A Partition Theorem for the Ordinal ω^ω

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Abstract

The theory of partition relations concerns generalisations of Ramsey's theorem. For any ordinal α, write $\alpha \to (\alpha, m)^2$ if for each function f from unordered pairs of elements of α into $\{0,1\}$, either there is a subset $X \subseteq \alpha$ order-isomorphic to α such that $f\{x,y\} = 0$ for all $\{x,y\} \subseteq X$, or there is an m element set $Y \subseteq \alpha$ such that $f\{x,y\} = 1$ for all $\{x,y\} \subseteq Y$. (In both cases, with $\{x,y\}$ we require $x \neq y$.) In particular, the infinite Ramsey theorem can be written in this notation as $\omega \to (\omega, \omega)^2$, or if we restrict m to the positive integers as above, then $\omega \to (\omega, m)^2$ for all m [3].

This entry formalises Larson's proof of $\omega^\omega \to (\omega^\omega, m)^2$ along with a similar proof of a result due to Specker: $\omega^2 \to (\omega^2, m)^2$. Also proved is a necessary result by Erdős and Milner [1, 2]: $\omega^{1+\alpha\cdot n} \to (\omega^{1+\alpha}, 2^n)^2$.

These examples demonstrate the use of Isabelle/HOL to formalise advanced results that combine ZF set theory with basic concepts like lists and natural numbers.

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1 Library additions

theory *Library-Additions*
imports *ZFC-in-HOL.Ordinal-Exp HOL−Library.Ramsey Nash-Williams.Nash-Williams*

begin

lemma *finite-enumerate-Diff-singleton*:
fixes S :: *'a::wellorder set*
assumes *finite* S **and** i: $i < card\ S$ *enumerate* $S\ i < x$
shows *enumerate* $(S - \{x\})\ i$ = *enumerate* $S\ i$
⟨*proof*⟩

lemma *hd-lex*: ⟦*hd ms* < *hd ns*; *length ms* = *length ns*; *ns* ≠ []⟧ ⟹ (*ms*, *ns*) ∈ *lex less-than*
⟨*proof*⟩

lemma *sorted-hd-le*:
assumes *sorted xs x* ∈ *list.set xs*
shows *hd xs* ≤ *x*
⟨*proof*⟩

lemma *sorted-le-last*:

assumes *sorted xs x* ∈ *list.set xs*
shows *x* ≤ *last xs*
⟨*proof*⟩

lemma *hd-list-of*:
assumes *finite A A* ≠ {}
shows *hd* (*sorted-list-of-set A*) = *Min A*
⟨*proof*⟩

lemma *sorted-hd-le-last*:
assumes *sorted xs xs* ≠ []
shows *hd xs* ≤ *last xs*
⟨*proof*⟩

lemma *sorted-list-of-set-set-of* [*simp*]: *strict-sorted l* ⟹ *sorted-list-of-set* (*list.set l*) = *l*
⟨*proof*⟩

lemma *range-strict-mono-ext*:
fixes *f*::*nat* ⇒ *'a*::*linorder*
assumes *eq*: *range f* = *range g*
and *sm*: *strict-mono f strict-mono g*
shows *f* = *g*
⟨*proof*⟩

1.1 Other material

definition *strict-mono-sets* :: [*'a*::*order set*, *'a*::*order* ⇒ *'b*::*order set*] ⇒ *bool* **where**
strict-mono-sets A f ≡ ∀ *x*∈*A*. ∀ *y*∈*A*. *x* < *y* ⟶ *less-sets* (*f x*) (*f y*)

lemma *strict-mono-setsD*:
assumes *strict-mono-sets A f x* < *y x* ∈ *A y* ∈ *A*
shows *less-sets* (*f x*) (*f y*)
⟨*proof*⟩

lemma *strict-mono-sets-imp-disjoint*:
fixes *A* :: *'a*::*linorder set*
assumes *strict-mono-sets A f*
shows *pairwise* (λ*x y*. *disjnt* (*f x*) (*f y*)) *A*
⟨*proof*⟩

lemma *strict-mono-sets-subset*:
assumes *strict-mono-sets B f A* ⊆ *B*
shows *strict-mono-sets A f*
⟨*proof*⟩

lemma *strict-mono-less-sets-Min*:
assumes *strict-mono-sets I f finite I I* ≠ {}
shows *less-sets* (*f* (*Min I*)) (⋃ (*f* ‘ (*I* − {*Min I*})))

⟨*proof*⟩

lemma *pair-less-iff1* [*simp*]: $((x,y), (x,z)) \in$ *pair-less* $\longleftrightarrow$ $y<z$
⟨*proof*⟩

lemma *infinite-finite-Inter*:
assumes *finite* $\mathcal{A}$ $\mathcal{A}\neq\{\}$ $\bigwedge A.\ A \in \mathcal{A} \Longrightarrow$ *infinite* A
and $\bigwedge A\ B.\ [\![A \in \mathcal{A};\ B \in \mathcal{A}]\!] \Longrightarrow A \cap B \in \mathcal{A}$
shows *infinite* $(\bigcap \mathcal{A})$
⟨*proof*⟩

lemma *atLeast-less-sets*: $[\![$*less-sets* A $\{x\};$ $B \subseteq \{x..\}]\!] \Longrightarrow$ *less-sets* A B
⟨*proof*⟩

1.2 The list-of function

lemma *sorted-list-of-set-insert-remove-cons*:
assumes *finite* A *less-sets* $\{a\}$ A
shows *sorted-list-of-set* (*insert* a A) = a # *sorted-list-of-set* A
⟨*proof*⟩

lemma *sorted-list-of-set-Un*:
assumes AB: *less-sets* A B **and** *fin*: *finite* A *finite* B
shows *sorted-list-of-set* $(A \cup B)$ = *sorted-list-of-set* A @ *sorted-list-of-set* B
⟨*proof*⟩

lemma *sorted-list-of-set-UN-lessThan*:
fixes k::*nat*
assumes *sm*: *strict-mono-sets* $\{..<k\}$ A **and** $\bigwedge i.\ i < k \Longrightarrow$ *finite* $(A\ i)$
shows *sorted-list-of-set* $(\bigcup i<k.\ A\ i)$ = *concat* (*map* (*sorted-list-of-set* $\circ$ A) (*sorted-list-of-set* $\{..<k\}$))
⟨*proof*⟩

lemma *sorted-list-of-set-UN-atMost*:
fixes k::*nat*
assumes *strict-mono-sets* $\{..k\}$ A **and** $\bigwedge i.\ i \leq k \Longrightarrow$ *finite* $(A\ i)$
shows *sorted-list-of-set* $(\bigcup i\leq k.\ A\ i)$ = *concat* (*map* (*sorted-list-of-set* $\circ$ A) (*sorted-list-of-set* $\{..k\}$))
⟨*proof*⟩

1.3 Monotonic enumeration of a countably infinite set

abbreviation *enum* ≡ *enumerate*

Could be generalised to infinite countable sets of any type

lemma *nat-infinite-iff*:
fixes N :: *nat set*
shows *infinite* $N \longleftrightarrow (\exists f$::*nat*⇒*nat*. N = *range* $f \wedge$ *strict-mono* $f)$
⟨*proof*⟩

lemma *enum-works*:
fixes $N :: nat\ set$
assumes *infinite* N
shows $N = range\ (enum\ N) \wedge strict\text{-}mono\ (enum\ N)$
⟨*proof*⟩

lemma *range-enum*: $range\ (enum\ N) = N$ **and** *strict-mono-enum*: *strict-mono* $(enum\ N)$
if *infinite* N **for** $N :: nat\ set$
⟨*proof*⟩

lemma *enum-0-eq-Inf*:
fixes $N :: nat\ set$
assumes *infinite* N
shows $enum\ N\ 0 = Inf\ N$
⟨*proof*⟩

lemma *enum-works-finite*:
fixes $N :: nat\ set$
assumes *finite* N
shows $N = enum\ N\ `\ \{..<card\ N\} \wedge strict\text{-}mono\text{-}on\ \{..<card\ N\}\ (enum\ N)$
⟨*proof*⟩

lemma *enum-obtain-index-finite*:
fixes $N :: nat\ set$
assumes $x \in N$ *finite* N
obtains i **where** $i < card\ N\ x = enum\ N\ i$
⟨*proof*⟩

lemma *enum-0-eq-Inf-finite*:
fixes $N :: nat\ set$
assumes *finite* $N\ N \neq \{\}$
shows $enum\ N\ 0 = Inf\ N$
⟨*proof*⟩

lemma *greaterThan-less-enum*:
fixes $N :: nat\ set$
assumes $N \subseteq \{x<..\}$ *infinite* N
shows $x < enum\ N\ i$
⟨*proof*⟩

lemma *atLeast-le-enum*:
fixes $N :: nat\ set$
assumes $N \subseteq \{x..\}$ *infinite* N
shows $x \leq enum\ N\ i$
⟨*proof*⟩

lemma *less-sets-empty1* [*simp*]: *less-sets* $\{\}\ A$ **and** *less-sets-empty2* [*simp*]: *less-sets* $A\ \{\}$

⟨*proof*⟩

lemma *less-sets-singleton1* [*simp*]: *less-sets* $\{a\}$ $A \longleftrightarrow (\forall x{\in}A.\ a < x)$
and *less-sets-singleton2* [*simp*]: *less-sets* A $\{a\} \longleftrightarrow (\forall x{\in}A.\ x < a)$
⟨*proof*⟩

lemma *less-sets-atMost* [*simp*]: *less-sets* $\{..a\}$ $A \longleftrightarrow (\forall x{\in}A.\ a < x)$
and *less-sets-alLeast* [*simp*]: *less-sets* A $\{a..\} \longleftrightarrow (\forall x{\in}A.\ x < a)$
⟨*proof*⟩

lemma *less-sets-imp-strict-mono-sets*:
assumes $\bigwedge i.$ *less-sets* $(A\ i)$ $(A\ (Suc\ i))$ $\bigwedge i.\ i{>}0 \Longrightarrow A\ i \neq \{\}$
shows *strict-mono-sets UNIV A*
⟨*proof*⟩

lemma *less-sets-Suc-Max*:
assumes *finite A*
shows *less-sets A* $\{Suc\ (Max\ A)..\}$
⟨*proof*⟩

lemma *infinite-nat-greaterThan*:
fixes *m*::*nat*
assumes *infinite N*
shows *infinite* $(N \cap \{m{<}..\})$
⟨*proof*⟩

end

2 Ordinal Partitions

Material from Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω. *Annals of Mathematical Logic*, 6:129–145, 1973. Also from "Partition Relations" by A. Hajnal and J. A. Larson, in *Handbook of Set Theory*, edited by Matthew Foreman and Akihiro Kanamori (Springer, 2010).

theory *Partitions*
imports *Library-Additions ZFC-in-HOL.ZFC-Typeclasses ZFC-in-HOL.Cantor-NF*

begin

abbreviation *tp* :: *V set* ⇒ *V*
where *tp A* ≡ *ordertype A VWF*

2.1 Ordinal Partitions: Definitions

definition *partn-lst* :: [('*a* × '*a*) *set*, '*a set*, *V list*, *nat*] ⇒ *bool*
where *partn-lst r B α n* ≡ ∀*f* ∈ $[B]^n$ → {..<*length α*}.
∃ *i* < *length α*. ∃ *H*. *H* ⊆ *B* ∧ *ordertype H r* = (*α*!*i*) ∧ *f* ‘ (*nsets H n*) ⊆ {*i*}

abbreviation *partn-lst-VWF* :: $V \Rightarrow V\ list \Rightarrow nat \Rightarrow bool$
  where $partn\text{-}lst\text{-}VWF\ \beta \equiv partn\text{-}lst\ VWF\ (elts\ \beta)$

lemma *partn-lst-E*:
  assumes $partn\text{-}lst\ r\ B\ \alpha\ n\ f \in nsets\ B\ n \rightarrow \{..<l\}\ l = length\ \alpha$
  obtains $i\ H$ **where** $i < l\ H \subseteq B$
    $ordertype\ H\ r = \alpha!i\ f\ `\ (nsets\ H\ n) \subseteq \{i\}$
  ⟨*proof*⟩

lemma *partn-lst-VWF-nontriv*:
  assumes $partn\text{-}lst\text{-}VWF\ \beta\ \alpha\ n\ l = length\ \alpha\ Ord\ \beta\ l > 0$
  obtains i **where** $i < l\ \alpha!i \leq \beta$
⟨*proof*⟩

lemma *partn-lst-triv0*:
  assumes $\alpha!i = 0\ i < length\ \alpha\ n \neq 0$
  shows $partn\text{-}lst\ r\ B\ \alpha\ n$
  ⟨*proof*⟩

lemma *partn-lst-triv1*:
  assumes $\alpha!i \leq 1\ i < length\ \alpha\ n > 1\ B \neq \{\}\ wf\ r$
  shows $partn\text{-}lst\ r\ B\ \alpha\ n$
  ⟨*proof*⟩

lemma *partn-lst-two-swap*:
  assumes $partn\text{-}lst\ r\ B\ [x,y]\ n$ **shows** $partn\text{-}lst\ r\ B\ [y,x]\ n$
⟨*proof*⟩

lemma *partn-lst-greater-resource*:
  assumes *M*: $partn\text{-}lst\ r\ B\ \alpha\ n$ **and** $B \subseteq C$
  shows $partn\text{-}lst\ r\ C\ \alpha\ n$
⟨*proof*⟩

lemma *partn-lst-less*:
  assumes *M*: $partn\text{-}lst\ r\ B\ \alpha\ n$ **and** *eq*: $length\ \alpha' = length\ \alpha$ **and** $List.set\ \alpha' \subseteq ON$
    and *le*: $\bigwedge i.\ i < length\ \alpha \Longrightarrow \alpha'!i \leq \alpha!i$
    and *r*: $wf\ r\ trans\ r\ total\text{-}on\ B\ r$ **and** $small\ B$
  shows $partn\text{-}lst\ r\ B\ \alpha'\ n$
⟨*proof*⟩

Holds because no n-sets exist!

lemma *partn-lst-VWF-degenerate*:
  assumes $k < n$
  shows $partn\text{-}lst\text{-}VWF\ \omega\ (ord\text{-}of\text{-}nat\ k\ \#\ \alpha s)\ n$
⟨*proof*⟩

lemma *partn-lst-VWF-ω-2*:
assumes *Ord* α
shows *partn-lst-VWF* $(\omega \uparrow (1{+}\alpha))$ $[2, \omega \uparrow (1{+}\alpha)]$ *2* (**is** *partn-lst-VWF ?β - -*)
⟨*proof*⟩

2.2 Relating partition properties on *VWF* to the general case

Two very similar proofs here!

lemma *partn-lst-imp-partn-lst-VWF-eq*:
assumes *part*: *partn-lst r U* α *n* **and** β: *ordertype U r* $= \beta$ **and** *small U*
and *r*: *wf r trans r total-on U r*
shows *partn-lst-VWF* β α *n*
⟨*proof*⟩

lemma *partn-lst-imp-partn-lst-VWF*:
assumes *part*: *partn-lst r U* α *n* **and** β: *ordertype U r* $\leq \beta$ *small U*
and *r*: *wf r trans r total-on U r*
shows *partn-lst-VWF* β α *n*
⟨*proof*⟩

lemma *partn-lst-VWF-imp-partn-lst-eq*:
assumes *part*: *partn-lst-VWF* β α *n* **and** β: *ordertype U r* $= \beta$ *small U*
and *r*: *wf r trans r total-on U r*
shows *partn-lst r U* α *n*
⟨*proof*⟩

corollary *partn-lst-VWF-imp-partn-lst*:
assumes *partn-lst-VWF* β α *n* **and** β: *ordertype U r* $\geq \beta$ *small U*
wf r trans r total-on U r
shows *partn-lst r U* α *n*
⟨*proof*⟩

2.3 Simple consequences of the definitions

A restatement of the infinite Ramsey theorem using partition notation

lemma *Ramsey-partn*: *partn-lst-VWF* ω $[\omega,\omega]$ *2*
⟨*proof*⟩

This is the counterexample sketched in Hajnal and Larson, section 9.1.

proposition *omega-basic-counterexample*:
assumes *Ord* α
shows $\neg$ *partn-lst-VWF* α $[succ\ (vcard\ \alpha), \omega]$ *2*
⟨*proof*⟩

2.4 Specker's theorem

definition *form-split* :: $[nat,nat,nat,nat,nat] \Rightarrow bool$ **where**
form-split a b c d i $\equiv a \leq c \wedge (i{=}0 \wedge a{<}b \wedge b{<}c \wedge c{<}d \vee$
$i{=}1 \wedge a{<}c \wedge c{<}b \wedge b{<}d \vee$

$i{=}2 \wedge a{<}c \wedge c{<}d \wedge d{<}b \vee$
$i{=}3 \wedge a{=}c \wedge b{\neq}d)$

definition *form* :: $[(nat{*}nat)set,\ nat] \Rightarrow bool$ **where**
form u i $\equiv \exists\, a\ b\ c\ d.\ u = \{(a,b),(c,d)\} \wedge$ *form-split a b c d i*

definition *scheme* :: $[(nat{*}nat)set] \Rightarrow nat\ set$ **where**
scheme u $\equiv$ *fst* ‘ *u* $\cup$ *snd* ‘ *u*

definition *UU* :: $(nat{*}nat)\ set$
where *UU* $\equiv \{(a,b).\ a < b\}$

lemma *ordertype-UNIV-ω2*: *ordertype UNIV pair-less* $= \omega{\uparrow}2$
⟨*proof*⟩

lemma *ordertype-UU-ge-ω2*: *ordertype UNIV pair-less* $\leq$ *ordertype UU pair-less*
⟨*proof*⟩

lemma *ordertype-UU-ω2*: *ordertype UU pair-less* $= \omega{\uparrow}2$
⟨*proof*⟩

Lemma 2.3 of Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω. *Annals of Mathematical Logic*, 6:129–145, 1973.

lemma *lemma-2-3*:
fixes *f* :: $(nat \times nat)\ set \Rightarrow nat$
assumes $f \in [UU]^2 \rightarrow \{..{<}Suc\ (Suc\ 0)\}$
obtains *N js* **where** *infinite N* **and** $\bigwedge k\ u.\ [\![k < 4;\ u \in [UU]^2;$ *form u k*; *scheme* $u \subseteq N]\!] \Longrightarrow f\ u = js!k$
⟨*proof*⟩

Lemma 2.4 of Jean A. Larson, ibid.

lemma *lemma-2-4*:
assumes *infinite N k* < 4
obtains *M* **where** $M \in [UU]^m\ \bigwedge u.\ u \in [M]^2 \Longrightarrow$ *form u k* $\bigwedge u.\ u \in [M]^2 \Longrightarrow$ *scheme* $u \subseteq N$
⟨*proof*⟩

Lemma 2.5 of Jean A. Larson, ibid.

lemma *lemma-2-5*:
assumes *infinite N*
obtains *X* **where** $X \subseteq$ *UU ordertype X pair-less* $= \omega{\uparrow}2$
$\bigwedge u.\ u \in [X]^2 \Longrightarrow (\exists\, k{<}4.$ *form u k*$) \wedge$ *scheme* $u \subseteq N$
⟨*proof*⟩

Theorem 2.1 of Jean A. Larson, ibid.

lemma *Specker-aux*:
assumes $\alpha \in$ *elts* ω
shows *partn-lst pair-less UU* $[\omega{\uparrow}2,\alpha]$ *2*
⟨*proof*⟩

theorem *Specker*: $\alpha \in elts\ \omega \Longrightarrow partn\text{-}lst\text{-}VWF\ (\omega\uparrow 2)\ [\omega\uparrow 2,\alpha]\ 2$
⟨*proof*⟩

end
theory *Erdos-Milner*
imports *Partitions*

begin

2.5 Erds-Milner theorem

P. Erds and E. C. Milner, A Theorem in the Partition Calculus. Canadian Math. Bull. 15:4 (1972), 501-505. Corrigendum, Canadian Math. Bull. 17:2 (1974), 305.

The paper defines strong types as satisfying the criteria below. It remarks that ordinals satisfy those criteria. Here is a (too complicated) proof.

proposition *strong-ordertype-eq*:
assumes *D*: $D \subseteq elts\ \beta$ **and** $Ord\ \beta$
obtains *L* **where** $\bigcup(List.set\ L) = D\ \bigwedge X.\ X \in List.set\ L \Longrightarrow indecomposable\ (tp\ X)$
and $\bigwedge M.\ [\![M \subseteq D; \bigwedge X.\ X \in List.set\ L \Longrightarrow tp\ (M \cap X) \geq tp\ X]\!] \Longrightarrow tp\ M = tp\ D$
⟨*proof*⟩

The "remark" of Erds and E. C. Milner, Canad. Math. Bull. Vol. 17 (2), 1974

proposition *indecomposable-imp-Ex-less-sets*:
assumes *indec*: $indecomposable\ \alpha$ **and** $\alpha \geq \omega$
and *A*: $tp\ A = \alpha\ small\ A\ A \subseteq ON$
and $x \in A$ **and** *A1*: $tp\ A1 = \alpha\ A1 \subseteq A$
obtains *A2* **where** $tp\ A2 = \alpha\ A2 \subseteq A1\ \{x\} \ll A2$
⟨*proof*⟩

the main theorem, from which they derive the headline result

theorem *Erdos-Milner-aux*:
assumes *part*: $partn\text{-}lst\text{-}VWF\ \alpha\ [k, \gamma]\ 2$
and *indec*: $indecomposable\ \alpha$ **and** $k > 1\ Ord\ \gamma$ **and** β: $\beta \in elts\ \omega 1$
shows $partn\text{-}lst\text{-}VWF\ (\alpha{*}\beta)\ [ord\text{-}of\text{-}nat\ (2{*}k), min\ \gamma\ (\omega{*}\beta)]\ 2$
⟨*proof*⟩

theorem *Erdos-Milner*:
assumes ν: $\nu \in elts\ \omega 1$
shows $partn\text{-}lst\text{-}VWF\ (\omega\uparrow(1 + \nu * n))\ [ord\text{-}of\text{-}nat\ (2\hat{}n), \omega\uparrow(1+\nu)]\ 2$
⟨*proof*⟩

corollary *remark-3*: *partn-lst-VWF* ($\omega\uparrow(Suc(4*k))$) [*4*, $\omega\uparrow(Suc(2*k))$] *2*
⟨*proof*⟩

Theorem 3.2 of Jean A. Larson, ibid.

corollary *Theorem-3-2*:
fixes *k n*::*nat*
shows *partn-lst-VWF* ($\omega\uparrow(n*k)$) [$\omega\uparrow n$, *ord-of-nat k*] *2*
⟨*proof*⟩

end

3 An ordinal partition theorem by Jean A. Larson

Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω. *Annals of Mathematical Logic*, 6:129–145, 1973.

theory *Omega-Omega*
imports *HOL−Library.Product-Lexorder Erdos-Milner*

begin

abbreviation *list-of* ≡ *sorted-list-of-set*

3.1 Cantor normal form for ordinals below $\omega \uparrow \omega$

Unlike *Cantor-sum*, there is no list of ordinal exponents, which are instead taken as consecutive. We obtain an order-isomorphism between $\omega \uparrow \omega$ and increasing lists of natural numbers (ordered lexicographically).

fun *omega-sum-aux* **where**
Nil: *omega-sum-aux 0 - = 0*
| *Suc*: *omega-sum-aux* (*Suc n*) [] = *0*
| *Cons*: *omega-sum-aux* (*Suc n*) (*m*#*ms*) = ($\omega\uparrow n$) ∗ (*ord-of-nat m*) + *omega-sum-aux n ms*

abbreviation *omega-sum* **where** *omega-sum ms* ≡ *omega-sum-aux* (*length ms*) *ms*

A normal expansion has no leading zeroes

inductive *normal*:: *nat list* ⇒ *bool* **where**
normal-Nil[*iff*]: *normal* []
| *normal-Cons*: *m > 0* ⟹ *normal* (*m*#*ms*)

inductive-simps *normal-Cons-iff* [*simp*]: *normal* (*m*#*ms*)

lemma *omega-sum-0-iff* [*simp*]: *normal ns* ⟹ *omega-sum ns = 0* ⟷ *ns* = []
⟨*proof*⟩

lemma *Ord-omega-sum-aux* [*simp*]: *Ord* (*omega-sum-aux k ms*)

⟨*proof*⟩

lemma *Ord-omega-sum*: *Ord* (*omega-sum ms*)
⟨*proof*⟩

lemma *omega-sum-less-ωω* [*intro*]: *omega-sum ms* < *ω*↑*ω*
⟨*proof*⟩

lemma *omega-sum-aux-less*: *omega-sum-aux k ms* < *ω* ↑ *k*
⟨*proof*⟩

lemma *omega-sum-less*: *omega-sum ms* < *ω* ↑ (*length ms*)
⟨*proof*⟩

lemma *omega-sum-ge*: *m* ≠ *0* ⟹ *ω* ↑ (*length ms*) ≤ *omega-sum* (*m*#*ms*)
⟨*proof*⟩

lemma *omega-sum-length-less*:
assumes *normal ns length ms* < *length ns*
shows *omega-sum ms* < *omega-sum ns*
⟨*proof*⟩

lemma *omega-sum-length-leD*:
assumes *omega-sum ms* ≤ *omega-sum ns normal ms*
shows *length ms* ≤ *length ns*
⟨*proof*⟩

lemma *omega-sum-less-eqlen-iff-cases* [*simp*]:
assumes *length ms* = *length ns*
shows *omega-sum* (*m*#*ms*) < *omega-sum* (*n*#*ns*) ⟷ *m*<*n* ∨ *m*=*n* ∧ *omega-sum ms* < *omega-sum ns*
⟨*proof*⟩

lemma *omega-sum-less-iff-cases*:
assumes *m* > *0 n* > *0*
shows *omega-sum* (*m*#*ms*) < *omega-sum* (*n*#*ns*)
⟷ *length ms* < *length ns*
∨ *length ms* = *length ns* ∧ *m*<*n*
∨ *length ms* = *length ns* ∧ *m*=*n* ∧ *omega-sum ms* < *omega-sum ns*
⟨*proof*⟩

lemma *omega-sum-less-iff*:
((*length ms*, *omega-sum ms*), (*length ns*, *omega-sum ns*)) ∈ *less-than* <∗*lex*∗> *VWF*
⟷ (*ms*,*ns*) ∈ *lenlex less-than*
⟨*proof*⟩

lemma *eq-omega-sum-less-iff*:
assumes *length ms* = *length ns*

shows (*omega-sum ms*, *omega-sum ns*) ∈ *VWF* ⟷ (*ms*,*ns*) ∈ *lenlex less-than*
⟨*proof*⟩

lemma *eq-omega-sum-eq-iff*:
assumes *length ms* = *length ns*
shows *omega-sum ms* = *omega-sum ns* ⟷ *ms*=*ns*
⟨*proof*⟩

lemma *inj-omega-sum*: *inj-on omega-sum* {*l*. *length l* = *n*}
⟨*proof*⟩

lemma *Ex-omega-sum*: γ ∈ *elts* (ω↑*n*) ⟹ ∃ *ns*. γ = *omega-sum ns* ∧ *length ns* = *n*
⟨*proof*⟩

lemma *omega-sum-drop* [*simp*]: *omega-sum* (*dropWhile* (λ*n*. *n*=*0*) *ns*) = *omega-sum ns*
⟨*proof*⟩

lemma *normal-drop* [*simp*]: *normal* (*dropWhile* (λ*n*. *n*=*0*) *ns*)
⟨*proof*⟩

lemma *omega-sum-ωω*:
assumes γ ∈ *elts* (ω↑ω)
obtains *ns* **where** γ = *omega-sum ns normal ns*
⟨*proof*⟩

definition *Cantor-ωω* :: *V* ⇒ *nat list*
where *Cantor-ωω* ≡ λ*x*. *SOME ns*. *x* = *omega-sum ns* ∧ *normal ns*

lemma
assumes γ ∈ *elts* (ω↑ω)
shows *Cantor-ωω*: *omega-sum* (*Cantor-ωω* γ) = γ
and *normal-Cantor-ωω*: *normal* (*Cantor-ωω* γ)
⟨*proof*⟩

3.2 Larson's set $W(n)$

definition *WW* :: *nat list set*
where *WW* ≡ {*l*. *strict-sorted l*}

fun *into-WW* :: *nat* ⇒ *nat list* ⇒ *nat list* **where**
into-WW k [] = []
| *into-WW k* (*n*#*ns*) = (*k*+*n*) # *into-WW* (*Suc* (*k*+*n*)) *ns*

fun *from-WW* :: *nat* ⇒ *nat list* ⇒ *nat list* **where**
from-WW k [] = []
| *from-WW k* (*n*#*ns*) = (*n* − *k*) # *from-WW* (*Suc n*) *ns*

lemma *from-into-WW* [*simp*]: *from-WW k* (*into-WW k ns*) = *ns*
⟨*proof*⟩

lemma *inj-into-WW*: *inj* (*into-WW k*)
⟨*proof*⟩

lemma *into-from-WW-aux*:
⟦*strict-sorted ns*; ∀ *n*∈*list.set ns*. *k* ≤ *n*⟧ ⟹ *into-WW k* (*from-WW k ns*) = *ns*
⟨*proof*⟩

lemma *into-from-WW* [*simp*]: *strict-sorted ns* ⟹ *into-WW 0* (*from-WW 0 ns*)
= *ns*
⟨*proof*⟩

lemma *into-WW-imp-ge*: *y* ∈ *List.set* (*into-WW x ns*) ⟹ *x* ≤ *y*
⟨*proof*⟩

lemma *strict-sorted-into-WW*: *strict-sorted* (*into-WW x ns*)
⟨*proof*⟩

lemma *length-into-WW*: *length* (*into-WW x ns*) = *length ns*
⟨*proof*⟩

lemma *WW-eq-range-into*: *WW* = *range* (*into-WW 0*)
⟨*proof*⟩

lemma *into-WW-lenlex-iff*: (*into-WW k ms*, *into-WW k ns*) ∈ *lenlex less-than*
⟷ (*ms*, *ns*) ∈ *lenlex less-than*
⟨*proof*⟩

lemma *wf-llt* [*simp*]: *wf* (*lenlex less-than*) **and** *trans-llt* [*simp*]: *trans* (*lenlex less-than*)
⟨*proof*⟩

lemma *total-llt* [*simp*]: *total-on A* (*lenlex less-than*)
⟨*proof*⟩

lemma *omega-sum-1-less*:
assumes (*ms*,*ns*) ∈ *lenlex less-than* **shows** *omega-sum* (*1*#*ms*) < *omega-sum*
(*1*#*ns*)
⟨*proof*⟩

lemma *ordertype-WW-1*: *ordertype WW* (*lenlex less-than*) ≤ *ordertype UNIV*
(*lenlex less-than*)
⟨*proof*⟩

lemma *ordertype-WW-2*: *ordertype UNIV* (*lenlex less-than*) ≤ ω↑ω
⟨*proof*⟩

lemma *ordertype-WW-3*: ω↑ω ≤ *ordertype WW* (*lenlex less-than*)

⟨*proof*⟩

lemma *ordertype-WW*: *ordertype WW* (*lenlex less-than*) = ω↑ω
and *ordertype-UNIV-ωω*: *ordertype UNIV* (*lenlex less-than*) = ω↑ω
⟨*proof*⟩

lemma *ordertype-ωω*:
fixes *F* :: *nat* ⇒ *nat list set*
assumes ⋀*j*::*nat*. *ordertype* (*F j*) (*lenlex less-than*) = ω↑*j*
shows *ordertype* (⋃*j*. *F j*) (*lenlex less-than*) = ω↑ω
⟨*proof*⟩

definition *WW-seg* :: *nat* ⇒ *nat list set*
where *WW-seg n* ≡ {*l* ∈ *WW*. *length l* = *n*}

lemma *WW-seg-subset-WW*: *WW-seg n* ⊆ *WW*
⟨*proof*⟩

lemma *WW-eq-UN-WW-seg*: *WW* = (⋃ *n*. *WW-seg n*)
⟨*proof*⟩

lemma *ordertype-list-seg*: *ordertype* {*l*. *length l* = *n*} (*lenlex less-than*) = ω↑*n*
⟨*proof*⟩

lemma *ordertype-WW-seg*: *ordertype* (*WW-seg n*) (*lenlex less-than*) = ω↑*n*
(**is** *ordertype ?W ?R* = ω↑*n*)
⟨*proof*⟩

3.3 Definitions required for the lemmas

3.3.1 Larson's "<"-relation on ordered lists

instantiation *list* :: (*ord*)*ord*
begin

definition *xs* < *ys* ≡ *xs* ≠ [] ∧ *ys* ≠ [] ⟶ *last xs* < *hd ys* **for** *xs ys* :: *'a list*
definition *xs* ≤ *ys* ≡ *xs* < *ys* ∨ *xs* = *ys* **for** *xs ys* :: *'a list*

instance
⟨*proof*⟩

end

lemma *less-Nil* [*simp*]: *xs* < [] [] < *xs*
⟨*proof*⟩

lemma *less-sets-imp-list-less*:

assumes *list.set xs* $\ll$ *list.set ys*
shows *xs* $<$ *ys*
⟨*proof*⟩

lemma *less-sets-imp-sorted-list-of-set*:
assumes *A* $\ll$ *B finite A finite B*
shows *list-of A* $<$ *list-of B*
⟨*proof*⟩

lemma *sorted-list-of-set-imp-less-sets*:
assumes *xs* $<$ *ys sorted xs sorted ys*
shows *list.set xs* $\ll$ *list.set ys*
⟨*proof*⟩

lemma *less-list-iff-less-sets*:
assumes *sorted xs sorted ys*
shows *xs* $<$ *ys* $\longleftrightarrow$ *list.set xs* $\ll$ *list.set ys*
⟨*proof*⟩

lemma *strict-sorted-append-iff*:
strict-sorted (*xs* @ *ys*) $\longleftrightarrow$ *xs* $<$ *ys* $\wedge$ *strict-sorted xs* $\wedge$ *strict-sorted ys*
⟨*proof*⟩

lemma *singleton-less-list-iff*: *sorted xs* $\Longrightarrow$ [*n*] $<$ *xs* $\longleftrightarrow$ {..*n*} $\cap$ *list.set xs* $=$ {}
⟨*proof*⟩

lemma *less-hd-imp-less*: *xs* $<$ [*hd ys*] $\Longrightarrow$ *xs* $<$ *ys*
⟨*proof*⟩

lemma *strict-sorted-concat-I*:
assumes $\bigwedge$*x*. *x* $\in$ *list.set xs* $\Longrightarrow$ *strict-sorted x*
$\bigwedge$*n*. *Suc n* $<$ *length xs* $\Longrightarrow$ *xs*!*n* $<$ *xs*!*Suc n*
xs $\in$ *lists* ($-$ {[]})
shows *strict-sorted* (*concat xs*)
⟨*proof*⟩

3.4 Nash Williams for lists

3.4.1 Thin sets of lists

inductive *initial-segment* :: *'a list* $\Rightarrow$ *'a list* $\Rightarrow$ *bool*
where *initial-segment xs* (*xs*@*ys*)

definition *thin* :: *'a list set* $\Rightarrow$ *bool*
where *thin A* $\equiv$ $\neg$ ($\exists$ *x y*. *x* $\in$ *A* $\wedge$ *y* $\in$ *A* $\wedge$ *x* $\neq$ *y* $\wedge$ *initial-segment x y*)

lemma *initial-segment-ne*:
assumes *initial-segment xs ys xs* $\neq$ []
shows *ys* $\neq$ [] $\wedge$ *hd ys* $=$ *hd xs*
⟨*proof*⟩

lemma *take-initial-segment*:
assumes *initial-segment xs ys k* ≤ *length xs*
shows *take k xs* = *take k ys*
⟨*proof*⟩

lemma *initial-segment-length-eq*:
assumes *initial-segment xs ys length xs* = *length ys*
shows *xs* = *ys*
⟨*proof*⟩

lemma *initial-segment-Nil* [*simp*]: *initial-segment* [] *ys*
⟨*proof*⟩

lemma *initial-segment-Cons* [*simp*]: *initial-segment* (*x*#*xs*) (*y*#*ys*) ⟷ *x*=*y* ∧ *initial-segment xs ys*
⟨*proof*⟩

lemma *init-segment-iff-initial-segment*:
assumes *strict-sorted xs strict-sorted ys*
shows *init-segment* (*list.set xs*) (*list.set ys*) ⟷ *initial-segment xs ys* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

theorem *Nash-Williams-WW*:
fixes *h* :: *nat list* ⇒ *nat*
assumes *infinite M* **and** *h*: *h* ‘ {*l* ∈ *A*. *List.set l* ⊆ *M*} ⊆ {..<*2*} **and** *thin A A* ⊆ *WW*
obtains *i N* **where** *i* < *2 infinite N N* ⊆ *M h* ‘ {*l* ∈ *A*. *List.set l* ⊆ *N*} ⊆ {*i*}
⟨*proof*⟩

3.5 Specialised functions on lists

lemma *mem-lists-non-Nil*: *xss* ∈ *lists* (− {[]}) ⟷ (∀ *x* ∈ *list.set xss*. *x* ≠ [])
⟨*proof*⟩

fun *acc-lengths* :: *nat* ⇒ *'a list list* ⇒ *nat list*
where *acc-lengths acc* [] = []
| *acc-lengths acc* (*l*#*ls*) = (*acc* + *length l*) # *acc-lengths* (*acc* + *length l*) *ls*

lemma *length-acc-lengths* [*simp*]: *length* (*acc-lengths acc ls*) = *length ls*
⟨*proof*⟩

lemma *acc-lengths-eq-Nil-iff* [*simp*]: *acc-lengths acc ls* = [] ⟷ *ls* = []
⟨*proof*⟩

lemma *set-acc-lengths*:
assumes *ls* ∈ *lists* (− {[]}) **shows** *list.set* (*acc-lengths acc ls*) ⊆ {*acc*<..}
⟨*proof*⟩

Useful because *acc-lengths.simps* will sometimes be deleted from the simpset.

lemma *hd-acc-lengths* [*simp*]: *hd* (*acc-lengths acc* (*l*#*ls*)) = *acc* + *length l*
⟨*proof*⟩

lemma *last-acc-lengths* [*simp*]:
ls ≠ [] ⟹ *last* (*acc-lengths acc ls*) = *acc* + *sum-list* (*map length ls*)
⟨*proof*⟩

lemma *nth-acc-lengths* [*simp*]:
⟦*ls* ≠ []; *k* < *length ls*⟧ ⟹ *acc-lengths acc ls* ! *k* = *acc* + *sum-list* (*map length* (*take* (*Suc k*) *ls*))
⟨*proof*⟩

lemma *acc-lengths-plus*: *acc-lengths* (*m*+*n*) *as* = *map* ((+)*m*) (*acc-lengths n as*)
⟨*proof*⟩

lemma *acc-lengths-shift*: *NO-MATCH 0 acc* ⟹ *acc-lengths acc as* = *map* ((+)*acc*) (*acc-lengths 0 as*)
⟨*proof*⟩

lemma *length-concat-acc-lengths*:
ls ≠ [] ⟹ *k* + *length* (*concat ls*) ∈ *list.set* (*acc-lengths k ls*)
⟨*proof*⟩

lemma *strict-sorted-acc-lengths*:
assumes *ls* ∈ *lists* (− {[]}) **shows** *strict-sorted* (*acc-lengths acc ls*)
⟨*proof*⟩

lemma *acc-lengths-append*:
acc-lengths acc (*xs* @ *ys*)
= *acc-lengths acc xs* @ *acc-lengths* (*acc* + *sum-list* (*map length xs*)) *ys*
⟨*proof*⟩

lemma *length-concat-ge*:
assumes *as* ∈ *lists* (− {[]})
shows *length* (*concat as*) ≥ *length as*
⟨*proof*⟩

fun *interact* :: *'a list list* ⇒ *'a list list* ⇒ *'a list*
where
interact [] *ys* = *concat ys*
| *interact xs* [] = *concat xs*
| *interact* (*x*#*xs*) (*y*#*ys*) = *x* @ *y* @ *interact xs ys*

lemma (**in** *monoid-add*) *length-interact*:
length (*interact xs ys*) = *sum-list* (*map length xs*) + *sum-list* (*map length ys*)

⟨*proof*⟩

lemma *length-interact-ge*:
assumes *xs* ∈ *lists* (− {[]}) *ys* ∈ *lists* (− {[]})
shows *length* (*interact xs ys*) ≥ *length xs* + *length ys*
⟨*proof*⟩

lemma *set-interact* [*simp*]:
shows *list.set* (*interact xs ys*) = *list.set* (*concat xs*) ∪ *list.set* (*concat ys*)
⟨*proof*⟩

lemma *interact-eq-Nil-iff* [*simp*]:
assumes *xs* ∈ *lists* (− {[]}) *ys* ∈ *lists* (− {[]})
shows *interact xs ys* = [] ⟷ *xs*=[] ∧ *ys*=[]
⟨*proof*⟩

lemma *interact-sing* [*simp*]: *interact* [*x*] *ys* = *x* @ *concat ys*
⟨*proof*⟩

lemma *hd-interact*: ⟦*xs* ≠ []; *hd xs* ≠ []⟧ ⟹ *hd* (*interact xs ys*) = *hd* (*hd xs*)
⟨*proof*⟩

lemma *acc-lengths-concat-injective*:
assumes *concat as′* = *concat as acc-lengths n as′* = *acc-lengths n as*
shows *as′* = *as*
⟨*proof*⟩

lemma *acc-lengths-interact-injective*:
assumes *interact as′ bs′* = *interact as bs acc-lengths a as′* = *acc-lengths a as*
acc-lengths b bs′ = *acc-lengths b bs*
shows *as′* = *as* ∧ *bs′* = *bs*
⟨*proof*⟩

lemma *strict-sorted-interact-I*:
assumes *length ys* ≤ *length xs length xs* ≤ *Suc* (*length ys*)
⋀*x*. *x* ∈ *list.set xs* ⟹ *strict-sorted x*
⋀*y*. *y* ∈ *list.set ys* ⟹ *strict-sorted y*
⋀*n*. *n* < *length ys* ⟹ *xs*!*n* < *ys*!*n*
⋀*n*. *Suc n* < *length xs* ⟹ *ys*!*n* < *xs*!*Suc n*
assumes *xs* ∈ *lists* (− {[]}) *ys* ∈ *lists* (− {[]})
shows *strict-sorted* (*interact xs ys*)
⟨*proof*⟩

3.6 Forms and interactions

3.6.1 Forms

inductive *Form-Body* :: [*nat*, *nat*, *nat list*, *nat list*, *nat list*] ⇒ *bool*
where *Form-Body ka kb xs ys zs*

if *length xs* < *length ys xs* = *concat* (*a*#*as*) *ys* = *concat* (*b*#*bs*)
a#*as* ∈ *lists* (− {[]}) *b*#*bs* ∈ *lists* (− {[]})
length (*a*#*as*) = *ka length* (*b*#*bs*) = *kb*
c = *acc-lengths 0* (*a*#*as*)
d = *acc-lengths 0* (*b*#*bs*)
zs = *concat* [*c*, *a*, *d*, *b*] @ *interact as bs*
strict-sorted zs

inductive *Form* :: [*nat*, *nat list set*] ⇒ *bool*
where *Form 0* {*xs*,*ys*} **if** *length xs* = *length ys xs* ≠ *ys*
| *Form* (*2*k*−*1*) {*xs*,*ys*} **if** *Form-Body k k xs ys zs k* > *0*
| *Form* (*2*k*) {*xs*,*ys*} **if** *Form-Body* (*Suc k*) *k xs ys zs k* > *0*

inductive-cases *Form-0-cases-raw*: *Form 0 u*

lemma *Form-elim-upair*:
assumes *Form l U*
obtains *xs ys* **where** *xs* ≠ *ys U* = {*xs*,*ys*} *length xs* ≤ *length ys*
⟨*proof*⟩

lemma assumes *Form-Body ka kb xs ys zs*
shows *Form-Body-WW*: *zs* ∈ *WW*
and *Form-Body-nonempty*: *length zs* > *0*
and *Form-Body-length*: *length xs* < *length ys*
⟨*proof*⟩

lemma *form-cases*:
fixes *l*::*nat*
obtains (*zero*) *l* = *0* | (*nz*) *ka kb* **where** *l* = *ka*+*kb* − *1 0* < *kb kb* ≤ *ka ka* ≤ *Suc kb*
⟨*proof*⟩

3.6.2 Interactions

lemma *interact*:
assumes *Form l U l*>*0*
obtains *ka kb xs ys zs* **where** *l* = *ka*+*kb* − *1 U* = {*xs*,*ys*} *Form-Body ka kb xs ys zs 0* < *kb kb* ≤ *ka ka* ≤ *Suc kb*
⟨*proof*⟩

definition *inter-scheme* :: *nat* ⇒ *nat list set* ⇒ *nat list*
where *inter-scheme l U* ≡
SOME zs. ∃ *k xs ys*. *U* = {*xs*,*ys*} ∧
(*l* = *2*k*−*1* ∧ *Form-Body k k xs ys zs* ∨ *l* = *2*k* ∧ *Form-Body* (*Suc k*) *k xs ys zs*)

lemma *inter-scheme*:
assumes *Form l U l>0*
obtains *ka kb xs ys* **where** *l = ka+kb − 1 U = {xs,ys} Form-Body ka kb xs ys (inter-scheme l U) 0 < kb kb ≤ ka ka ≤ Suc kb*
⟨*proof*⟩

lemma *inter-scheme-strict-sorted*:
assumes *Form l U l>0*
shows *strict-sorted (inter-scheme l U)*
⟨*proof*⟩

lemma *inter-scheme-simple*:
assumes *Form l U l>0*
shows *inter-scheme l U ∈ WW ∧ length (inter-scheme l U) > 0*
⟨*proof*⟩

3.6.3 Injectivity of interactions

proposition *inter-scheme-injective*:
assumes *Form l U Form l U′ l > 0* **and** *eq*: *inter-scheme l U′ = inter-scheme l U*
shows *U′ = U*
⟨*proof*⟩

lemma *strict-sorted-interact-imp-concat*:
strict-sorted (interact as bs) ⟹ strict-sorted (concat as) ∧ strict-sorted (concat bs)
⟨*proof*⟩

lemma *strict-sorted-interact-hd*:
⟦*strict-sorted (interact cs ds)*; *cs ≠ []*; *ds ≠ []*; *hd cs ≠ []*; *hd ds ≠ []*⟧
⟹ hd (hd cs) < hd (hd ds)
⟨*proof*⟩

the lengths of the two lists can differ by one

proposition *interaction-scheme-unique-aux*:
assumes *concat as = concat as′* **and** *ys′*: *concat bs = concat bs′*
and *as ∈ lists (− {[]}) bs ∈ lists (− {[]})*
and *strict-sorted (interact as bs)*
and *length bs ≤ length as length as ≤ Suc (length bs)*
and *as′ ∈ lists (− {[]}) bs′ ∈ lists (− {[]})*
and *strict-sorted (interact as′ bs′)*
and *length bs′ ≤ length as′ length as′ ≤ Suc (length bs′)*
and *length as = length as′ length bs = length bs′*
shows *as = as′ ∧ bs = bs′*
⟨*proof*⟩

proposition *Form-Body-unique*:
assumes *Form-Body ka kb xs ys zs Form-Body ka kb xs ys zs'* **and** $kb \leq ka\ ka \leq Suc\ kb$
shows $zs' = zs$
⟨*proof*⟩

lemma *Form-Body-imp-inter-scheme*:
assumes *FB*: *Form-Body ka kb xs ys zs* **and** $0 < kb\ kb \leq ka\ ka \leq Suc\ kb$
shows $zs = inter\text{-}scheme\ ((ka+kb) - Suc\ 0)\ \{xs,ys\}$
⟨*proof*⟩

3.7 For Lemma 3.8 AND PROBABLY 3.7

definition *grab* :: *nat set* ⇒ *nat* ⇒ *nat set* × *nat set*
where $grab\ N\ n \equiv (N \cap enumerate\ N\ `\ \{..<n\},\ N \cap \{enumerate\ N\ n..\})$

lemma *grab-0* [*simp*]: $grab\ N\ 0 = (\{\},\ N)$
⟨*proof*⟩

lemma *less-sets-grab*:
$infinite\ N \Longrightarrow fst\ (grab\ N\ n) \ll snd\ (grab\ N\ n)$
⟨*proof*⟩

lemma *finite-grab* [*iff*]: $finite\ (fst\ (grab\ N\ n))$
⟨*proof*⟩

lemma *card-grab* [*simp*]:
assumes *infinite N* **shows** $card\ (fst\ (grab\ N\ n)) = n$
⟨*proof*⟩

lemma *fst-grab-subset*: $fst\ (grab\ N\ n) \subseteq N$
⟨*proof*⟩

lemma *snd-grab-subset*: $snd\ (grab\ N\ n) \subseteq N$
⟨*proof*⟩

lemma *grab-Un-eq*:
assumes *infinite N* **shows** $fst\ (grab\ N\ n) \cup snd\ (grab\ N\ n) = N$
⟨*proof*⟩

lemma *finite-grab-iff* [*simp*]: $finite\ (snd\ (grab\ N\ n)) \longleftrightarrow finite\ N$
⟨*proof*⟩

lemma *grab-eqD*:
$[\![grab\ N\ n = (A,M);\ infinite\ N]\!]$
$\Longrightarrow A \ll M \wedge finite\ A \wedge card\ A = n \wedge infinite\ M \wedge A \subseteq N \wedge M \subseteq N$
⟨*proof*⟩

lemma *less-sets-fst-grab*: $A \ll N \Longrightarrow A \ll fst\ (grab\ N\ n)$
⟨*proof*⟩

Possibly redundant, given *grab*

definition *nxt* **where** $nxt \equiv \lambda N.\ \lambda n{::}nat.\ N \cap \{n{<}..\}$

lemma *infinite-nxtN*: *infinite* $N \Longrightarrow$ *infinite* $(nxt\ N\ n)$
⟨*proof*⟩

lemma *nxt-subset*: $nxt\ N\ n \subseteq N$
⟨*proof*⟩

lemma *nxt-subset-greaterThan*: $m \leq n \Longrightarrow nxt\ N\ n \subseteq \{m{<}..\}$
⟨*proof*⟩

lemma *nxt-subset-atLeast*: $m \leq n \Longrightarrow nxt\ N\ n \subseteq \{m..\}$
⟨*proof*⟩

lemma *enum-nxt-ge*: *infinite* $N \Longrightarrow a \leq enum\ (nxt\ N\ a)\ n$
⟨*proof*⟩

lemma *inj-enum-nxt*: *infinite* $N \Longrightarrow$ *inj-on* $(enum\ (nxt\ N\ a))\ A$
⟨*proof*⟩

3.8 Larson's Lemma 3.11

Again from Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω. *Annals of Mathematical Logic*, 6:129–145, 1973.

lemma *lemma-3-11*:
assumes $l > 0$
shows *thin* $(inter\text{-}scheme\ l\ `\ \{U.\ Form\ l\ U\})$
⟨*proof*⟩

3.9 Larson's Lemma 3.6

proposition *lemma-3-6*:
fixes $g :: nat\ list\ set \Rightarrow nat$
assumes *g*: $g \in [WW]^2 \to \{0{,}1\}$
obtains $N\ j$ **where** *infinite N*
and $\bigwedge k\ u.\ [\![k > 0;\ u \in [WW]^2;\ Form\ k\ u;\ [enum\ N\ k] < inter\text{-}scheme\ k\ u;\ List.set\ (inter\text{-}scheme\ k\ u) \subseteq N]\!] \Longrightarrow g\ u = j\ k$
⟨*proof*⟩

3.10 Larson's Lemma 3.7

3.10.1 Preliminaries

Analogous to *ordered-nsets-2-eq*, but without type classes

lemma *total-order-nsets-2-eq*:
assumes *tot*: *total-on A r* **and** *irr*: *irrefl r*
shows *nsets A 2* = {{*x*,*y*} | *x y*. *x* ∈ *A* ∧ *y* ∈ *A* ∧ (*x*,*y*) ∈ *r*}
(**is** - = *?rhs*)
⟨*proof*⟩

lemma *lenlex-nsets-2-eq*: *nsets A 2* = {{*x*,*y*} | *x y*. *x* ∈ *A* ∧ *y* ∈ *A* ∧ (*x*,*y*) ∈ *lenlex less-than*}
⟨*proof*⟩

lemma *sum-sorted-list-of-set-map*: *finite I* ⟹ *sum-list* (*map f* (*list-of I*)) = *sum f I*
⟨*proof*⟩

lemma *sorted-list-of-set-UN-eq-concat*:
assumes *I*: *strict-mono-sets I f finite I* **and** *fin*: ⋀*i*. *finite* (*f i*)
shows *list-of* (⋃ *i* ∈ *I*. *f i*) = *concat* (*map* (*list-of* ∘ *f*) (*list-of I*))
⟨*proof*⟩

3.10.2 Lemma 3.7 of Jean A. Larson, ibid.

proposition *lemma-3-7*:
assumes *infinite N l* > *0*
obtains *M* **where** *M* ∈ [*WW*]^m
⋀*U*. *U* ∈ [*M*]^2 ⟹ *Form l U* ∧ *List.set* (*inter-scheme l U*) ⊆ *N*
⟨*proof*⟩

3.11 Larson's Lemma 3.8

3.11.1 Primitives needed for the inductive construction of *b*

definition *IJ* **where** *IJ* ≡ λ*k*. *Sigma* {..*k*} (λ*j*::*nat*. {..<*j*})

lemma *IJ-iff*: *u* ∈ *IJ k* ⟷ (∃ *j i*. *u* = (*j*,*i*) ∧ *i*<*j* ∧ *j*≤*k*)
⟨*proof*⟩

lemma *finite-IJ*: *finite* (*IJ k*)
⟨*proof*⟩

fun *prev* **where**
prev 0 0 = *None*
| *prev* (*Suc 0*) *0* = *Some* (*j*, *j* − *Suc 0*)
| *prev j* (*Suc i*) = *Some* (*j*,*i*)

lemma *prev-eq-None-iff*: *prev j i* = *None* ⟷ *j* ≤ *Suc 0* ∧ *i* = *0*
⟨*proof*⟩

lemma *prev-pair-less*:

prev j i = Some ji′ ⟹ (ji′, (j,i)) ∈ pair-less
⟨*proof*⟩

lemma *prev-Some-less*: ⟦*prev j i = Some (j′,i′); i ≤ j*⟧ ⟹ *i′ < j′*
⟨*proof*⟩

lemma *prev-maximal*:
⟦*prev j i = Some (j′,i′); (ji″, (j,i)) ∈ pair-less; ji″ ∈ IJ k*⟧
⟹ *(ji″, (j′,i′)) ∈ pair-less ∨ ji″ = (j′,i′)*
⟨*proof*⟩

lemma *pair-less-prev*:
assumes *(u, (j,i)) ∈ pair-less u ∈ IJ k*
shows *prev j i = Some u ∨ (∃ x. (u, x) ∈ pair-less ∧ prev j i = Some x)*
⟨*proof*⟩

3.11.2 Special primitives for the ordertype proof

definition *USigma* :: *'a set set ⇒ ('a set ⇒ 'a set) ⇒ 'a set set*
where *USigma 𝒜 B ≡ ⋃X∈𝒜. ⋃y ∈ B X. {insert y X}*

definition *usplit*
where *usplit f A ≡ f (A − {Max A}) (Max A)*

lemma *USigma-empty* [*simp*]: *USigma {} B = {}*
⟨*proof*⟩

lemma *USigma-iff*:
assumes *⋀I j. I ∈ ℐ ⟹ I ≪ J I ∧ finite I*
shows *x ∈ USigma ℐ J ⟷ usplit (λI j. I ∈ ℐ ∧ j ∈ J I ∧ x = insert j I) x*
⟨*proof*⟩

proposition *ordertype-append-image-IJ*:
assumes *lenB* [*simp*]: *⋀i j. i ∈ ℐ ⟹ j ∈ J i ⟹ length (B j) = c*
and *AB*: *⋀i j. i ∈ ℐ ⟹ j ∈ J i ⟹ A i < B j*
and *IJ*: *⋀i. i ∈ ℐ ⟹ i ≪ J i ∧ finite i*
and *β*: *⋀i. i ∈ ℐ ⟹ ordertype (B ‘ J i) (lenlex less-than) = β*
and *A*: *inj-on A ℐ*
shows *ordertype (usplit (λi j. A i @ B j) ‘ USigma ℐ J) (lenlex less-than)*
= β ∗ ordertype (A ‘ ℐ) (lenlex less-than)
(**is** *ordertype ?AB ?R = - ∗ ?α*)
⟨*proof*⟩

3.11.3 The final part of 3.8, where two sequences are merged

inductive *merge* :: *[nat list list,nat list list,nat list list,nat list list] ⇒ bool*
where *NullNull*: *merge [] [] [] []*
| *Null*: *as ≠ [] ⟹ merge as [] [concat as] []*
| *App*: *⟦as1 ≠ []; bs1 ≠ [];*

concat as1 < concat bs1; *concat bs1 < concat as2*; *merge as2 bs2 as bs*⟧
⟹ *merge* (*as1*@*as2*) (*bs1*@*bs2*) (*concat as1* # *as*) (*concat bs1* # *bs*)

inductive-simps *Null1* [*simp*]: *merge* [] *bs us vs*
inductive-simps *Null2* [*simp*]: *merge as* [] *us vs*

lemma *merge-single*:
⟦*concat as < concat bs*; *concat as* ≠ []; *concat bs* ≠ []⟧ ⟹ *merge as bs* [*concat as*] [*concat bs*]
⟨*proof*⟩

lemma *merge-length1-nonempty*:
assumes *merge as bs us vs as* ∈ *lists* (− {[]})
shows *us* ∈ *lists* (− {[]})
⟨*proof*⟩

lemma *merge-length2-nonempty*:
assumes *merge as bs us vs bs* ∈ *lists* (− {[]})
shows *vs* ∈ *lists* (− {[]})
⟨*proof*⟩

lemma *merge-length1-gt-0*:
assumes *merge as bs us vs as* ≠ []
shows *length us* > *0*
⟨*proof*⟩

lemma *merge-length-le*:
assumes *merge as bs us vs*
shows *length vs* ≤ *length us*
⟨*proof*⟩

lemma *merge-length-le-Suc*:
assumes *merge as bs us vs*
shows *length us* ≤ *Suc* (*length vs*)
⟨*proof*⟩

lemma *merge-length-less2*:
assumes *merge as bs us vs*
shows *length vs* ≤ *length as*
⟨*proof*⟩

lemma *merge-preserves*:
assumes *merge as bs us vs*
shows *concat as* = *concat us* ∧ *concat bs* = *concat vs*
⟨*proof*⟩

lemma *merge-interact*:
assumes *merge as bs us vs strict-sorted* (*concat as*) *strict-sorted* (*concat bs*)

bs ∈ *lists* (− {[]})
shows *strict-sorted* (*interact us vs*)
⟨*proof*⟩

lemma *acc-lengths-merge1*:
assumes *merge as bs us vs*
shows *list.set* (*acc-lengths k us*) ⊆ *list.set* (*acc-lengths k as*)
⟨*proof*⟩

lemma *acc-lengths-merge2*:
assumes *merge as bs us vs*
shows *list.set* (*acc-lengths k vs*) ⊆ *list.set* (*acc-lengths k bs*)
⟨*proof*⟩

lemma *length-hd-le-concat*:
assumes *as* ≠ [] **shows** *length* (*hd as*) ≤ *length* (*concat as*)
⟨*proof*⟩

lemma *length-hd-merge2*:
assumes *merge as bs us vs*
shows *length* (*hd bs*) ≤ *length* (*hd vs*)
⟨*proof*⟩

lemma *merge-less-sets-hd*:
assumes *merge as bs us vs strict-sorted* (*concat as*) *strict-sorted* (*concat bs*) *bs* ∈ *lists* (− {[]})
shows *list.set* (*hd us*) ≪ *list.set* (*concat vs*)
⟨*proof*⟩

lemma *set-takeWhile*:
assumes *strict-sorted* (*concat as*) *as* ∈ *lists* (− {[]})
shows *list.set* (*takeWhile* (λ*x*. *x* < *y*) *as*) = {*x* ∈ *list.set as*. *x* < *y*}
⟨*proof*⟩

proposition *merge-exists*:
assumes *strict-sorted* (*concat as*) *strict-sorted* (*concat bs*)
as ∈ *lists* (− {[]}) *bs* ∈ *lists* (− {[]})
hd as < *hd bs as* ≠ [] *bs* ≠ []
and *disj*: ⋀*a b*. ⟦*a* ∈ *list.set as*; *b* ∈ *list.set bs*⟧ ⟹ *a*<*b* ∨ *b*<*a*
shows ∃ *us vs*. *merge as bs us vs*
⟨*proof*⟩

3.11.4 Actual proof of Larson's Lemma 3.8

proposition *lemma-3-8*:
assumes *infinite N*
obtains *X* **where** *X* ⊆ *WW ordertype X* (*lenlex less-than*) = ω↑ω
⋀*u*. *u* ∈ [*X*]^2 ⟹

$\exists\, l.\ Form\ l\ u \wedge (l > 0 \longrightarrow [enum\ N\ l] < inter\text{-}scheme\ l\ u \wedge List.set\ (inter\text{-}scheme\ l\ u) \subseteq N)$
⟨*proof*⟩

3.12 The main partition theorem for $\omega \uparrow \omega$

definition *iso-ll* **where** *iso-ll A B* ≡ *iso* (*lenlex less-than* ∩ (*A*×*A*)) (*lenlex less-than* ∩ (*B*×*B*))

corollary *ordertype-eq-ordertype-iso-ll*:
  assumes *Field* (*Restr* (*lenlex less-than*) *A*) = *A Field* (*Restr* (*lenlex less-than*) *B*) = *B*
  shows (*ordertype A* (*lenlex less-than*) = *ordertype B* (*lenlex less-than*))
      ⟷ (∃*f*. *iso-ll A B f*)
⟨*proof*⟩

theorem *partition-ωω-aux*:
  assumes α ∈ *elts* ω
  shows *partn-lst* (*lenlex less-than*) *WW* [ω↑ω,α] *2* (**is** *partn-lst ?R WW* [ω↑ω,α] *2*)
⟨*proof*⟩

Theorem 3.1 of Jean A. Larson, ibid.

theorem *partition-ωω*: α ∈ *elts* ω ⟹ *partn-lst-VWF* (ω↑ω) [ω↑ω,α] *2*
  ⟨*proof*⟩

end

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