

A Partition Theorem for the Ordinal ω^ω

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Abstract

The theory of partition relations concerns generalisations of Ramsey's theorem. For any ordinal α , write $\alpha \rightarrow (\alpha, m)^2$ if for each function f from unordered pairs of elements of α into $\{0, 1\}$, either there is a subset $X \subseteq \alpha$ order-isomorphic to α such that $f\{x, y\} = 0$ for all $\{x, y\} \subseteq X$, or there is an m element set $Y \subseteq \alpha$ such that $f\{x, y\} = 1$ for all $\{x, y\} \subseteq Y$. (In both cases, with $\{x, y\}$ we require $x \neq y$.) In particular, the infinite Ramsey theorem can be written in this notation as $\omega \rightarrow (\omega, \omega)^2$, or if we restrict m to the positive integers as above, then $\omega \rightarrow (\omega, m)^2$ for all m [3].

This entry formalises Larson's proof of $\omega^\omega \rightarrow (\omega^\omega, m)^2$ along with a similar proof of a result due to Specker: $\omega^2 \rightarrow (\omega^2, m)^2$. Also proved is a necessary result by Erdős and Milner [1, 2]: $\omega^{1+\alpha \cdot n} \rightarrow (\omega^{1+\alpha}, 2^n)^2$.

These examples demonstrate the use of Isabelle/HOL to formalise advanced results that combine ZF set theory with basic concepts like lists and natural numbers.

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1 Library additions

```

theory Library-Additions
  imports ZFC-in-HOL.Ordinal-Exp HOL-Library.Ramsey Nash-Williams.Nash-Williams

begin

lemma finite-enumerate-Diff-singleton:
  fixes  $S :: 'a::wellorder\ set$ 
  assumes finite S and  $i: i < card\ S\ enumerate\ S\ i < x$ 
  shows enumerate (S - {x}) i = enumerate S i
  using  $i$ 
proof (induction i)
  case 0
  have  $(LEAST\ i.\ i \in S \wedge i \neq x) = (LEAST\ i.\ i \in S)$ 
  proof (rule Least-equality)
    have  $\exists t. t \in S \wedge t \neq x$ 
      using 0  $\langle finite\ S \rangle$  finite-enumerate-in-set by blast
    then show  $(LEAST\ i.\ i \in S) \in S \wedge (LEAST\ i.\ i \in S) \neq x$ 
      by (metis 0.prem(2) LeastI enumerate-0 not-less-Least)
  qed (simp add: Least-le)
then show ?case
  by (auto simp: enumerate-0)

```

next
case (*Suc i*)
then have *x*: *enumerate S i < x*
by (*meson enumerate-step finite-enumerate-step less-trans*)
have *cardSx*: *Suc i < card (S - {x})* **and** *i < card S*
using *Suc* \langle *finite S* \rangle *card-Diff-singleton-if*[*of S*] *finite-enumerate-Ex* **by** *fast-force+*
have (*LEAST s. s ∈ S ∧ s ≠ x ∧ enumerate (S - {x}) i < s*) = (*LEAST s. s ∈ S ∧ enumerate S i < s*)
(is - = ?r)
proof (*intro Least-equality conjI*)
show *?r ∈ S*
by (*metis (lifting) LeastI Suc.prem(1) assms(1) finite-enumerate-in-set finite-enumerate-step*)
show *?r ≠ x*
using *Suc.prem not-less-Least* [*of - λt. t ∈ S ∧ enumerate S i < t*]
 \langle *finite S* \rangle *finite-enumerate-in-set finite-enumerate-step* **by** *blast*
show *enumerate (S - {x}) i < ?r*
by (*metis (full-types) Suc.IH Suc.prem(1) <i < card S> enumerate-Suc''*
enumerate-step finite-enumerate-Suc'' finite-enumerate-step x)
show $\bigwedge y. y ∈ S ∧ y ≠ x ∧ enumerate (S - {x}) i < y \implies ?r \leq y$
by (*simp add: Least-le Suc.IH <i < card S> x*)
qed
then show *?case*
using *Suc assms* **by** (*simp add: finite-enumerate-Suc'' cardSx*)
qed

lemma *hd-lex*: $\llbracket hd\ ms < hd\ ns; length\ ms = length\ ns; ns \neq [] \rrbracket \implies (ms, ns) \in lex\ less-than$
by (*metis hd-Cons-tl length-0-conv less-than-iff lexord-cons-cons lexord-lex*)

lemma *sorted-hd-le*:
assumes *sorted xs x ∈ list.set xs*
shows *hd xs ≤ x*
using *assms* **by** (*induction xs*) (*auto simp: less-imp-le*)

lemma *sorted-le-last*:
assumes *sorted xs x ∈ list.set xs*
shows *x ≤ last xs*
using *assms* **by** (*induction xs*) (*auto simp: less-imp-le*)

lemma *hd-list-of*:
assumes *finite A A ≠ {}*
shows *hd (sorted-list-of-set A) = Min A*
proof (*rule antisym*)
have *Min A ∈ A*
by (*simp add: assms*)
then show *hd (sorted-list-of-set A) ≤ Min A*
by (*simp add: sorted-hd-le <finite A>*)

next
show $Min\ A \leq hd\ (sorted-list-of-set\ A)$
by (*metis Min-le assms hd-in-set set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff*)
qed

lemma *sorted-hd-le-last*:
assumes *sorted xs xs \neq []*
shows $hd\ xs \leq last\ xs$
using *assms* **by** (*simp add: sorted-hd-le*)

lemma *sorted-list-of-set-set-of [simp]*: $strict-sorted\ l \implies sorted-list-of-set\ (list.set\ l) = l$
by (*simp add: strict-sorted-equal*)

lemma *range-strict-mono-ext*:
fixes $f::nat \Rightarrow 'a::linorder$
assumes $eq: range\ f = range\ g$
and $sm: strict-mono\ f\ strict-mono\ g$
shows $f = g$

proof
fix n
show $f\ n = g\ n$
proof (*induction n rule: less-induct*)
case (*less n*)
obtain $x\ y$ **where** $xy: f\ n = g\ y\ f\ x = g\ n$
by (*metis eq imageE rangeI*)
then have $n = y$
by (*metis (no-types) less.IH neq-iff sm strict-mono-less xy*)
then show *?case* **using** xy **by** *auto*
qed
qed

1.1 Other material

definition *strict-mono-sets* :: $['a::order\ set, 'a::order \Rightarrow 'b::order\ set] \Rightarrow bool$ **where**
 $strict-mono-sets\ A\ f \equiv \forall x \in A. \forall y \in A. x < y \longrightarrow less-sets\ (f\ x)\ (f\ y)$

lemma *strict-mono-setsD*:
assumes $strict-mono-sets\ A\ f\ x < y\ x \in A\ y \in A$
shows $less-sets\ (f\ x)\ (f\ y)$
using *assms* **by** (*auto simp: strict-mono-sets-def*)

lemma *strict-mono-sets-imp-disjoint*:
fixes $A :: 'a::linorder\ set$
assumes $strict-mono-sets\ A\ f$
shows $pairwise\ (\lambda x\ y. disjoint\ (f\ x)\ (f\ y))\ A$
using *assms* **unfolding** *strict-mono-sets-def pairwise-def*
by (*meson antisym-conv3 disjoint-sym less-sets-imp-disjnt*)

lemma *strict-mono-sets-subset*:

assumes *strict-mono-sets* $B f A \subseteq B$

shows *strict-mono-sets* $A f$

using *assms* **by** (*auto simp: strict-mono-sets-def*)

lemma *strict-mono-less-sets-Min*:

assumes *strict-mono-sets* $I f$ *finite* $I I \neq \{\}$

shows *less-sets* $(f (Min I)) (\cup (f ' (I - \{Min I\})))$

using *assms* **by** (*simp add: strict-mono-sets-def less-sets-UN2 dual-order.strict-iff-order*)

lemma *pair-less-iff1* [*simp*]: $((x,y), (x,z)) \in \text{pair-less} \longleftrightarrow y < z$

by (*simp add: pair-less-def*)

lemma *infinite-finite-Inter*:

assumes *finite* $\mathcal{A} \mathcal{A} \neq \{\}$ $\bigwedge A. A \in \mathcal{A} \implies \text{infinite } A$

and $\bigwedge A B. \llbracket A \in \mathcal{A}; B \in \mathcal{A} \rrbracket \implies A \cap B \in \mathcal{A}$

shows *infinite* $(\bigcap \mathcal{A})$

by (*simp add: assms finite-Inf-in*)

lemma *atLeast-less-sets*: $\llbracket \text{less-sets } A \{x\}; B \subseteq \{x..\} \rrbracket \implies \text{less-sets } A B$

by (*force simp: less-sets-def subset-iff*)

1.2 The list-of function

lemma *sorted-list-of-set-insert-remove-cons*:

assumes *finite* A *less-sets* $\{a\} A$

shows *sorted-list-of-set* $(\text{insert } a A) = a \# \text{sorted-list-of-set } A$

proof –

have *strict-sorted* $(a \# \text{sorted-list-of-set } A)$

using *assms less-setsD* **by** *auto*

moreover **have** *list.set* $(a \# \text{sorted-list-of-set } A) = \text{insert } a A$

using *assms* **by** *force*

moreover **have** *length* $(a \# \text{sorted-list-of-set } A) = \text{card } (\text{insert } a A)$

using *assms card-insert-if less-setsD* **by** *fastforce*

ultimately **show** *?thesis*

by (*metis* $\langle \text{finite } A \rangle$ *finite-insert sorted-list-of-set-unique*)

qed

lemma *sorted-list-of-set-Un*:

assumes AB : *less-sets* $A B$ **and** *fin*: *finite* A *finite* B

shows *sorted-list-of-set* $(A \cup B) = \text{sorted-list-of-set } A @ \text{sorted-list-of-set } B$

proof –

have *strict-sorted* $(\text{sorted-list-of-set } A @ \text{sorted-list-of-set } B)$

using AB **unfolding** *less-sets-def*

by (*metis* *fin set-sorted-list-of-set sorted-wrt-append strict-sorted-list-of-set*)

moreover **have** *card* $A + \text{card } B = \text{card } (A \cup B)$

using *less-sets-imp-disjnt* [$OF AB$]

by (*simp add: assms card-Un-disjoint disjnt-def*)

ultimately **show** *?thesis*

by (*simp add: assms strict-sorted-equal*)
qed

lemma *sorted-list-of-set-UN-lessThan*:

fixes $k::nat$
assumes $sm: strict\ mono\ sets \{..<k\} A$ and $\bigwedge i. i < k \implies finite (A i)$
shows $sorted\ list\ of\ set (\bigcup_{i<k}. A i) = concat (map (sorted\ list\ of\ set \circ A) (sorted\ list\ of\ set \{..<k\}))$
using *assms*
proof (*induction k*)
case 0
then show ?*case*
by *auto*
next
case (*Suc k*)
have $ls: less\ sets (\bigcup (A ' \{..<k\})) (A k)$
using $sm\ Suc.prem\ s(1)\ strict\ mono\ setsD$ by (*force simp: less-sets-UN1*)
have $sorted\ list\ of\ set (\bigcup (A ' \{..<Suc\ k\})) = sorted\ list\ of\ set (\bigcup (A ' \{..<k\}) \cup A k)$
by (*simp add: Un-commute lessThan-Suc*)
also have $\dots = sorted\ list\ of\ set (\bigcup (A ' \{..<k\})) @ sorted\ list\ of\ set (A k)$
by (*rule sorted-list-of-set-Un (auto simp: Suc.prem\ s ls)*)
also have $\dots = concat (map (sorted\ list\ of\ set \circ A) (sorted\ list\ of\ set \{..<k\})) @ sorted\ list\ of\ set (A k)$
using $Suc\ strict\ mono\ sets\ def$ by *fastforce*
also have $\dots = concat (map (sorted\ list\ of\ set \circ A) (sorted\ list\ of\ set \{..<Suc\ k\}))$
using $strict\ mono\ sets\ def$ by *fastforce*
finally show ?*case* .
qed

lemma *sorted-list-of-set-UN-atMost*:

fixes $k::nat$
assumes $strict\ mono\ sets \{..k\} A$ and $\bigwedge i. i \leq k \implies finite (A i)$
shows $sorted\ list\ of\ set (\bigcup_{i\leq k}. A i) = concat (map (sorted\ list\ of\ set \circ A) (sorted\ list\ of\ set \{..k\}))$
by (*metis assms lessThan-Suc-atMost less-Suc-eq-le sorted-list-of-set-UN-lessThan*)

1.3 Monotonic enumeration of a countably infinite set

abbreviation $enum \equiv enumerate$

Could be generalised to infinite countable sets of any type

lemma *nat-infinite-iff*:

fixes $N :: nat\ set$
shows $infinite\ N \iff (\exists f::nat \Rightarrow nat. N = range\ f \wedge strict\ mono\ f)$
proof *safe*
assume $infinite\ N$
then show $\exists f. N = range (f::nat \Rightarrow nat) \wedge strict\ mono\ f$
by (*metis bij-betw-imp-surj-on bij-enumerate enumerate-mono strict-mono-def*)

```

next
  fix  $f :: \text{nat} \Rightarrow \text{nat}$ 
  assume  $\text{strict-mono } f$  and  $N = \text{range } f$  and  $\text{finite } (\text{range } f)$ 
  then show  $\text{False}$ 
    using  $\text{range-inj-infinite strict-mono-imp-inj-on}$  by  $\text{blast}$ 
qed

lemma  $\text{enum-works}$ :
  fixes  $N :: \text{nat set}$ 
  assumes  $\text{infinite } N$ 
  shows  $N = \text{range } (\text{enum } N) \wedge \text{strict-mono } (\text{enum } N)$ 
  by  $(\text{metis } \text{assms } \text{bij-betw-imp-surj-on } \text{bij-enumerate } \text{enumerate-mono } \text{strict-monoI})$ 

lemma  $\text{range-enum}$ :  $\text{range } (\text{enum } N) = N$  and  $\text{strict-mono-enum}$ :  $\text{strict-mono } (\text{enum } N)$ 
  if  $\text{infinite } N$  for  $N :: \text{nat set}$ 
  using  $\text{enum-works [OF that]}$  by  $\text{auto}$ 

lemma  $\text{enum-0-eq-Inf}$ :
  fixes  $N :: \text{nat set}$ 
  assumes  $\text{infinite } N$ 
  shows  $\text{enum } N \ 0 = \text{Inf } N$ 
proof –
  have  $\text{enum } N \ 0 \in N$ 
    using  $\text{assms } \text{range-enum}$  by  $\text{auto}$ 
  moreover have  $\bigwedge x. x \in N \implies \text{enum } N \ 0 \leq x$ 
    by  $(\text{metis } (\text{mono-tags}, \text{opaque-lifting}) \text{assms } \text{imageE } \text{le0 } \text{less-mono-imp-le-mono } \text{range-enum } \text{strict-monoD } \text{strict-mono-enum})$ 
  ultimately show  $?thesis$ 
    by  $(\text{metis } \text{cInf-eq-minimum})$ 
qed

lemma  $\text{enum-works-finite}$ :
  fixes  $N :: \text{nat set}$ 
  assumes  $\text{finite } N$ 
  shows  $N = \text{enum } N \ ' \ \{..<\text{card } N\} \wedge \text{strict-mono-on } \{..<\text{card } N\} (\text{enum } N)$ 
  using  $\text{assms}$ 
  by  $(\text{metis } \text{bij-betw-imp-surj-on } \text{finite-bij-enumerate } \text{finite-enumerate-mono } \text{lessThan-iff } \text{strict-mono-onI})$ 

lemma  $\text{enum-obtain-index-finite}$ :
  fixes  $N :: \text{nat set}$ 
  assumes  $x \in N$   $\text{finite } N$ 
  obtains  $i$  where  $i < \text{card } N$   $x = \text{enum } N \ i$ 
  by  $(\text{metis } \langle x \in N \rangle \langle \text{finite } N \rangle \text{enum-works-finite } \text{imageE } \text{lessThan-iff})$ 

lemma  $\text{enum-0-eq-Inf-finite}$ :
  fixes  $N :: \text{nat set}$ 
  assumes  $\text{finite } N$   $N \neq \{\}$ 

```

shows $\text{enum } N \ 0 = \text{Inf } N$
proof –
have $\text{enum } N \ 0 \in N$
by (*metis* *Nat.neq0-conv* *assms* *empty-is-image* *enum-works-finite* *image-eqI* *lessThan-empty-iff* *lessThan-iff*)
moreover have $\text{enum } N \ 0 \leq x$ **if** $x \in N$ **for** x
proof –
obtain i **where** $i < \text{card } N$ $x = \text{enum } N \ i$
by (*metis* $\langle x \in N \rangle$ $\langle \text{finite } N \rangle$ *enum-obtain-index-finite*)
with *assms* **show** *?thesis*
by (*metis* *Nat.neq0-conv* *finite-enumerate-mono* *less-or-eq-imp-le*)
qed
ultimately show *?thesis*
by (*metis* *cInf-eq-minimum*)
qed

lemma *greaterThan-less-enum*:
fixes $N :: \text{nat set}$
assumes $N \subseteq \{x < ..\}$ *infinite* N
shows $x < \text{enum } N \ i$
using *assms* *range-enum* **by** *fastforce*

lemma *atLeast-le-enum*:
fixes $N :: \text{nat set}$
assumes $N \subseteq \{x ..\}$ *infinite* N
shows $x \leq \text{enum } N \ i$
using *assms* *range-enum* **by** *fastforce*

lemma *less-sets-empty1* [*simp*]: *less-sets* $\{\}$ A **and** *less-sets-empty2* [*simp*]: *less-sets* $A \ \{\}$
by (*simp-all* *add*: *less-sets-def*)

lemma *less-sets-singleton1* [*simp*]: *less-sets* $\{a\}$ $A \longleftrightarrow (\forall x \in A. a < x)$
and *less-sets-singleton2* [*simp*]: *less-sets* $A \ \{a\} \longleftrightarrow (\forall x \in A. x < a)$
by (*simp-all* *add*: *less-sets-def*)

lemma *less-sets-atMost* [*simp*]: *less-sets* $\{..a\}$ $A \longleftrightarrow (\forall x \in A. a < x)$
and *less-sets-atLeast* [*simp*]: *less-sets* $A \ \{a.. \}$ $\longleftrightarrow (\forall x \in A. x < a)$
by (*auto simp*: *less-sets-def*)

lemma *less-sets-imp-strict-mono-sets*:
assumes $\bigwedge i. \text{less-sets } (A \ i) \ (A \ (\text{Suc } i)) \ \bigwedge i. i > 0 \implies A \ i \neq \{\}$
shows *strict-mono-sets* *UNIV* A
proof (*clarsimp simp*: *strict-mono-sets-def*)
fix $i \ j :: \text{nat}$
assume $i < j$
then show *less-sets* $(A \ i) \ (A \ j)$
proof (*induction* $j - i$ *arbitrary*: $i \ j$)
case (*Suc* x)


```

    then show ?case
      by (metis Suc-diff-Suc Suc-inject Suc-mono assms less-Suc-eq less-sets-trans
zero-less-Suc)
    qed auto
  qed

```

```

lemma less-sets-Suc-Max:
  assumes finite A
  shows less-sets A {Suc (Max A)..}
proof (cases A = {})
  case False
  then show ?thesis
    by (simp add: assms less-Suc-eq-le)
qed auto

```

```

lemma infinite-nat-greaterThan:
  fixes m::nat
  assumes infinite N
  shows infinite (N ∩ {m<..})
proof -
  have N ⊆ -{m<..} ∪ (N ∩ {m<..})
    by blast
  moreover have finite (-{m<..})
    by simp
  ultimately show ?thesis
    using assms finite-subset by blast
qed

```

end

2 Ordinal Partitions

Material from Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω . *Annals of Mathematical Logic*, 6:129–145, 1973. Also from “Partition Relations” by A. Hajnal and J. A. Larson, in *Handbook of Set Theory*, edited by Matthew Foreman and Akihiro Kanamori (Springer, 2010).

```

theory Partitions
  imports Library-Additions ZFC-in-HOL.ZFC-Typeclasses ZFC-in-HOL.Cantor-NF

```

```
begin
```

```

abbreviation tp :: V set ⇒ V
  where tp A ≡ ordertype A VWF

```

2.1 Ordinal Partitions: Definitions

```

definition partn-1st :: [('a × 'a) set, 'a set, V list, nat] ⇒ bool
  where partn-1st r B α n ≡ ∀ f ∈ [B]n → {..length α}.

```

$\exists i < \text{length } \alpha. \exists H. H \subseteq B \wedge \text{ordertype } H r = (\alpha!i) \wedge f' (nsets H n) \subseteq \{i\}$

abbreviation *partn-lst-VWF* :: $V \Rightarrow V \text{ list} \Rightarrow \text{nat} \Rightarrow \text{bool}$
where *partn-lst-VWF* $\beta \equiv \text{partn-lst VWF } (\text{elts } \beta)$

lemma *partn-lst-E*:

assumes *partn-lst* $r B \alpha n f \in nsets B n \rightarrow \{..l\} l = \text{length } \alpha$
obtains $i H$ **where** $i < l H \subseteq B$
 $\text{ordertype } H r = \alpha!i f' (nsets H n) \subseteq \{i\}$
using *assms* **by** (*auto simp: partn-lst-def*)

lemma *partn-lst-VWF-nontriv*:

assumes *partn-lst-VWF* $\beta \alpha n l = \text{length } \alpha \text{ Ord } \beta l > 0$
obtains i **where** $i < l \alpha!i \leq \beta$

proof –

have $\{..l\} \neq \{\}$
by (*simp add: <l > 0 lessThan-empty-iff*)
then obtain f **where** $f \in nsets (\text{elts } \beta) n \rightarrow \{..l\}$
by (*meson Pi-eq-empty equals0I*)
then obtain $i H$ **where** $i < l H \subseteq \text{elts } \beta$ **and** *eq: tp* $H = \alpha!i$
using *assms* **by** (*metis partn-lst-E*)
then have $\alpha!i \leq \beta$
by (*metis <H ⊆ elts β> <Ord β> eq ordertype-le-Ord*)
then show *thesis*
using $\langle i < l \rangle$ **that** **by** *auto*

qed

lemma *partn-lst-triv0*:

assumes $\alpha!i = 0 i < \text{length } \alpha n \neq 0$
shows *partn-lst* $r B \alpha n$
by (*metis partn-lst-def assms bot-least image-empty nsets-empty-iff ordertype-empty*)

lemma *partn-lst-triv1*:

assumes $\alpha!i \leq 1 i < \text{length } \alpha n > 1 B \neq \{\}$ *wf* r
shows *partn-lst* $r B \alpha n$
unfolding *partn-lst-def*

proof *clarsimp*

obtain γ **where** $\gamma \in B \alpha \neq []$
using *assms mem-0-Ord* **by** *fastforce*
have *01*: $\alpha!i = 0 \vee \alpha!i = 1$
using *assms* **by** (*fastforce simp: one-V-def*)
fix f
assume $f: f \in [B]^n \rightarrow \{..length } \alpha\}$
with *assms* **have** $\text{ordertype } \{\gamma\} r = 1 \wedge f' [\{\gamma\}]^n \subseteq \{i\}$
 $\text{ordertype } \{\} r = 0 \wedge f' [\{\}]^n \subseteq \{i\}$
by (*auto simp: one-V-def ordertype-insert nsets-eq-empty*)
with *assms 01* **show** $\exists i < \text{length } \alpha. \exists H \subseteq B. \text{ordertype } H r = \alpha!i \wedge f' [H]^n \subseteq \{i\}$

using $\langle \gamma \in B \rangle$ by auto
qed

lemma *partn-lst-two-swap*:

assumes *partn-lst* r B $[x,y]$ n shows *partn-lst* r B $[y,x]$ n

proof –

{ **fix** $f :: 'a \text{ set} \Rightarrow \text{nat}$
assume $f: f \in [B]^n \rightarrow \{..<2\}$
then have $f': (\lambda i. 1 - i) \circ f \in [B]^n \rightarrow \{..<2\}$
by (*auto simp: Pi-def*)
obtain i H **where** $i < 2$ $H \subseteq B$ *ordertype* H $r = ([x,y]!i) ((\lambda i. 1 - i) \circ f)'$
 $(nsets\ H\ n) \subseteq \{i\}$
by (*auto intro: partn-lst-E [OF assms f']*)
moreover have $f\ x = Suc\ 0$ **if** $Suc\ 0 \leq f\ x$ $x \in [H]^n$ **for** x
using f **that** $\langle H \subseteq B \rangle$ *nsets-mono* **by** (*fastforce simp: Pi-iff*)
ultimately have *ordertype* H $r = [y,x]! (1-i) \wedge f' [H]^n \subseteq \{1-i\}$
by (*force simp: eval-nat-numeral less-Suc-eq*)
then have $\exists i$ $H. i < 2 \wedge H \subseteq B \wedge$ *ordertype* H $r = [y,x]! i \wedge f' [H]^n \subseteq \{i\}$
by (*metis Suc-1 <H ⊆ B> diff-less-Suc*) }
then show *?thesis*
by (*auto simp: partn-lst-def eval-nat-numeral*)
qed

lemma *partn-lst-greater-resource*:

assumes $M: \text{partn-lst } r\ B\ \alpha\ n$ **and** $B \subseteq C$

shows *partn-lst* r C α n

proof (*clarsimp simp: partn-lst-def*)

fix f

assume $f \in [C]^n \rightarrow \{..<\text{length } \alpha\}$

then have $f \in [B]^n \rightarrow \{..<\text{length } \alpha\}$

by (*metis <B ⊆ C> part-fn-def part-fn-subset*)

then obtain i H **where** $i < \text{length } \alpha$

and $H \subseteq B$ *ordertype* H $r = (\alpha!i)$

and $f' nsets\ H\ n \subseteq \{i\}$

using M *partn-lst-def* **by** *metis*

then show $\exists i < \text{length } \alpha. \exists H \subseteq C. \text{ordertype } H\ r = \alpha! i \wedge f' [H]^n \subseteq \{i\}$

using $\langle B \subseteq C \rangle$ **by** *blast*

qed

lemma *partn-lst-less*:

assumes $M: \text{partn-lst } r\ B\ \alpha\ n$ **and** $eq: \text{length } \alpha' = \text{length } \alpha$ **and** $List.set\ \alpha' \subseteq ON$

and $le: \bigwedge i. i < \text{length } \alpha \implies \alpha!i \leq \alpha'!i$

and $r: wf\ r\ \text{trans } r\ \text{total-on } B\ r$ **and** *small B*

shows *partn-lst* r B α' n

proof (*clarsimp simp: partn-lst-def*)

fix f

assume $f \in [B]^n \rightarrow \{..<\text{length } \alpha'\}$

then obtain $i \in H$ **where** $i < \text{length } \alpha$
and $H \subseteq B$ **small** H **and** $H: \text{ordertype } H \ r = (\alpha!i)$
and $f: \text{‘ } n\text{-sets } H \ n \subseteq \{i\}$
using *assms* **by** (*auto simp: partn-lst-def smaller-than-small*)
then have $\text{bij: } \text{bij-betw } (\text{ordermap } H \ r) \ H \ (\text{elts } (\alpha!i))$
using *ordermap-bij [of r H] r*
by (*smt <small B> in-mono smaller-than-small total-on-def*)
define H' **where** $H' = \text{inv-into } H \ (\text{ordermap } H \ r) \ \text{‘ } (\text{elts } (\alpha!i))$
have $H' \subseteq H$
using $\text{bij } \langle i < \text{length } \alpha \rangle \text{ bij-betw-imp-surj-on } le$
by (*force simp: H'-def image-subset-iff intro: inv-into-into*)
moreover have $ot: \text{ordertype } H' \ r = (\alpha!i)$
proof (*subst ordertype-eq-iff*)
show $\text{Ord } (\alpha!i)$
using *assms* **by** (*simp add: <i < length alpha> subset-eq*)
show *small H'*
by (*simp add: H'-def*)
show $\exists f. \text{bij-betw } f \ H' \ (\text{elts } (\alpha!i)) \wedge (\forall x \in H'. \forall y \in H'. (f \ x < f \ y) = ((x, y) \in r))$
proof (*intro exI conjI ballI*)
show $\text{bij-betw } (\text{ordermap } H \ r) \ H' \ (\text{elts } (\alpha!i))$
using $\langle H' \subseteq H \rangle$
by (*metis H'-def <i < length alpha> bij bij-betw-inv-into-RIGHT bij-betw-subset le less-eq-V-def*)
show $(\text{ordermap } H \ r \ x < \text{ordermap } H \ r \ y) = ((x, y) \in r)$
if $x \in H' \ y \in H'$ **for** $x \ y$
proof (*intro iffI ordermap-mono-less*)
assume $\text{ordermap } H \ r \ x < \text{ordermap } H \ r \ y$
then show $(x, y) \in r$
by (*metis <H subseteq B> <small H> <H' subseteq H> leD ordermap-mono-le r subsetD that total-on-def*)
qed (*use assms that <H' subseteq H> <small H> in auto*)
qed
show *total-on H' r*
using r **by** (*meson <H subseteq B> <H' subseteq H> subsetD total-on-def*)
qed (*use r in auto*)
ultimately show $\exists i < \text{length } \alpha'. \exists H \subseteq B. \text{ordertype } H \ r = \alpha!i \wedge f \ \text{‘ } [H]^n \subseteq \{i\}$
using $\langle H \subseteq B \rangle \langle i < \text{length } \alpha \rangle f$ *assms*
by (*metis image-mono nsets-mono subset-trans*)
qed

Holds because no n -sets exist!

lemma *partn-lst-VWF-degenerate*:

assumes $k < n$

shows *partn-lst-VWF* $\omega \ (\text{ord-of-nat } k \ \# \ \alpha \ s) \ n$

proof (*clarsimp simp: partn-lst-def*)

fix $f :: V \ \text{set} \Rightarrow \text{nat}$

have $[\text{elts } (\text{ord-of-nat } k)]^n = \{\}$

by (*simp add: nsets-eq-empty assms finite-Ord-omega*)

then have $f \text{ ' } [elts \text{ (ord-of-nat } k)]^n \subseteq \{0\}$
by *auto*
then show $\exists i < Suc \text{ (length } \alpha s). \exists H \subseteq elts \ \omega. tp \ H = \text{ (ord-of-nat } k \ \# \ \alpha s) ! \ i \wedge$
 $f \text{ ' } [H]^n \subseteq \{i\}$
using *assms ordertype-eq-Ord [of ord-of-nat k] elts-ord-of-nat less-Suc-eq-0-disj*
by *fastforce*
qed

lemma *partn-lst-VWF- ω -2:*

assumes *Ord* α

shows *partn-lst-VWF* $(\omega \uparrow (1+\alpha)) [2, \omega \uparrow (1+\alpha)] \ 2$ (**is** *partn-lst-VWF* $? \beta$ - -)

proof (*clarsimp simp: partn-lst-def*)

fix f

assume $f: f \in [elts \ ?\beta]^2 \rightarrow \{.. < Suc \text{ (Suc } 0)\}$

show $\exists i < Suc \text{ (Suc } 0). \exists H \subseteq elts \ ?\beta. tp \ H = [2, \ ?\beta] ! \ i \wedge f \text{ ' } [H]^2 \subseteq \{i\}$

proof (*cases* $\exists x \in elts \ ?\beta. \exists y \in elts \ ?\beta. x \neq y \wedge f\{x,y\} = 0$)

case *True*

then obtain $x \ y$ **where** $x \in elts \ ?\beta \ y \in elts \ ?\beta \ x \neq y \ f \{x, y\} = 0$

by *auto*

then have $\{x,y\} \subseteq elts \ ?\beta \ tp \ \{x,y\} = 2 \ f \text{ ' } [\{x, y\}]^2 \subseteq \{0\}$

by *auto (simp add: eval-nat-numeral ordertype-VWF-finite-nat)*

with $\langle x \neq y \rangle$ **show** *?thesis*

by (*metis nth-Cons-0 zero-less-Suc*)

next

case *False*

with f **have** $\forall x \in elts \ ?\beta. \forall y \in elts \ ?\beta. x \neq y \longrightarrow f \{x, y\} = 1$

unfolding *Pi-iff* **using** *lessThan-Suc* **by** *force*

then have $tp \ (elts \ ?\beta) = ?\beta \ f \text{ ' } [elts \ ?\beta]^2 \subseteq \{Suc \ 0\}$

by (*auto simp: assms nsets-2-eq*)

then show *?thesis*

by (*metis lessI nth-Cons-0 nth-Cons-Suc subsetI*)

qed

qed

2.2 Relating partition properties on VWF to the general case

Two very similar proofs here!

lemma *partn-lst-imp-partn-lst-VWF-eq:*

assumes *part: partn-lst* $r \ U \ \alpha \ n$ **and** $\beta: ordertype \ U \ r = \beta$ **and** *small* U

and $r: wf \ r \ trans \ r \ total-on \ U \ r$

shows *partn-lst-VWF* $\beta \ \alpha \ n$

unfolding *partn-lst-def*

proof *clarsimp*

fix f

assume $f: f \in [elts \ \beta]^n \rightarrow \{.. < length \ \alpha\}$

define cv **where** $cv \equiv \lambda X. ordermap \ U \ r \text{ ' } X$

have *bij: bij-betw* $(ordermap \ U \ r) \ U \ (elts \ \beta)$

using *ordermap-bij [of r U] assms* **by** *blast*

then have *bij-cv: bij-betw* $cv \ ([U]^n) \ ([elts \ \beta]^n)$

```

    using bij-betw-nsets cv-def by blast
  then have func:  $f \circ cv \in [U]^n \rightarrow \{..<length\ \alpha\}$  and inj-on (ordermap  $U\ r$ )  $U$ 
    using bij bij-betw-def bij-betw-apply f by fastforce+
  then have cv-part:  $\llbracket \forall x \in [X]^n. f\ (cv\ x) = i; X \subseteq U; a \in [cv\ X]^n \rrbracket \implies f\ a = i$ 
for a X i n
  by (force simp: cv-def nsets-def subset-image-iff inj-on-subset finite-image-iff card-image)
  have ot-eq [simp]:  $tp\ (cv\ X) = ordertype\ X\ r$  if  $X \subseteq U$  for  $X$ 
    unfolding cv-def
    by (meson <small U> ordertype-image-ordermap r that total-on-subset)
  obtain  $X\ i$  where  $X \subseteq U$  and  $X$ :  $ordertype\ X\ r = \alpha!i$  ( $f \circ cv$ ) ' $[X]^n \subseteq \{i\}$ '
    and  $i < length\ \alpha$ 
    using part func by (auto simp: partn-lst-def)
  show  $\exists i < length\ \alpha. \exists H \subseteq elts\ \beta. tp\ H = \alpha!i \wedge f\ '[H]^n \subseteq \{i\}$ 
  proof (intro exI conjI)
    show  $i < length\ \alpha$ 
      by (simp add: <i < length alpha>)
    show  $cv\ X \subseteq elts\ \beta$ 
      using  $\langle X \subseteq U \rangle$  bij bij-betw-imp-surj-on cv-def by blast
    show  $tp\ (cv\ X) = \alpha!i$ 
      by (simp add: X(1) <X subset U>)
    show  $f\ '[cv\ X]^n \subseteq \{i\}$ 
      using  $X\ \langle X \subseteq U \rangle$  cv-part unfolding image-subset-iff cv-def
      by (metis comp-apply insertCI singletonD)
  qed
qed

```

```

lemma partn-lst-imp-partn-lst-VWF:
  assumes part: partn-lst  $r\ U\ \alpha\ n$  and  $\beta$ :  $ordertype\ U\ r \leq \beta$  small U
    and  $r$ : wf r trans r total-on U r
  shows partn-lst-VWF  $\beta\ \alpha\ n$ 
  by (metis assms less-eq-V-def partn-lst-imp-partn-lst-VWF-eq partn-lst-greater-resource)

```

```

lemma partn-lst-VWF-imp-partn-lst-eq:
  assumes part: partn-lst-VWF  $\beta\ \alpha\ n$  and  $\beta$ :  $ordertype\ U\ r = \beta$  small U
    and  $r$ : wf r trans r total-on U r
  shows partn-lst  $r\ U\ \alpha\ n$ 
  unfolding partn-lst-def
proof clarsimp
  fix  $f$ 
  assume  $f$ :  $f \in [U]^n \rightarrow \{..<length\ \alpha\}$ 
  define cv where  $cv \equiv \lambda X. inv-into\ U\ (ordermap\ U\ r)\ '[X$ 
  have bij: bij-betw (ordermap  $U\ r$ )  $U\ (elts\ \beta)$ 
    using ordermap-bij [of r U] assms by blast
  then have bij-cv: bij-betw  $cv\ ([elts\ \beta]^n)\ ([U]^n)$ 
    using bij-betw-nsets bij-betw-inv-into unfolding cv-def by blast
  then have func:  $f \circ cv \in [elts\ \beta]^n \rightarrow \{..<length\ \alpha\}$ 
    using bij-betw-apply f by fastforce
  have inj-on (ordermap  $U\ r$ )  $U$ 

```

using *bij bij-betw-def* **by** *blast*
then have *cv-part*: $\llbracket \forall x \in [X]^n. f (cv\ x) = i; X \subseteq \text{elts } \beta; a \in [cv\ X]^n \rrbracket \implies f\ a =$
i **for** *a X i n*
by (*metis bij bij-betw-imp-surj-on cv-def inj-on-inv-into nset-image-obtains*)
have *ot-eq [simp]*: *ordertype (cv X) r = tp X* **if** $X \subseteq \text{elts } \beta$ **for** *X*
unfolding *cv-def*
proof (*rule ordertype-inc-eq*)
show *small X*
using *down that by auto*
show (*inv-into U (ordermap U r) x, inv-into U (ordermap U r) y*) $\in r$
if $x \in X\ y \in X$ **and** $(x,y) \in VWF$ **for** $x\ y$
proof –
have *xy*: $x \in \text{ordermap } U\ r\ \text{‘ } U\ y \in \text{ordermap } U\ r\ \text{‘ } U$
using $\langle X \subseteq \text{elts } \beta \rangle\ \langle x \in X \rangle\ \langle y \in X \rangle$ *bij bij-betw-imp-surj-on* **by** *blast+*
then have *eq*: $\text{ordermap } U\ r\ (\text{inv-into } U\ (\text{ordermap } U\ r)\ x) = x\ \text{ordermap } U$
r (inv-into U (ordermap U r) y) = y
by (*meson f-inv-into-f*)
then have $y \notin \text{elts } x$
by (*metis (no-types) VWF-non-refl mem-imp-VWF that(3) trans-VWF*
trans-def)
with *r* **show** *?thesis*
by (*metis (no-types) VWF-non-refl xy eq assms(3) inv-into-into ordermap-mono*
that(3) total-on-def)
qed
qed (*use r in auto*)
obtain *X i* **where** $X \subseteq \text{elts } \beta$ **and** $X: tp\ X = \alpha!i\ (f \circ cv)\ \text{‘ } [X]^n \subseteq \{i\}$
and $i < \text{length } \alpha$
using *part func* **by** (*auto simp: partn-lst-def*)
show $\exists i < \text{length } \alpha. \exists H \subseteq U. \text{ordertype } H\ r = \alpha!i \wedge f\ \text{‘ } [H]^n \subseteq \{i\}$
proof (*intro exI conjI*)
show $i < \text{length } \alpha$
by (*simp add: <i < length alpha>*)
show $cv\ X \subseteq U$
using $\langle X \subseteq \text{elts } \beta \rangle$ *bij bij-betw-imp-surj-on bij-betw-inv-into cv-def* **by** *blast*
show *ordertype (cv X) r = alpha ! i*
by (*simp add: X(1) <X subset elts beta>*)
show $f\ \text{‘ } [cv\ X]^n \subseteq \{i\}$
using $X\ \langle X \subseteq \text{elts } \beta \rangle$ *cv-part* **unfolding** *image-subset-iff cv-def*
by (*metis comp-apply insertCI singletonD*)
qed
qed

corollary *partn-lst-VWF-imp-partn-lst*:
assumes *partn-lst-VWF beta alpha n* **and** $\beta: \text{ordertype } U\ r \geq \beta$ *small U*
wf r trans r total-on U r
shows *partn-lst r U alpha n*
by (*metis assms less-eq-V-def partn-lst-VWF-imp-partn-lst-eq partn-lst-greater-resource*)

2.3 Simple consequences of the definitions

A restatement of the infinite Ramsey theorem using partition notation

lemma *Ramsey-partn: partn-lst-VWF* ω $[\omega, \omega]$ 2

proof (*clarsimp simp: partn-lst-def*)

fix f

assume $f \in [\text{elts } \omega]^2 \rightarrow \{..< \text{Suc } (\text{Suc } 0)\}$

then have $*$: $\forall x \in \text{elts } \omega. \forall y \in \text{elts } \omega. x \neq y \longrightarrow f \{x, y\} < 2$

by (*auto simp: nsets-def eval-nat-numeral*)

obtain H i **where** $H: H \subseteq \text{elts } \omega$ **and** *infinite* H

and $t: i < \text{Suc } (\text{Suc } 0)$

and $\text{teq}: \forall x \in H. \forall y \in H. x \neq y \longrightarrow f \{x, y\} = i$

using *Ramsey2 [OF infinite- ω $*$]* **by** (*auto simp: eval-nat-numeral*)

then have $\text{tp } H = [\omega, \omega] ! i$

using *less-2-cases eval-nat-numeral ordertype-infinite- ω* **by force**

moreover have $f' \{N. N \subseteq H \wedge \text{finite } N \wedge \text{card } N = 2\} \subseteq \{i\}$

by (*force simp: teq card-2-iff*)

ultimately have $f' [H]^2 \subseteq \{i\}$

by (*metis (no-types) nsets-def numeral-2-eq-2*)

then show $\exists i < \text{Suc } (\text{Suc } 0). \exists H \subseteq \text{elts } \omega. \text{tp } H = [\omega, \omega] ! i \wedge f' [H]^2 \subseteq \{i\}$

using $H \langle \text{tp } H = [\omega, \omega] ! i \rangle t$ **by blast**

qed

This is the counterexample sketched in Hajnal and Larson, section 9.1.

proposition *omega-basic-counterexample:*

assumes *Ord* α

shows $\neg \text{partn-lst-VWF } \alpha$ $[\text{succ } (\text{vcard } \alpha), \omega]$ 2

proof –

obtain π **where** $\text{fun } \pi: \pi \in \text{elts } \alpha \rightarrow \text{elts } (\text{vcard } \alpha)$ **and** $\text{inj } \pi: \text{inj-on } \pi$ $(\text{elts } \alpha)$

using *inj-into-vcard* **by auto**

have *Ord* π : *Ord* (πx) **if** $x \in \text{elts } \alpha$ **for** x

using *Ord-in-Ord fun* π **that** **by fastforce**

define f **where** $f A \equiv @i::\text{nat}. \exists x y. A = \{x, y\} \wedge x < y \wedge (\pi x < \pi y \wedge i=0 \vee \pi y < \pi x \wedge i=1)$ **for** A

have $f\text{-Pi}: f \in [\text{elts } \alpha]^2 \rightarrow \{..< \text{Suc } (\text{Suc } 0)\}$

proof

fix A

assume $A \in [\text{elts } \alpha]^2$

then obtain $x y$ **where** $xy: x \in \text{elts } \alpha \ y \in \text{elts } \alpha \ x < y$ **and** $A: A = \{x, y\}$

apply (*clarsimp simp: nsets-2-eq*)

by (*metis Ord-in-Ord Ord-linear-lt assms insert-commute*)

consider $\pi x < \pi y \mid \pi y < \pi x$

by (*metis Ord* π *Ord-linear-lt inj* π *inj-onD less-imp-not-eq2* xy)

then show $f A \in \{..< \text{Suc } (\text{Suc } 0)\}$

by (*metis A One-nat-def lessI lessThan-iff zero-less-Suc* $\langle x < y \rangle A \text{ exE-some}$ [*OF - f-def*])

qed

have $\text{fiff}: \pi x < \pi y \wedge i=0 \vee \pi y < \pi x \wedge i=1$

if $f: f \{x, y\} = i$ **and** $xy: x \in \text{elts } \alpha \ y \in \text{elts } \alpha \ x < y$ **for** $x y i$

proof –
consider $\pi x < \pi y \mid \pi y < \pi x$
using xy **by** (*metis Ord π Ord-linear-lt inj π inj-onD less-V-def*)
then show *?thesis*
proof cases
case 1
then have $f\{x,y\} = 0$
using $\langle x < y \rangle$ **by** (*force simp: f-def Set.doubleton-eq-iff*)
then show *?thesis*
using $1 f$ **by** *auto*
next
case 2
then have $f\{x,y\} = 1$
using $\langle x < y \rangle$ **by** (*force simp: f-def Set.doubleton-eq-iff*)
then show *?thesis*
using $2 f$ **by** *auto*
qed
qed
have *False*
if $eq: tp\ H = succ\ (vcard\ \alpha)$ **and** $H: H \subseteq elts\ \alpha$
and $0: \bigwedge A. A \in [H]^2 \implies f\ A = 0$ **for** H
proof –
have [*simp*]: *small H*
using H **down by** *auto*
have $OH: Ord\ x$ **if** $x \in H$ **for** x
using H *Ord-in-Ord* $\langle Ord\ \alpha \rangle$ **that by** *blast*
have $\pi: \pi\ x < \pi\ y$ **if** $x \in H\ y \in H\ x < y$ **for** $x\ y$
using 0 [*of* $\{x,y\}$] *that H fiff* **by** (*fastforce simp: nsets-2-eq*)
have *sub-vcad*: $\pi\ ' H \subseteq elts\ (vcard\ \alpha)$
using H *fun π* **by** *auto*
have $tp\ H = tp\ (\pi\ ' H)$
proof (*rule ordertype-VWF-inc-eq [symmetric]*)
show $\pi\ ' H \subseteq ON$
using H *Ord π* **by** *blast*
qed (*auto simp: π OH subsetI*)
also have $\dots \leq vcard\ \alpha$
by (*simp add: H sub-vcad assms ordertype-le-Ord*)
finally show *False*
by (*simp add: eq succ-le-iff*)
qed
moreover have *False*
if $eq: tp\ H = \omega$ **and** $H: H \subseteq elts\ \alpha$
and $1: \bigwedge A. A \in [H]^2 \implies f\ A = Suc\ 0$ **for** H
proof –
have [*simp*]: *small H*
using H **down by** *auto*
define η **where** $\eta \equiv inv\ into\ H\ (ordermap\ H\ VWF) \circ ord\ of\ nat$
have *bij*: *bij-betw* (*ordermap H VWF*) H (*elts* ω)
by (*metis ordermap-bij* $\langle small\ H \rangle$ *eq total-on-VWF wf-VWF*)

then have $\text{bij-betw } (\text{inv-into } H \text{ (ordermap } H \text{ VWF)}) \text{ (elts } \omega) \text{ } H$
by (*simp add: bij-betw-inv-into*)
then have $\eta: \text{bij-betw } \eta \text{ UNIV } H$
unfolding $\eta\text{-def}$
by (*metis $\omega\text{-def}$ bij-betw-comp-iff2 bij-betw-def elts-of-set inf inj-ord-of-nat order-refl*)
moreover have $\text{Ord}\eta: \text{Ord } (\eta \text{ } k) \text{ for } k$
by (*meson H Ord-in-Ord UNIV-I η assms bij-betw-apply subsetD*)
moreover obtain $k \text{ where } k: (\pi (\eta(\text{Suc } k)), \pi (\eta \text{ } k)) \notin \text{VWF}$
by (*meson wf-VWF wf-iff-no-infinite-down-chain*)
moreover have $\pi: \pi \text{ } y < \pi \text{ } x \text{ if } x \in H \text{ } y \in H \text{ } x < y \text{ for } x \text{ } y$
using 1 [*of* $\{x,y\}$] **that** $H \text{ fiff}$ **by** (*fastforce simp: nsets-2-eq*)
ultimately have $\eta (\text{Suc } k) \leq \eta \text{ } k$
by (*metis H Ord π Ord-linear2 VWF-iff-Ord-less bij-betw-def rangeI subset-iff*)
then have $(\eta (\text{Suc } k), \eta \text{ } k) \in \text{VWF} \vee \eta (\text{Suc } k) = \eta \text{ } k$
using $\text{Ord}\eta \text{ Ord-mem-iff-lt}$ **by** *auto*
then have $\text{ordermap } H \text{ VWF } (\eta (\text{Suc } k)) \leq \text{ordermap } H \text{ VWF } (\eta \text{ } k)$
by (*metis $\eta \text{ } \langle \text{small } H \rangle$ bij-betw-imp-surj-on ordermap-mono-le rangeI trans-VWF wf-VWF*)
moreover have $\text{ordermap } H \text{ VWF } (\eta (\text{Suc } k)) = \text{succ } (\text{ord-of-nat } k)$
unfolding $\eta\text{-def}$ **using** *bij bij-betw-inv-into-right* **by** *force*
moreover have $\text{ordermap } H \text{ VWF } (\eta \text{ } k) = \text{ord-of-nat } k$
using $\eta\text{-def}$ *bij bij-betw-inv-into-right* **by** *force*
ultimately show *False*
by (*simp add: less-eq-V-def*)
qed
ultimately show *?thesis*
apply (*simp add: partn-lst-def image-subset-iff*)
by (*metis f-Pi less-2-cases nth-Cons-0 nth-Cons-Suc numeral-2-eq-2*)
qed

2.4 Specker's theorem

definition *form-split* :: $[\text{nat}, \text{nat}, \text{nat}, \text{nat}, \text{nat}] \Rightarrow \text{bool}$ **where**
 $\text{form-split } a \text{ } b \text{ } c \text{ } d \text{ } i \equiv a \leq c \wedge (i=0 \wedge a < b \wedge b < c \wedge c < d \vee$
 $i=1 \wedge a < c \wedge c < b \wedge b < d \vee$
 $i=2 \wedge a < c \wedge c < d \wedge d < b \vee$
 $i=3 \wedge a=c \wedge b \neq d)$

definition *form* :: $[(\text{nat} * \text{nat}) \text{set}, \text{nat}] \Rightarrow \text{bool}$ **where**
 $\text{form } u \text{ } i \equiv \exists a \text{ } b \text{ } c \text{ } d. u = \{(a,b), (c,d)\} \wedge \text{form-split } a \text{ } b \text{ } c \text{ } d \text{ } i$

definition *scheme* :: $[(\text{nat} * \text{nat}) \text{set}] \Rightarrow \text{nat set}$ **where**
 $\text{scheme } u \equiv \text{fst } ' u \cup \text{snd } ' u$

definition *UU* :: $(\text{nat} * \text{nat}) \text{set}$
where $\text{UU} \equiv \{(a,b). a < b\}$

lemma *ordertype-UNIV- ω 2*: $\text{ordertype UNIV pair-less} = \omega \uparrow 2$

using *ordertype-Times* [of concl: UNIV UNIV less-than less-than]
by (*simp add: total-less-than pair-less-def ordertype-nat- ω numeral-2-eq-2*)

lemma *ordertype-UU-ge- ω 2*: *ordertype UNIV pair-less \leq ordertype UU pair-less*
proof (*rule ordertype-inc-le*)

define π **where** $\pi \equiv \lambda(m,n). (m, \text{Suc } (m+n))$
show $(\pi (x::\text{nat} \times \text{nat}), \pi y) \in \text{pair-less}$ **if** $(x, y) \in \text{pair-less}$ **for** $x y$
using *that by (auto simp: π -def pair-less-def split: prod.split)*
show $\text{range } \pi \subseteq \text{UU}$
by (*auto simp: π -def UU-def*)

qed *auto*

lemma *ordertype-UU- ω 2*: *ordertype UU pair-less = $\omega \uparrow 2$*

by (*metis eq-iff ordertype-UNIV- ω 2 ordertype-UU-ge- ω 2 ordertype-mono small top-greatest trans-pair-less wf-pair-less*)

Lemma 2.3 of Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω . *Annals of Mathematical Logic*, 6:129–145, 1973.

lemma *lemma-2-3*:

fixes $f :: (\text{nat} \times \text{nat}) \text{ set} \Rightarrow \text{nat}$
assumes $f \in [\text{UU}]^2 \rightarrow \{..<\text{Suc } (\text{Suc } 0)\}$
obtains $N \text{ js}$ **where** *infinite N and $\bigwedge k u. [k < 4; u \in [\text{UU}]^2; \text{form } u \text{ } k; \text{scheme } u \subseteq N] \implies f u = \text{js}!k$*

proof –

have *f-less2*: $f \{p,q\} < \text{Suc } (\text{Suc } 0)$ **if** $p \neq q$ $p \in \text{UU}$ $q \in \text{UU}$ **for** $p q$

proof –

have $\{p,q\} \in [\text{UU}]^2$
using *that by (simp add: nsets-def)*
then show *?thesis*
using *assms by (simp add: Pi-iff)*

qed

define $f0$ **where** $f0 \equiv (\lambda A::\text{nat set. THE } x. \exists a b c d. A = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d \wedge x = f \{(a,b),(c,d)\})$

have $f0$: $f0 \{a,b,c,d\} = f \{(a,b),(c,d)\}$ **if** $a < b$ $b < c$ $c < d$ **for** $a b c d$
unfolding *f0-def*

apply (*rule theI2 [where a = f {(a,b), (c,d)}]*)
using *that by (force simp: insert-eq-iff split: if-split-asm)+*

have $f0 X < \text{Suc } (\text{Suc } 0)$
if *finite X and card X = 4* **for** X

proof –

have $X \in [X]^4$
using *that by (auto simp: nsets-def)*
then obtain $a b c d$ **where** $X = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d$
by (*auto simp: ordered-nsets-4-eq*)
then show *?thesis*

using $f0$ *f-less2* **by** (*auto simp: UU-def*)

qed

then have $\exists N t. \text{infinite } N \wedge t < \text{Suc } (\text{Suc } 0)$

$\wedge (\forall X. X \subseteq N \wedge \text{finite } X \wedge \text{card } X = 4 \longrightarrow f0 X = t)$

using *Ramsey* [of *UNIV 4 f0 2*] **by** (*simp add: eval-nat-numeral*)
then obtain $N0\ j0$ **where** *infinite* $N0$ **and** $j0: j0 < \text{Suc}(\text{Suc } 0)$ **and** $N0: \bigwedge A. A \in [N0]^4 \implies f0\ A = j0$
by (*auto simp: nsets-def*)

define $f1$ **where** $f1 \equiv (\lambda A::\text{nat set. THE } x. \exists a\ b\ c\ d. A = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d \wedge x = f\ \{(a,c),(b,d)\})$
have $f1: f1\ \{a,b,c,d\} = f\ \{(a,c),(b,d)\}$ **if** $a < b\ b < c\ c < d$ **for** $a\ b\ c\ d$
unfolding *f1-def*
apply (*rule theI2 [where a = f\ \{(a,c), (b,d)\}]*)
using *that by (force simp: insert-eq-iff split: if-split-asm)+*
have $f1\ X < \text{Suc}(\text{Suc } 0)$
if *finite* X **and** $\text{card } X = 4$ **for** X
proof –
have $X \in [X]^4$
using *that by (auto simp: nsets-def)*
then obtain $a\ b\ c\ d$ **where** $X = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d$
by (*auto simp: ordered-nsets-4-eq*)
then show *?thesis*
using $f1\ f\text{-less2}$ **by** (*auto simp: UU-def*)

qed

then have $\exists N\ t. N \subseteq N0 \wedge \text{infinite } N \wedge t < \text{Suc}(\text{Suc } 0)$
 $\wedge (\forall X. X \subseteq N \wedge \text{finite } X \wedge \text{card } X = 4 \implies f1\ X = t)$
using $\langle \text{infinite } N0 \rangle$ *Ramsey* [of *N0 4 f1 2*] **by** (*simp add: eval-nat-numeral*)
then obtain $N1\ j1$ **where** $N1 \subseteq N0$ *infinite* $N1$ **and** $j1: j1 < \text{Suc}(\text{Suc } 0)$ **and**
 $N1: \bigwedge A. A \in [N1]^4 \implies f1\ A = j1$
by (*auto simp: nsets-def*)

define $f2$ **where** $f2 \equiv (\lambda A::\text{nat set. THE } x. \exists a\ b\ c\ d. A = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d \wedge x = f\ \{(a,d),(b,c)\})$
have $f2: f2\ \{a,b,c,d\} = f\ \{(a,d),(b,c)\}$ **if** $a < b\ b < c\ c < d$ **for** $a\ b\ c\ d$
unfolding *f2-def*
apply (*rule theI2 [where a = f\ \{(a,d), (b,c)\}]*)
using *that by (force simp: insert-eq-iff split: if-split-asm)+*
have $f2\ X < \text{Suc}(\text{Suc } 0)$
if *finite* X **and** $\text{card } X = 4$ **for** X
proof –
have $X \in [X]^4$
using *that by (auto simp: nsets-def)*
then obtain $a\ b\ c\ d$ **where** $X = \{a,b,c,d\} \wedge a < b \wedge b < c \wedge c < d$
by (*auto simp: ordered-nsets-4-eq*)
then show *?thesis*
using $f2\ f\text{-less2}$ **by** (*auto simp: UU-def*)

qed

then have $\exists N\ t. N \subseteq N1 \wedge \text{infinite } N \wedge t < \text{Suc}(\text{Suc } 0)$
 $\wedge (\forall X. X \subseteq N \wedge \text{finite } X \wedge \text{card } X = 4 \implies f2\ X = t)$
using $\langle \text{infinite } N1 \rangle$ *Ramsey* [of *N1 4 f2 2*] **by** (*simp add: eval-nat-numeral*)
then obtain $N2\ j2$ **where** $N2 \subseteq N1$ *infinite* $N2$ **and** $j2: j2 < \text{Suc}(\text{Suc } 0)$ **and**
 $N2: \bigwedge A. A \in [N2]^4 \implies f2\ A = j2$

by (*auto simp: nsets-def*)

define $f3$ **where** $f3 \equiv (\lambda A::nat\ set.\ THE\ x.\ \exists\ a\ b\ c.\ A = \{a,b,c\} \wedge a < b \wedge b < c$
 $\wedge\ x = f\ \{(a,b),(a,c)\})$

have $f3: f3\ \{a,b,c\} = f\ \{(a,b),(a,c)\}$ **if** $a < b\ b < c$ **for** $a\ b\ c$
unfolding $f3\text{-def}$

apply (*rule theI2 [where a = f {(a,b), (a,c)}]*)

using *that by (force simp: insert-eq-iff split: if-split-asm)+*

have $f3': f3\ \{a,b,c\} = f\ \{(a,b),(a,c)\}$ **if** $a < c\ c < b$ **for** $a\ b\ c$
using $f3$ [*of a c b*] *that*

by (*simp add: insert-commute*)

have $f3\ X < Suc\ (Suc\ 0)$

if *finite X* **and** $card\ X = 3$ **for** X

proof –

have $X \in [X]^3$

using *that by (auto simp: nsets-def)*

then obtain $a\ b\ c$ **where** $X = \{a,b,c\} \wedge a < b \wedge b < c$
by (*auto simp: ordered-nsets-3-eq*)

then show *?thesis*

using $f3\ f\text{-less2}$ **by** (*auto simp: UU-def*)

qed

then have $\exists\ N\ t.\ N \subseteq N2 \wedge infinite\ N \wedge t < Suc\ (Suc\ 0)$
 $\wedge (\forall\ X.\ X \subseteq N \wedge finite\ X \wedge card\ X = 3 \longrightarrow f3\ X = t)$

using $\langle infinite\ N2 \rangle$ *Ramsey* [*of N2 3 f3 2*] **by** (*simp add: eval-nat-numeral*)

then obtain $N3\ j3$ **where** $N3 \subseteq N2$ *infinite N3* **and** $j3: j3 < Suc\ (Suc\ 0)$ **and**
 $N3: \bigwedge A.\ A \in [N3]^3 \implies f3\ A = j3$

by (*auto simp: nsets-def*)

show *thesis*

proof

fix $k\ u$

assume $k < 4$

and $u:$ *form u k scheme u* $\subseteq N3$

and $UU:$ $u \in [UU]^2$

then consider $(0)\ k=0 \mid (1)\ k=1 \mid (2)\ k=2 \mid (3)\ k=3$

by *linarith*

then show $f\ u = [j0,j1,j2,j3] ! k$

proof *cases*

case 0

have $N3 \subseteq N0$

using $\langle N1 \subseteq N0 \rangle \langle N2 \subseteq N1 \rangle \langle N3 \subseteq N2 \rangle$ **by** *auto*

then show *?thesis*

using $u\ 0$

apply (*auto simp: form-def form-split-def scheme-def simp flip: f0*)

apply (*force simp: nsets-def intro: N0*)

done

next

case 1

have $N3 \subseteq N1$

```

    using ⟨N2 ⊆ N1⟩ ⟨N3 ⊆ N2⟩ by auto
  then show ?thesis
    using u 1
    apply (auto simp: form-def form-split-def scheme-def simp flip: f1)
    apply (force simp: nsets-def intro: N1)
    done
next
case 2
then show ?thesis
  using u ⟨N3 ⊆ N2⟩
  apply (auto simp: form-def form-split-def scheme-def nsets-def simp flip: f2)
  apply (force simp: nsets-def intro: N2)
  done
next
case 3
{ fix a b d
  assume {(a, b), (a, d)} ∈ [UU]2
  and *: a ∈ N3 b ∈ N3 d ∈ N3 b ≠ d
  then have a < b a < d
    by (auto simp: UU-def nsets-def)
  then have f {(a, b), (a, d)} = j3
    using *
    apply (auto simp: neq-iff simp flip: f3 f3')
    apply (force simp: nsets-def intro: N3)+
    done
}
then show ?thesis
  using u UU 3
  by (auto simp: form-def form-split-def scheme-def)
qed
qed (rule ⟨infinite N3⟩)
qed

```

Lemma 2.4 of Jean A. Larson, *ibid.*

lemma *lemma-2-4*:

```

  assumes infinite N k < 4
  obtains M where M ∈ [UU]m ∧ u. u ∈ [M]2 ⇒ form u k ∧ u. u ∈ [M]2 ⇒
  scheme u ⊆ N
proof –
  obtain f:: nat ⇒ nat where bij-betw f UNIV N strict-mono f
  using assms by (meson bij-enumerate enumerate-mono strict-monoI)
  then have iff[simp]: f x = f y ↔ x=y f x < f y ↔ x < y for x y
  by (simp-all add: strict-mono-eq strict-mono-less)
  have [simp]: f x ∈ N for x
  using bij-betw-apply [OF ⟨bij-betw f UNIV N⟩] by blast
  define M0 where M0 = (λi. (f(2*i), f(Suc(2*i)))) ‘{.. $m$ }
  define M1 where M1 = (λi. (f i, f(m+i))) ‘{.. $m$ }
  define M2 where M2 = (λi. (f i, f(2*m-i))) ‘{.. $m$ }
  define M3 where M3 = (λi. (f 0, f (Suc i))) ‘{.. $m$ }

```

```

consider (0)  $k=0$  | (1)  $k=1$  | (2)  $k=2$  | (3)  $k=3$ 
  using assms by linarith
then show thesis
proof cases
  case 0
  show ?thesis
  proof
    have inj-on ( $\lambda i. (f (2 * i), f (Suc (2 * i)))$ ) {.. $m$ }
      by (auto simp: inj-on-def)
    then show  $M0 \in [UU]^m$ 
      by (simp add: M0-def nsets-def card-image UU-def image-subset-iff)
  next
  fix  $u$ 
  assume  $u: (u: (nat \times nat) set) \in [M0]^2$ 
  then obtain  $x y$  where  $u = \{x, y\}$   $x \neq y$   $x \in M0$   $y \in M0$ 
    by (auto simp: nsets-2-eq)
  then obtain  $i j$  where  $i < j$   $j < m$  and  $ueq: u = \{(f(2*i), f(Suc(2*i))),$ 
( $f(2*j), f(Suc(2*j))\}$ 
    apply (clarsimp simp: M0-def)
    apply (metis (full-types) insert-commute less-linear) +
    done
  moreover have  $f (2 * i) \leq f (2 * j)$ 
    by (simp add: <i<j> less-imp-le-nat)
  ultimately show form  $u$   $k$ 
    apply (simp add: 0 form-def form-split-def nsets-def)
    apply (rule-tac x=f (2 * i) in exI)
    apply (rule-tac x=f (Suc (2 * i)) in exI)
    apply (rule-tac x=f (2 * j) in exI)
    apply (rule-tac x=f (Suc (2 * j)) in exI)
    apply auto
    done
  show scheme  $u \subseteq N$ 
    using ueq by (auto simp: scheme-def)
  qed
next
  case 1
  show ?thesis
  proof
    have inj-on ( $\lambda i. (f i, f(m+i))$ ) {.. $m$ }
      by (auto simp: inj-on-def)
    then show  $M1 \in [UU]^m$ 
      by (simp add: M1-def nsets-def card-image UU-def image-subset-iff)
  next
  fix  $u$ 
  assume  $u: (u: (nat \times nat) set) \in [M1]^2$ 
  then obtain  $x y$  where  $u = \{x, y\}$   $x \neq y$   $x \in M1$   $y \in M1$ 
    by (auto simp: nsets-2-eq)
  then obtain  $i j$  where  $i < j$   $j < m$  and  $ueq: u = \{(f i, f(m+i)), (f j, f(m+j))\}$ 
    apply (auto simp: M1-def)

```

```

    apply (metis (full-types) insert-commute less-linear)+
  done
then show form u k
  apply (simp add: 1 form-def form-split-def nsets-def)
  by (metis iff(2) nat-add-left-cancel-less nat-less-le trans-less-add1)
show scheme u  $\subseteq$  N
  using ueq by (auto simp: scheme-def)
qed
next
case 2
show ?thesis
proof
  have inj-on ( $\lambda i. (f\ i, f(2*m-i))$ )  $\{..<m\}$ 
    by (auto simp: inj-on-def)
  then show  $M2 \in [UU]^m$ 
    by (auto simp: M2-def nsets-def card-image UU-def image-subset-iff)
next
fix u
assume u: ( $u::(nat \times nat)$  set)  $\in [M2]^2$ 
then obtain x y where  $u = \{x,y\}$   $x \neq y$   $x \in M2$   $y \in M2$ 
  by (auto simp: nsets-2-eq)
then obtain i j where  $i < j < m$  and ueq:  $u = \{(f\ i, f(2*m-i)), (f\ j,$ 
 $f(2*m-j))\}$ 
  apply (auto simp: M2-def)
  apply (metis (full-types) insert-commute less-linear)+
  done
then show form u k
  apply (simp add: 2 form-def form-split-def nsets-def)
  apply (rule-tac  $x=f\ i$  in exI)
  apply (rule-tac  $x=f\ (2 * m - i)$  in exI)
  apply (rule-tac  $x=f\ j$  in exI)
  apply (rule-tac  $x=f\ (2 * m - j)$  in exI)
  apply (auto simp: less-imp-le-nat)
  done
show scheme u  $\subseteq$  N
  using ueq by (auto simp: scheme-def)
qed
next
case 3
show ?thesis
proof
  have inj-on ( $\lambda i. (f\ 0, f\ (Suc\ i))$ )  $\{..<m\}$ 
    by (auto simp: inj-on-def)
  then show  $M3 \in [UU]^m$ 
    by (auto simp: M3-def nsets-def card-image UU-def image-subset-iff)
next
fix u
assume u: ( $u::(nat \times nat)$  set)  $\in [M3]^2$ 
then obtain x y where  $u = \{x,y\}$   $x \neq y$   $x \in M3$   $y \in M3$ 

```



```

    by (auto simp: nsets-2-eq)
  then obtain  $i j$  where  $i < j < m$  and  $ueq: u = \{(f\ 0, f(Suc\ i)), (f\ 0, f(Suc\ j))\}$ 
    apply (auto simp: M3-def)
    apply (metis (full-types) insert-commute less-linear)+
    done
  then show  $form\ u\ k$ 
    by (fastforce simp: 3 form-def form-split-def nsets-def)
  show  $scheme\ u \subseteq N$ 
    using  $ueq$  by (auto simp: scheme-def)
qed
qed
qed

```

Lemma 2.5 of Jean A. Larson, *ibid.*

lemma *lemma-2-5:*

```

  assumes  $infinite\ N$ 
  obtains  $X$  where  $X \subseteq UU$   $ordertype\ X\ pair-less = \omega \uparrow 2$ 
     $\wedge u. u \in [X]^2 \implies (\exists k < 4. form\ u\ k) \wedge scheme\ u \subseteq N$ 
proof –
  obtain  $C$ 
    where  $dis: pairwise\ (\lambda i\ j. disjnt\ (C\ i)\ (C\ j))\ UNIV$ 
      and  $N: (\bigcup i. C\ i) \subseteq N$  and  $infC: \bigwedge i::nat. infinite\ (C\ i)$ 
    using  $assms\ infinite-infinite-partition$  by blast
  then have  $\exists \varphi::nat \implies nat. inj\ \varphi \wedge range\ \varphi = C\ i \wedge strict-mono\ \varphi$  for  $i$ 
    by (metis  $nat-infinite-iff\ strict-mono-on-imp-inj-on$ )
  then obtain  $\varphi::[nat, nat] \implies nat$ 
    where  $\varphi: \bigwedge i. inj\ (\varphi\ i) \wedge range\ (\varphi\ i) = C\ i \wedge strict-mono\ (\varphi\ i)$ 
    by metis
  then have  $\pi-in-C\ [simp]: \varphi\ i\ j \in C\ i' \longleftrightarrow i'=i$  for  $i\ i'\ j$ 
    using  $dis$  by (fastforce simp: pairwise-def disjnt-def)
  have  $less-iff\ [simp]: \varphi\ i\ j' < \varphi\ i\ j \longleftrightarrow j' < j$  for  $i\ j'\ j$ 
    by (simp add:  $\varphi\ strict-mono-less$ )
  let  $?a = \varphi\ 0$ 
  define  $X$  where  $X \equiv \{(?a\ i, b) \mid i\ b. ?a\ i < b \wedge b \in C\ (Suc\ i)\}$ 
  show  $thesis$ 
proof
  show  $X \subseteq UU$ 
    by (auto simp:  $X-def\ UU-def$ )
  show  $ordertype\ X\ pair-less = \omega \uparrow 2$ 
proof (rule antisym)
  have  $ordertype\ X\ pair-less \leq ordertype\ UU\ pair-less$ 
    by ( $simp\ add: \langle X \subseteq UU \rangle\ ordertype-mono$ )
  then show  $ordertype\ X\ pair-less \leq \omega \uparrow 2$ 
    using  $ordertype-UU-\omega 2$  by auto
  define  $\pi$  where  $\pi \equiv \lambda(i, j::nat). (?a\ i, \varphi\ (Suc\ i)\ (?a\ j))$ 
  have  $\bigwedge i\ j. i < j \implies \varphi\ 0\ i < \varphi\ (Suc\ i)\ (\varphi\ 0\ j)$ 
    by ( $meson\ \varphi\ le-less-trans\ less-iff\ strict-mono-imp-increasing$ )
  then have  $subX: \pi\ ' UU \subseteq X$ 

```

```

    by (auto simp: UU-def  $\pi$ -def X-def)
  then have ordertype ( $\pi$  ' UU) pair-less  $\leq$  ordertype X pair-less
    by (simp add: ordertype-mono)
  moreover have ordertype ( $\pi$  ' UU) pair-less = ordertype UU pair-less
  proof (rule ordertype-inc-eq)
    show ( $\pi$  x,  $\pi$  y)  $\in$  pair-less
      if  $x \in UU$   $y \in UU$  and  $(x, y) \in$  pair-less for  $x$   $y$ 
      using that by (auto simp: UU-def  $\pi$ -def pair-less-def)
  qed auto
  ultimately show  $\omega \uparrow 2 \leq$  ordertype X pair-less
    using ordertype-UU- $\omega 2$  by simp
qed
next
fix U
assume  $U \in [X]^2$ 
then obtain a b c d where Ueq:  $U = \{(a,b),(c,d)\}$  and ne:  $(a,b) \neq (c,d)$  and
inX:  $(a,b) \in X$   $(c,d) \in X$  and  $a \leq c$ 
apply (auto simp: nsets-def subset-iff eval-nat-numeral card-Suc-eq Set.doubleton-eq-iff)
  apply (metis nat-le-linear)+
done
show  $(\exists k < 4. \text{form } U k) \wedge \text{scheme } U \subseteq N$ 
proof
  show scheme  $U \subseteq N$ 
    using inX N  $\varphi$  by (fastforce simp: scheme-def Ueq X-def)
next
consider  $a < c \mid a = c \wedge b \neq d$ 
  using  $\langle a \leq c \rangle$  ne nat-less-le by blast
then show  $\exists k < 4. \text{form } U k$ 
proof cases
  case 1
  have *:  $a < b$   $b \neq c$   $c < d$ 
    using inX by (auto simp: X-def)
  moreover have  $\llbracket a < c; c < b; \neg d < b \rrbracket \implies b < d$ 
    using inX apply (clarsimp simp: X-def not-less)
    by (metis  $\varphi$   $\pi$ -in-C imageE nat.inject nat-less-le)
  ultimately consider (k0)  $a < b \wedge b < c \wedge c < d \mid$  (k1)  $a < c \wedge c < b \wedge b < d \mid$ 
(k2)  $a < c \wedge c < d \wedge d < b$ 
    using 1 less-linear by blast
  then show ?thesis
proof cases
  case k0
  then have form U 0
    unfolding form-def form-split-def using Ueq  $\langle a \leq c \rangle$  by blast
  then show ?thesis by force
next
  case k1
  then have form U 1
    unfolding form-def form-split-def using Ueq  $\langle a \leq c \rangle$  by blast
  then show ?thesis by force

```

```

next
  case k2
  then have form U 2
    unfolding form-def form-split-def using Ueq ⟨a ≤ c⟩ by blast
  then show ?thesis by force
qed
next
  case 2
  then have form-split a b c d 3
    by (auto simp: form-split-def)
  then show ?thesis
    using Ueq form-def leI by force
qed
qed
qed
qed

```

Theorem 2.1 of Jean A. Larson, *ibid.*

lemma *Specker-aux*:

```

assumes α ∈ elts ω
shows partn-1st pair-less UU [ω↑2, α] 2
unfolding partn-1st-def
proof clarsimp
  fix f
  assume f: f ∈ [UU]2 → {..Suc (Suc 0)}
  let ?P0 = ∃ X ⊆ UU. ordertype X pair-less = ω↑2 ∧ f ‘ [X]2 ⊆ {0}
  let ?P1 = ∃ M ⊆ UU. ordertype M pair-less = α ∧ f ‘ [M]2 ⊆ {1}
  have †: ?P0 ∨ ?P1
  proof (rule disjCI)
    assume ¬ ?P1
    then have not1: ∧M. [M ⊆ UU; ordertype M pair-less = α] ⇒ ∃ x ∈ [M]2. f
      x ≠ Suc 0
      by auto
    obtain m where m: α = ord-of-nat m
      using assms elts-ω by auto
    then have f-eq-0: M ∈ [UU]m ⇒ ∃ x ∈ [M]2. f x = 0 for M
      using not1 [of M] finite-ordertype-eq-card [of M pair-less m] f
      apply (clarsimp simp: nsets-def eval-nat-numeral Pi-def)
      by (meson less-Suc0 not-less-less-Suc-eq subset-trans)
    obtain N js where infinite N and N: ∧k u. [k < 4; u ∈ [UU]2; form u k;
      scheme u ⊆ N] ⇒ f u = js!k
      using f lemma-2-3 by blast
    obtain M0 where M0: M0 ∈ [UU]m ∧ u. u ∈ [M0]2 ⇒ form u 0 ∧ u. u ∈
      [M0]2 ⇒ scheme u ⊆ N
      by (rule lemma-2-4 [OF ⟨infinite N⟩]) auto
    obtain M1 where M1: M1 ∈ [UU]m ∧ u. u ∈ [M1]2 ⇒ form u 1 ∧ u. u ∈
      [M1]2 ⇒ scheme u ⊆ N
      by (rule lemma-2-4 [OF ⟨infinite N⟩]) auto
    obtain M2 where M2: M2 ∈ [UU]m ∧ u. u ∈ [M2]2 ⇒ form u 2 ∧ u. u ∈

```

```

[M2]2 ⇒ scheme u ⊆ N
  by (rule lemma-2-4 [OF <infinite N>]) auto
  obtain M3 where M3: M3 ∈ [UU]m ∧ u. u ∈ [M3]2 ⇒ form u 3 ∧ u. u ∈
[M3]2 ⇒ scheme u ⊆ N
  by (rule lemma-2-4 [OF <infinite N>]) auto
  have js!0 = 0
  using N [of 0] M0 f-eq-0 [of M0] by (force simp: nsets-def eval-nat-numeral)
  moreover have js!1 = 0
  using N [of 1] M1 f-eq-0 [of M1] by (force simp: nsets-def eval-nat-numeral)
  moreover have js!2 = 0
  using N [of 2] M2 f-eq-0 [of M2] by (force simp: nsets-def eval-nat-numeral)
  moreover have js!3 = 0
  using N [of 3] M3 f-eq-0 [of M3] by (force simp: nsets-def eval-nat-numeral)
  ultimately have js0: js!k = 0 if k < 4 for k
  using that by (auto simp: eval-nat-numeral less-Suc-eq)
  obtain X where X ⊆ UU and otX: ordertype X pair-less = ω↑2
  and X: ∧u. u ∈ [X]2 ⇒ (∃ k < 4. form u k) ∧ scheme u ⊆ N
  using <infinite N> lemma-2-5 by auto
  moreover have f ' [X]2 ⊆ {0}
  proof (clarsimp simp: image-subset-iff)
    fix u
    assume u: u ∈ [X]2
    then have u-UU2: u ∈ [UU]2
      using <X ⊆ UU> nsets-mono by blast
    show f u = 0
      using X u N [OF - u-UU2] js0 by auto
  qed
  ultimately show ∃ X ⊆ UU. ordertype X pair-less = ω↑2 ∧ f ' [X]2 ⊆ {0}
  by blast
qed
then show ∃ i < Suc (Suc 0). ∃ H ⊆ UU. ordertype H pair-less = [ω↑2, α] ! i ∧ f
' [H]2 ⊆ {i}
  by (metis One-nat-def lessI nth-Cons-0 nth-Cons-Suc zero-less-Suc)
qed

theorem Specker: α ∈ elts ω ⇒ partn-lst-VWF (ω↑2) [ω↑2,α] 2
  using partn-lst-imp-partn-lst-VWF-eq [OF Specker-aux] ordertype-UU-ω2 wf-pair-less
  by blast

end
theory Erdos-Milner
  imports Partitions

begin

```

2.5 Erdos-Milner theorem

P. Erds and E. C. Milner, A Theorem in the Partition Calculus. Canadian Math. Bull. 15:4 (1972), 501-505. Corrigendum, Canadian Math. Bull.

17:2 (1974), 305.

The paper defines strong types as satisfying the criteria below. It remarks that ordinals satisfy those criteria. Here is a (too complicated) proof.

proposition *strong-ordertype-eq*:

assumes $D: D \subseteq \text{elts } \beta$ **and** $\text{Ord } \beta$

obtains L **where** $\bigcup (\text{List.set } L) = D \wedge X. X \in \text{List.set } L \implies \text{indecomposable}$
($tp\ X$)

and $\bigwedge M. \llbracket M \subseteq D; \bigwedge X. X \in \text{List.set } L \implies tp\ (M \cap X) \geq tp\ X \rrbracket \implies tp\ M = tp\ D$

proof –

define φ **where** $\varphi \equiv \text{inv-into } D\ (\text{ordermap } D\ \text{VWF})$

then have $\text{bij-}\varphi: \text{bij-betw } \varphi\ (\text{elts } (tp\ D))\ D$

using D *bij-betw-inv-into down ordermap-bij* **by** *blast*

have $\varphi\text{-cancel-left}: \bigwedge d. d \in D \implies \varphi\ (\text{ordermap } D\ \text{VWF } d) = d$

by (*metis* D $\varphi\text{-def}$ *bij-betw-inv-into-left down ordermap-bij total-on-VWF wf-VWF*)

have $\varphi\text{-cancel-right}: \bigwedge \gamma. \gamma \in \text{elts } (tp\ D) \implies \text{ordermap } D\ \text{VWF } (\varphi\ \gamma) = \gamma$

by (*metis* $\varphi\text{-def}$ *f-inv-into-f ordermap-surj subsetD*)

have *small* $D\ D \subseteq ON$

using *assms down elts-subset-ON [of β]* **by** *auto*

then have $\varphi\text{-less-iff}: \bigwedge \gamma\ \delta. \llbracket \gamma \in \text{elts } (tp\ D); \delta \in \text{elts } (tp\ D) \rrbracket \implies \varphi\ \gamma < \varphi\ \delta$
 $\longleftrightarrow \gamma < \delta$

by (*metis* $\varphi\text{-def}$ *inv-ordermap-VWF-mono-iff inv-ordermap-mono-eq less-V-def*)

obtain αs **where** $\text{List.set } \alpha s \subseteq ON$ **and** $\omega\text{-dec } \alpha s$ **and** $tpD\text{-eq}: tp\ D = \omega\text{-sum } \alpha s$

using *Ord-ordertype $\omega\text{-nf-exists}$* **by** *blast* — Cantor normal form of the ordertype

have *ord [simp]: Ord* $(\alpha s ! k)\ \text{Ord } (\omega\text{-sum } (\text{take } k\ \alpha s))$ **if** $k < \text{length } \alpha s$ **for** k

using *that* $\langle \text{list.set } \alpha s \subseteq ON \rangle$

by (*auto simp: dual-order.trans set-take-subset elim: ON-imp-Ord*)

define E **where** $E \equiv \lambda k. \text{lift } (\omega\text{-sum } (\text{take } k\ \alpha s))\ (\omega \uparrow (\alpha s ! k))$

define L **where** $L \equiv \text{map } (\lambda k. \varphi\ ' (\text{elts } (E\ k)))\ [0..<\text{length } \alpha s]$

have *lengthL [simp]: length* $L = \text{length } \alpha s$

by (*simp add: L-def*)

have *in-elts-E-less: elts* $(E\ k') \ll \text{elts } (E\ k)$ **if** $k' < k < \text{length } \alpha s$ **for** $k\ k'$

— The ordinals have been partitioned into disjoint intervals

proof –

have *ord ω : Ord* $(\omega \uparrow \alpha s ! k')$

using *that* **by** *auto*

from *that id-take-nth-drop [of k' take k αs]*

obtain l **where** $\text{take } k\ \alpha s = \text{take } k'\ \alpha s @ (\alpha s ! k') \# l$

by (*simp add: min-def*)

then show *?thesis*

using *that unfolding* $E\text{-def}$ *lift-def less-sets-def*

by (*auto dest!: OrdmemD [OF ord ω] intro: less-le-trans*)

qed

have *elts-E: elts* $(E\ k) \subseteq \text{elts } (\omega\text{-sum } \alpha s)$

if $k < \text{length } \alpha s$ **for** k

proof –

have $\omega \uparrow (\alpha s!k) \leq \omega\text{-sum } (\text{drop } k \ \alpha s)$
by (*metis that Cons-nth-drop-Suc $\omega\text{-sum-Cons add-le-cancel-left0}$*)
then have $(+) (\omega\text{-sum } (\text{take } k \ \alpha s)) \text{ 'elts } (\omega \uparrow (\alpha s!k)) \subseteq \text{elts } (\omega\text{-sum } (\text{take } k \ \alpha s) + \omega\text{-sum } (\text{drop } k \ \alpha s))$
by *blast*
also have $\dots = \text{elts } (\omega\text{-sum } \alpha s)$
using $\omega\text{-sum-take-drop}$ **by** *auto*
finally show *?thesis*
by (*simp add: lift-def E-def*)
qed
have $\omega\text{-sum-in-tpD: } \omega\text{-sum } (\text{take } k \ \alpha s) + \gamma \in \text{elts } (\text{tp } D)$
if $\gamma \in \text{elts } (\omega \uparrow \alpha s!k) \ k < \text{length } \alpha s$ **for** $\gamma \ k$
using $\text{elts-E lift-def that tpD-eq}$ **by** (*auto simp: E-def*)
have $\text{Ord-}\varphi: \text{Ord } (\varphi (\omega\text{-sum } (\text{take } k \ \alpha s) + \gamma))$
if $\gamma \in \text{elts } (\omega \uparrow \alpha s!k) \ k < \text{length } \alpha s$ **for** $\gamma \ k$
using $\omega\text{-sum-in-tpD } \langle D \subseteq ON \rangle \text{ bij-}\varphi \text{ bij-betw-imp-surj-on that}$ **by** *fastforce*
define π **where** $\pi \equiv \lambda k. ((\lambda y. \text{odiff } y (\omega\text{-sum } (\text{take } k \ \alpha s))) \circ \text{ordermap } D \text{ VWF})$
— mapping the segments of D into some power of ω
have $\text{bij-}\pi: \text{bij-betw } (\pi \ k) (\varphi \text{ 'elts } (E \ k)) (\text{elts } (\omega \uparrow (\alpha s!k)))$
if $k < \text{length } \alpha s$ **for** k
using *that* **by** (*auto simp: bij-betw-def π -def E-def inj-on-def lift-def image-comp $\omega\text{-sum-in-tpD } \varphi\text{-cancel-right that}$*)
have $\pi\text{-iff: } \pi \ k \ x < \pi \ k \ y \longleftrightarrow (x, y) \in \text{VWF}$
if $x \in \varphi \text{ 'elts } (E \ k) \ y \in \varphi \text{ 'elts } (E \ k) \ k < \text{length } \alpha s$ **for** $k \ x \ y$
using *that*
by (*auto simp: π -def E-def lift-def $\omega\text{-sum-in-tpD } \varphi\text{-cancel-right Ord-}\varphi \varphi\text{-less-iff}$*)
have $\text{tp-E-eq [simp]: } \text{tp } (\varphi \text{ 'elts } (E \ k)) = \omega \uparrow (\alpha s!k)$
if $k: k < \text{length } \alpha s$ **for** k
by (*metis Ord- ω Ord- $\text{oexp } \pi\text{-iff bij-}\pi \text{ ord}(1) \text{ ordertype-VWF-eq-iff replacement small-elts that}$*)
have $\text{tp-L-eq [simp]: } \text{tp } (L!k) = \omega \uparrow (\alpha s!k)$ **if** $k < \text{length } \alpha s$ **for** k
by (*simp add: L-def that*)
have $\text{UL-eq-D: } \bigcup (\text{list.set } L) = D$
proof (*rule antisym*)
show $\bigcup (\text{list.set } L) \subseteq D$
by (*force simp: L-def tpD-eq bij-betw-apply [OF bij- φ] dest: elts-E*)
show $D \subseteq \bigcup (\text{list.set } L)$
proof
fix δ
assume $\delta \in D$
then have $\text{ordermap } D \text{ VWF } \delta \in \text{elts } (\omega\text{-sum } \alpha s)$
by (*metis $\langle \text{small } D \rangle \text{ ordermap-in-ordertype tpD-eq}$*)
then show $\delta \in \bigcup (\text{list.set } L)$
using $\langle \delta \in D \rangle \varphi\text{-cancel-left in-elts-}\omega\text{-sum}$
by (*fastforce simp: L-def E-def image-iff lift-def*)
qed
qed
show *thesis*
proof

```

show indecomposable (tp X) if X ∈ list.set L for X
  using that by (auto simp: L-def indecomposable- $\omega$ -power)
next
fix M
assume M ⊆ D and *:  $\bigwedge X. X \in \text{list.set } L \implies \text{tp } X \leq \text{tp } (M \cap X)$ 
show tp M = tp D
proof (rule antisym)
  show tp M ≤ tp D
    by (simp add:  $\langle M \subseteq D \rangle \langle \text{small } D \rangle \text{ordertype-VWF-mono}$ )
  define  $\sigma$  where  $\sigma \equiv \lambda X. \text{inv-into } (M \cap X) (\text{ordermap } (M \cap X) \text{ VWF})$ 
    — The bijection from an  $\omega$ -power into the appropriate
segment of M
  have bij- $\sigma$ : bij-betw ( $\sigma$  X) (elts (tp (M ∩ X))) (M ∩ X) for X
    unfolding  $\sigma$ -def
    by (meson  $\langle M \subseteq D \rangle \langle \text{small } D \rangle \text{bij-betw-inv-into inf-le1 ordermap-bij}$ 
smaller-than-small total-on-VWF wf-VWF)
  have Ord- $\sigma$ : Ord ( $\sigma$  X  $\alpha$ ) if  $\alpha \in \text{elts } (\text{tp } (M \cap X))$  for  $\alpha$  X
    using  $\langle D \subseteq \text{ON} \rangle \langle M \subseteq D \rangle$  bij-betw-apply [OF bij- $\sigma$ ] that by blast
  have  $\sigma$ -cancel-right:  $\bigwedge \gamma X. \gamma \in \text{elts } (\text{tp } (M \cap X)) \implies \text{ordermap } (M \cap X)$ 
VWF ( $\sigma$  X  $\gamma$ ) =  $\gamma$ 
    by (metis  $\sigma$ -def f-inv-into-f ordermap-surj subsetD)
  have smM: small (M ∩ X) for X
    by (meson  $\langle M \subseteq D \rangle \langle \text{small } D \rangle \text{inf-le1 subset-iff-less-eq-V}$ )
  have  $\exists k < \text{length } \alpha s. \gamma \in \text{elts } (E k)$  if  $\gamma: \gamma \in \text{elts } (\text{tp } D)$  for  $\gamma$ 
proof —
  obtain X where X ∈ set L and X:  $\varphi \gamma \in X$ 
    by (metis UL-eq-D  $\gamma$  Union-iff  $\varphi$ -def in-mono inv-into-into ordermap-surj)
  then have  $\exists k < \text{length } \alpha s. \gamma \in \text{elts } (E k) \wedge X = \varphi \text{ ' } \text{elts } (E k)$ 
    apply (clarsimp simp: L-def)
    by (metis  $\gamma$   $\varphi$ -cancel-right elts-E in-mono tpD-eq)
  then show ?thesis
    by blast
qed
then obtain K where K:  $\bigwedge \gamma. \gamma \in \text{elts } (\text{tp } D) \implies K \gamma < \text{length } \alpha s \wedge \gamma \in$ 
elts (E (K  $\gamma$ ))
    by metis — The index from tp D to the appropriate segment number
  define  $\psi$  where  $\psi \equiv \lambda d. \sigma (L ! K (\text{ordermap } D \text{ VWF } d)) (\pi (K (\text{ordermap}$ 
D VWF d)) d)
  show tp D ≤ tp M
proof (rule ordertype-inc-le)
  show small D small M
    using  $\langle M \subseteq D \rangle \langle \text{small } D \rangle$  subset-iff-less-eq-V by auto
next
fix d' d
assume xy: d' ∈ D d ∈ D and (d', d) ∈ VWF
then have d' < d
    using ON-imp-Ord  $\langle D \subseteq \text{ON} \rangle$  by auto
let ? $\gamma'$  = ordermap D VWF d'
let ? $\gamma$  = ordermap D VWF d

```

have len' : $K \ ?\gamma' < length \ \alpha s$ **and** $elts'$: $?\gamma' \in elts \ (E \ (K \ ?\gamma'))$
and len : $K \ ?\gamma < length \ \alpha s$ **and** $elts$: $?\gamma \in elts \ (E \ (K \ ?\gamma))$
using $K \ \langle d' \in D \rangle \ \langle d \in D \rangle$ **by** (*auto simp*: $\langle small \ D \rangle$ *ordermap-in-ordertype*)
have $Ord\text{-}\sigma L$: $Ord \ (\sigma \ (L!k) \ (\pi \ k \ d))$ **if** $d \in \varphi \ \text{'} \ elts \ (E \ k) \ k < length \ \alpha s$ **for**
 $k \ d$
by (*metis* (*mono-tags*) * $Ord\text{-}\sigma$ *bij- π* *bij-betw-apply lengthL nth-mem* *that tp-L-eq vsubsetD*)
have $?\gamma' < ?\gamma$
by (*simp add*: $\langle (d', d) \in VWF \rangle \ \langle small \ D \rangle$ *ordermap-mono-less xy*)
then have $K \ ?\gamma' \leq K \ ?\gamma$
using $elts' \ elts \ in\text{-}elts\text{-}E\text{-}less$ **by** (*meson leI len' less-asymp less-sets-def*)
then consider (*less*) $K \ ?\gamma' < K \ ?\gamma \mid$ (*equal*) $K \ ?\gamma' = K \ ?\gamma$
by *linarith*
then have $\sigma \ (L!K \ ?\gamma') \ (\pi \ (K \ ?\gamma') \ d') < \sigma \ (L!K \ ?\gamma) \ (\pi \ (K \ ?\gamma) \ d)$
proof cases
case less
obtain \dagger : $\sigma \ (L!K \ ?\gamma') \ (\pi \ (K \ ?\gamma') \ d') \in M \cap L!K \ ?\gamma' \ \sigma \ (L!K \ ?\gamma) \ (\pi \ (K \ ?\gamma) \ d) \in M \cap L!K \ ?\gamma$
using $elts' \ elts \ len' \ len$ * [*THEN vsubsetD*]
by (*metis lengthL φ -cancel-left bij- π bij- σ bij-betw-imp-surj-on imageI nth-mem tp-L-eq xy*)
then have *ordermap D VWF* $(\sigma \ (L!K \ ?\gamma') \ (\pi \ (K \ ?\gamma') \ d')) \in elts \ (E \ (K \ ?\gamma'))$ *ordermap D VWF* $(\sigma \ (L!K \ ?\gamma) \ (\pi \ (K \ ?\gamma) \ d)) \in elts \ (E \ (K \ ?\gamma))$
using $len' \ len \ elts\text{-}E \ tpD\text{-}eq$
by (*fastforce simp*: *L-def φ -cancel-right*)
then have *ordermap D VWF* $(\sigma \ (L!K \ ?\gamma') \ (\pi \ (K \ ?\gamma') \ d')) < \text{ordermap } D \ VWF \ (\sigma \ (L!K \ ?\gamma) \ (\pi \ (K \ ?\gamma) \ d))$
using *in-elts-E-less len less* **by** (*meson less-sets-def*)
moreover have $\sigma \ (L!K \ ?\gamma') \ (\pi \ (K \ ?\gamma') \ d') \in D \ \sigma \ (L!K \ ?\gamma) \ (\pi \ (K \ ?\gamma) \ d) \in D$
using $\langle M \subseteq D \rangle \ \dagger$ **by** *auto*
ultimately show *?thesis*
by (*metis* $\langle small \ D \rangle$ *φ -cancel-left φ -less-iff ordermap-in-ordertype*)
next
case equal
have $\sigma\text{-less}$: $\bigwedge X \ \gamma \ \delta. \llbracket \gamma < \delta; \gamma \in elts \ (tp \ (M \cap X)); \delta \in elts \ (tp \ (M \cap X)) \rrbracket$
 $\implies \sigma \ X \ \gamma < \sigma \ X \ \delta$
by (*metis* $\langle D \subseteq ON \rangle \ \langle M \subseteq D \rangle$ *σ -def dual-order.trans inv-ordermap-VWF-strict-mono-iff le-infI1 smM*)
have $\pi \ (K \ ?\gamma) \ d' < \pi \ (K \ ?\gamma) \ d$
by (*metis equal* $\langle (d', d) \in VWF \rangle$ *φ -cancel-left π -iff elts elts' imageI len xy*)
then show *?thesis*
unfolding equal
by (*metis* * [*THEN vsubsetD*] *lengthL φ -cancel-left σ -less bij- π bij-betw-imp-surj-on elts elts' image-eqI len local.equal nth-mem tp-L-eq xy*)
qed

moreover have $Ord (\sigma (L ! K ?\gamma') (\pi (K ?\gamma') d')) Ord (\sigma (L ! K ?\gamma) (\pi (K ?\gamma) d))$
using *Ord- σ L φ -cancel-left elts len elts' len' xy by fastforce+*
ultimately show $(\psi d', \psi d) \in VWF$
by (*simp add: ψ -def*)
next
show $\psi \langle D \subseteq M$
proof (*clarsimp simp: ψ -def*)
fix d
assume $d \in D$
let $?\gamma = ordermap D VWF d$
have $len: K ?\gamma < length \alpha$ **and** $elts: ?\gamma \in elts (E (K ?\gamma))$
using $K \langle d \in D \rangle$ **by** (*auto simp: \langle small $D \rangle$ ordermap-in-ordertype*)
have $\pi (K ?\gamma) d \in elts (tp (L! (K ?\gamma)))$
using $bij-\pi [OF len] \langle d \in D \rangle$
by (*metis φ -cancel-left bij-betw-apply elts imageI len tp-L-eq*)
then show $\sigma (L ! K (ordermap D VWF d)) (\pi (K (ordermap D VWF d))) d \in M$
by (*metis * lengthL Int-iff bij- σ bij-betw-apply len nth-mem vsubsetD*)
qed
qed *auto*
qed
qed (*simp add: UL-eq-D*)
qed

The “remark” of Erds and E. C. Milner, *Canad. Math. Bull.* Vol. 17 (2), 1974

proposition *indecomposable-imp-Ex-less-sets:*

assumes *indec: indecomposable α and $\alpha \geq \omega$*

and $A: tp A = \alpha$ *small A* $A \subseteq ON$

and $x \in A$ **and** $A1: tp A1 = \alpha$ $A1 \subseteq A$

obtains $A2$ **where** $tp A2 = \alpha$ $A2 \subseteq A1$ $\{x\} \ll A2$

proof –

have $Ord \alpha$

using *indec indecomposable-imp-Ord by blast*

have $Limit \alpha$

by (*meson ω -gt1 assms indec indecomposable-imp-Limit less-le-trans*)

define φ **where** $\varphi \equiv inv-into A (ordermap A VWF)$

then have $bij-\varphi: bij-betw \varphi (elts \alpha) A$

using A *bij-betw-inv-into down ordermap-bij by blast*

have $bij-om: bij-betw (ordermap A VWF) A (elts \alpha)$

using A *down ordermap-bij by blast*

define γ **where** $\gamma \equiv ordermap A VWF x$

have $\gamma: \gamma \in elts \alpha$

unfolding γ -def **using** $A \langle x \in A \rangle$ *down by auto*

then have $Ord \gamma$

using *Ord-in-Ord \langle Ord $\alpha \rangle$ by blast*

define B **where** $B \equiv \varphi \langle elts (succ \gamma) \rangle$

have $B: \{y \in A. ordermap A VWF y \leq ordermap A VWF x\} \subseteq B$

```

apply (clarsimp simp add: B-def  $\gamma$ -def image-iff  $\varphi$ -def)
by (metis Ord-linear Ord-ordermap OrdmemD bij-betw-inv-into-left bij-om leD)
show thesis
proof
  have small A1
    by (meson  $\langle$ small A $\rangle$  A1 smaller-than-small)
  then have tp (A1 - B)  $\leq$  tp A1
    by (simp add: ordertype-VWF-mono)
  moreover have tp (A1 - B)  $\geq$   $\alpha$ 
  proof -
    have  $\neg$  ( $\alpha \leq$  tp B)
      unfolding B-def
    proof (subst ordertype-VWF-inc-eq)
      show elts (succ  $\gamma$ )  $\subseteq$  ON
        by (auto simp:  $\gamma$ -def ordertype-VWF-inc-eq intro: Ord-in-Ord)
      have sub: elts (succ  $\gamma$ )  $\subseteq$  elts  $\alpha$ 
        using Ord-trans  $\gamma$   $\langle$ Ord  $\alpha$  $\rangle$  by auto
      then show  $\varphi$  ' elts (succ  $\gamma$ )  $\subseteq$  ON
        using  $\langle$ A  $\subseteq$  ON $\rangle$  bij- $\varphi$  bij-betw-imp-surj-on by blast
      have succ  $\gamma \in$  elts  $\alpha$ 
        using  $\gamma$  Limit-def  $\langle$ Limit  $\alpha$  $\rangle$  by blast
      with A sub show  $\varphi$  u <  $\varphi$  v
        if u  $\in$  elts (succ  $\gamma$ ) and v  $\in$  elts (succ  $\gamma$ ) and u < v for u v
        by (metis  $\varphi$ -def inv-ordermap-VWF-strict-mono-iff subsetD that)
      show  $\neg$   $\alpha \leq$  tp (elts (succ  $\gamma$ ))
        by (metis Limit-def Ord-succ  $\gamma$   $\langle$ Limit  $\alpha$  $\rangle$   $\langle$ Ord  $\gamma$  $\rangle$  mem-not-refl order-
type-eq-Ord vsubsetD)
    qed auto
  then show ?thesis
    using indecomposable-ordertype-ge [OF indec, of A1 B]  $\langle$ small A1 $\rangle$  A1 by
(auto simp: B-def)
  qed
  ultimately show tp (A1 - B) =  $\alpha$ 
    using A1 by blast
  have {x}  $\ll$  (A - B)
    using assms B
  apply (clarsimp simp: less-sets-def  $\varphi$ -def subset-iff)
  by (metis Ord-not-le VWF-iff-Ord-less less-V-def order-refl ordermap-mono-less
trans-VWF wf-VWF)
  with  $\langle$ A1  $\subseteq$  A $\rangle$  show {x}  $\ll$  (A1 - B) by auto
  qed auto
qed

```

the main theorem, from which they derive the headline result

theorem Erdos-Milner-aux:

```

assumes part: partn-lst-VWF  $\alpha$  [k,  $\gamma$ ] 2
and indec: indecomposable  $\alpha$  and k > 1 Ord  $\gamma$  and  $\beta$ :  $\beta \in$  elts  $\omega$ 1
shows partn-lst-VWF ( $\alpha * \beta$ ) [ord-of-nat (2*k), min  $\gamma$  ( $\omega * \beta$ )] 2
proof (cases  $\alpha \leq 1 \vee \beta = 0$ )

```

```

case True
have Ord  $\alpha$ 
  using indec indecomposable-def by blast
show ?thesis
proof (cases  $\beta=0$ )
  case True then show ?thesis
    by (simp add: partn-lst-triv0 [where  $i=1$ ])
  next
  case False
  then consider (0)  $\alpha=0$  | (1)  $\alpha=1$ 
  by (metis Ord-0 OrdmemD True  $\langle$ Ord  $\alpha$  $\rangle$  mem-0-Ord one-V-def order.antisym
succ-le-iff)
  then show ?thesis
  proof cases
    case 0
    with part show ?thesis
      by (force simp add: partn-lst-def nsets-empty-iff less-Suc-eq)
    next
    case 1
    then obtain Ord  $\beta$ 
      by (meson ON-imp-Ord Ord- $\omega$ 1 True  $\beta$  elts-subset-ON)
    then obtain  $i$  where  $i < \text{Suc } (\text{Suc } 0)$  [ord-of-nat  $k, \gamma$ ] !  $i \leq \alpha$ 
      using partn-lst-VWF-nontriv [OF part] 1 by auto
    then have  $\gamma \leq 1$ 
      using  $\langle \alpha=1 \rangle \langle k > 1 \rangle$  by (fastforce simp: less-Suc-eq)
    then have  $\min \gamma (\omega * \beta) \leq 1$ 
      by (metis Ord-1 Ord- $\omega$  Ord-linear-le Ord-mult  $\langle$ Ord  $\beta$  $\rangle$  min-def order-trans)
    then show ?thesis
      using False by (auto simp: True  $\langle$ Ord  $\beta$  $\rangle \langle \beta \neq 0 \rangle \langle \alpha=1 \rangle$  intro!: partn-lst-triv1
[where  $i=1$ ])
  qed
qed
next
case False
then have  $\alpha \geq \omega$ 
  by (meson Ord-1 Ord-not-less indec indecomposable-def indecomposable-imp-Limit
omega-le-Limit)
have  $0 \in \text{elts } \beta$   $\beta \neq 0$ 
  using False Ord- $\omega$ 1 Ord-in-Ord  $\beta$  mem-0-Ord by blast+
show ?thesis
  unfolding partn-lst-def
proof clarsimp
  fix  $f$ 
  assume  $f \in [\text{elts } (\alpha * \beta)]^2 \rightarrow \{.. < \text{Suc } (\text{Suc } 0)\}$ 
  then have  $f: f \in [\text{elts } (\alpha * \beta)]^2 \rightarrow \{.. < 2::\text{nat}\}$ 
    by (simp add: eval-nat-numeral)
  obtain ord [iff]: Ord  $\alpha$  Ord  $\beta$  Ord  $(\alpha * \beta)$ 
    using Ord- $\omega$ 1 Ord-in-Ord  $\beta$  indec indecomposable-imp-Ord Ord-mult by blast
  have *: False

```

if i [rule-format]: $\forall H. tp\ H = ord\text{-of-nat}\ (2*k) \longrightarrow H \subseteq elts\ (\alpha*\beta) \longrightarrow \neg f$
 $\text{' } [H]^2 \subseteq \{0\}$
and ii [rule-format]: $\forall H. tp\ H = \gamma \longrightarrow H \subseteq elts\ (\alpha*\beta) \longrightarrow \neg f \text{' } [H]^2 \subseteq$
 $\{1\}$
and iii [rule-format]: $\forall H. tp\ H = (\omega*\beta) \longrightarrow H \subseteq elts\ (\alpha*\beta) \longrightarrow \neg f \text{' } [H]^2$
 $\subseteq \{1\}$
proof –
have $Ak0$: $\exists X \in [A]^k. f \text{' } [X]^2 \subseteq \{0\}$ — remark (8) about $A \subseteq S$
if $A-\alpha\beta$: $A \subseteq elts\ (\alpha*\beta)$ **and** ot : $tp\ A \geq \alpha$ **for** A
proof –
let $?g = inv\text{-into}\ A\ (ordermap\ A\ VWF)$
have $small\ A$
using $down\ that\ by\ auto$
then **have** $inj\text{-}g$: $inj\text{-on}\ ?g\ (elts\ \alpha)$
by $(meson\ inj\text{-on}\text{-}inv\text{-into}\ less\text{-eq}\text{-}V\text{-def}\ ordermap\text{-}surj\ ot\ subset\text{-}trans)$
have $om\text{-}A\text{-}less$: $\bigwedge x\ y. \llbracket x \in A; y \in A; x < y \rrbracket \implies ordermap\ A\ VWF\ x <$
 $ordermap\ A\ VWF\ y$
using ord
by $(meson\ A-\alpha\beta\ Ord\text{-in}\text{-}Ord\ VWF\text{-iff}\text{-}Ord\text{-}less\ \langle small\ A \rangle\ in\text{-}mono\ or\text{-}$
 $dermap\text{-}mono\text{-}less\ trans\text{-}VWF\ wf\text{-}VWF)$
have $\alpha\text{-}sub$: $elts\ \alpha \subseteq ordermap\ A\ VWF \text{' } A$
by $(metis\ \langle small\ A \rangle\ elts\text{-of}\text{-}set\ less\text{-eq}\text{-}V\text{-def}\ ordertype\text{-}def\ ot\ replacement)$
have g : $?g \in elts\ \alpha \rightarrow elts\ (\alpha*\beta)$
by $(meson\ A-\alpha\beta\ Pi\text{-}I'\ \alpha\text{-}sub\ inv\text{-into}\text{-}into\ subset\text{-}eq)$
then **have** fg : $f \circ (\lambda X. ?g \text{' } X) \in [elts\ \alpha]^2 \rightarrow \{..<2\}$
by $(rule\ nsets\text{-}compose\text{-}image\text{-}funcset\ [OF\ f\ \text{-}\ inj\text{-}g])$
obtain $i\ H$ **where** $i < 2\ H \subseteq elts\ \alpha$
and $ot\text{-}eq$: $tp\ H = [k,\gamma]!i\ (f \circ (\lambda X. ?g \text{' } X)) \text{' } (nsets\ H\ 2) \subseteq \{i\}$
using $ii\ partn\text{-}lst\text{-}E\ [OF\ part\ fg]$ **by** $(auto\ simp: eval\text{-}nat\text{-}numeral)$
then **consider** $(0)\ i=0 \mid (1)\ i=1$
by $linarith$
then **show** $?thesis$
proof $cases$
case 0
then **have** $f \text{' } [inv\text{-into}\ A\ (ordermap\ A\ VWF) \text{' } H]^2 \subseteq \{0\}$
using $ot\text{-}eq\ \langle H \subseteq elts\ \alpha \rangle\ \alpha\text{-}sub$ **by** $(auto\ simp: nsets\text{-}def\ [of\ \text{-}\ k])$
 $inj\text{-on}\text{-}inv\text{-into}\ elim!$: $nset\text{-image}\text{-}obtains)$
moreover **have** $finite\ H \wedge card\ H = k$
using $0\ ot\text{-}eq\ \langle H \subseteq elts\ \alpha \rangle\ down$ **by** $(simp\ add: finite\text{-}ordertype\text{-}eq\text{-}card)$
then **have** $inv\text{-into}\ A\ (ordermap\ A\ VWF) \text{' } H \in [A]^k$
using $\langle H \subseteq elts\ \alpha \rangle\ \alpha\text{-}sub$ **by** $(auto\ simp: nsets\text{-}def\ [of\ \text{-}\ k]\ card\text{-}image$
 $inj\text{-on}\text{-}inv\text{-into}\ inv\text{-into}\text{-}into)$
ultimately **show** $?thesis$
by $blast$
next
case 1
have gH : $?g \text{' } H \subseteq elts\ (\alpha*\beta)$
by $(metis\ A-\alpha\beta\ \alpha\text{-}sub\ \langle H \subseteq elts\ \alpha \rangle\ image\text{-}subsetI\ inv\text{-into}\text{-}into\ subset\text{-}eq)$
have $g\text{-}less$: $?g\ x < ?g\ y$ **if** $x < y\ x \in elts\ \alpha\ y \in elts\ \alpha$ **for** $x\ y$

using *Pi-mem* [*OF g*] *ord that*
by (*meson* $A\text{-}\alpha\beta$ *Ord-in-Ord Ord-not-le* $\langle \text{small } A \rangle$ *dual-order.trans*
elts-subset-ON inv-ordermap-VWF-mono-le ot vsubsetD)
have [*simp*]: $tp (?g \text{ ' } H) = tp H$
by (*meson* $\langle H \subseteq \text{elts } \alpha \rangle$ *ord down dual-order.trans elts-subset-ON gH*
g-less ordertype-VWF-inc-eq subsetD)
show *?thesis*
using *ii* [*of ?g ' H*] *subset-inj-on* [*OF inj-g* $\langle H \subseteq \text{elts } \alpha \rangle$] *ot-eq 1*
by (*auto simp: gH elim!: nset-image-obtains*)
qed
qed
define *K* **where** $K \equiv \lambda i x. \{y \in \text{elts } (\alpha*\beta). x \neq y \wedge f\{x,y\} = i\}$
have *small-K*: *small* ($K i x$) **for** $i x$
by (*simp add: K-def*)
define *KI* **where** $KI \equiv \lambda i X. (\bigcap x \in X. K i x)$
have *KI-disj-self*: $X \cap KI i X = \{\}$ **for** $i X$
by (*auto simp: KI-def K-def*)
define *M* **where** $M \equiv \lambda D \mathfrak{A} x. \{\nu::V. \nu \in D \wedge tp (K 1 x \cap \mathfrak{A} \nu) \geq \alpha\}$
have *M-sub-D*: $M D \mathfrak{A} x \subseteq D$ **for** $D \mathfrak{A} x$
by (*auto simp: M-def*)
have *small-M* [*simp*]: *small* ($M D \mathfrak{A} x$) **if** *small D* **for** $D \mathfrak{A} x$
by (*simp add: M-def that*)
have *9*: $tp \{x \in A. tp (M D \mathfrak{A} x) \geq tp D\} \geq \alpha$ (**is** *ordertype ?AD - $\geq \alpha$*)
if *inD*: *indecomposable* ($tp D$) **and** *D*: $D \subseteq \text{elts } \beta$ **and** *A*: $A \subseteq \text{elts } (\alpha*\beta)$
and *tpA*: $tp A = \alpha$
and \mathfrak{A} : $\mathfrak{A} \in D \rightarrow \{X. X \subseteq \text{elts } (\alpha*\beta) \wedge tp X = \alpha\}$ **for** $D A \mathfrak{A}$
— *remark (9), assuming an indecomposable order type*
proof (*rule ccontr*)
define *A'* **where** $A' \equiv \{x \in A. \neg tp (M D \mathfrak{A} x) \geq tp D\}$
have *small [iff]*: *small A small D*
using *A D down* **by** (*auto simp: M-def*)
have *small- \mathfrak{A}* : *small* ($\mathfrak{A} \delta$) **if** $\delta \in D$ **for** δ
using *that \mathfrak{A}* **by** (*auto simp: Pi-iff subset-iff-less-eq-V*)
assume *not- α -le*: $\neg \alpha \leq tp \{x \in A. tp (M D \mathfrak{A} x) \geq tp D\}$
moreover
obtain *small A small A'* $A' \subseteq A$ **and** *A'-sub*: $A' \subseteq \text{elts } (\alpha*\beta)$
using *A'-def A down* **by** *auto*
moreover **have** $A' = A - ?AD$
by (*force simp: A'-def*)
ultimately **have** *A'-ge*: $tp A' \geq \alpha$
by (*metis (no-types, lifting) dual-order.refl indec indecomposable-ordertype-eq*
mem-Collect-eq subsetI tpA)
obtain *X* **where** $X \subseteq A'$ *finite X card X = k* **and** *fX0*: $f \text{ ' } [X]^2 \subseteq \{0\}$
using *Ak0* [*OF A'-sub A'-ge*] **by** (*auto simp: nsets-def [of - k]*)
then **have** \ddagger : $\neg tp (M D \mathfrak{A} x) \geq tp D$ **if** $x \in X$ **for** x
using *that* **by** (*auto simp: A'-def*)
obtain *x* **where** $x \in X$
using $\langle \text{card } X = k \rangle \langle k > 1 \rangle$ **by** *fastforce*
have $\neg D \subseteq (\bigcup x \in X. M D \mathfrak{A} x)$

proof
assume $\text{not}: D \subseteq (\bigcup_{x \in X}. M D \mathfrak{A} x)$
have $\exists X \in M D \mathfrak{A} \text{ ' } X. \text{tp } D \leq \text{tp } X$
proof (*rule indecomposable-ordertype-finite-ge [OF inD]*)
show $M D \mathfrak{A} \text{ ' } X \neq \{\}$
using $A'\text{-def } A'\text{-ge not not-}\alpha\text{-le}$ **by** *auto*
show $\text{small} (\bigcup (M D \mathfrak{A} \text{ ' } X))$
using $\langle \text{finite } X \rangle$ **by** (*simp add: finite-imp-small*)
qed (*use* $\langle \text{finite } X \rangle$ **not in** *auto*)
then show *False*
by (*simp add: †*)
qed
then obtain ν **where** $\nu \in D$ **and** $\nu: \nu \notin (\bigcup_{x \in X}. M D \mathfrak{A} x)$
by *blast*
define \mathcal{A} **where** $\mathcal{A} \equiv \{KI 0 X \cap \mathfrak{A} \nu, \bigcup_{x \in X}. K 1 x \cap \mathfrak{A} \nu, X \cap \mathfrak{A} \nu\}$
have $\alpha\beta: X \subseteq \text{elts} (\alpha*\beta) \mathfrak{A} \nu \subseteq \text{elts} (\alpha*\beta)$
using $A'\text{-sub} \langle X \subseteq A' \rangle \mathfrak{A} \langle \nu \in D \rangle$ **by** *auto*
then have $KI 0 X \cup (\bigcup_{x \in X}. K 1 x) \cup X = \text{elts} (\alpha*\beta)$
using $\langle x \in X \rangle f$ **by** (*force simp: K-def KI-def Pi-iff less-2-cases-iff*)
with $\alpha\beta$ **have** $\mathfrak{A}\nu\text{-}\mathcal{A}: \text{finite } \mathcal{A} \mathfrak{A} \nu \subseteq \bigcup \mathcal{A}$
by (*auto simp: A-def*)
then have $\neg \text{tp} (K 1 x \cap \mathfrak{A} \nu) \geq \alpha$ **if** $x \in X$ **for** x
using *that* $\langle \nu \in D \rangle \nu \langle k > 1 \rangle \langle \text{card } X = k \rangle$ **by** (*fastforce simp: M-def*)
moreover have $\text{sm-K1}: \text{small} (\bigcup_{x \in X}. K 1 x \cap \mathfrak{A} \nu)$
by (*meson Finite-V Int-lower2* $\langle \nu \in D \rangle \langle \text{finite } X \rangle \text{small-}\mathfrak{A} \text{small-UN}$
smaller-than-small)
ultimately have $\text{not1}: \neg \text{tp} (\bigcup_{x \in X}. K 1 x \cap \mathfrak{A} \nu) \geq \alpha$
using $\langle \text{finite } X \rangle \langle x \in X \rangle$ *indecomposable-ordertype-finite-ge [OF indec, of*
 $(\lambda x. K 1 x \cap \mathfrak{A} \nu) \text{ ' } X]$ **by** *blast*
moreover have $\neg \text{tp} (X \cap \mathfrak{A} \nu) \geq \alpha$
using $\langle \text{finite } X \rangle \langle \alpha \geq \omega \rangle$
by (*meson finite-Int mem-not-refl ordertype-VWF- ω vsubsetD*)
moreover have $\alpha \leq \text{tp} (\mathfrak{A} \nu)$
using $\mathfrak{A} \langle \nu \in D \rangle \text{small-}\mathfrak{A}$ **by** *fastforce+*
moreover have $\text{small} (\bigcup \mathcal{A})$
using $\langle \nu \in D \rangle \text{small-}\mathfrak{A}$ **by** (*fastforce simp: A-def intro: smaller-than-small*
sm-K1)
ultimately have $K0\mathfrak{A}\text{-ge}: \text{tp} (KI 0 X \cap \mathfrak{A} \nu) \geq \alpha$
using *indecomposable-ordertype-finite-ge [OF indec* $\mathfrak{A}\nu\text{-}\mathcal{A}]$ **by** (*auto simp:*
A-def)
have $\mathfrak{A}\nu: \mathfrak{A} \nu \subseteq \text{elts} (\alpha*\beta) \text{tp} (\mathfrak{A} \nu) = \alpha$
using $\langle \nu \in D \rangle \mathfrak{A}$ **by** *blast+*
then obtain Y **where** $Y_{\text{sub}}: Y \subseteq KI 0 X \cap \mathfrak{A} \nu$ **and** *finite* Y $\text{card } Y = k$
and $fY0: f \text{ ' } [Y]^2 \subseteq \{0\}$
using $Ak0$ [*OF - K0A-ge*] **by** (*auto simp: nsets-def [of - k]*)
have $\dagger: X \cap Y = \{\}$
using Y_{sub} *KI-disj-self* **by** *blast*
then have $\text{card} (X \cup Y) = 2*k$
by (*simp add:* $\langle \text{card } X = k \rangle \langle \text{card } Y = k \rangle \langle \text{finite } X \rangle \langle \text{finite } Y \rangle$)

card-Un-disjoint)
moreover have $X \cup Y \subseteq \text{elts } (\alpha * \beta)$
using A' -sub $\langle X \subseteq A' \rangle$ $\mathfrak{A} \nu(1)$ $\langle Y \subseteq KI\ 0\ X \cap \mathfrak{A}\ \nu \rangle$ **by** *auto*
moreover have $f '[X \cup Y]^2 \subseteq \{0\}$
using $fX0\ fY0\ Ysub$ **by** (*auto simp: † nsets-disjoint-2 image-Un image-UN*)
KI-def K-def)
ultimately show *False*
using i $\langle finite\ X \rangle$ $\langle finite\ Y \rangle$ *ordertype-VWF-finite-nat* **by** *auto*
qed
have $IX: tp \{x \in A. tp (M\ D\ \mathfrak{A}\ x) \geq tp\ D\} \geq \alpha$
if $D: D \subseteq \text{elts } \beta$ **and** $A: A \subseteq \text{elts } (\alpha * \beta)$ **and** $tpA: tp\ A = \alpha$
and $\mathfrak{A}: \mathfrak{A} \in D \rightarrow \{X. X \subseteq \text{elts } (\alpha * \beta) \wedge tp\ X = \alpha\}$ **for** $D\ A\ \mathfrak{A}$
— remark (9) for any order type
proof –
obtain L **where** $UL: \bigcup (List.set\ L) \subseteq D$
and $indL: \bigwedge X. X \in List.set\ L \implies indecomposable (tp\ X)$
and $eqL: \bigwedge M. \llbracket M \subseteq D; \bigwedge X. X \in List.set\ L \implies tp (M \cap X) \geq tp\ X \rrbracket$
 $\implies tp\ M = tp\ D$
using *ord* **by** (*metis strong-ordertype-eq D order-refl*)
obtain A'' **where** $A'': A'' \subseteq A\ tp\ A'' \geq \alpha$
and $\bigwedge x\ X. \llbracket x \in A''; X \in List.set\ L \rrbracket \implies tp (M\ D\ \mathfrak{A}\ x \cap X) \geq tp\ X$
using $UL\ indL$
proof (*induction L arbitrary: thesis*)
case ($Cons\ X\ L$)
then obtain A'' **where** $A'': A'' \subseteq A\ tp\ A'' \geq \alpha$ **and** $X \subseteq D$
and $ge-X: \bigwedge x\ X. \llbracket x \in A''; X \in List.set\ L \rrbracket \implies tp (M\ D\ \mathfrak{A}\ x \cap X) \geq tp\ X$
 X **by** *auto*
then have $tp-A'': tp\ A'' = \alpha$
by (*metis A antisym down ordertype-VWF-mono tpA*)
have $ge-\alpha: tp \{x \in A''. tp (M\ X\ \mathfrak{A}\ x) \geq tp\ X\} \geq \alpha$
by (*rule 9*) (*use A A'' tp-A'' Cons.premis* $\langle D \subseteq \text{elts } \beta \rangle$ $\langle X \subseteq D \rangle$ \mathfrak{A} **in**
auto)
let $?A = \{x \in A''. tp (M\ D\ \mathfrak{A}\ x \cap X) \geq tp\ X\}$
have $M\text{-eq}: M\ D\ \mathfrak{A}\ x \cap X = M\ X\ \mathfrak{A}\ x$ **if** $x \in A''$ **for** x
using *that* $\langle X \subseteq D \rangle$ **by** (*auto simp: M-def*)
show *thesis*
proof (*rule Cons.premis*)
show $\alpha \leq tp\ ?A$
using $ge-\alpha$ **by** (*simp add: M-eq cong: conj-cong*)
show $tp\ Y \leq tp (M\ D\ \mathfrak{A}\ x \cap Y)$ **if** $x \in ?A\ Y \in list.set (X \# L)$ **for** $x\ Y$
using *that* $ge-X$ **by** *force*
qed (*use A'' in auto*)
qed (*use tpA in auto*)
then have $tp-M-ge: tp (M\ D\ \mathfrak{A}\ x) \geq tp\ D$ **if** $x \in A''$ **for** x
using eqL *that* **by** (*auto simp: M-def*)
have $\alpha \leq tp\ A''$
by (*simp add: A''*)
also have $\dots \leq tp \{x \in A''. tp (M\ D\ \mathfrak{A}\ x) \geq tp\ D\}$
by (*metis (mono-tags, lifting) tp-M-ge eq-iff mem-Collect-eq subsetI*)

also have $\dots \leq tp \{x \in A. tp D \leq tp (M D \mathfrak{A} x)\}$
by (rule ordertype-mono) (use $A'' A$ down in auto)
finally show ?thesis .
qed
have IX' : $tp \{x \in A'. tp (K 1 x \cap A) \geq \alpha\} \geq \alpha$
if $A: A \subseteq elts (\alpha*\beta)$ $tp A = \alpha$ **and** $A': A' \subseteq elts (\alpha*\beta)$ $tp A' = \alpha$ **for** $A A'$
proof –
have $\ddagger: \alpha \leq tp (K 1 t \cap A)$ **if** $1 \leq tp \{\nu. \nu = 0 \wedge \alpha \leq tp (K 1 t \cap A)\}$ **for**
 t
using that
by (metis Collect-empty-eq less-eq-V-0-iff ordertype-empty zero-neq-one)
have $tp \{x \in A'. 1 \leq tp \{\nu. \nu = 0 \wedge \alpha \leq tp (K 1 x \cap A)\}\}$
 $\leq tp \{x \in A'. \alpha \leq tp (K 1 x \cap A)\}$
by (rule ordertype-mono) (use $\ddagger A'$ in ‹auto simp: down›)
then show ?thesis
using IX [of $\{0\} A' \lambda x. A$] that ‹ $0 \in elts \beta$ › **by** (force simp: M-def)
qed
have $10: \exists x0 \in A. \exists g \in elts \beta \rightarrow elts \beta. strict-mono-on (elts \beta) g \wedge (\forall \nu \in F. g \nu = \nu)$
 $\wedge (\forall \nu \in elts \beta. tp (K 1 x0 \cap \mathfrak{A} (g \nu)) \geq \alpha)$
if $F: finite F F \subseteq elts \beta$
and $A: A \subseteq elts (\alpha*\beta)$ $tp A = \alpha$
and $\mathfrak{A}: \mathfrak{A} \in elts \beta \rightarrow \{X. X \subseteq elts (\alpha*\beta) \wedge tp X = \alpha\}$
for $F A \mathfrak{A}$
proof –
define p where $p \equiv card F$
have $\beta \notin F$
using that **by** auto
then obtain $\iota :: nat \Rightarrow V$ **where** $bij\iota: bij-betw \iota \{..p\}$ (insert βF) **and**
 $monu: strict-mono-on \{..p\} \iota$
using ZFC-Cardinals.ex-bij-betw-strict-mono-card [of insert βF] $elts-subset-ON$
‹ $Ord \beta$ › F
by (simp add: p-def lessThan-Suc-atMost) blast
have $less-\iota-I: \iota k < \iota l$ **if** $k < l l \leq p$ **for** $k l$
using $monu$ that **by** (auto simp: strict-mono-on-def)
then have $less-\iota-D: k < l$ **if** $\iota k < \iota l k \leq p$ **for** $k l$
by (metis less-asym linorder-neqE-nat that)
have $Ord-\iota: Ord (\iota k)$ **if** $k \leq p$ **for** k
by (metis (no-types, lifting) ON-imp-Ord atMost-iff insert-subset mem-Collect-eq
order-trans ‹ $F \subseteq elts \beta$ › $bij\iota$ $bij-betwE$ $elts-subset-ON$ ‹ $Ord \beta$ › that)
have $le-\iota 0$ [simp]: $\bigwedge j. j \leq p \implies \iota 0 \leq \iota j$
by (metis eq-refl leI le-0-eq less-\iota-I less-imp-le)
have $le-\iota: \iota i \leq \iota (j - Suc 0)$ **if** $i < j j \leq p$ **for** $i j$
proof (cases i)
case 0 **then show** ?thesis
using $le-\iota 0$ that **by** auto
next
case (Suc i') **then show** ?thesis

by (metis (no-types, opaque-lifting) Suc-pred le-less less-Suc-eq less-Suc-eq-0-disj
less-ι-I not-less-eq that)

qed

have [simp]: $\iota p = \beta$

proof –

obtain k where $k: \iota k = \beta k \leq p$

by (meson atMost-iff bij ι bij-betw-iff-bijections insertI1)

then have $k = p \vee k < p$

by linarith

then show ?thesis

using bij ι ord k that(2)

by (metis OrdmemD atMost-iff bij-betw-iff-bijections insert-iff leD less-ι-D
order-refl subsetD)

qed

have F -imp-Ex: $\exists k < p. \xi = \iota k$ if $\xi \in F$ for ξ

proof –

obtain k where $k: k \leq p \xi = \iota k$

by (metis $\langle \xi \in F \rangle$ atMost-iff bij ι bij-betw-def imageE insert-iff)

then show ?thesis

using $\langle \beta \notin F \rangle \langle \iota p = \beta \rangle$ le-imp-less-or-eq that by blast

qed

have F -imp-ge: $\xi \geq \iota 0$ if $\xi \in F$ for ξ

using F -imp-Ex [OF that] by (metis dual-order.order-iff-strict le0 less-ι-I)

define D where $D \equiv \lambda k. (if\ k=0\ then\ \{..<\iota\ 0\}\ else\ \{\iota\ (k-1)<..<\iota\ k\}) \cap$
elts β

have $D\beta: D\ k \subseteq$ elts β for k

by (auto simp: D-def)

then have small-D [simp]: small (D k) for k

by (meson down)

have M -Int-D: M (elts β) $\mathcal{A} x \cap D\ k = M$ (D k) $\mathcal{A} x$ if $k \leq p$ for $x\ k$

using $D\beta$ by (auto simp: M-def)

have ι -le-if-D: $\iota k \leq \mu$ if $\mu \in D$ (Suc k) for $\mu\ k$

using that by (simp add: D-def order.order-iff-strict)

have mono-D: $D\ i \ll D\ j$ if $i < j\ j \leq p$ for $i\ j$

proof (cases j)

case (Suc j')

with that show ?thesis

apply (simp add: less-sets-def D-def Ball-def)

by (metis One-nat-def diff-Suc-1 le-ι less-le-trans less-trans)

qed (use that in auto)

then have disjnt-DD: disjnt (D i) (D j) if $i \neq j\ i \leq p\ j \leq p$ for $i\ j$

by (meson disjnt-sym less-linear less-sets-imp-disjnt that)

have UN-D-eq: $(\bigcup l \leq k. D\ l) = \{..<\iota\ k\} \cap$ (elts $\beta - F$) if $k \leq p$ for k

using that

proof (induction k)

case 0

then show ?case

```

    by (auto simp: D-def F-imp-ge leD)
  next
    case (Suc k)
    have D (Suc k)  $\cup$   $\{..<\iota k\} \cap (\text{elts } \beta - F) = \{..<\iota (Suc k)\} \cap (\text{elts } \beta - F)$ 
      (is ?lhs = ?rhs)
    proof
      show ?lhs  $\subseteq$  ?rhs
      using Suc.prem1
      by (auto simp: D-def if-split-mem2 intro: less- $\iota$ -I less-trans dest!: less- $\iota$ -D
F-imp-Ex)
      have  $\bigwedge x. [x < \iota (Suc k); x \in \text{elts } \beta; x \notin F; \iota k \leq x] \implies \iota k < x$ 
        using Suc.prem1  $\langle F \subseteq \text{elts } \beta \rangle$  bij le-imp-less-or-eq
        by (fastforce simp: bij-betw-iff-bijections)
      then show ?rhs  $\subseteq$  ?lhs
        using Suc.prem1 by (auto simp: D-def Ord-not-less Ord-in-Ord [OF
 $\langle \text{Ord } \beta \rangle$ ] Ord- $\iota$  if-split-mem2)
      qed
    then
      show ?case
      using Suc by (simp add: atMost-Suc)
    qed
  have  $\beta$ -decomp:  $\text{elts } \beta = F \cup (\bigcup_{k \leq p} D k)$ 
    using  $\langle F \subseteq \text{elts } \beta \rangle$  OrdmemD [OF  $\langle \text{Ord } \beta \rangle$ ] by (auto simp: UN-D-eq)
  define  $\beta$ idx where  $\beta$ idx  $\equiv \lambda \nu. @k. \nu \in D k \wedge k \leq p$ 
  have  $\beta$ idx:  $\nu \in D (\beta$ idx  $\nu) \wedge \beta$ idx  $\nu \leq p$  if  $\nu \in \text{elts } \beta - F$  for  $\nu$ 
    using that by (force simp:  $\beta$ idx-def  $\beta$ -decomp intro: someI-ex del: conjI)
  have any-imp- $\beta$ idx:  $k = \beta$ idx  $\nu$  if  $\nu \in D k$   $k \leq p$  for  $k \nu$ 
  proof (rule ccontr)
    assume non:  $k \neq \beta$ idx  $\nu$ 
    have  $\nu \notin F$ 
      using that UN-D-eq by auto
    then show False
      using disjnt-DD [OF non] by (metis D $\beta$  Diff-iff  $\beta$ idx disjnt-iff subsetD
that)
    qed
  have  $\exists A'. A' \subseteq A \wedge \text{tp } A' = \alpha \wedge (\forall x \in A'. F \subseteq M (\text{elts } \beta) \mathfrak{A} x)$ 
    using F
  proof induction
    case (insert  $\nu$  F)
    then obtain  $A'$  where  $A' \subseteq A$  and  $A'$ :  $A' \subseteq \text{elts } (\alpha * \beta)$   $\text{tp } A' = \alpha$  and
FN:  $\bigwedge x. x \in A' \implies F \subseteq M (\text{elts } \beta) \mathfrak{A} x$ 
      using A(1) by auto
    define  $A''$  where  $A'' \equiv \{x \in A'. \alpha \leq \text{tp } (K 1 x \cap \mathfrak{A} \nu)\}$ 
    have  $\nu \in \text{elts } \beta$   $F \subseteq \text{elts } \beta$ 
      using insert by auto
    note ordertype-eq-Ord [OF  $\langle \text{Ord } \beta \rangle$ , simp]
    show ?case
    proof (intro exI conjI)
      show  $A'' \subseteq A$ 

```

```

    using  $\langle A' \subseteq A \rangle$  by (auto simp:  $A''$ -def)
  show  $tp\ A'' = \alpha$ 
  proof (rule antisym)
    show  $tp\ A'' \leq \alpha$ 
      using  $\langle A'' \subseteq A \rangle$  down ordertype-VWF-mono  $A$  by blast
    have  $\mathfrak{A}\ \nu \subseteq elts\ (\alpha * \beta)$   $tp\ (\mathfrak{A}\ \nu) = \alpha$ 
      using  $\mathfrak{A}\ \langle \nu \in elts\ \beta \rangle$  by auto
    then show  $\alpha \leq tp\ A''$ 
      using  $IX'\ [OF\ -\ -\ A']$  by (simp add:  $A''$ -def)
  qed
  show  $\forall x \in A''.\ insert\ \nu\ F \subseteq M\ (elts\ \beta)\ \mathfrak{A}\ x$ 
    using  $A''$ -def  $FN\ M$ -def  $\langle \nu \in elts\ \beta \rangle$  by blast
  qed
  qed (use  $A$  in auto)
  then obtain  $A'$  where  $A': A' \subseteq A\ tp\ A' = \alpha$  and  $FN: \bigwedge x.\ x \in A' \implies F$ 
 $\subseteq M\ (elts\ \beta)\ \mathfrak{A}\ x$ 
    by metis
  have False
    if *:  $\bigwedge x0\ g.\ [\![x0 \in A; g \in elts\ \beta \rightarrow elts\ \beta; strict\ mono\ on\ (elts\ \beta)\ g]\!] \implies (\exists \nu \in F.\ g\ \nu \neq \nu) \vee (\exists \nu \in elts\ \beta.\ tp\ (K\ 1\ x0 \cap \mathfrak{A}\ (g\ \nu)) < \alpha)$ 
  proof -
    { fix  $x$  — construction of the monotone map  $g$  mentioned above
      assume  $x \in A'$ 
      with  $A'$  have  $x \in A$  by blast
      have  $\exists k.\ k \leq p \wedge tp\ (M\ (D\ k)\ \mathfrak{A}\ x) < tp\ (D\ k)$  (is ?P)
      proof (rule ccontr)
        assume  $\neg ?P$ 
        then have  $le: tp\ (D\ k) \leq tp\ (M\ (D\ k)\ \mathfrak{A}\ x)$  if  $k \leq p$  for  $k$ 
          by (meson Ord-linear2 Ord-ordertype that)
        have  $\exists f \in D\ k \rightarrow M\ (D\ k)\ \mathfrak{A}\ x.\ inj\ on\ f\ (D\ k) \wedge (strict\ mono\ on\ (D\ k)\ f)$ 
          if  $k \leq p$  for  $k$ 
            using le [OF that] that VWF-iff-Ord-less
            apply (clarsimp simp: ordertype-le-ordertype strict-mono-on-def)
            by (metis (full-types)  $D\beta\ M$ -sub- $D\ Ord$ -in- $Ord\ PiE\ VWF$ -iff-Ord-less
            ord(2) subsetD)
        then obtain  $h$  where  $fun\ h: \bigwedge k.\ k \leq p \implies h\ k \in D\ k \rightarrow M\ (D\ k)\ \mathfrak{A}\ x$ 
          and  $inj\ h: \bigwedge k.\ k \leq p \implies inj\ on\ (h\ k)\ (D\ k)$ 
          and  $mono\ h: \bigwedge k\ x\ y.\ k \leq p \implies strict\ mono\ on\ (D\ k)\ (h\ k)$ 
          by metis
        then have  $fun\ hD: \bigwedge k.\ k \leq p \implies h\ k \in D\ k \rightarrow D\ k$ 
          by (auto simp:  $M$ -def)
        have  $h$ -increasing:  $\nu \leq h\ k\ \nu$ 
          if  $k \leq p\ \nu \in D\ k$  for  $k\ \nu$ 
            by (meson  $D\beta\ Ord$ -mono-imp-increasing ord dual-order.trans
            elts-subset-ON  $fun\ hD\ mono\ h$  that)
        define  $g$  where  $g \equiv \lambda \nu.\ if\ \nu \in F\ then\ \nu\ else\ h\ (\beta id\ x\ \nu)\ \nu$ 
        have [simp]:  $g\ \nu = \nu$  if  $\nu \in F$  for  $\nu$ 
          using that by (auto simp:  $g$ -def)

```

have *fun-g*: $g \in \text{elts } \beta \rightarrow \text{elts } \beta$
proof (*rule Pi-I*)
fix *x* **assume** $x \in \text{elts } \beta$
then have $x \in D (\beta \text{idx } x) \beta \text{idx } x \leq p$ **if** $x \notin F$
using *that* **by** (*auto simp: beta*)
then show $g x \in \text{elts } \beta$
by (*metis fun-h D beta M-sub-D x in elts beta PiE g-def subsetD*)
qed
have *h-in-D*: $h (\beta \text{idx } \nu) \nu \in D (\beta \text{idx } \nu)$ **if** $\nu \notin F$ $\nu \in \text{elts } \beta$ **for** ν
using *beta fun-hD* **that** **by** *fastforce*
have *1*: $\iota k < h (\beta \text{idx } \nu) \nu$
if $k < p$ **and** $\nu: \nu \notin F$ $\nu \in \text{elts } \beta$ **and** $\iota k < \nu$ **for** $k \nu$
by (*meson that h-in-D [OF nu] beta DiffI h-increasing order-less-le-trans*)
moreover have *2*: $h (\beta \text{idx } \mu) \mu < \iota k$
if $\mu: \mu \notin F$ $\mu \in \text{elts } \beta$ **and** $k < p$ $\mu < \iota k$ **for** μk
proof –
have $\beta \text{idx } \mu \leq k$
proof (*rule ccontr*)
assume $\neg \beta \text{idx } \mu \leq k$
then have $k < \beta \text{idx } \mu$
by *linarith*
then show *False*
using *iota-le-if-D beta* **that** **by** (*metis Diff-iff Suc-pred le0 leD le-
le-less-trans*)
qed
then show *?thesis*
using *that h-in-D [OF mu]*
by (*smt (verit, best) Int-lower1 UN-D-eq UN-I atMost-iff lessThan-iff
less-imp-le subset-eq*)
qed
moreover have $h (\beta \text{idx } \mu) \mu < h (\beta \text{idx } \nu) \nu$
if $\mu: \mu \notin F$ $\mu \in \text{elts } \beta$ **and** $\nu: \nu \notin F$ $\nu \in \text{elts } \beta$ **and** $\mu < \nu$ **for** $\mu \nu$
proof –
have *le*: $\beta \text{idx } \mu \leq \beta \text{idx } \nu$ **if** $\iota (\beta \text{idx } \mu - \text{Suc } 0) < h (\beta \text{idx } \mu) \mu$ h
 $(\beta \text{idx } \nu) \nu < \iota (\beta \text{idx } \nu)$
by (*metis 2 DiffI beta mu nu mu < nu order.strict-trans h-increasing
leI le- ι order-less-*asym* order-less-le-trans that*)
have $h 0 \mu < h 0 \nu$ **if** $\beta \text{idx } \mu = 0$ $\beta \text{idx } \nu = 0$
using *that mono-h unfolding strict-mono-on-def*
by (*metis Diff-iff beta mu nu mu < nu*)
moreover have $h 0 \mu < h (\beta \text{idx } \nu) \nu$
if $0 < \beta \text{idx } \nu$ $h 0 \mu < \iota 0$ **and** $\iota (\beta \text{idx } \nu - \text{Suc } 0) < h (\beta \text{idx } \nu) \nu$
by (*meson DiffI beta nu le- ι le-less-trans less-le-not-le that*)
moreover have $\beta \text{idx } \nu \neq 0$
if $0 < \beta \text{idx } \mu$ $h 0 \nu < \iota 0$ $\iota (\beta \text{idx } \mu - \text{Suc } 0) < h (\beta \text{idx } \mu) \mu$
using *le le-0-eq* **that** **by** *fastforce*
moreover have $h (\beta \text{idx } \mu) \mu < h (\beta \text{idx } \nu) \nu$
if $\iota (\beta \text{idx } \mu - \text{Suc } 0) < h (\beta \text{idx } \mu) \mu$ $h (\beta \text{idx } \nu) \nu < \iota (\beta \text{idx } \nu)$
 $h (\beta \text{idx } \mu) \mu < \iota (\beta \text{idx } \mu)$ $\iota (\beta \text{idx } \nu - \text{Suc } 0) < h (\beta \text{idx } \nu) \nu$

using *mono-h unfolding strict-mono-on-def*
by (*metis le Diff-iff $\beta \text{id}_x \mu \nu \langle \mu < \nu \rangle \text{le-}l \text{le-less le-less-trans}$ that*)
ultimately show *?thesis*
using *h-in-D [OF μ] h-in-D [OF ν] by (simp add: D-def split:*
if-split-asm)
qed
ultimately have *sm-g: strict-mono-on (elts β) g*
by (*auto simp: g-def strict-mono-on-def dest!: F-imp-Ex*)
have *False if $\nu \in \text{elts } \beta$ and ν : $tp (K 1 x \cap \mathfrak{A} (g \nu)) < \alpha$ for ν*
proof –
have *$F \subseteq M (\text{elts } \beta) \mathfrak{A} x$*
by (*meson FN $\langle x \in A' \rangle$*)
then have *False if $tp (K (Suc 0) x \cap \mathfrak{A} \nu) < \alpha \nu \in F$*
using *that by (auto simp: M-def)*
moreover have *False if $tp (K (Suc 0) x \cap \mathfrak{A} (g \nu)) < \alpha \nu \in D k k$*
 $\leq p \nu \notin F$ **for** *k*
proof –
have *$h (\beta \text{id}_x \nu) \nu \in M (D (\beta \text{id}_x \nu)) \mathfrak{A} x$*
using *fun-h $\beta \text{id}_x \langle \nu \in \text{elts } \beta \rangle \langle \nu \notin F \rangle$ by auto*
then show *False*
using *that by (simp add: M-def g-def leD)*
qed
ultimately show *False*
using *$\langle \nu \in \text{elts } \beta \rangle \nu$ by (force simp: β -decomp)*
qed
then show *False*
using ** [OF $\langle x \in A \rangle$ fun-g sm-g] by auto*
qed
then have $\exists l. l \leq p \wedge tp (M (\text{elts } \beta) \mathfrak{A} x \cap D l) < tp (D l)$
using *M-Int-D by auto*
}
then obtain *l where $lp: \bigwedge x. x \in A' \implies l x \leq p$*
and *lless: $\bigwedge x. x \in A' \implies tp (M (\text{elts } \beta) \mathfrak{A} x \cap D (l x)) < tp (D (l x))$*
by *metis*
obtain *$A'' L$ where $A'' \subseteq A'$ and $A'': A'' \subseteq \text{elts } (\alpha * \beta) tp A'' = \alpha$ and*
 $lL: \bigwedge x. x \in A'' \implies l x = L$
proof –
have *eq: $A' = (\bigcup_{i \leq p}. \{x \in A'. l x = i\})$*
using *lp by auto*
have $\exists X \in (\lambda i. \{x \in A'. l x = i\}) ' \{..p\}. \alpha \leq tp X$
proof (*rule indecomposable-ordertype-finite-ge [OF indec]*)
show *small $(\bigcup_{i \leq p}. \{x \in A'. l x = i\})$*
by (*metis $A'(1) A(1)$ eq down smaller-than-small*)
qed (*use A' eq in auto*)
then show *thesis*
proof
fix *A''*
assume *$A'': A'' \in (\lambda i. \{x \in A'. l x = i\}) ' \{..p\}$ and $\alpha \leq tp A''$*
then obtain *L where $L: \bigwedge x. x \in A'' \implies l x = L$*

```

    by auto
    have  $A'' \subseteq A'$ 
    using  $A''$  by force
    then have  $tp\ A'' \leq tp\ A'$ 
    by (meson  $A'\ A$  down order-trans ordertype-VWF-mono)
    with  $\langle \alpha \leq tp\ A'' \rangle$  have  $tp\ A'' = \alpha$ 
    using  $A'(2)$  by auto
    moreover have  $A'' \subseteq elts\ (\alpha * \beta)$ 
    using  $A'\ A\ \langle A'' \subseteq A' \rangle$  by auto
    ultimately show thesis
    using  $L$  that  $[OF\ \langle A'' \subseteq A' \rangle]$  by blast
  qed
  qed
  have  $\mathfrak{A}D: \mathfrak{A} \in D\ L \rightarrow \{X. X \subseteq elts\ (\alpha * \beta) \wedge tp\ X = \alpha\}$ 
  using  $\mathfrak{A}\ D\beta$  by blast
  have  $\alpha: \alpha \leq tp\ \{x \in A''. tp\ (D\ L) \leq tp\ (M\ (D\ L)\ \mathfrak{A}\ x)\}$ 
  using  $IX\ [OF\ D\beta\ A''\ \mathfrak{A}D]$  by simp
  have  $M\ (elts\ \beta)\ \mathfrak{A}\ x \cap D\ L = M\ (D\ L)\ \mathfrak{A}\ x$  for  $x$ 
  using  $D\beta$  by (auto simp:  $M$ -def)
  then have  $tp\ (M\ (D\ L)\ \mathfrak{A}\ x) < tp\ (D\ L)$  if  $x \in A''$  for  $x$ 
  using less that  $\langle A'' \subseteq A' \rangle\ lL$  by force
  then have  $[simp]: \{x \in A''. tp\ (D\ L) \leq tp\ (M\ (D\ L)\ \mathfrak{A}\ x)\} = \{\}$ 
  using  $leD$  by blast
  show False
  using  $\alpha\ \langle \alpha \geq \omega \rangle$  by simp
  qed
  then show ?thesis
  by (meson  $Ord$ -linear2  $Ord$ -ordertype  $\langle Ord\ \alpha \rangle$ )
  qed
  let  $?U = UNIV :: nat\ set$ 
  define  $\mu$  where  $\mu \equiv fst \circ from\ nat\ into\ (elts\ \beta \times ?U)$ 
  define  $q$  where  $q \equiv to\ nat\ on\ (elts\ \beta \times ?U)$ 
  have  $co\text{-}\beta U: countable\ (elts\ \beta \times ?U)$ 
  by (simp add:  $\beta$  less- $\omega$ 1-imp-countable)
  moreover have  $elts\ \beta \times ?U \neq \{\}$ 
  using  $\langle 0 \in elts\ \beta \rangle$  by blast
  ultimately have  $range\ (from\ nat\ into\ (elts\ \beta \times ?U)) = (elts\ \beta \times ?U)$ 
  by (metis range-from-nat-into)
  then have  $\mu$ -in- $\beta$   $[simp]: \mu\ i \in elts\ \beta$  for  $i$ 
  by (metis  $SigmaE\ \mu$ -def comp-apply fst-conv range-eqI)

  then have  $Ord$ - $\mu$   $[simp]: Ord\ (\mu\ i)$  for  $i$ 
  using  $Ord$ -in- $Ord$  by blast

  have  $inf$ - $\beta U: infinite\ (elts\ \beta \times ?U)$ 
  using  $\langle 0 \in elts\ \beta \rangle$  finite-cartesian-productD2 by auto

  have 11  $[simp]: \mu\ (q\ (\nu, n)) = \nu$  if  $\nu \in elts\ \beta$  for  $\nu\ n$ 
  by (simp add:  $\mu$ -def  $q$ -def that  $co\text{-}\beta U$ )

```

have *range- μ [simp]*: *range* $\mu = \text{elts } \beta$
by (*auto simp: image-iff*) (*metis 11*)
have [simp]: $KI\ i\ \{\} = UNIV\ KI\ i\ (\text{insert } a\ X) = K\ i\ a \cap KI\ i\ X$ **for** $i\ a\ X$
by (*auto simp: KI-def*)
define Φ **where** $\Phi \equiv \lambda n::nat. \lambda \mathfrak{A}\ x. (\forall \nu \in \text{elts } \beta. \mathfrak{A}\ \nu \subseteq \text{elts } (\alpha * \beta) \wedge tp\ (\mathfrak{A}\ \nu) = \alpha)$

$$\wedge x\ '\ \{\dots < n\} \subseteq \text{elts } (\alpha * \beta)$$

$$\wedge (\bigcup \nu \in \text{elts } \beta. \mathfrak{A}\ \nu) \subseteq KI\ 1\ (x\ '\ \{\dots < n\})$$

$$\wedge \text{strict-mono-sets } (\text{elts } \beta)\ \mathfrak{A}$$
define Ψ **where** $\Psi \equiv \lambda n::nat. \lambda g\ \mathfrak{A}\ \mathfrak{A}'\ xn. g \in \text{elts } \beta \rightarrow \text{elts } \beta \wedge \text{strict-mono-on } (\text{elts } \beta)\ g$

$$\wedge (\forall i \leq n. g\ (\mu\ i) = \mu\ i)$$

$$\wedge (\forall \nu \in \text{elts } \beta. \mathfrak{A}'\ \nu \subseteq K\ 1\ xn \cap \mathfrak{A}\ (g\ \nu))$$

$$\wedge \{xn\} \ll (\mathfrak{A}'\ (\mu\ n)) \wedge xn \in \mathfrak{A}\ (\mu\ n)$$
let $\mathfrak{A}0 = \lambda \nu. \text{plus } (\alpha * \nu)\ '\ \text{elts } \alpha$
have *base*: $\Phi\ 0\ \mathfrak{A}0\ x$ **for** x
by (*auto simp: Φ -def add-mult-less add-mult-less-add-mult ordertype-image-plus strict-mono-sets-def less-sets-def*)
have *step*: $Ex\ (\lambda(g, \mathfrak{A}', xn). \Psi\ n\ g\ \mathfrak{A}\ \mathfrak{A}'\ xn \wedge \Phi\ (Suc\ n)\ \mathfrak{A}'\ (x(n:=xn)))$ **if** $\Phi\ n\ \mathfrak{A}\ x$ **for** $n\ \mathfrak{A}\ x$
proof –
have \mathfrak{A} : $\bigwedge \nu. \nu \in \text{elts } \beta \implies \mathfrak{A}\ \nu \subseteq \text{elts } (\alpha * \beta) \wedge tp\ (\mathfrak{A}\ \nu) = \alpha$
and x : $x\ '\ \{\dots < n\} \subseteq \text{elts } (\alpha * \beta)$
and *sub*: $\bigcup (\mathfrak{A}\ '\ \text{elts } \beta) \subseteq KI\ (Suc\ 0)\ (x\ '\ \{\dots < n\})$
and *sm*: *strict-mono-sets* $(\text{elts } \beta)\ \mathfrak{A}$
and $\mu\beta$: $\mu\ '\ \{\dots n\} \subseteq \text{elts } \beta$ **and** \mathfrak{A}_{sub} : $\mathfrak{A}\ (\mu\ n) \subseteq \text{elts } (\alpha * \beta)$
and \mathfrak{A}_{fun} : $\mathfrak{A} \in \text{elts } \beta \rightarrow \{X. X \subseteq \text{elts } (\alpha * \beta) \wedge tp\ X = \alpha\}$
using *that* **by** (*auto simp: Φ -def*)
then **obtain** $xn\ g$ **where** $xn: xn \in \mathfrak{A}\ (\mu\ n)$ **and** $g: g \in \text{elts } \beta \rightarrow \text{elts } \beta$
and *sm-g*: *strict-mono-on* $(\text{elts } \beta)\ g$ **and** $g\text{-}\mu$: $\forall \nu \in \mu\ '\ \{\dots n\}. g\ \nu = \nu$
and $g\text{-}\alpha$: $\forall \nu \in \text{elts } \beta. \alpha \leq tp\ (K\ 1\ xn \cap \mathfrak{A}\ (g\ \nu))$
using 10 [*OF* - $\mu\beta\ \mathfrak{A}_{sub}$ - \mathfrak{A}_{fun}] **by** (*auto simp: \mathfrak{A}*)
have *tp1*: $tp\ (K\ 1\ xn \cap \mathfrak{A}\ (g\ \nu)) = \alpha$ **if** $\nu \in \text{elts } \beta$ **for** ν
by (*metis antisym Int-lower2 PiE \mathfrak{A} down g g- α ordertype-VWF-mono*)
that)
have *tp2*: $tp\ (\mathfrak{A}\ (\mu\ n)) = \alpha$
by (*auto simp: \mathfrak{A}*)
obtain *small* $(\mathfrak{A}\ (\mu\ n))\ \mathfrak{A}\ (\mu\ n) \subseteq ON$
by (*meson \mathfrak{A}_{sub} ord down elts-subset-ON subset-trans*)
then **obtain** $A2$ **where** $A2: tp\ A2 = \alpha\ A2 \subseteq K\ 1\ xn \cap \mathfrak{A}\ (\mu\ n)\ \{xn\} \ll$
 $A2$
using *indecomposable-imp-Ex-less-sets* [*OF indec* $\langle \alpha \geq \omega \rangle\ tp2$]
by (*metis μ -in- β atMost-iff image-eqI inf-le2 le-refl xn tp1 g- μ*)
then **have** $A2\text{-sub}$: $A2 \subseteq \mathfrak{A}\ (\mu\ n)$ **by** *simp*
let $\mathfrak{A} = \lambda \nu. \text{if } \nu = \mu\ n \text{ then } A2 \text{ else } K\ 1\ xn \cap \mathfrak{A}\ (g\ \nu)$
have [simp]: $(\{\dots < Suc\ n\} \cap \{x. x \neq n\}) = (\{\dots < n\})$
by *auto*
have $K\ (Suc\ 0)\ xn \cap (\bigcup x \in \text{elts } \beta \cap \{\nu. \nu \neq \mu\ n\}. \mathfrak{A}\ (g\ x)) \subseteq KI\ (Suc\ 0)$
 $(x\ '\ \{\dots < n\})$

using *sub g* **by** (*auto simp: KI-def*)
moreover have $A2 \subseteq KI (Suc\ 0) (x \text{ ' } \{..<n\}) A2 \subseteq elts (\alpha*\beta) xn \in elts$
 $(\alpha*\beta)$
using \mathfrak{A}_{sub} **sub** $A2\ xn$ **by** *fastforce+*
moreover have *strict-mono-sets* ($elts\ \beta$) \mathfrak{A}
using *sm sm-g g g- μ A2-sub*
unfolding *strict-mono-sets-def strict-mono-on-def less-sets-def Pi-iff sub-*
set-iff Ball-def Bex-def image-iff
by (*simp (no-asm-use) add: if-split-mem2*) (*smt order-refl*)
ultimately have $\Phi (Suc\ n) \mathfrak{A} (x(n := xn))$
using *tp1 x A2* **by** (*auto simp: Φ -def K-def*)
with $A2$ **show** *?thesis*
by (*rule-tac x=(g, \mathfrak{A} ,xn) in exI*) (*simp add: Ψ -def g sm-g g- μ xn*)
qed
define G **where** $G \equiv \lambda n\ \mathfrak{A}\ x.\ @ (g, \mathfrak{A}', x'). \exists xn. \Psi\ n\ g\ \mathfrak{A}\ \mathfrak{A}'\ xn \wedge x' =$
 $(x(n:=xn)) \wedge \Phi (Suc\ n) \mathfrak{A}'\ x'$
have $G\Phi: (\lambda (g, \mathfrak{A}', x'). \Phi (Suc\ n) \mathfrak{A}'\ x') (G\ n\ \mathfrak{A}\ x)$
and $G\Psi: (\lambda (g, \mathfrak{A}', x'). \Psi\ n\ g\ \mathfrak{A}\ \mathfrak{A}'\ (x'\ n)) (G\ n\ \mathfrak{A}\ x)$ **if** $\Phi\ n\ \mathfrak{A}\ x$ **for** $n\ \mathfrak{A}\ x$
using *step [OF that] by (force simp: G-def dest: some-eq-imp)+*
define H **where** $H \equiv rec\ nat\ (id, \mathfrak{A}0, undefined) (\lambda n\ (g0, \mathfrak{A}, x0). G\ n\ \mathfrak{A}\ x0)$
have $(\lambda (g, \mathfrak{A}, x). \Phi\ n\ \mathfrak{A}\ x) (H\ n)$ **for** n
unfolding *H-def* **by** (*induction n*) (*use G Φ base in fastforce*)
then have $H\text{-imp-}\Phi: \Phi\ n\ \mathfrak{A}\ x$ **if** $H\ n = (g, \mathfrak{A}, x)$ **for** $g\ \mathfrak{A}\ x\ n$
by (*metis case-prodD that*)
then have $H\text{-imp-}\Psi: (\lambda (g, \mathfrak{A}', x'). let\ (g0, \mathfrak{A}, x) = H\ n\ in\ \Psi\ n\ g\ \mathfrak{A}\ \mathfrak{A}'\ (x'\ n))$
 $(H (Suc\ n))$ **for** n
using $G\Psi$ **by** (*fastforce simp: H-def split: prod.split*)
define g **where** $g \equiv \lambda n. (\lambda (g, \mathfrak{A}, x). g) (H (Suc\ n))$
have $g: g\ n \in elts\ \beta \rightarrow elts\ \beta$ **and** *sm-g: strict-mono-on* ($elts\ \beta$) ($g\ n$)
and $13: \bigwedge i. i \leq n \implies g\ n\ (\mu\ i) = \mu\ i$ **for** n
using $H\text{-imp-}\Psi$ [of n] **by** (*auto simp: g-def Ψ -def*)
define \mathfrak{A} **where** $\mathfrak{A} \equiv \lambda n. (\lambda (g, \mathfrak{A}, x). \mathfrak{A}) (H\ n)$
define x **where** $x \equiv \lambda n. (\lambda (g, \mathfrak{A}, x). x\ n) (H (Suc\ n))$
have $14: \mathfrak{A} (Suc\ n)\ \nu \subseteq K\ 1\ (x\ n) \cap \mathfrak{A}\ n\ (g\ n\ \nu)$ **if** $\nu \in elts\ \beta$ **for** $\nu\ n$
using $H\text{-imp-}\Psi$ [of n] **that** **by** (*force simp: Ψ -def \mathfrak{A} -def x-def g-def*)
then have $x14: \mathfrak{A} (Suc\ n)\ \nu \subseteq \mathfrak{A}\ n\ (g\ n\ \nu)$ **if** $\nu \in elts\ \beta$ **for** $\nu\ n$
using *that* **by** *blast*
have $15: x\ n \in \mathfrak{A}\ n\ (\mu\ n)$ **and** $16: \{x\ n\} \ll (\mathfrak{A} (Suc\ n)\ (\mu\ n))$ **for** n
using $H\text{-imp-}\Psi$ [of n] **by** (*force simp: Ψ -def \mathfrak{A} -def x-def*)
have $\mathfrak{A}\text{-}\alpha\beta: \mathfrak{A}\ n\ \nu \subseteq elts\ (\alpha*\beta)$ **if** $\nu \in elts\ \beta$ **for** $\nu\ n$
using $H\text{-imp-}\Phi$ [of n] **that** **by** (*auto simp: Φ -def \mathfrak{A} -def split: prod.split*)
have $12: strict-mono-sets$ ($elts\ \beta$) ($\mathfrak{A}\ n$) **for** n
using $H\text{-imp-}\Phi$ [of n] **that** **by** (*auto simp: Φ -def \mathfrak{A} -def split: prod.split*)
let $?Z = range\ x$
have $S\text{-dec}: \bigcup (\mathfrak{A}\ (m+k) \text{ ' } elts\ \beta) \subseteq \bigcup (\mathfrak{A}\ m \text{ ' } elts\ \beta)$ **for** $k\ m$
by (*induction k*) (*use 14 g in <fastforce+>*)
have $x\ n \in K\ 1\ (x\ m)$ **if** $m < n$ **for** $m\ n$
proof -
have $x\ n \in (\bigcup \nu \in elts\ \beta. \mathfrak{A}\ n\ \nu)$

by (*meson 15 UN-I μ -in- β*)
 also have $\dots \subseteq (\bigcup \nu \in \text{elts } \beta. \mathfrak{A} (\text{Suc } m) \nu)$
 using *S-dec [of Suc m] less-iff-Suc-add* that by *auto*
 also have $\dots \subseteq K 1 (x m)$
 using *14* by *auto*
 finally show *?thesis* .
 qed
 then have $f\{x m, x n\} = 1$ if $m < n$ for $m n$
 using that by (*auto simp: K-def*)
 then have *Z-K1*: $f ' [?Z]^2 \subseteq \{1\}$
 by (*clarsimp simp: nsets-2-eq*) (*metis insert-commute less-linear*)
 moreover have *Z-sub*: $?Z \subseteq \text{elts } (\alpha * \beta)$
 using *15 \mathfrak{A} - $\alpha\beta$ μ -in- β* by *blast*
 moreover have *tp* $?Z \geq \omega * \beta$
 proof –
 define **g** where $\mathbf{g} \equiv \lambda i j x. \text{wfrec } (\lambda k. j - k) (\lambda \mathbf{g} k. \text{if } k < j \text{ then } \mathbf{g} k (\text{Suc } k) \text{ else } x) i$
 have **g**: $\mathbf{g} i j x = (\text{if } i < j \text{ then } \mathbf{g} i (\mathbf{g} (\text{Suc } i) j x) \text{ else } x)$ for $i j x$
 by (*simp add: g-def wfrec cut-apply*)
 have *17*: $\mathbf{g} k j (\mu i) = \mu i$ if $i \leq k$ for $i j k$
 using *wf-measure [of $\lambda k. j - k$]* that
 by (*induction k rule: wf-induct-rule*) (*simp add: 13 g le-imp-less-Suc*)
 have **g-in- β** : $\mathbf{g} i j \nu \in \text{elts } \beta$ if $\nu \in \text{elts } \beta$ for $i j \nu$
 using *wf-measure [of $\lambda k. j - k$]* that
 proof (*induction i rule: wf-induct-rule*)
 case (*less i*)
 with *g* show *?case* by (*force simp: g [of i]*)
 qed
 then have **g-fun**: $\mathbf{g} i j \in \text{elts } \beta \rightarrow \text{elts } \beta$ for $i j$
 by *simp*
 have **sm-g**: *strict-mono-on* ($\text{elts } \beta$) ($\mathbf{g} i j$) for $i j$
 using *wf-measure [of $\lambda k. j - k$]*
 proof (*induction i rule: wf-induct-rule*)
 case (*less i*)
 with *sm-g* show *?case*
 by (*auto simp: g [of i] strict-mono-on-def g-in- β*)
 qed
 have *: $\mathfrak{A} j (\mu j) \subseteq \mathfrak{A} i (\mathbf{g} i j (\mu j))$ if $i < j$ for $i j$
 using *wf-measure [of $\lambda k. j - k$]* that
 proof (*induction i rule: wf-induct-rule*)
 case (*less i*)
 then have $j - \text{Suc } i < j - i$
 by (*metis (no-types) Suc-diff-Suc lessI*)
 with *less g-in- β* show *?case*
 by (*simp add: g [of i]*) (*metis 17 Suc-lessI μ -in- β order-refl order-trans*)
 x14)

by (metis 17 Ord-in-Ord Ord-linear2 μ -in- β leD le-refl less-V-def \langle Ord β \rangle)
 then have less-iff: $\mathbf{g} \ i \ j \ (\mu \ j) < \mu \ i \longleftrightarrow \mu \ j < \mu \ i$ for $i \ j$
 by (metis (no-types, lifting) 17 μ -in- β less-V-def order-refl sm-g strict-mono-on-def)
 have eq-iff: $\mathbf{g} \ i \ j \ (\mu \ j) = \mu \ i \longleftrightarrow \mu \ j = \mu \ i$ for $i \ j$
 by (metis eq-refl le-iff less-iff less-le)
 have μ -if-ne: $\mu \ m < \mu \ n$ if mn : $\mathfrak{A} \ m \ (\mu \ m) \ll \mathfrak{A} \ n \ (\mu \ n) \ m \neq n$ for $m \ n$
 proof –
 have xmn : $x \ m < x \ n$
 using 15 less-setsD that(1) by blast
 have Ordg: Ord (g n m ($\mu \ m$))
 using Ord-in-Ord g-in- β μ -in- β ord(2) by presburger
 have $\neg \mathfrak{A} \ m \ (\mu \ m) \ll \mathfrak{A} \ n \ (\mu \ n)$ if $\mu \ n = \mu \ m$
 using * 15 eq-iff that unfolding less-sets-def
 by (metis in-mono less-irrefl not-less-iff-gr-or-eq)
 moreover
 have $\mathfrak{A} \ n \ (\mu \ n) \subseteq \mathfrak{A} \ m \ (\mathbf{g} \ m \ n \ (\mu \ n)) \vee \mathfrak{A} \ m \ (\mu \ m) \subseteq \mathfrak{A} \ n \ (\mathbf{g} \ n \ m \ (\mu \ m))$
 using * mn
 by (meson antisym-conv3)
 then have False if $\mu \ n < \mu \ m$
 using strict-mono-setsD [OF 12] 15 xmn g-in- β μ -in- β that
 by (smt (verit, best) Ordg Ord- μ Ord-linear2 leD le-iff less-asym less-iff
 less-setsD subset-iff)
 ultimately show $\mu \ m < \mu \ n$
 by (meson that(1) Ord- μ Ord-linear-lt)
 qed
 have 18: $\mathfrak{A} \ m \ (\mu \ m) \ll \mathfrak{A} \ n \ (\mu \ n) \longleftrightarrow \mu \ m < \mu \ n$ for $m \ n$
 proof (cases $n \ m$ rule: linorder-cases)
 case less
 show ?thesis
 proof (intro iffI)
 assume $\mu \ m < \mu \ n$
 then have $\mathfrak{A} \ n \ (\mathbf{g} \ n \ m \ (\mu \ m)) \ll \mathfrak{A} \ n \ (\mu \ n)$
 by (metis 12 g-in- β μ -in- β eq-iff le-iff less-V-def strict-mono-sets-def)
 then show $\mathfrak{A} \ m \ (\mu \ m) \ll \mathfrak{A} \ n \ (\mu \ n)$
 by (meson * less less-sets-weaken1)
 qed (use μ -if-ne less in blast)
 next
 case equal
 with 15 show ?thesis by auto
 next
 case greater
 show ?thesis
 proof (intro iffI)
 assume $\mu \ m < \mu \ n$
 then have $\mathfrak{A} \ m \ (\mu \ m) \ll (\mathfrak{A} \ m \ (\mathbf{g} \ m \ n \ (\mu \ n)))$
 by (meson 12 Ord-in-Ord Ord-linear2 g-in- β μ -in- β le-iff leD ord(2)
 strict-mono-sets-def)
 then show $\mathfrak{A} \ m \ (\mu \ m) \ll \mathfrak{A} \ n \ (\mu \ n)$
 by (meson * greater less-sets-weaken2)

qed (use μ -if-ne greater in blast)
qed
have \mathfrak{A} -increasing- μ : $\mathfrak{A} \ n \ (\mu \ n) \subseteq \mathfrak{A} \ m \ (\mu \ m)$ **if** $m \leq n \ \mu \ m = \mu \ n$ **for** $m \ n$
by (metis * 17 dual-order.order-iff-strict that)
moreover have INF : $infinite \ \{n. \ n \geq m \wedge \mu \ m = \mu \ n\}$ **for** m
proof –
have $infinite \ (range \ (\lambda n. \ q \ (\mu \ m, \ n)))$
unfolding q -def
using $to\text{-nat-on-infinite} \ [OF \ co\text{-}\beta U \ inf\text{-}\beta U] \ finite\text{-image-iff}$
by ($simp \ add: \ finite\text{-image-iff} \ inj\text{-on-def}$)
moreover have $(range \ (\lambda n. \ q \ (\mu \ m, \ n))) \subseteq \{n. \ \mu \ m = \mu \ n\}$
using 11 [of $\mu \ m$] **by** $auto$
ultimately have $infinite \ \{n. \ \mu \ m = \mu \ n\}$
using $finite\text{-subset}$ **by** $auto$
then have $infinite \ (\{n. \ \mu \ m = \mu \ n\} - \{..<m\})$
by $simp$
then show ?thesis
by ($auto \ simp: \ finite\text{-nat-set-iff-bounded} \ Bex\text{-def} \ not\text{-less}$)
qed
let $?eqv = \lambda m. \ \{n. \ m \leq n \wedge \mu \ m = \mu \ n\}$
have $sm\text{-}x$: $strict\text{-mono-on} \ (?eqv \ m) \ x$ **for** m
proof ($clarsimp \ simp: \ strict\text{-mono-on-def}$)
fix $n \ p$
assume $m \leq n \ \mu \ p = \mu \ n \ \mu \ m = \mu \ n \ n < p$
with 16 [of n] **show** $x \ n < x \ p$
by (metis * 15 17 $Suc\text{-lessI} \ insert\text{-absorb} \ insert\text{-subset} \ le\text{-SucI} \ less\text{-sets-singleton1}$)
qed
then have $inj\text{-}x$: $inj\text{-on} \ x \ (?eqv \ m)$ **for** m
using $strict\text{-mono-on-imp-inj-on}$ **by** $blast$
define ZA **where** $ZA \equiv \lambda m. \ ?Z \cap \mathfrak{A} \ m \ (\mu \ m)$
have $small\text{-}ZA$ [$simp$]: $small \ (ZA \ m)$ **for** m
by (metis $ZA\text{-def} \ inf\text{-le1} \ small\text{-image-nat} \ smaller\text{-than-small}$)
have 19: $tp \ (ZA \ m) \geq \omega$ **for** m
proof –
have $x \ ' \ \{n. \ m \leq n \wedge \mu \ m = \mu \ n\} \subseteq ZA \ m$
unfolding $ZA\text{-def}$ **using** 15 \mathfrak{A} -increasing- μ **by** $blast$
then have $infinite \ (ZA \ m)$
using INF [of m] $finite\text{-image-iff} \ [OF \ inj\text{-}x]$ **by** ($meson \ finite\text{-subset}$)
then show ?thesis
by ($simp \ add: \ ordertype\text{-infinite-ge-}\omega$)
qed
have $\exists f \in \text{elts} \ \omega \rightarrow ZA \ m. \ strict\text{-mono-on} \ (\text{elts} \ \omega) \ f$ **for** m
proof –
obtain Z **where** $Z \subseteq ZA \ m \ tp \ Z = \omega$
by ($meson \ 19 \ Ord\text{-}\omega \ le\text{-ordertype-obtains-subset} \ small\text{-}ZA$)
moreover have $ZA \ m \subseteq ON$
using $Ord\text{-in-Ord} \ \mathfrak{A}\text{-}\alpha\beta \ \mu\text{-in-}\beta$ **unfolding** $ZA\text{-def}$ **by** $blast$
ultimately show ?thesis
by (metis $strict\text{-mono-on-ordertype} \ Pi\text{-mono} \ small\text{-}ZA \ smaller\text{-than-small}$)

subset-iff)
qed
then obtain φ **where** $\varphi: \bigwedge m. \varphi m \in \text{elts } \omega \rightarrow ZA m$
and $sm\text{-}\varphi: \bigwedge m. \text{strict-mono-on } (\text{elts } \omega) (\varphi m)$
by *metis*
have $Ex(\lambda(m,\nu). \nu \in \text{elts } \beta \wedge \gamma = \omega * \nu + \text{ord-of-nat } m)$ **if** $\gamma \in \text{elts } (\omega * \beta)$ **for** γ
using *that by (auto simp: mult [of $\omega \beta$] lift-def elts- ω)*
then obtain *split* **where** $split: \bigwedge \gamma. \gamma \in \text{elts } (\omega * \beta) \implies (\lambda(m,\nu). \nu \in \text{elts } \beta \wedge \gamma = \omega * \nu + \text{ord-of-nat } m)(split \gamma)$
by *meson*
have $split\text{-eq [simp]: split } (\omega * \nu + \text{ord-of-nat } m) = (m,\nu)$ **if** $\nu \in \text{elts } \beta$ **for** νm
proof –
have $[simp]: \omega * \nu + \text{ord-of-nat } m = \omega * \xi + \text{ord-of-nat } n \iff \xi = \nu \wedge n = m$ **if** $\xi \in \text{elts } \beta$ **for** ξn
by (*metis Ord- ω that Ord-mem-iff-less-TC mult-cancellation-lemma ord-of-nat- ω ord-of-nat-inject*)
show *?thesis*
using *split [of $\omega * \nu + m$] that by (auto simp: mult [of $\omega \beta$] lift-def cong: conj-cong)*
qed
define π **where** $\pi \equiv \lambda \gamma. (\lambda(m,\nu). \varphi (q(\nu,0)) m)(split \gamma)$
have $\pi\text{-Pi}: \pi \in \text{elts } (\omega * \beta) \rightarrow (\bigcup m. ZA m)$
using φ **by** (*fastforce simp: $\pi\text{-def mult [of $\omega \beta$] lift-def elts- $\omega$$*)
moreover $\text{have } (\bigcup m. ZA m) \subseteq ON$
unfolding *ZA-def* **using** $\mathfrak{A}\text{-}\alpha\beta \mu\text{-in-}\beta \text{elts-subset-ON}$ **by** *blast*
ultimately $\text{have } \text{Ord-}\pi\text{-Pi}: \pi \in \text{elts } (\omega * \beta) \rightarrow ON$
by *fastforce*
show $tp \text{ ?}Z \geq \omega * \beta$
proof –
have $\dagger: (\bigcup m. ZA m) = \text{?}Z$
using *15 by (force simp: ZA-def)*
moreover
have $tp (\text{elts } (\omega * \beta)) \leq tp (\bigcup m. ZA m)$
proof (*rule ordertype-inc-le*)
show $\pi \text{ ' } \text{elts } (\omega * \beta) \subseteq (\bigcup m. ZA m)$
using $\pi\text{-Pi}$ **by** *blast*
next
fix $u v$
assume $x: u \in \text{elts } (\omega * \beta)$ **and** $y: v \in \text{elts } (\omega * \beta)$ **and** $(u,v) \in VWF$
then $\text{have } u < v$
by (*meson Ord- ω Ord-in-Ord Ord-mult VWF-iff-Ord-less ord(2)*)
moreover
obtain $m \nu n \xi$ **where** $ueq: u = \omega * \nu + \text{ord-of-nat } m$ **and** $\nu: \nu \in \text{elts } \beta$
and $veq: v = \omega * \xi + \text{ord-of-nat } n$ **and** $\xi: \xi \in \text{elts } \beta$
using $x y$ **by** (*auto simp: mult [of $\omega \beta$] lift-def elts- ω*)
ultimately $\text{have } \nu \leq \xi$
by (*meson Ord- ω Ord-in-Ord Ord-linear2 <Ord β > add-mult-less-add-mult*)

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less-asym ord-of-nat- $\omega$ 
  consider (eq)  $\nu = \xi \mid$  (lt)  $\nu < \xi$ 
    using  $\langle \nu \leq \xi \rangle$  le-neq-trans by blast
  then have  $\pi u < \pi v$ 
  proof cases
    case eq
      then have  $m < n$ 
        using ueq veq  $\langle u < v \rangle$  by simp
      then have  $\varphi (q (\xi, 0)) m < \varphi (q (\xi, 0)) n$ 
        using sm- $\varphi$  strict-mono-onD by blast
      then show ?thesis
        using eq ueq veq  $\nu \langle m < n \rangle$  by (simp add:  $\pi$ -def)
    next
      case lt
        have  $\varphi (q(\nu, 0)) m \in \mathfrak{A} (q(\nu, 0)) (\mu(q(\nu, 0))) \varphi (q (\xi, 0)) n \in \mathfrak{A} (q(\xi, 0))$ 
          ( $\mu(q(\xi, 0))$ )
          using  $\varphi$  unfolding ZA-def by blast+
        then show ?thesis
          using lt ueq veq  $\nu \xi$  18 [of  $q(\nu, 0)$   $q(\xi, 0)$ ]
          by (simp add:  $\pi$ -def less-sets-def)
        qed
        then show  $(\pi u, \pi v) \in VWF$ 
          using  $\pi$ -Pi by (metis Ord- $\pi$ -Pi PiE VWF-iff-Ord-less x y mem-Collect-eq)
        qed (use  $\dagger$  in auto)
        ultimately show ?thesis by simp
      qed
    qed
  then obtain  $Z$  where  $Z \subseteq ?Z$  tp  $Z = \omega * \beta$ 
    by (meson Ord- $\omega$  Ord-mult ord Z-sub down le-ordertype-obtains-subset)
  ultimately show False
    using iii [of  $Z$ ] by (meson dual-order.trans image-mono nsets-mono)
  qed
  have False
    if 0:  $\forall H. \text{tp } H = \text{ord-of-nat } (2*k) \longrightarrow H \subseteq \text{elts } (\alpha*\beta) \longrightarrow \neg f ' [H]^2 \subseteq \{0\}$ 
    and 1:  $\forall H. \text{tp } H = \text{min } \gamma (\omega * \beta) \longrightarrow H \subseteq \text{elts } (\alpha*\beta) \longrightarrow \neg f ' [H]^2 \subseteq$ 
    {1}
  proof (cases  $\omega*\beta \leq \gamma$ )
    case True
      then have  $\dagger: \exists H' \subseteq H. \text{tp } H' = \omega * \beta$  if tp  $H = \gamma$  small  $H$  for  $H$ 
        by (metis Ord- $\omega$  Ord- $\omega$ 1 Ord-in-Ord Ord-mult  $\beta$  le-ordertype-obtains-subset
        that)
      have [simp]:  $\text{min } \gamma (\omega*\beta) = \omega*\beta$ 
        by (simp add: min-absorb2 that True)
      then show ?thesis
        using * [OF 0] 1 True
        by simp (meson  $\dagger$  down image-mono nsets-mono subset-trans)
    next
      case False
        then have  $\dagger: \exists H' \subseteq H. \text{tp } H' = \gamma$  if tp  $H = \omega * \beta$  small  $H$  for  $H$ 

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by (metis Ord-linear-le Ord-ordertype ‹Ord γ › le-ordertype-obtains-subset
 that)
 then have $\gamma \leq \omega * \beta$
 by (meson Ord- ω Ord- $\omega 1$ Ord-in-Ord Ord-linear-le Ord-mult β ‹Ord γ ›
 False)
 then have [simp]: $\min \gamma (\omega * \beta) = \gamma$
 by (simp add: min-absorb1)
 then show ?thesis
 using * [OF 0] 1 False
 by simp (meson † down image-mono nsets-mono subset-trans)
 qed
 then show $\exists i < \text{Suc } (\text{Suc } 0). \exists H \subseteq \text{elts } (\alpha * \beta). \text{tp } H = [\text{ord-of-nat } (2 * k), \min$
 $\gamma (\omega * \beta)] ! i \wedge f ' [H]^2 \subseteq \{i\}$
 by force
 qed
 qed

theorem Erdos-Milner:

assumes $\nu: \nu \in \text{elts } \omega 1$
 shows partn-lst-VWF ($\omega \uparrow (1 + \nu * n)$) [ord-of-nat ($2 \hat{n}$), $\omega \uparrow (1 + \nu)$] 2
 proof (induction n)
 case 0
 then show ?case
 using partn-lst-VWF-degenerate [of 1 2] by simp
 next
 case (Suc n)
 have Ord ν
 using Ord- $\omega 1$ Ord-in-Ord assms by blast
 have $1 + \nu \leq \nu + 1$
 by (simp add: ‹Ord ν › one-V-def plus-Ord-le)
 then have [simp]: $\min (\omega \uparrow (1 + \nu)) (\omega * \omega \uparrow \nu) = \omega \uparrow (1 + \nu)$
 by (simp add: ‹Ord ν › oexp-add min-def)
 have ind: indecomposable ($\omega \uparrow (1 + \nu * \text{ord-of-nat } n)$)
 by (simp add: ‹Ord ν › indecomposable- ω -power)
 show ?case
 proof (cases n = 0)
 case True
 then show ?thesis
 using partn-lst-VWF- ω -2 ‹Ord ν › one-V-def by auto
 next
 case False
 then have $\text{Suc } 0 < 2 \hat{n}$
 using less-2-cases not-less-eq by fastforce
 then have partn-lst-VWF ($\omega \uparrow (1 + \nu * n) * \omega \uparrow \nu$) [ord-of-nat ($2 * 2 \hat{n}$),
 $\omega \uparrow (1 + \nu)$] 2
 using Erdos-Milner-aux [OF Suc ind, where $\beta = \omega \uparrow \nu$] ‹Ord ν › ν
 by (auto simp: countable-oexp)
 then show ?thesis

```

    using ⟨Ord ν⟩ by (simp add: mult-succ mult.assoc oexp-add)
  qed
qed

```

```

corollary remark-3: partn-lst-VWF (ω↑(Suc(4*k))) [4, ω↑(Suc(2*k))] 2
  using Erdos-Milner [of 2*k 2]
  apply (simp flip: ord-of-nat-mult ord-of-nat.simps)
  by (simp add: one-V-def)

```

Theorem 3.2 of Jean A. Larson, *ibid.*

corollary Theorem-3-2:

```

  fixes k n::nat
  shows partn-lst-VWF (ω↑(n*k)) [ω↑n, ord-of-nat k] 2
proof (cases n=0 ∨ k=0)
  case True
  then show ?thesis
    by (auto intro: partn-lst-triv0 [where i=1] partn-lst-triv1 [where i=0])
  next
  case False
  then have n > 0 k > 0
    by auto
  from ⟨k > 0⟩ less-exp [of ⟨k - 1⟩] have ⟨k ≤ 2 ^ (k - 1)⟩
    by (cases k) (simp-all add: less-eq-Suc-le)
  have PV: partn-lst-VWF (ω ↑ (1 + ord-of-nat (n-1) * ord-of-nat (k-1)))
    [ord-of-nat (2 ^ (k-1)), ω ↑ (1 + ord-of-nat (n-1))] 2
    using Erdos-Milner [of ord-of-nat (n-1) k-1] Ord-ω1 Ord-mem-iff-lt less-imp-le
  by blast
  have k+n ≤ Suc (k * n)
    using False not0-implies-Suc by fastforce
  then have 1 + (n - 1) * (k - 1) ≤ n*k
    using False by (auto simp: algebra-simps)
  then have (1 + ord-of-nat (n - 1) * ord-of-nat (k - 1)) ≤ ord-of-nat(n*k)
    by (metis (mono-tags, lifting) One-nat-def one-V-def ord-of-nat.simps ord-of-nat-add
    ord-of-nat-mono-iff ord-of-nat-mult)
  then have x: ω ↑ (1 + ord-of-nat (n - 1) * ord-of-nat (k - 1)) ≤ ω↑(n*k)
    by (simp add: oexp-mono-le)
  then have partn-lst-VWF (ω↑(n*k)) [ω ↑ (1 + ord-of-nat (n-1)), ord-of-nat (2
    ^ (k-1))] 2
    by (metis PV partn-lst-two-swap Partitions.partn-lst-greater-resource less-eq-V-def)
  then have partn-lst-VWF (ω↑(n*k)) [ω ↑ n, ord-of-nat (2 ^ (k-1))] 2
    using ord-of-minus-1 [OF ⟨n > 0⟩] by (simp add: one-V-def)
  then show ?thesis
    using ⟨k ≤ 2 ^ (k - 1)⟩
    by (auto elim!: partn-lst-less simp add: less-Suc-eq)
qed
end

```

3 An ordinal partition theorem by Jean A. Larson

Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω .
Annals of Mathematical Logic, 6:129–145, 1973.

theory *Omega-Omega*
imports *HOL-Library.Product-Lexorder Erdos-Milner*

begin

abbreviation *list-of* \equiv *sorted-list-of-set*

3.1 Cantor normal form for ordinals below $\omega \uparrow \omega$

Unlike *Cantor-sum*, there is no list of ordinal exponents, which are instead taken as consecutive. We obtain an order-isomorphism between $\omega \uparrow \omega$ and increasing lists of natural numbers (ordered lexicographically).

fun *omega-sum-aux* **where**
Nil: *omega-sum-aux* 0 = 0
| *Suc*: *omega-sum-aux* (Suc n) [] = 0
| *Cons*: *omega-sum-aux* (Suc n) (m#ms) = ($\omega \uparrow n$) * (*ord-of-nat* m) + *omega-sum-aux* n ms

abbreviation *omega-sum* **where** *omega-sum* ms \equiv *omega-sum-aux* (length ms) ms

A normal expansion has no leading zeroes

inductive *normal*:: nat list \Rightarrow bool **where**
normal-Nil[*iff*]: *normal* []
| *normal-Cons*: $m > 0 \implies \text{normal } (m\#ms)$

inductive-simps *normal-Cons-iff* [*simp*]: *normal* (m#ms)

lemma *omega-sum-0-iff* [*simp*]: *normal* ns \implies *omega-sum* ns = 0 \iff ns = []
by (*induction* ns rule: *normal.induct*) *auto*

lemma *Ord-omega-sum-aux* [*simp*]: *Ord* (*omega-sum-aux* k ms)
by (*induction* rule: *omega-sum-aux.induct*) *auto*

lemma *Ord-omega-sum*: *Ord* (*omega-sum* ms)
by *simp*

lemma *omega-sum-less- $\omega\omega$* [*intro*]: *omega-sum* ms < $\omega \uparrow \omega$

proof (*induction* ms)

case (*Cons* m ms)

have $\omega \uparrow (\text{length } ms) * \text{ord-of-nat } m \in \text{elts } (\omega \uparrow \text{Suc } (\text{length } ms))$

using *Ord-mem-iff-lt* **by** *auto*

then have $\omega \uparrow (\text{length } ms) * \text{ord-of-nat } m \in \text{elts } (\omega \uparrow \omega)$

using *Ord-ord-of-nat oexp-mono-le omega-nonzero ord-of-nat-le-omega* **by** *blast*

with *Cons* **show** ?*case*
by (*auto simp: mult-succ OrdmemD oexp-less indecomposableD indecomposable- ω -power*)
qed (*auto simp: zero-less-Limit*)

lemma *omega-sum-aux-less: omega-sum-aux k ms < $\omega \uparrow k$*
proof (*induction rule: omega-sum-aux.induct*)
case ($\exists n m ms$)
have $\omega \uparrow n * \text{ord-of-nat } m + \omega \uparrow n < \omega \uparrow n * \omega$
by (*metis Ord-ord-of-nat ω -power-succ-gtr mult-succ oexp-succ ord-of-nat.simps(2)*)
with \exists **show** ?*case*
using *dual-order.strict-trans* **by** *force*
qed *auto*

lemma *omega-sum-less: omega-sum ms < $\omega \uparrow (\text{length } ms)$*
by (*rule omega-sum-aux-less*)

lemma *omega-sum-ge: m \neq 0 \implies $\omega \uparrow (\text{length } ms) \leq \text{omega-sum } (m\#ms)$*
apply *clarsimp*
by (*metis Ord-ord-of-nat add-le-cancel-left0 le-mult Nat.neq0-conv ord-of-eq-0-iff vsubsetD*)

lemma *omega-sum-length-less:*
assumes *normal ns length ms < length ns*
shows *omega-sum ms < omega-sum ns*
using *assms*
proof (*induction rule: normal.induct*)
case (*normal-Cons n ns'*)
have $\omega \uparrow \text{length } ms \leq \omega \uparrow \text{length } ns'$
using *normal-Cons oexp-mono-le* **by** *auto*
then show ?*case*
by (*metis gr-implies-not-zero less-le-trans normal-Cons.hyps omega-sum-aux-less omega-sum-ge*)
qed *auto*

lemma *omega-sum-length-leD:*
assumes *omega-sum ms \leq omega-sum ns normal ms*
shows *length ms \leq length ns*
by (*meson assms leD leI omega-sum-length-less*)

lemma *omega-sum-less-eqlen-iff-cases [simp]:*
assumes *length ms = length ns*
shows *omega-sum (m#ms) < omega-sum (n#ns) \iff m < n \vee m = n \wedge omega-sum ms < omega-sum ns*
using *omega-sum-less [of ms] omega-sum-less [of ns]*
by (*metis Kirby.add-less-cancel-left Omega-Omega.Cons Ord- ω Ord-oexp Ord-omega-sum Ord-ord-of-nat assms length-Cons linorder-neqE-nat mult-nat-less-add-less order-less-asm*)

lemma *omega-sum-less-iff-cases:*

assumes $m > 0 \ n > 0$
shows $\text{omega-sum } (m\#ms) < \text{omega-sum } (n\#ns)$
 $\longleftrightarrow \text{length } ms < \text{length } ns$
 $\vee \text{length } ms = \text{length } ns \wedge m < n$
 $\vee \text{length } ms = \text{length } ns \wedge m = n \wedge \text{omega-sum } ms < \text{omega-sum } ns$
by (*smt (verit) assms length-Cons less-eq-Suc-le less-le-not-le nat-less-le normal-Cons not-le*
 $\text{omega-sum-length-less omega-sum-less-eqlen-iff-cases}$)

lemma *omega-sum-less-iff*:
 $((\text{length } ms, \text{omega-sum } ms), (\text{length } ns, \text{omega-sum } ns)) \in \text{less-than } \langle *lex* \rangle$
VWF
 $\longleftrightarrow (ms, ns) \in \text{lenlex less-than}$
proof (*induction ms arbitrary: ns*)
case (*Cons m ms*)
then show *?case*
proof (*induction ns*)
case (*Cons n ns'*)
show *?case*
using *Cons.premis Cons-lenlex-iff omega-sum-less-eqlen-iff-cases* **by** *fastforce*
qed *auto*
qed *auto*

lemma *eq-omega-sum-less-iff*:
assumes $\text{length } ms = \text{length } ns$
shows $(\text{omega-sum } ms, \text{omega-sum } ns) \in \text{VWF} \longleftrightarrow (ms, ns) \in \text{lenlex less-than}$
by (*metis assms in-lex-prod less-not-refl less-than-iff omega-sum-less-iff*)

lemma *eq-omega-sum-eq-iff*:
assumes $\text{length } ms = \text{length } ns$
shows $\text{omega-sum } ms = \text{omega-sum } ns \longleftrightarrow ms = ns$
proof
assume $\text{omega-sum } ms = \text{omega-sum } ns$
then have $(\text{omega-sum } ms, \text{omega-sum } ns) \notin \text{VWF} (\text{omega-sum } ns, \text{omega-sum } ms) \notin \text{VWF}$
by *auto*
then obtain $(ms, ns) \notin \text{lenlex less-than} (ns, ms) \notin \text{lenlex less-than}$
using *assms eq-omega-sum-less-iff* **by** *metis*
moreover have *total (lenlex less-than)*
by (*simp add: total-lenlex total-less-than*)
ultimately show $ms = ns$
by (*meson UNIV-I total-on-def*)
qed *auto*

lemma *inj-omega-sum*: *inj-on omega-sum {l. length l = n}*
unfolding *inj-on-def* **using** *eq-omega-sum-eq-iff* **by** *fastforce*

lemma *Ex-omega-sum*: $\gamma \in \text{elts } (\omega \uparrow n) \implies \exists ns. \gamma = \text{omega-sum } ns \wedge \text{length } ns = n$

```

proof (induction n arbitrary:  $\gamma$ )
  case 0
  then show ?case
    by (rule-tac x=[] in exI) auto
next
  case (Suc n)
  then obtain k::nat where k:  $\gamma \in \text{elts } (\omega \uparrow n * k)$ 
    and kmin:  $\bigwedge k'. k' < k \implies \gamma \notin \text{elts } (\omega \uparrow n * k')$ 
    by (metis Ord-ord-of-nat elts-mult- $\omega E$  oexp-succ ord-of-nat.simps(2))
  show ?case
  proof (cases k)
    case (Suc k')
    then obtain  $\delta$  where  $\delta: \gamma = (\omega \uparrow n * k') + \delta$ 
      by (metis lessI mult-succ ord-of-nat.simps(2) k kmin mem-plus-V-E)
    then have  $\delta \text{in}: \delta \in \text{elts } (\omega \uparrow n)$ 
      using Suc k mult-succ by auto
    then obtain ns where ns:  $\delta = \text{omega-sum } ns$  and len:  $\text{length } ns = n$ 
      using Suc.IH by auto
    moreover have  $\text{omega-sum } ns < \omega \uparrow n$ 
      using OrdmemD ns  $\delta \text{in}$  by auto
    ultimately show ?thesis
      by (rule-tac x=k'#ns in exI) (simp add:  $\delta$ )
    qed (use k in auto)
  qed

lemma omega-sum-drop [simp]:  $\text{omega-sum } (\text{dropWhile } (\lambda n. n=0) ns) = \text{omega-sum } ns$ 
  by (induction ns) auto

lemma normal-drop [simp]:  $\text{normal } (\text{dropWhile } (\lambda n. n=0) ns)$ 
  by (induction ns) auto

lemma omega-sum- $\omega\omega$ :
  assumes  $\gamma \in \text{elts } (\omega \uparrow \omega)$ 
  obtains ns where  $\gamma = \text{omega-sum } ns$  normal ns
proof –
  obtain ms where  $\gamma = \text{omega-sum } ms$ 
    using assms Ex-omega-sum by (auto simp: oexp-Limit elts- $\omega$ )
  then show thesis
    by (metis normal-drop omega-sum-drop that)
  qed

definition Cantor- $\omega\omega$  ::  $V \Rightarrow \text{nat list}$ 
  where Cantor- $\omega\omega \equiv \lambda x. \text{SOME } ns. x = \text{omega-sum } ns \wedge \text{normal } ns$ 

lemma
  assumes  $\gamma \in \text{elts } (\omega \uparrow \omega)$ 
  shows Cantor- $\omega\omega$ :  $\text{omega-sum } (\text{Cantor-}\omega\omega \ \gamma) = \gamma$ 
    and normal-Cantor- $\omega\omega$ :  $\text{normal } (\text{Cantor-}\omega\omega \ \gamma)$ 

```

by (metis (mono-tags, lifting) Cantor- ω -def assms omega-sum- ω someI)+

3.2 Larson's set $W(n)$

definition $WW :: \text{nat list set}$
where $WW \equiv \{l. \text{strict-sorted } l\}$

fun $\text{into-}WW :: \text{nat} \Rightarrow \text{nat list} \Rightarrow \text{nat list}$ **where**
 $\text{into-}WW \ k \ [] = []$
 $|\ \text{into-}WW \ k \ (n\#ns) = (k+n) \# \text{into-}WW \ (\text{Suc } (k+n)) \ ns$

fun $\text{from-}WW :: \text{nat} \Rightarrow \text{nat list} \Rightarrow \text{nat list}$ **where**
 $\text{from-}WW \ k \ [] = []$
 $|\ \text{from-}WW \ k \ (n\#ns) = (n - k) \# \text{from-}WW \ (\text{Suc } n) \ ns$

lemma $\text{from-into-}WW$ [simp]: $\text{from-}WW \ k \ (\text{into-}WW \ k \ ns) = ns$
by (induction ns arbitrary: k) auto

lemma $\text{inj-into-}WW$: $\text{inj} \ (\text{into-}WW \ k)$
by (metis from-into- WW injI)

lemma $\text{into-from-}WW\text{-aux}$:
 $\llbracket \text{strict-sorted } ns; \forall n \in \text{list.set } ns. k \leq n \rrbracket \Longrightarrow \text{into-}WW \ k \ (\text{from-}WW \ k \ ns) = ns$
by (induction ns arbitrary: k) (auto simp: Suc-leI)

lemma $\text{into-from-}WW$ [simp]: $\text{strict-sorted } ns \Longrightarrow \text{into-}WW \ 0 \ (\text{from-}WW \ 0 \ ns) = ns$
by (simp add: into-from- WW -aux)

lemma $\text{into-}WW\text{-imp-ge}$: $y \in \text{List.set} \ (\text{into-}WW \ x \ ns) \Longrightarrow x \leq y$
by (induction ns arbitrary: x) fastforce+

lemma $\text{strict-sorted-into-}WW$: $\text{strict-sorted} \ (\text{into-}WW \ x \ ns)$
by (induction ns arbitrary: x) (auto simp: dest: into- WW -imp-ge)

lemma $\text{length-into-}WW$: $\text{length} \ (\text{into-}WW \ x \ ns) = \text{length } ns$
by (induction ns arbitrary: x) auto

lemma $WW\text{-eq-range-into}$: $WW = \text{range} \ (\text{into-}WW \ 0)$

proof –

have $\bigwedge ns. \text{strict-sorted } ns \Longrightarrow ns \in \text{range} \ (\text{into-}WW \ 0)$

by (metis into-from- WW rangeI)

then show ?thesis **by** (auto simp: WW -def strict-sorted-into- WW)

qed

lemma $\text{into-}WW\text{-lenlex-iff}$: $(\text{into-}WW \ k \ ms, \text{into-}WW \ k \ ns) \in \text{lenlex less-than} \iff (ms, ns) \in \text{lenlex less-than}$

proof (induction ms arbitrary: ns k)

case Nil

```

then show ?case
  by simp (metis length-0-conv length-into-WW)
next
  case (Cons m ms)
  then show ?case
    by (induction ns) (auto simp: Cons-lenlex-iff length-into-WW)
qed

lemma wf-llt [simp]: wf (lenlex less-than) and trans-llt [simp]: trans (lenlex less-than)
  by blast+

lemma total-llt [simp]: total-on A (lenlex less-than)
  by (meson UNIV-I total-lenlex total-less-than total-on-def)

lemma omega-sum-1-less:
  assumes (ms,ns) ∈ lenlex less-than shows omega-sum (1#ms) < omega-sum
  (1#ns)
proof –
  have omega-sum (1#ms) < omega-sum (1#ns) if length ms < length ns
    using omega-sum-less-iff-cases that zero-less-one by blast
  then show ?thesis
    using assms by (auto simp: mult-succ simp flip: omega-sum-less-iff)
qed

lemma ordertype-WW-1: ordertype WW (lenlex less-than) ≤ ordertype UNIV
  (lenlex less-than)
  by (rule ordertype-mono) auto

lemma ordertype-WW-2: ordertype UNIV (lenlex less-than) ≤ ω↑ω
proof (rule ordertype-inc-le-Ord)
  show range (λms. omega-sum (1#ms)) ⊆ elts (ω↑ω)
    by (meson Ord-ω Ord-mem-iff-lt Ord-oexp Ord-omega-sum image-subset-iff
  omega-sum-less-ωω)
qed (use omega-sum-1-less in auto)

lemma ordertype-WW-3: ω↑ω ≤ ordertype WW (lenlex less-than)
proof –
  define π where π ≡ into-WW 0 ∘ Cantor-ωω
  have ωω: ω↑ω = tp (elts (ω↑ω))
    by simp
  also have ... ≤ ordertype WW (lenlex less-than)
proof (rule ordertype-inc-le)
  fix α β
  assume α: α ∈ elts (ω↑ω) and β: β ∈ elts (ω↑ω) and (α, β) ∈ VWF
  then obtain *: Ord α Ord β α < β
    by (metis Ord-in-Ord Ord-ordertype VWF-iff-Ord-less ωω)
  then have length (Cantor-ωω α) ≤ length (Cantor-ωω β)
    using α β by (simp add: Cantor-ωω normal-Cantor-ωω omega-sum-length-leD)
  with α β * have (Cantor-ωω α, Cantor-ωω β) ∈ lenlex less-than

```

by (auto simp: Cantor- ω simp flip: omega-sum-less-iff)
 then show $(\pi \alpha, \pi \beta) \in \text{lenlex less-than}$
 by (simp add: π -def into-WW-lenlex-iff)
 qed (auto simp: π -def WW-def strict-sorted-into-WW)
 finally show $\omega \uparrow \omega \leq \text{ordertype WW (lenlex less-than)}$.
 qed

lemma ordertype-WW: $\text{ordertype WW (lenlex less-than)} = \omega \uparrow \omega$
and ordertype-UNIV- ω : $\text{ordertype UNIV (lenlex less-than)} = \omega \uparrow \omega$
using ordertype-WW-1 ordertype-WW-2 ordertype-WW-3 **by** auto

lemma ordertype- ω :
 fixes $F :: \text{nat} \Rightarrow \text{nat list set}$
 assumes $\bigwedge j :: \text{nat}. \text{ordertype (F j) (lenlex less-than)} = \omega \uparrow j$
 shows $\text{ordertype } (\bigcup j. F j) \text{ (lenlex less-than)} = \omega \uparrow \omega$
proof (rule antisym)
 show $\text{ordertype } (\bigcup (\text{range } F)) \text{ (lenlex less-than)} \leq \omega \uparrow \omega$
 by (metis ordertype-UNIV- ω ordertype-mono small top-greatest trans-llt wf-llt)
 have $\bigwedge n. \omega \uparrow \text{ord-of-nat } n \leq \text{ordertype } (\bigcup (\text{range } F)) \text{ (lenlex less-than)}$
 by (metis TC-small Union-upper assms ordertype-mono rangeI trans-llt wf-llt)
 then show $\omega \uparrow \omega \leq \text{ordertype } (\bigcup (\text{range } F)) \text{ (lenlex less-than)}$
 by (auto simp: oexp- ω -Limit ZFC-in-HOL.SUP-le-iff elts- ω)
 qed

definition WW-seg $:: \text{nat} \Rightarrow \text{nat list set}$
 where $\text{WW-seg } n \equiv \{l \in \text{WW}. \text{length } l = n\}$

lemma WW-seg-subset-WW: $\text{WW-seg } n \subseteq \text{WW}$
by (auto simp: WW-seg-def)

lemma WW-eq-UN-WW-seg: $\text{WW} = (\bigcup n. \text{WW-seg } n)$
by (auto simp: WW-seg-def)

lemma ordertype-list-seg: $\text{ordertype } \{l. \text{length } l = n\} \text{ (lenlex less-than)} = \omega \uparrow n$
proof –
 have *bij-betw omega-sum* $\{l. \text{length } l = n\} \text{ (elts } (\omega \uparrow n))$
 unfolding WW-seg-def *bij-betw-def*
 by (auto simp: *inj-omega-sum Ord-mem-iff-lt omega-sum-less dest: Ex-omega-sum*)
 then show *?thesis*
 by (force simp: ordertype-eq-iff simp flip: eq-omega-sum-less-iff)
 qed

lemma ordertype-WW-seg: $\text{ordertype (WW-seg } n) \text{ (lenlex less-than)} = \omega \uparrow n$
 (is $\text{ordertype } ?W ?R = \omega \uparrow n$)

proof –
 have $\text{ordertype } \{l. \text{length } l = n\} ?R = \text{ordertype } ?W ?R$

```

proof (subst ordertype-eq-ordertype)
  show  $\exists f. \text{bij-betw } f \{l. \text{length } l = n\} ?W \wedge (\forall x \in \{l. \text{length } l = n\}. \forall y \in \{l. \text{length } l = n\}. ((f x, f y) \in \text{lenlex less-than}) = ((x, y) \in \text{lenlex less-than}))$ 
proof (intro exI conjI)
  have inj-on (into-WW 0) {l. length l = n}
  by (metis from-into-WW inj-onI)
  then show bij-betw (into-WW 0) {l. length l = n} ?W
  by (auto simp: bij-betw-def WW-seg-def WW-eq-range-into length-into-WW)
qed (simp add: into-WW-lenlex-iff)
qed auto
then show ?thesis
  using ordertype-list-seg by auto
qed

```

3.3 Definitions required for the lemmas

3.3.1 Larson's "<"-relation on ordered lists

```

instantiation list :: (ord)ord
begin

```

```

definition  $xs < ys \equiv xs \neq [] \wedge ys \neq [] \longrightarrow \text{last } xs < \text{hd } ys$  for  $xs \ ys :: 'a \text{ list}$ 

```

```

definition  $xs \leq ys \equiv xs < ys \vee xs = ys$  for  $xs \ ys :: 'a \text{ list}$ 

```

```

instance

```

```

  by standard

```

```

end

```

```

lemma less-Nil [simp]:  $xs < [] \ \ [] < xs$ 
  by (auto simp: less-list-def)

```

```

lemma less-sets-imp-list-less:

```

```

  assumes  $list.set \ xs \ll list.set \ ys$ 

```

```

  shows  $xs < ys$ 

```

```

  by (metis assms last-in-set less-list-def less-sets-def list.set-sel(1))

```

```

lemma less-sets-imp-sorted-list-of-set:

```

```

  assumes  $A \ll B \ \text{finite } A \ \text{finite } B$ 

```

```

  shows  $list-of \ A < list-of \ B$ 

```

```

  by (simp add: assms less-sets-imp-list-less)

```

```

lemma sorted-list-of-set-imp-less-sets:

```

```

  assumes  $xs < ys \ \text{sorted } xs \ \text{sorted } ys$ 

```

```

  shows  $list.set \ xs \ll list.set \ ys$ 

```

```

  using assms sorted-hd-le sorted-le-last

```

```

  by (force simp: less-list-def less-sets-def intro: order.trans)

```

```

lemma less-list-iff-less-sets:

```

```

  assumes  $\text{sorted } xs \ \text{sorted } ys$ 

```

shows $xs < ys \longleftrightarrow list.set\ xs \ll list.set\ ys$
using *assms sorted-hd-le sorted-le-last*
by (*force simp: less-list-def less-sets-def intro: order.trans*)

lemma *strict-sorted-append-iff*:
 $strict\ sorted\ (xs\ @\ ys) \longleftrightarrow xs < ys \wedge strict\ sorted\ xs \wedge strict\ sorted\ ys$
by (*metis less-list-iff-less-sets less-setsD sorted-wrt-append strict-sorted-imp-less-sets strict-sorted-imp-sorted*)

lemma *singleton-less-list-iff*: $sorted\ xs \implies [n] < xs \longleftrightarrow \{..n\} \cap list.set\ xs = \{\}$
apply (*simp add: less-list-def disjoint-iff*)
by (*metis empty-iff less-le-trans list.set(1) list.set-sel(1) not-le sorted-hd-le*)

lemma *less-hd-imp-less*: $xs < [hd\ ys] \implies xs < ys$
by (*simp add: less-list-def*)

lemma *strict-sorted-concat-I*:
assumes $\bigwedge x. x \in list.set\ xs \implies strict\ sorted\ x$
 $\bigwedge n. Suc\ n < length\ xs \implies xs!n < xs!Suc\ n$
 $xs \in lists\ (-\ \{\}\}$
shows $strict\ sorted\ (concat\ xs)$
using *assms*
proof (*induction xs*)
case (*Cons x xs*)
then have $x < concat\ xs$
apply (*simp add: less-list-def*)
by (*metis Compl-iff hd-concat insertI1 length-greater-0-conv length-pos-if-in-set list.sel(1) lists.cases nth-Cons-0*)
with *Cons show ?case*
by (*force simp: strict-sorted-append-iff*)
qed *auto*

3.4 Nash Williams for lists

3.4.1 Thin sets of lists

inductive *initial-segment* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where *initial-segment xs (xs@ys)*

definition *thin* :: $'a\ list\ set \Rightarrow bool$
where $thin\ A \equiv \neg (\exists x\ y. x \in A \wedge y \in A \wedge x \neq y \wedge initial\ segment\ x\ y)$

lemma *initial-segment-ne*:
assumes $initial\ segment\ xs\ ys\ xs \neq []$
shows $ys \neq [] \wedge hd\ ys = hd\ xs$
using *assms* **by** (*auto elim!: initial-segment.cases*)

lemma *take-initial-segment*:
assumes $initial\ segment\ xs\ ys\ k \leq length\ xs$
shows $take\ k\ xs = take\ k\ ys$


```

by (metis append-eq-conv-conj assms initial-segment.cases min-def take-take)

lemma initial-segment-length-eq:
  assumes initial-segment xs ys length xs = length ys
  shows xs = ys
  using assms initial-segment.cases by fastforce

lemma initial-segment-Nil [simp]: initial-segment [] ys
  by (simp add: initial-segment.simps)

lemma initial-segment-Cons [simp]: initial-segment (x#xs) (y#ys)  $\longleftrightarrow$  x=y  $\wedge$ 
  initial-segment xs ys
  by (metis append-Cons initial-segment.simps list.inject)

lemma init-segment-iff-initial-segment:
  assumes strict-sorted xs strict-sorted ys
  shows init-segment (list.set xs) (list.set ys)  $\longleftrightarrow$  initial-segment xs ys (is ?lhs =
  ?rhs)
proof
  assume ?lhs
  then obtain S' where S': list.set ys = list.set xs  $\cup$  S' list.set xs  $\ll$  S'
    by (auto simp: init-segment-def)
  then have finite S'
    by (metis List.finite-set finite-Un)
  have ys = xs @ list-of S'
    using S' <strict-sorted xs>
  proof (induction xs)
    case Nil
    with <strict-sorted ys> show ?case
      by auto
    next
    case (Cons a xs)
    with <finite S'> have ys = a # xs @ list-of S'
      by (metis List.finite-set append-Cons assms(2) sorted-list-of-set-Un sorted-list-of-set-set-of)
    then show ?case
      by (auto simp: Cons)
  qed
  then show ?rhs
    using initial-segment.intros by blast
next
  assume ?rhs
  then show ?lhs
  proof cases
    case (1 ys)
    with assms show ?thesis
      by (simp add: init-segment-Un strict-sorted-imp-less-sets)
  qed
qed

```

```

theorem Nash-Williams-WW:
  fixes  $h :: \text{nat list} \Rightarrow \text{nat}$ 
  assumes infinite M and  $h : \{l \in A. \text{List.set } l \subseteq M\} \subseteq \{..<2\}$  and thin A A
   $\subseteq WW$ 
  obtains  $i N$  where  $i < 2$  infinite N  $N \subseteq M$   $h : \{l \in A. \text{List.set } l \subseteq N\} \subseteq \{i\}$ 
proof –
  define AM where  $AM \equiv \{l \in A. \text{List.set } l \subseteq M\}$ 
  have thin-set (list.set ‘A)
    using  $\langle \text{thin } A \rangle \langle A \subseteq WW \rangle$  unfolding thin-def thin-set-def WW-def
    by (auto simp: subset-iff init-segment-iff-initial-segment)
  then have thin-set (list.set ‘AM)
    by (simp add: AM-def image-subset-iff thin-set-def)
  then have Ramsey (list.set ‘AM) 2
    using Nash-Williams-2 by metis
  moreover have  $(h \circ \text{list-of}) \in \text{list.set } ‘AM \rightarrow \{..<2\}$ 
    unfolding AM-def
  proof clarsimp
    fix  $l$ 
    assume  $l \in A$  list.set  $l \subseteq M$ 
    then have strict-sorted  $l$ 
      using WW-def  $\langle A \subseteq WW \rangle$  by blast
    then show  $h (\text{list-of } (\text{list.set } l)) < 2$ 
      using  $h \langle l \in A \rangle \langle \text{list.set } l \subseteq M \rangle$  by auto
  qed
  ultimately obtain  $N i$  where  $N : N \subseteq M$  infinite N  $i < 2$ 
    and list.set ‘AM  $\cap \text{Pow } N \subseteq (h \circ \text{list-of}) - \{i\}$ 
    unfolding Ramsey-eq by (metis  $\langle \text{infinite } M \rangle$ )
  then have N-disjoint:  $(h \circ \text{list-of}) - \{1-i\} \cap (\text{list.set } ‘AM) \cap \text{Pow } N = \{\}$ 
    unfolding subset-vimage-iff less-2-cases-iff by force
  have  $h : \{l \in A. \text{list.set } l \subseteq N\} \subseteq \{i\}$ 
proof clarify
  fix  $l$ 
  assume  $l \in A$  and list.set  $l \subseteq N$ 
  then have  $h l < 2$ 
    using  $h \langle N \subseteq M \rangle$  by force
  with  $\langle i < 2 \rangle$  have  $h l \neq \text{Suc } 0 - i \implies h l = i$ 
    by (auto simp: eval-nat-numeral less-Suc-eq)
  moreover have strict-sorted  $l$ 
    using  $\langle A \subseteq WW \rangle \langle l \in A \rangle$  unfolding WW-def by blast
  moreover have  $h (\text{list-of } (\text{list.set } l)) = 1 - i \longrightarrow \neg (\text{list.set } l \subseteq N)$ 
    using N-disjoint  $\langle N \subseteq M \rangle \langle l \in A \rangle$  by (auto simp: AM-def)
  ultimately show  $h l = i$ 
    using  $N \langle N \subseteq M \rangle \langle l \in A \rangle \langle \text{list.set } l \subseteq N \rangle$ 
    by (auto simp: vimage-def set-eq-iff AM-def WW-def subset-iff)
  qed
  then show thesis
    using that  $\langle i < 2 \rangle N$  by auto
qed

```

3.5 Specialised functions on lists

lemma *mem-lists-non-Nil*: $xss \in \text{lists } (- \{\}\}) \longleftrightarrow (\forall x \in \text{list.set } xss. x \neq \{\})$
by *auto*

fun *acc-lengths* :: $\text{nat} \Rightarrow 'a \text{ list list} \Rightarrow \text{nat list}$
where *acc-lengths* *acc* $\{\} = \{\}$
| *acc-lengths* *acc* $(l\#ls) = (\text{acc} + \text{length } l) \# \text{acc-lengths } (\text{acc} + \text{length } l) \text{ } ls$

lemma *length-acc-lengths* [*simp*]: $\text{length } (\text{acc-lengths } \text{acc } ls) = \text{length } ls$
by (*induction* *ls* *arbitrary*: *acc*) *auto*

lemma *acc-lengths-eq-Nil-iff* [*simp*]: $\text{acc-lengths } \text{acc } ls = \{\} \longleftrightarrow ls = \{\}$
by (*metis* *length-0-conv* *length-acc-lengths*)

lemma *set-acc-lengths*:
assumes $ls \in \text{lists } (- \{\}\})$ **shows** $\text{list.set } (\text{acc-lengths } \text{acc } ls) \subseteq \{\text{acc} < ..\}$
using *assms* **by** (*induction* *ls* *rule*: *acc-lengths.induct*) *fastforce*+

Useful because *acc-lengths.simps* will sometimes be deleted from the simpset.

lemma *hd-acc-lengths* [*simp*]: $\text{hd } (\text{acc-lengths } \text{acc } (l\#ls)) = \text{acc} + \text{length } l$
by *simp*

lemma *last-acc-lengths* [*simp*]:
 $ls \neq \{\} \implies \text{last } (\text{acc-lengths } \text{acc } ls) = \text{acc} + \text{sum-list } (\text{map } \text{length } ls)$
by (*induction* *acc* *ls* *rule*: *acc-lengths.induct*) *auto*

lemma *nth-acc-lengths* [*simp*]:
 $\llbracket ls \neq \{\}; k < \text{length } ls \rrbracket \implies \text{acc-lengths } \text{acc } ls ! k = \text{acc} + \text{sum-list } (\text{map } \text{length } (\text{take } (\text{Suc } k) \text{ } ls))$
by (*induction* *acc* *ls* *arbitrary*: *k* *rule*: *acc-lengths.induct*) (*fastforce* *simp*: *less-Suc-eq* *nth-Cons*)+

lemma *acc-lengths-plus*: $\text{acc-lengths } (m+n) \text{ } as = \text{map } ((+)m) (\text{acc-lengths } n \text{ } as)$
by (*induction* *n* *as* *arbitrary*: *m* *rule*: *acc-lengths.induct*) (*auto* *simp*: *add.assoc*)

lemma *acc-lengths-shift*: *NO-MATCH* $0 \text{ } \text{acc} \implies \text{acc-lengths } \text{acc } as = \text{map } ((+)\text{acc}) (\text{acc-lengths } 0 \text{ } as)$
by (*metis* *acc-lengths-plus* *add.comm-neutral*)

lemma *length-concat-acc-lengths*:
 $ls \neq \{\} \implies k + \text{length } (\text{concat } ls) \in \text{list.set } (\text{acc-lengths } k \text{ } ls)$
by (*metis* *acc-lengths-eq-Nil-iff* *last-acc-lengths* *last-in-set* *length-concat*)

lemma *strict-sorted-acc-lengths*:
assumes $ls \in \text{lists } (- \{\}\})$ **shows** *strict-sorted* $(\text{acc-lengths } \text{acc } ls)$
using *assms*
proof (*induction* *ls* *rule*: *acc-lengths.induct*)
case $(2 \text{ } \text{acc } l \text{ } ls)$

then have *strict-sorted* (*acc-lengths* (*acc* + *length* *l*) *ls*)
by *auto*
then show *?case*
using *set-acc-lengths 2.premis* **by** *auto*
qed *auto*

lemma *acc-lengths-append*:
acc-lengths *acc* (*xs* @ *ys*)
= *acc-lengths* *acc* *xs* @ *acc-lengths* (*acc* + *sum-list* (*map length xs*)) *ys*
by (*induction acc xs rule: acc-lengths.induct*) (*auto simp: add.assoc*)

lemma *length-concat-ge*:
assumes *as* ∈ *lists* (− {[]})
shows *length* (*concat as*) ≥ *length as*
using *assms*
proof (*induction as*)
case (*Cons a as*)
then have *length a* ≥ *Suc 0* ∧ *l. l* ∈ *list.set as* ⇒ *length l* ≥ *Suc 0*
by (*auto simp: Suc-leI*)
then show *?case*
using *Cons.IH* **by** *force*
qed *auto*

fun *interact* :: 'a list list ⇒ 'a list list ⇒ 'a list
where
interact [] *ys* = *concat ys*
| *interact xs* [] = *concat xs*
| *interact (x#xs)* (*y#ys*) = *x* @ *y* @ *interact xs ys*

lemma (*in monoid-add*) *length-interact*:
length (*interact xs ys*) = *sum-list* (*map length xs*) + *sum-list* (*map length ys*)
by (*induction rule: interact.induct*) (*auto simp: length-concat*)

lemma *length-interact-ge*:
assumes *xs* ∈ *lists* (− {[]}) *ys* ∈ *lists* (− {[]})
shows *length* (*interact xs ys*) ≥ *length xs* + *length ys*
by (*metis add-mono assms length-concat length-concat-ge length-interact*)

lemma *set-interact [simp]*:
shows *list.set* (*interact xs ys*) = *list.set* (*concat xs*) ∪ *list.set* (*concat ys*)
by (*induction rule: interact.induct*) *auto*

lemma *interact-eq-Nil-iff [simp]*:
assumes *xs* ∈ *lists* (− {[]}) *ys* ∈ *lists* (− {[]})
shows *interact xs ys* = [] ⇔ *xs* = [] ∧ *ys* = []
using *length-interact-ge [OF assms]* **by** *fastforce*

lemma *interact-sing* [*simp*]: $interact [x] ys = x @ concat ys$
by (*metis* (*no-types*) *concat.simps(2)* *interact.simps* *neq-Nil-conv*)

lemma *hd-interact*: $\llbracket xs \neq []; hd xs \neq [] \rrbracket \implies hd (interact xs ys) = hd (hd xs)$
by (*smt* (*verit*, *best*) *hd-append2* *hd-concat* *interact.elims* *list.sel(1)*)

lemma *acc-lengths-concat-injective*:
assumes $concat as' = concat as$ $acc-lengths n as' = acc-lengths n as$
shows $as' = as$
using *assms*
proof (*induction as arbitrary: n as'*)
case *Nil*
then show *?case*
by (*metis* *acc-lengths-eq-Nil-iff*)
next
case (*Cons a as*)
then obtain $a' bs$ **where** $as' = a' \# bs$
by (*metis* *Suc-length-conv* *length-acc-lengths*)
with *Cons* **show** *?case*
by *simp*
qed

lemma *acc-lengths-interact-injective*:
assumes $interact as' bs' = interact as bs$ $acc-lengths a as' = acc-lengths a as$
 $acc-lengths b bs' = acc-lengths b bs$
shows $as' = as \wedge bs' = bs$
using *assms*
proof (*induction as bs arbitrary: a b as' bs' rule: interact.induct*)
case (*1 cs*) **then show** *?case*
by (*metis* *acc-lengths-concat-injective* *acc-lengths-eq-Nil-iff* *interact.simps(1)*)
next
case (*2 c cs*)
then show *?case*
by (*metis* *acc-lengths-concat-injective* *acc-lengths-eq-Nil-iff* *interact.simps(2)* *list.exhaust*)
next
case (*3 x xs y ys*)
then obtain $a' us b' vs$ **where** $as' = a' \# us$ $bs' = b' \# vs$
by (*metis* *length-Suc-conv* *length-acc-lengths*)
with *3* **show** *?case*
by *auto*
qed

lemma *strict-sorted-interact-I*:
assumes $length ys \leq length xs$ $length xs \leq Suc (length ys)$
 $\bigwedge x. x \in list.set xs \implies strict-sorted x$
 $\bigwedge y. y \in list.set ys \implies strict-sorted y$
 $\bigwedge n. n < length ys \implies xs!n < ys!n$

```

   $\wedge n. \text{Suc } n < \text{length } xs \implies ys!n < xs!\text{Suc } n$ 
assumes  $xs \in \text{lists } (- \{\}\}$   $ys \in \text{lists } (- \{\}\}$ 
shows strict-sorted (interact  $xs$   $ys$ )
using assms
proof (induction rule: interact.induct)
case ( $\exists x$   $xs$   $y$   $ys$ )
then have  $x < y$ 
  by force
moreover have strict-sorted (interact  $xs$   $ys$ )
  using  $\exists$  by simp (metis Suc-less-eq nth-Cons-Suc)
moreover have  $y < \text{interact } xs$   $ys$ 
  using  $\exists$  apply (simp add: less-list-def)
  by (metis hd-interact le-zero-eq length-greater-0-conv list.sel(1) list.set-sel(1)
list.size(3) lists.simps mem-lists-non-Nil nth-Cons-0)
  ultimately show ?case
  using  $\exists$  by (simp add: strict-sorted-append-iff less-list-def)
qed auto

```

3.6 Forms and interactions

3.6.1 Forms

```

inductive Form-Body :: [nat, nat, nat list, nat list, nat list]  $\Rightarrow$  bool
where Form-Body  $ka$   $kb$   $xs$   $ys$   $zs$ 
  if  $\text{length } xs < \text{length } ys$   $xs = \text{concat } (a\#as)$   $ys = \text{concat } (b\#bs)$ 
     $a\#as \in \text{lists } (- \{\}\}$   $b\#bs \in \text{lists } (- \{\}\}$ 
     $\text{length } (a\#as) = ka$   $\text{length } (b\#bs) = kb$ 
     $c = \text{acc-lengths } 0$   $(a\#as)$ 
     $d = \text{acc-lengths } 0$   $(b\#bs)$ 
     $zs = \text{concat } [c, a, d, b]$  @ interact  $as$   $bs$ 
    strict-sorted  $zs$ 

```

```

inductive Form :: [nat, nat list set]  $\Rightarrow$  bool
where Form  $0$   $\{xs, ys\}$  if  $\text{length } xs = \text{length } ys$   $xs \neq ys$ 
  | Form  $(2*k-1)$   $\{xs, ys\}$  if Form-Body  $k$   $k$   $xs$   $ys$   $zs$   $k > 0$ 
  | Form  $(2*k)$   $\{xs, ys\}$  if Form-Body  $(\text{Suc } k)$   $k$   $xs$   $ys$   $zs$   $k > 0$ 

```

```

inductive-cases Form-0-cases-raw: Form  $0$   $u$ 

```

```

lemma Form-elim-upair:

```

```

assumes Form  $l$   $U$ 

```

```

obtains  $xs$   $ys$  where  $xs \neq ys$   $U = \{xs, ys\}$   $\text{length } xs \leq \text{length } ys$ 

```

```

using assms

```

```

by (smt (verit, best) Form.simps Form-Body.cases less-or-eq-imp-le nat-neq-iff)

```

```

lemma assumes Form-Body  $ka$   $kb$   $xs$   $ys$   $zs$ 

```

```

shows Form-Body-WW:  $zs \in WW$ 

```

```

and Form-Body-nonempty:  $\text{length } zs > 0$ 

```

and *Form-Body-length*: $\text{length } xs < \text{length } ys$
using *Form-Body.cases* [*OF assms*] **by** (*fastforce simp: WW-def*)+

lemma *form-cases*:

fixes $l::\text{nat}$

obtains $(\text{zero}) \ l = 0 \mid (\text{nz}) \ ka \ kb$ **where** $l = ka+kb - 1 \ 0 < kb \ kb \leq ka \ ka \leq \text{Suc } kb$

proof –

have $l = 0 \vee (\exists ka \ kb. \ l = ka+kb - 1 \wedge 0 < kb \wedge kb \leq ka \wedge ka \leq \text{Suc } kb)$

by *presburger*

then show *thesis*

using *nz zero* **by** *blast*

qed

3.6.2 Interactions

lemma *interact*:

assumes *Form l U l>0*

obtains $ka \ kb \ xs \ ys \ zs$ **where** $l = ka+kb - 1 \ U = \{xs,ys\}$ *Form-Body ka kb xs ys zs* $0 < kb \ kb \leq ka \ ka \leq \text{Suc } kb$

using *assms*

unfolding *Form.simps*

by (*smt (verit, best) add-Suc diff-Suc-1 lessI mult-2 nat-less-le order-refl*)

definition *inter-scheme* :: $\text{nat} \Rightarrow \text{nat list set} \Rightarrow \text{nat list}$

where *inter-scheme l U* \equiv

SOME zs. $\exists k \ xs \ ys. \ U = \{xs,ys\} \wedge$

*($l = 2*k-1 \wedge \text{Form-Body } k \ k \ xs \ ys \ zs \vee l = 2*k \wedge \text{Form-Body } (\text{Suc } k)$*

k xs ys zs)

lemma *inter-scheme*:

assumes *Form l U l>0*

obtains $ka \ kb \ xs \ ys$ **where** $l = ka+kb - 1 \ U = \{xs,ys\}$ *Form-Body ka kb xs ys* $(\text{inter-scheme } l \ U) \ 0 < kb \ kb \leq ka \ ka \leq \text{Suc } kb$

using *interact* [*OF $\langle \text{Form } l \ U \rangle$*]

proof *cases*

case $(2 \ ka \ kb \ xs \ ys \ zs)$

then have $\S: \bigwedge ka \ kb \ zs. \ \neg \text{Form-Body } ka \ kb \ ys \ xs \ zs$

using *Form-Body-length less-asymp'* **by** *blast*

have *Form-Body ka kb xs ys* (*inter-scheme l U*)

proof (*cases ka = kb*)

case *True*

with 2 **have** $l: \forall k. \ l \neq k * 2$

by *presburger*

have [*simp*]: $\bigwedge k. \ kb + kb - \text{Suc } 0 = k * 2 - \text{Suc } 0 \longleftrightarrow k=kb$

by *auto*

show *?thesis*

```

    unfolding inter-scheme-def using 2 l True
    by (auto simp: § ⟨l > 0⟩ Set.doubleton-eq-iff conj-disj-distribR ex-disj-distrib
algebra-simps some-eq-ex)
  next
    case False
    with 2 have l: ∀k. l ≠ k * 2 - Suc 0 and [simp]: ka = Suc kb
      by presburger+
    have [simp]: ∧k. kb + kb = k * 2 ⟷ k=kb
      by auto
    show ?thesis
      unfolding inter-scheme-def using 2 l False
      by (auto simp: § ⟨l > 0⟩ Set.doubleton-eq-iff conj-disj-distribR ex-disj-distrib
algebra-simps some-eq-ex)
    qed
  then show ?thesis
    by (simp add: 2 that)
qed (use ⟨l > 0⟩ in auto)

```

```

lemma inter-scheme-strict-sorted:
  assumes Form l U l>0
  shows strict-sorted (inter-scheme l U)
  using Form-Body.simps assms inter-scheme by fastforce

```

```

lemma inter-scheme-simple:
  assumes Form l U l>0
  shows inter-scheme l U ∈ WW ∧ length (inter-scheme l U) > 0
  using inter-scheme [OF assms] by (meson Form-Body-WW Form-Body-nonempty)

```

3.6.3 Injectivity of interactions

```

proposition inter-scheme-injective:
  assumes Form l U Form l U' l > 0 and eq: inter-scheme l U' = inter-scheme l
U

```

```

  shows U' = U

```

```

proof -

```

```

  obtain ka kb xs ys

```

```

  where l: l = ka+kb - 1 and U: U = {xs,ys}
    and FB: Form-Body ka kb xs ys (inter-scheme l U)
    and kb: 0 < kb kb ≤ ka ka ≤ Suc kb

```

```

  using assms inter-scheme by blast

```

```

then obtain a as b bs c d

```

```

  where xs: xs = concat (a#as) and ys: ys = concat (b#bs)
    and len: length (a#as) = ka length (b#bs) = kb
    and c: c = acc-lengths 0 (a#as)
    and d: d = acc-lengths 0 (b#bs)
    and Ueq: inter-scheme l U = concat [c, a, d, b] @ interact as bs
  by (auto simp: Form-Body.simps)

```

```

obtain ka' kb' xs' ys'

```

```

  where l': l = ka'+kb' - 1 and U': U' = {xs',ys'}

```


and FB' : *Form-Body* $ka' kb' xs' ys'$ (*inter-scheme* $l U'$)
and kb' : $0 < kb' kb' \leq ka' ka' \leq \text{Suc } kb'$
using *assms inter-scheme* **by** *blast*
then obtain $a' as' b' bs' c' d'$
where xs' : $xs' = \text{concat } (a' \# as')$ **and** ys' : $ys' = \text{concat } (b' \# bs')$
and len' : $\text{length } (a' \# as') = ka' \text{ length } (b' \# bs') = kb'$
and c' : $c' = \text{acc-lengths } 0 (a' \# as')$
and d' : $d' = \text{acc-lengths } 0 (b' \# bs')$
and Ueq' : *inter-scheme* $l U' = \text{concat } [c', a', d', b'] @ \text{interact } as' bs'$
using *Form-Body.simps* **by** *auto*
have $[simp]$: $ka' = ka \wedge kb' = kb$
using $\langle l > 0 \rangle l l' kb kb' le\text{-SucE } le\text{-antisym } mult\text{-2}$ **by** *linarith*
have $[simp]$: $\text{length } c = \text{length } c' \text{ length } d = \text{length } d'$
using $c c' d d' len' len$ **by** *auto*
have $c\text{-off}$: $c' = c a' @ d' @ b' @ \text{interact } as' bs' = a @ d @ b @ \text{interact } as bs$
using *eq* **by** (*auto simp: Ueq Ueq'*)
then have $len\text{-a}$: $\text{length } a' = \text{length } a$
by (*metis acc-lengths.simps(2) add.left-neutral c c' nth-Cons-0*)
with $c\text{-off}$ **have** \S : $a' = a d' = d b' @ \text{interact } as' bs' = b @ \text{interact } as bs$
by *auto*
then have $\text{length } (\text{interact } as' bs') = \text{length } (\text{interact } as bs)$
by (*metis acc-lengths.simps(2) add-left-cancel append-eq-append-conv d d' list.inject*)
with \S **have** $b' = b \text{ interact } as' bs' = \text{interact } as bs$
by *auto*
moreover have $\text{acc-lengths } 0 as' = \text{acc-lengths } 0 as$
using $\langle a' = a \rangle \langle c' = c \rangle$ **by** (*simp add: c' c acc-lengths-shift*)
moreover have $\text{acc-lengths } 0 bs' = \text{acc-lengths } 0 bs$
using $\langle b' = b \rangle \langle d' = d \rangle$ **by** (*simp add: d' d acc-lengths-shift*)
ultimately have $as' = as \wedge bs' = bs$
using *acc-lengths-interact-injective* **by** *blast*
then show *?thesis*
by (*simp add: \langle a' = a \rangle U U' \langle b' = b \rangle xs xs' ys ys'*)

qed

lemma *strict-sorted-interact-imp-concat*:

strict-sorted (interact as bs) \implies strict-sorted (concat as) \wedge strict-sorted (concat bs)

proof (*induction as bs rule: interact.induct*)

case $(\exists x xs y ys)$

have $x < \text{concat } xs$

using $\exists.\text{prems}$

by (*smt (verit, del-insts) Un-iff hd-in-set interact.simps(3) last-in-set less-list-def set-append set-interact sorted-wrt-append*)

moreover have $y < \text{concat } ys$

using \exists *sorted-wrt-append strict-sorted-append-iff* **by** *fastforce*

ultimately show *?case*

using \exists **by** (*auto simp add: strict-sorted-append-iff*)

qed *auto*

lemma *strict-sorted-interact-hd*:
 $\llbracket \text{strict-sorted } (\text{interact } cs \ ds); \ cs \neq []; \ ds \neq []; \ \text{hd } cs \neq []; \ \text{hd } ds \neq [] \rrbracket$
 $\implies \text{hd } (\text{hd } cs) < \text{hd } (\text{hd } ds)$
by (*metis append-is-Nil-conv hd-append2 hd-in-set interact.simps(3) list.exhaust-sel sorted-wrt-append*)

the lengths of the two lists can differ by one

proposition *interaction-scheme-unique-aux*:
assumes $\text{concat } as = \text{concat } as'$ **and** ys' : $\text{concat } bs = \text{concat } bs'$
and $as \in \text{lists } (- \{\}\}$ $bs \in \text{lists } (- \{\}\}$
and *strict-sorted* (*interact* $as \ bs$)
and $\text{length } bs \leq \text{length } as$ $\text{length } as \leq \text{Suc } (\text{length } bs)$
and $as' \in \text{lists } (- \{\}\}$ $bs' \in \text{lists } (- \{\}\}$
and *strict-sorted* (*interact* $as' \ bs'$)
and $\text{length } bs' \leq \text{length } as'$ $\text{length } as' \leq \text{Suc } (\text{length } bs')$
and $\text{length } as = \text{length } as'$ $\text{length } bs = \text{length } bs'$
shows $as = as' \wedge bs = bs'$
using *assms*
proof (*induction length as arbitrary: as bs as' bs'*)
case 0 then show *?case*
by *auto*
next
case SUC: (*Suc k*)
show *?case*
proof (*cases k*)
case 0
with SUC obtain $a \ a'$ **where** aa' : $as = [a]$ $as' = [a']$
by (*metis Suc-length-conv length-0-conv*)
show *?thesis*
proof
show $as = as'$
using aa' $\langle \text{concat } as = \text{concat } as' \rangle$ **by** *force*
with SUC 0 show $bs = bs'$
by (*metis Suc-leI append-Nil2 concat.simps impossible-Cons le-antisym length-greater-0-conv list.exhaust*)
qed
next
case (*Suc k'*)
then obtain $a \ cs \ b \ ds$ **where** eq : $as = a \# \ cs$ $bs = b \# \ ds$
using *SUC*
by (*metis le0 list.exhaust list.size(3) not-less-eq-eq*)
have $\text{length } as' \neq 0$
using *SUC* **by** *force*
then obtain $a' \ cs' \ b' \ ds'$ **where** eq' : $as' = a' \# \ cs'$ $bs' = b' \# \ ds'$
by (*metis* $\langle \text{length } bs = \text{length } bs' \rangle$ $eq(2)$ *length-0-conv list.exhaust*)
obtain k : $k = \text{length } cs$ $k \leq \text{Suc } (\text{length } ds)$
using eq *SUC* **by** *auto*

```

case (Suc k')
obtain [simp]: b ≠ [] b' ≠ [] a ≠ [] a' ≠ []
  using SUC by (simp add: eq eq')
then have hd b' = hd b
  using SUC by (metis concat.simps(2) eq'(2) eq(2) hd-append2)
have ss-ab: strict-sorted (concat as) strict-sorted (concat bs)
  using strict-sorted-interact-imp-concat SUC.prem(5) by blast+
have sw-ab: strict-sorted (a @ b @ interact cs ds)
  by (metis SUC.prem(5) eq interact.simps(3))
then obtain a < b strict-sorted a strict-sorted b
  by (metis append-assoc strict-sorted-append-iff)
have b-cs: strict-sorted (concat (b # cs))
  by (metis append.simps(1) concat.simps(2) interact.simps(3) strict-sorted-interact-imp-concat
sw-ab)
then have b < concat cs
  using strict-sorted-append-iff by auto
have strict-sorted (a @ concat cs)
  using eq(1) ss-ab(1) by force
have list.set a = list.set (concat as) ∩ {..

```

```

    if  $x < \text{hd } b'$  and  $l \in \text{list.set } cs'$  and  $x \in \text{list.set } l$  for  $x l$ 
    using  $b\text{-}cs'$  sorted-hd-le strict-sorted-imp-sorted that by fastforce
    then show ?thesis
    using  $\langle b' \neq [] \rangle$  sw-ab' by (force simp: strict-sorted-append-iff sorted-wrt-append
eq^)
  qed
  ultimately have  $a = a'$ 
    by (simp add: SUC.prems(1)  $\langle \text{hd } b' = \text{hd } b \rangle$   $\langle \text{strict-sorted } a' \rangle$   $\langle \text{strict-sorted } a \rangle$ 
    strict-sorted-equal)
  moreover
  have ccat-cs-cs':  $\text{concat } cs = \text{concat } cs'$ 
    using SUC.prems(1)  $\langle a = a' \rangle$  eq'(1) eq(1) by fastforce
  have  $b = b'$ 
  proof (cases  $ds = [] \vee ds' = []$ )
    case True
    then show ?thesis
      using SUC eq'(2) eq(2) by fastforce
    next
    case False
    then have  $ds \neq []$   $ds' \neq []$  sorted (concat ds) sorted (concat ds')
      using eq(2) ss-ab(2) eq'(2) ss-ab'(2) strict-sorted-append-iff strict-sorted-imp-sorted
    by auto
    have strict-sorted b strict-sorted b'
      using b-cs b-cs' sorted-wrt-append by auto
    moreover
    have  $cs \neq []$ 
      using k local.Suc by auto
    then obtain  $\text{hd } cs \neq []$   $\text{hd } ds \neq []$ 
      using SUC.prems(3) SUC.prems(4) eq list.set-sel(1)
      by (simp add:  $\langle ds \neq [] \rangle$  mem-lists-non-Nil)
    then have  $\text{concat } cs \neq []$ 
      using  $\langle cs \neq [] \rangle$  hd-in-set by auto
    have  $\text{hd } (\text{concat } cs) < \text{hd } (\text{concat } ds)$ 
      using strict-sorted-interact-hd
    by (metis  $\langle cs \neq [] \rangle$   $\langle ds \neq [] \rangle$   $\langle \text{hd } cs \neq [] \rangle$   $\langle \text{hd } ds \neq [] \rangle$  hd-concat sorted-wrt-append
sw-ab)

    have  $\text{list.set } b = \text{list.set } (\text{concat } bs) \cap \{.. < \text{hd } (\text{concat } cs)\}$ 
    proof -
      have 1:  $x \in \text{list.set } b$ 
        if  $x < \text{hd } (\text{concat } cs)$  and  $l \in \text{list.set } ds$  and  $x \in \text{list.set } l$  for  $x l$ 
        using  $\langle \text{hd } (\text{concat } cs) < \text{hd } (\text{concat } ds) \rangle$   $\langle \text{sorted } (\text{concat } ds) \rangle$  sorted-hd-le
      that by fastforce
      have 2:  $l < \text{hd } (\text{concat } cs)$  if  $l \in \text{list.set } b$  for  $l$ 
      by (metis  $\langle \text{concat } cs \neq [] \rangle$  b-cs concat.simps(2) list.set-sel(1) sorted-wrt-append
      that)
      show ?thesis
        using 1 2 by (auto simp: strict-sorted-append-iff sorted-wrt-append eq)
    qed
  qed

```

```

moreover
have  $cs' \neq []$ 
  using  $k \text{ Suc } \langle \text{concat } cs \neq [] \rangle \text{ ccat-cs-cs' by auto}$ 
then obtain  $hd \ cs' \neq [] \ hd \ ds' \neq []$ 
  using  $SUC.prem5(8,9) \langle ds' \neq [] \rangle \text{ eq'(1) eq'(2) list.set-sel(1) by auto}$ 
then have  $\text{concat } cs' \neq []$ 
  using  $\langle cs' \neq [] \rangle \text{ hd-in-set by auto}$ 
have  $hd \ (\text{concat } cs') < hd \ (\text{concat } ds')$ 
  using  $\text{strict-sorted-interact-hd}$ 
  by  $(\text{metis } \langle cs' \neq [] \rangle \langle ds' \neq [] \rangle \langle hd \ cs' \neq [] \rangle \langle hd \ ds' \neq [] \rangle \text{ hd-concat}$ 
 $\text{sorted-wrt-append sw-ab'})$ 
have  $\text{list.set } b' = \text{list.set } (\text{concat } bs') \cap \{..< hd \ (\text{concat } cs')\}$ 
proof –
  have  $1: x \in \text{list.set } b'$ 
    if  $x < hd \ (\text{concat } cs')$  and  $l \in \text{list.set } ds'$  and  $x \in \text{list.set } l$  for  $x \ l$ 
    using  $\langle hd \ (\text{concat } cs') < hd \ (\text{concat } ds') \rangle \langle \text{sorted } (\text{concat } ds') \rangle \text{ sorted-hd-le}$ 
that by  $\text{fastforce}$ 
    have  $2: l < hd \ (\text{concat } cs')$  if  $l \in \text{list.set } b'$  for  $l$ 
    by  $(\text{metis } \langle \text{concat } cs' \neq [] \rangle \text{ b-cs' list.set-sel(1) sorted-wrt-append that})$ 
    show  $?thesis$ 
    using  $1 \ 2 \text{ by (auto simp: strict-sorted-append-iff sorted-wrt-append eq')}$ 
qed
ultimately show  $b = b'$ 
  by  $(\text{simp add: } SUC.prem5(2) \text{ ccat-cs-cs' strict-sorted-equal})$ 
qed
moreover
have  $cs = cs' \wedge ds = ds'$ 
proof  $(\text{rule } SUC.hyps)$ 
  show  $k = \text{length } cs$ 
    using  $\text{eq } SUC.hyps(2) \text{ by auto}[1]$ 
  show  $\text{concat } ds = \text{concat } ds'$ 
    using  $SUC.prem5(2) \langle b = b' \rangle \text{ eq'(2) eq(2) by auto}$ 
  show  $\text{strict-sorted } (\text{interact } cs \ ds)$ 
    using  $\text{eq } SUC.prem5(5) \text{ strict-sorted-append-iff by auto}$ 
  show  $\text{length } ds \leq \text{length } cs \ \text{length } cs \leq \text{Suc } (\text{length } ds)$ 
    using  $\text{eq } SUC \ k \text{ by auto}$ 
  show  $\text{strict-sorted } (\text{interact } cs' \ ds')$ 
    using  $\text{eq' } SUC.prem5(10) \text{ strict-sorted-append-iff by auto}$ 
  show  $\text{length } cs = \text{length } cs'$ 
    using  $SUC \ \text{eq'(1) } k(1) \text{ by force}$ 
qed  $(\text{use ccat-cs-cs' eq eq' } SUC.prem5 \text{ in auto})$ 
ultimately show  $?thesis$ 
  by  $(\text{simp add: } \langle a = a' \rangle \langle b = b' \rangle \text{ eq eq'})$ 
qed
qed

```

proposition *Form-Body-unique:*

assumes *Form-Body* $ka \ kb \ xs \ ys \ zs$ *Form-Body* $ka \ kb \ xs \ ys \ zs'$ **and** $kb \leq ka \ ka \leq$

```

Suc kb
shows  $zs' = zs$ 
proof –
  obtain  $a\ as\ b\ bs\ c\ d$ 
    where  $xs: xs = \text{concat } (a\#\ as)$  and  $ys: ys = \text{concat } (b\#\ bs)$ 
    and  $ne: a\#\ as \in \text{lists } (-\ \{\ \})$   $b\#\ bs \in \text{lists } (-\ \{\ \})$ 
    and  $len: \text{length } (a\#\ as) = ka$   $\text{length } (b\#\ bs) = kb$ 
    and  $c: c = \text{acc-lengths } 0\ (a\#\ as)$ 
    and  $d: d = \text{acc-lengths } 0\ (b\#\ bs)$ 
    and  $Ueq: zs = \text{concat } [c, a, d, b] \text{ @ interact } as\ bs$ 
    and  $ss-zs: \text{strict-sorted } zs$ 
  using Form-Body.cases [OF assms(1)] by (metis (no-types))
  obtain  $a'\ as'\ b'\ bs'\ c'\ d'$ 
    where  $xs': xs = \text{concat } (a'\#\ as')$  and  $ys': ys = \text{concat } (b'\#\ bs')$ 
    and  $ne': a'\#\ as' \in \text{lists } (-\ \{\ \})$   $b'\#\ bs' \in \text{lists } (-\ \{\ \})$ 
    and  $len': \text{length } (a'\#\ as') = ka$   $\text{length } (b'\#\ bs') = kb$ 
    and  $c': c' = \text{acc-lengths } 0\ (a'\#\ as')$ 
    and  $d': d' = \text{acc-lengths } 0\ (b'\#\ bs')$ 
    and  $Ueq': zs' = \text{concat } [c', a', d', b'] \text{ @ interact } as'\ bs'$ 
    and  $ss-zs': \text{strict-sorted } zs'$ 
  using Form-Body.cases [OF assms(2)] by (metis (no-types))
  have [simp]:  $\text{length } c = \text{length } c'$   $\text{length } d = \text{length } d'$ 
    using  $c\ c'\ d\ d'\ len'\ len$  by auto
  note acc-lengths.simps [simp del]
  have  $a < b$ 
    using  $ss-zs$  by (auto simp: Ueq strict-sorted-append-iff less-list-def c d)
  have  $a' < b'$ 
    using  $ss-zs'$  by (auto simp: Ueq' strict-sorted-append-iff less-list-def c' d')
  have  $a\#\ as = a'\#\ as' \wedge b\#\ bs = b'\#\ bs'$ 
  proof (rule interaction-scheme-unique-aux)
    show strict-sorted (interact ( $a\ \#\ as$ ) ( $b\ \#\ bs$ ))
      using  $ss-zs\ \langle a < b \rangle$  by (auto simp: Ueq strict-sorted-append-iff less-list-def d)
    show strict-sorted (interact ( $a'\ \#\ as'$ ) ( $b'\ \#\ bs'$ ))
      using  $ss-zs'\ \langle a' < b' \rangle$  by (auto simp: Ueq' strict-sorted-append-iff less-list-def
d')
    show  $\text{length } (b\ \#\ bs) \leq \text{length } (a\ \#\ as)$   $\text{length } (b'\ \#\ bs') \leq \text{length } (a'\ \#\ as')$ 
      using  $\langle kb \leq ka \rangle\ len\ len'$  by auto
    show  $\text{length } (a\ \#\ as) \leq \text{Suc } (\text{length } (b\ \#\ bs))$ 
      using  $\langle ka \leq \text{Suc } kb \rangle\ len$  by linarith
    then show  $\text{length } (a'\ \#\ as') \leq \text{Suc } (\text{length } (b'\ \#\ bs'))$ 
      using  $len\ len'$  by fastforce
  qed (use len len' xs xs' ys ys' ne ne' in fastforce)
  then show ?thesis
    using  $Ueq\ Ueq'\ c\ c'\ d\ d'$  by blast
qed

```

lemma *Form-Body-imp-inter-scheme*:

assumes *FB: Form-Body* $ka\ kb\ xs\ ys\ zs$ **and** $0 < kb\ kb \leq ka\ ka \leq \text{Suc } kb$

```

shows  $zs = \text{inter-scheme } ((ka+kb) - \text{Suc } 0) \{xs,ys\}$ 
proof -
  have  $\text{length } xs < \text{length } ys$ 
    by (meson Form-Body-length assms(1))
  have [simp]:  $a + a = b + b \longleftrightarrow a=b \quad a + a - \text{Suc } 0 = b + b - \text{Suc } 0 \longleftrightarrow$ 
 $a=b$  for  $a \ b::\text{nat}$ 
    by auto
  show ?thesis
  proof (cases ka = kb)
    case True
      show ?thesis
        unfolding inter-scheme-def
        apply (rule some-equality [symmetric],metis One-nat-def True FB mult-2)
        using assms  $\langle \text{length } xs < \text{length } ys \rangle$ 
        by (auto simp: True mult-2 Set.doubleton-eq-iff Form-Body-unique dest:
Form-Body-length, presburger)
      next
        case False
        then have  $eq: ka = \text{Suc } kb$ 
          using assms by linarith
        show ?thesis
          unfolding inter-scheme-def
          apply (rule some-equality [symmetric], use assms False mult-2 one-is-add eq)
in fastforce)
    using assms  $\langle \text{length } xs < \text{length } ys \rangle$ 
    by (auto simp: eq mult-2 Set.doubleton-eq-iff Form-Body-unique dest: Form-Body-length,
presburger)
  qed
qed

```

3.7 For Lemma 3.8 AND PROBABLY 3.7

definition $\text{grab} :: \text{nat set} \Rightarrow \text{nat} \Rightarrow \text{nat set} \times \text{nat set}$
where $\text{grab } N \ n \equiv (N \cap \text{enumerate } N \text{ ' } \{..<n\}, N \cap \{\text{enumerate } N \ n..\})$

lemma *grab-0* [*simp*]: $\text{grab } N \ 0 = (\{\}, N)$
by (*fastforce simp: grab-def enumerate-0 Least-le*)

lemma *less-sets-grab*:
 $\text{infinite } N \Longrightarrow \text{fst } (\text{grab } N \ n) \ll \text{snd } (\text{grab } N \ n)$
by (*auto simp: grab-def less-sets-def intro: enumerate-mono less-le-trans*)

lemma *finite-grab* [*iff*]: $\text{finite } (\text{fst } (\text{grab } N \ n))$
by (*simp add: grab-def*)

lemma *card-grab* [*simp*]:
assumes $\text{infinite } N$ **shows** $\text{card } (\text{fst } (\text{grab } N \ n)) = n$
proof -
have $N \cap \text{enumerate } N \text{ ' } \{..<n\} = \text{enumerate } N \text{ ' } \{..<n\}$

using *assms* **by** (*auto simp: enumerate-in-set*)
with *assms* **show** *?thesis*
by (*simp add: card-image grab-def strict-mono-enum strict-mono-imp-inj-on*)
qed

lemma *fst-grab-subset*: $\text{fst } (\text{grab } N \ n) \subseteq N$
using *grab-def range-enum* **by** *fastforce*

lemma *snd-grab-subset*: $\text{snd } (\text{grab } N \ n) \subseteq N$
by (*auto simp: grab-def*)

lemma *grab-Un-eq*:
assumes *infinite N* **shows** $\text{fst } (\text{grab } N \ n) \cup \text{snd } (\text{grab } N \ n) = N$
proof
show $N \subseteq \text{fst } (\text{grab } N \ n) \cup \text{snd } (\text{grab } N \ n)$
unfolding *grab-def*
using *assms enumerate-Ex le-less-linear strict-mono-enum strict-mono-less* **by**
fastforce
qed (*simp add: grab-def*)

lemma *finite-grab-iff* [*simp*]: $\text{finite } (\text{snd } (\text{grab } N \ n)) \longleftrightarrow \text{finite } N$
by (*metis finite-grab grab-Un-eq infinite-Un infinite-super snd-grab-subset*)

lemma *grab-eqD*:
 $\llbracket \text{grab } N \ n = (A, M); \text{infinite } N \rrbracket$
 $\implies A \ll M \wedge \text{finite } A \wedge \text{card } A = n \wedge \text{infinite } M \wedge A \subseteq N \wedge M \subseteq N$
using *card-grab grab-def less-sets-grab finite-grab-iff* **by** *auto*

lemma *less-sets-fst-grab*: $A \ll N \implies A \ll \text{fst } (\text{grab } N \ n)$
by (*simp add: fst-grab-subset less-sets-weaken2*)

Possibly redundant, given *grab*

definition *nxt* **where** $\text{nxt} \equiv \lambda N. \lambda n::\text{nat}. N \cap \{n<..\}$

lemma *infinite-nxtN*: $\text{infinite } N \implies \text{infinite } (\text{nxt } N \ n)$
by (*simp add: infinite-nat-greaterThan nxt-def*)

lemma *nxt-subset*: $\text{nxt } N \ n \subseteq N$
unfolding *nxt-def* **by** *blast*

lemma *nxt-subset-greaterThan*: $m \leq n \implies \text{nxt } N \ n \subseteq \{m<..\}$
by (*auto simp: nxt-def*)

lemma *nxt-subset-atLeast*: $m \leq n \implies \text{nxt } N \ n \subseteq \{m..\}$
by (*auto simp: nxt-def*)

lemma *enum-nxt-ge*: $\text{infinite } N \implies a \leq \text{enum } (\text{nxt } N \ a) \ n$
by (*simp add: atLeast-le-enum infinite-nxtN nxt-subset-atLeast*)

lemma *inj-enum-nxt*: $\text{infinite } N \implies \text{inj-on } (\text{enum } (\text{nxt } N \ a)) \ A$
by (*simp add: infinite-nxtN strict-mono-enum strict-mono-imp-inj-on*)

3.8 Larson's Lemma 3.11

Again from Jean A. Larson, A short proof of a partition theorem for the ordinal ω^ω . *Annals of Mathematical Logic*, 6:129–145, 1973.

lemma *lemma-3-11*:

assumes $l > 0$

shows *thin* (*inter-scheme* $l \ \{U. \text{Form } l \ U\}$)

using *form-cases* [of l]

proof *cases*

case *zero*

then show *?thesis*

using *assms* **by** *auto*

next

case (*nz ka kb*)

note *acc-lengths.simps* [*simp del*]

show *?thesis*

unfolding *thin-def*

proof *clarify*

fix $U \ U'$

assume *ne*: *inter-scheme* $l \ U \neq \text{inter-scheme } l \ U'$ **and** *init*: *initial-segment* (*inter-scheme* $l \ U$) (*inter-scheme* $l \ U'$)

assume *Form* $l \ U$

then obtain $kp \ kq \ xs \ ys$ **where** $l = kp+kq - 1 \ U = \{xs, ys\}$

and U : *Form-Body* $kp \ kq \ xs \ ys$ (*inter-scheme* $l \ U$) **and** $0 < kq \ kq \leq kp$
 $kp \leq \text{Suc } kq$

using *assms* *inter-scheme* **by** *blast*

then have $kp = ka \wedge kq = kb$

using *nz* **by** *linarith*

then obtain $a \ as \ b \ bs \ c \ d$

where *len*: $\text{length } (a\#as) = ka \ \text{length } (b\#bs) = kb$

and c : $c = \text{acc-lengths } 0 \ (a\#as)$

and d : $d = \text{acc-lengths } 0 \ (b\#bs)$

and Ueq : *inter-scheme* $l \ U = \text{concat } [c, a, d, b] \ @ \ \text{interact } as \ bs$

using U **by** (*auto simp: Form-Body.simps*)

assume *Form* $l \ U'$

then obtain $kp' \ kq' \ xs' \ ys'$ **where** $l = kp'+kq' - 1 \ U' = \{xs', ys'\}$

and U' : *Form-Body* $kp' \ kq' \ xs' \ ys'$ (*inter-scheme* $l \ U'$) **and** $0 < kq' \ kq' \leq kp' \ kp' \leq \text{Suc } kq'$

using *assms* *inter-scheme* **by** *blast*

then have $kp' = ka \wedge kq' = kb$

using *nz* **by** *linarith*

then obtain $a' \ as' \ b' \ bs' \ c' \ d'$

where *len'*: $\text{length } (a'\#as') = ka \ \text{length } (b'\#bs') = kb$

and c' : $c' = \text{acc-lengths } 0 \ (a'\#as')$

and d' : $d' = \text{acc-lengths } 0 \ (b'\#bs')$

and Ueq' : *inter-scheme* $l \ U' = \text{concat } [c', a', d', b'] \ @ \ \text{interact } as' \ bs'$

```

    using U' by (auto simp: Form-Body.simps)
  have [simp]: length bs' = length bs length as' = length as
    using len len' by auto
  have inter-scheme l U ≠ [] inter-scheme l U' ≠ []
    using Form-Body-nonempty U U' by auto
  define u1 where u1 ≡ hd (inter-scheme l U)
  have u1-eq': u1 = hd (inter-scheme l U')
    using ⟨inter-scheme l U ≠ []⟩ init u1-def initial-segment-ne by fastforce
  have au1: u1 = length a
    by (simp add: u1-def Ueq c)
  have au1': u1 = length a'
    by (simp add: u1-eq' Ueq' c')
  have len-eqk: length c' = ka length d' = kb length c' = ka length d' = kb
    using c d len c' d' len' by auto
  have take: take (ka + u1 + kb) (c @ a @ d @ l) = c @ a @ d
    take (ka + u1 + kb) (c' @ a' @ d' @ l) = c' @ a' @ d' for l
    using c d c' d' len by (simp-all flip: au1 au1')
  have leU: ka + u1 + kb ≤ length (inter-scheme l U)
    using c d len by (simp add: au1 Ueq)
  then have take (ka + u1 + kb) (inter-scheme l U) = take (ka + u1 + kb)
(inter-scheme l U')
    using take-initial-segment init by blast
  then have §: c @ a @ d = c' @ a' @ d'
    by (metis Ueq Ueq' append.assoc concat.simps(2) take)
  have length (inter-scheme l U) = ka + (c @ a @ d)!(ka-1) + kb + last d
    by (simp add: Ueq c d length-interact nth-append flip: len)
  moreover have length (inter-scheme l U') = ka + (c' @ a' @ d')!(ka-1) +
kb + last d'
    by (simp add: Ueq' c' d' length-interact nth-append flip: len')
  moreover have last d = last d'
    using § c d d' len'(1) len-eqk(1) by auto
  ultimately have length (inter-scheme l U) = length (inter-scheme l U')
    by (simp add: §)
  then show False
    using init initial-segment-length-eq ne by blast
qed
qed

```

3.9 Larson's Lemma 3.6

proposition lemma-3-6:

fixes $g :: \text{nat list set} \Rightarrow \text{nat}$

assumes $g: g \in [WW]^2 \rightarrow \{0,1\}$

obtains $N j$ **where** *infinite* N

and $\bigwedge k u. \llbracket k > 0; u \in [WW]^2; \text{Form } k u; [\text{enum } N k] < \text{inter-scheme } k u; \text{List.set } (\text{inter-scheme } k u) \subseteq N \rrbracket \Longrightarrow g u = j k$

proof –

define Φ **where** $\Phi \equiv \lambda m::\text{nat}. \lambda M. \text{infinite } M \wedge m < \text{Inf } M$

define Ψ **where** $\Psi \equiv \lambda l m n::\text{nat}. \lambda M N j. n > m \wedge N \subseteq M \wedge n \in M$

$\wedge (\forall U. \text{Form } l \ U \wedge U \subseteq WW \wedge [n] < \text{inter-scheme } l \ U \wedge \text{list.set } (\text{inter-scheme } l \ U) \subseteq N \longrightarrow g \ U = j)$
have *: $\exists n \ N \ j. \Phi \ n \ N \wedge \Psi \ l \ m \ n \ M \ N \ j$ **if** $l > 0$ $\Phi \ m \ M$ **for** $l \ m :: \text{nat}$ **and** $M :: \text{nat set}$
proof –
define FF **where** $FF \equiv \{U \in [WW]^2. \text{Form } l \ U\}$
define h **where** $h \equiv \lambda z s. g \ (\text{inv-into } FF \ (\text{inter-scheme } l) \ z s)$
have $\text{thin} \ (\text{inter-scheme } l \ ' \ FF)$
using $\langle l > 0 \rangle$ lemma-3-11 **by** $(\text{simp add: thin-def } FF\text{-def})$
moreover
have $\text{inter-scheme } l \ ' \ FF \subseteq WW$
using $\text{inter-scheme-simple } \langle 0 < l \rangle$ $FF\text{-def}$ **by** blast
moreover
have $h \ ' \ \{xs \in \text{inter-scheme } l \ ' \ FF. \text{List.set } xs \subseteq M\} \subseteq \{.. < 2\}$
using $g \ \text{inv-into-into}[\text{of concl: } FF \ \text{inter-scheme } l]$
by $(\text{force simp: } h\text{-def } FF\text{-def } Pi\text{-iff})$
ultimately
obtain $j \ N$ **where** $j < 2$ $\text{infinite } N \ N \subseteq M$ **and** $h j: h \ ' \ \{xs \in \text{inter-scheme } l \ ' \ FF. \text{List.set } xs \subseteq N\} \subseteq \{j\}$
using $\langle \Phi \ m \ M \rangle$ **unfolding** $\Phi\text{-def}$ **by** $(\text{blast intro: Nash-Williams-WW } [\text{of } M])$
let $?n = \text{Inf } N$
have $?n > m$
using $\langle \Phi \ m \ M \rangle$ $\langle \text{infinite } N \rangle$ **unfolding** $\Phi\text{-def}$ Inf-nat-def $\text{infinite-nat-iff-unbounded}$
by $(\text{metis LeastI-ex } \langle N \subseteq M \rangle \ \text{le-less-trans not-less not-less-Least subsetD})$
have $g \ U = j$ **if** $\text{Form } l \ U \ U \subseteq WW \ [?n] < \text{inter-scheme } l \ U \ \text{list.set } (\text{inter-scheme } l \ U) \subseteq N - \{?n\}$ **for** U
proof –
obtain $xs \ ys$ **where** $xy s: xs \neq ys \ U = \{xs, ys\}$
using $\text{Form-elim-upair } \langle \text{Form } l \ U \rangle$ **by** blast
moreover **have** $\text{inj-on} \ (\text{inter-scheme } l) \ FF$
using $\langle 0 < l \rangle$ inj-on-def $\text{inter-scheme-injective } FF\text{-def}$ **by** blast
moreover
have $g \ (\text{inv-into } FF \ (\text{inter-scheme } l) \ (\text{inter-scheme } l \ U)) = j$
using $h j$ **that** $xy s \ \text{subset-Diff-insert}$ **by** $(\text{fastforce simp: } h\text{-def } FF\text{-def } \text{image-iff})$
ultimately show $?thesis$
using $\text{that } FF\text{-def}$ **by** auto
qed
moreover **have** $?n < \text{Inf} \ (N - \{?n\})$
by $(\text{metis Diff-iff Inf-nat-def Inf-nat-def1 } \langle \text{infinite } N \rangle \ \text{finite.emptyI infinite-remove linorder-neqE-nat not-less-Least singletonI})$
moreover **have** $?n \in M$
by $(\text{metis Inf-nat-def1 } \langle N \subseteq M \rangle \langle \text{infinite } N \rangle \ \text{finite.emptyI subsetD})$
ultimately **have** $\Phi \ ?n \ (N - \{?n\}) \wedge \Psi \ l \ m \ ?n \ M \ (N - \{?n\}) \ j$
using $\langle \Phi \ m \ M \rangle$ $\langle \text{infinite } N \rangle$ $\langle N \subseteq M \rangle$ $\langle ?n > m \rangle$ **by** $(\text{auto simp: } \Phi\text{-def } \Psi\text{-def})$
then show $?thesis$
by blast
qed
have $\text{base: } \Phi \ 0 \ \{0 < ..\}$

unfolding Φ -def **by** (*metis infinite-Ioi Inf-nat-def1 greaterThan-iff greaterThan-non-empty*)
have *step*: $Ex (\lambda(n,N,j). \Phi n N \wedge \Psi l m n M N j)$ **if** $\Phi m M l > 0$ **for** $m M l$
using * [of $l m M$] **that by** (*auto simp: Φ -def*)
define G **where** $G \equiv \lambda l m M. @ (n,N,j). \Phi n N \wedge \Psi (Suc l) m n M N j$
have $G\Phi$: $(\lambda(n,N,j). \Phi n N) (G l m M)$ **and** $G\Psi$: $(\lambda(n,N,j). \Psi (Suc l) m n M N j) (G l m M)$
if $\Phi m M$ **for** $l m M$
using *step* [OF *that, of Suc l*] **by** (*force simp: G-def dest: some-eq-imp*)
have *G-increasing*: $(\lambda(n,N,j). n > m \wedge N \subseteq M \wedge n \in M) (G l m M)$ **if** $\Phi m M$ **for** $l m M$
using $G\Psi$ [OF *that, of l*] **that by** (*simp add: Ψ -def split: prod.split-asm*)
define H **where** $H \equiv rec\text{-nat } (0, \{0<..\}, 0) (\lambda l (m,M,j). G l m M)$
have *H-simps*: $H 0 = (0, \{0<..\}, 0) \wedge l. H (Suc l) = (case H l of (m,M,j) \Rightarrow G l m M)$
by (*simp-all add: H-def*)
have $H\Phi$: $(\lambda(n,N,j). \Phi n N) (H l)$ **for** l
by (*induction l*) (*use base G Φ in \langle auto simp: H-simps split: prod.split-asm \rangle*)
define ν **where** $\nu \equiv (\lambda l. case H l of (n,M,j) \Rightarrow n)$
have *H-inc*: $\nu l \geq l$ **for** l
proof (*induction l*)
case (*Suc l*)
then show ?*case*
using $H\Phi$ [of l] *G-increasing* [of νl]
apply (*clarsimp simp: H-simps ν -def split: prod.split*)
by (*metis (no-types, lifting) case-prodD leD le-less-trans not-less-eq-eq*)
qed auto
let ? $N = range \nu$
define j **where** $j \equiv \lambda l. case H l of (n,M,j) \Rightarrow j$
have *H-increasing-Suc*: $(case H k of (n, N, j') \Rightarrow N) \supseteq (case H (Suc k) of (n, N, j') \Rightarrow insert n N)$ **for** k
using $H\Phi$ [of k]
by (*force simp: H-simps split: prod.split dest: G-increasing [where $l=k$]*)
have *H-increasing-superset*: $(case H k of (n, N, j') \Rightarrow N) \supseteq (case H (n+k) of (n, N, j') \Rightarrow N)$ **for** $k n$
proof (*induction n*)
case (*Suc n*)
then show ?*case*
using *H-increasing-Suc* [of $n+k$] **by** (*auto split: prod.split-asm*)
qed auto
then have *H-increasing-less*: $(case H k of (n, N, j') \Rightarrow N) \supseteq (case H l of (n, N, j') \Rightarrow insert n N)$
if $k < l$ **for** $k l$
by (*smt (verit, best) H-increasing-Suc add.commute less-natE order-trans that*)
have $\nu k < \nu (Suc k)$ **for** k
using $H\Phi$ [of k] **unfolding** ν -def
by (*auto simp: H-simps split: prod.split dest: G-increasing [where $l=k$]*)
then have *strict-mono- ν* : *strict-mono ν*
by (*simp add: strict-mono-Suc-iff*)
then have *enum-N*: *enum ? $N = \nu$*

```

  by (metis enum-works nat-infinite-iff range-strict-mono-ext)
have **:  $?N \cap \{n < ..\} \subseteq N'$  if  $H: H k = (n, N', j)$  for  $n N' k j$ 
proof clarify
  fix l
  assume  $n < \nu l$ 
  then have  $False$  if  $l \leq k$ 
    using that strict-monoD [OF strict-mono- $\nu$ , of l k] H by (force simp:  $\nu$ -def)
  then have  $k < l$  using not-less by blast
  then obtain M j where Mj:  $H l = (\nu l, M, j)$ 
    unfolding  $\nu$ -def
    by (metis (mono-tags, lifting) case-prod-conv old.prod.exhaust)
  then show  $\nu l \in N'$ 
    using that H-increasing-less [OF  $\langle k < l \rangle$ ] Mj by auto
qed
show thesis
proof
  show infinite ( $?N::nat$  set)
    using H-inc infinite-nat-iff-unbounded-le by auto
next
fix l U
assume  $0 < l$  and  $U: U \in [WW]^2$ 
  and interU:  $[enum ?N l] < inter\ scheme\ l\ U\ Form\ l\ U$ 
  and sub:  $list.set (inter\ scheme\ l\ U) \subseteq ?N$ 
obtain k where k:  $l = Suc\ k$ 
  using  $\langle 0 < l \rangle$  gr0-conv-Suc by blast
have  $g\ U = v$  if  $H\ k = (m, M, j0)$  and  $G\ k\ m\ M = (n, N', v)$ 
  for  $m\ M\ j0\ n\ N'\ v$ 
proof -
  have  $n: \nu (Suc\ k) = n$ 
    using that by (simp add:  $\nu$ -def H-simps)
  have  $\{..enum (range\ \nu)\ l\} \cap list.set (inter\ scheme\ l\ U) = \{\}$ 
  using inter-scheme-strict-sorted  $\langle 0 < l \rangle$  interU singleton-less-list-iff strict-sorted-iff
by blast
  then have  $list.set (inter\ scheme (Suc\ k)\ U) \subseteq N'$ 
    using that sub ** [of Suc k n N' v] Suc-le-eq not-less-eq-eq
    by (fastforce simp: k n enum-N H-simps)
  then show ?thesis
    using that interU U G $\Psi$  [of m M k] H $\Phi$  [of k]
    by (auto simp:  $\Psi$ -def k enum-N H-simps n nsets-def)
qed
with U show  $g\ U = j\ l$ 
  by (auto simp: k j-def H-simps split: prod.split)
qed
qed

```

3.10 Larson's Lemma 3.7

3.10.1 Preliminaries

Analogous to *ordered-nsets-2-eq*, but without type classes

lemma *total-order-nsets-2-eq*:
assumes *tot*: *total-on A r* **and** *irr*: *irrefl r*
shows $nsets\ A\ 2 = \{\{x,y\} \mid x\ y.\ x \in A \wedge y \in A \wedge (x,y) \in r\}$
(is - = ?rhs)
proof
show $nsets\ A\ 2 \subseteq ?rhs$
using *tot*
unfolding *numeral-nat total-on-def nsets-def*
by (*fastforce simp: card-Suc-eq Set.doubleton-eq-iff not-less*)
show $?rhs \subseteq nsets\ A\ 2$
using *irr unfolding numeral-nat* **by** (*force simp: nsets-def card-Suc-eq irrefl-def*)
qed

lemma *lenlex-nsets-2-eq*: $nsets\ A\ 2 = \{\{x,y\} \mid x\ y.\ x \in A \wedge y \in A \wedge (x,y) \in lenlex\ less-than\}$
using *total-order-nsets-2-eq* **by** (*simp add: total-order-nsets-2-eq irrefl-def*)

lemma *sum-sorted-list-of-set-map*: $finite\ I \implies sum-list\ (map\ f\ (list-of\ I)) = sum\ f\ I$
proof (*induction card I arbitrary: I*)
case (*Suc n I*)
then have [*simp*]: $I \neq \{\}$
by *auto*
have $sum-list\ (map\ f\ (list-of\ (I - \{Min\ I\}))) = sum\ f\ (I - \{Min\ I\})$
using *Suc* **by** *auto*
then show *?case*
using *Suc.prem1 sum.remove [of I Min I f]*
by (*simp add: sorted-list-of-set-nonempty Suc*)
qed *auto*

lemma *sorted-list-of-set-UN-eq-concat*:
assumes *I*: *strict-mono-sets I f* *finite I* **and** *fin*: $\bigwedge i.\ finite\ (f\ i)$
shows $list-of\ (\bigcup i \in I.\ f\ i) = concat\ (map\ (list-of\ \circ\ f)\ (list-of\ I))$
using *I*
proof (*induction card I arbitrary: I*)
case (*Suc n I*)
then have $I \neq \{\}$ **and** *Iexp*: $I = insert\ (Min\ I)\ (I - \{Min\ I\})$
using *Min-in Suc.hyps(2) Suc.prem1(2)* **by** *fastforce+*
have *IH*: $list-of\ (\bigcup (f\ ' (I - \{Min\ I\}))) = concat\ (map\ (list-of\ \circ\ f)\ (list-of\ (I - \{Min\ I\})))$
using *Suc unfolding strict-mono-sets-def*
by (*metis DiffE Iexp card-Diff-singleton diff-Suc-1 finite-Diff insertI1*)
have $list-of\ (\bigcup (f\ ' I)) = list-of\ (\bigcup (f\ ' (insert\ (Min\ I)\ (I - \{Min\ I\}))))$
using *Iexp* **by** *auto*
also have $\dots = list-of\ (f\ (Min\ I) \cup \bigcup (f\ ' (I - \{Min\ I\})))$
by (*metis Union-image-insert*)
also have $\dots = list-of\ (f\ (Min\ I)) @ list-of\ (\bigcup (f\ ' (I - \{Min\ I\})))$

```

proof (rule sorted-list-of-set-Un)
  show  $f (Min I) \ll \bigcup (f \text{ ` } (I - \{Min I\}))$ 
    using Suc.prems  $\langle I \neq \{\} \rangle$  strict-mono-less-sets-Min by blast
  show finite  $(\bigcup (f \text{ ` } (I - \{Min I\})))$ 
    by (simp add:  $\langle \text{finite } I \rangle$  fin)
  qed (use fin in auto)
  also have  $\dots = \text{list-of } (f (Min I)) @ \text{concat } (\text{map } (\text{list-of } \circ f) (\text{list-of } (I - \{Min I\})))$ 
    using IH by metis
  also have  $\dots = \text{concat } (\text{map } (\text{list-of } \circ f) (\text{list-of } I))$ 
    by (simp add: Suc.prems(2)  $\langle I \neq \{\} \rangle$  sorted-list-of-set-nonempty)
  finally show ?case .
qed auto

```

3.10.2 Lemma 3.7 of Jean A. Larson, *ibid*.

proposition *lemma-3-7*:

assumes *infinite* N $l > 0$

obtains M **where** $M \in [WW]^m$

$$\bigwedge U. U \in [M]^2 \implies \text{Form } l U \wedge \text{List.set } (\text{inter-scheme } l U) \subseteq N$$

proof (*cases* $m < 2$)

case *True*

obtain w **where** $w \in WW$

using *WW-def strict-sorted-into-WW* **by** *auto*

define M **where** $M \equiv \text{if } m=0 \text{ then } \{\} \text{ else } \{w\}$

have $M: M \in [WW]^m$

using *True* **by** (*auto simp*: *M-def nsets-def* w)

have [*simp*]: $[M]^2 = \{\}$

using *True* **by** (*auto simp*: *M-def nsets-def* w *dest*: *subset-singletonD*)

show ?*thesis*

using M **that** **by** *fastforce*

next

case *False*

then have $m \geq 2$

by *auto*

have *nonz*: $(\text{enum } N \circ \text{Suc}) i > 0$ **for** i

using *assms*(1) *le-enumerate less-le-trans* **by** *fastforce*

note *infinite-nxt-N* = *infinite-nxtN* [*OF* $\langle \text{infinite } N \rangle$, *iff*]

note $\langle \text{infinite } N \rangle$ [*iff*]

have [*simp*]: $\{n <..< \text{Suc } n\} = \{\}$ $\{..< 1::\text{nat}\} = \{0\}$ **for** n

by *auto*

note *One-nat-def* [*simp del*]

define *DF-Suc* **where** $DF\text{-}Suc \equiv \lambda k D. \text{enum } (\text{nxt } N (\text{enum } (\text{nxt } N (\text{Max } D)) (\text{Inf } D - 1))) \text{ ` } \{..< \text{Suc } k\}$

define *DF* **where** $DF \equiv \lambda k n. (DF\text{-}Suc k \rightsquigarrow n) ((\text{enum } N \circ \text{Suc}) \text{ ` } \{..< \text{Suc } k\})$

have *DF-simps*: $DF k 0 = (\text{enum } N \circ \text{Suc}) \text{ ` } \{..< \text{Suc } k\}$ $DF k (\text{Suc } i) = DF\text{-}Suc k (DF k i)$ **for** $i k$

by (*auto simp*: *DF-def*)

```

have card-DF:  $\text{card } (DF\ k\ i) = \text{Suc } k$  for  $i\ k$ 
proof (induction i)
  case 0
  have inj-on ( $\text{enum } N \circ \text{Suc}$ )  $\{..<\text{Suc } k\}$ 
    by (simp add: assms(1) strict-mono-def strict-mono-imp-inj-on)
  with 0 show ?case
    using card-image DF-simps by fastforce
next
  case ( $\text{Suc } i$ )
  then show ?case
    by (simp add: <infinite N> DF-simps DF-Suc-def card-image infinite-nxtN
strict-mono-enum strict-mono-imp-inj-on)
qed
have DF-ne:  $DF\ k\ i \neq \{\}$  for  $i\ k$ 
  by (metis card-DF card-lessThan lessThan-empty-iff nat.simps(3))

have finite-DF:  $\text{finite } (DF\ k\ i)$  for  $i\ k$ 
  by (induction i) (auto simp: DF-simps DF-Suc-def)
have DF-Suc:  $DF\ k\ i \ll DF\ k\ (\text{Suc } i)$  for  $i\ k$ 
  unfolding less-sets-def
  by (force simp: finite-DF DF-simps DF-Suc-def
    intro!: greaterThan-less-enum nxt-subset-greaterThan atLeast-le-enum nxt-subset-atLeast
infinite-nxtN [OF <infinite N>])
  have DF-DF:  $DF\ k\ i \ll DF\ k\ j$  if  $i < j$  for  $i\ j\ k$ 
  by (meson DF-Suc DF-ne UNIV-I less-sets-imp-strict-mono-sets strict-mono-setsD
that)
  then have sm-DF: strict-mono-sets UNIV ( $DF\ k$ ) for  $k$ 
    by (simp add: strict-mono-sets-def)

have DF-gt0:  $0 < \text{Inf } (DF\ k\ i)$  for  $i\ k$ 
proof (cases i)
  case 0
  then show ?thesis
    by (metis DF-ne DF-simps(1) Inf-nat-def1 imageE nonz)
next
  case ( $\text{Suc } n$ )
  then show ?thesis
    by (metis DF-Suc DF-ne Inf-nat-def1 grOI gr-implies-not0 less-sets-def)
qed
have DF-N:  $DF\ k\ i \subseteq N$  for  $i\ k$ 
proof (induction i)
  case 0
  then show ?case
    using  $\langle \text{infinite } N \rangle$  range-enum by (auto simp: DF-simps)
next
  case ( $\text{Suc } i$ )
  then show ?case
    unfolding DF-simps DF-Suc-def image-subset-iff

```


by (metis IntE <infinite N> enumerate-in-set infinite-nxtN nxt-def)
 qed

have sm-enum-DF: strict-mono-on {..k} (enum (DF k i)) for k i
 by (metis card-DF enum-works-finite finite-DF lessThan-Suc-atMost)

define AF where AF $\equiv \lambda k i. \text{enum } (nxt N (Max (DF k i))) \text{ ' } \{..<Inf (DF k i)\}$
 have AF-ne: AF k i $\neq \{\}$ for i k
 by (auto simp: AF-def lessThan-empty-iff DF-gt0)
 have finite-AF [simp]: finite (AF k i) for i k
 by (simp add: AF-def)
 have card-AF: card (AF k i) = \prod (DF k i) for k i
 by (simp add: AF-def card-image inj-enum-nxt)

have DF-AF: DF k i \ll AF k i for i k
 unfolding less-sets-def AF-def
 by (simp add: finite-DF greaterThan-less-enum nxt-subset-greaterThan)

have E: $[x \leq y; \text{infinite } M] \implies \text{enum } M x < \text{enum } (nxt N (\text{enum } M y)) z$ for
 x y z M
 by (metis infinite-nxt-N dual-order.eq-iff enumerate-mono greaterThan-less-enum
 nat-less-le nxt-subset-greaterThan)

have AF-DF-Suc: AF k i \ll DF k (Suc i) for i k
 by (auto simp: DF-simps DF-Suc-def less-sets-def AF-def E)

have AF-DF: AF k p \ll DF k q if p < q for k p q
 by (metis AF-DF-Suc DF-ne Suc-lessI UNIV-I less-sets-trans sm-DF strict-mono-sets-def
 that)

have AF-Suc: AF k i \ll AF k (Suc i) for i k
 using AF-DF-Suc DF-AF DF-ne less-sets-trans by blast
 then have sm-AF: strict-mono-sets UNIV (AF k) for k
 by (simp add: AF-ne less-sets-imp-strict-mono-sets)

define del where del $\equiv \lambda k i j. \text{enum } (DF k i) j - \text{enum } (DF k i) (j - 1)$

define QF where QF k $\equiv \text{wfrec pair-less } (\lambda f (j, i).$
 if j=0 then AF k i
 else let r = (if i=0 then f (j-1, m-1) else f (j, i-1)) in
 enum (nxt N (Suc (Max r))) ' {..< del k (if j=k then m - Suc i else
 i) j})
 for k
 note cut-apply [simp]

have finite-QF [simp]: finite (QF k p) for p k
 using wf-pair-less
 proof (induction p rule: wf-induct-rule)

```

    case (less p)
  then show ?case
    by (simp add: def-wfrec [OF QF-def, of k p] split: prod.split)
qed

have del-gt-0:  $\llbracket j < \text{Suc } k; 0 < j \rrbracket \implies 0 < \text{del } k \ i \ j$  for  $i \ j \ k$ 
  by (simp add: card-DF del-def finite-DF)

have QF-ne [simp]:  $QF \ k \ (j, i) \neq \{\}$  if  $j: j < \text{Suc } k$  for  $j \ i \ k$ 
  using wf-pair-less j
proof (induction (j,i) rule: wf-induct-rule)
  case less
  then show ?case
    by (auto simp: def-wfrec [OF QF-def, of k (j,i)] AF-ne lessThan-empty-iff
del-gt-0)
qed

have QF-0 [simp]:  $QF \ k \ (0, i) = AF \ k \ i$  for  $i \ k$ 
  by (simp add: def-wfrec [OF QF-def])

have QF-Suc:  $QF \ k \ (\text{Suc } j, 0) = \text{enum } (\text{nxt } N \ (\text{Suc } (\text{Max } (QF \ k \ (j, m - 1))))$ 
  ‘  $\{.. < \text{del } k \ (\text{if } \text{Suc } j = k \ \text{then } m - 1 \ \text{else } 0) \ (\text{Suc } j)\}$  for  $j \ k$ 
  apply (simp add: def-wfrec [OF QF-def, of k (Suc j, 0)] One-nat-def)
  apply (simp add: pair-less-def cut-def)
  done

have QF-Suc-Suc:  $QF \ k \ (\text{Suc } j, \text{Suc } i)$ 
  =  $\text{enum } (\text{nxt } N \ (\text{Suc } (\text{Max } (QF \ k \ (\text{Suc } j, i))))$  ‘  $\{.. < \text{del } k \ (\text{if } \text{Suc}$ 
 $j = k \ \text{then } m - \text{Suc}(\text{Suc } i) \ \text{else } \text{Suc } i) \ (\text{Suc } j)\}$ 
  for  $i \ j \ k$ 
  by (simp add: def-wfrec [OF QF-def, of k (Suc j, Suc i)])

have less-QF1:  $QF \ k \ (j, m - 1) \ll QF \ k \ (\text{Suc } j, 0)$  for  $j \ k$ 
  by (auto simp: def-wfrec [OF QF-def, of k (Suc j, 0)] pair-lessI1 enum-nxt-ge
  intro!: less-sets-weaken2 [OF less-sets-Suc-Max])

have less-QF2:  $QF \ k \ (j, i) \ll QF \ k \ (j, \text{Suc } i)$  for  $j \ i \ k$ 
  by (auto simp: def-wfrec [OF QF-def, of k (j, Suc i)] pair-lessI2 enum-nxt-ge
  intro: less-sets-weaken2 [OF less-sets-Suc-Max] strict-mono-setsD [OF
sm-AF])

have less-QF-same:  $QF \ k \ (j, i') \ll QF \ k \ (j, i)$ 
  if  $i' < i \leq k$  for  $i' \ i \ j \ k$ 
proof (rule strict-mono-setsD [OF less-sets-imp-strict-mono-sets])
  show  $QF \ k \ (j, i) \ll QF \ k \ (j, \text{Suc } i)$  for  $i$ 
    by (simp add: less-QF2)
  show  $QF \ k \ (j, i) \neq \{\}$  if  $0 < i$  for  $i$ 
    using that by (simp add:  $\langle j \leq k \rangle \text{le-imp-less-Suc}$ )

```

qed (use that in auto)

have less-QF-step: $QF\ k\ (j-1, i') \ll QF\ k\ (j, i)$
 if $0 < j\ j \leq k\ i' < m$ for $j\ i'\ i\ k$

proof -
 have less-QF1': $QF\ k\ (j-1, m-1) \ll QF\ k\ (j, 0)$ if $j > 0$ for j
 by (metis less-QF1 that Suc-pred One-nat-def)
 have $QF\ k\ (j-1, i') \ll QF\ k\ (j, 0)$
 proof (cases $i' = m - 1$)
 case True
 then show ?thesis
 using less-QF1' $\langle 0 < j \rangle$ by blast
 next
 case False
 show ?thesis
 using False that less-sets-trans [OF less-QF-same less-QF1' QF-ne] by auto
 qed
 then show ?thesis
 by (metis QF-ne less-QF-same less-Suc-eq-le less-sets-trans $\langle j \leq k \rangle$ zero-less-iff-neq-zero)
 qed

have less-QF: $QF\ k\ (j', i') \ll QF\ k\ (j, i)$
 if $j: j' < j\ j \leq k$ and $i: i' < m\ i < m$ for $j'\ j\ i'\ i\ k$
 using j

proof (induction $j-j'$ arbitrary: j)
 case (Suc d)
 show ?case
 proof (cases $j' < j - 1$)
 case True
 then have $QF\ k\ (j', i') \ll QF\ k\ (j - 1, i)$
 using Suc.hyps Suc.prem(2) by force
 then show ?thesis
 by (rule less-sets-trans [OF less-QF-step QF-ne]) (use Suc i in auto)
 next
 case False
 then have $j' = j - 1$
 using $\langle j' < j \rangle$ by linarith
 then show ?thesis
 using Suc.hyps $\langle j \leq k \rangle$ less-QF-step i by auto
 qed
 qed auto

have sm-QF: strict-mono-sets $(\{..k\} \times \{..<m\})$ (QF k) for k
 unfolding strict-mono-sets-def
 proof (intro strip)
 fix p q
 assume $p: p \in \{..k\} \times \{..<m\}$ and $q: q \in \{..k\} \times \{..<m\}$ and $p < q$
 then obtain $j'\ i'\ j\ i$ where $\S: p = (j', i')\ q = (j, i)\ i' < m\ i < m\ j' \leq k\ j \leq k$
 using surj-pair [of p] surj-pair [of q] by blast

with $\langle p < q \rangle$ **have** $j' < j \vee j' = j \wedge i' < i$
by *auto*
then show $QF\ k\ p \ll QF\ k\ q$
using § *less-QF less-QF-same* **by** *presburger*
qed
then have *sm-QF1: strict-mono-sets* $\{..<ka\}$ $(\lambda j. QF\ k\ (j,i))$
if $i < m$ **Suc** $k \geq ka$ $ka \geq k$ **for** $ka\ k\ i$
proof –
have $\{..<ka\} \subseteq \{..k\}$
by (*metis lessThan-Suc-atMost lessThan-subset-iff* $\langle Suc\ k \geq ka \rangle$)
then show *?thesis*
by (*simp add: less-QF strict-mono-sets-def subset-iff that*)
qed

have *disjoint-QF: $i'=i \wedge j'=j$ if \neg disjoint* $(QF\ k\ (j', i')) (QF\ k\ (j, i))$ $j' \leq k\ j \leq$
 $k\ i' < m\ i < m$ **for** $i'\ i\ j'\ j\ k$
using *that strict-mono-sets-imp-disjoint* [*OF sm-QF*]
by (*force simp: pairwise-def*)

have *card-QF: card* $(QF\ k\ (j, i)) = (if\ j=0\ then\ \square\ (DF\ k\ i)\ else\ del\ k\ (if\ j = k$
then $m - Suc\ i\ else\ i)\ j)$
for $i\ k\ j$
proof (*cases j*)
case 0
then show *?thesis*
by (*simp add: AF-def card-image inj-enum-nxt*)
next
case $(Suc\ j')$
show *?thesis*
by (*cases i; simp add: Suc One-nat-def QF-Suc QF-Suc-Suc card-image*
inj-enum-nxt)
qed
have *AF-non-Nil: list-of* $(AF\ k\ i) \neq \square$ **for** $k\ i$
by (*simp add: AF-ne*)
have *QF-non-Nil: list-of* $(QF\ k\ (j, i)) \neq \square$ **if** $j < Suc\ k$ **for** $i\ j\ k$
by (*simp add: that*)

have *AF-subset-N: AF* $k\ i \subseteq N$ **for** $i\ k$
unfolding *AF-def image-subset-iff*
using *nxt-subset enumerate-in-set infinite-nxtN* $\langle infinite\ N \rangle$ **by** *blast*

have *QF-subset-N: QF* $k\ (j, i) \subseteq N$ **for** $i\ j\ k$
proof (*induction j*)
case $(Suc\ j)$
show *?case*
by (*cases i*) (*use nxt-subset enumerate-in-set in* $\langle (force\ simp: QF-Suc\ QF-Suc-Suc) + \rangle$)
qed (*use AF-subset-N in auto*)

obtain $ka\ k$ **where** $k > 0$ **and** $kka: k \leq ka\ ka \leq Suc\ k\ l = ((ka+k) - 1)$

```

  by (metis One-nat-def assms(2) diff-add-inverse form-cases le0 le-refl)
then have ka > 0
  using dual-order.strict-trans1 by blast
have ka-k-or-Suc: ka = k  $\vee$  ka = Suc k
  using kka by linarith
have lessThan-k: {.. $k$ } = insert 0 {0<.. $k$ } if  $k > 0$  for  $k::nat$ 
  using that by auto
then have sorted-list-of-set-k: list-of {.. $k$ } = 0 # list-of {0<.. $k$ } if  $k > 0$  for
 $k::nat$ 
  using sorted-list-of-set-insert-remove-cons [of concl: 0 {0<.. $k$ }] that by simp

define RF where RF  $\equiv \lambda j i.$  if  $j = k$  then QF k (j, m - Suc i) else QF k (j, i)
have RF-subset-N: RF j i  $\subseteq$  N if  $i < m$  for i j
  using that QF-subset-N by (simp add: RF-def)
have finite-RF [simp]: finite (RF k p) for p k
  by (simp add: RF-def)
have RF-0: RF 0 i = AF k i for i
  using RF-def <0 < k> by auto
have disjoint-RF:  $i'=i \wedge j'=j$  if  $\neg$  disjnt (RF j' i') (RF j i)  $j' \leq k$   $j \leq k$   $i' < m$ 
 $i < m$  for i' i j' j
  using disjoint-QF that
  by (auto simp: RF-def split: if-split-asm dest: disjoint-QF)

have sum-card-RF [simp]: ( $\sum j \leq n.$  card (RF j i)) = enum (DF k i) n if  $n \leq k$ 
 $i < m$  for i n
  using that
proof (induction n)
case 0
then show ?case
  using DF-ne [of k i] finite-DF [of k i] <k>0>
  by (simp add: RF-def AF-def card-image inj-enum-nxt enum-0-eq-Inf-finite)
next
case (Suc n)
then have enum (DF k i) 0  $\leq$  enum (DF k i) n  $\wedge$  enum (DF k i) n  $\leq$  enum
(DF k i) (Suc n)
  using sm-enum-DF [of k i]
  by (metis card-DF finite-DF finite-enumerate-mono-iff le-imp-less-Suc less-nat-zero-code
linorder-not-le nless-le)
with Suc show ?case
  by (auto simp: RF-def card-QF del-def)
qed
have DF-in-N: enum (DF k i) j  $\in$  N if  $j \leq k$  for i j
  by (metis DF-N card-DF finite-DF finite-enumerate-in-set le-imp-less-Suc sub-
setD that)
have Inf-DF-N:  $\prod$  (DF k p)  $\in$  N for k p
  using DF-N DF-ne Inf-nat-def1 by blast
have RF-in-N: ( $\sum j \leq n.$  card (RF j i))  $\in$  N if  $n \leq k$   $i < m$  for i n
  by (auto simp: DF-in-N that)

```

have $ka - 1 \leq k$
using $kka(2)$ **by** *linarith*
then have *sum-card-RF'* [*simp*]:
 $(\sum_{j < ka}. \text{card} (RF\ j\ i)) = \text{enum} (DF\ k\ i) (ka - 1)$ **if** $i < m$ **for** i
using *sum-card-RF* [*of ka - 1 i*]
by (*metis Suc-diff-1* $\langle 0 < ka \rangle$ *lessThan-Suc-atMost* *that*)

have *enum-DF-le-iff* [*simp*]:
 $\text{enum} (DF\ k\ i)\ j \leq \text{enum} (DF\ k\ i')\ j \iff i \leq i'$ (**is** *?lhs = -*)
if $j \leq k$ **for** $i'\ i\ j\ k$
proof
show $i \leq i'$ **if** *?lhs*
proof $-$
have $\text{enum} (DF\ k\ i)\ j \in DF\ k\ i$
by (*simp add: card-DF finite-enumerate-in-set finite-DF le-imp-less-Suc* $\langle j \leq k \rangle$)
moreover have $\text{enum} (DF\ k\ i')\ j \in DF\ k\ i'$
by (*simp add: card-DF finite-enumerate-in-set finite-DF le-imp-less-Suc* *that*)
ultimately have $\text{enum} (DF\ k\ i')\ j < \text{enum} (DF\ k\ i)\ j$ **if** $i' < i$
using *sm-DF* [*of k*] **by** (*meson UNIV-I less-sets-def strict-mono-setsD* *that*)
then show *?thesis*
using *not-less* **that by** *blast*
qed
show *?lhs* **if** $i \leq i'$
by (*metis DF-DF* $\langle j \leq k \rangle$ *card-DF finite-DF finite-enumerate-in-set le-eq-less-or-eq* *less-Suc-eq-le less-sets-def* *that*)
qed

then have *enum-DF-eq-iff* [*simp*]:
 $\text{enum} (DF\ k\ i)\ j = \text{enum} (DF\ k\ i')\ j \iff i = i'$ **if** $j \leq k$ **for** $i'\ i\ j\ k$
by (*metis le-antisym order-refl* *that*)
have *enum-DF-less-iff* [*simp*]:
 $\text{enum} (DF\ k\ i)\ j < \text{enum} (DF\ k\ i')\ j \iff i < i'$ **if** $j \leq k$ **for** $i'\ i\ j\ k$
by (*meson enum-DF-le-iff not-less* *that*)

have *card-AF-sum*: $\text{card} (AF\ k\ i) + (\sum_{j \in \{0 <..<ka\}}. \text{card} (RF\ j\ i)) = \text{enum} (DF\ k\ i) (ka - 1)$
if $i < m$ **for** i
using *that* $\langle k > 0 \rangle$ $\langle k \leq ka \rangle$ $\langle ka \leq \text{Suc}\ k \rangle$
by (*simp add: lessThan-k RF-0 flip: sum-card-RF'*)

have *sorted-list-of-set-iff* [*simp*]: $\text{list-of} \{0 <..<k\} = [] \iff k = 1$ **if** $k > 0$ **for** $k::\text{nat}$
using *atLeastSucLessThan-greaterThanLessThan* *that by fastforce*
show *thesis* $-$ *proof of main result*
proof
have *inj*: $\text{inj-on} (\lambda i. \text{list-of} (\bigcup_{j < ka}. RF\ j\ i)) \{..<m\}$
proof (*clarsimp simp: inj-on-def*)

fix $x y$
assume $x < m y < m$ *list-of* $(\bigcup_{j < ka}. RF\ j\ x) = \text{list-of } (\bigcup_{j < ka}. RF\ j\ y)$
then have $eq: (\bigcup_{j < ka}. RF\ j\ x) = (\bigcup_{j < ka}. RF\ j\ y)$
by (*simp add: sorted-list-of-set-inject*)
show $x = y$
proof –
obtain n **where** $n: n \in RF\ 0\ x$
using *AF-ne QF-0* $\langle 0 < k \rangle$ *Inf-nat-def1* $\langle k \leq ka \rangle$ **by** (*force simp: RF-def*)
with $eq\ \langle ka > 0 \rangle$ **obtain** j' **where** $j' < ka\ n \in RF\ j'\ y$
by *blast*
then show *?thesis*
using *disjoint-QF* $[of\ k\ 0\ x\ j']\ n\ \langle x < m \rangle\ \langle y < m \rangle\ \langle ka \leq Suc\ k \rangle\ \langle 0 < k \rangle$
by (*force simp: RF-def disjnt-iff simp del: QF-0 split: if-split-asm*)
qed
qed

define M **where** $M \equiv (\lambda i. \text{list-of } (\bigcup_{j < ka}. RF\ j\ i))\ \text{'}\{..<m\}$
have *finite* M
unfolding *M-def* **by** *blast*
moreover have $\text{card } M = m$
by (*simp add: M-def* $\langle k \leq ka \rangle$ *card-image inj*)
moreover have $M \subseteq WW$
by (*force simp: M-def WW-def*)
ultimately show $M \in [WW]^m$
by (*simp add: nsets-def*)

have *sm-RF: strict-mono-sets* $\{..<ka\}$ $(\lambda j. RF\ j\ i)$ **if** $i < m$ **for** i
using *sm-QF1* *that kka*
by (*simp add: less-QF RF-def strict-mono-sets-def*)

have *RF-non-Nil: list-of* $(RF\ j\ i) \neq []$ **if** $j < Suc\ k$ **for** $i\ j$
using *that* **by** (*simp add: RF-def*)

have *less-RF-same: RF* $j\ i' \ll RF\ j\ i$
if $i' < i\ j < k$ **for** $i'\ i\ j$
using *that* **by** (*simp add: less-QF-same RF-def*)

have *less-RF-same-k: RF* $k\ i' \ll RF\ k\ i$ — reversed version for k
if $i < i'\ i' < m$ **for** $i'\ i$
using *that* **by** (*simp add: less-QF-same RF-def*)

show *Form* $l\ U \wedge \text{list.set } (\text{inter-scheme } l\ U) \subseteq N$ **if** $U \in [M]^2$ **for** U

proof –
from *that* **obtain** $x\ y$ **where** $U = \{x, y\}\ x \in M\ y \in M$ **and** $xy: (x, y) \in \text{lenlex}$
less-than
by (*auto simp: lenlex-nsets-2-eq*)
let $?R = \lambda p. \text{list-of } \circ (\lambda j. RF\ j\ p)$
obtain $p\ q$ **where** $x: x = \text{list-of } (\bigcup_{j < ka}. RF\ j\ p)$
and $y: y = \text{list-of } (\bigcup_{j < ka}. RF\ j\ q)$ **and** $p < m\ q < m$

```

using  $\langle x \in M \rangle \langle y \in M \rangle$  by (auto simp: M-def)
then have  $p < q$  length  $x < \text{length } y$ 
using  $\langle k \leq ka \rangle \langle ka \leq \text{Suc } k \rangle$  le1-not-refl [OF irrefl-less-than]
by (auto simp: lenlex-def sm-RF sorted-list-of-set-UN-lessThan length-concat
sum-sorted-list-of-set-map)
have  $xc: x = \text{concat } (\text{map } (?R \ p) (\text{list-of } \{..<ka\}))$ 
by (simp add: x sorted-list-of-set-UN-eq-concat  $\langle k \leq ka \rangle \langle ka \leq \text{Suc } k \rangle \langle p <$ 
 $m \rangle$  sm-RF)
have  $yc: y = \text{concat } (\text{map } (?R \ q) (\text{list-of } \{..<ka\}))$ 
by (simp add: y sorted-list-of-set-UN-eq-concat  $\langle k \leq ka \rangle \langle ka \leq \text{Suc } k \rangle \langle q <$ 
 $m \rangle$  sm-RF)
have enum-DF-AF: enum (DF k p)  $(ka - 1) < \text{hd } (\text{list-of } (AF \ k \ p))$  for p
proof (rule less-setsD [OF DF-AF])
show enum (DF k p)  $(ka - 1) \in DF \ k \ p$ 
using  $\langle ka \leq \text{Suc } k \rangle$  card-DF finite-DF by (auto simp: finite-enumerate-in-set)
show  $\text{hd } (\text{list-of } (AF \ k \ p)) \in AF \ k \ p$ 
using AF-non-Nil finite-AF hd-in-set set-sorted-list-of-set by blast
qed

have less-RF-RF:  $RF \ n \ p \ll RF \ n \ q$  if  $n < k$  for n
using that  $\langle p < q \rangle$  by (simp add: less-RF-same)
have less-RF-Suc:  $RF \ n \ q \ll RF \ (\text{Suc } n) \ q$  if  $n < k$  for n
using  $\langle q < m \rangle$  that by (auto simp: RF-def less-QF)
have less-RF-k:  $RF \ k \ q \ll RF \ k \ p$ 
using  $\langle q < m \rangle$  less-RF-same-k  $\langle p < q \rangle$  by blast
have less-RF-k-ka:  $RF \ (k-1) \ p \ll RF \ (ka - 1) \ q$ 
using ka-k-or-Suc less-RF-RF
by (metis One-nat-def RF-def  $\langle 0 < k \rangle \langle ka - 1 \leq k \rangle \langle p < m \rangle$  diff-Suc-1
diff-Suc-less less-QF-step)
have Inf-DF-eq-enum:  $\sqcap (DF \ k \ i) = \text{enum } (DF \ k \ i) \ 0$  for k i
by (simp add: Inf-nat-def enumerate-0)

have Inf-DF-less:  $\sqcap (DF \ k \ i') < \sqcap (DF \ k \ i)$  if  $i' < i$  for  $i' \ i \ k$ 
by (metis DF-ne enum-0-eq-Inf enum-0-eq-Inf-finite enum-DF-less-iff le0
that)
have AF-Inf-DF-less:  $\bigwedge x. x \in AF \ k \ i \implies \sqcap (DF \ k \ i') < x$  if  $i' \leq i$  for  $i' \ i \ k$ 
using less-setsD [OF DF-AF] DF-ne that
by (metis Inf-DF-less Inf-nat-def1 dual-order.order-iff-strict dual-order.strict-trans)

show ?thesis — The general case requires  $1 < k$ , necessitating a painful special
case
proof (cases  $k=1$ )
case True
with kka consider  $ka=1 \mid ka=2$  by linarith
then show ?thesis
proof cases
case 1
define zs where  $zs = \text{card } (AF \ 1 \ p) \ \# \ \text{list-of } (AF \ 1 \ p)$ 
 $\ @ \ \text{card } (AF \ 1 \ q) \ \# \ \text{list-of } (AF \ 1 \ q)$ 

```



```

have zs: Form-Body ka k x y zs
proof (intro that exI conjI Form-Body.intros)
  show  $x = \text{concat} ([\text{list-of } (AF\ k\ p)])\ y = \text{concat} ([\text{list-of } (AF\ k\ q)])$ 
    by (simp-all add: x y 1 lessThan-Suc RF-0)
  have  $AF\ k\ p \ll \text{insert} (\sqcap (DF\ k\ q)) (AF\ k\ q)$ 
  by (metis AF-DF DF-ne Inf-nat-def1 RF-0 <0 < k> insert-iff less-RF-RF
less-sets-def pq(1))
  then have  $\text{strict-sorted} (\text{list-of } (AF\ k\ p) @ \sqcap (DF\ k\ q) \# \text{list-of } (AF\ k\ q))$ 
    by (auto simp: strict-sorted-append-iff intro: less-sets-imp-list-less
AF-Inf-DF-less)
  moreover have  $\bigwedge x. x \in AF\ k\ q \implies \sqcap (DF\ k\ p) < x$ 
    by (meson AF-Inf-DF-less less-imp-le-nat <p < q>)
  moreover have  $\bigwedge x. x \in AF\ 1\ p \implies \sqcap (DF\ 1\ p) < x$ 
    by (meson DF-AF DF-ne Inf-nat-def1 less-setsD)
  ultimately show strict-sorted zs
    using  $\langle p < q \rangle$  True Inf-DF-less DF-AF DF-ne
    by (auto simp: zs-def less-sets-def card-AF AF-Inf-DF-less)
qed (auto simp: <k=1> <ka=1> zs-def AF-ne <length x < length y>)
have zs-N:  $\text{list.set } zs \subseteq N$ 
  using AF-subset-N by (auto simp: zs-def card-AF Inf-DF-N <k=1>)
show ?thesis
proof
  have  $l = 1$ 
    using kka <k=1> <ka=1> by auto
  have Form  $(2*1-1)\ \{x,y\}$ 
    using 1 Form.intros(2) True zs by fastforce
  then show Form l U
    by (simp add: <U = {x,y}> <l = 1> One-nat-def)
  show  $\text{list.set } (\text{inter-scheme } l\ U) \subseteq N$ 
    using kka zs zs-N <k=1> Form-Body-imp-inter-scheme by (fastforce
simp: <U = {x,y}>)
  qed
next
  case 2 — Still in our painful special case
  note True [simp] note 2 [simp]
  have [simp]:  $\{0 < .. < 2\} = \{1 :: \text{nat}\}$ 
    by auto

  have enum-DF1-eq:  $\text{enum } (DF\ 1\ i)\ 1 = \text{card } (AF\ 1\ i) + \text{card } (RF\ 1\ i)$ 
    if  $i < m$  for  $i$ 
    using card-AF-sum that by (simp add: One-nat-def)
  have card-RF:  $\text{card } (RF\ 1\ i) = \text{enum } (DF\ 1\ i)\ 1 - \text{enum } (DF\ 1\ i)\ 0$  if  $i$ 
   $< m$  for  $i$ 
    using that by (auto simp: RF-def card-QF del-def)
  have list-of-AF-RF:  $\text{list-of } (AF\ 1\ q \cup RF\ 1\ q) = \text{list-of } (AF\ 1\ q) @ \text{list-of}$ 
   $(RF\ 1\ q)$ 
    by (metis One-nat-def RF-0 True <0 < k> finite-RF less-RF-Suc
sorted-list-of-set-Un)

```

```

define zs where zs = card (AF 1 p) # (card (AF 1 p) + card (RF 1 p))
# list-of (AF 1 p)
      @ (card (AF 1 q) + card (RF 1 q)) # list-of (AF 1 q) @ list-of (RF
1 q) @ list-of (RF 1 p)
have zs: Form-Body ka k x y zs
proof (intro that exI conjI Form-Body.intros)
      have x = list-of (RF 0 p ∪ RF 1 p)
      by (simp add: x eval-nat-numeral lessThan-Suc RF-0 Un-commute
One-nat-def)
      also have ... = list-of (RF 0 p) @ list-of (RF 1 p)
      using RF-def True ⟨p < m⟩ less-QF-step
      by (metis QF-0 RF-0 diff-self-eq-0 finite-RF le-refl sorted-list-of-set-Un
zero-less-one)
      finally show x = concat ([list-of (AF 1 p),list-of (RF 1 p)])
      by (simp add: RF-0)

      have *: i ∈ AF 1 q ∨ i ∈ RF 1 q ∨ i ∈ RF 1 p ⇒ enum (DF 1 q) 1
< i for i
      using True card-DF finite-enumerate-in-set[OF finite-DF]
      by (metis AF-ne DF-AF One-nat-def RF-0 RF-non-Nil finite-RF lessI
less-RF-Suc less-RF-k less-setsD
      less-sets-trans sorted-list-of-set.sorted-key-list-of-set-eq-Nil-iff)

      show y = concat [list-of (RF 1 q ∪ AF 1 q)]
      by (simp add: y eval-nat-numeral lessThan-Suc RF-0 One-nat-def)
      show zs: zs = concat [[card (AF 1 p), card (AF 1 p) + card (RF 1 p)],
list-of (AF 1 p),
      [ card (AF 1 q) + card (RF 1 q), list-of (RF 1 q ∪ AF
1 q) ] @ interact [list-of (RF 1 p) ] []
      using list-of-AF-RF by (simp add: zs-def Un-commute)

      show strict-sorted zs
      proof (simp add: ⟨p < m⟩ ⟨q < m⟩ ⟨p < q⟩ zs-def strict-sorted-append-iff,
intro conjI strip)
      show 0 < card (RF 1 p)
      using ⟨p < m⟩ by (simp add: card-RF card-DF finite-DF)
      show G1: card (AF 1 p) < card (AF 1 q) + card (RF 1 q)
      by (simp add: Inf-DF-less card-AF ⟨p < q⟩ trans-less-add1)
      show card (AF 1 p) < x
      if x ∈ AF 1 p ∪ (AF 1 q ∪ (RF 1 q ∪ RF 1 p)) for x
      using that ⟨q < m⟩ *
      by (metis (no-types) order-refl AF-Inf-DF-less Un-iff G1 card-AF
order.strict-trans enum-DF1-eq that)
      show G2: card (AF 1 p) + card (RF 1 p) < card (AF 1 q) + card (RF
1 q)
      using ⟨p < q⟩ ⟨p < m⟩ ⟨q < m⟩ by (metis enum-DF1-eq enum-DF-less-iff
le-refl)
      show card (AF 1 q) + card (RF 1 q) < x

```

```

      if  $x \in AF\ 1\ q \cup (RF\ 1\ q \cup RF\ 1\ p)$  for  $x$ 
      using that  $\langle q < m \rangle * enum\text{-}DF1\text{-}eq$  by force
      then show  $card\ (AF\ 1\ p) + card\ (RF\ 1\ p) < x$ 
      if  $x \in AF\ 1\ p \cup (AF\ 1\ q \cup (RF\ 1\ q \cup RF\ 1\ p))$  for  $x$ 
      using that  $\langle p < m \rangle finite\text{-}enumerate\text{-}in\text{-}set[OF\ finite\text{-}DF]$ 
      by (metis DF-AF G2 Un-iff card-DF dual-order.strict-trans enum-DF1-eq
lessI less-setsD)
      have  $list\text{-}of\ (AF\ 1\ p) < list\text{-}of\ \{enum\ (DF\ 1\ q)\ 1\}$ 
      proof (rule less-sets-imp-sorted-list-of-set)
      show  $AF\ 1\ p \ll \{enum\ (DF\ 1\ q)\ 1\}$ 
      by (metis AF-DF card-DF empty-subsetI finite-DF finite-enumerate-in-set
insert-subset
      less-Suc-eq less-sets-weaken2  $\langle p < q \rangle$ )
      qed auto
      then show  $list\text{-}of\ (AF\ 1\ p) < (card\ (AF\ 1\ q) + card\ (RF\ 1\ q)) \#$ 
list-of  $(AF\ 1\ q) @ list\text{-}of\ (RF\ 1\ q) @ list\text{-}of\ (RF\ 1\ p)$ 
      using  $\langle q < m \rangle$  by (simp add: less-list-def enum-DF1-eq)
      have  $list\text{-}of\ (AF\ 1\ q) < list\text{-}of\ (RF\ 1\ q)$ 
      by (metis One-nat-def RF-0 True  $\langle 0 < k \rangle$  finite-RF less-RF-Suc
less-sets-imp-sorted-list-of-set)
      then show  $list\text{-}of\ (AF\ 1\ q) < list\text{-}of\ (RF\ 1\ q) @ list\text{-}of\ (RF\ 1\ p)$ 
      using RF-non-Nil by (auto simp: less-list-def)
      show  $list\text{-}of\ (RF\ 1\ q) < list\text{-}of\ (RF\ 1\ p)$ 
      using True finite-RF less-RF-k less-sets-imp-sorted-list-of-set by metis
      qed
      show  $[list\text{-}of\ (AF\ 1\ p), list\text{-}of\ (RF\ 1\ p)] \in lists\ (-\ \{\ \})$ 
      using RF-non-Nil  $\langle 0 < k \rangle$  by (auto simp: zs-def AF-ne)
      show  $[card\ (AF\ 1\ q) + card\ (RF\ 1\ q)] = acc\text{-}lengths\ 0\ [list\text{-}of\ (RF\ 1\ q) \cup AF\ 1\ q]$ 
      using list-of-AF-RF by (auto simp: zs-def AF-ne sup-commute)
      qed (auto simp: zs-def AF-ne  $\langle length\ x < length\ y \rangle$ )
      have  $zs\text{-}N: list.set\ zs \subseteq N$ 
      using  $\langle p < m \rangle \langle q < m \rangle$  DF-in-N enum-DF1-eq [symmetric]
      by (auto simp: zs-def card-AF AF-subset-N RF-subset-N Inf-DF-N)
      show ?thesis
      proof
      have Form  $(2*1)\ \{x,y\}$ 
      by (metis 2 Form.simps Suc-1 True zero-less-one zs)
      with kka show Form  $l\ U$ 
      by (simp add:  $\langle U = \{x,y\} \rangle$ )
      show  $list.set\ (inter\text{-}scheme\ l\ U) \subseteq N$ 
      using kka zs zs-N  $\langle k=1 \rangle$  Form-Body-imp-inter-scheme by (fastforce
simp:  $\langle U = \{x, y\} \rangle$ )
      qed
      qed
      next
      case False
      then have  $k \geq 2$   $ka \geq 2$ 
      using kka  $\langle k > 0 \rangle$  by auto

```

then have *k-minus-1* [*simp*]: $Suc (k - Suc (Suc 0)) = k - Suc 0$
by *auto*
have [*simp*]: $Suc (k - 2) = k - 1$
using $\langle k \geq 2 \rangle$ **by** *linarith*
define *PP* **where** $PP \equiv map (?R p) (list-of \{0 <..<ka\})$
define *QQ* **where** $QQ \equiv map (?R q) (list-of \{0 <..<k-1\}) @ ([list-of (RF$
 $(k-1) q \cup RF (ka-1) q])$
let *?INT* = *interact PP QQ*
— No separate sets A and B as in the text, but instead we treat both cases
as once
have [*simp*]: $length PP = ka - 1$
by (*simp add: PP-def*)
have [*simp*]: $length QQ = k-1$
using $\langle k \geq 2 \rangle$ **by** (*simp add: QQ-def*)

have *PP-n*: $PP ! n = list-of (RF (Suc n) p)$
if $n < ka-1$ **for** n
using *that kka* **by** (*auto simp: PP-def nth-sorted-list-of-set-greaterThanLessThan*)

have *QQ-n*: $QQ ! n = (if n < k-2 then list-of (RF (Suc n) q)$
 $else list-of (RF (k-1) q \cup RF (ka - 1) q))$
if $n < k-1$ **for** n
using *that kka* **by** (*auto simp: QQ-def nth-append nth-sorted-list-of-set-greaterThanLessThan*)

have *QQ-n-same*: $QQ ! n = list-of (RF (Suc n) q)$
if $n < k-1$ $k=ka$ **for** n
using *that kka Suc-diff-Suc*
by (*fastforce simp: One-nat-def QQ-def nth-append nth-sorted-list-of-set-greaterThanLessThan*)

have *split-nat-interval*: $\{0 <..<n\} = insert (n-1) \{0 <..<n-1\}$ **if** $n \geq 2$
for $n::nat$
using *that* **by** *auto*
have *split-list-interval*: $list-of\{0 <..<n\} = list-of\{0 <..<n-1\} @ [n-1]$ **if** n
 ≥ 2 **for** $n::nat$
proof (*intro sorted-list-of-set-unique [THEN iffD1] conjI*)
have $list-of \{0 <..<n - 1\} < [n - 1]$
by (*auto intro: less-sets-imp-list-less*)
then show *strict-sorted* ($list-of \{0 <..<n - 1\} @ [n - 1]$)
by (*auto simp: strict-sorted-append-iff*)
qed (*use* $\langle n \geq 2 \rangle$ **in** *auto*)

have *list-of-RF-Un*: $list-of (RF (k-1) q \cup RF k q) = list-of (RF (k-1) q)$
 $@ list-of (RF k q)$
by (*metis Suc-diff-1* $\langle 0 < k \rangle$ *finite-RF lessI less-RF-Suc sorted-list-of-set-Un*)

have *card-AF-sum-QQ*: $card (AF k q) + sum-list (map length QQ) =$
 $(\sum j < ka. card (RF j q))$
proof (*cases ka = Suc k*)
case *True*

```

have  $RF\ (k-1)\ q \cap RF\ k\ q = \{\}$ 
  using less-RF-Suc [of  $k-1$ ]  $\langle k > 0 \rangle$  by (auto simp: less-sets-def)
  then have  $card\ (RF\ (k-1)\ q \cup RF\ k\ q) = card\ (RF\ (k-1)\ q) + card$ 
( $RF\ k\ q$ )
  by (simp add: card-Un-disjoint)
then show ?thesis
  using  $\langle k \geq 2 \rangle\ \langle q < m \rangle$ 
  apply (simp add: QQ-def True flip: RF-0)
  apply (simp add: lessThan-k split-nat-interval sum-sorted-list-of-set-map)
  done
next
  case False
  with  $kka$  have  $ka=k$  by linarith
with  $\langle k \geq 2 \rangle$  show ?thesis by (simp add: QQ-def lessThan-k split-nat-interval
sum-sorted-list-of-set-map flip: RF-0)
qed

define LENS where  $LENS \equiv \lambda i. acc-lengths\ 0\ (list-of\ (AF\ k\ i)\ \# \ map$ 
( $?R\ i$ ) (list-of  $\{0 < .. < ka\}$ ))
have LENS-subset-N:  $list.set\ (LENS\ i) \subseteq N$  if  $i < m$  for  $i$ 
proof -
  have eq: (list-of  $(AF\ k\ i)\ \# \ map\ (?R\ i)\ (list-of\ \{0 < .. < ka\})$ ) = map ( $?R$ 
 $i$ ) (list-of  $\{.. < ka\}$ )
  using RF-0  $\langle 0 < ka \rangle$  sorted-list-of-set-k by auto
  let  $?f = rec-nat\ [card\ (AF\ k\ i)]\ (\lambda n\ r. r\ @\ [(\sum\ j \leq Suc\ n. card\ (RF\ j\ i))])$ 
  have  $f$ :  $acc-lengths\ 0\ (map\ (?R\ i)\ (list-of\ \{..v\})) = ?f\ v$  for  $v$ 
by (induction v) (auto simp: RF-0 acc-lengths-append sum-sorted-list-of-set-map)
  have  $\exists$ :  $list.set\ (?f\ v) \subseteq N$  if  $v \leq k$  for  $v$ 
  using that
  proof (induction v)
  case  $0$ 
  have  $card\ (AF\ k\ i) \in N$ 
  by (metis DF-N DF-ne Inf-nat-def1 card-AF subsetD)
  with  $0$  show ?case by simp
next
  case (Suc v)
  then have  $enum\ (DF\ k\ i)\ (Suc\ v) \in N$ 
  by (metis DF-N card-DF finite-enumerate-in-set finite-DF in-mono
le-imp-less-Suc)
  with  $Suc\ \langle i < m \rangle$  show ?case
  by (simp del: sum.atMost-Suc)
qed
show ?thesis
  unfolding LENS-def
  by (metis  $\exists\ Suc-pred'\ \langle 0 < ka \rangle\ \langle ka - 1 \leq k \rangle$  eq f lessThan-Suc-atMost)
qed
define LENS-QQ where  $LENS-QQ \equiv acc-lengths\ 0\ (list-of\ (AF\ k\ q)\ \#$ 
 $QQ)$ 
have LENS-QQ-subset:  $list.set\ LENS-QQ \subseteq list.set\ (LENS\ q)$ 

```

```

proof (cases ka = Suc k)
  case True
    with ⟨k ≥ 2⟩ show ?thesis
      unfolding QQ-def LENS-QQ-def LENS-def
      by (auto simp: list-of-RF-Un split-list-interval acc-lengths-append)
  next
    case False
      then have ka=k
        using kka by linarith
      with ⟨k ≥ 2⟩ show ?thesis
        by (simp add: QQ-def LENS-QQ-def LENS-def split-list-interval)
  qed
have ss-INT: strict-sorted ?INT
proof (rule strict-sorted-interact-I)
  fix n
  assume n < length QQ
  then have n: n < k-1
    by simp
  have n = k - 2 if ¬ n < k - 2
    using n that by linarith
  moreover have list-of (RF (Suc (k - 2)) p) < list-of (RF (k-1) q) ∪
RF (ka - 1) q)
    by (auto simp: less-sets-imp-sorted-list-of-set less-sets-Un2 less-RF-RF
less-RF-k-ka ⟨0 < k⟩)
  ultimately show PP ! n < QQ ! n
  using ⟨k ≤ ka⟩ n by (auto simp: PP-n QQ-n less-sets-imp-sorted-list-of-set
less-RF-RF)
  next
    fix n
    have V: [Suc n < ka - 1] ⇒ list-of (RF (Suc n) q) < list-of (RF (Suc
(Suc n)) p) for n
      by (smt RF-def Suc-leI ⟨ka - 1 ≤ k⟩ ⟨q < m⟩ diff-Suc-1 finite-RF
less-QF-step less-le-trans less-sets-imp-sorted-list-of-set nat-neq-iff zero-less-Suc)
    have RF (k - 1) q ≪ RF k p
      by (metis One-nat-def RF-non-Nil Suc-pred ⟨0 < k⟩ finite-RF lessI
less-RF-Suc less-RF-k less-sets-trans sorted-list-of-set-eq-Nil-iff)
    with kka have RF (k-1) q ∪ RF (ka - 1) q ≪ RF k p
      by (metis less-RF-k One-nat-def less-sets-Un1 antisym-conv2 diff-Suc-1
le-less-Suc-eq)
    then have VI: list-of (RF (k-1) q) ∪ RF (ka - 1) q < list-of (RF k p)
      by (rule less-sets-imp-sorted-list-of-set) auto
    assume Suc n < length PP
    with ⟨ka ≤ Suc k⟩ VI
    show QQ ! n < PP ! Suc n
      apply (clarsimp simp: PP-n QQ-n V)
      by (metis One-nat-def Suc-1 Suc-lessI add.right-neutral add-Suc-right
diff-Suc-Suc ka-k-or-Suc less-diff-conv)
  next
    show PP ∈ lists (- {[]})

```

```

    using RF-non-Nil kka
    by (clarsimp simp: PP-def) (metis RF-non-Nil less-le-trans)
  show QQ ∈ lists (- {})
    using RF-non-Nil kka
    by (clarsimp simp: QQ-def) (metis RF-non-Nil Suc-pred ⟨0 < k⟩ less-SucI
One-nat-def)
  qed (use kka PP-def QQ-def in auto)
  then have ss-QQ: strict-sorted (concat QQ)
    using strict-sorted-interact-imp-concat by blast

obtain zs where zs: Form-Body ka k x y zs and zs-N: list.set zs ⊆ N
proof (intro that exI conjI Form-Body.intros [OF ⟨length x < length y⟩])
  show x = concat (list-of (AF k p) # PP)
    using ⟨ka > 0⟩ by (simp add: PP-def RF-0 xc sorted-list-of-set-k)
  let ?YR = (map (list-of ∘ (λj. RF j q)) (list-of {0<..

```

using *RF-non-Nil kka* **by** (*auto simp: AF-ne PP-def QQ-def eq-commute*
[of []])
show *list.set ((LENS p @ list-of (AF k p) @ LENS-QQ @ list-of (AF k*
q) @ ?INT)) $\subseteq N$
using *AF-subset-N RF-subset-N LENS-subset-N* $\langle p < m \rangle \langle q < m \rangle$
LENS-QQ-subset
by (*auto simp: subset-iff PP-def QQ-def*)
show *length (list-of (AF k p) # PP) = ka length (list-of (AF k q) # QQ)*
 $= k$
using $\langle 0 < ka \rangle \langle 0 < k \rangle$ **by** *auto*
show *LENS p = acc-lengths 0 (list-of (AF k p) # PP)*
by (*auto simp: LENS-def PP-def*)
show *strict-sorted (LENS p @ list-of (AF k p) @ LENS-QQ @ list-of (AF*
k q) @ ?INT)
unfolding *strict-sorted-append-iff*
proof (*intro conjI ss-INT*)
show *LENS p < list-of (AF k p) @ LENS-QQ @ list-of (AF k q) @*
?INT
using *AF-non-Nil [of k p] <k ≤ ka> <ka ≤ Suc k> <p < m> card-AF-sum*
enum-DF-AF
by (*simp add: enum-DF-AF less-list-def card-AF-sum LENS-def*
sum-sorted-list-of-set-map
del: acc-lengths.simps)
show *strict-sorted (LENS p)*
unfolding *LENS-def*
by (*rule strict-sorted-acc-lengths*) (*use RF-non-Nil AF-non-Nil kka in*
<auto simp: in-lists-conv-set>)
show *strict-sorted LENS-QQ*
unfolding *LENS-QQ-def QQ-def*
by (*rule strict-sorted-acc-lengths*) (*use RF-non-Nil AF-non-Nil kka in*
<auto simp: in-lists-conv-set>)
have *last-AF-DF: last (list-of (AF k p)) < [] (DF k q)*
using *AF-DF [OF <p < q>, of k] AF-non-Nil [of k p] DF-ne [of k q]*
by (*metis Inf-nat-def1 finite-AF last-in-set less-sets-def set-sorted-list-of-set*)
then show *list-of (AF k p) < LENS-QQ @ list-of (AF k q) @ ?INT*
by (*simp add: less-list-def card-AF LENS-QQ-def*)
show *LENS-QQ < list-of (AF k q) @ ?INT*
using *AF-non-Nil [of k q] <q < m> card-AF-sum enum-DF-AF*
card-AF-sum-QQ
by (*auto simp: less-list-def AF-ne hd-append card-AF-sum LENS-QQ-def*)
show *list-of (AF k q) < ?INT*
proof –
have *AF k q ≪ RF 1 p*
using $\langle 0 < k \rangle \langle p < m \rangle \langle q < m \rangle$ **by** (*simp add: RF-def less-QF flip:*
QF-0)
then have *last (list-of (AF k q)) < hd (list-of (RF 1 p))*
proof (*rule less-setsD*)
show *last (list-of (AF k q)) ∈ AF k q*
using *AF-non-Nil finite-AF last-in-set set-sorted-list-of-set* **by** *blast*


```

      show hd (list-of (RF 1 p)) ∈ RF 1 p
      by (metis One-nat-def RF-non-Nil ⟨0 < k⟩ finite-RF hd-in-set
not-less-eq set-sorted-list-of-set)
    qed
    with ⟨k > 0⟩ ⟨ka ≥ 2⟩ RF-non-Nil show ?thesis
    by (simp add: One-nat-def hd-interact less-list-def sorted-list-of-set-greaterThanLessThan
PP-def QQ-def)
  qed
  qed auto
  qed (auto simp: LENS-QQ-def)
  show ?thesis
  proof (cases ka = k)
    case True
    then have l = 2*k-1
      by (simp add: kka(3) mult-2)
    then show ?thesis
      by (metis One-nat-def Form.intros(2) Form-Body-imp-inter-scheme True
⟨0 < k⟩ ⟨U = {x, y}⟩ kka zs zs-N)
  next
    case False
    then have l = 2*k
      using kka by linarith
    then show ?thesis
      by (metis One-nat-def False Form.intros(3) Form-Body-imp-inter-scheme
⟨0 < k⟩ ⟨U = {x, y}⟩ antisym kka le-SucE zs zs-N)
  qed
  qed
  qed
  qed
  qed

```

3.11 Larson's Lemma 3.8

3.11.1 Primitives needed for the inductive construction of b

definition IJ where $IJ \equiv \lambda k. \text{Sigma } \{..k\} (\lambda j::\text{nat}. \{..<j\})$

lemma $IJ\text{-iff}$: $u \in IJ\ k \longleftrightarrow (\exists j\ i. u = (j, i) \wedge i < j \wedge j \leq k)$

by (auto simp: $IJ\text{-def}$)

lemma $\text{finite-}IJ$: $\text{finite } (IJ\ k)$

by (auto simp: $IJ\text{-def}$)

fun prev where

```

  prev 0 0 = None
| prev (Suc 0) 0 = None
| prev (Suc j) 0 = Some (j, j - Suc 0)
| prev j (Suc i) = Some (j, i)

```

lemma prev-eq-None-iff : $\text{prev } j\ i = \text{None} \longleftrightarrow j \leq \text{Suc } 0 \wedge i = 0$

by (auto simp: le-Suc-eq elim: prev.elims)

lemma *prev-pair-less*:

$prev\ j\ i = Some\ ji' \implies (ji', (j,i)) \in pair-less$

by (auto simp: pair-lessI1 elim: prev.elims)

lemma *prev-Some-less*: $\llbracket prev\ j\ i = Some\ (j',i'); i \leq j \rrbracket \implies i' < j'$

by (auto elim: prev.elims)

lemma *prev-maximal*:

$\llbracket prev\ j\ i = Some\ (j',i'); (ji'', (j,i)) \in pair-less; ji'' \in IJ\ k \rrbracket$

$\implies (ji'', (j',i')) \in pair-less \vee ji'' = (j',i')$

by (force simp: IJ-def pair-less-def elim: prev.elims)

lemma *pair-less-prev*:

assumes $(u, (j,i)) \in pair-less\ u \in IJ\ k$

shows $prev\ j\ i = Some\ u \vee (\exists x. (u, x) \in pair-less \wedge prev\ j\ i = Some\ x)$

using *assms*

proof (cases *prev j i*)

case *None*

then show *?thesis*

using *assms* by (force simp: prev-eq-None-iff pair-less-def IJ-def split: prod.split)

next

case (*Some a*)

then show *?thesis*

by (*metis* *assms* *prev-maximal* *prod.exhaust-sel*)

qed

3.11.2 Special primitives for the ordertype proof

definition *USigma* :: 'a set set \Rightarrow ('a set \Rightarrow 'a set) \Rightarrow 'a set set

where $USigma\ A\ B \equiv \bigcup X \in A. \bigcup y \in B\ X. \{insert\ y\ X\}$

definition *usplit*

where $usplit\ f\ A \equiv f\ (A - \{Max\ A\})\ (Max\ A)$

lemma *USigma-empty* [*simp*]: $USigma\ \{\}\ B = \{\}$

by (auto simp: *USigma-def*)

lemma *USigma-iff*:

assumes $\bigwedge I\ j. I \in \mathcal{I} \implies I \ll J\ I \wedge finite\ I$

shows $x \in USigma\ \mathcal{I}\ J \iff usplit\ (\lambda I\ j. I \in \mathcal{I} \wedge j \in J\ I \wedge x = insert\ j\ I)\ x$

proof –

have [*simp*]: $\bigwedge I\ j. \llbracket I \in \mathcal{I}; j \in J\ I \rrbracket \implies Max\ (insert\ j\ I) = j$

by (*meson* *Max-insert2* *assms* *less-imp-le* *less-sets-def*)

show *?thesis*

proof –

have $\S: j \notin I$ if $I \in \mathcal{I}\ j \in J\ I$ for $I\ j$

using *that* by (*metis* *assms* *less-irrefl* *less-sets-def*)

have $\exists I \in \mathcal{I}. \exists j \in J I. x = \text{insert } j I$
if $x - \{\text{Max } x\} \in \mathcal{I}$ **and** $\text{Max } x \in J (x - \{\text{Max } x\}) x \neq \{\}$
using that by (*metis Max-in assms infinite-remove insert-Diff*)
then show *?thesis*
by (*auto simp: USigma-def usplit-def §*)
qed
qed

proposition *ordertype-append-image-IJ:*

assumes *lenB [simp]: $\bigwedge i j. i \in \mathcal{I} \implies j \in J i \implies \text{length } (B j) = c$*
and *AB: $\bigwedge i j. i \in \mathcal{I} \implies j \in J i \implies A i < B j$*
and *IJ: $\bigwedge i. i \in \mathcal{I} \implies i \ll J i \wedge \text{finite } i$*
and $\beta: \bigwedge i. i \in \mathcal{I} \implies \text{ordertype } (B \text{ ' } J i) (\text{lenlex less-than}) = \beta$
and *A: inj-on A \mathcal{I}*
shows $\text{ordertype } (\text{usplit } (\lambda i j. A i @ B j) \text{ ' } \text{USigma } \mathcal{I} J) (\text{lenlex less-than})$
 $= \beta * \text{ordertype } (A \text{ ' } \mathcal{I}) (\text{lenlex less-than})$
(is ordertype ?AB ?R = - * ? α)

proof (*cases $\mathcal{I} = \{\}$*)

case *False*

have *Ord β*

using *False wf-Ord-ordertype by fastforce*

show *?thesis*

proof (*subst ordertype-eq-iff*)

define *split where split $\equiv \lambda l::\text{nat list}. (\text{take } (\text{length } l - c) l, (\text{drop } (\text{length } l - c) l))$*

have *oB: ordermap (B ' J i) ?R (B j) $\sqsubset \beta$ if $\langle i \in \mathcal{I} \rangle \langle j \in J i \rangle$ for $i j$*

using β *less-TC-iff that by fastforce*

then show *Ord ($\beta * ?\alpha$)*

by (*intro $\langle \text{Ord } \beta \rangle$ wf-Ord-ordertype Ord-mult; simp*)

define *f where f $\equiv \lambda u. \text{let } (x,y) = \text{split } u \text{ in let } i = \text{inv-into } \mathcal{I} A x \text{ in}$*

$\beta * \text{ordermap } (A \text{ ' } \mathcal{I}) ?R x + \text{ordermap } (B \text{ ' } J i) ?R y$

have *inv-into-IA [simp]: inv-into $\mathcal{I} A (A i) = i$ if $i \in \mathcal{I}$ for i*

by (*simp add: A that*)

show $\exists f. \text{bij-betw } f ?AB (\text{elts } (\beta * ?\alpha)) \wedge (\forall x \in ?AB. \forall y \in ?AB. (f x < f y) = ((x, y) \in ?R))$

unfolding *bij-betw-def*

proof (*intro exI conjI strip*)

show *inj-on f ?AB*

proof (*clarsimp simp: f-def inj-on-def split-def USigma-iff IJ usplit-def*)

fix *x y*

assume $\S: \beta * \text{ordermap } (A \text{ ' } \mathcal{I}) ?R (A (x - \{\text{Max } x\})) + \text{ordermap } (B \text{ ' } J (x - \{\text{Max } x\})) ?R (B (\text{Max } x))$

$= \beta * \text{ordermap } (A \text{ ' } \mathcal{I}) ?R (A (y - \{\text{Max } y\})) + \text{ordermap } (B \text{ ' } J$

$(y - \{\text{Max } y\})) ?R (B (\text{Max } y))$

and *x: $x - \{\text{Max } x\} \in \mathcal{I}$*

and *y: $y - \{\text{Max } y\} \in \mathcal{I}$*

and *mx: $\text{Max } x \in J (x - \{\text{Max } x\})$*

and *x = insert (Max x) x*

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    and my: Max y ∈ J (y - {Max y})
    have ordermap (A'ℐ) ?R (A (x - {Max x})) = ordermap (A'ℐ) ?R (A (y
- {Max y}))
    and B-eq: ordermap (B'J (x - {Max x})) ?R (B (Max x)) = ordermap
(B'J (y - {Max y})) ?R (B (Max y))
    using mult-cancellation-lemma [OF §] oB mx my x y by blast+
    then have A (x - {Max x}) = A (y - {Max y})
    using x y by auto
    then have x - {Max x} = y - {Max y}
    by (metis x y inv-into-IA)
    then show A (x - {Max x}) = A (y - {Max y}) ∧ B (Max x) = B (Max
y)
    using B-eq mx my by auto
qed
show f' ?AB = elts (β * ?α)
proof
  show f' ?AB ⊆ elts (β * ?α)
  using ⟨Ord β⟩
  apply (clarsimp simp add: f-def split-def USigma-iff IJ usplit-def)
  by (metis TC-small β add-mult-less image-eqI ordermap-in-ordertype
trans-llt wf-Ord-ordertype wf-llt)
  show elts (β * ?α) ⊆ f' ?AB
  proof (clarsimp simp: f-def split-def image-iff USigma-iff IJ usplit-def Bex-def
elim!: elts-multE split: prod.split)
    fix γ δ
    assume δ: δ ∈ elts β and γ: γ ∈ elts ?α
    have γ ∈ ordermap (A'ℐ) (lenlex less-than) 'A'ℐ
    by (meson γ ordermap-surj subset-iff)
    then obtain i where i ∈ ℐ and yv: γ = ordermap (A'ℐ) ?R (A i)
    by blast
    have δ ∈ ordermap (B'J i) (lenlex less-than) 'B'J i
    by (metis (no-types) β δ ⟨i ∈ ℐ⟩ in-mono ordermap-surj)
    then obtain j where j ∈ J i and xu: δ = ordermap (B'J i) ?R (B j)
    by blast
    then have mji: Max (insert j i) = j
    by (meson IJ Max-insert2 ⟨i ∈ ℐ⟩ less-imp-le less-sets-def)
    have [simp]: i - {j} = i
    using IJ ⟨i ∈ ℐ⟩ ⟨j ∈ J i⟩ less-setsD by fastforce
    show ∃ l. (∃ K. K - {Max K} ∈ ℐ ∧ Max K ∈ J (K - {Max K})) ∧ K
= insert (Max K) K ∧
      l = A (K - {Max K}) @ B (Max K) ∧ β * γ + δ =
      β *
      ordermap (A'ℐ) ?R (take (length l - c) l) +
      ordermap (B'J (inv-into ℐ A (take (length l - c) l)))
      ?R (drop (length l - c) l)
  proof (intro conjI exI)
    let ?ji = insert j i
    show A i @ B j = A (?ji - {Max ?ji}) @ B (Max ?ji)
    by (auto simp: mji)
  end
end

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    qed (use ⟨i ∈ I⟩ ⟨j ∈ J i⟩ mji xu yv in auto)
  qed
  qed
next
fix p q
assume p ∈ ?AB and q ∈ ?AB
then obtain x y where peq: p = A (x - {Max x}) @ B (Max x)
    and qeq: q = A (y - {Max y}) @ B (Max y)
    and x: x - {Max x} ∈ I
    and y: y - {Max y} ∈ I
    and mx: Max x ∈ J (x - {Max x})
    and my: Max y ∈ J (y - {Max y})
  by (auto simp: USigma-iff IJ usplit-def)
let ?mx = x - {Max x}
let ?my = y - {Max y}
show (f p < f q) ↔ ((p, q) ∈ ?R)
proof
  assume f p < f q
  then
    consider ordermap (A I) ?R (A (x - {Max x})) < ordermap (A I) ?R (A
(y - {Max y}))
    | ordermap (A I) ?R (A (x - {Max x})) = ordermap (A I) ?R (A (y -
{Max y}))
    ordermap (B J (x - {Max x})) ?R (B (Max x)) < ordermap (B J (y -
{Max y})) ?R (B (Max y))
    using x y mx my
  by (auto dest: mult-cancellation-less simp: f-def split-def peq qeq oB)
then have (A ?mx @ B (Max x), A ?my @ B (Max y)) ∈ ?R
proof cases
  case 1
  then have (A ?mx, A ?my) ∈ ?R
  using x y by (force simp: Ord-mem-iff-lt intro: converse-ordermap-mono)
  then show ?thesis
  using x y mx my lenB lenlex-append1 by blast
  case 2
  then have A ?mx = A ?my
  using ⟨?my ∈ I⟩ ⟨?mx ∈ I⟩ by auto
  then have eq: ?mx = ?my
  by (metis ⟨?my ∈ I⟩ ⟨?mx ∈ I⟩ inv-into-IA)
  then have (B (Max x), B (Max y)) ∈ ?R
  using mx my 2 by (force simp: Ord-mem-iff-lt intro: converse-ordermap-mono)
  with 2 show ?thesis
  by (simp add: eq irreft-less-than)
qed
then show (p, q) ∈ ?R
  by (simp add: peq qeq f-def split-def sorted-list-of-set-Un AB)
next
assume pqR: (p, q) ∈ ?R

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then have §: (A ?mx @ B (Max x), A ?my @ B (Max y)) ∈ ?R
  using peq qeq by blast
then consider (A ?mx, A ?my) ∈ ?R | A ?mx = A ?my ∧ (B (Max x), B
(Max y)) ∈ ?R
  proof (cases (A ?mx, A ?my) ∈ ?R)
    case False
      have False if (A ?my, A ?mx) ∈ ?R
        by (metis ‹?my ∈ ℐ› ‹?mx ∈ ℐ› § ‹(Max y) ∈ J ?my› ‹(Max x) ∈ J
?mx› lenB lenlex-append1 omega-sum-1-less order.asym that)
      then have A ?mx = A ?my
        by (meson False UNIV-I total-llt total-on-def)
      then show ?thesis
        using § irrefl-less-than that by auto
    qed (use that in blast)
then have β * ordermap (A ℐ) ?R (A ?mx) + ordermap (B ‘ J ?mx) ?R (B
(Max x))
  < β * ordermap (A ℐ) ?R (A ?my) + ordermap (B ‘ J ?my) ?R (B
(Max y))
  proof cases
    case 1
      show ?thesis
      proof (rule add-mult-less-add-mult)
        show ordermap (A ℐ) (lenlex less-than) (A ?mx) < ordermap (A ℐ)
(lenlex less-than) (A ?my)
          by (simp add: 1 ‹?my ∈ ℐ› ‹?mx ∈ ℐ› ordermap-mono-less)
        show Ord (ordertype (A ℐ) ?R)
          using wf-Ord-ordertype by blast
        show ordermap (B ‘ J ?mx) ?R (B (Max x)) ∈ elts β
          using Ord-less-TC-mem ‹Ord β› ‹?mx ∈ ℐ› ‹(Max x) ∈ J ?mx› oB
by blast
        show ordermap (B ‘ J ?my) ?R (B (Max y)) ∈ elts β
          using Ord-less-TC-mem ‹Ord β› ‹?my ∈ ℐ› ‹(Max y) ∈ J ?my› oB
by blast
      qed (use ‹?my ∈ ℐ› ‹?mx ∈ ℐ› ‹Ord β› in auto)
    next
      case 2
        with ‹?mx ∈ ℐ› show ?thesis
          using ‹(Max y) ∈ J ?my› ‹(Max x) ∈ J ?mx› ordermap-mono-less
          by (metis (no-types, opaque-lifting) Kirby.add-less-cancel-left TC-small
image-iff inv-into-IA trans-llt wf-llt y)
        qed
        then show f p < f q
          using ‹?my ∈ ℐ› ‹?mx ∈ ℐ› ‹(Max y) ∈ J ?my› ‹(Max x) ∈ J ?mx›
          by (auto simp: peq qeq f-def split-def AB)
        qed
      qed
    qed auto
  qed auto

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3.11.3 The final part of 3.8, where two sequences are merged

inductive *merge* :: [nat list list, nat list list, nat list list, nat list list] \Rightarrow bool
where *NullNull*: *merge* [] [] [] []
| *Null*: *as* \neq [] \implies *merge as* [] [concat *as*] []
| *App*: [*as1* \neq []; *bs1* \neq [];
concat *as1* < concat *bs1*; concat *bs1* < concat *as2*; *merge as2 bs2 as*
bs]
 \implies *merge (as1@as2) (bs1@bs2) (concat as1 # as) (concat bs1 # bs)*

inductive-simps *Null1* [*simp*]: *merge* [] *bs us vs*

inductive-simps *Null2* [*simp*]: *merge as* [] *us vs*

lemma *merge-single*:

[concat *as* < concat *bs*; concat *as* \neq []; concat *bs* \neq []] \implies *merge as bs* [concat
as] [concat *bs*]

using *merge.App* [of *as bs* [] []]

by (*fastforce simp: less-list-def*)

lemma *merge-length1-nonempty*:

assumes *merge as bs us vs as* \in *lists* (- {[]})

shows *us* \in *lists* (- {[]})

using *assms* **by** *induction (auto simp: mem-lists-non-Nil)*

lemma *merge-length2-nonempty*:

assumes *merge as bs us vs bs* \in *lists* (- {[]})

shows *vs* \in *lists* (- {[]})

using *assms* **by** *induction (auto simp: mem-lists-non-Nil)*

lemma *merge-length1-gt-0*:

assumes *merge as bs us vs as* \neq []

shows *length us* > 0

using *assms* **by** *induction auto*

lemma *merge-length-le*:

assumes *merge as bs us vs*

shows *length vs* \leq *length us*

using *assms* **by** *induction auto*

lemma *merge-length-le-Suc*:

assumes *merge as bs us vs*

shows *length us* \leq *Suc (length vs)*

using *assms* **by** *induction auto*

lemma *merge-length-less2*:

assumes *merge as bs us vs*

shows *length vs* \leq *length as*

using *assms*

proof *induction*

case (*App as1 bs1 as2 bs2 as bs*)

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then show ?case
  using length-greater-0-conv [of as1] by (simp, presburger)
qed auto

lemma merge-preserves:
  assumes merge as bs us vs
  shows concat as = concat us  $\wedge$  concat bs = concat vs
  using assms by induction auto

lemma merge-interact:
  assumes merge as bs us vs strict-sorted (concat as) strict-sorted (concat bs)
    bs  $\in$  lists ( $-$  {[]})
  shows strict-sorted (interact us vs)
  using assms
proof induction
  case (App as1 bs1 as2 bs2 as bs)
  then have bs: concat bs1 < concat bs concat bs1 < concat as
    and nonmt: concat bs1  $\neq$  []
    using merge-preserves strict-sorted-append-iff by fastforce+
  then have concat bs1 < interact as bs
    unfolding less-list-def using App bs
    by (metis (no-types, lifting) Un-iff concat-append hd-in-set last-in-set merge-preserves
set-interact sorted-wrt-append strict-sorted-append-iff)
  with App show ?case
    by (metis append-in-lists-conv concat-append hd-append2 interact.simps(3)
less-list-def strict-sorted-append-iff nonmt)
qed auto

lemma acc-lengths-merge1:
  assumes merge as bs us vs
  shows list.set (acc-lengths k us)  $\subseteq$  list.set (acc-lengths k as)
  using assms
proof (induction arbitrary: k)
  case (App as1 bs1 as2 bs2 as bs)
  then show ?case
    apply (simp add: acc-lengths-append strict-sorted-append-iff length-concat-acc-lengths)
    by (simp add: le-supI2 length-concat)
qed (auto simp: length-concat-acc-lengths)

lemma acc-lengths-merge2:
  assumes merge as bs us vs
  shows list.set (acc-lengths k vs)  $\subseteq$  list.set (acc-lengths k bs)
  using assms
proof (induction arbitrary: k)
  case (App as1 bs1 as2 bs2 as bs)
  then show ?case
    apply (simp add: acc-lengths-append strict-sorted-append-iff length-concat-acc-lengths)
    by (simp add: le-supI2 length-concat)
qed (auto simp: length-concat-acc-lengths)

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lemma length-hd-le-concat:
  assumes  $as \neq []$  shows  $length\ (hd\ as) \leq length\ (concat\ as)$ 
  by (metis (no-types) add.commute assms concat.simps(2) le-add2 length-append
list.exhaust-sel)

lemma length-hd-merge2:
  assumes merge  $as\ bs\ us\ vs$ 
  shows  $length\ (hd\ bs) \leq length\ (hd\ vs)$ 
  using assms by induction (auto simp: length-hd-le-concat)

lemma merge-less-sets-hd:
  assumes merge  $as\ bs\ us\ vs$  strict-sorted (concat  $as$ ) strict-sorted (concat  $bs$ )  $bs$ 
 $\in lists\ (-\ \{\}\}$ 
  shows  $list.set\ (hd\ us) \ll list.set\ (concat\ vs)$ 
  using assms
proof induction
  case (App  $as1\ bs1\ as2\ bs2\ as\ bs$ )
  then have  $\S$ :  $list.set\ (concat\ bs1) \ll list.set\ (concat\ bs2)$ 
    by (force simp: dest: strict-sorted-imp-less-sets)
  have  $*$ :  $list.set\ (concat\ as1) \ll list.set\ (concat\ bs1)$ 
    using App by (metis concat-append strict-sorted-append-iff strict-sorted-imp-less-sets)
  then have  $list.set\ (concat\ as1) \ll list.set\ (concat\ bs)$ 
    using App  $\S$  less-sets-trans merge-preserves
  by (metis List.set-empty append-in-lists-conv le-zero-eq length-0-conv length-concat-ge)
  with  $*$  App.hyps show ?case
    by (fastforce simp: less-sets-UN1 less-sets-UN2 less-sets-Un2)
qed auto

lemma set-takeWhile:
  assumes strict-sorted (concat  $as$ )  $as \in lists\ (-\ \{\}\}$ 
  shows  $list.set\ (takeWhile\ (\lambda x.\ x < y)\ as) = \{x \in list.set\ as.\ x < y\}$ 
  using assms
proof (induction  $as$ )
  case (Cons  $a\ as$ )
  have  $a < y$ 
    if  $a: a < concat\ as$  strict-sorted  $a$  strict-sorted (concat  $as$ )  $x < y$   $x \neq []$   $x \in$ 
list.set  $as$ 
    for  $x$ 
  proof –
  have  $\S$ :  $last\ x \in list.set\ (concat\ as)$ 
    using set-concat that by fastforce
  have  $last\ a < hd\ (concat\ as)$ 
    using Cons.prems that by (auto simp: less-list-def)
  also have  $\dots \leq hd\ y$  if  $y \neq []$ 
    using that  $a$ 
  by (meson  $\S$  order.strict-trans less-list-def not-le sorted-hd-le strict-sorted-imp-sorted)
  finally show ?thesis

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    by (simp add: less-list-def)
  qed
  then show ?case
    using Cons by (auto simp: strict-sorted-append-iff)
  qed auto

proposition merge-exists:
  assumes strict-sorted (concat as) strict-sorted (concat bs)
    as ∈ lists (- {}) bs ∈ lists (- {})
    hd as < hd bs as ≠ [] bs ≠ []
  and disj:  $\bigwedge a b. [a \in \text{list.set } as; b \in \text{list.set } bs] \implies a < b \vee b < a$ 
  shows  $\exists us \text{ vs. merge } as \ bs \ us \ vs$ 
  using assms
proof (induction length as + length bs arbitrary: as bs rule: less-induct)
  case (less as bs)
  obtain as1 as2 bs1 bs2
    where A: as1 ≠ [] bs1 ≠ [] concat as1 < concat bs1 concat bs1 < concat as2
      and B: as = as1@as2 bs = bs1@bs2 and C: bs2 = []  $\vee$  (as2 ≠ []  $\wedge$  hd as2
  < hd bs2)
  proof
  define as1 where as1  $\equiv$  takeWhile ( $\lambda x. x < \text{hd } bs$ ) as
  define as2 where as2  $\equiv$  dropWhile ( $\lambda x. x < \text{hd } bs$ ) as
  define bs1 where bs1  $\equiv$  if as2=[] then bs else takeWhile ( $\lambda x. x < \text{hd } as2$ ) bs
  define bs2 where bs2  $\equiv$  if as2=[] then [] else dropWhile ( $\lambda x. x < \text{hd } as2$ ) bs

  have as1: as1 = takeWhile ( $\lambda x. \text{last } x < \text{hd } (\text{hd } bs)$ ) as
    using less.prem1 by (auto simp: as1-def less-list-def cong: takeWhile-cong)
  have as2: as2 = dropWhile ( $\lambda x. \text{last } x < \text{hd } (\text{hd } bs)$ ) as
    using less.prem2 by (auto simp: as2-def less-list-def cong: dropWhile-cong)

  have hd-as2: as2 ≠ []  $\implies \neg \text{hd } as2 < \text{hd } bs$ 
    using as2-def hd-dropWhile by metis
  have hd-bs2: bs2 ≠ []  $\implies \neg \text{hd } bs2 < \text{hd } as2$ 
    using bs2-def hd-dropWhile by metis
  show as1 ≠ []
    by (simp add: as1-def less.prem1 takeWhile-eq-Nil-iff)
  show bs1 ≠ []
    by (metis as2 bs1-def hd-as2 hd-in-set less.prem1(7) less.prem1(8) set-dropWhileD
  takeWhile-eq-Nil-iff)
  show bs2 = []  $\vee$  (as2 ≠ []  $\wedge$  hd as2 < hd bs2)
    by (metis as2-def bs2-def hd-bs2 less.prem1(8) list.set-sel(1) set-dropWhileD)
  have AB: list.set A  $\ll$  list.set B
    if A ∈ list.set as1 B ∈ list.set bs for A B
  proof -
  have A ∈ list.set as
    using that by (metis as1 set-takeWhileD)
  then have sorted A
    by (metis concat.simps(2) concat-append less.prem1(1) sorted-append
  split-list-last strict-sorted-imp-sorted)

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moreover have sorted (hd bs)
by (metis concat.simps(2) hd-Cons-tl less.prem(2) less.prem(7) strict-sorted-append-iff
strict-sorted-imp-sorted)
ultimately show ?thesis
using that less.prem
apply (clarsimp simp add: as1-def set-takeWhile less-list-iff-less-sets less-sets-def)
by (metis (full-types) UN-I hd-concat less-le-trans list.set-sel(1) set-concat
sorted-hd-le strict-sorted-imp-sorted)
qed
show as = as1@as2
by (simp add: as1-def as2-def)
show bs = bs1@bs2
by (simp add: bs1-def bs2-def)
have list.set (concat as1)  $\ll$  list.set (concat bs1)
using AB set-takeWhileD by (fastforce simp: as1-def bs1-def less-sets-UN1
less-sets-UN2)
then show concat as1 < concat bs1
by (rule less-sets-imp-list-less)
have list.set (concat bs1)  $\ll$  list.set (concat as2) if as2  $\neq$  []
proof (clarsimp simp add: bs1-def less-sets-UN1 less-sets-UN2 set-takeWhile
less.prem)
fix A B
assume A  $\in$  list.set as2 B  $\in$  list.set bs B < hd as2
with that show list.set B  $\ll$  list.set A
using hd-as2 less.prem(1,2)
apply (clarsimp simp add: less-sets-def less-list-def)
apply (auto simp: as2-def)
apply (simp flip: as2-def)
by (smt (verit, ccfv-SIG) UN-I  $\langle$ as = as1 @ as2 $\rangle$  concat.simps(2) con-
cat-append hd-concat in-set-conv-decomp-first le-less-trans less-le-trans set-concat
sorted-append sorted-hd-le sorted-le-last strict-sorted-imp-sorted that)
qed
then show concat bs1 < concat as2
by (simp add: bs1-def less-sets-imp-list-less)
qed
obtain cs ds where merge as2 bs2 cs ds
proof (cases as2 = []  $\vee$  bs2 = [])
case True
then show thesis
using that C NullNull Null by metis
next
have †: length as2 + length bs2 < length as + length bs
by (simp add: A B)
case False
moreover have strict-sorted (concat as2) strict-sorted (concat bs2)
as2  $\in$  lists (- {[]}) bs2  $\in$  lists (- {[]})
 $\wedge$  a b. [a  $\in$  list.set as2; b  $\in$  list.set bs2]  $\implies$  a < b  $\vee$  b < a
using B less.prem strict-sorted-append-iff by auto
ultimately show ?thesis

```

```

    using C less.hyps [OF †] False that by force
  qed
  then obtain cs where merge (as1 @ as2) (bs1 @ bs2) (concat as1 # cs) (concat
bs1 # ds)
    using A merge.App by blast
  then show ?case
    using B by blast
  qed

```

3.11.4 Actual proof of Larson's Lemma 3.8

proposition *lemma-3-8*:

assumes *infinite N*

obtains *X* where $X \subseteq WW$ *ordertype X (lenlex less-than) = $\omega \uparrow \omega$*

$\bigwedge u. u \in [X]^2 \implies$

$\exists l. \text{Form } l \ u \wedge (l > 0 \longrightarrow [\text{enum } N \ l] < \text{inter-scheme } l \ u \wedge \text{List.set}$

$(\text{inter-scheme } l \ u) \subseteq N)$

proof –

let *?LL = lenlex less-than*

define *bf* where $bf \equiv \lambda M \ q. \text{wfrec pair-less } (\lambda f \ (j, i).$

$\text{let } R = (\text{case prev } j \ i \ \text{of None} \Rightarrow M \mid \text{Some } u \Rightarrow \text{snd } (f$

$u))$

$\text{in grab } R \ (q \ j \ i))$

have *bf-rec*: $bf \ M \ q \ (j, i) =$

$(\text{let } R = (\text{case prev } j \ i \ \text{of None} \Rightarrow M \mid \text{Some } u \Rightarrow \text{snd } (bf \ M \ q \ u))$

$\text{in grab } R \ (q \ j \ i)) \ \text{for } M \ q \ j \ i$

by (*subst (1) bf-def*) (*simp add: Let-def wfrec bf-def cut-apply prev-pair-less cong: conj-cong split: option.split*)

have *infinite (snd (bf M q u)) = infinite M \wedge fst (bf M q u) \subseteq M \wedge snd (bf M q u) \subseteq M* **for** *M q u*

using *wf-pair-less*

proof (*induction u rule: wf-induct-rule*)

case (*less u*)

then show *?case*

proof (*cases u*)

case (*Pair j i*)

with *less.IH prev-pair-less* **show** *?thesis*

apply (*simp add: bf-rec [of M q j i] split: option.split*)

using *fst-grab-subset snd-grab-subset* **by** *blast*

qed

qed

then have *infinite-bf [simp]: infinite (snd (bf M q u)) = infinite M*

and *bf-subset: fst (bf M q u) \subseteq M \wedge snd (bf M q u) \subseteq M* **for** *M q u*

by *auto*

have *bf-less-sets: fst (bf M q ij) \ll snd (bf M q ij)* **if** *infinite M* **for** *M q ij*

using *wf-pair-less*

```

proof (induction ij rule: wf-induct-rule)
  case (less u)
  then show ?case
    by (metis bf-rec finite-grab-iff infinite-bf less-sets-grab prod.exhaust-sel that)
qed

have card-fst-bf: finite (fst (bf M q (j,i))) ∧ card (fst (bf M q (j,i))) = q j i if
infinite M for M q j i
  by (simp add: that bf-rec [of M q j i] split: option.split)

have bf-cong: bf M q u = bf M q' u
  if snd u ≤ fst u and eq: ∧y x. [x ≤ y; y ≤ fst u] ⇒ q' y x = q y x for M q q' u
  using wf-pair-less that
proof (induction u rule: wf-induct-rule)
  case (less u)
  show ?case
  proof (cases u)
  case (Pair j i)
  with less.premis show ?thesis
  proof (clarsimp simp add: bf-rec [of M - j i] split: option.split)
    fix j' i'
    assume *: prev j i = Some (j',i')
    then have **: ((j', i'), u) ∈ pair-less
      by (simp add: Pair prev-pair-less)
    moreover have i' < j'
      using Pair less.premis by (simp add: prev-Some-less [OF *])
    moreover have ∧x y. [x ≤ y; y ≤ j'] ⇒ q' y x = q y x
      using ** less.premis by (auto simp: pair-less-def Pair)
    ultimately show grab (snd (bf M q (j',i'))) (q j i) = grab (snd (bf M q'
(j',i'))) (q j i)
      using less.IH by auto
  qed
qed
qed

define ediff where ediff ≡ λD:: nat ⇒ nat set. λj i. enum (D j) (Suc i) – enum
(D j) i
define F where F ≡ λl (dl,a0::nat set,b0::nat × nat ⇒ nat set,M).
  let (d,Md) = grab (nxt M (enum N (Suc (2 * Suc l)))) (Suc l) in
  let (a,Ma) = grab Md (Min d) in
  let Gb = bf Ma (ediff (dl(l := d))) in
  let dl' = dl(l := d) in
  (dl', a, fst ∘ Gb, snd (Gb(l, l-1)))
define DF where DF ≡ rec-nat (λi∈{..<0}. {}, {}, λp. {}, N) F
have DF-simps: DF 0 = (λi∈{..<0}. {}, {}, λp. {}, N)
  DF (Suc l) = F l (DF l) for l
  by (auto simp: DF-def)
note cut-apply [simp]

```

have *inf [rule-format]: $\forall dl\ al\ bl\ L. DF\ l = (dl,al,bl,L) \longrightarrow infinite\ L$ for l*
by (*induction l*) (*auto simp: DF-simps F-def Let-def grab-eqD infinite-nxtN*
assms split: prod.split)

define Ψ **where**
 $\Psi \equiv \lambda(dl, a, b, M). \lambda l::nat.$
 $dl\ l \ll a \wedge card\ a > 0 \wedge$
 $(\forall j \leq l. card\ (dl\ j) = Suc\ j) \wedge a \ll \bigcup (range\ b) \wedge range\ b \subseteq Collect\ finite$
 \wedge
 $a \subseteq N \wedge \bigcup (range\ b) \subseteq N \wedge infinite\ M \wedge b(l,l-1) \ll M \wedge M \subseteq N$

have Ψ -DF: $\Psi\ (DF\ (Suc\ l))\ l$ **for** l
proof (*induction l*)
case 0
show ?case
using *assms*
apply (*clarsimp simp add: bf-rec F-def DF-simps Ψ -def split: prod.split*)
apply (*drule grab-eqD, blast dest: grab-eqD infinite-nxtN*)
apply (*auto simp: less-sets-UN2 less-sets-grab card-fst-bf elim!: less-sets-weaken2*)
apply (*metis card-1-singleton-iff Min-singleton greaterThan-iff insertI1 le0*
nxt-subset-greaterThan subsetD)
using *nxt-subset snd-grab-subset bf-subset* **by** *blast+*

next
case (*Suc l*)
then show ?case
using *assms*
unfolding *Let-def DF-simps(2)[of Suc l] F-def Ψ -def*
apply (*clarsimp simp add: bf-rec DF-simps split: prod.split*)
apply (*drule grab-eqD, metis grab-eqD infinite-nxtN*)
apply (*safe, simp-all add: less-sets-UN2 less-sets-grab card-fst-bf card-Suc-eq-finite*)
apply (*meson less-sets-weaken2*)
apply (*metis Min-in grOI greaterThan-iff insert-not-empty le-inf-iff*
less-asm nxt-def subsetD)
apply (*meson bf-subset less-sets-weaken2*)
apply (*meson nxt-subset subset-eq*)
apply (*meson bf-subset nxt-subset subset-eq*)
using *bf-rec infinite-bf* **apply** *force*
using *bf-less-sets bf-rec* **apply** *force*
by (*metis bf-rec bf-subset nxt-subset subsetD*)

qed

define d **where** $d \equiv \lambda k. let\ (dk,ak,bk,M) = DF(Suc\ k)\ in\ dk\ k$
define a **where** $a \equiv \lambda k. let\ (dk,ak,bk,M) = DF(Suc\ k)\ in\ ak$
define b **where** $b \equiv \lambda k. let\ (dk,ak,bk,M) = DF(Suc\ k)\ in\ bk$
define M **where** $M \equiv \lambda k. let\ (dk,ak,bk,M) = DF\ k\ in\ M$

have *infinite-M [simp]: infinite (M k) for k*
by (*auto simp: M-def inf split: prod.split*)

have *M-Suc-subset: M (Suc k) \subseteq M k for k*

apply (*clarsimp simp add: Let-def M-def F-def DF-simps split: prod.split*)
by (*metis bf-subset in-mono nxt-subset snd-conv snd-grab-subset*)

have *Inf-M-Suc-ge*: $\text{Inf } (M \ k) \leq \text{Inf } (M \ (\text{Suc } k))$ **for** *k*
by (*simp add: M-Suc-subset cInf-superset-mono infinite-imp-nonempty*)

have *Inf-M-telescoping*: $\{\text{Inf } (M \ k)..\} \subseteq \{\text{Inf } (M \ k')..\}$ **if** $k': k' \leq k$ **for** *k k'*
using *that Inf-nat-def1 infinite-M unfolding Inf-nat-def atLeast-subset-iff*
by (*metis M-Suc-subset finite.emptyI le-less-linear lift-Suc-antimono-le not-less-Least subsetD*)

have *d-eq*: $d \ k = \text{fst } (\text{grab } (\text{nxt } (M \ k) \ (\text{enum } N \ (\text{Suc } (2 * \text{Suc } k)))) \ (\text{Suc } k))$ **for** *k*
by (*simp add: d-def M-def Let-def DF-simps F-def split: prod.split*)
then have *finite-d* [*simp*]: *finite* (*d k*) **for** *k*
by *simp*
then have *d-ne* [*simp*]: $d \ k \neq \{\}$ **for** *k*
by (*metis card.empty card-grab d-eq infinite-M infinite-nxtN nat.distinct(1)*)
have *a-eq*: $\exists M. a \ k = \text{fst } (\text{grab } M \ (\text{Min } (d \ k))) \wedge \text{infinite } M$ **for** *k*
apply (*simp add: a-def d-def M-def Let-def DF-simps F-def split: prod.split*)
by (*metis fst-conv grab-eqD infinite-nxtN local.inf*)
then have *card-a*: $\text{card } (a \ k) = \text{Inf } (d \ k)$ **for** *k*
by (*metis cInf-eq-Min card-grab d-ne finite-d*)

have *d-eq-dl*: $d \ k = dl \ k$ **if** $(dl, a, b, P) = DF \ l \ k < l$ **for** *k l dl a b P*
using *that*
by (*induction l arbitrary: dl a b P*) (*simp-all add: d-def DF-simps F-def Let-def split: prod.split-asm prod.split*)

have *card-d* [*simp*]: $\text{card } (d \ k) = \text{Suc } k$ **for** *k*
by (*auto simp: d-eq infinite-nxtN*)

have *d-ne* [*simp*]: $d \ j \neq \{\}$ **and** *a-ne* [*simp*]: $a \ j \neq \{\}$
and *finite-d* [*simp*]: *finite* (*d j*) **and** *finite-a* [*simp*]: *finite* (*a j*) **for** *j*
using $\Psi\text{-DF}$ [*of j*] **by** (*auto simp: $\Psi\text{-def a-def d-def card-gt-0-iff split: prod.split-asm}$*)

have *da*: $d \ k \ll a \ k$ **for** *k*
using $\Psi\text{-DF}$ [*of k*] **by** (*simp add: $\Psi\text{-def a-def d-def split: prod.split-asm}$*)

have *ab-same*: $a \ k \ll \bigcup (\text{range } (b \ k))$ **for** *k*
using $\Psi\text{-DF}$ [*of k*] **by** (*simp add: $\Psi\text{-def a-def b-def M-def split: prod.split-asm}$*)

have *snd-bf-subset*: $\text{snd } (bf \ M \ r \ (j, i)) \subseteq \text{snd } (bf \ M \ r \ (j', i'))$
if *ji*: $((j', i'), (j, i)) \in \text{pair-less } (j', i') \in IJ \ k$
for $M \ r \ k \ j \ i \ j' \ i'$
using *wf-pair-less ji*
proof (*induction rule: wf-induct-rule [where a = (j, i)]*)
case (*less u*)
show *?case*

```

proof (cases u)
  case (Pair j i)
    then consider prev j i = Some (j', i') | x where ((j', i'), x) ∈ pair-less prev
j i = Some x
    using less.premis pair-less-prev by blast
    then show ?thesis
    proof cases
      case 2 with less.IH show ?thesis
      unfolding bf-rec Pair
      by (metis in-mono option.simps(5) prev-pair-less snd-grab-subset subsetI
that(2))
    qed (simp add: Pair bf-rec snd-grab-subset)
  qed
qed

have less-bf: fst (bf M r (j', i')) ≤ fst (bf M r (j, i))
  if ji: ((j', i'), (j, i)) ∈ pair-less (j', i') ∈ IJ k and infinite M
  for M r k j i j' i'
proof -
  consider prev j i = Some (j', i') | j'' i'' where ((j', i'), (j'', i'')) ∈ pair-less
prev j i = Some (j'', i'')
  by (metis pair-less-prev ji prod.exhaust-sel)
  then show ?thesis
  proof cases
    case 1
      then show ?thesis
      using bf-less-sets bf-rec less-sets-fst-grab ⟨infinite M⟩ by force
    next
      case 2
        then have fst (bf M r (j', i')) ≤ snd (bf M r (j'', i''))
          by (meson bf-less-sets snd-bf-subset less-sets-weaken2 that)
        with 2 show ?thesis
        using bf-rec bf-subset less-sets-fst-grab ⟨infinite M⟩ by auto
  qed
qed

have aM: a k ≤ M (Suc k) for k
  apply (clarsimp simp add: a-def M-def DF-simps F-def Let-def split: prod.split)
  by (meson bf-subset grab-eqD infinite-nxtN less-sets-weaken2 local.inf)
then have a k ≤ a (Suc k) for k
  by (metis IntE card-d card.empty d-eq da fst-grab-subset less-sets-trans less-sets-weaken2
nat.distinct(1) nxt-def subsetI)
then have aa: a j ≤ a k if j < k for k j
  by (meson UNIV-I a-ne less-sets-imp-strict-mono-sets strict-mono-sets-def that)
then have ab: a k' ≤ b k (j, i) if k' ≤ k for k k' j i
  by (metis a-ne ab-same le-less less-sets-UN2 less-sets-trans rangeI that)
have db: d j ≤ b k (j, i) if j ≤ k for k j i
  by (meson a-ne ab da less-sets-trans that)

```



```

have bMkk: b k (k,k-1) << M (Suc k) for k
  using Ψ-DF [of k]
  by (simp add: Ψ-def b-def d-def M-def split: prod.split-asm)

have b: ∃ P ⊆ M k. infinite P ∧ (∀ j i. i ≤ j → j ≤ k → b k (j,i) = fst (bf P
(ediff d) (j,i))) for k
proof (clarsimp simp: b-def DF-simps F-def Let-def split: prod.split)
  fix a a' d' dl bb P M' M''
  assume gr: grab M'' (Min d') = (a', M') grab (nxt P (enum N (Suc (Suc (Suc
(2 * k)))))) (Suc k) = (d', M'')
  and DF: DF k = (dl, a, bb, P)
  have deg: d j = (if j = k then d' else dl j) if j ≤ k for j
  proof (cases j < k)
    case True
      then show ?thesis by (metis DF d-eq-dl less-not-refl)
    next
      case False
        then show ?thesis
          using that DF gr by (auto simp: d-def DF-simps F-def Let-def split: prod.split)
  qed
have M' ⊆ P
  by (metis gr in-mono nxt-subset snd-conv snd-grab-subset subsetI)
also have P ⊆ M k
  using DF by (simp add: M-def)
finally have M' ⊆ M k .
moreover have infinite M'
  using DF by (metis (mono-tags) finite-grab-iff gr infinite-nxtN local.inf
snd-conv)
moreover
have ediff (dl(k := d')) j i = ediff d j i if j ≤ k for j i
  by (simp add: deg that ediff-def)
then have bf M' (ediff (dl(k := d'))) (j,i)
  = bf M' (ediff d) (j,i) if i ≤ j j ≤ k for j i
  using bf-cong that by fastforce
ultimately show ∃ P ⊆ M k. infinite P ∧
  (∀ j i. i ≤ j → j ≤ k
  → fst (bf M' (ediff (dl(k := d'))) (j,i))
  = fst (bf P (ediff d) (j,i)))

  by auto
qed

have card-b: card (b k (j,i)) = enum (d j) (Suc i) - enum (d j) i if j ≤ k for k j
i
  — there's a short proof of this from the previous result but it would need i ≤ j
proof (clarsimp simp: b-def DF-simps F-def Let-def split: prod.split)
  fix dl
  and a a' d': nat set
  and bb M M' M''
  assume gr: grab M'' (Min d') = (a', M') grab (nxt M (enum N (Suc (Suc (Suc

```

```

(2 * k)))))) (Suc k) = (d', M'')
  and DF: DF k = (dl, a, bb, M)
  have d j = (if j = k then d' else dl j)
  proof (cases j < k)
    case True
    then show ?thesis by (metis DF d-eq-dl less-not-refl)
  next
    case False
    then show ?thesis
    using that DF gr by (auto simp: d-def DF-simps F-def Let-def split: prod.split)
  qed
  then show card (fst (bf M' (ediff (dl(k := d')) (j, i)))
    = enum (d j) (Suc i) - enum (d j) i
    using DF gr card-fst-bf grab-eqD infinite-nextN local.inf ediff-def by auto
  qed

have card-b-pos: card (b k (j, i)) > 0 if i < j j ≤ k for k j i
  by (simp add: card-b that finite-enumerate-step)
have b-ne [simp]: b k (j, i) ≠ {} if i < j j ≤ k for k j i
  using card-b-pos [OF that] less-imp-neq by fastforce+

have card-b-finite [simp]: finite (b k u) for k u
  using Ψ-DF [of k] by (fastforce simp: Ψ-def b-def)

have bM: b k (j, i) ≪ M (Suc k) if i < j j ≤ k for i j k
  proof -
    obtain M' where M' ⊆ M k infinite M'
    and bk: ⋀ j i. i ≤ j ⇒ j ≤ k ⇒ b k (j, i) = fst (bf M' (ediff d) (j, i))
    using b by (metis (no-types, lifting))
    show ?thesis
    proof (cases j=k ∧ i = k-1)
      case False
      show ?thesis
      proof (rule less-sets-trans [OF - bMkk])
        show b k (j, i) ≪ b k (k, k-1)
          using that ⟨infinite M'⟩ False
          by (force simp: bk pair-less-def IJ-def intro: less-bf)
        show b k (k, k-1) ≠ {}
          using b-ne that by auto
      qed
    qed (use bMkk in auto)
  qed

have b-InfM: ⋃ (range (b k)) ⊆ {⋂ (M k)..} for k
  proof (clarify simp add: Ψ-def b-def M-def DF-simps F-def Let-def split:
  prod.split)
    fix r dl :: nat ⇒ nat set
    and a b and d' a' M'' M' P and x j' i' :: nat
    assume gr: grab M'' (Min d') = (a', M')

```

grab (nxt P (enum N (Suc (Suc (Suc (2 * k)))))) (Suc k) = (d', M'')
 and DF: DF k = (dl, a, b, P)
 and x: x ∈ fst (bf M' (ediff (dl(k := d'))) (j', i'))
 have infinite P
 using DF local.inf by blast
 then have M' ⊆ P
 by (meson gr grab-eqD infinite-nxtN nxt-subset order.trans)
 with bf-subset show □ P ≤ x
 using Inf-nat-def x le-less-linear not-less-Least by fastforce
 qed

have b-Inf-M-Suc: b k (j, i) ≪ {Inf(M (Suc k))} if i < j j ≤ k for k j i
 using bMkk [of k] that
 by (metis Inf-nat-def1 bM finite.emptyI infinite-M less-setsD less-sets-singleton2)

have bb-same: b k (j', i') ≪ b k (j, i)
 if ((j', i'), (j, i)) ∈ pair-less (j', i') ∈ IJ k for k j i j' i'
 using that
 unfolding b-def DF-simps F-def Let-def
 by (auto simp: less-bf grab-eqD infinite-nxtN local.inf split: prod.split)

have bb: b k' (j', i') ≪ b k (j, i)
 if j: i' < j' j' ≤ k' and k: k' < k for i i' j j' k' k
 proof (rule atLeast-less-sets)
 show b k' (j', i') ≪ {Inf(M (Suc k'))}
 using Suc-lessD b-Inf-M-Suc nat-less-le j by blast
 show b k (j, i) ⊆ {Inf(M (Suc k'))..
 by (meson Inf-M-telescoping Suc-leI UnionI b-InfM rangeI subset-eq k)
 qed

have M-subset-N: M k ⊆ N for k
 proof (cases k)
 case (Suc k')
 with Ψ-DF [of k'] show ?thesis
 by (auto simp: M-def Let-def Ψ-def split: prod.split)
 qed (auto simp: M-def DF-simps)
 have a-subset-N: a k ⊆ N for k
 using Ψ-DF [of k] by (simp add: a-def Ψ-def split: prod.split prod.split-asm)
 have d-subset-N: d k ⊆ N for k
 using M-subset-N [of k] d-eq fst-grab-subset nxt-subset by blast
 have b-subset-N: b k (j, i) ⊆ N for k j i
 using Ψ-DF [of k] by (force simp: b-def Ψ-def)

define K:: [nat, nat] ⇒ nat set set
 where K ≡ λj0 j. nsets {j0 < ..} j
 have K-finite: finite K and K-card: card K = j if K ∈ K j0 j for K j0 j
 using that by (auto simp add: K-def nsets-def)
 have K-enum: j0 < enum K i if K ∈ K j0 j i < card K for K j0 j i
 using that by (auto simp: K-def nsets-def finite-enumerate-in-set subset-eq)

have $\mathcal{K}\text{-}0$ [simp]: $\mathcal{K} \ k \ 0 = \{\{\}\}$ **for** k
by (auto simp: $\mathcal{K}\text{-def}$)

have $\mathcal{K}\text{-Suc}$: $\mathcal{K} \ j0 \ (Suc \ j) = USigma \ (\mathcal{K} \ j0 \ j) \ (\lambda K. \ \{Max \ (insert \ j0 \ K) <..\})$ (is
 $?lhs = ?rhs$)
for $j \ j0$
proof
show $\mathcal{K} \ j0 \ (Suc \ j) \subseteq USigma \ (\mathcal{K} \ j0 \ j) \ (\lambda K. \ \{Max \ (insert \ j0 \ K) <..\})$
unfolding $\mathcal{K}\text{-def}$ $nsets\text{-def}$ $USigma\text{-def}$
proof clarsimp
fix K
assume $K: K \subseteq \{j0 <..\}$ $finite \ K$ $card \ K = Suc \ j$
then have $Max \ K \in K$
by (metis $Max\text{-in}$ $card\text{-}0\text{-eq}$ $nat.\text{distinct}(1)$)
then obtain i **where** $Max \ (insert \ j0 \ (K - \{Max \ K\})) < i \ K = insert \ i \ (K - \{Max \ K\})$
using K
by (simp add: $subset\text{-iff}$) (metis $DiffE$ $Max.\text{cobounded}I$ $insertCI$ $insert\text{-Diff}$ $le\text{-neq}\text{-implies}\text{-less}$)
then show $\exists L \subseteq \{j0 <..\}. \ finite \ L \wedge \ card \ L = j \wedge (\exists i \in \{Max \ (insert \ j0 \ L) <..\}. \ K = insert \ i \ L)$
using K
by (metis $\langle Max \ K \in K \rangle \ card\text{-Diff}\text{-singleton}\text{-if}$ $diff\text{-Suc}\text{-}1$ $finite\text{-Diff}$ $greaterThan\text{-iff}$ $insert\text{-subset}$)
qed
show $?rhs \subseteq \mathcal{K} \ j0 \ (Suc \ j)$
by (force simp: $\mathcal{K}\text{-def}$ $nsets\text{-def}$ $USigma\text{-def}$)
qed

define BB **where** $BB \equiv \lambda j0 \ j \ K. \ list\text{-of} \ (a \ j0 \cup (\bigcup i < j. \ b \ (enum \ K \ i) \ (j0, i)))$
define XX **where** $XX \equiv \lambda j. \ BB \ j \ j \ ' \ \mathcal{K} \ j \ j$

have $less\text{-list}\text{-of}$: $BB \ j \ i \ K < list\text{-of} \ (b \ l \ (j, i))$
if $K: K \in \mathcal{K} \ j \ i \ \forall j \in K. \ j < l$ **and** $i \leq j \ j \leq l$ **for** $j \ i \ K \ l$
unfolding $BB\text{-def}$
proof (rule $less\text{-sets}\text{-imp}\text{-sorted}\text{-list}\text{-of}\text{-set}$)
have $\bigwedge i. \ i < card \ K \implies b \ (enum \ K \ i) \ (j, i) \ll b \ l \ (j, card \ K)$
using that **by** (metis $\mathcal{K}\text{-card}$ $\mathcal{K}\text{-enum}$ $\mathcal{K}\text{-finite}$ bb $finite\text{-enumerate}\text{-in}\text{-set}$ $nat\text{-less}\text{-le}$ $less\text{-le}\text{-trans}$)
then show $a \ j \cup (\bigcup i < i. \ b \ (enum \ K \ i) \ (j, i)) \ll b \ l \ (j, i)$
using that **unfolding** $\mathcal{K}\text{-def}$ $nsets\text{-def}$
by (auto simp: $less\text{-sets}\text{-Un}1$ $less\text{-sets}\text{-UN}1$ ab $finite\text{-enumerate}\text{-in}\text{-set}$ $subset\text{-eq}$)
qed auto
have $BB\text{-Suc}$: $BB \ j0 \ (Suc \ j) \ K = usplit \ (\lambda L \ k. \ BB \ j0 \ j \ L \ @ \ list\text{-of} \ (b \ k \ (j0, j)))$
 K
if $j: j \leq j0$ **and** $K: K \in \mathcal{K} \ j0 \ (Suc \ j)$ **for** $j0 \ j \ K$
— towards the ordertype proof
proof —
have $Kj: K \subseteq \{j0 <..\}$ **and** [simp]: $finite \ K$ **and** $cardK: card \ K = Suc \ j$

```

    using K by (auto simp: K-def nsets-def)
  have KMK:  $K - \{Max\ K\} \in \mathcal{K}\ j0\ j$ 
    using that by (simp add: K-Suc USigma-iff K-finite less-sets-def usplit-def)
  have  $j0 < Max\ K$ 
    by (metis Kj Max-in cardK card-gt-0-iff greaterThan-iff subsetD zero-less-Suc)
  have MaxK:  $Max\ K = enum\ K\ j$ 
  proof (rule Max-eqI)
    fix k
    assume  $k \in K$ 
    with K cardK show  $k \leq enum\ K\ j$ 
      by (metis ⟨finite K⟩ finite-enumerate-Ex finite-enumerate-mono-iff leI lessI
not-less-eq)
    qed (auto simp: cardK finite-enumerate-in-set)
  have ene:  $i < j \implies enum\ (K - \{enum\ K\ j\})\ i = enum\ K\ i$  for i
    using finite-enumerate-Diff-singleton [OF ⟨finite K⟩] by (simp add: cardK)
  have BB j0 (Suc j)  $K = list-of\ ((a\ j0 \cup (\bigcup_{x < j}. b\ (enum\ K\ x)\ (j0, x))) \cup b$ 
  (enum K j) (j0, j))
    by (simp add: BB-def lessThan-Suc Un-ac)
  also have ... = list-of ((a j0  $\cup (\bigcup_{i < j}. b\ (enum\ K\ i)\ (j0, i))$ )) @ list-of (b
  (enum K j) (j0, j))
  proof (rule sorted-list-of-set-Un)
    have  $b\ (enum\ K\ i)\ (j0, i) \ll b\ (enum\ K\ j)\ (j0, j)$  if  $i < j$  for i
      using K K-enum bb cardK j le-eq-less-or-eq that by auto
    moreover have  $a\ j0 \ll b\ (enum\ K\ j)\ (j0, j)$ 
      using MaxK ⟨j0 < Max K⟩ ab by auto
    ultimately show  $a\ j0 \cup (\bigcup_{x < j}. b\ (enum\ K\ x)\ (j0, x)) \ll b\ (enum\ K\ j)\ (j0,$ 
j)
      by (simp add: less-sets-Un1 less-sets-UN1)
    qed (auto simp: finite-UnI)
  also have ... = BB j0 j (K - {Max K}) @ list-of (b (Max K) (j0, j))
    by (simp add: BB-def MaxK ene)
  also have ... = usplit ( $\lambda L\ k. BB\ j0\ j\ L$  @ list-of (b k (j0, j))) K
    by (simp add: usplit-def)
  finally show ?thesis .
qed

have enum-d-0:  $enum\ (d\ j)\ 0 = Inf\ (d\ j)$  for j
  using enum-0-eq-Inf-finite by auto

have Inf-b-less:  $\prod (b\ k'\ (j', i')) < \prod (b\ k\ (j, i))$ 
  if j:  $i' < j'$   $i < j$   $j' \leq k'$   $j \leq k$  and k:  $k' < k$  for i i' j j' k' k
  using bb [of i' j' k' k j i] that b-ne [of i' j' k'] b-ne [of i j k]
  by (simp add: less-sets-def Inf-nat-def1)

have b-ge-k:  $\prod (b\ k\ (k, k-1)) \geq k-1$  for k
  proof (induction k)
    case (Suc k)
    show ?case
    proof (cases k=0)

```

```

case False
then have  $\prod (b\ k\ (k, k - 1)) < \prod (b\ (Suc\ k)\ (Suc\ k, k))$ 
  using Inf-b-less by auto
with Suc show ?thesis
  by simp
qed auto
qed auto

have b-ge:  $\prod (b\ k\ (j, i)) \geq k - 1$  if  $k \geq j\ j > i$  for  $k\ j\ i$ 
  by (metis Inf-b-less Suc-leI b-ge-k diff-Suc-1 lessI not-less that diff-le-mono)
have hd-b:  $hd\ (list-of\ (b\ k\ (j, i))) = \prod (b\ k\ (j, i))$ 
  if  $i < j\ j \leq k$  for  $k\ j\ i$ 
  using that by (simp add: hd-list-of cInf-eq-Min)

have b-disjoint-less:  $b\ (enum\ K\ i)\ (j0, i) \cap b\ (enum\ K\ i')\ (j0, i') = \{\}$ 
  if  $K: K \subseteq \{j0 <..\}$  finite  $K\ card\ K \geq j0\ i < j\ i' < j\ i \neq i'\ j \leq j0$  for  $i\ i'\ j\ j0\ K$ 
proof (intro bb less-sets-imp-disjnt [unfolded disjnt-def])
  show  $i < j0$ 
    using that by linarith
  then show  $j0 \leq enum\ K\ i$ 
    by (meson K finite-enumerate-in-set greaterThan-iff less-imp-le-nat less-le-trans subsetD)
  show  $enum\ K\ i < enum\ K\ i'$ 
    using  $K\ \langle j \leq j0 \rangle$  that by auto
qed

have b-disjoint:  $b\ (enum\ K\ i)\ (j0, i) \cap b\ (enum\ K\ i')\ (j0, i') = \{\}$ 
  if  $K: K \subseteq \{j0 <..\}$  finite  $K\ card\ K \geq j0\ i < j\ i' < j\ i \neq i'\ j \leq j0$  for  $i\ i'\ j\ j0\ K$ 
  using that b-disjoint-less inf-commute neq-iff by metis

have otw: ordertype  $((\lambda k. list-of\ (b\ k\ (j, i)))\ \text{'}\ \{Max\ (insert\ j\ K) <..\})\ ?LL = \omega$ 
  (is ?lhs = -)
  if  $K: K \in \mathcal{K}\ j\ i\ j > i$  for  $j\ i\ K$ 
proof -
  have Sucj:  $Suc\ (Max\ (insert\ j\ K)) \geq j$ 
    using K-finite that(1) le-Suc-eq by auto
  let ?N =  $\{Inf(b\ k\ (j, i)) \mid k. Max\ (insert\ j\ K) < k\}$ 
  have infN: infinite ?N
  proof (clarsimp simp add: infinite-nat-iff-unbounded-le)
    fix  $m$ 
    show  $\exists n \geq m. \exists k. n = \prod (b\ k\ (j, i)) \wedge Max\ (insert\ j\ K) < k$ 
      using b-ge  $\langle j > i \rangle$  Sucj
    by (metis (no-types, lifting) diff-Suc-1 le-SucI le-trans less-Suc-eq-le nat-le-linear)
  qed
have [simp]:  $Max\ (insert\ j\ K) < k \iff j < k \wedge (\forall a \in K. a < k)$  for  $k$ 
  using that by (auto simp: K-finite)
have ?lhs = ordertype ?N less-than
proof (intro ordertype-eqI strip)
  have  $list-of\ (b\ k\ (j, i)) = list-of\ (b\ k'\ (j, i))$ 

```

if $j \leq k$ **for** k **for** k' $hd (list-of (b k (j,i))) = hd (list-of (b k' (j,i)))$
for $k k'$
by (*metis Inf-b-less* $\langle i < j \rangle$ *hd-b nat-less-le not-le that*)
moreover have $\exists k' j' i'. hd (list-of (b k (j,i))) = \sqcap (b k' (j', i')) \wedge i' < j'$
 $\wedge j' \leq k'$
if $j \leq k$ **for** k
using that $\langle i < j \rangle$ *hd-b less-imp-le-nat* **by** *blast*
moreover have $\exists k'. hd (list-of (b k (j,i))) = \sqcap (b k' (j,i)) \wedge j < k' \wedge$
 $(\forall a \in K. a < k')$
if $j < k \forall a \in K. a < k$ **for** k
using that K *hd-b less-imp-le-nat* **by** *blast*
moreover have $\sqcap (b k (j,i)) \in hd \text{ ' } (\lambda k. list-of (b k (j,i))) \text{ ' } \{Max (insert j$
 $K) < ..\}$
if $j < k \forall a \in K. a < k$ **for** k
using that K **by** (*auto simp: hd-b image-iff*)
ultimately
show *bij-betw* $hd ((\lambda k. list-of (b k (j,i))) \text{ ' } \{Max (insert j K) < ..\}) \{ \sqcap (b k$
 $(j,i)) | k. Max (insert j K) < k \}$
by (*auto simp: bij-betw-def inj-on-def*)
next
fix $ms ns$
assume $ms \in (\lambda k. list-of (b k (j,i))) \text{ ' } \{Max (insert j K) < ..\}$
and $ns \in (\lambda k. list-of (b k (j,i))) \text{ ' } \{Max (insert j K) < ..\}$
with that obtain $k k'$ **where**
 $ms: ms = list-of (b k (j,i))$ **and** $ns: ns = list-of (b k' (j,i))$
and $j < k$ $j < k'$ **and** $lt-k: \forall a \in K. a < k$ **and** $lt-k': \forall a \in K. a < k'$
by (*auto simp: K-finite*)
then have *len-eq [simp]: length ns = length ms*
by (*simp add: card-b*)
have $nz: length ns \neq 0$
using *b-ne* $\langle i < j \rangle \langle j < k' \rangle$ ns **by** *auto*
show $(hd ms, hd ns) \in less-than \longleftrightarrow (ms, ns) \in ?LL$
proof
assume $(hd ms, hd ns) \in less-than$
then show $(ms, ns) \in ?LL$
using that nz
by (*fastforce simp: lenlex-def K-finite card-b intro: hd-lex*)
next
assume $\S: (ms, ns) \in ?LL$
then have $(list-of (b k' (j,i)), list-of (b k (j,i))) \notin ?LL$
using *less-asym ms ns omega-sum-1-less* **by** *blast*
then show $(hd ms, hd ns) \in less-than$
using $\langle j < k \rangle \langle j < k' \rangle$ *Inf-b-less [of i j i j]* $ms ns$
by (*metis Cons-lenlex-iff* \S *len-eq b-ne card-b-finite diff-Suc-1 hd-Cons-tl hd-b*
length-Cons less-or-eq-imp-le less-than-iff linorder-neqE-nat sorted-list-of-set-eq-Nil-iff
that(2))
qed
qed *auto*
also have $\dots = \omega$

```

    using infN ordertype-nat- $\omega$  by blast
  finally show ?thesis .
qed

have otwj: ordertype (BB j0 j '  $\mathcal{K}$  j0 j) ?LL =  $\omega \uparrow j$  if  $j \leq j0$  for  $j j0$ 
  using that
proof (induction j) — a difficult proof, but no hints in Larson's text
  case 0
  then show ?case
    by (auto simp: XX-def)
next
  case (Suc j)
  then have ih: ordertype (BB j0 j '  $\mathcal{K}$  j0 j) ?LL =  $\omega \uparrow j$ 
    by simp
  have  $j \leq j0$ 
    by (simp add: Suc.prem1 Suc-leD)
  have inj-BB: inj-on (BB j0 j) ( $\{\{j0 < ..\}\}^j$ )
proof (clarsimp simp: inj-on-def BB-def nsets-def sorted-list-of-set-Un less-sets-UN2)
  fix X Y
  assume X:  $X \subseteq \{j0 < ..\}$  and Y:  $Y \subseteq \{j0 < ..\}$ 
    and finite X finite Y
    and jeq:  $j = \text{card } X$ 
    and card Y = card X
    and eq: list-of (a j0  $\cup$  ( $\bigcup_{i < \text{card } X} b (\text{enum } X i) (j0, i)$ ))
      = list-of (a j0  $\cup$  ( $\bigcup_{i < \text{card } X} b (\text{enum } Y i) (j0, i)$ ))
  have enumX:  $\bigwedge n. \llbracket n < \text{card } X \rrbracket \implies j0 \leq \text{enum } X n$ 
    using X  $\langle \text{finite } X \rangle$  finite-enumerate-in-set less-imp-le-nat by blast
  have enumY:  $\bigwedge n. \llbracket n < \text{card } X \rrbracket \implies j0 \leq \text{enum } Y n$ 
    using subsetD [OF Y]
  by (metis  $\langle \text{card } Y = \text{card } X \rangle \langle \text{finite } Y \rangle$  finite-enumerate-in-set greaterThan-iff
less-imp-le-nat)
  have smX: strict-mono-sets  $\{.. < \text{card } X\}$  ( $\lambda i. b (\text{enum } X i) (j0, i)$ )
    and smY: strict-mono-sets  $\{.. < \text{card } X\}$  ( $\lambda i. b (\text{enum } Y i) (j0, i)$ )
    using Suc.prem1  $\langle \text{card } Y = \text{card } X \rangle \langle \text{finite } X \rangle \langle \text{finite } Y \rangle$  bb enumX enumY
  jeq
    by (auto simp: strict-mono-sets-def)

  have len-eq: length ms = length ns
    if (ms, ns)  $\in$  list.set (zip (map (list-of  $\circ$  ( $\lambda i. b (\text{enum } X i) (j0, i)$ ))) (list-of
 $\{.. < n\}$ ))
      (map (list-of  $\circ$  ( $\lambda i. b (\text{enum } Y i) (j0, i)$ ))) (list-of
 $\{.. < n\}$ ))
      n  $\leq$  card X
    for ms ns n
    using that
  by (induction n rule: nat.induct) (auto simp: card-b enumX enumY)
  have concat (map (list-of  $\circ$  ( $\lambda i. b (\text{enum } X i) (j0, i)$ ))) (list-of  $\{.. < \text{card } X\}$ )
    = concat (map (list-of  $\circ$  ( $\lambda i. b (\text{enum } Y i) (j0, i)$ ))) (list-of  $\{.. < \text{card } X\}$ )
    using eq

```


by (*simp add: sorted-list-of-set-Un less-sets-UN2 sorted-list-of-set-UN-lessThan*
ab enumX enumY smX smY)
then have *map-eq: map (list-of ∘ (λi. b (enum X i) (j0, i))) (list-of {..<card*
X})

$$= \text{map } (\text{list-of} \circ (\lambda i. b (\text{enum } Y \ i) (j0, i))) (\text{list-of } \{..<\text{card } X\})$$
by (*rule concat-injective*) (*auto simp: len-eq split: prod.split*)
have *enum X i = enum Y i if i < card X for i*
proof –
have *Inf (b (enum X i) (j0,i)) = Inf (b (enum Y i) (j0,i))*
using *iffD1 [OF map-eq-conv, OF map-eq] Suc.premis that*
by (*metis (mono-tags, lifting) card-b-finite comp-apply finite-lessThan*
lessThan-iff set-sorted-list-of-set)
moreover have *Inf (b (enum X i) (j0,i)) ∈ (b (enum X i) (j0,i))*
Inf (b (enum Y i) (j0,i)) ∈ (b (enum Y i) (j0,i)) i < j0
using *Inf-nat-def1 Suc.premis b-ne enumX enumY jeq that by auto*
ultimately show *?thesis*
by (*metis Inf-b-less enumX enumY leI nat-less-le that*)
qed
then show *X = Y*
by (*simp add: <card Y = card X> <finite X> <finite Y> finite-enum-ext*)
qed
have *BB-Suc': BB j0 (Suc j) X = usplit (λL k. BB j0 j L @ list-of (b k (j0,*
j))) X
if *X ∈ USigma (K j0 j) (λK. {Max (insert j0 K)<..}) for X*
using *that*
by (*simp add: USigma-iff K-finite less-sets-def usplit-def K-Suc BB-Suc <j ≤*
j0>)
have *ordertype (BB j0 (Suc j) 'K j0 (Suc j)) ?LL*

$$= \text{ordertype}$$

$$(\text{usplit } (\lambda L k. \text{BB } j0 \ j \ L \ @ \ \text{list-of } (b \ k \ (j0, \ j))) \ ' \ \text{USigma } (K \ j0 \ j) \ (\lambda K. \ \{Max \ (insert \ j0 \ K) \ <.. \})) \ ?LL$$
by (*simp add: BB-Suc' K-Suc*)
also have $\dots = \omega * \text{ordertype } (BB \ j0 \ j \ ' \ K \ j0 \ j) \ ?LL$
proof (*intro ordertype-append-image-IJ*)
fix *L k*
assume *L ∈ K j0 j and k ∈ {Max (insert j0 L)<..}*
then have *j0 < k and L: ∧a. a ∈ L ⇒ a < k*
by (*simp-all add: K-finite*)
then show *BB j0 j L < list-of (b k (j0, j))*
by (*simp add: <L ∈ K j0 j> <j ≤ j0> K-finite less-list-of*)
next
show *inj-on (BB j0 j) (K j0 j)*
by (*simp add: K-def inj-BB*)
next
fix *L*
assume *L: L ∈ K j0 j*
then show *L ≪ {Max (insert j0 L)<..} ∧ finite L*
by (*simp add: K-finite less-sets-def*)
show *ordertype ((λi. list-of (b i (j0, j))) ' {Max (insert j0 L)<..}) ?LL = ω*

```

    using L Suc.premS Suc-le-lessD otw by blast
  qed (auto simp: K-finite card-b)
  also have ... =  $\omega \uparrow \text{ord-of-nat } (\text{Suc } j)$ 
    by (simp add: oexp-mult-commute ih)
  finally show ?case .
qed

define seqs where seqs  $\equiv \lambda j0 j K. \text{list-of } (a j0) \# (\text{map } (\text{list-of } \circ (\lambda i. b (\text{enum } K i) (j0, i))) (\text{list-of } \{..<j\}))$ 

have length-seqs [simp]:  $\text{length } (\text{seqs } j0 j K) = \text{Suc } j$  for  $j0 j K$ 
  by (simp add: seqs-def)

have BB-eq-concat-seqs:  $BB j0 j K = \text{concat } (\text{seqs } j0 j K)$ 
  and seqs-ne:  $\text{seqs } j0 j K \in \text{lists } (- \{\{\}\})$ 
  if  $K: K \in \mathcal{K} j0 j$  and  $j \leq j0$  for  $K j j0$ 
proof -
  have  $j0: \bigwedge i. i < \text{card } K \implies j0 \leq \text{enum } K i$  and  $le-j0: \text{card } K \leq j0$ 
    using finite-enumerate-in-set that unfolding K-def nsets-def by fastforce+
  show  $BB j0 j K = \text{concat } (\text{seqs } j0 j K)$ 
    using that unfolding BB-def K-def nsets-def seqs-def
    by (fastforce simp:  $j0$  ab bb less-sets-UN2 sorted-list-of-set-Un
      strict-mono-sets-def sorted-list-of-set-UN-lessThan)
  have  $b (\text{enum } K i) (j0, i) \neq \{\}$  if  $i < \text{card } K$  for  $i$ 
    using  $j0$  le-j0 less-le-trans that by simp
  moreover have  $\text{card } K = j$ 
    using K K-card by blast
  ultimately show  $\text{seqs } j0 j K \in \text{lists } (- \{\{\}\})$ 
    by (clarsimp simp: seqs-def) (metis card-b-finite sorted-list-of-set-eq-Nil-iff)
qed

have BB-decomp:  $\exists cs. BB j0 j K = \text{concat } cs \wedge cs \in \text{lists } (- \{\{\}\})$ 
  if  $K: K \in \mathcal{K} j0 j$  and  $j \leq j0$  for  $K j j0$ 
  using BB-eq-concat-seqs seqs-ne K that(2) by blast

have a-subset-M:  $a k \subseteq M k$  for  $k$ 
  apply (clarsimp simp: a-def M-def DF-simps F-def Let-def split: prod.split-asm)
  by (metis (no-types) fst-conv fst-grab-subset nst-subset snd-conv snd-grab-subset
    subsetD)
have ba-Suc:  $b k (j, i) \ll a (\text{Suc } k)$  if  $i < j j \leq k$  for  $i j k$ 
  by (meson a-subset-M bM less-sets-weaken2 nat-less-le that)
have ba:  $b k (j, i) \ll a r$  if  $i < j j \leq k k < r$  for  $i j k r$ 
  by (metis Suc-lessI a-ne aa ba-Suc less-sets-trans that)

have disjnt-ba:  $\text{disjnt } (b k (j, i)) (a r)$  if  $i < j j \leq k$  for  $i j k r$ 
  by (meson ab ba disjnt-sym less-sets-imp-disjnt not-le that)

have bb-disjnt:  $\text{disjnt } (b k (j, i)) (b l (r, q))$ 
  if  $q < r i < j j \leq k r \leq l j < r$  for  $i j q r k l$ 

```

```

proof (cases k=l)
  case True
    with that show ?thesis
      by (force simp: pair-less-def IJ-def intro: bb-same less-sets-imp-disjnt)
  next
    case False
      with that show ?thesis
        by (metis bb less-sets-imp-disjnt disjnt-sym nat-neq-iff)
qed

have sum-card-b:  $(\sum i < j. \text{card } (b \text{ (enum } K \ i) \ (j0, \ i))) = \text{enum } (d \ j0) \ j - \text{enum } (d \ j0) \ 0$ 
  if  $K: K \subseteq \{j0 < ..\}$  finite  $K$   $\text{card } K \geq j0$  and  $j \leq j0$  for  $j0 \ j \ K$ 
  using  $\langle j \leq j0 \rangle$ 
proof (induction j)
  case 0
    then show ?case
      by auto
  next
    case (Suc j)
      then have  $j < \text{card } K$ 
        using that(3) by linarith
      have dis:  $\text{disjnt } (b \text{ (enum } K \ j) \ (j0, \ j)) \ (\bigcup i < j. b \text{ (enum } K \ i) \ (j0, \ i))$ 
        unfolding disjoint-UN-iff
      by (meson Suc.prem1s b-disjoint-less disjnt-def disjnt-sym lessThan-iff less-Suc-eq that)
      have j0-less:  $j0 < \text{enum } K \ j$ 
        using  $K \ \langle j < \text{card } K \rangle$  by (force simp: finite-enumerate-in-set)
      have  $(\sum i < \text{Suc } j. \text{card } (b \text{ (enum } K \ i) \ (j0, \ i))) = \text{card } (b \text{ (enum } K \ j) \ (j0, \ j)) + (\sum i < j. \text{card } (b \text{ (enum } K \ i) \ (j0, \ i)))$ 
        by (simp add: lessThan-Suc card-Un-disjnt [OF - - dis])
      also have  $\dots = \text{card } (b \text{ (enum } K \ j) \ (j0, \ j)) + \text{enum } (d \ j0) \ j - \text{enum } (d \ j0) \ 0$ 
        using  $\langle \text{Suc } j \leq j0 \rangle$  by (simp add: Suc.IH split: nat-diff-split)
      also have  $\dots = \text{enum } (d \ j0) \ (\text{Suc } j) - \text{enum } (d \ j0) \ 0$ 
        using j0-less Suc.prem1s card-b less-or-eq-imp-le by force
      finally show ?case .
qed

have card-UN-b:  $\text{card } (\bigcup i < j. b \text{ (enum } K \ i) \ (j0, \ i)) = \text{enum } (d \ j0) \ j - \text{enum } (d \ j0) \ 0$ 
  if  $K: K \subseteq \{j0 < ..\}$  finite  $K$   $\text{card } K \geq j0$  and  $j \leq j0$  for  $j0 \ j \ K$ 
  using that by (simp add: card-UN-disjnt sum-card-b b-disjoint)

have len-BB:  $\text{length } (BB \ j \ j \ K) = \text{enum } (d \ j) \ j$ 
  if  $K: K \in \mathcal{K} \ j \ j$  and  $j \leq j$  for  $j \ K$ 
proof -
  have dis-ab:  $\bigwedge i. i < j \implies \text{disjnt } (a \ j) \ (b \text{ (enum } K \ i) \ (j, \ i))$ 
    using  $K \ \mathcal{K}\text{-card } \mathcal{K}\text{-enum } ab \ \text{less-sets-imp-disjnt } \text{nat-less-le}$  by blast
  show ?thesis

```

```

    using K unfolding BB-def K-def nsets-def
    by (simp add: card-UN-b card-Un-disjnt dis-ab card-a cInf-le-finite finite-enumerate-in-set
enum-0-eq-Inf-finite)
qed

have d k << d (Suc k) for k
  by (metis aM a-ne d-eq da less-sets-fst-grab less-sets-trans less-sets-weaken2
next-subset)
then have dd: d k' << d k if k' < k for k' k
  by (meson UNIV-I d-ne less-sets-imp-strict-mono-sets strict-mono-sets-def that)

show thesis
proof
  show (⋃ (range XX)) ⊆ WW
    by (auto simp: XX-def BB-def WW-def)
  show ordertype (⋃ (range XX)) (?LL) = ω ↑ ω
    using otwj by (simp add: XX-def ordertype-ωω)
next
fix U
assume U: U ∈ [⋃ (range XX)]2
then obtain x y where Ueq: U = {x,y} and len-xy: length x ≤ length y
  by (auto simp: lenlex-nsets-2-eq lenlex-length)

  show ∃ l. Form l U ∧ (0 < l ⟶ [enum N l] < inter-scheme l U ∧ list.set
(inter-scheme l U) ⊆ N)
  proof (cases length x = length y)
    case True
    then show ?thesis
      using Form.intros(1) U Ueq by fastforce
  next
  case False
  then have xy: length x < length y
    using len-xy by auto
  obtain j r K L where K: K ∈ K j j and xeq: x = BB j j K and ne: BB j j
K ≠ BB r r L
    and L: L ∈ K r r and yeq: y = BB r r L
    using U by (auto simp: Ueq XX-def)
  then have length x = enum (d j) j length y = enum (d r) r
    by (auto simp: len-BB)
  then have j < r
    using xy dd
    by (metis card-d finite-enumerate-in-set finite-d lessI less-asym less-setsD
linorder-neqE-nat)
  then have aj-ar: a j << a r
    using aa by auto
  have Ksub: K ⊆ {j<..} and finite K card K ≥ j
    using K by (auto simp: K-def nsets-def)
  have Lsub: L ⊆ {r<..} and finite L card L ≥ r
    using L by (auto simp: K-def nsets-def)

```

```

have enumK: enum K i > j if i < j for i
  using K K-card K-enum that by blast
have enumL: enum L i > r if i < r for i
  using L K-card K-enum that by blast
have list.set (acc-lengths w (seqs j0 j K))  $\subseteq$  (+) w ‘ d j0
  if K: K  $\subseteq$  {j0<..} finite K card K  $\geq$  j0 and j  $\leq$  j0 for j0 j K w
  using ⟨j  $\leq$  j0⟩
proof (induction j arbitrary: w)
  case 0
  then show ?case
    by (simp add: seqs-def Inf-nat-def1 card-a)
  next
  case (Suc j)
  let ?db =  $\prod$  (d j0) + (( $\sum$  i<j. card (b (enum K i) (j0,i))) + card (b (enum
K j) (j0,j)))
  have j0 < enum K j
    by (meson Suc.premS Suc-le-lessD finite-enumerate-in-set greaterThan-iff
le-trans subsetD K)
  then have enum (d j0) j  $\geq$   $\prod$  (d j0)
  using Suc.premS card-d by (simp add: cInf-le-finite finite-enumerate-in-set)
  then have ?db = enum (d j0) (Suc j)
  using Suc.premS that
  by (simp add: cInf-le-finite finite-enumerate-in-set sum-card-b card-b
enum-d-0 ⟨j0 < enum K j⟩ less-or-eq-imp-le)
  then have ?db  $\in$  d j0
  using Suc.premS finite-enumerate-in-set by (auto simp: finite-enumerate-in-set)
  moreover have list.set (acc-lengths w (seqs j0 j K))  $\subseteq$  (+) w ‘ d j0
  by (simp add: Suc Suc-leD)
  then have list.set (acc-lengths (w +  $\prod$  (d j0))
    (map (list-of  $\circ$  ( $\lambda$ i. b (enum K i) (j0,i))) (list-of {..<j})))
     $\subseteq$  (+) w ‘ d j0
  by (simp add: seqs-def card-a subset-insertI)
  ultimately show ?case
  by (simp add: seqs-def acc-lengths-append image-iff Inf-nat-def1
sum-sorted-list-of-set-map card-a)
qed
then have acc-lengths-subset-d: list.set (acc-lengths 0 (seqs j0 j K))  $\subseteq$  d j0
  if K: K  $\subseteq$  {j0<..} finite K card K  $\geq$  j0 and j  $\leq$  j0 for j0 j K
  by (metis image-add-0 that)

have strict-sorted x strict-sorted y
  by (auto simp: xeq yeq BB-def)
have disjnt-xy: disjnt (list.set x) (list.set y)
proof –
  have disjnt (a j) (a r)
    using ⟨j < r⟩ aa less-sets-imp-disjnt by blast
  moreover have disjnt (b (enum K i) (j,i)) (a r) if i < j for i
    by (simp add: disjnt-ba enumK less-imp-le-nat that)
  moreover have disjnt (a j) (b (enum L q) (r,q)) if q < r for q

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```

    by (meson disjnt-ba disjnt-sym enumL less-imp-le-nat that)
  moreover have disjnt (b (enum K i) (j,i)) (b (enum L q) (r,q)) if i < j q
< r for i q
  by (meson ⟨j < r⟩ bb-disjnt enumK enumL less-imp-le that)
ultimately show ?thesis
  by (simp add: xeq yeq BB-def)
qed
have ∃ us vs. merge (seqs j j K) (seqs r r L) us vs
proof (rule merge-exists)
  show strict-sorted (concat (seqs j j K))
    using BB-eq-concat-seqs K ⟨strict-sorted x⟩ xeq by auto
  show strict-sorted (concat (seqs r r L))
    using BB-eq-concat-seqs L ⟨strict-sorted y⟩ yeq by auto
  show seqs j j K ∈ lists (− {[]}) seqs r r L ∈ lists (− {[]})
    by (auto simp: K L seqs-ne)
  show hd (seqs j j K) < hd (seqs r r L)
    by (simp add: aj-ar less-sets-imp-list-less seqs-def)
  show seqs j j K ≠ [] seqs r r L ≠ []
    using seqs-def by blast+
  have less-bb: b (enum K i) (j,i) ≪ b (enum L p) (r, p)
    if ¬ b (enum L p) (r, p) ≪ b (enum K i) (j,i) and i < j p < r
    for i p
    by (metis IJ-iff ⟨j < r⟩ bb bb-same enumK enumL less-imp-le-nat
linorder-neqE-nat pair-lessI1 that)
  show u < v ∨ v < u
    if u ∈ list.set (seqs j j K) and v ∈ list.set (seqs r r L) for u v
    using that enumK enumL unfolding seqs-def
    apply (auto simp: seqs-def aj-ar intro!: less-bb less-sets-imp-list-less)
    apply (meson ab ba less-imp-le-nat not-le)+
    done
qed
then obtain uus vvs where merge: merge (seqs j j K) (seqs r r L) uus vvs
  by metis
then have uus ≠ []
  using merge-length1-gt-0 by (auto simp: seqs-def)
then obtain u1 us where us: u1 # us = uus
  by (metis neq-Nil-conv)
define ku where ku ≡ length (u1 # us)
define ps where ps ≡ acc-lengths 0 (u1 # us)
have us-ne: u1 # us ∈ lists (− {[]})
  using merge-length1-nonempty seqs-ne us merge us K by auto
have xu-eq: x = concat (u1 # us)
  using BB-eq-concat-seqs K merge merge-preserves us xeq by auto
then have strict-sorted u1
  using ⟨strict-sorted x⟩ strict-sorted-append-iff by auto
have u-sub: list.set ps ⊆ list.set (acc-lengths 0 (seqs j j K))
  using acc-lengths-merge1 merge ps-def us by blast
have vvs ≠ []
  using merge BB-eq-concat-seqs L merge-preserves xy yeq by auto

```

```

then obtain  $v1\ vs$  where  $vs: v1\ \#vs = vvs$ 
  by (metis neq-Nil-conv)
define  $kv$  where  $kv \equiv length\ (v1\ \#vs)$ 
define  $qs$  where  $qs \equiv acc-lengths\ 0\ (v1\ \#vs)$ 
have  $vs-ne: v1\ \#vs \in lists\ (-\ \{\ \})$ 
  using  $L\ merge\ merge-length2-nonempty\ seqs-ne\ vs$  by auto
have  $yv-eq: y = concat\ (v1\ \#vs)$ 
  using  $BB-eq-concat-seqs\ L\ merge\ merge-preserves\ vs\ yeq$  by auto
then have strict-sorted  $v1$ 
  using  $\langle strict-sorted\ y \rangle\ strict-sorted-append-iff$  by auto
have  $v-sub: list.set\ qs \subseteq list.set\ (acc-lengths\ 0\ (seqs\ r\ r\ L))$ 
  using  $acc-lengths-merge2\ merge\ qs-def\ vs$  by blast

have  $ss-concat-jj: strict-sorted\ (concat\ (seqs\ j\ j\ K))$ 
  using  $BB-eq-concat-seqs\ K\ \langle strict-sorted\ x \rangle\ xeq$  by auto
then obtain  $k: 0 < kv\ kv \leq ku\ ku \leq Suc\ kv\ kv \leq Suc\ j$ 
  using  $us\ vs\ merge-length-le\ merge-length-le-Suc\ merge-length-less2\ merge$ 
  unfolding  $ku-def\ kv-def$  by fastforce

define  $zs$  where  $zs \equiv concat\ [ps, u1, qs, v1] @ interact\ us\ vs$ 
have  $ss: strict-sorted\ zs$ 
proof –
  have  $ssp: strict-sorted\ ps$ 
    unfolding  $ps-def$  by (meson strict-sorted-acc-lengths us-ne)
  have  $ssq: strict-sorted\ qs$ 
    unfolding  $qs-def$  by (meson strict-sorted-acc-lengths vs-ne)

  have  $d\ j \ll list.set\ x$ 
    using  $da\ [of\ j]\ db\ [of\ j]\ K\ \mathcal{K}\text{-card}\ \mathcal{K}\text{-enum}\ nat\text{-less-le}$ 
    by (auto simp: xeq BB-def less-sets-Un2 less-sets-UN2)
  then have  $ac-x: acc-lengths\ 0\ (seqs\ j\ j\ K) < x$ 
    by (meson Ksub  $\langle finite\ K \rangle\ \langle j \leq card\ K \rangle\ acc-lengths-subset-d\ le-refl$ 
less-sets-imp-list-less less-sets-weaken1)
  then have  $ps < x$ 
    by (meson Ksub  $\langle d\ j \ll list.set\ x \rangle\ \langle finite\ K \rangle\ \langle j \leq card\ K \rangle\ acc-lengths-subset-d$ 
le-refl less-sets-imp-list-less less-sets-weaken1 u-sub)
  then have  $ps < u1$ 
    by (metis Nil-is-append-conv concat.simps(2) hd-append2 less-list-def xu-eq)

  have  $d\ r \ll list.set\ y$ 
    using  $da\ [of\ r]\ db\ [of\ r]\ L\ \mathcal{K}\text{-card}\ \mathcal{K}\text{-enum}\ nat\text{-less-le}$ 
    by (auto simp: yeq BB-def less-sets-Un2 less-sets-UN2)
  then have  $acc-lengths\ 0\ (seqs\ r\ r\ L) < y$ 
    by (meson Lsub  $\langle finite\ L \rangle\ \langle r \leq card\ L \rangle\ acc-lengths-subset-d\ le-refl$ 
less-sets-imp-list-less less-sets-weaken1)
  then have  $qs < y$ 
    by (metis L Lsub  $\mathcal{K}\text{-card}\ \langle d\ r \ll list.set\ y \rangle\ \langle finite\ L \rangle\ acc-lengths-subset-d$ 
less-sets-imp-list-less less-sets-weaken1 order-refl v-sub)
  then have  $qs < v1$ 

```

by (*metis concat.simps(2) gr-implies-not0 hd-append2 less-list-def list.size(3) xy yv-eq*)

have *carda-v1*: $\text{card } (a \ r) \leq \text{length } v1$
using *length-hd-merge2 [OF merge] unfolding vs [symmetric]* **by** (*simp add: seqs-def*)

have *ab-enumK*: $\bigwedge i. i < j \implies a \ j \ll b \ (\text{enum } K \ i) \ (j, i)$
by (*meson ab enumK le-trans less-imp-le-nat*)

have *ab-enumL*: $\bigwedge q. q < r \implies a \ j \ll b \ (\text{enum } L \ q) \ (r, q)$
by (*meson <j < r> ab enumL le-trans less-imp-le-nat*)

then have *ay*: $a \ j \ll \text{list.set } y$
by (*auto simp: yeq BB-def less-sets-Un2 less-sets-UN2 aj-ar*)

have *disjnt-hd-last-K-y*: $\text{disjnt } \{\text{hd } l.. \text{last } l\} \ (\text{list.set } y)$
if $l \in \text{list.set } (\text{seqs } j \ j \ K)$ **for** l

proof (*clarsimp simp add: yeq BB-def disjnt-iff Ball-def, intro conjI strip*)
fix u
assume $u \leq \text{last } l$ **and** $\text{hd } l \leq u$
with l **consider** $u \leq \text{last } (\text{list-of } (a \ j)) \ \text{hd } (\text{list-of } (a \ j)) \leq u$
| i **where** $i < j \ u \leq \text{last } (\text{list-of } (b \ (\text{enum } K \ i) \ (j, i))) \ \text{hd } (\text{list-of } (b \ (\text{enum } K \ i) \ (j, i))) \leq u$

by (*force simp: seqs-def*)
note *l-cases = this*
then show $u \notin a \ r$

proof *cases*
case 1
then show *?thesis*
by (*metis a-ne aj-ar finite-a last-in-set leD less-setsD set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff*)

next
case 2
then show *?thesis*
by (*metis enumK ab ba Inf-nat-def1 b-ne card-b-finite hd-b last-in-set less-asym less-setsD not-le set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff*)

qed
fix q
assume $q < r$
show $u \notin b \ (\text{enum } L \ q) \ (r, q)$
using *l-cases*

proof *cases*
case 1
then show *?thesis*
by (*metis <q < r> a-ne ab-enumL finite-a last-in-set leD less-setsD set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff*)

next
case 2
show *?thesis*
proof (*cases enum K i = enum L q*)


```

    case True
    then show ?thesis
      using 2 bb-same [of concl: enum L q j i r q] ⟨j < r⟩ u
      by (metis JJ-iff b-ne card-b-finite enumK last-in-set leD less-imp-le-nat
less-setsD pair-lessI1 set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff)
    next
    case False
    with 2 bb enumK enumL show ?thesis
      unfolding less-sets-def
      by (metis ⟨q < r⟩ b-ne card-b-finite last-in-set leD less-imp-le-nat
list.set-sel(1) nat-neq-iff set-sorted-list-of-set sorted-list-of-set-eq-Nil-iff)
    qed
  qed
qed

have u1-y: list.set u1 ≪ list.set y
  using vs yv-eq L ⟨strict-sorted y⟩ merge merge-less-sets-hd merge-preserves
seqs-ne ss-concat-jj us by fastforce
have u1-subset-seqs: list.set u1 ⊆ list.set (concat (seqs j j K))
  using merge-preserves [OF merge] us by auto

have b k (j,i) ≪ d (Suc k) if j ≤ k i < j for k j i
  by (metis bM d-eq less-sets-fst-grab less-sets-weaken2 nxt-subset that)
then have bd: b k (j,i) ≪ d k' if j ≤ k i < j k < k' for k k' j i
  by (metis Suc-lessI d-ne dd less-sets-trans that)

have a k ≪ d (Suc k) for k
  by (metis aM d-eq less-sets-fst-grab less-sets-weaken2 nxt-subset)
then have ad: a k ≪ d k' if k < k' for k k'
  by (metis Suc-lessI d-ne dd less-sets-trans that)

have u1 < y
  by (simp add: u1-y less-sets-imp-list-less)
have n < Inf (d r) if n: n ∈ list.set u1 for n
proof -
  obtain l where l: l ∈ list.set (seqs j j K) and n: n ∈ list.set l
    using n u1-subset-seqs by auto
  then consider l = list-of (a j) | i where l = list-of (b (enum K i) (j,i))
i < j
    by (force simp: seqs-def)
  then show ?thesis
proof cases
  case 1
  then show ?thesis
by (metis Inf-nat-def1 ⟨j < r⟩ ad d-ne finite-a less-setsD n set-sorted-list-of-set)
next
  case 2
  then have hd (list-of (b (enum K i) (j,i))) = Min (b (enum K i) (j,i))
    by (meson b-ne card-b-finite enumK hd-list-of less-imp-le-nat)

```

also have $\dots \leq n$
using $2\ n$ **by** (*simp add: less-list-def disjnt-iff less-sets-def*)
also have $f8: n < \text{hd } y$
using *less-setsD that u1-y*
by (*metis gr-implies-not0 list.set-sel(1) list.size(3) xy*)
finally have $l < y$
using $2\ \text{disjnt-hd-last-}K\text{-}y$ [*OF l*]
by (*simp add: disjnt-iff*) (*metis leI less-imp-le-nat less-list-def*
list.set-sel(1))
moreover have $\text{last } (\text{list-of } (b\ (\text{enum } K\ i)\ (j,i))) < \text{hd } (\text{list-of } (a\ r))$
using $\langle l < y \rangle\ L\ n$ **by** (*auto simp: 2yeq BB-eq-concat-seqs seqs-def*
less-list-def)
then have $\text{enum } K\ i < r$
by (*metis 2(1) a-ne ab card-b-finite empty-iff finite.emptyI finite-a*
last-in-set leI less-asm less-setsD list.set-sel(1) n set-sorted-list-of-set)
moreover have $j \leq \text{enum } K\ i$
by (*simp add: 2(2) enumK less-imp-le-nat*)
ultimately show *?thesis*
using $2\ n\ \text{bd}$ [*of j enum K i i r*] *Inf-nat-def1 less-setsD* **by** *force*
qed
then have $\text{last } u1 < \text{Inf } (d\ r)$
using $\langle u1 \neq [] \rangle\ \text{us-ne}$ **by** *auto*
also have $\dots \leq \text{length } v1$
using *card-a carda-v1* **by** *auto*
finally have $\text{last } u1 < \text{length } v1$.
then have $u1 < qs$
by (*simp add: qs-def less-list-def*)

have *strict-sorted* (*interact* ($u1 \# us$) ($v1 \# vs$))
using $L\ \langle \text{strict-sorted } x \rangle\ \langle \text{strict-sorted } y \rangle$ *merge merge-interact merge-preserves*
seqs-ne us vs xu-eq yv-eq **by** *auto*
then have *strict-sorted* (*interact us vs*) $v1 < \text{interact } us\ vs$
by (*auto simp: strict-sorted-append-iff*)
moreover have $ps < u1\ @\ qs\ @\ v1\ @\ \text{interact } us\ vs$
using $\langle ps < u1 \rangle\ \text{us-ne}$ **unfolding** *less-list-def* **by** *auto*
moreover have $u1 < qs\ @\ v1\ @\ \text{interact } us\ vs$
by (*metis* $\langle u1 < qs \rangle\ \langle v1 \neq [] \rangle\ \text{acc-lengths-eq-Nil-iff hd-append less-list-def}$
qs-def vs)
moreover have $qs < v1\ @\ \text{interact } us\ vs$
using $\langle qs < v1 \rangle\ \text{us-ne}\ \langle \text{last } u1 < \text{length } v1 \rangle\ \text{vs-ne}$ **by** (*auto simp:*
less-list-def)
ultimately show *?thesis*
by (*simp add: zs-def strict-sorted-append-iff ssp ssq* $\langle \text{strict-sorted } u1 \rangle$
 $\langle \text{strict-sorted } v1 \rangle$)
qed
have *ps-subset-d: list.set ps* $\subseteq d\ j$
using $K\ Ksub\ \mathcal{K}\text{-card}$ $\langle \text{finite } K \rangle\ \text{acc-lengths-subset-d } u\text{-sub}$ **by** *blast*
have *ps-less-u1: ps* $< u1$

```

    by (metis append.assoc concat.simps(2) ss strict-sorted-append-iff zs-def)
  have qs-subset-d: list.set qs  $\subseteq$  d r
    using L Lsub K-card ⟨finite L⟩ acc-lengths-subset-d v-sub by blast
  have qs-less-v1: qs < v1
    by (metis append.assoc concat.simps(2) ss strict-sorted-append-iff zs-def)
  have FB: Form-Body ku kv x y zs
    unfolding Form-Body.simps ku-def kv-def
    using ps-def qs-def ss us-ne vs-ne xu-eq xy yv-eq zs-def by blast
  then have zs = (inter-scheme ((ku+kv) - Suc 0) {x,y})
    by (simp add: Form-Body-imp-inter-scheme k)
  obtain l where l ≤ 2 * (Suc j) and l: Form l U and zs-eq-interact: zs =
inter-scheme l {x,y}
  proof
    show ku+kv-1 ≤ 2 * (Suc j)
      using k by auto
    show Form (ku+kv-1) U
    proof (cases ku=kv)
      case True
        then show ?thesis
          using FB Form.simps Ueq ⟨0 < kv⟩ by (auto simp: mult-2)
      next
        case False
          then have ku = Suc kv
            using k by auto
          then show ?thesis
            using FB Form.simps Ueq ⟨0 < kv⟩ by auto
    qed
    show zs = inter-scheme (ku + kv - 1) {x, y}
      using Form-Body-imp-inter-scheme by (simp add: FB k)
    qed
  then have enum N l ≤ enum N (Suc (2 * Suc j))
    by (simp add: assms less-imp-le-nat)
  also have ... < Min (d j)
  by (smt (verit, best) Min-gr-iff d-eq d-ne finite-d fst-grab-subset greaterThan-iff
in-mono le-inf-iff nxt-def)
  finally have ls: {enum N l}  $\ll$  d j
    by simp
  have l > 0
    by (metis l False Form-0-cases-raw Set.doubleton-eq-iff Ueq gr0I)
  show ?thesis
    unfolding Ueq
  proof (intro exI conjI impI)
    have zs-subset: list.set zs  $\subseteq$  list.set (acc-lengths 0 (seqs j j K))  $\cup$  list.set
(acc-lengths 0 (seqs r r L))  $\cup$  list.set x  $\cup$  list.set y
      using u-sub v-sub by (auto simp: zs-def xu-eq yv-eq)
    also have ...  $\subseteq$  N
  proof (simp, intro conjI)
    show list.set (acc-lengths 0 (seqs j j K))  $\subseteq$  N
      using d-subset-N Ksub ⟨finite K⟩ ⟨j ≤ card K⟩ acc-lengths-subset-d by

```

```

blast
  show list.set (acc-lengths 0 (seqs r r L))  $\subseteq$  N
    using d-subset-N Lsub  $\langle$ finite L $\rangle$   $\langle$ r  $\leq$  card L $\rangle$  acc-lengths-subset-d by
blast
  show list.set x  $\subseteq$  N list.set y  $\subseteq$  N
    by (simp-all add: xeq yeq BB-def a-subset-N UN-least b-subset-N)
qed
finally show list.set (inter-scheme l {x, y})  $\subseteq$  N
  using zs-eq-interact by blast
have [enum N l] < ps
  using ps-subset-d ls
  by (metis empty-set less-sets-imp-list-less less-sets-weaken2 list.simps(15))
then show [enum N l] < inter-scheme l {x, y}
  by (simp add: zs-def less-list-def ps-def flip: zs-eq-interact)
qed (use Ueq l in blast)
qed
qed
qed

```

3.12 The main partition theorem for $\omega \uparrow \omega$

definition *iso-ll* where $iso-ll\ A\ B \equiv iso\ (lenlex\ less-than \cap (A \times A))\ (lenlex\ less-than \cap (B \times B))$

corollary *ordertype-eq-ordertype-iso-ll*:

assumes $Field\ (Restr\ (lenlex\ less-than)\ A) = A\ Field\ (Restr\ (lenlex\ less-than)\ B) = B$

shows $(ordertype\ A\ (lenlex\ less-than) = ordertype\ B\ (lenlex\ less-than))$
 $\longleftrightarrow (\exists f. iso-ll\ A\ B\ f)$

proof –

have $total-on\ A\ (lenlex\ less-than) \wedge total-on\ B\ (lenlex\ less-than)$

by (*meson UNIV-I total-lenlex total-on-def total-on-less-than*)

then show *?thesis*

by (*simp add: assms wf-lenlex lenlex-transI iso-ll-def ordertype-eq-ordertype-iso-Restr*)

qed

theorem *partition- $\omega\omega$ -aux*:

assumes $\alpha \in elts\ \omega$

shows $partn-lst\ (lenlex\ less-than)\ WW\ [\omega \uparrow \omega, \alpha]\ 2\ (is\ partn-lst\ ?R\ WW\ [\omega \uparrow \omega, \alpha]\ 2)$

proof (*cases* $\alpha \leq 1$)

case *True*

then show *?thesis*

using *strict-sorted-into-WW unfolding WW-def* by (*auto intro!: partn-lst-triv1 [where $i=1$]*)

next

case *False*

obtain *m* where $m: \alpha = ord-of-nat\ m$

using *assms elts- ω* by *auto*

```

then have m>1
  using False by auto
show ?thesis
  unfolding partn-lst-def
proof clarsimp
  fix f
  assume f: f ∈ [WW]2 → {..Suc (Suc 0)}
  let ?P0 = ∃ X ⊆ WW. ordertype X ?R = ω↑ω ∧ f ' [X]2 ⊆ {0}
  let ?P1 = ∃ M ⊆ WW. ordertype M ?R = α ∧ f ' [M]2 ⊆ {1}
  have †: ?P0 ∨ ?P1
  proof (rule disjCI)
    assume not1: ¬ ?P1
    have ∃ W'. ordertype W' ?R = ω↑n ∧ f ' [W']2 ⊆ {0} ∧ W' ⊆ WW-seg
  (n*m) for n::nat
    proof -
      have fnm: f ∈ [WW-seg (n*m)]2 → {..Suc (Suc 0)}
      using f WW-seg-subset-WW [of n*m] by (meson in-mono nsets-Pi-contr)
      have *: partn-lst ?R (WW-seg (n*m)) [ω↑n, ord-of-nat m] 2
        using ordertype-WW-seg [of n*m]
        by (simp add: partn-lst-VWF-imp-partn-lst [OF Theorem-3-2])
      show ?thesis
        using partn-lst-E [OF * fnm, simplified]
        by (metis One-nat-def WW-seg-subset-WW less-2-cases m not1 nth-Cons-0
nth-Cons-Suc numeral-2-eq-2 subset-trans)
      qed
    then obtain W':: nat ⇒ nat list set
      where otW': ∧n. ordertype (W' n) ?R = ω↑n
      and f-W': ∧n. f ' [W' n]2 ⊆ {0}
      and seg-W': ∧n. W' n ⊆ WW-seg (n*m)
      by metis
    define WW' where WW' ≡ (∪ n. W' n)
    have WW' ⊆ WW
      using seg-W' WW-seg-subset-WW by (force simp: WW'-def)
    with f have f': f ∈ [WW]2 → {..Suc (Suc 0)}
      using nsets-mono by fastforce
    have ot': ordertype WW' ?R = ω↑ω
    proof (rule antisym)
      have ordertype WW' ?R ≤ ordertype WW ?R
        by (simp add: ‹WW' ⊆ WW› lenlex-transI ordertype-mono wf-lenlex)
      with ordertype-WW
      show ordertype WW' ?R ≤ ω ↑ ω
        by simp
      have ω ↑ n ≤ ordertype (∪ (range W')) ?R for n::nat
        using oexp-Limit ordertype-ω otW' by auto
      then show ω ↑ ω ≤ ordertype WW' ?R
        by (auto simp: elts-ω oexp-Limit ZFC-in-HOL.SUP-le-iff WW'-def)
    qed
    have FR-WW: Field (Restr (lenlex less-than) WW) = WW
      by (simp add: Limit-omega-oexp Limit-ordertype-imp-Field-Restr order-

```

```

type-WW)
  have FR-WW': Field (Restr (lenlex less-than) WW') = WW'
    by (simp add: Limit-omega-oexp Limit-ordertype-imp-Field-Restr ot')
  have FR-W: Field (Restr (lenlex less-than) (WW-seg n)) = WW-seg n if
n>0 for n
  by (simp add: Limit-omega-oexp ordertype-WW-seg that Limit-ordertype-imp-Field-Restr)
  have FR-W': Field (Restr (lenlex less-than) (W' n)) = W' n if n>0 for n
    by (simp add: Limit-omega-oexp otW' that Limit-ordertype-imp-Field-Restr)
  have  $\exists h. \text{iso-ll } (WW\text{-seg } n) (W' n) h$  if n>0 for n
  proof (subst ordertype-eq-ordertype-iso-ll [symmetric])
    show ordertype (WW-seg n) (lenlex less-than) = ordertype (W' n) (lenlex
less-than)
      by (simp add: ordertype-WW-seg otW')
  qed (auto simp: FR-W FR-W' that)
  then obtain h-seg where h-seg:  $\bigwedge n. n > 0 \implies \text{iso-ll } (WW\text{-seg } n) (W' n)$ 
(h-seg n)
    by metis
  define h where h  $\equiv \lambda l. \text{if } l = [] \text{ then } [] \text{ else } h\text{-seg } (\text{length } l) l$ 

  have bij-h-seg:  $\bigwedge n. n > 0 \implies \text{bij-betw } (h\text{-seg } n) (WW\text{-seg } n) (W' n)$ 
    using h-seg by (simp add: iso-ll-def iso-iff2 FR-W FR-W')
  have len-h-seg:  $\text{length } (h\text{-seg } (\text{length } l) l) = \text{length } l * m$ 
    if  $\text{length } l > 0 \ l \in WW$  for l
    using bij-betwE [OF bij-h-seg] seg-W' that by (simp add: WW-seg-def
subset-iff)
  have hlen:  $\text{length } (h x) = \text{length } (h y) \iff \text{length } x = \text{length } y$  if  $x \in WW \ y$ 
 $\in WW$  for x y
    using that <1 < m> h-def len-h-seg by force

  have h: iso-ll WW WW' h
    unfolding iso-ll-def iso-iff2 FR-WW FR-WW'
  proof (intro conjI strip)
    have W'-ne:  $W' n \neq \{\}$  for n
      using otW' [of n] by auto
    then have  $[] \in WW'$ 
      using seg-W' [of 0] by (auto simp: WW'-def WW-seg-def)
    let ?g =  $\lambda l. \text{if } l = [] \text{ then } l \text{ else } \text{inv-into } (WW\text{-seg } (\text{length } l \text{ div } m)) (h\text{-seg }
(\text{length } l \text{ div } m)) l$ 
    have h-seg-iff:  $\bigwedge n \ a \ b. [a \in WW\text{-seg } n; b \in WW\text{-seg } n; n > 0] \implies$ 
 $(a, b) \in \text{lenlex less-than} \iff$ 
 $(h\text{-seg } n \ a, h\text{-seg } n \ b) \in \text{lenlex less-than} \wedge h\text{-seg } n \ a \in W' n$ 
 $\wedge h\text{-seg } n \ b \in W' n$ 
      using h-seg by (auto simp: iso-ll-def iso-iff2 FR-W FR-W')

  show bij-betw h WW WW'
    unfolding bij-betw-iff-bijections
  proof (intro exI conjI ballI)
    fix l
    assume l  $\in WW$ 

```

```

then have  $l \in WW\text{-seg } (\text{length } l)$ 
  by (simp add: WW-seg-def)
have  $h \ l \in W' (\text{length } l)$ 
proof (cases l=[])
  case True
    with seg-W' [of 0] W'-ne show ?thesis
      by (auto simp: WW-seg-def h-def)
  next
    case False
      then show ?thesis
        using bij-betwE bij-h-seg h-def l by fastforce
qed
show  $h \ l \in WW'$ 
  using WW'-def ⟨h l ∈ W' (length l)⟩ by blast
show  $?g (h \ l) = l$ 
proof (cases l=[])
  case False
    then have  $\text{length } l > 0$ 
      by auto
    then have  $h\text{-seg } (\text{length } l) \ l \neq []$ 
      using  $\langle 1 < m \rangle \langle l \in WW \rangle \text{len-h-seg}$  by fastforce
      moreover have  $\text{bij-betw } (h\text{-seg } (\text{length } l)) (WW\text{-seg } (\text{length } l)) (W'$ 
(length l)
        using  $\langle 0 < \text{length } l \rangle \text{bij-h-seg}$  by presburger
        ultimately show ?thesis
          using  $\langle l \in WW \rangle \text{bij-betw-inv-into-left } h\text{-def } l \text{len-h-seg}$  by fastforce
qed (auto simp: h-def)
next
fix  $l$ 
assume  $l \in WW'$ 
then have  $l \in W' (\text{length } l \text{ div } m)$ 
  using WW-seg-def ⟨1 < m⟩ seg-W' by (fastforce simp: WW'-def)
show  $?g \ l \in WW$ 
proof (cases l=[])
  case False
    then have  $l \notin W' \ 0$ 
      using WW-seg-def seg-W' by fastforce
      with  $l$  have  $\text{inv-into } (WW\text{-seg } (\text{length } l \text{ div } m)) (h\text{-seg } (\text{length } l \text{ div } m))$ 
 $l \in WW\text{-seg } (\text{length } l \text{ div } m)$ 
      by (metis Nat.neq0-conv bij-betwE bij-betw-inv-into bij-h-seg)
      then show ?thesis
        using False WW-seg-subset-WW by auto
qed (auto simp: WW-def)

show  $h (?g \ l) = l$ 
proof (cases l=[])
  case False
    then have  $0 < \text{length } l \text{ div } m$ 
      using WW-seg-def l seg-W' by fastforce

```

then have $inv\text{-}into$ ($WW\text{-}seg$ ($length\ l\ div\ m$)) ($h\text{-}seg$ ($length\ l\ div\ m$)) l
 $\in WW\text{-}seg$ ($length\ l\ div\ m$)
by ($metis\ bij\text{-}betw\text{-}imp\text{-}surj\text{-}on\ bij\text{-}h\text{-}seg\ inv\text{-}into\text{-}into\ l$)
then show $?thesis$
using $bij\text{-}h\text{-}seg$ [$of\ length\ l\ div\ m$] $WW\text{-}seg\text{-}def$ $\langle 0 < length\ l\ div\ m \rangle$
 $bij\text{-}betw\text{-}inv\text{-}into\text{-}right\ l$
by ($fastforce\ simp: h\text{-}def$)
qed ($auto\ simp: h\text{-}def$)
qed
fix $a\ b$
assume $a \in WW\ b \in WW$
show $(a, b) \in Restr$ ($lenlex\ less\text{-}than$) $WW \longleftrightarrow (h\ a, h\ b) \in Restr$ ($lenlex$
 $less\text{-}than$) WW'
(is $?lhs = ?rhs$)
proof
assume $L: ?lhs$
then consider $length\ a < length\ b \mid length\ a = length\ b$ $(a, b) \in lex$
 $less\text{-}than$
by ($auto\ simp: lenlex\text{-}conv$)
then show $?rhs$
proof cases
case 1
then have $length\ (h\ a) < length\ (h\ b)$
using $\langle 1 < m \rangle \langle a \in WW \rangle \langle b \in WW \rangle h\text{-}def\ len\text{-}h\text{-}seg$ **by** $auto$
then have $(h\ a, h\ b) \in lenlex\ less\text{-}than$
by ($auto\ simp: lenlex\text{-}conv$)
then show $?thesis$
using $\langle a \in WW \rangle \langle b \in WW \rangle \langle bij\text{-}betw\ h\ WW\ WW' \rangle bij\text{-}betwE$ **by**
 $fastforce$
next
case 2
then have $ab: a \in WW\text{-}seg$ ($length\ a$) $b \in WW\text{-}seg$ ($length\ a$)
using $\langle a \in WW \rangle \langle b \in WW \rangle$ **by** ($auto\ simp: WW\text{-}seg\text{-}def$)
have $length\ (h\ a) = length\ (h\ b)$
using $2 \langle a \in WW \rangle \langle b \in WW \rangle h\text{-}def\ len\text{-}h\text{-}seg$ **by** $force$
moreover have $(a, b) \in lenlex\ less\text{-}than$
using L **by** $blast$
then have $(h\text{-}seg$ ($length\ a$) $a, h\text{-}seg$ ($length\ a$) $b) \in lenlex\ less\text{-}than$
using $2\ ab\ h\text{-}seg\text{-}iff$ **by** $blast$
ultimately show $?thesis$
using $2 \langle a \in WW \rangle \langle b \in WW \rangle \langle bij\text{-}betw\ h\ WW\ WW' \rangle bij\text{-}betwE\ h\text{-}def$
by $fastforce$
qed
next
assume $R: ?rhs$
then have $R': (h\ a, h\ b) \in lenlex\ less\text{-}than$
by $blast$
then consider $length\ a < length\ b$
 $\mid length\ a = length\ b$ $(h\ a, h\ b) \in lex\ less\text{-}than$


```

    using ⟨a ∈ WW⟩ ⟨b ∈ WW⟩ ⟨m > 1⟩
    by (auto simp: lenlex-conv h-def len-h-seg split: if-split-asm)
  then show ?lhs
  proof cases
    case 1
    then show ?thesis
      using omega-sum-less-iff ⟨a ∈ WW⟩ ⟨b ∈ WW⟩ by auto
    next
    case 2
    then have ab: a ∈ WW-seg (length a) b ∈ WW-seg (length a)
      using ⟨a ∈ WW⟩ ⟨b ∈ WW⟩ by (auto simp: WW-seg-def)
    then have (a, b) ∈ lenlex less-than
      using bij-betwE [OF bij-h-seg] ⟨a ∈ WW⟩ ⟨b ∈ WW⟩ R' 2
      by (simp add: h-def h-seg-iff split: if-split-asm)
    then show ?thesis
      using ⟨a ∈ WW⟩ ⟨b ∈ WW⟩ by blast
  qed
qed
qed

let ?fh = f ∘ image h
have bij-betw h WW WW'
  using h unfolding iso-ll-def iso-iff2 by (fastforce simp: FR-WW FR-WW')
moreover have {..Suc (Suc 0)} = {0,1}
  by auto
ultimately have fh: ?fh ∈ [WW]2 → {0,1}
unfolding Pi-iff using bij-betwE f' bij-betw-nsets by (metis PiE comp-apply)
have f{x,y} = 0 if x ∈ WW' y ∈ WW' length x = length y x ≠ y for x y
proof -
  obtain p q where x ∈ W' p and y ∈ W' q
    using WW'-def ⟨x ∈ WW'⟩ ⟨y ∈ WW'⟩ by blast
  then obtain n where {x,y} ∈ [W' n]2
    using seg-W' ⟨1 < m⟩ ⟨length x = length y⟩ ⟨x ≠ y⟩
    by (auto simp: WW'-def WW-seg-def subset-iff)
  then show f{x,y} = 0
    using f-W' by blast
qed
then have fh-eq-0-eqlen: ?fh{x,y} = 0 if x ∈ WW y ∈ WW length x = length
y x ≠ y for x y
  using ⟨bij-betw h WW WW'⟩ that hlen by (simp add: bij-betw-iff-bijections)
metis
have m-f-0: ∃ x ∈ [M]2. f x = 0 if M ⊆ WW card M = m for M
proof -
  have finite M
    using False m that by auto
  with not1 [simplified, rule-format, of M] that
  have ∃ x ∈ [M]2. f x ≠ Suc 0
    by (simp add: image-subset-iff finite-ordertype-eq-card m)
  with that show ?thesis

```

by (metis PiE f lessThan-iff less-2-cases nsets-mono numeral-2-eq-2
 subset-iff)
 qed
 have m-fh-0: $\exists x \in [M]^2. ?fh\ x = 0$ if $M \subseteq WW$ card $M = m$ for M
 proof –
 have h ‘ $M \subseteq WW$
 using $\langle WW' \subseteq WW \rangle \langle \text{bij-betw } h\ WW\ WW' \rangle \text{bij-betwE that(1)}$ by fastforce
 moreover have card (h ‘ M) = m
 by (metis $\langle \text{bij-betw } h\ WW\ WW' \rangle \text{bij-betw-def bij-betw-subset card-image}$
 that)
 ultimately have $\exists x \in [h\ 'M]^2. f\ x = 0$
 by (metis m-f-0)
 then obtain Y where $Y: f\ (h\ 'Y) = 0$ $Y \subseteq M$ and finite (h ‘ Y) card
 (h ‘ Y) = 2
 by (auto simp: nsets-def subset-image-iff)
 then have card $Y = 2$
 using $\langle \text{bij-betw } h\ WW\ WW' \rangle \langle M \subseteq WW \rangle$
 by (metis bij-betw-def card-image inj-on-subset)
 with Y card.infinite[of Y] show ?thesis
 by (auto simp: nsets-def)
 qed

 obtain N j where infinite N
 and N: $\bigwedge k\ u. [k > 0; u \in [WW]^2; \text{Form } k\ u; [\text{enum } N\ k] < \text{inter-scheme}$
 $k\ u; \text{List.set } (\text{inter-scheme } k\ u) \subseteq N] \implies ?fh\ u = j\ k$
 using lemma-3-6 [OF fh] by blast

 have infN': infinite (enum N ‘ {k<..}) for k
 by (simp add: infinite N) enum-works finite-image-iff infinite-Ioi strict-mono-imp-inj-on)
 have j-0: $j\ k = 0$ if $k > 0$ for k
 proof –
 obtain M where $M: M \in [WW]^m$
 and MF: $\bigwedge u. u \in [M]^2 \implies \text{Form } k\ u$
 and Mi: $\bigwedge u. u \in [M]^2 \implies \text{List.set } (\text{inter-scheme } k\ u) \subseteq \text{enum } N\ ' \{k<..\}$
 {k<..}
 using lemma-3-7 [OF infN' $\langle k > 0 \rangle$] by metis
 obtain u where $u: u \in [M]^2$?fh u = 0
 using m-fh-0 [of M] M [unfolded nsets-def] by force
 moreover
 have §: $\text{Form } k\ u$ List.set (inter-scheme k u) $\subseteq \text{enum } N\ ' \{k<..\}$
 by (simp-all add: MF Mi $\langle u \in [M]^2 \rangle$)
 then have hd (inter-scheme k u) $\in \text{enum } N\ ' \{k<..\}$
 using hd-in-set inter-scheme-simple that by blast
 then have $[\text{enum } N\ k] < \text{inter-scheme } k\ u$
 using strict-mono-enum [OF $\langle \text{infinite } N \rangle$] by (auto simp: less-list-def
 strict-mono-def)
 moreover have $u \in [WW]^2$
 using M u by (auto simp: nsets-def)
 moreover have $\text{enum } N\ ' \{k<..\} \subseteq N$

```

    using ⟨infinite N⟩ range-enum by auto
  ultimately show ?thesis
  using N § that by auto
qed
obtain X where X ⊆ WW and otX: ordertype X (lenlex less-than) = ω↑ω
  and X: ⋀u. u ∈ [X]2 ⇒
    ∃l. Form l u ∧ (l > 0 → [enum N l] < inter-scheme l u ∧ List.set
(inter-scheme l u) ⊆ N)
  using lemma-3-8 [OF ⟨infinite N⟩] ot' by blast
  have 0: ?fh ' [X]2 ⊆ {0}
  proof clarsimp
    fix u
    assume u: u ∈ [X]2
    obtain l where Form l u and l: l > 0 → [enum N l] < inter-scheme l u
  ∧ List.set (inter-scheme l u) ⊆ N
    using u X by blast
    have ?fh u = 0
    proof (cases l = 0)
      case True
      then show ?thesis
        by (metis Form-0-cases-raw ⟨Form l u⟩ ⟨X ⊆ WW⟩ doubleton-in-nsets-2
fh-eq-0-eqlen subset-iff u)
      next
      case False
      then obtain [enum N l] < inter-scheme l u List.set (inter-scheme l u) ⊆
N j l = 0
        using Nat.neq0-conv j-0 l by blast
      with False show ?thesis
        using ⟨X ⊆ WW⟩ N inter-scheme ⟨Form l u⟩ doubleton-in-nsets-2 u by
(auto simp: nsets-def)
    qed
    then show f (h ' u) = 0
      by auto
  qed
show ?P0
proof (intro exI conjI)
  show h ' X ⊆ WW
  using ⟨WW' ⊆ WW⟩ ⟨X ⊆ WW⟩ ⟨bij-betw h WW WW'⟩ bij-betw-imp-surj-on
by fastforce
  show ordertype (h ' X) (lenlex less-than) = ω ↑ ω
  proof (subst ordertype-inc-eq)
    show (h x, h y) ∈ lenlex less-than
      if x ∈ X y ∈ X (x, y) ∈ lenlex less-than for x y
      using that h ⟨X ⊆ WW⟩ by (auto simp: FR-WW FR-WW' iso-iff2
iso-ll-def)
  qed (use otX in auto)
  show f ' [h ' X]2 ⊆ {0}
  proof (clarsimp simp: image-subset-iff nsets-def)
    fix Y

```

```

assume  $Y: Y \subseteq h \text{ ' } X$  finite  $Y$   $\text{card } Y = 2$ 
then have inv-into  $WW$   $h \text{ ' } Y \subseteq X$ 
  by (metis  $\langle X \subseteq WW \rangle$   $\langle \text{bij-betw } h \text{ } WW \text{ } WW \rangle$  bij-betw-inv-into-LEFT
image-mono)
  moreover have finite (inv-into  $WW$   $h \text{ ' } Y$ )
    using  $\langle \text{finite } Y \rangle$  by blast
  moreover have  $\text{card} (\text{inv-into } WW \text{ } h \text{ ' } Y) = 2$ 
  using  $Y$  by (metis  $\langle X \subseteq WW \rangle$  card-image inj-on-inv-into subset-image-iff
subset-trans)
  ultimately have  $f (h \text{ ' } \text{inv-into } WW \text{ } h \text{ ' } Y) = 0$ 
    using  $0$  by (auto simp: image-subset-iff nsets-def)
  then show  $f Y = 0$ 
    by (metis  $\langle X \subseteq WW \rangle$   $\langle Y \subseteq h \text{ ' } X \rangle$  image-inv-into-cancel image-mono
order-trans)
  qed
qed
qed
then show  $\exists i < \text{Suc } 0. \exists H \subseteq WW. \text{ordertype } H \text{ } ?R = [\omega \uparrow \omega, \alpha] ! i \wedge f \text{ '}$ 
 $[H]^2 \subseteq \{i\}$ 
  by (metis One-nat-def lessI nth-Cons-0 nth-Cons-Suc zero-less-Suc)
qed
qed

```

Theorem 3.1 of Jean A. Larson, *ibid*.

```

theorem partition- $\omega\omega$ :  $\alpha \in \text{elts } \omega \implies \text{partn-lst-VWF } (\omega \uparrow \omega) [\omega \uparrow \omega, \alpha] 2$ 
  using partn-lst-imp-partn-lst-VWF-eq [OF partition- $\omega\omega$ -aux] ordertype-WW by
auto
end

```

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