹*llist-polynomial* ⇒ *llist-polynomial*› **where**
‹*merge-coeffs0* [] = []› |
‹*merge-coeffs0* [(*xs*, *n*)] = (*if n* = *0 then* [] *else* [(*xs*, *n*)])› |
‹*merge-coeffs0* ((*xs*, *n*) # (*ys*, *m*) # *p*) =
(*if xs* = *ys*
*then if n* + *m* ≠ *0 then merge-coeffs0* ((*xs*, *n* + *m*) # *p*) *else merge-coeffs0 p*
*else if n* = *0 then merge-coeffs0* ((*ys*, *m*) # *p*)
*else*(*xs*, *n*) # *merge-coeffs0* ((*ys*, *m*) # *p*))›

**lemma** *sorted-repeat-poly-list-rel-with0-wrt-ConsD*:
‹((*ys*, *n*) # *p*, *a*) ∈ *sorted-repeat-poly-list-rel-with0-wrt S term-poly-list-rel* ⟹

(p, remove1-mset (mset ys, n) a) ∈ sorted-repeat-poly-list-rel-with0-wrt S term-poly-list-rel ∧
(mset ys, n) ∈# a ∧ (∀ x ∈ set p. S ys (fst x)) ∧ sorted-wrt (rel2p var-order-rel) ys ∧
distinct ys›
⟨proof⟩

**lemma** *sorted-repeat-poly-list-rel-with0-wrtl-Cons-iff*:
‹((ys, n) # p, a) ∈ sorted-repeat-poly-list-rel-with0-wrt S term-poly-list-rel ⟷
(p, remove1-mset (mset ys, n) a) ∈ sorted-repeat-poly-list-rel-with0-wrt S term-poly-list-rel ∧
(mset ys, n) ∈# a ∧ (∀ x ∈ set p. S ys (fst x)) ∧ sorted-wrt (rel2p var-order-rel) ys ∧
distinct ys›
⟨proof⟩

**lemma** *fst-normalize0-polynomial-subsetD*:
‹(a, b) ∈ set (merge-coeffs0 p) ⟹ a ∈ mononoms p›
⟨proof⟩

**lemma** *in-set-merge-coeffs0D*:
‹(a, b) ∈ set (merge-coeffs0 p) ⟹∃ b. (a, b) ∈ set p›
⟨proof⟩

**lemma** *merge-coeffs0-is-normalize-poly-p*:
‹(xs, ys) ∈ sorted-repeat-poly-rel-with0 ⟹ ∃ r. (merge-coeffs0 xs, r) ∈ sorted-poly-rel ∧ normalize-poly-p** ys r›
⟨proof⟩

**definition** *full-normalize-poly* **where**
‹full-normalize-poly p = do {
p ← sort-all-coeffs p;
p ← sort-poly-spec p;
RETURN (merge-coeffs0 p)
}›

**fun** *sorted-remdups* **where**
‹sorted-remdups (x # y # zs) =
(if x = y then sorted-remdups (y # zs) else x # sorted-remdups (y # zs))› |
‹sorted-remdups zs = zs›

**lemma** *set-sorted-remdups*[*simp*]:
‹set (sorted-remdups xs) = set xs›
⟨proof⟩

**lemma** *distinct-sorted-remdups*:
‹sorted-wrt R xs ⟹ antisymp R ⟹ distinct (sorted-remdups xs)›
⟨proof⟩

**lemma** *full-normalize-poly-normalize-poly-p*:
**assumes** ‹(p, p′) ∈ fully-unsorted-poly-rel›
**shows** ‹full-normalize-poly p ≤ ⇓ (sorted-poly-rel) (SPEC (λr. normalize-poly-p** p′ r))›
(**is** ‹?A ≤ ⇓ ?R ?B›)
⟨proof⟩

**definition** *mult-poly-full* :: ‹-› **where**
‹mult-poly-full p q = do {
let pq = mult-poly-raw p q;

*normalize-poly pq*
}›

**lemma** *normalize-poly-normalize-poly-p*:
**assumes** ‹(*p*, *p'*) ∈ *unsorted-poly-rel*›
**shows** ‹*normalize-poly p* ≤ ⇓ (*sorted-poly-rel*) (*SPEC* (λ*r*. *normalize-poly-p*** *p' r*))›
⟨*proof*⟩

## 8.6 Multiplication and normalisation

**definition** *mult-poly-p'* :: ‹-› **where**
‹*mult-poly-p' p' q'* = *do* {
*pq* ← *SPEC*(λ*r*. (*mult-poly-p q'*)** (*p'*, {#}) ({#}, *r*));
*SPEC* (λ*r*. *normalize-poly-p*** *pq r*)
}›

**lemma** *unsorted-poly-rel-fully-unsorted-poly-rel*:
‹*unsorted-poly-rel* ⊆ *fully-unsorted-poly-rel*›
⟨*proof*⟩

**lemma** *mult-poly-full-mult-poly-p'*:
**assumes** ‹(*p*, *p'*) ∈ *sorted-poly-rel*› ‹(*q*, *q'*) ∈ *sorted-poly-rel*›
**shows** ‹*mult-poly-full p q* ≤ ⇓ (*sorted-poly-rel*) (*mult-poly-p' p' q'*)›
⟨*proof*⟩

**definition** *add-poly-spec* :: ‹-› **where**
‹*add-poly-spec p q* = *SPEC* (λ*r*. *p* + *q* − *r* ∈ *ideal polynomial-bool*)›

**definition** *add-poly-p'* :: ‹-› **where**
‹*add-poly-p' p q* = *SPEC*(λ*r*. *add-poly-p*** (*p*, *q*, {#}) ({#}, {#}, *r*))›

**lemma** *add-poly-l-add-poly-p'*:
**assumes** ‹(*p*, *p'*) ∈ *sorted-poly-rel*› ‹(*q*, *q'*) ∈ *sorted-poly-rel*›
**shows** ‹*add-poly-l* (*p*, *q*) ≤ ⇓ (*sorted-poly-rel*) (*add-poly-p' p' q'*)›
⟨*proof*⟩

## 8.7 Correctness

**context** *poly-embed*
**begin**

**definition** *mset-poly-rel* **where**
‹*mset-poly-rel* = {(*a*, *b*). *b* = *polynomial-of-mset a*}›

**definition** *var-rel* **where**
‹*var-rel* = *br* φ (λ-. *True*)›

**lemma** *normalize-poly-p-normalize-poly-spec*:
‹(*p*, *p'*) ∈ *mset-poly-rel* ⟹
*SPEC* (λ*r*. *normalize-poly-p*** *p r*) ≤ ⇓*mset-poly-rel* (*normalize-poly-spec p'*)›
⟨*proof*⟩

**lemma** *mult-poly-p'-mult-poly-spec*:
‹(*p*, *p'*) ∈ *mset-poly-rel* ⟹ (*q*, *q'*) ∈ *mset-poly-rel* ⟹
*mult-poly-p' p q* ≤ ⇓*mset-poly-rel* (*mult-poly-spec p' q'*)›

⟨*proof*⟩

**lemma** *add-poly-p'-add-poly-spec*:
‹(*p*, *p'*) ∈ *mset-poly-rel* ⟹ (*q*, *q'*) ∈ *mset-poly-rel* ⟹
*add-poly-p'* *p* *q* ≤ ⇓*mset-poly-rel* (*add-poly-spec* *p'* *q'*)›
⟨*proof*⟩

**end**

**definition** *weak-equality-l* :: ‹*llist-polynomial* ⇒ *llist-polynomial* ⇒ *bool nres*› **where**
‹*weak-equality-l* *p* *q* = *RETURN* (*p* = *q*)›

**definition** *weak-equality* :: ‹*int mpoly* ⇒ *int mpoly* ⇒ *bool nres*› **where**
‹*weak-equality* *p* *q* = *SPEC* (λ*r*. *r* ⟷ *p* = *q*)›

**definition** *weak-equality-spec* :: ‹*mset-polynomial* ⇒ *mset-polynomial* ⇒ *bool nres*› **where**
‹*weak-equality-spec* *p* *q* = *SPEC* (λ*r*. *r* ⟷ *p* = *q*)›

**lemma** *term-poly-list-rel-same-rightD*:
‹(*a*, *aa*) ∈ *term-poly-list-rel* ⟹ (*a*, *ab*) ∈ *term-poly-list-rel* ⟹ *aa* = *ab*›
⟨*proof*⟩

**lemma** *list-rel-term-poly-list-rel-same-rightD*:
‹(*xa*, *y*) ∈ ⟨*term-poly-list-rel* ×$_r$ *int-rel*⟩*list-rel* ⟹
(*xa*, *ya*) ∈ ⟨*term-poly-list-rel* ×$_r$ *int-rel*⟩*list-rel* ⟹
*y* = *ya*›
⟨*proof*⟩

**lemma** *weak-equality-l-weak-equality-spec*:
‹(*uncurry* *weak-equality-l*, *uncurry* *weak-equality-spec*) ∈
*sorted-poly-rel* ×$_r$ *sorted-poly-rel* →$_f$ ⟨*bool-rel*⟩*nres-rel*›
⟨*proof*⟩

**end**
**theory** *PAC-Misc*
**imports** *Main*
**begin**

I believe this should be added to the simplifier by default...

**lemma** *Collect-eq-comp'*:
{(*x*, *y*). *P* *x* *y*} *O* {(*c*, *a*). *c* = *f* *a*} = {(*x*, *a*). *P* *x* (*f* *a*)}
⟨*proof*⟩

**lemma** *in-set-conv-iff*:
*x* ∈ *set* (*take* *n* *xs*) ⟷ (∃ *i*<*n*. *i* < *length* *xs* ∧ *xs* ! *i* = *x*)
⟨*proof*⟩

**lemma** *in-set-take-conv-nth*:
*x* ∈ *set* (*take* *n* *xs*) ⟷ (∃ *i*<*min* *n* (*length* *xs*). *xs* ! *i* = *x*)
⟨*proof*⟩

**lemma** *in-set-remove1D*:
*a* ∈ *set* (*remove1* *x* *xs*) ⟹ *a* ∈ *set* *xs*

⟨*proof*⟩

**end**

**theory** *PAC-Checker*
**imports** *PAC-Polynomials-Operations*
*PAC-Checker-Specification*
*PAC-Map-Rel*
*Show.Show*
*Show.Show-Instances*
*PAC-Misc*
**begin**

# 9 Executable Checker

In this layer we finally refine the checker to executable code.

## 9.1 Definitions

Compared to the previous layer, we add an error message when an error is discovered. We do not attempt to prove anything on the error message (neither that there really is an error, nor that the error message is correct).

**Extended error message** **datatype** *'a code-status* =
*is-cfailed*: *CFAILED* (*the-error*: *'a*) |
*CSUCCESS* |
*is-cfound*: *CFOUND*

In the following function, we merge errors. We will never merge an error message with an another error message; hence we do not attempt to concatenate error messages.

**fun** *merge-cstatus* **where**
‹*merge-cstatus* (*CFAILED a*) - = *CFAILED a*› |
‹*merge-cstatus* - (*CFAILED a*) = *CFAILED a*› |
‹*merge-cstatus CFOUND* - = *CFOUND*› |
‹*merge-cstatus* - *CFOUND* = *CFOUND*› |
‹*merge-cstatus* - - = *CSUCCESS*›

**definition** *code-status-status-rel* :: ‹(*'a code-status* × *status*) *set*› **where**
‹*code-status-status-rel* =
{(*CFOUND*, *FOUND*), (*CSUCCESS*, *SUCCESS*)} ∪
{(*CFAILED a*, *FAILED*)| *a*. *True*}›

**lemma** *in-code-status-status-rel-iff*[*simp*]:
‹(*CFOUND*, *b*) ∈ *code-status-status-rel* ⟷ *b* = *FOUND*›
‹(*a*, *FOUND*) ∈ *code-status-status-rel* ⟷ *a* = *CFOUND*›
‹(*CSUCCESS*, *b*) ∈ *code-status-status-rel* ⟷ *b* = *SUCCESS*›
‹(*a*, *SUCCESS*) ∈ *code-status-status-rel* ⟷ *a* = *CSUCCESS*›
‹(*a*, *FAILED*) ∈ *code-status-status-rel* ⟷ *is-cfailed a*›
‹(*CFAILED C*, *b*) ∈ *code-status-status-rel* ⟷ *b* = *FAILED*›
⟨*proof*⟩

**Refinement relation** **fun** *pac-step-rel-raw* :: ‹(*'olbl* × *'lbl*) *set* ⇒ (*'a* × *'b*) *set* ⇒ (*'c* × *'d*) *set* ⇒ (*'a*, *'c*, *'olbl*) *pac-step* ⇒ (*'b*, *'d*, *'lbl*) *pac-step* ⇒ *bool*› **where**

‹*pac-step-rel-raw R1 R2 R3* (*Add p1 p2 i r*) (*Add p1′ p2′ i′ r′*) ⟷
(*p1*, *p1′*) ∈ *R1* ∧ (*p2*, *p2′*) ∈ *R1* ∧ (*i*, *i′*) ∈ *R1* ∧
(*r*, *r′*) ∈ *R2*› |
‹*pac-step-rel-raw R1 R2 R3* (*Mult p1 p2 i r*) (*Mult p1′ p2′ i′ r′*) ⟷
(*p1*, *p1′*) ∈ *R1* ∧ (*p2*, *p2′*) ∈ *R2* ∧ (*i*, *i′*) ∈ *R1* ∧
(*r*, *r′*) ∈ *R2*› |
‹*pac-step-rel-raw R1 R2 R3* (*Del p1*) (*Del p1′*) ⟷
(*p1*, *p1′*) ∈ *R1*› |
‹*pac-step-rel-raw R1 R2 R3* (*Extension i x p1*) (*Extension j x′ p1′*) ⟷
(*i*, *j*) ∈ *R1* ∧ (*x*, *x′*) ∈ *R3* ∧ (*p1*, *p1′*) ∈ *R2*› |
‹*pac-step-rel-raw R1 R2 R3* - - ⟷ *False*›

**fun** *pac-step-rel-assn* :: ‹(*'olbl* ⇒ *'lbl* ⇒ *assn*) ⇒ (*'a* ⇒ *'b* ⇒ *assn*) ⇒ (*'c* ⇒ *'d* ⇒ *assn*) ⇒ (*'a*, *'c*, *'olbl*) *pac-step* ⇒ (*'b*, *'d*, *'lbl*) *pac-step* ⇒ *assn*› **where**
‹*pac-step-rel-assn R1 R2 R3* (*Add p1 p2 i r*) (*Add p1′ p2′ i′ r′*) =
*R1 p1 p1′* ∗ *R1 p2 p2′* ∗ *R1 i i′* ∗
*R2 r r′*› |
‹*pac-step-rel-assn R1 R2 R3* (*Mult p1 p2 i r*) (*Mult p1′ p2′ i′ r′*) =
*R1 p1 p1′* ∗ *R2 p2 p2′* ∗ *R1 i i′* ∗
*R2 r r′*› |
‹*pac-step-rel-assn R1 R2 R3* (*Del p1*) (*Del p1′*) =
*R1 p1 p1′*› |
‹*pac-step-rel-assn R1 R2 R3* (*Extension i x p1*) (*Extension i′ x′ p1′*) =
*R1 i i′* ∗ *R3 x x′* ∗ *R2 p1 p1′*› |
‹*pac-step-rel-assn R1 R2* - - - = *false*›

**lemma** *pac-step-rel-assn-alt-def*:
‹*pac-step-rel-assn R1 R2 R3 x y* = (
*case* (*x*, *y*) *of*
(*Add p1 p2 i r*, *Add p1′ p2′ i′ r′*) ⇒
*R1 p1 p1′* ∗ *R1 p2 p2′* ∗ *R1 i i′* ∗ *R2 r r′*
| (*Mult p1 p2 i r*, *Mult p1′ p2′ i′ r′*) ⇒
*R1 p1 p1′* ∗ *R2 p2 p2′* ∗ *R1 i i′* ∗ *R2 r r′*
| (*Del p1*, *Del p1′*) ⇒ *R1 p1 p1′*
| (*Extension i x p1*, *Extension i′ x′ p1′*) ⇒ *R1 i i′* ∗ *R3 x x′* ∗ *R2 p1 p1′*
| - ⇒ *false*)›
⟨*proof*⟩

**Addition checking** **definition** *error-msg* **where**
‹*error-msg i msg* = *CFAILED* (*''s CHECKING failed at line ''* @ *show i* @ *'' with error ''* @ *msg*)›

**definition** *error-msg-notin-dom-err* **where**
‹*error-msg-notin-dom-err* = *'' notin domain''*›

**definition** *error-msg-notin-dom* :: ‹*nat* ⇒ *string*› **where**
‹*error-msg-notin-dom i* = *show i* @ *error-msg-notin-dom-err*›

**definition** *error-msg-reused-dom* **where**
‹*error-msg-reused-dom i* = *show i* @ *'' already in domain''*›

**definition** *error-msg-not-equal-dom* **where**
‹*error-msg-not-equal-dom p q pq r* = *show p* @ *''* + *''* @ *show q* @ *''* = *''* @ *show pq* @ *'' not equal''* @ *show r*›

**definition** *check-not-equal-dom-err* :: ‹*llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *string nres*› **where**
‹*check-not-equal-dom-err p q pq r* = *SPEC* (λ-. *True*)›

**definition** *vars-llist* :: ‹*llist-polynomial* ⇒ *string set*› **where**
‹*vars-llist xs* = ⋃(*set* ‘ *fst* ‘ *set xs*)›

**definition** *check-addition-l* :: ‹- ⇒ - ⇒ *string set* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *llist-polynomial* ⇒ *string code-status nres*› **where**
‹*check-addition-l spec A* $\mathcal{V}$ *p q i r* = *do* {
*let b* = *p* ∈# *dom-m A* ∧ *q* ∈# *dom-m A* ∧ *i* ∉# *dom-m A* ∧ *vars-llist r* ⊆ $\mathcal{V}$;
*if* ¬*b*
*then RETURN* (*error-msg i* ((*if p* ∉# *dom-m A then error-msg-notin-dom p else* []) @ (*if q* ∉# *dom-m A then error-msg-notin-dom p else* []) @
(*if i* ∈# *dom-m A then error-msg-reused-dom p else* [])))
*else do* {
*ASSERT* (*p* ∈# *dom-m A*);
*let p* = *the* (*fmlookup A p*);
*ASSERT* (*q* ∈# *dom-m A*);
*let q* = *the* (*fmlookup A q*);
*pq* ← *add-poly-l* (*p*, *q*);
*b* ← *weak-equality-l pq r*;
*b*′ ← *weak-equality-l r spec*;
*if b then* (*if b*′ *then RETURN CFOUND else RETURN CSUCCESS*)
*else do* {
*c* ← *check-not-equal-dom-err p q pq r*;
*RETURN* (*error-msg i c*)}
}
}›

**Multiplication checking** **definition** *check-mult-l-dom-err* :: ‹*bool* ⇒ *nat* ⇒ *bool* ⇒ *nat* ⇒ *string nres*› **where**
‹*check-mult-l-dom-err p-notin p i-already i* = *SPEC* (λ-. *True*)›

**definition** *check-mult-l-mult-err* :: ‹*llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *string nres*› **where**
‹*check-mult-l-mult-err p q pq r* = *SPEC* (λ-. *True*)›

**definition** *check-mult-l* :: ‹- ⇒ - ⇒ - ⇒ *nat* ⇒*llist-polynomial* ⇒ *nat* ⇒ *llist-polynomial* ⇒ *string code-status nres*› **where**
‹*check-mult-l spec A* $\mathcal{V}$ *p q i r* = *do* {
*let b* = *p* ∈# *dom-m A* ∧ *i* ∉# *dom-m A* ∧ *vars-llist q* ⊆ $\mathcal{V}$∧ *vars-llist r* ⊆ $\mathcal{V}$;
*if* ¬*b*
*then do* {
*c* ← *check-mult-l-dom-err* (*p* ∉# *dom-m A*) *p* (*i* ∈# *dom-m A*) *i*;
*RETURN* (*error-msg i c*)}
*else do* {
*ASSERT* (*p* ∈# *dom-m A*);
*let p* = *the* (*fmlookup A p*);
*pq* ← *mult-poly-full p q*;

```
    b ← weak-equality-l pq r;
    b' ← weak-equality-l r spec;
    if b then (if b' then RETURN CFOUND else RETURN CSUCCESS) else do {
      c ← check-mult-l-mult-err p q pq r;
      RETURN (error-msg i c)
    }
  }
}›
```

**Deletion checking** **definition** *check-del-l* :: ‹- ⇒ - ⇒ *nat* ⇒ *string code-status nres*› **where**
‹*check-del-l spec A p* = *RETURN CSUCCESS*›

**Extension checking** **definition** *check-extension-l-dom-err* :: ‹*nat* ⇒ *string nres*› **where**
‹*check-extension-l-dom-err p* = *SPEC* (λ-. *True*)›

**definition** *check-extension-l-no-new-var-err* :: ‹*llist-polynomial* ⇒ *string nres*› **where**
‹*check-extension-l-no-new-var-err p* = *SPEC* (λ-. *True*)›

**definition** *check-extension-l-new-var-multiple-err* :: ‹*string* ⇒ *llist-polynomial* ⇒ *string nres*› **where**
‹*check-extension-l-new-var-multiple-err v p* = *SPEC* (λ-. *True*)›

**definition** *check-extension-l-side-cond-err*
:: ‹*string* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *llist-polynomial* ⇒ *string nres*›
**where**
‹*check-extension-l-side-cond-err v p p' q* = *SPEC* (λ-. *True*)›

**definition** *check-extension-l*
:: ‹- ⇒ - ⇒ *string set* ⇒ *nat* ⇒ *string* ⇒ *llist-polynomial* ⇒ (*string code-status*) *nres*›
**where**

```
‹check-extension-l spec A V i v p = do {
  let b = i ∉# dom-m A ∧ v ∉ V ∧ ([v], −1) ∈ set p;
  if ¬b
  then do {
    c ← check-extension-l-dom-err i;
    RETURN (error-msg i c)
  } else do {
      let p' = remove1 ([v], −1) p;
      let b = vars-llist p' ⊆ V;
      if ¬b
      then do {
        c ← check-extension-l-new-var-multiple-err v p';
        RETURN (error-msg i c)
      }
      else do {
         p2 ← mult-poly-full p' p';
         let p' = map (λ(a,b). (a, −b)) p';
         q ← add-poly-l (p2, p');
         eq ← weak-equality-l q [];
         if eq then do {
           RETURN (CSUCCESS)
         } else do {
          c ← check-extension-l-side-cond-err v p p' q;
          RETURN (error-msg i c)
        }
```

```
    }
   }
  }›
```

**lemma** *check-extension-alt-def*:

```
  ‹check-extension A 𝒱 i v p ≥ do {
    b ← SPEC(λb. b ⟶ i ∉# dom-m A ∧ v ∉ 𝒱);
    if ¬b
    then RETURN (False)
    else do {
        p' ← RETURN (p + Var v);
        b ← SPEC(λb. b ⟶ vars p' ⊆ 𝒱);
        if ¬b
        then RETURN (False)
        else do {
          pq ← mult-poly-spec p' p';
          let p' = - p';
          p ← add-poly-spec pq p';
          eq ← weak-equality p 0;
          if eq then RETURN(True)
          else RETURN (False)
      }
    }
  }›
⟨proof⟩
```

**lemma** *RES-RES-RETURN-RES*: ‹*RES A* ⤜ (λ*T*. *RES* (*f T*)) = *RES* (⋃(*f* ‘ *A*))›

⟨*proof*⟩

**lemma** *check-add-alt-def*:

```
  ‹check-add A 𝒱 p q i r ≥
    do {
     b ← SPEC(λb. b ⟶ p ∈# dom-m A ∧ q ∈# dom-m A ∧ i ∉# dom-m A ∧ vars r ⊆ 𝒱);
     if ¬b
     then RETURN False
     else do {
       ASSERT (p ∈# dom-m A);
       let p = the (fmlookup A p);
       ASSERT (q ∈# dom-m A);
       let q = the (fmlookup A q);
       pq ← add-poly-spec p q;
       eq ← weak-equality pq r;
       RETURN eq
     }
  }› (is ‹- ≥ ?A›)
```

⟨*proof*⟩

**lemma** *check-mult-alt-def*:

```
  ‹check-mult A 𝒱 p q i r ≥
    do {
     b ← SPEC(λb. b ⟶ p ∈# dom-m A ∧ i ∉# dom-m A ∧ vars q ⊆ 𝒱  ∧ vars r ⊆ 𝒱);
     if ¬b
```

```
    then RETURN False
    else do {
      ASSERT (p ∈# dom-m A);
      let p = the (fmlookup A p);
      pq ← mult-poly-spec p q;
      p ← weak-equality pq r;
      RETURN p
    }
  }›
  ⟨proof⟩

primrec insort-key-rel :: ('b ⇒ 'b ⇒ bool) ⇒ 'b ⇒ 'b list ⇒ 'b list where
insort-key-rel f x [] = [x] |
insort-key-rel f x (y#ys) =
  (if f x y then (x#y#ys) else y#(insort-key-rel f x ys))

lemma set-insort-key-rel[simp]: ‹set (insort-key-rel R x xs) = insert x (set xs)›
  ⟨proof⟩

lemma sorted-wrt-insort-key-rel:
   ‹totalp-on (insert x (set xs)) R ⟹ transp R ⟹ reflp R ⟹
    sorted-wrt R xs ⟹ sorted-wrt R (insort-key-rel R x xs)›
  ⟨proof⟩

lemma sorted-wrt-insort-key-rel2:
   ‹totalp-on (insert x (set xs)) R ⟹ transp R ⟹ x ∉ set xs ⟹
    sorted-wrt R xs ⟹ sorted-wrt R (insort-key-rel R x xs)›
  ⟨proof⟩
```

**Step checking** **definition** *PAC-checker-l-step* :: ‹- ⇒ *string code-status* × *string set* × - ⇒ (*llist-polynomial*, *string*, *nat*) *pac-step* ⇒ -› **where**

```
  ‹PAC-checker-l-step = (λspec (st', V, A) st. case st of
    Add - - - - ⇒
      do {
        r ← full-normalize-poly (pac-res st);
        eq ← check-addition-l spec A V (pac-src1 st) (pac-src2 st) (new-id st) r;
        let - = eq;
        if ¬is-cfailed eq
        then RETURN (merge-cstatus st' eq,
          V, fmupd (new-id st) r A)
        else RETURN (eq, V, A)
    }
  | Del - ⇒
      do {
        eq ← check-del-l spec A (pac-src1 st);
        let - = eq;
        if ¬is-cfailed eq
        then RETURN (merge-cstatus st' eq, V, fmdrop (pac-src1 st) A)
        else RETURN (eq, V, A)
    }
  | Mult - - - - ⇒
      do {
        r ← full-normalize-poly (pac-res st);
        q ← full-normalize-poly (pac-mult st);
        eq ← check-mult-l spec A V (pac-src1 st) q (new-id st) r;
```

```
      let - = eq;
      if ¬is-cfailed eq
      then RETURN (merge-cstatus st' eq,
        V, fmupd (new-id st) r A)
      else RETURN (eq, V, A)
  }
  | Extension - - - ⇒
     do {
       r ← full-normalize-poly (([new-var st], −1) # (pac-res st));
      (eq) ← check-extension-l spec A V (new-id st) (new-var st) r;
      if ¬is-cfailed eq
      then do {
        RETURN (st',
          insert (new-var st) V, fmupd (new-id st) r A)}
      else RETURN (eq, V, A)
  }
)›

lemma pac-step-rel-raw-def:
  ‹⟨K, V, R⟩ pac-step-rel-raw = pac-step-rel-raw K V R›
  ⟨proof⟩

definition mononoms-equal-up-to-reorder where
  ‹mononoms-equal-up-to-reorder xs ys ⟷
     map (λ(a, b). (mset a, b)) xs = map (λ(a, b). (mset a, b)) ys›


 definition normalize-poly-l where
  ‹normalize-poly-l p = SPEC (λp'.
     normalize-poly-p** ((λ(a, b). (mset a, b)) `# mset p) ((λ(a, b). (mset a, b)) `# mset p') ∧
     0 ∉# snd `# mset p' ∧
     sorted-wrt (rel2p (term-order-rel ×r int-rel)) p' ∧
     (∀ x ∈ mononoms p'. sorted-wrt (rel2p var-order-rel) x))›


definition remap-polys-l-dom-err :: ‹string nres› where
  ‹remap-polys-l-dom-err = SPEC (λ-. True)›


definition remap-polys-l :: ‹llist-polynomial ⇒ string set ⇒ (nat, llist-polynomial) fmap ⇒
   (- code-status × string set × (nat, llist-polynomial) fmap) nres› where
  ‹remap-polys-l spec = (λV A. do{
   dom ← SPEC(λdom. set-mset (dom-m A) ⊆ dom ∧ finite dom);
   failed ← SPEC(λ-::bool. True);
   if failed
   then do {
     c ← remap-polys-l-dom-err;
     RETURN (error-msg (0 :: nat) c, V, fmempty)
   }
   else do {
    (b, V, A) ← FOREACH dom
     (λi (b, V,  A').
       if i ∈# dom-m A
       then  do {
         p ← full-normalize-poly (the (fmlookup A i));
```

```
          eq ← weak-equality-l p spec;
          V ← RETURN(V ∪ vars-llist (the (fmlookup A i)));
          RETURN(b ∨ eq, V, fmupd i p A')
        } else RETURN (b, V, A'))
      (False, V, fmempty);
    RETURN (if b then CFOUND else CSUCCESS, V, A)
}})›

definition PAC-checker-l where
  ‹PAC-checker-l spec A b st = do {
    (S, -) ← WHILE_T
      (λ((b, A), n). ¬is-cfailed b ∧ n ≠ [])
      (λ((bA), n). do {
        ASSERT(n ≠ []);
        S ← PAC-checker-l-step spec bA (hd n);
        RETURN (S, tl n)
       })
     ((b, A), st);
    RETURN S
  }›
```

## 9.2 Correctness

We now enter the locale to reason about polynomials directly.

```
context poly-embed
begin

abbreviation pac-step-rel where
  ‹pac-step-rel ≡ p2rel (⟨Id, fully-unsorted-poly-rel O mset-poly-rel, var-rel⟩ pac-step-rel-raw)›

abbreviation fmap-polys-rel where
  ‹fmap-polys-rel ≡ ⟨nat-rel, sorted-poly-rel O mset-poly-rel⟩fmap-rel›

lemma
  ‹normalize-poly-p s0 s ⟹
      (s0, p) ∈ mset-poly-rel ⟹
      (s, p) ∈ mset-poly-rel›
  ⟨proof⟩

lemma vars-poly-of-vars:
  ‹vars (poly-of-vars a :: int mpoly) ⊆ (φ ‘ set-mset a)›
  ⟨proof⟩

lemma vars-polynomial-of-mset:
  ‹vars (polynomial-of-mset za) ⊆ ⋃(image φ ‘ (set-mset o fst) ‘ set-mset za)›
  ⟨proof⟩

lemma fully-unsorted-poly-rel-vars-subset-vars-llist:
  ‹(A, B) ∈ fully-unsorted-poly-rel O mset-poly-rel ⟹ vars B ⊆ φ ‘ vars-llist A›
  ⟨proof⟩

lemma fully-unsorted-poly-rel-extend-vars:
  ‹(A, B) ∈ fully-unsorted-poly-rel O mset-poly-rel ⟹
  (x1c, x1a) ∈ ⟨var-rel⟩set-rel ⟹
   RETURN (x1c ∪ vars-llist A)
```

$\leq \Downarrow$ ($\langle$*var-rel*$\rangle$*set-rel*)
(*SPEC* ((⊆) (*x1a* ∪ *vars* (*B*))))›
⟨*proof*⟩

**lemma** *remap-polys-l-remap-polys*:
**assumes**
*AB*: ‹(*A*, *B*) ∈ ⟨*nat-rel*, *fully-unsorted-poly-rel O mset-poly-rel*⟩*fmap-rel*› **and**
*spec*: ‹(*spec*, *spec'*) ∈ *sorted-poly-rel O mset-poly-rel*› **and**
*V*: ‹(𝒱, 𝒱') ∈ ⟨*var-rel*⟩*set-rel*›
**shows** ‹*remap-polys-l spec* 𝒱 *A* ≤
⇓(*code-status-status-rel* ×$_r$ ⟨*var-rel*⟩*set-rel* ×$_r$ *fmap-polys-rel*) (*remap-polys spec'* 𝒱' *B*)›
(**is** ‹- ≤ ⇓ *?R* -›)
⟨*proof*⟩

**lemma** *fref-to-Down-curry*:
‹(*uncurry f*, *uncurry g*) ∈ [*P*]$_f$ *A* → ⟨*B*⟩*nres-rel* ⟹
(⋀*x x' y y'*. *P* (*x'*, *y'*) ⟹ ((*x*, *y*), (*x'*, *y'*)) ∈ *A* ⟹ *f x y* ≤ ⇓ *B* (*g x' y'*))›
⟨*proof*⟩

**lemma** *weak-equality-spec-weak-equality*:
‹(*p*, *p'*) ∈ *mset-poly-rel* ⟹
(*r*, *r'*) ∈ *mset-poly-rel* ⟹
*weak-equality-spec p r* ≤ *weak-equality p' r'*›
⟨*proof*⟩

**lemma** *weak-equality-l-weak-equality-l'*[*refine*]:
‹*weak-equality-l p q* ≤ ⇓ *bool-rel* (*weak-equality p' q'*)›
**if** ‹(*p*, *p'*) ∈ *sorted-poly-rel O mset-poly-rel*›
‹(*q*, *q'*) ∈ *sorted-poly-rel O mset-poly-rel*›
**for** *p p' q q'*
⟨*proof*⟩

**lemma** *error-msg-ne-SUCCES*[*iff*]:
‹*error-msg i m* ≠ *CSUCCESS*›
‹*error-msg i m* ≠ *CFOUND*›
‹*is-cfailed* (*error-msg i m*)›
‹¬*is-cfound* (*error-msg i m*)›
⟨*proof*⟩

**lemma** *sorted-poly-rel-vars-llist*:
‹(*r*, *r'*) ∈ *sorted-poly-rel O mset-poly-rel* ⟹
*vars r'* ⊆ φ ‘ *vars-llist r*›
⟨*proof*⟩

**lemma** *check-addition-l-check-add*:
**assumes** ‹(*A*, *B*) ∈ *fmap-polys-rel*› **and** ‹(*r*, *r'*) ∈ *sorted-poly-rel O mset-poly-rel*›
‹(*p*, *p'*) ∈ *Id*› ‹(*q*, *q'*) ∈ *Id*› ‹(*i*, *i'*) ∈ *nat-rel*›
‹(𝒱', 𝒱) ∈ ⟨*var-rel*⟩*set-rel*›
**shows**
‹*check-addition-l spec A* 𝒱' *p q i r* ≤ ⇓ {(*st*, *b*). (¬*is-cfailed st* ⟷ *b*) ∧
(*is-cfound st* ⟶ *spec* = *r*)} (*check-add B* 𝒱 *p' q' i' r'*)›
⟨*proof*⟩

**lemma** *check-del-l-check-del*:
‹(*A*, *B*) ∈ *fmap-polys-rel* ⟹ (*x3*, *x3a*) ∈ *Id* ⟹ *check-del-l spec A* (*pac-src1* (*Del x3*))
≤ ⇓ {(*st*, *b*). (¬*is-cfailed st* ⟷ *b*) ∧ (*b* ⟶ *st* = *CSUCCESS*)} (*check-del B* (*pac-src1* (*Del x3a*)))›
⟨*proof*⟩

**lemma** *check-mult-l-check-mult*:
**assumes** ‹(*A*, *B*) ∈ *fmap-polys-rel*› **and** ‹(*r*, *r'*) ∈ *sorted-poly-rel O mset-poly-rel*› **and**
‹(*q*, *q'*) ∈ *sorted-poly-rel O mset-poly-rel*›
‹(*p*, *p'*) ∈ *Id*› ‹(*i*, *i'*) ∈ *nat-rel*› ‹(𝒱, 𝒱') ∈ ⟨*var-rel*⟩*set-rel*›
**shows**
‹*check-mult-l spec A* 𝒱 *p q i r* ≤ ⇓ {(*st*, *b*). (¬*is-cfailed st* ⟷ *b*) ∧
(*is-cfound st* ⟶ *spec* = *r*)} (*check-mult B* 𝒱' *p' q' i' r'*)›
⟨*proof*⟩

**lemma** *normalize-poly-normalize-poly-spec*:
**assumes** ‹(*r*, *t*) ∈ *unsorted-poly-rel O mset-poly-rel*›
**shows**
‹*normalize-poly r* ≤ ⇓(*sorted-poly-rel O mset-poly-rel*) (*normalize-poly-spec t*)›
⟨*proof*⟩

**lemma** *remove1-list-rel*:
‹(*xs*, *ys*) ∈ ⟨*R*⟩ *list-rel* ⟹
(*a*, *b*) ∈ *R* ⟹
*IS-RIGHT-UNIQUE R* ⟹
*IS-LEFT-UNIQUE R* ⟹
(*remove1 a xs*, *remove1 b ys*) ∈ ⟨*R*⟩*list-rel*›
⟨*proof*⟩

**lemma** *remove1-list-rel2*:
‹(*xs*, *ys*) ∈ ⟨*R*⟩ *list-rel* ⟹
(*a*, *b*) ∈ *R* ⟹
(⋀*c*. (*a*, *c*) ∈ *R* ⟹ *c* = *b*) ⟹
(⋀*c*. (*c*, *b*) ∈ *R* ⟹ *c* = *a*) ⟹
(*remove1 a xs*, *remove1 b ys*) ∈ ⟨*R*⟩*list-rel*›
⟨*proof*⟩

**lemma** *remove1-sorted-poly-rel-mset-poly-rel*:
**assumes**
‹(*r*, *r'*) ∈ *sorted-poly-rel O mset-poly-rel*› **and**
‹([*a*], *1*) ∈ *set r*›
**shows**
‹(*remove1* ([*a*], *1*) *r*, *r'* − *Var* (*φ a*))
∈ *sorted-poly-rel O mset-poly-rel*›
⟨*proof*⟩

**lemma** *remove1-sorted-poly-rel-mset-poly-rel-minus*:
**assumes**
‹(*r*, *r'*) ∈ *sorted-poly-rel O mset-poly-rel*› **and**
‹([*a*], −*1*) ∈ *set r*›
**shows**
‹(*remove1* ([*a*], −*1*) *r*, *r'* + *Var* (*φ a*))
∈ *sorted-poly-rel O mset-poly-rel*›
⟨*proof*⟩

**lemma** *insert-var-rel-set-rel*:
‹$(\mathcal{V}, \mathcal{V}') \in \langle \textit{var-rel} \rangle \textit{set-rel} \Longrightarrow$
$(yb, x2) \in \textit{var-rel} \Longrightarrow$
$(\textit{insert } yb\ \mathcal{V}, \textit{insert } x2\ \mathcal{V}') \in \langle \textit{var-rel} \rangle \textit{set-rel}$›
⟨*proof*⟩

**lemma** *var-rel-set-rel-iff*:
‹$(\mathcal{V}, \mathcal{V}') \in \langle \textit{var-rel} \rangle \textit{set-rel} \Longrightarrow$
$(yb, x2) \in \textit{var-rel} \Longrightarrow$
$yb \in \mathcal{V} \longleftrightarrow x2 \in \mathcal{V}'$›
⟨*proof*⟩

**lemma** *check-extension-l-check-extension*:
**assumes** ‹$(A, B) \in \textit{fmap-polys-rel}$› **and** ‹$(r, r') \in \textit{sorted-poly-rel } O \textit{ mset-poly-rel}$› **and**
‹$(i, i') \in \textit{nat-rel}$› ‹$(\mathcal{V}, \mathcal{V}') \in \langle \textit{var-rel} \rangle \textit{set-rel}$› ‹$(x, x') \in \textit{var-rel}$›
**shows**
‹$\textit{check-extension-l spec } A\ \mathcal{V}\ i\ x\ r \leq$
$\Downarrow\{((st), (b)).$
$(\neg \textit{is-cfailed } st \longleftrightarrow b) \wedge$
$(\textit{is-cfound } st \longrightarrow \textit{spec} = r)\}\ (\textit{check-extension } B\ \mathcal{V}'\ i'\ x'\ r')$›
⟨*proof*⟩

**lemma** *full-normalize-poly-diff-ideal*:
**fixes** *dom*
**assumes** ‹$(p, p') \in \textit{fully-unsorted-poly-rel } O \textit{ mset-poly-rel}$›
**shows**
‹$\textit{full-normalize-poly } p$
$\leq \Downarrow (\textit{sorted-poly-rel } O \textit{ mset-poly-rel})$
$(\textit{normalize-poly-spec } p')$›
⟨*proof*⟩

**lemma** *insort-key-rel-decomp*:
‹$\exists\, ys\ zs.\ xs = ys\ @\ zs \wedge \textit{insort-key-rel } R\ x\ xs = ys\ @\ x\ \#\ zs$›
⟨*proof*⟩

**lemma** *list-rel-append-same-length*:
‹$\textit{length } xs = \textit{length } xs' \Longrightarrow (xs\ @\ ys, xs'\ @\ ys') \in \langle R \rangle \textit{list-rel} \longleftrightarrow (xs, xs') \in \langle R \rangle \textit{list-rel} \wedge (ys, ys') \in \langle R \rangle \textit{list-rel}$›
⟨*proof*⟩

**lemma** *term-poly-list-rel-list-relD*: ‹$(ys, cs) \in \langle \textit{term-poly-list-rel} \times_r \textit{int-rel} \rangle \textit{list-rel} \Longrightarrow$
$cs = \textit{map } (\lambda(a, y).\ (\textit{mset } a, y))\ ys$›
⟨*proof*⟩

**lemma** *term-poly-list-rel-single*: ‹$([x32], \{\#x32\#\}) \in \textit{term-poly-list-rel}$›
⟨*proof*⟩

**lemma** *unsorted-poly-rel-list-rel-list-rel-uminus*:
‹$(\textit{map } (\lambda(a, b).\ (a, -\ b))\ r, yc)$
$\in \langle \textit{unsorted-term-poly-list-rel} \times_r \textit{int-rel} \rangle \textit{list-rel} \Longrightarrow$
$(r, \textit{map } (\lambda(a, b).\ (a, -\ b))\ yc)$

$\in \langle$*unsorted-term-poly-list-rel* $\times_r$ *int-rel*$\rangle$*list-rel*$\rangle$
⟨*proof*⟩

**lemma** *mset-poly-rel-minus*: ‹$(\{\#(a, b)\#\}, v') \in$ *mset-poly-rel* $\Longrightarrow$
$(mset\ yc, r') \in$ *mset-poly-rel* $\Longrightarrow$
$(r, yc)$
$\in \langle$*unsorted-term-poly-list-rel* $\times_r$ *int-rel*$\rangle$*list-rel* $\Longrightarrow$
$(add\text{-}mset\ (a, b)\ (mset\ yc),$
$v' + r')$
$\in$ *mset-poly-rel*›
⟨*proof*⟩

**lemma** *fully-unsorted-poly-rel-diff*:
‹$([v], v') \in$ *fully-unsorted-poly-rel O mset-poly-rel* $\Longrightarrow$
$(r, r') \in$ *fully-unsorted-poly-rel O mset-poly-rel* $\Longrightarrow$
$(v \# r,$
$v' + r')$
$\in$ *fully-unsorted-poly-rel O mset-poly-rel*›
⟨*proof*⟩

**lemma** *PAC-checker-l-step-PAC-checker-step*:
**assumes**
‹$(Ast, Bst) \in$ *code-status-status-rel* $\times_r \langle$*var-rel*$\rangle$*set-rel* $\times_r$ *fmap-polys-rel*› **and**
‹$(st, st') \in$ *pac-step-rel*› **and**
*spec*: ‹$(spec, spec') \in$ *sorted-poly-rel O mset-poly-rel*›
**shows**
‹*PAC-checker-l-step spec Ast st* $\leq \Downarrow$ (*code-status-status-rel* $\times_r \langle$*var-rel*$\rangle$*set-rel* $\times_r$ *fmap-polys-rel*)
(*PAC-checker-step spec′ Bst st′*)›
⟨*proof*⟩

**lemma** *code-status-status-rel-discrim-iff*:
‹$(x1a, x1c) \in$ *code-status-status-rel* $\Longrightarrow$ *is-cfailed x1a* $\longleftrightarrow$ *is-failed x1c*›
‹$(x1a, x1c) \in$ *code-status-status-rel* $\Longrightarrow$ *is-cfound x1a* $\longleftrightarrow$ *is-found x1c*›
⟨*proof*⟩

**lemma** *PAC-checker-l-PAC-checker*:
**assumes**
‹$(A, B) \in \langle$*var-rel*$\rangle$*set-rel* $\times_r$ *fmap-polys-rel*› **and**
‹$(st, st') \in \langle$*pac-step-rel*$\rangle$*list-rel*› **and**
‹$(spec, spec') \in$ *sorted-poly-rel O mset-poly-rel*› **and**
‹$(b, b') \in$ *code-status-status-rel*›
**shows**
‹*PAC-checker-l spec A b st* $\leq \Downarrow$ (*code-status-status-rel* $\times_r \langle$*var-rel*$\rangle$*set-rel* $\times_r$ *fmap-polys-rel*) (*PAC-checker spec′ B b′ st′*)›
⟨*proof*⟩

**end**

**lemma** *less-than-char-of-char*[*code-unfold*]:
‹$(x, y) \in$ *less-than-char* $\longleftrightarrow$ (*of-char x* :: *nat*) $<$ *of-char y*›
⟨*proof*⟩

**lemmas** [*code*] =
*add-poly-l′.simps*[*unfolded var-order-rel-def*]

**export-code** *add-poly-l′* **in** *SML* **module-name** *test*

**definition** *full-checker-l*
:: ‹*llist-polynomial* ⇒ (*nat*, *llist-polynomial*) *fmap* ⇒ (-, *string*, *nat*) *pac-step list* ⇒
(*string code-status* × -) *nres*›
**where**
‹*full-checker-l spec A st* = *do* {
*spec′* ← *full-normalize-poly spec*;
(*b*, $\mathcal{V}$, *A*) ← *remap-polys-l spec′* {} *A*;
*if is-cfailed b*
*then RETURN* (*b*, $\mathcal{V}$, *A*)
*else do* {
*let* $\mathcal{V}$ = $\mathcal{V}$ ∪ *vars-llist spec*;
*PAC-checker-l spec′* ($\mathcal{V}$, *A*) *b st*
}
}›

**context** *poly-embed*
**begin**

**term** *normalize-poly-spec*
**thm** *full-normalize-poly-diff-ideal*[*unfolded normalize-poly-spec-def*[*symmetric*]]
**abbreviation** *unsorted-fmap-polys-rel* **where**
‹*unsorted-fmap-polys-rel* ≡ ⟨*nat-rel*, *fully-unsorted-poly-rel O mset-poly-rel*⟩*fmap-rel*›

**lemma** *full-checker-l-full-checker*:
**assumes**
‹(*A*, *B*) ∈ *unsorted-fmap-polys-rel*› **and**
‹(*st*, *st′*) ∈ ⟨*pac-step-rel*⟩*list-rel*› **and**
‹(*spec*, *spec′*) ∈ *fully-unsorted-poly-rel O mset-poly-rel*›
**shows**
‹*full-checker-l spec A st* ≤ ⇓ (*code-status-status-rel* ×$_r$ ⟨*var-rel*⟩*set-rel* ×$_r$ *fmap-polys-rel*) (*full-checker spec′ B st′*)›
⟨*proof*⟩

**lemma** *full-checker-l-full-checker′*:
‹(*uncurry2 full-checker-l*, *uncurry2 full-checker*) ∈
((*fully-unsorted-poly-rel O mset-poly-rel*) ×$_r$ *unsorted-fmap-polys-rel*) ×$_r$ ⟨*pac-step-rel*⟩*list-rel* →$_f$
⟨(*code-status-status-rel* ×$_r$ ⟨*var-rel*⟩*set-rel* ×$_r$ *fmap-polys-rel*)⟩*nres-rel*›
⟨*proof*⟩

**end**

**definition** *remap-polys-l2* :: ‹*llist-polynomial* ⇒ *string set* ⇒ (*nat*, *llist-polynomial*) *fmap* ⇒ - *nres*›
**where**
‹*remap-polys-l2 spec* = (λ$\mathcal{V}$ *A*. *do*{
*n* ← *upper-bound-on-dom A*;
*b* ← *RETURN* (*n* ≥ 2^64);
*if b*
*then do* {
*c* ← *remap-polys-l-dom-err*;

```
    RETURN (error-msg (0 ::nat) c, 𝒱, fmempty)
  }
  else do {
    (b, 𝒱, A) ← nfoldli ([0..<n]) (λ-. True)
    (λi (b, 𝒱, A').
      if i ∈# dom-m A
      then do {
        ASSERT(fmlookup A i ≠ None);
        p ← full-normalize-poly (the (fmlookup A i));
        eq ← weak-equality-l p spec;
        𝒱 ← RETURN (𝒱 ∪ vars-llist (the (fmlookup A i)));
        RETURN(b ∨ eq, 𝒱, fmupd i p A')
      } else RETURN (b, 𝒱, A')
     )
    (False, 𝒱, fmempty);
   RETURN (if b then CFOUND else CSUCCESS, 𝒱, A)
  }
})›

lemma remap-polys-l2-remap-polys-l:
  ‹remap-polys-l2 spec 𝒱 A ≤ ⇓ Id (remap-polys-l spec 𝒱 A)›
⟨proof⟩

end

theory PAC-Checker-Relation
  imports PAC-Checker WB-Sort Native-Word.Uint64
begin
```

# 10 Various Refinement Relations

When writing this, it was not possible to share the definition with the IsaSAT version.

```
definition uint64-nat-rel :: (uint64 × nat) set where
 ‹uint64-nat-rel = br nat-of-uint64 (λ-. True)›

abbreviation uint64-nat-assn where
  ‹uint64-nat-assn ≡ pure uint64-nat-rel›

instantiation uint32 :: hashable
begin
definition hashcode-uint32 :: ‹uint32 ⇒ uint32› where
  ‹hashcode-uint32 n = n›

definition def-hashmap-size-uint32 :: ‹uint32 itself ⇒ nat› where
  ‹def-hashmap-size-uint32 = (λ-. 16)›
  — same as nat
instance
  ⟨proof⟩
end

instantiation uint64 :: hashable
begin

context
```

**includes** *bit-operations-syntax*
**begin**

**definition** *hashcode-uint64* :: ‹*uint64* ⇒ *uint32*› **where**
‹*hashcode-uint64 n* = (*uint32-of-nat* (*nat-of-uint64* ((*n*) *AND* ((*2* :: *uint64*) ^ *32* −*1*))))›

**end**

**definition** *def-hashmap-size-uint64* :: ‹*uint64 itself* ⇒ *nat*› **where**
‹*def-hashmap-size-uint64* = (λ-. *16*)›
— same as *nat*
**instance**
⟨*proof*⟩
**end**

**lemma** *word-nat-of-uint64-Rep-inject*[*simp*]: ‹*nat-of-uint64 ai* = *nat-of-uint64 bi* ⟷ *ai* = *bi*›
⟨*proof*⟩

**instance** *uint64* :: *heap*
⟨*proof*⟩

**instance** *uint64* :: *semiring-numeral*
⟨*proof*⟩

**lemma** *nat-of-uint64-012*[*simp*]: ‹*nat-of-uint64 0* = *0*› ‹*nat-of-uint64 2* = *2*› ‹*nat-of-uint64 1* = *1*›
⟨*proof*⟩

**definition** *uint64-of-nat-conv* **where**
[*simp*]: ‹*uint64-of-nat-conv* (*x* :: *nat*) = *x*›

**lemma** *less-upper-bintrunc-id*: ‹*n* < *2* ^ *b* ⟹ *n* ≥ *0* ⟹ *take-bit b n* = *n*› **for** *n* :: *int*
⟨*proof*⟩

**lemma** *nat-of-uint64-uint64-of-nat-id*: ‹*n* < *2*^*64* ⟹ *nat-of-uint64* (*uint64-of-nat n*) = *n*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:
‹(*return o uint64-of-nat*, *RETURN o uint64-of-nat-conv*) ∈ [λ*a*. *a* < *2* ^ *64*]$_a$ *nat-assn*$^k$ → *uint64-nat-assn*›
⟨*proof*⟩

**definition** *string-rel* :: ‹(*String.literal* × *string*) *set*› **where**
‹*string-rel* = {(*x*, *y*). *y* = *String.explode x*}›

**abbreviation** *string-assn* :: ‹*string* ⇒ *String.literal* ⇒ *assn*› **where**
‹*string-assn* ≡ *pure string-rel*›

**lemma** *eq-string-eq*:
‹((=), (=)) ∈ *string-rel* → *string-rel* → *bool-rel*›
⟨*proof*⟩

**lemmas** *eq-string-eq-hnr* =
*eq-string-eq*[*sepref-import-param*]

**definition** *string2-rel* :: ‹(*string* × *string*) *set*› **where**
‹*string2-rel* ≡ ⟨*Id*⟩*list-rel*›

**abbreviation** *string2-assn* :: ‹*string* ⇒ *string* ⇒ *assn*› **where**
‹*string2-assn* ≡ *pure string2-rel*›

**abbreviation** *monom-rel* **where**
‹*monom-rel* ≡ ⟨*string-rel*⟩*list-rel*›

**abbreviation** *monom-assn* **where**
‹*monom-assn* ≡ *list-assn string-assn*›

**abbreviation** *monomial-rel* **where**
‹*monomial-rel* ≡ *monom-rel* ×$_r$ *int-rel*›

**abbreviation** *monomial-assn* **where**
‹*monomial-assn* ≡ *monom-assn* ×$_a$ *int-assn*›

**abbreviation** *poly-rel* **where**
‹*poly-rel* ≡ ⟨*monomial-rel*⟩*list-rel*›

**abbreviation** *poly-assn* **where**
‹*poly-assn* ≡ *list-assn monomial-assn*›

**lemma** *poly-assn-alt-def*:
‹*poly-assn* = *pure poly-rel*›
⟨*proof*⟩

**abbreviation** *polys-assn* **where**
‹*polys-assn* ≡ *hm-fmap-assn uint64-nat-assn poly-assn*›

**lemma** *string-rel-string-assn*:
‹(↑ ((*c*, *a*) ∈ *string-rel*)) = *string-assn a c*›
⟨*proof*⟩

**lemma** *single-valued-string-rel*:
‹*single-valued string-rel*›
⟨*proof*⟩

**lemma** *IS-LEFT-UNIQUE-string-rel*:
‹*IS-LEFT-UNIQUE string-rel*›
⟨*proof*⟩

**lemma** *IS-RIGHT-UNIQUE-string-rel*:
‹*IS-RIGHT-UNIQUE string-rel*›
⟨*proof*⟩

**lemma** *single-valued-monom-rel*: ‹*single-valued monom-rel*›
⟨*proof*⟩

**lemma** *single-valued-monomial-rel*:
‹*single-valued monomial-rel*›
⟨*proof*⟩

**lemma** *single-valued-monom-rel′*: ‹*IS-LEFT-UNIQUE monom-rel*›
⟨*proof*⟩

**lemma** *single-valued-monomial-rel′*:
‹*IS-LEFT-UNIQUE monomial-rel*›
⟨*proof*⟩

**lemma** [*safe-constraint-rules*]:
‹*Sepref-Constraints.CONSTRAINT single-valued string-rel*›
‹*Sepref-Constraints.CONSTRAINT IS-LEFT-UNIQUE string-rel*›
⟨*proof*⟩

**lemma** *eq-string-monom-hnr*[*sepref-fr-rules*]:
‹(*uncurry* (*return oo* (=)), *uncurry* (*RETURN oo* (=))) ∈ *monom-assn*$^k$ ∗$_a$ *monom-assn*$^k$ →$_a$ *bool-assn*›
⟨*proof*⟩

**definition** *term-order-rel′* **where**
[*simp*]: ‹*term-order-rel′ x y* = ((*x*, *y*) ∈ *term-order-rel*)›

**lemma** *term-order-rel*[*def-pat-rules*]:
‹(∈)$(*x*,*y*)$*term-order-rel* ≡ *term-order-rel′*$*x*$*y*›
⟨*proof*⟩

**lemma** *term-order-rel-alt-def*:
‹*term-order-rel* = *lexord* (*p2rel char.lexordp*)›
⟨*proof*⟩

**instantiation** *char* :: *linorder*
**begin**
**definition** *less-char* **where** [*symmetric*, *simp*]: *less-char* = *PAC-Polynomials-Term.less-char*
**definition** *less-eq-char* **where** [*symmetric*, *simp*]: *less-eq-char* = *PAC-Polynomials-Term.less-eq-char*
**instance**
⟨*proof*⟩
**end**

**instantiation** *list* :: (*linorder*) *linorder*
**begin**
**definition** *less-list* **where** *less-list* = *lexordp* (<)
**definition** *less-eq-list* **where** *less-eq-list* = *lexordp-eq*

**instance**
⟨*proof*⟩

**end**

**lemma** *term-order-rel′-alt-def-lexord*:
‹*term-order-rel′ x y* = *ord-class.lexordp x y*› **and**
*term-order-rel′-alt-def*:
‹*term-order-rel′ x y* ⟷ *x* < *y*›
⟨*proof*⟩

**lemma** *list-rel-list-rel-order-iff*:

**assumes** ‹(*a*, *b*) ∈ ⟨*string-rel*⟩*list-rel*› ‹(*a'*, *b'*) ∈ ⟨*string-rel*⟩*list-rel*›
**shows** ‹*a* < *a'* ⟷ *b* < *b'*›
⟨*proof*⟩

**lemma** *string-rel-le*[*sepref-import-param*]:
**shows** ‹((<), (<)) ∈ ⟨*string-rel*⟩*list-rel* → ⟨*string-rel*⟩*list-rel* → *bool-rel*›
⟨*proof*⟩

**lemma** [*sepref-import-param*]:
**assumes** ‹*CONSTRAINT IS-LEFT-UNIQUE R*› ‹*CONSTRAINT IS-RIGHT-UNIQUE R*›
**shows** ‹(*remove1*, *remove1*) ∈ *R* → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*›
⟨*proof*⟩

**instantiation** *pac-step* :: (*heap*, *heap*, *heap*) *heap*
**begin**

**instance**
⟨*proof*⟩

**end**

**end**
**theory** *PAC-Assoc-Map-Rel*
**imports** *PAC-Map-Rel*
**begin**

# 11 Hash Map as association list

**type-synonym** ('*k*, '*v*) *hash-assoc* = ‹('*k* × '*v*) *list*›

**definition** *hassoc-map-rel-raw* :: ‹(('*k*, '*v*) *hash-assoc* × -) *set*› **where**
‹*hassoc-map-rel-raw* = *br map-of* (λ-. *True*)›

**abbreviation** *hassoc-map-assn* :: ‹('*k* ⇒ '*v option*) ⇒ ('*k*, '*v*) *hash-assoc* ⇒ *assn*› **where**
‹*hassoc-map-assn* ≡ *pure* (*hassoc-map-rel-raw*)›

**lemma** *hassoc-map-rel-raw-empty*[*simp*]:
‹([], *m*) ∈ *hassoc-map-rel-raw* ⟷ *m* = *Map.empty*›
‹(*p*, *Map.empty*) ∈ *hassoc-map-rel-raw* ⟷ *p* = []›
‹*hassoc-map-assn Map.empty* [] = *emp*›
⟨*proof*⟩

**definition** *hassoc-new* :: ‹('*k*, '*v*) *hash-assoc Heap*›**where**
‹*hassoc-new* = *return* []›

**lemma** *precise-hassoc-map-assn*: ‹*precise hassoc-map-assn*›
⟨*proof*⟩

**definition** *hassoc-isEmpty* :: ('*k* × '*v*) *list* ⇒ *bool Heap* **where**
*hassoc-isEmpty ht* ≡ *return* (*length ht* = *0*)

**interpretation** *hassoc*: *bind-map-empty hassoc-map-assn hassoc-new*

⟨*proof*⟩

**interpretation** *hassoc*: *bind-map-is-empty hassoc-map-assn hassoc-isEmpty*
⟨*proof*⟩

**definition** *op-assoc-empty* ≡ *IICF-Map.op-map-empty*

**interpretation** *hassoc*: *map-custom-empty op-assoc-empty*
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] = *hassoc.empty-hnr*[*folded op-assoc-empty-def*]

**definition** *hassoc-update* :: *'k* ⇒ *'v* ⇒ (*'k*, *'v*) *hash-assoc* ⇒ (*'k*, *'v*) *hash-assoc Heap* **where**
*hassoc-update k v ht* = *return* ((*k*, *v* ) # *ht*)

**lemma** *hassoc-map-assn-Cons*:
‹*hassoc-map-assn* (*m*) (*p*) ⟹$_A$ *hassoc-map-assn* (*m*(*k* ↦ *v*)) ((*k*, *v*) # *p*) ∗ *true*›
⟨*proof*⟩

**interpretation** *hassoc*: *bind-map-update hassoc-map-assn hassoc-update*
⟨*proof*⟩

**definition** *hassoc-delete* :: ‹*'k* ⇒ (*'k*, *'v*) *hash-assoc* ⇒ (*'k*, *'v*) *hash-assoc Heap*› **where**
‹*hassoc-delete k ht* = *return* (*filter* (λ(*a*, *b*). *a* ≠ *k*) *ht*)›

**lemma** *hassoc-map-of-filter-all*:
‹*map-of p* |‘ (− {*k*}) = *map-of* (*filter* (λ(*a*, *b*). *a* ≠ *k*) *p*)›
⟨*proof*⟩

**lemma** *hassoc-map-assn-hassoc-delete*: ‹<*hassoc-map-assn m p*> *hassoc-delete k p* <*hassoc-map-assn* (*m* |‘ (− {*k*}))>$_t$›
⟨*proof*⟩

**interpretation** *hassoc*: *bind-map-delete hassoc-map-assn hassoc-delete*
⟨*proof*⟩

**definition** *hassoc-lookup* :: ‹*'k* ⇒ (*'k*, *'v*) *hash-assoc* ⇒ *'v option Heap*› **where**
‹*hassoc-lookup k ht* = *return* (*map-of ht k*)›

**lemma** *hassoc-map-assn-hassoc-lookup*:
‹<*hassoc-map-assn m p*> *hassoc-lookup k p* <λ*r*. *hassoc-map-assn m p* ∗ ↑ (*r* = *m k*)>$_t$›
⟨*proof*⟩

**interpretation** *hassoc*: *bind-map-lookup hassoc-map-assn hassoc-lookup*
⟨*proof*⟩

⟨*ML*⟩
**interpretation** *hassoc*: *gen-contains-key-by-lookup hassoc-map-assn hassoc-lookup*
⟨*proof*⟩
⟨*ML*⟩

**interpretation** *hassoc*: *bind-map-contains-key hassoc-map-assn hassoc.contains-key*
⟨*proof*⟩

### 11.1 Conversion from assoc to other map

**definition** *hash-of-assoc-map* **where**
‹*hash-of-assoc-map xs* = *fold* (λ(*k*, *v*) *m*. *if m k* ≠ *None then m else m*(*k* ↦ *v*)) *xs Map.empty*›

**lemma** *map-upd-map-add-left*:
‹*m*(*a* ↦ *b*) ++ *m*′ = *m* ++ (*if a* ∉ *dom m*′ *then m*′(*a* ↦ *b*) *else m*′)›
⟨*proof*⟩

**lemma** *fold-map-of-alt*:
‹*fold* (λ(*k*, *v*) *m*. *if m k* ≠ *None then m else m*(*k* ↦ *v*)) *xs m*′ = *map-of xs* ++ *m*′›
⟨*proof*⟩

**lemma** *map-of-alt-def*:
‹*map-of xs* = *hash-of-assoc-map xs*›
⟨*proof*⟩

**definition** *hashmap-conv* **where**
[*simp*]: ‹*hashmap-conv x* = *x*›

**lemma** *hash-of-assoc-map-id*:
‹(*hash-of-assoc-map*, *hashmap-conv*) ∈ *hassoc-map-rel-raw* → *Id*›
⟨*proof*⟩

**definition** *hassoc-map-rel* **where**
*hassoc-map-rel-internal-def*:
‹*hassoc-map-rel K V* = *hassoc-map-rel-raw O* ⟨*K*, *V*⟩*map-rel*›

**lemma** *hassoc-map-rel-def*:
‹⟨*K*, *V*⟩ *hassoc-map-rel* = *hassoc-map-rel-raw O* ⟨*K*, *V*⟩*map-rel*›
⟨*proof*⟩

**end**

**theory** *PAC-Checker-Init*
**imports** *PAC-Checker WB-Sort PAC-Checker-Relation*
**begin**

# 12 Initial Normalisation of Polynomials

### 12.1 Sorting

Adapted from the theory *HOL−ex.MergeSort* by Tobias Nipkow. We did not change much, but we refine it to executable code and try to improve efficiency.

**fun** *merge* :: - ⇒ *'a list* ⇒ *'a list* ⇒ *'a list*
**where**
*merge f* (*x*#*xs*) (*y*#*ys*) =
(*if f x y then x* # *merge f xs* (*y*#*ys*) *else y* # *merge f* (*x*#*xs*) *ys*)
| *merge f xs* [] = *xs*
| *merge f* [] *ys* = *ys*

**lemma** *mset-merge* [*simp*]:
*mset* (*merge f xs ys*) = *mset xs* + *mset ys*
⟨*proof*⟩

**lemma** *set-merge* [*simp*]:
*set* (*merge f xs ys*) = *set xs* ∪ *set ys*
⟨*proof*⟩

**lemma** *sorted-merge*:
*transp f* ⟹ (⋀*x y*. *f x y* ∨ *f y x*) ⟹
*sorted-wrt f* (*merge f xs ys*) ⟷ *sorted-wrt f xs* ∧ *sorted-wrt f ys*
⟨*proof*⟩

**fun** *msort* :: - ⇒ *'a list* ⇒ *'a list*
**where**
*msort f* [] = []
| *msort f* [*x*] = [*x*]
| *msort f xs* = *merge f*
(*msort f* (*take* (*size xs div 2*) *xs*))
(*msort f* (*drop* (*size xs div 2*) *xs*))

**fun** *swap-ternary* :: ‹-⇒*nat*⇒*nat*⇒ (*'a* × *'a* × *'a*) ⇒ (*'a* × *'a* × *'a*)› **where**
‹*swap-ternary f m n* =
(*if* (*m* = *0* ∧ *n* = *1*)
*then* (λ(*a*, *b*, *c*). *if f a b then* (*a*, *b*, *c*)
*else* (*b*,*a*,*c*))
*else if* (*m* = *0* ∧ *n* = *2*)
*then* (λ(*a*, *b*, *c*). *if f a c then* (*a*, *b*, *c*)
*else* (*c*,*b*,*a*))
*else if* (*m* = *1* ∧ *n* = *2*)
*then* (λ(*a*, *b*, *c*). *if f b c then* (*a*, *b*, *c*)
*else* (*a*,*c*,*b*))
*else* (λ(*a*, *b*, *c*). (*a*,*b*,*c*)))›

**fun** *msort2* :: - ⇒ *'a list* ⇒ *'a list*
**where**
*msort2 f* [] = []
| *msort2 f* [*x*] = [*x*]
| *msort2 f* [*x*,*y*] = (*if f x y then* [*x*,*y*] *else* [*y*,*x*])
| *msort2 f xs* = *merge f*
(*msort f* (*take* (*size xs div 2*) *xs*))
(*msort f* (*drop* (*size xs div 2*) *xs*))

**lemmas** [*code del*] =
*msort2.simps*

**declare** *msort2.simps*[*simp del*]
**lemmas** [*code*] =
*msort2.simps*[*unfolded swap-ternary.simps*, *simplified*]

**declare** *msort2.simps*[*simp*]

**lemma** *msort-msort2*:
**fixes** *xs* :: ‹*'a* :: *linorder list*›
**shows** ‹*msort* (≤) *xs* = *msort2* (≤) *xs*›

⟨*proof*⟩

**lemma** *sorted-msort*:
*transp f* ⟹ (⋀*x y*. *f x y* ∨ *f y x*) ⟹
*sorted-wrt f* (*msort f xs*)
⟨*proof*⟩

**lemma** *mset-msort*[*simp*]:
*mset* (*msort f xs*) = *mset xs*
⟨*proof*⟩

## 12.2 Sorting applied to monomials

**lemma** *merge-coeffs-alt-def*:
‹(*RETURN o merge-coeffs*) *p* =
*REC*$_T$(λ*f p*.
(*case p of*
[] ⇒ *RETURN* []
| [-] => *RETURN p*
| ((*xs*, *n*) # (*ys*, *m*) # *p*) ⇒
(*if xs* = *ys*
*then if n* + *m* ≠ *0 then f* ((*xs*, *n* + *m*) # *p*) *else f p*
*else do* {*p* ← *f* ((*ys*, *m*) # *p*); *RETURN* ((*xs*, *n*) # *p*)})))
*p*›
⟨*proof*⟩

**lemma** *hn-invalid-recover*:
‹*is-pure R* ⟹ *hn-invalid R* = (λ*x y*. *R x y* ∗ *true*)›
‹*is-pure R* ⟹ *invalid-assn R* = (λ*x y*. *R x y* ∗ *true*)›
⟨*proof*⟩

**lemma** *safe-poly-vars*:
**shows**
[*safe-constraint-rules*]:
*is-pure* (*poly-assn*) **and**
[*safe-constraint-rules*]:
*is-pure* (*monom-assn*) **and**
[*safe-constraint-rules*]:
*is-pure* (*monomial-assn*) **and**
[*safe-constraint-rules*]:
*is-pure string-assn*
⟨*proof*⟩

**lemma** *invalid-assn-distrib*:
‹*invalid-assn monom-assn* ×$_a$ *invalid-assn int-assn* = *invalid-assn* (*monom-assn* ×$_a$ *int-assn*)›
⟨*proof*⟩

**lemma** *WTF-RF-recover*:
‹*hn-ctxt* (*invalid-assn monom-assn* ×$_a$ *invalid-assn int-assn*) *xb*
*x*′*a* ∨$_A$
*hn-ctxt monomial-assn xb x*′*a* ⟹$_t$
*hn-ctxt* (*monomial-assn*) *xb x*′*a*›
⟨*proof*⟩

**lemma** *WTF-RF*:
‹*hn-ctxt* (*invalid-assn monom-assn* ×$_a$ *invalid-assn int-assn*) *xb x*′*a* ∗

(*hn-invalid poly-assn la l'a* ∗ *hn-invalid int-assn a2' a2* ∗
*hn-invalid monom-assn a1' a1* ∗
*hn-invalid poly-assn l l'* ∗
*hn-invalid monomial-assn xa x'* ∗
*hn-invalid poly-assn ax px*) $\Longrightarrow_t$
*hn-ctxt* (*monomial-assn*) *xb x'a* ∗
*hn-ctxt poly-assn*
*la l'a* ∗
*hn-ctxt poly-assn l l'* ∗
(*hn-invalid int-assn a2' a2* ∗
*hn-invalid monom-assn a1' a1* ∗
*hn-invalid monomial-assn xa x'* ∗
*hn-invalid poly-assn ax px*)›
‹*hn-ctxt* (*invalid-assn monom-assn* $\times_a$ *invalid-assn int-assn*) *xa x'* ∗
(*hn-ctxt poly-assn l l'* ∗ *hn-invalid poly-assn ax px*) $\Longrightarrow_t$
*hn-ctxt* (*monomial-assn*) *xa x'* ∗
*hn-ctxt poly-assn l l'* ∗
*hn-ctxt poly-assn ax px* ∗
*emp*›
⟨*proof*⟩

The refinement frameword is completely lost here when synthesizing the constants – it does not understant what is pure (actually everything) and what must be destroyed.

**sepref-definition** *merge-coeffs-impl*
**is** ‹*RETURN o merge-coeffs*›
:: ‹*poly-assn*$^d$ $\rightarrow_a$ *poly-assn*›
⟨*proof*⟩

**definition** *full-quicksort-poly* **where**
‹*full-quicksort-poly* = *full-quicksort-ref* (λ*x y*. *x* = *y* ∨ (*x*, *y*) ∈ *term-order-rel*) *fst*›

**lemma** *down-eq-id-list-rel*: ‹⇓(⟨*Id*⟩*list-rel*) *x* = *x*›
⟨*proof*⟩

**definition** *quicksort-poly*:: ‹*nat* ⇒ *nat* ⇒ *llist-polynomial* ⇒ (*llist-polynomial*) *nres*› **where**
‹*quicksort-poly x y z* = *quicksort-ref* (≤) *fst* (*x*, *y*, *z*)›

**term** *partition-between-ref*

**definition** *partition-between-poly* :: ‹*nat* ⇒ *nat* ⇒ *llist-polynomial* ⇒ (*llist-polynomial* × *nat*) *nres*› **where**
‹*partition-between-poly* = *partition-between-ref* (≤) *fst*›

**definition** *partition-main-poly* :: ‹*nat* ⇒ *nat* ⇒ *llist-polynomial* ⇒ (*llist-polynomial* × *nat*) *nres*› **where**
‹*partition-main-poly* = *partition-main* (≤) *fst*›

**lemma** *string-list-trans*:
‹(*xa* ::*char list list*, *ya*) ∈ *lexord* (*lexord* {(*x*, *y*). *x* < *y*}) ⟹
(*ya*, *z*) ∈ *lexord* (*lexord* {(*x*, *y*). *x* < *y*}) ⟹
(*xa*, *z*) ∈ *lexord* (*lexord* {(*x*, *y*). *x* < *y*})›
⟨*proof*⟩

**lemma** *full-quicksort-sort-poly-spec*:
‹(*full-quicksort-poly*, *sort-poly-spec*) ∈ ⟨*Id*⟩*list-rel* $\rightarrow_f$ ⟨⟨*Id*⟩*list-rel*⟩*nres-rel*›
⟨*proof*⟩

## 12.3 Lifting to polynomials

**definition** *merge-sort-poly* :: ‹-› **where**
‹*merge-sort-poly* = *msort* (λ*a b*. *fst a* ≤ *fst b*)›

**definition** *merge-monoms-poly* :: ‹-› **where**
‹*merge-monoms-poly* = *msort* (≤)›

**definition** *merge-poly* :: ‹-› **where**
‹*merge-poly* = *merge* (λ*a b*. *fst a* ≤ *fst b*)›

**definition** *merge-monoms* :: ‹-› **where**
‹*merge-monoms* = *merge* (≤)›

**definition** *msort-poly-impl* :: ‹(*String.literal list* × *int*) *list* ⇒ -› **where**
‹*msort-poly-impl* = *msort* (λ*a b*. *fst a* ≤ *fst b*)›

**definition** *msort-monoms-impl* :: ‹(*String.literal list*) ⇒ -› **where**
‹*msort-monoms-impl* = *msort* (≤)›

**lemma** *msort-poly-impl-alt-def*:
‹*msort-poly-impl xs* =
(*case xs of*
[] ⇒ []
| [*a*] ⇒ [*a*]
| [*a*,*b*] ⇒ *if fst a* ≤ *fst b then* [*a*,*b*]*else* [*b*,*a*]
| *xs* ⇒ *merge-poly*
(*msort-poly-impl* (*take* ((*length xs*) *div* 2) *xs*))
(*msort-poly-impl* (*drop* ((*length xs*) *div* 2) *xs*)))›
⟨*proof*⟩

**lemma** *le-term-order-rel′*:
‹(≤) = (λ*x y*. *x* = *y* ∨ *term-order-rel′ x y*)›
⟨*proof*⟩

**fun** *lexord-eq* **where**
‹*lexord-eq* [] - = *True*› |
‹*lexord-eq* (*x* # *xs*) (*y* # *ys*) = (*x* < *y* ∨ (*x* = *y* ∧ *lexord-eq xs ys*))› |
‹*lexord-eq* - - = *False*›

**lemma** [*simp*]:
‹*lexord-eq* [] [] = *True*›
‹*lexord-eq* (*a* # *b*)[] = *False*›
‹*lexord-eq* [] (*a* # *b*) = *True*›
⟨*proof*⟩

**lemma** *var-order-rel′*:
‹(≤) = (λ*x y*. *x* = *y* ∨ (*x*,*y*) ∈ *var-order-rel*)›
⟨*proof*⟩

**lemma** *var-order-rel′′*:
‹(*x*,*y*) ∈ *var-order-rel* ⟷ *x* < *y*›
⟨*proof*⟩

**lemma** *lexord-eq-alt-def1*:

‹$a \leq b = lexord\text{-}eq\ a\ b$› **for** $a\ b$ :: ‹*String.literal list*›
⟨*proof*⟩

**lemma** *lexord-eq-alt-def2*:
‹(*RETURN oo lexord-eq*) *xs ys* =
*REC*$_T$ (λ*f* (*xs*, *ys*).
*case* (*xs*, *ys*) *of*
([], -) ⇒ *RETURN True*
| (*x* # *xs*, *y* # *ys*) ⇒
*if x* < *y then RETURN True*
*else if x* = *y then f* (*xs*, *ys*) *else RETURN False*
| - ⇒ *RETURN False*)
(*xs*, *ys*)›
⟨*proof*⟩

**definition** *var-order′* **where**
[*simp*]: ‹*var-order′* = *var-order*›

**lemma** *var-order-rel*[*def-pat-rules*]:
‹(∈)$(*x*,*y*)$*var-order-rel* ≡ *var-order′*$*x*$*y*›
⟨*proof*⟩

**lemma** *var-order-rel-alt-def*:
‹*var-order-rel* = *p2rel char.lexordp*›
⟨*proof*⟩

**lemma** *var-order-rel-var-order*:
‹(*x*, *y*) ∈ *var-order-rel* ⟷ *var-order x y*›
⟨*proof*⟩

**lemma** *var-order-string-le*[*sepref-import-param*]:
‹((<), *var-order′*) ∈ *string-rel* → *string-rel* → *bool-rel*›
⟨*proof*⟩

**lemma** [*sepref-import-param*]:
‹( (≤), (≤)) ∈ *monom-rel* → *monom-rel* →*bool-rel*›
⟨*proof*⟩

**lemma** [*sepref-import-param*]:
‹( (<), (<)) ∈ *string-rel* → *string-rel* →*bool-rel*›
⟨*proof*⟩

**lemma** *lexordp-char-char*: ‹*ord-class.lexordp* = *char.lexordp*›
⟨*proof*⟩

**lemma** [*sepref-import-param*]:
‹( (≤), (≤)) ∈ *string-rel* → *string-rel* →*bool-rel*›
⟨*proof*⟩

**sepref-register** *lexord-eq*
**sepref-definition** *lexord-eq-term*
**is** ‹*uncurry* (*RETURN oo lexord-eq*)›
:: ‹*monom-assn*$^k$ ∗$_a$ *monom-assn*$^k$ →$_a$ *bool-assn*›

⟨*proof*⟩

**declare** *lexord-eq-term.refine*[*sepref-fr-rules*]

**lemmas** [*code del*] = *msort-poly-impl-def msort-monoms-impl-def*
**lemmas** [*code*] =
*msort-poly-impl-def*[*unfolded lexord-eq-alt-def1*[*abs-def*]]
*msort-monoms-impl-def*[*unfolded msort-msort2*]

**lemma** *term-order-rel-trans*:
‹(*a*, *aa*) ∈ *term-order-rel* ⟹
(*aa*, *ab*) ∈ *term-order-rel* ⟹ (*a*, *ab*) ∈ *term-order-rel*›
⟨*proof*⟩

**lemma** *merge-sort-poly-sort-poly-spec*:
‹(*RETURN o merge-sort-poly*, *sort-poly-spec*) ∈ ⟨*Id*⟩*list-rel* →$_f$ ⟨⟨*Id*⟩*list-rel*⟩*nres-rel*›
⟨*proof*⟩

**lemma** *msort-alt-def*:
‹*RETURN o* (*msort f*) =
*REC*$_T$ (λ*g xs*.
*case xs of*
[] ⇒ *RETURN* []
| [*x*] ⇒ *RETURN* [*x*]
| - ⇒ *do* {
*a* ← *g* (*take* (*size xs div* 2) *xs*);
*b* ← *g* (*drop* (*size xs div* 2) *xs*);
*RETURN* (*merge f a b*)})›
⟨*proof*⟩

**lemma** *monomial-rel-order-map*:
‹(*x*, *a*, *b*) ∈ *monomial-rel* ⟹
(*y*, *aa*, *bb*) ∈ *monomial-rel* ⟹
*fst x* ≤ *fst y* ⟷ *a* ≤ *aa*›
⟨*proof*⟩

**lemma** *step-rewrite-pure*:
**fixes** *K* :: ‹('*olbl* × '*lbl*) *set*›
**shows**
‹*pure* (*p2rel* (⟨*K*, *V*, *R*⟩*pac-step-rel-raw*)) = *pac-step-rel-assn* (*pure K*) (*pure V*) (*pure R*)›
‹*monomial-assn* = *pure* (*monom-rel* ×$_r$ *int-rel*)› **and**
*poly-assn-list*:
‹*poly-assn* = *pure* (⟨*monom-rel* ×$_r$ *int-rel*⟩*list-rel*)›
⟨*proof*⟩

**lemma** *safe-pac-step-rel-assn*[*safe-constraint-rules*]:
*is-pure K* ⟹ *is-pure V* ⟹ *is-pure R* ⟹ *is-pure* (*pac-step-rel-assn K V R*)
⟨*proof*⟩

**lemma** *merge-poly-merge-poly*:
‹(*merge-poly*, *merge-poly*)
∈ *poly-rel* → *poly-rel* → *poly-rel*›

⟨*proof*⟩

**lemmas** [*fcomp-norm-unfold*] =
*poly-assn-list*[*symmetric*]
*step-rewrite-pure*(*1*)

**lemma** *merge-poly-merge-poly2*:
‹(*a*, *b*) ∈ *poly-rel* ⟹ (*a'*, *b'*) ∈ *poly-rel* ⟹
(*merge-poly a a'*, *merge-poly b b'*) ∈ *poly-rel*›
⟨*proof*⟩

**lemma** *list-rel-takeD*:
‹(*a*, *b*) ∈ ⟨*R*⟩*list-rel* ⟹ (*n*, *n'*)∈ *Id* ⟹ (*take n a*, *take n' b*) ∈ ⟨*R*⟩*list-rel*›
⟨*proof*⟩

**lemma** *list-rel-dropD*:
‹(*a*, *b*) ∈ ⟨*R*⟩*list-rel* ⟹ (*n*, *n'*)∈ *Id* ⟹ (*drop n a*, *drop n' b*) ∈ ⟨*R*⟩*list-rel*›
⟨*proof*⟩

**lemma** *merge-sort-poly*[*sepref-import-param*]:
‹(*msort-poly-impl*, *merge-sort-poly*)
∈ *poly-rel* → *poly-rel*›
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] = *merge-sort-poly*[*FCOMP merge-sort-poly-sort-poly-spec*]

**sepref-definition** *partition-main-poly-impl*
**is** ‹*uncurry2 partition-main-poly*›
:: ‹*nat-assn*$^k$ *$*_a$ *nat-assn*$^k$ *$*_a$ *poly-assn*$^k$ →$_a$ *prod-assn poly-assn nat-assn* ›
⟨*proof*⟩

**declare** *partition-main-poly-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *partition-between-poly-impl*
**is** ‹*uncurry2 partition-between-poly*›
:: ‹*nat-assn*$^k$ *$*_a$ *nat-assn*$^k$ *$*_a$ *poly-assn*$^k$ →$_a$ *prod-assn poly-assn nat-assn* ›
⟨*proof*⟩

**declare** *partition-between-poly-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *quicksort-poly-impl*
**is** ‹*uncurry2 quicksort-poly*›
:: ‹*nat-assn*$^k$ *$*_a$ *nat-assn*$^k$ *$*_a$ *poly-assn*$^k$ →$_a$ *poly-assn*›
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] = *quicksort-poly-impl.refine*

**sepref-register** *quicksort-poly*
**sepref-definition** *full-quicksort-poly-impl*
**is** ‹*full-quicksort-poly*›
:: ‹*poly-assn*$^k$ →$_a$ *poly-assn*›
⟨*proof*⟩

**lemmas** *sort-poly-spec-hnr* =
*full-quicksort-poly-impl.refine*[*FCOMP full-quicksort-sort-poly-spec*]

**declare** *merge-coeffs-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *normalize-poly-impl*
**is** ‹*normalize-poly*›
:: ‹*poly-assn*$^k$ $\rightarrow_a$ *poly-assn*›
⟨*proof*⟩

**declare** *normalize-poly-impl.refine*[*sepref-fr-rules*]

**definition** *full-quicksort-vars* **where**
‹*full-quicksort-vars* = *full-quicksort-ref* (λ*x y*. *x* = *y* ∨ (*x*, *y*) ∈ *var-order-rel*) *id*›

**definition** *quicksort-vars*:: ‹*nat* ⇒ *nat* ⇒ *string list* ⇒ (*string list*) *nres*› **where**
‹*quicksort-vars x y z* = *quicksort-ref* (≤) *id* (*x*, *y*, *z*)›

**definition** *partition-between-vars* :: ‹*nat* ⇒ *nat* ⇒ *string list* ⇒ (*string list* × *nat*) *nres*› **where**
‹*partition-between-vars* = *partition-between-ref* (≤) *id*›

**definition** *partition-main-vars* :: ‹*nat* ⇒ *nat* ⇒ *string list* ⇒ (*string list* × *nat*) *nres*› **where**
‹*partition-main-vars* = *partition-main* (≤) *id*›

**lemma** *total-on-lexord-less-than-char-linear2*:
‹*xs* ≠ *ys* ⟹ (*xs*, *ys*) ∉ *lexord* (*less-than-char*) ⟷
(*ys*, *xs*) ∈ *lexord less-than-char*›
⟨*proof*⟩

**lemma** *string-trans*:
‹(*xa*, *ya*) ∈ *lexord* {(*x*::*char*, *y*::*char*). *x* < *y*} ⟹
(*ya*, *z*) ∈ *lexord* {(*x*::*char*, *y*::*char*). *x* < *y*} ⟹
(*xa*, *z*) ∈ *lexord* {(*x*::*char*, *y*::*char*). *x* < *y*}›
⟨*proof*⟩

**lemma** *full-quicksort-sort-vars-spec*:
‹(*full-quicksort-vars*, *sort-coeff*) ∈ ⟨*Id*⟩*list-rel* →$_f$ ⟨⟨*Id*⟩*list-rel*⟩*nres-rel*›
⟨*proof*⟩

**sepref-definition** *partition-main-vars-impl*
**is** ‹*uncurry2 partition-main-vars*›
:: ‹*nat-assn*$^k$ *$_a$ *nat-assn*$^k$ *$_a$ (*monom-assn*)$^k$ $\rightarrow_a$ *prod-assn* (*monom-assn*) *nat-assn*›
⟨*proof*⟩

**declare** *partition-main-vars-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *partition-between-vars-impl*
**is** ‹*uncurry2 partition-between-vars*›
:: ‹*nat-assn*$^k$ *$_a$ *nat-assn*$^k$ *$_a$ *monom-assn*$^k$ $\rightarrow_a$ *prod-assn monom-assn nat-assn* ›
⟨*proof*⟩

**declare** *partition-between-vars-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *quicksort-vars-impl*
  **is** ‹*uncurry2 quicksort-vars*›
  :: ‹*nat-assn*$^k$ $*_a$ *nat-assn*$^k$ $*_a$ *monom-assn*$^k$ $\rightarrow_a$ *monom-assn*›
  ⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] = *quicksort-vars-impl.refine*

**sepref-register** *quicksort-vars*

**lemma** *le-var-order-rel*:
  ‹$(\leq) = (\lambda x\ y.\ x = y \vee (x, y) \in$ *var-order-rel*)›
  ⟨*proof*⟩

**sepref-definition** *full-quicksort-vars-impl*
  **is** ‹*full-quicksort-vars*›
  :: ‹*monom-assn*$^k$ $\rightarrow_a$ *monom-assn*›
  ⟨*proof*⟩

**lemmas** *sort-vars-spec-hnr* =
  *full-quicksort-vars-impl.refine*[*FCOMP full-quicksort-sort-vars-spec*]

**lemma** *string-rel-order-map*:
  ‹$(x, a) \in$ *string-rel* $\Longrightarrow$
      $(y, aa) \in$ *string-rel* $\Longrightarrow$
      $x \leq y \longleftrightarrow a \leq aa$›
  ⟨*proof*⟩

**lemma** *merge-monoms-merge-monoms*:
  ‹(*merge-monoms*, *merge-monoms*) $\in$ *monom-rel* $\rightarrow$ *monom-rel* $\rightarrow$ *monom-rel*›
  ⟨*proof*⟩

**lemma** *merge-monoms-merge-monoms2*:
  ‹$(a, b) \in$ *monom-rel* $\Longrightarrow$ $(a', b') \in$ *monom-rel* $\Longrightarrow$
    (*merge-monoms a a'*, *merge-monoms b b'*) $\in$ *monom-rel*›
  ⟨*proof*⟩

**lemma** *msort-monoms-impl*:
  ‹(*msort-monoms-impl*, *merge-monoms-poly*)
  $\in$ *monom-rel* $\rightarrow$ *monom-rel*›
  ⟨*proof*⟩

**lemma** *merge-sort-monoms-sort-monoms-spec*:
  ‹(*RETURN o merge-monoms-poly*, *sort-coeff*) $\in \langle$*Id*$\rangle$*list-rel* $\rightarrow_f \langle\langle$*Id*$\rangle$*list-rel*$\rangle$*nres-rel*›
  ⟨*proof*⟩

**sepref-register** *sort-coeff*
**lemma** [*sepref-fr-rules*]:
  ‹(*return o msort-monoms-impl*, *sort-coeff*) $\in$ *monom-assn*$^k$ $\rightarrow_a$ *monom-assn*›
  ⟨*proof*⟩

**sepref-definition** *sort-all-coeffs-impl*
  **is** ‹*sort-all-coeffs*›
  :: ‹*poly-assn*$^k$ →$_a$ *poly-assn*›
  ⟨*proof*⟩

**declare** *sort-all-coeffs-impl.refine*[*sepref-fr-rules*]

**lemma** *merge-coeffs0-alt-def*:
  ‹(*RETURN o merge-coeffs0*) *p* =
   *REC*$_T$(λ*f p*.
    (*case p of*
      [] ⇒ *RETURN* []
    | [*p*] => *if snd p* = *0 then RETURN* [] *else RETURN* [*p*]
    | ((*xs*, *n*) # (*ys*, *m*) # *p*) ⇒
     (*if xs* = *ys*
      *then if n* + *m* ≠ *0 then f* ((*xs*, *n* + *m*) # *p*) *else f p*
      *else if n* = *0 then*
         *do* {*p* ← *f* ((*ys*, *m*) # *p*);
           *RETURN p*}
      *else do* {*p* ← *f* ((*ys*, *m*) # *p*);
           *RETURN* ((*xs*, *n*) # *p*)})))
   *p*›
  ⟨*proof*⟩

Again, Sepref does not understand what is going here.

**sepref-definition** *merge-coeffs0-impl*
  **is** ‹*RETURN o merge-coeffs0*›
  :: ‹*poly-assn*$^k$ →$_a$ *poly-assn*›
  ⟨*proof*⟩

**declare** *merge-coeffs0-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *fully-normalize-poly-impl*
  **is** ‹*full-normalize-poly*›
  :: ‹*poly-assn*$^k$ →$_a$ *poly-assn*›
  ⟨*proof*⟩

**declare** *fully-normalize-poly-impl.refine*[*sepref-fr-rules*]

**end**

**theory** *PAC-Version*
  **imports** *Main*
**begin**

This code was taken from IsaFoR. However, for the AFP, we use the version name *AFP*, instead of a mercurial version.

⟨*ML*⟩

**declare** *version-def* [*code*]

**end**

**theory** *PAC-Checker-Synthesis*
**imports** *PAC-Checker WB-Sort PAC-Checker-Relation*
*PAC-Checker-Init More-Loops PAC-Version*
**begin**

# 13 Code Synthesis of the Complete Checker

We here combine refine the full checker, using the initialisation provided in another file and adding more efficient data structures (mostly replacing the set of variables by a more efficient hash map).

**abbreviation** *vars-assn* **where**
‹*vars-assn* ≡ *hs.assn string-assn*›

**fun** *vars-of-monom-in* **where**
‹*vars-of-monom-in* [] - = *True*› |
‹*vars-of-monom-in* (*x* # *xs*) $\mathcal{V}$ ⟷ *x* ∈ $\mathcal{V}$ ∧ *vars-of-monom-in xs* $\mathcal{V}$›

**fun** *vars-of-poly-in* **where**
‹*vars-of-poly-in* [] - = *True*› |
‹*vars-of-poly-in* ((*x*, -) # *xs*) $\mathcal{V}$ ⟷ *vars-of-monom-in x* $\mathcal{V}$ ∧ *vars-of-poly-in xs* $\mathcal{V}$›

**lemma** *vars-of-monom-in-alt-def*:
‹*vars-of-monom-in xs* $\mathcal{V}$ ⟷ *set xs* ⊆ $\mathcal{V}$›
⟨*proof*⟩

**lemma** *vars-llist-alt-def*:
‹*vars-llist xs* ⊆ $\mathcal{V}$ ⟷ *vars-of-poly-in xs* $\mathcal{V}$›
⟨*proof*⟩

**lemma** *vars-of-monom-in-alt-def2*:
‹*vars-of-monom-in xs* $\mathcal{V}$ ⟷ *fold* (λ*x b*. *b* ∧ *x* ∈ $\mathcal{V}$) *xs True*›
⟨*proof*⟩

**sepref-definition** *vars-of-monom-in-impl*
**is** ‹*uncurry* (*RETURN oo vars-of-monom-in*)›
:: ‹(*list-assn string-assn*)$^k$ *∗$_a$ vars-assn*$^k$ →$_a$ *bool-assn*›
⟨*proof*⟩

**declare** *vars-of-monom-in-impl.refine*[*sepref-fr-rules*]

**lemma** *vars-of-poly-in-alt-def2*:
‹*vars-of-poly-in xs* $\mathcal{V}$ ⟷ *fold* (λ(*x*, -) *b*. *b* ∧ *vars-of-monom-in x* $\mathcal{V}$) *xs True*›
⟨*proof*⟩

**sepref-definition** *vars-of-poly-in-impl*
**is** ‹*uncurry* (*RETURN oo vars-of-poly-in*)›
:: ‹(*poly-assn*)$^k$ *∗$_a$ vars-assn*$^k$ →$_a$ *bool-assn*›
⟨*proof*⟩

**declare** *vars-of-poly-in-impl.refine*[*sepref-fr-rules*]

**definition** *union-vars-monom* :: ‹*string list* ⇒ *string set* ⇒ *string set*› **where**
‹*union-vars-monom xs* $\mathcal{V}$ = *fold insert xs* $\mathcal{V}$›

**definition** *union-vars-poly* :: ‹*llist-polynomial* ⇒ *string set* ⇒ *string set*› **where**
‹*union-vars-poly xs* $\mathcal{V}$ = *fold* (λ(*xs*, -) $\mathcal{V}$. *union-vars-monom xs* $\mathcal{V}$) *xs* $\mathcal{V}$›

**lemma** *union-vars-monom-alt-def*:
‹*union-vars-monom xs* $\mathcal{V}$ = $\mathcal{V}$ ∪ *set xs*›
⟨*proof*⟩

**lemma** *union-vars-poly-alt-def*:
‹*union-vars-poly xs* $\mathcal{V}$ = $\mathcal{V}$ ∪ *vars-llist xs*›
⟨*proof*⟩

**sepref-definition** *union-vars-monom-impl*
**is** ‹*uncurry* (*RETURN oo union-vars-monom*)›
:: ‹*monom-assn*$^k$ $*_a$ *vars-assn*$^d$ $\rightarrow_a$ *vars-assn*›
⟨*proof*⟩

**declare** *union-vars-monom-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *union-vars-poly-impl*
**is** ‹*uncurry* (*RETURN oo union-vars-poly*)›
:: ‹*poly-assn*$^k$ $*_a$ *vars-assn*$^d$ $\rightarrow_a$ *vars-assn*›
⟨*proof*⟩

**declare** *union-vars-poly-impl.refine*[*sepref-fr-rules*]

**hide-const** (**open**) *Autoref-Fix-Rel.CONSTRAINT*

**fun** *status-assn* **where**
‹*status-assn - CSUCCESS CSUCCESS* = *emp*› |
‹*status-assn - CFOUND CFOUND* = *emp*› |
‹*status-assn R* (*CFAILED a*) (*CFAILED b*) = *R a b*› |
‹*status-assn - - -* = *false*›

**lemma** *SUCCESS-hnr*[*sepref-fr-rules*]:
‹(*uncurry0* (*return CSUCCESS*), *uncurry0* (*RETURN CSUCCESS*)) ∈ *unit-assn*$^k$ $\rightarrow_a$ *status-assn R*›
⟨*proof*⟩

**lemma** *FOUND-hnr*[*sepref-fr-rules*]:
‹(*uncurry0* (*return CFOUND*), *uncurry0* (*RETURN CFOUND*)) ∈ *unit-assn*$^k$ $\rightarrow_a$ *status-assn R*›
⟨*proof*⟩

**lemma** *is-success-hnr*[*sepref-fr-rules*]:
‹*CONSTRAINT is-pure R* ⟹
((*return o is-cfound*), (*RETURN o is-cfound*)) ∈ (*status-assn R*)$^k$ $\rightarrow_a$ *bool-assn*›
⟨*proof*⟩

**lemma** *is-cfailed-hnr*[*sepref-fr-rules*]:
‹*CONSTRAINT is-pure R* ⟹
((*return o is-cfailed*), (*RETURN o is-cfailed*)) ∈ (*status-assn R*)$^k$ $\rightarrow_a$ *bool-assn*›
⟨*proof*⟩

**lemma** *merge-cstatus-hnr*[*sepref-fr-rules*]:
‹*CONSTRAINT is-pure R* ⟹
(*uncurry* (*return oo merge-cstatus*), *uncurry* (*RETURN oo merge-cstatus*)) ∈
(*status-assn R*)$^k$ *$_a$ (*status-assn R*)$^k$ →$_a$ *status-assn R*›
⟨*proof*⟩

**sepref-definition** *add-poly-impl*
**is** ‹*add-poly-l*›
:: ‹(*poly-assn* ×$_a$ *poly-assn*)$^k$ →$_a$ *poly-assn*›
⟨*proof*⟩

**declare** *add-poly-impl.refine*[*sepref-fr-rules*]

**sepref-register** *mult-monomials*
**lemma** *mult-monoms-alt-def*:
‹(*RETURN oo mult-monoms*) *x y* = *REC*$_T$
(λ*f* (*p*, *q*).
*case* (*p*, *q*) *of*
([], -) ⇒ *RETURN q*
| (-, []) ⇒ *RETURN p*
| (*x* # *p*, *y* # *q*) ⇒
(*if x* = *y then do* {
*pq* ← *f* (*p*, *q*);
*RETURN* (*x* # *pq*)}
*else if* (*x*, *y*) ∈ *var-order-rel*
*then do* {
*pq* ← *f* (*p*, *y* # *q*);
*RETURN* (*x* # *pq*)}
*else do* {
*pq* ← *f* (*x* # *p*, *q*);
*RETURN* (*y* # *pq*)}))
(*x*, *y*)›
⟨*proof*⟩

**sepref-definition** *mult-monoms-impl*
**is** ‹*uncurry* (*RETURN oo mult-monoms*)›
:: ‹(*monom-assn*)$^k$ *$_a$ (*monom-assn*)$^k$ →$_a$ (*monom-assn*)›
⟨*proof*⟩

**declare** *mult-monoms-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *mult-monomials-impl*
**is** ‹*uncurry* (*RETURN oo mult-monomials*)›
:: ‹(*monomial-assn*)$^k$ *$_a$ (*monomial-assn*)$^k$ →$_a$ (*monomial-assn*)›
⟨*proof*⟩

**lemma** *map-append-alt-def2*:
‹(*RETURN o* (*map-append f b*)) *xs* = *REC*$_T$
(λ*g xs*. *case xs of* [] ⇒ *RETURN b*
| *x* # *xs* ⇒ *do* {
*y* ← *g xs*;

*RETURN* $(f\ x\ \#\ y)$
}) *xs*›
⟨*proof*⟩

**definition** *map-append-poly-mult* **where**
‹*map-append-poly-mult x* = *map-append* (*mult-monomials x*)›

**declare** *mult-monomials-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *map-append-poly-mult-impl*
**is** ‹*uncurry2* (*RETURN ooo map-append-poly-mult*)›
:: ‹*monomial-assn*$^k$ $*_a$ *poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $\to_a$ *poly-assn*›
⟨*proof*⟩

**declare** *map-append-poly-mult-impl.refine*[*sepref-fr-rules*]

TODO *foldl* $(\lambda l\ x.\ l\ @\ [?f\ x])\ []\ ?l$ = *map* $?f\ ?l$ is the worst possible implementation of map!

**sepref-definition** *mult-poly-raw-impl*
**is** ‹*uncurry* (*RETURN oo mult-poly-raw*)›
:: ‹*poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $\to_a$ *poly-assn*›
⟨*proof*⟩

**declare** *mult-poly-raw-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *mult-poly-impl*
**is** ‹*uncurry mult-poly-full*›
:: ‹*poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $\to_a$ *poly-assn*›
⟨*proof*⟩

**declare** *mult-poly-impl.refine*[*sepref-fr-rules*]

**lemma** *inverse-monomial*:
‹*monom-rel*$^{-1}$ $\times_r$ *int-rel* = (*monom-rel* $\times_r$ *int-rel*)$^{-1}$›
⟨*proof*⟩

**lemma** *eq-poly-rel-eq*[*sepref-import-param*]:
‹((=), (=)) ∈ *poly-rel* → *poly-rel* → *bool-rel*›
⟨*proof*⟩

**sepref-definition** *weak-equality-l-impl*
**is** ‹*uncurry weak-equality-l*›
:: ‹*poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $\to_a$ *bool-assn*›
⟨*proof*⟩

**declare** *weak-equality-l-impl.refine*[*sepref-fr-rules*]
**sepref-register** *add-poly-l mult-poly-full*

**abbreviation** *raw-string-assn* :: ‹*string* ⇒ *string* ⇒ *assn*› **where**
‹*raw-string-assn* ≡ *list-assn id-assn*›

**definition** *show-nat* :: ‹*nat* ⇒ *string*› **where**
‹*show-nat i* = *show i*›

**lemma** [*sepref-import-param*]:
‹(*show-nat*, *show-nat*) ∈ *nat-rel* → ⟨*Id*⟩*list-rel*›
⟨*proof*⟩

**lemma** *status-assn-pure-conv*:
‹*status-assn* (*id-assn*) *a b* = *id-assn a b*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:
‹(*uncurry3* (λ*x y*. *return oo* (*error-msg-not-equal-dom x y*)), *uncurry3 check-not-equal-dom-err*) ∈
*poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ →$_a$ *raw-string-assn*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:
‹(*return o* (*error-msg-notin-dom o nat-of-uint64*), *RETURN o error-msg-notin-dom*)
∈ *uint64-nat-assn*$^k$ →$_a$ *raw-string-assn*›
‹(*return o* (*error-msg-reused-dom o nat-of-uint64*), *RETURN o error-msg-reused-dom*)
∈ *uint64-nat-assn*$^k$ →$_a$ *raw-string-assn*›
‹(*uncurry* (*return oo* (λ*i*. *error-msg* (*nat-of-uint64 i*))), *uncurry* (*RETURN oo error-msg*))
∈ *uint64-nat-assn*$^k$ ∗$_a$ *raw-string-assn*$^k$ →$_a$ *status-assn raw-string-assn*›
‹(*uncurry* (*return oo error-msg*), *uncurry* (*RETURN oo error-msg*))
∈ *nat-assn*$^k$ ∗$_a$ *raw-string-assn*$^k$ →$_a$ *status-assn raw-string-assn*›
⟨*proof*⟩

**sepref-definition** *check-addition-l-impl*
**is** ‹*uncurry6 check-addition-l*›
:: ‹*poly-assn*$^k$ ∗$_a$ *polys-assn*$^k$ ∗$_a$ *vars-assn*$^k$ ∗$_a$ *uint64-nat-assn*$^k$ ∗$_a$ *uint64-nat-assn*$^k$ ∗$_a$
*uint64-nat-assn*$^k$ ∗$_a$ *poly-assn*$^k$ →$_a$ *status-assn raw-string-assn*›
⟨*proof*⟩

**declare** *check-addition-l-impl.refine*[*sepref-fr-rules*]

**sepref-register** *check-mult-l-dom-err*

**definition** *check-mult-l-dom-err-impl* **where**
‹*check-mult-l-dom-err-impl pd p ia i* =
(*if pd then* ''*The polynomial with id* '' @ *show* (*nat-of-uint64 p*) @ '' *was not found*'' *else* ''''') @
(*if ia then* ''*The id of the resulting id* '' @ *show* (*nat-of-uint64 i*) @ '' *was already given*'' *else* ''''')›

**definition** *check-mult-l-mult-err-impl* **where**
‹*check-mult-l-mult-err-impl p q pq r* =
''*Multiplying* '' @ *show p* @ '' *by* '' @ *show q* @ '' *gives* '' @ *show pq* @ '' *and not* '' @ *show r*›

**lemma** [*sepref-fr-rules*]:
‹(*uncurry3* ((λ*x y*. *return oo* (*check-mult-l-dom-err-impl x y*))),
*uncurry3* (*check-mult-l-dom-err*)) ∈ *bool-assn*$^k$ ∗$_a$ *uint64-nat-assn*$^k$ ∗$_a$ *bool-assn*$^k$ ∗$_a$ *uint64-nat-assn*$^k$
→$_a$ *raw-string-assn*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:
‹(*uncurry3* ((λ*x y*. *return oo* (*check-mult-l-mult-err-impl x y*))),
*uncurry3* (*check-mult-l-mult-err*)) ∈ *poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ ∗$_a$ *poly-assn*$^k$ →$_a$ *raw-string-assn*›

⟨*proof*⟩

**sepref-definition** *check-mult-l-impl*
**is** ‹*uncurry6 check-mult-l*›
:: ‹*poly-assn*$^k$ $*_a$ *polys-assn*$^k$ $*_a$ *vars-assn*$^k$ $*_a$ *uint64-nat-assn*$^k$ $*_a$ *poly-assn*$^k$ $*_a$ *uint64-nat-assn*$^k$ $*_a$ *poly-assn*$^k$ $\rightarrow_a$ *status-assn raw-string-assn*›
⟨*proof*⟩

**declare** *check-mult-l-impl.refine*[*sepref-fr-rules*]

**definition** *check-ext-l-dom-err-impl* :: ‹*uint64* ⇒ -› **where**
‹*check-ext-l-dom-err-impl p* =
*''There is already a polynomial with index ''* @ *show* (*nat-of-uint64 p*)›

**lemma** [*sepref-fr-rules*]:
‹(((*return o* (*check-ext-l-dom-err-impl*))),
(*check-extension-l-dom-err*)) ∈ *uint64-nat-assn*$^k$ $\rightarrow_a$ *raw-string-assn*›
⟨*proof*⟩

**definition** *check-extension-l-no-new-var-err-impl* :: ‹- ⇒ -› **where**
‹*check-extension-l-no-new-var-err-impl p* =
*''No new variable could be found in polynomial ''* @ *show p*›

**lemma** [*sepref-fr-rules*]:
‹(((*return o* (*check-extension-l-no-new-var-err-impl*))),
(*check-extension-l-no-new-var-err*)) ∈ *poly-assn*$^k$ $\rightarrow_a$ *raw-string-assn*›
⟨*proof*⟩

**definition** *check-extension-l-side-cond-err-impl* :: ‹- ⇒ -› **where**
‹*check-extension-l-side-cond-err-impl v p r s* =
*''Error while checking side conditions of extensions polynow, var is ''* @ *show v* @
*'' polynomial is ''* @ *show p* @ *''side condition p∗p − p = ''* @ *show s* @ *'' and should be 0''*›

**lemma** [*sepref-fr-rules*]:
‹((*uncurry3* (λ*x y*. *return oo* (*check-extension-l-side-cond-err-impl x y*))),
*uncurry3* (*check-extension-l-side-cond-err*)) ∈ *string-assn*$^k$ $*_a$ *poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $*_a$ *poly-assn*$^k$ $\rightarrow_a$ *raw-string-assn*›
⟨*proof*⟩

**definition** *check-extension-l-new-var-multiple-err-impl* :: ‹- ⇒ -› **where**
‹*check-extension-l-new-var-multiple-err-impl v p* =
*''Error while checking side conditions of extensions polynow, var is ''* @ *show v* @
*'' but it either appears at least once in the polynomial or another new variable is created ''* @
*show p* @ *'' but should not.''*›

**lemma** [*sepref-fr-rules*]:
‹((*uncurry* (*return oo* (*check-extension-l-new-var-multiple-err-impl*))),
*uncurry* (*check-extension-l-new-var-multiple-err*)) ∈ *string-assn*$^k$ $*_a$ *poly-assn*$^k$ $\rightarrow_a$ *raw-string-assn*›
⟨*proof*⟩

**sepref-register** *check-extension-l-dom-err fmlookup′*
*check-extension-l-side-cond-err check-extension-l-no-new-var-err*
*check-extension-l-new-var-multiple-err*

**definition** *uminus-poly* :: ‹*llist-polynomial* ⇒ *llist-polynomial*› **where**
‹*uminus-poly* $p'$ = *map* (λ(*a*, *b*). (*a*, − *b*)) $p'$›

**sepref-register** *uminus-poly*
**lemma** [*sepref-import-param*]:
‹(*map* (λ(*a*, *b*). (*a*, − *b*)), *uminus-poly*) ∈ *poly-rel* → *poly-rel*›
⟨*proof*⟩

**sepref-register** *vars-of-poly-in*
*weak-equality-l*

**lemma** [*safe-constraint-rules*]:
‹*Sepref-Constraints.CONSTRAINT single-valued* (*the-pure monomial-assn*)› **and**
*single-valued-the-monomial-assn*:
‹*single-valued* (*the-pure monomial-assn*)›
‹*single-valued* ((*the-pure monomial-assn*)$^{-1}$)›
⟨*proof*⟩

**sepref-definition** *check-extension-l-impl*
**is** ‹*uncurry5 check-extension-l*›
:: ‹*poly-assn*$^k$ $*_a$ *polys-assn*$^k$ $*_a$ *vars-assn*$^k$ $*_a$ *uint64-nat-assn*$^k$ $*_a$ *string-assn*$^k$ $*_a$ *poly-assn*$^k$ $\to_a$
*status-assn raw-string-assn*›
⟨*proof*⟩

**declare** *check-extension-l-impl.refine*[*sepref-fr-rules*]

**sepref-definition** *check-del-l-impl*
**is** ‹*uncurry2 check-del-l*›
:: ‹*poly-assn*$^k$ $*_a$ *polys-assn*$^k$ $*_a$ *uint64-nat-assn*$^k$ $\to_a$ *status-assn raw-string-assn*›
⟨*proof*⟩

**lemmas** [*sepref-fr-rules*] = *check-del-l-impl.refine*

**abbreviation** *pac-step-rel* **where**
‹*pac-step-rel* ≡ *p2rel* (⟨*Id*, ⟨*monomial-rel*⟩*list-rel*, *Id*⟩ *pac-step-rel-raw*)›

**sepref-register** *PAC-Polynomials-Operations.normalize-poly*
*pac-src1 pac-src2 new-id pac-mult case-pac-step check-mult-l*
*check-addition-l check-del-l check-extension-l*

**lemma** *pac-step-rel-assn-alt-def2*:
‹*hn-ctxt* (*pac-step-rel-assn nat-assn poly-assn id-assn*) *b bi* =
*hn-val*
(*p2rel*
(⟨*nat-rel*, *poly-rel*, *Id* :: (*string* × -) *set*⟩*pac-step-rel-raw*)) *b bi*›
⟨*proof*⟩

**lemma** *is-AddD-import*[*sepref-fr-rules*]:
**assumes** ‹*CONSTRAINT is-pure K*› ‹*CONSTRAINT is-pure V*›
**shows**
‹(*return o pac-res*, *RETURN o pac-res*) ∈ [λ*x*. *is-Add x* ∨ *is-Mult x* ∨ *is-Extension x*]$_a$
(*pac-step-rel-assn K V R*)$^k$ → *V*›

‹(*return o pac-src1*, *RETURN o pac-src1*) ∈ [λ*x*. *is-Add x* ∨ *is-Mult x* ∨ *is-Del x*]$_a$ (*pac-step-rel-assn K V R*)$^k$ → *K*›
‹(*return o new-id*, *RETURN o new-id*) ∈ [λ*x*. *is-Add x* ∨ *is-Mult x* ∨ *is-Extension x*]$_a$ (*pac-step-rel-assn K V R*)$^k$ → *K*›
‹(*return o is-Add*, *RETURN o is-Add*) ∈ (*pac-step-rel-assn K V R*)$^k$ →$_a$ *bool-assn*›
‹(*return o is-Mult*, *RETURN o is-Mult*) ∈ (*pac-step-rel-assn K V R*)$^k$ →$_a$ *bool-assn*›
‹(*return o is-Del*, *RETURN o is-Del*) ∈ (*pac-step-rel-assn K V R*)$^k$ →$_a$ *bool-assn*›
‹(*return o is-Extension*, *RETURN o is-Extension*) ∈ (*pac-step-rel-assn K V R*)$^k$ →$_a$ *bool-assn*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:
‹*CONSTRAINT is-pure K* ⟹
(*return o pac-src2*, *RETURN o pac-src2*) ∈ [λ*x*. *is-Add x*]$_a$ (*pac-step-rel-assn K V R*)$^k$ → *K*›
‹*CONSTRAINT is-pure V* ⟹
(*return o pac-mult*, *RETURN o pac-mult*) ∈ [λ*x*. *is-Mult x*]$_a$ (*pac-step-rel-assn K V R*)$^k$ → *V*›
‹*CONSTRAINT is-pure R* ⟹
(*return o new-var*, *RETURN o new-var*) ∈ [λ*x*. *is-Extension x*]$_a$ (*pac-step-rel-assn K V R*)$^k$ → *R*›
⟨*proof*⟩

**lemma** *is-Mult-lastI*:
‹¬ *is-Add b* ⟹ ¬*is-Mult b* ⟹ ¬*is-Extension b* ⟹ *is-Del b*›
⟨*proof*⟩

**sepref-register** *is-cfailed is-Del*

**definition** *PAC-checker-l-step′* :: - **where**
‹*PAC-checker-l-step′ a b c d* = *PAC-checker-l-step a* (*b*, *c*, *d*)›

**lemma** *PAC-checker-l-step-alt-def*:
‹*PAC-checker-l-step a bcd e* = (*let* (*b*,*c*,*d*) = *bcd in PAC-checker-l-step′ a b c d e*)›
⟨*proof*⟩

**sepref-decl-intf** ('*k*) *acode-status* **is** ('*k*) *code-status*
**sepref-decl-intf** ('*k*, '*b*, '*lbl*) *apac-step* **is** ('*k*, '*b*, '*lbl*) *pac-step*

**sepref-register** *merge-cstatus full-normalize-poly new-var is-Add*

**lemma** *poly-rel-the-pure*:
‹*poly-rel* = *the-pure poly-assn*› **and**
*nat-rel-the-pure*:
‹*nat-rel* = *the-pure nat-assn*› **and**
*WTF-RF*: ‹*pure* (*the-pure nat-assn*) = *nat-assn*›
⟨*proof*⟩

**lemma** [*safe-constraint-rules*]:
‹*CONSTRAINT IS-LEFT-UNIQUE uint64-nat-rel*› **and**
*single-valued-uint64-nat-rel*[*safe-constraint-rules*]:
‹*CONSTRAINT single-valued uint64-nat-rel*›
⟨*proof*⟩

**sepref-definition** *check-step-impl*
**is** ‹*uncurry4 PAC-checker-l-step′*›
:: ‹*poly-assn*$^k$ ∗$_a$ (*status-assn raw-string-assn*)$^d$ ∗$_a$ *vars-assn*$^d$ ∗$_a$ *polys-assn*$^d$ ∗$_a$ (*pac-step-rel-assn* (*uint64-nat-assn*) *poly-assn* (*string-assn* :: *string* ⇒ -))$^d$ →$_a$
*status-assn raw-string-assn* ×$_a$ *vars-assn* ×$_a$ *polys-assn*›

⟨*proof*⟩

**declare** *check-step-impl.refine*[*sepref-fr-rules*]

**sepref-register** *PAC-checker-l-step PAC-checker-l-step′ fully-normalize-poly-impl*

**definition** *PAC-checker-l′* **where**
‹*PAC-checker-l′ p* $\mathcal{V}$ *A status steps* = *PAC-checker-l p* ($\mathcal{V}$, *A*) *status steps*›

**lemma** *PAC-checker-l-alt-def*:
‹*PAC-checker-l p* $\mathcal{V}A$ *status steps* =
(*let* ($\mathcal{V}$, *A*) = $\mathcal{V}A$ *in PAC-checker-l′ p* $\mathcal{V}$ *A status steps*)›
⟨*proof*⟩

**sepref-definition** *PAC-checker-l-impl*
**is** ‹*uncurry4 PAC-checker-l′*›
:: ‹*poly-assn*$^k$ $*_a$ *vars-assn*$^d$ $*_a$ *polys-assn*$^d$ $*_a$ (*status-assn raw-string-assn*)$^d$ $*_a$
(*list-assn* (*pac-step-rel-assn* (*uint64-nat-assn*) *poly-assn string-assn*))$^k$ $\rightarrow_a$
*status-assn raw-string-assn* $\times_a$ *vars-assn* $\times_a$ *polys-assn*›
⟨*proof*⟩

**declare** *PAC-checker-l-impl.refine*[*sepref-fr-rules*]

**abbreviation** *polys-assn-input* **where**
‹*polys-assn-input* ≡ *iam-fmap-assn nat-assn poly-assn*›

**definition** *remap-polys-l-dom-err-impl* :: ‹-› **where**
‹*remap-polys-l-dom-err-impl* =
*''Error during initialisation. Too many polynomials where provided. If this happens,''* @
*''please report the example to the authors, because something went wrong during ''* @
*''code generation (code generation to arrays is likely to be broken).''*›

**lemma** [*sepref-fr-rules*]:
‹((*uncurry0* (*return* (*remap-polys-l-dom-err-impl*))),
*uncurry0* (*remap-polys-l-dom-err*)) ∈ *unit-assn*$^k$ $\rightarrow_a$ *raw-string-assn*›
⟨*proof*⟩

MLton is not able to optimise the calls to pow.

**lemma** *pow-2-64*: ‹(2::*nat*) ^ 64 = 18446744073709551616›
⟨*proof*⟩

**sepref-register** *upper-bound-on-dom op-fmap-empty*

**sepref-definition** *remap-polys-l-impl*
**is** ‹*uncurry2 remap-polys-l2*›
:: ‹*poly-assn*$^k$ $*_a$ *vars-assn*$^d$ $*_a$ *polys-assn-input*$^d$ $\rightarrow_a$
*status-assn raw-string-assn* $\times_a$ *vars-assn* $\times_a$ *polys-assn*›
⟨*proof*⟩

**lemma** *remap-polys-l2-remap-polys-l*:
‹(*uncurry2 remap-polys-l2*, *uncurry2 remap-polys-l*) ∈ (*Id* $\times_r$ ⟨*Id*⟩*set-rel*) $\times_r$ *Id* $\rightarrow_f$ ⟨*Id*⟩*nres-rel*›
⟨*proof*⟩

**lemma** [*sepref-fr-rules*]:

‹(*uncurry2 remap-polys-l-impl*,
*uncurry2 remap-polys-l*) ∈ *poly-assn*$^k$ $*_a$ *vars-assn*$^d$ $*_a$ *polys-assn-input*$^d$ $\to_a$
*status-assn raw-string-assn* $\times_a$ *vars-assn* $\times_a$ *polys-assn*›
⟨*proof*⟩

**sepref-register** *remap-polys-l*

**sepref-definition** *full-checker-l-impl*
**is** ‹*uncurry2 full-checker-l*›
:: ‹*poly-assn*$^k$ $*_a$ *polys-assn-input*$^d$ $*_a$ (*list-assn* (*pac-step-rel-assn* (*uint64-nat-assn*) *poly-assn string-assn*))$^k$
$\to_a$
*status-assn raw-string-assn* $\times_a$ *vars-assn* $\times_a$ *polys-assn*›
⟨*proof*⟩

**sepref-definition** *PAC-update-impl*
**is** ‹*uncurry2* (*RETURN ooo fmupd*)›
:: ‹*nat-assn*$^k$ $*_a$ *poly-assn*$^k$ $*_a$ (*polys-assn-input*)$^d$ $\to_a$ *polys-assn-input*›
⟨*proof*⟩

**sepref-definition** *PAC-empty-impl*
**is** ‹*uncurry0* (*RETURN fmempty*)›
:: ‹*unit-assn*$^k$ $\to_a$ *polys-assn-input*›
⟨*proof*⟩

**sepref-definition** *empty-vars-impl*
**is** ‹*uncurry0* (*RETURN* {})›
:: ‹*unit-assn*$^k$ $\to_a$ *vars-assn*›
⟨*proof*⟩

This is a hack for performance. There is no need to recheck that that a char is valid when working on chars coming from strings... It is not that important in most cases, but in our case the preformance difference is really large.

**definition** *unsafe-asciis-of-literal* :: ‹- ⇒ -› **where**
‹*unsafe-asciis-of-literal xs* = *String.asciis-of-literal xs*›

**definition** *unsafe-asciis-of-literal′* :: ‹- ⇒ -› **where**
[*simp*, *symmetric*, *code*]: ‹*unsafe-asciis-of-literal′* = *unsafe-asciis-of-literal*›

**code-printing**
**constant** *unsafe-asciis-of-literal′* ⇀
(*SML*) !(*List.map* (*fn c => let val k* = *Char.ord c in IntInf.fromInt k end*) */o String.explode*)

Now comes the big and ugly and unsafe hack.

Basically, we try to avoid the conversion to IntInf when calculating the hash. The performance gain is roughly 40%, which is a LOT and definitively something we need to do. We are aware that the SML semantic encourages compilers to optimise conversions, but this does not happen here, corroborating our early observation on the verified SAT solver IsaSAT.x

**definition** *raw-explode* **where**
[*simp*]: ‹*raw-explode* = *String.explode*›
**code-printing**
**constant** *raw-explode* ⇀
(*SML*) *String.explode*

**definition** ‹*hashcode-literal′ s* ≡

*foldl* (λ*h x*. *h* ∗ *33* + *uint32-of-int* (*of-char x*)) *5381*
(*raw-explode s*)›

**lemmas** [*code*] =
*hashcode-literal-def*[*unfolded String.explode-code*
*unsafe-asciis-of-literal-def*[*symmetric*]]

**definition** *uint32-of-char* **where**
[*symmetric*, *code-unfold*]: ‹*uint32-of-char x* = *uint32-of-int* (*int-of-char x*)›

**code-printing**
**constant** *uint32-of-char* ⇀
(*SML*) !(*Word32.fromInt* /*o* (*Char.ord*))

**lemma** [*code*]: ‹*hashcode s* = *hashcode-literal′ s*›
⟨*proof*⟩

We compile Pastèque in `PAC_Checker_MLton.thy`.

**export-code** *PAC-checker-l-impl PAC-update-impl PAC-empty-impl the-error is-cfailed is-cfound*
*int-of-integer Del Add Mult nat-of-integer String.implode remap-polys-l-impl*
*fully-normalize-poly-impl union-vars-poly-impl empty-vars-impl*
*full-checker-l-impl check-step-impl CSUCCESS*
*Extension hashcode-literal′ version*
**in** *SML-imp* **module-name** *PAC-Checker*

# 14 Correctness theorem

**context** *poly-embed*
**begin**

**definition** *full-poly-assn* **where**
‹*full-poly-assn* = *hr-comp poly-assn* (*fully-unsorted-poly-rel O mset-poly-rel*)›

**definition** *full-poly-input-assn* **where**
‹*full-poly-input-assn* = *hr-comp*
(*hr-comp polys-assn-input*
(⟨*nat-rel*, *fully-unsorted-poly-rel O mset-poly-rel*⟩*fmap-rel*))
*polys-rel*›

**definition** *fully-pac-assn* **where**
‹*fully-pac-assn* = (*list-assn*
(*hr-comp* (*pac-step-rel-assn uint64-nat-assn poly-assn string-assn*)
(*p2rel*
(⟨*nat-rel*,
*fully-unsorted-poly-rel O*
*mset-poly-rel*, *var-rel*⟩*pac-step-rel-raw*))))›

**definition** *code-status-assn* **where**
‹*code-status-assn* = *hr-comp* (*status-assn raw-string-assn*)
*code-status-status-rel*›

**definition** *full-vars-assn* **where**
‹*full-vars-assn* = *hr-comp* (*hs.assn string-assn*)
(⟨*var-rel*⟩*set-rel*)›

**lemma** *polys-rel-full-polys-rel*:
‹*polys-rel-full* = *Id* ×$_r$ *polys-rel*›
⟨*proof*⟩

**definition** *full-polys-assn* :: ‹-› **where**
‹*full-polys-assn* = *hr-comp* (*hr-comp polys-assn*
(⟨*nat-rel*,
*sorted-poly-rel O mset-poly-rel*⟩*fmap-rel*))
*polys-rel*›

Below is the full correctness theorems. It basically states that:

1. assuming that the input polynomials have no duplicate variables

Then:

1. if the checker returns *CFOUND*, the spec is in the ideal and the PAC file is correct
2. if the checker returns *CSUCCESS*, the PAC file is correct (but there is no information on the spec, aka checking failed)
3. if the checker return *CFAILED err*, then checking failed (and *err might* give you an indication of the error, but the correctness theorem does not say anything about that).

   The input parameters are:
4. the specification polynomial represented as a list
5. the input polynomials as hash map (as an array of option polynomial)
6. a represention of the PAC proofs.

**lemma** *PAC-full-correctness*:
‹(*uncurry2 full-checker-l-impl*,
*uncurry2* (λ*spec A* -. *PAC-checker-specification spec A*))
∈ (*full-poly-assn*)$^k$ ∗$_a$ (*full-poly-input-assn*)$^d$ ∗$_a$ (*fully-pac-assn*)$^k$ →$_a$ *hr-comp*
(*code-status-assn* ×$_a$ *full-vars-assn* ×$_a$ *hr-comp polys-assn*
(⟨*nat-rel*, *sorted-poly-rel O mset-poly-rel*⟩*fmap-rel*))
{((*st*, *G*), *st*′, *G*′).
*st* = *st*′ ∧ (*st* ≠ *FAILED* ⟶ (*G*, *G*′) ∈ *Id* ×$_r$ *polys-rel*)}›
⟨*proof*⟩

It would be more efficient to move the parsing to Isabelle, as this would be more memory efficient (and also reduce the TCB). But now comes the fun part: It cannot work. A stream (of a file) is consumed by side effects. Assume that this would work. The code could look like:
*Let* (*read-file file*) *f*
This code is equal to (in the HOL sense of equality): *let* - = *read-file file in Let* (*read-file file*) *f*
However, as an hypothetical *read-file* changes the underlying stream, we would get the next token. Remark that this is already a weird point of ML compilers. Anyway, I see currently two solutions to this problem:

1. The meta-argument: use it only in the Refinement Framework in a setup where copies are disallowed. Basically, this works because we can express the non-duplication constraints on the type level. However, we cannot forbid people from expressing things directly at the HOL level.

2. On the target language side, model the stream as the stream and the position. Reading takes two arguments. First, the position to read. Second, the stream (and the current position) to read. If the position to read does not match the current position, return an error. This would fit the correctness theorem of the code generation (roughly "if it terminates without exception, the answer is the same"), but it is still unsatisfactory.

**end**

**definition** $\varphi$ :: ‹*string* ⇒ *nat*› **where**
‹$\varphi$ = (*SOME* $\varphi$. *bij* $\varphi$)›

**lemma** *bij-φ*: ‹*bij* $\varphi$›
⟨*proof*⟩

**global-interpretation** *PAC*: *poly-embed* **where**
$\varphi = \varphi$
⟨*proof*⟩

The full correctness theorem is (*uncurry2 full-checker-l-impl*, *uncurry2* (λ*spec A* -. *PAC-checker-specification spec A*)) ∈ *PAC.full-poly-assn*$^k$ ∗$_a$ *PAC.full-poly-input-assn*$^d$ ∗$_a$ *PAC.fully-pac-assn*$^k$ →$_a$ *hr-comp* (*PAC.code-status-assn* ×$_a$ *PAC.full-vars-assn* ×$_a$ *hr-comp* (*hm-fmap-assn uint64-nat-assn* (*list-assn* (*monom-assn* ×$_a$ *id-assn*))) (⟨*nat-rel*, *sorted-poly-rel O PAC.mset-poly-rel*⟩*fmap-rel*)) {((*st*, *G*), *st*′, *G*′). *st* = *st*′ ∧ (*st* ≠ *FAILED* ⟶ (*G*, *G*′) ∈ *Id* ×$_r$ *polys-rel*)}.

**end**

**theory** *PAC-Checker-MLton*
**imports** *PAC-Checker-Synthesis*
**begin**

**export-code** *PAC-checker-l-impl PAC-update-impl PAC-empty-impl the-error is-cfailed is-cfound*
*int-of-integer Del Add Mult nat-of-integer String.implode remap-polys-l-impl*
*fully-normalize-poly-impl union-vars-poly-impl empty-vars-impl*
*full-checker-l-impl check-step-impl CSUCCESS*
*Extension hashcode-literal*′ *version*
**in** *SML-imp* **module-name** *PAC-Checker*
**file-prefix** *checker*

Here is how to compile it:

```
compile-generated-files -
  external-files
    ‹code/parser.sml›
    ‹code/pasteque.sml›
    ‹code/pasteque.mlb›
  where ‹fn dir =>
    let
      val exec = Generated-Files.execute (dir + Path.basic code);
      val - =
        exec ‹Compilation›
          (verbatim ‹$ISABELLE-MLTON $ISABELLE-MLTON-OPTIONS › ^
            −const 'MLton.safe false' −verbose 1 −default−type int64 −output pasteque ^
            −codegen native −inline 700 −cc−opt −O3 pasteque.mlb);
    in () end›

end
```

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## References

[1] D. Kaufmann, M. Fleury, and A. Biere. The proof checkers pacheck and pasteque for the practical algebraic calculus. In O. Strichman and A. Ivrii, editors, *Formal Methods in Computer-Aided Design, FMCAD 2020, September 21-24, 2020.* IEEE, 2020.