Octonions

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Abstract

We develop the basic theory of Octonions, including various identities and properties of the octonions and of the octonionic product, a description of 7D isometries and representations of orthogonal transformations. To this end we first develop the theory of the vector cross product in 7 dimensions. The development of the theory of Octonions is inspired by that of the theory of Quaternions by Lawrence Paulson. However, we do not work within the type class *real_algebra_1* because the octonionic product is not associative.

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1 Vector Cross Product in 7 Dimensions

theory *Cross-Product-7*
  imports *HOL−Analysis.Multivariate-Analysis*
begin

1.1 Elementary auxiliary lemmas.

lemma *exhaust-7*:
  fixes $x :: 7$
  shows $x = 1 \vee x = 2 \vee x = 3 \vee x = 4 \vee x = 5 \vee x = 6 \ \vee x = 7$
proof (*induct x*)
  case (*of-int z*)
  then have $0 \leq z$ **and** $z < 7$ **by** *simp-all*
  then have $z = 0 \vee z = 1 \vee z = 2 \vee z = 3 \vee z = 4 \ \vee z = 5 \vee z = 6 \ \vee z = 7$
by *arith*
  then show *?case* **by** *auto*
qed

lemma *forall-7*: $(\forall i{::}7.\ P\ i) \longleftrightarrow P\ 1 \wedge P\ 2 \wedge P\ 3 \wedge P\ 4 \wedge P\ 5 \wedge P\ 6 \wedge P\ 7$
  by (*metis exhaust-7*)

lemma *vector-7* [*simp*]:
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$1 = x1$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$2 = x2$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$3 = x3$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$4 = x4$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$5 = x5$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$6 = x6$
 $(vector\ [x1,x2,x3,x4,x5,x6,x7] ::('a::zero)\hat{}7)\$7 = x7$
  unfolding *vector-def* **by** *auto*

lemma *forall-vector-7*:
 $(\forall v{::}'a{::}zero\hat{}7.\ P\ v) \longleftrightarrow (\forall x1\ x2\ x3\ x4\ x5\ x6\ x7.\ P(vector[x1,\ x2,\ x3,\ x4,\ x5,\ x6,\ x7]))$
  apply *auto*
  apply (*erule-tac* $x{=}v\$1$ **in** *allE*)
  apply (*erule-tac* $x{=}v\$2$ **in** *allE*)
  apply (*erule-tac* $x{=}v\$3$ **in** *allE*)
  apply (*erule-tac* $x{=}v\$4$ **in** *allE*)
  apply (*erule-tac* $x{=}v\$5$ **in** *allE*)
  apply (*erule-tac* $x{=}v\$6$ **in** *allE*)

apply (*erule-tac x=v\$7* **in** *allE*)
apply (*subgoal-tac vector* [*v\$1*, *v\$2*, *v\$3*, *v\$4*, *v\$5*, *v\$6*, *v\$7*] = *v*)
apply *simp*
apply (*vector vector-def*)
apply (*simp add*: *forall-7*)
done

lemma *UNIV-7*: *UNIV* = {*1::7*, *2::7*, *3::7*, *4::7*, *5::7*, *6::7*, *7::7*}
using *exhaust-7* **by** *auto*

lemma *sum-7*: *sum f* (*UNIV::7 set*) = *f 1* + *f 2* + *f 3* + *f 4* + *f 5* + *f 6* + *f 7*
unfolding *UNIV-7* **by** (*simp add*: *ac-simps*)

lemma *not-equal-vector7* :
fixes *x::real^7* **and** *y::real^7*
assumes *x* = *vector*[*x1*,*x2*,*x3*,*x4*,*x5*,*x6*,*x7*] **and** *y*= *vector* [*y1*,*y2*,*y3*,*y4*,*y5*,*y6*,*y7*]
and *x\$1* ≠ *y\$1* ∨ *x\$2* ≠ *y\$2* ∨ *x\$3* ≠ *y\$3* ∨ *x\$4* ≠ *y\$4* ∨ *x\$5* ≠ *y\$5* ∨ *x\$6* ≠ *y\$6* ∨ *x\$7* ≠ *y\$7*
shows *x* ≠ *y* **using** *assms* **by** *blast*

lemma *equal-vector7*:
fixes *x::real^7* **and** *y::real^7*
assumes *x* = *vector*[*x1*,*x2*,*x3*,*x4*,*x5*,*x6*,*x7*] **and** *y*= *vector* [*y1*,*y2*,*y3*,*y4*,*y5*,*y6*,*y7*]
and *x* = *y*
shows *x\$1* = *y\$1* ∧ *x\$2* = *y\$2* ∧ *x\$3* = *y\$3* ∧ *x\$4* = *y\$4* ∧ *x\$5* = *y\$5* ∧ *x\$6* = *y\$6* ∧ *x\$7* = *y\$7*
using *assms* **by** *blast*

lemma *numeral-4-eq-4*: *4* = *Suc*(*Suc* (*Suc* (*Suc 0*)))
by (*simp add*: *eval-nat-numeral*)
lemma *numeral-5-eq-5*: *5* = *Suc*(*Suc*(*Suc* (*Suc* (*Suc 0*))))
by (*simp add*: *eval-nat-numeral*)
lemma *numeral-6-eq-6*: *6* = *Suc*(*Suc*(*Suc*(*Suc* (*Suc* (*Suc 0*)))))
by (*simp add*: *eval-nat-numeral*)
lemma *numeral-7-eq-7*: *7* = *Suc*(*Suc*(*Suc*(*Suc*(*Suc* (*Suc* (*Suc 0*))))))
by (*simp add*: *eval-nat-numeral*)

1.2 The definition of the 7D cross product and related lemmas

Note: in total there exist 480 equivalent multiplication tables for the definition, the following is based on the one most widely used:

definition *cross7* :: [*real^7*, *real^7*] ⇒ *real^7* (**infixr** ‹$\times_7$› *80*)
where $a \times_7 b \equiv$
vector [*a\$2* ∗ *b\$4* − *a\$4* ∗ *b\$2* + *a\$3* ∗ *b\$7* − *a\$7* ∗ *b\$3* + *a\$5* ∗ *b\$6* − *a\$6* ∗ *b\$5* ,
a\$3 ∗ *b\$5* − *a\$5* ∗ *b\$3* + *a\$4* ∗ *b\$1* − *a\$1* ∗ *b\$4* + *a\$6* ∗ *b\$7* − *a\$7* ∗ *b\$6* ,
a\$4 ∗ *b\$6* − *a\$6* ∗ *b\$4* + *a\$5* ∗ *b\$2* − *a\$2* ∗ *b\$5* + *a\$7* ∗ *b\$1* −

$a\$1 * b\$7$,
$a\$5 * b\$7 - a\$7 * b\$5 + a\$6 * b\$3 - a\$3 * b\$6 + a\$1 * b\$2 - a\$2 * b\$1$,
$a\$6 * b\$1 - a\$1 * b\$6 + a\$7 * b\$4 - a\$4 * b\$7 + a\$2 * b\$3 - a\$3 * b\$2$,
$a\$7 * b\$2 - a\$2 * b\$7 + a\$1 * b\$5 - a\$5 * b\$1 + a\$3 * b\$4 - a\$4 * b\$3$,
$a\$1 * b\$3 - a\$3 * b\$1 + a\$2 * b\$6 - a\$6 * b\$2 + a\$4 * b\$5 - a\$5 * b\$4$]

lemmas *cross7-simps* = *cross7-def inner-vec-def sum-7 det-def vec-eq-iff vector-def algebra-simps*

lemma *dot-cross7-self*: $x \cdot (x \times_7 y) = 0$ $x \cdot (y \times_7 x) = 0$ $(x \times_7 y) \cdot y = 0$ $(y \times_7 x) \cdot y = 0$
by (*simp-all add*: *orthogonal-def cross7-simps*)

lemma *orthogonal-cross7*: *orthogonal* $(x \times_7 y)$ x *orthogonal* $(x \times_7 y)$ y
orthogonal y $(x \times_7 y)$ *orthogonal* $(x \times_7 y)$ x
by (*simp-all add*: *orthogonal-def dot-cross7-self*)

lemma *cross7-zero-left* [*simp*]: $0 \times_7 x = 0$
and *cross7-zero-right* [*simp*]: $x \times_7 0 = 0$
by (*simp-all add*: *cross7-simps*)

lemma *cross7-skew*: $(x \times_7 y) = -(y \times_7 x)$
by (*simp add*: *cross7-simps*)

lemma *cross7-refl* [*simp*]: $x \times_7 x = 0$
by (*simp add*: *cross7-simps*)

lemma *cross7-add-left*: $(x + y) \times_7 z = (x \times_7 z) + (y \times_7 z)$
and *cross7-add-right*: $x \times_7 (y + z) = (x \times_7 y) + (x \times_7 z)$
by (*simp-all add*: *cross7-simps*)

lemma *cross7-mult-left*: $(c *_R x) \times_7 y = c *_R (x \times_7 y)$
and *cross7-mult-right*: $x \times_7 (c *_R y) = c *_R (x \times_7 y)$
by (*simp-all add*: *cross7-simps*)

lemma *cross7-minus-left* [*simp*]: $(-x) \times_7 y = - (x \times_7 y)$
and *cross7-minus-right* [*simp*]: $x \times_7 -y = - (x \times_7 y)$
by (*simp-all add*: *cross7-simps*)

lemma *left-diff-distrib*: $(x - y) \times_7 z = x \times_7 z - y \times_7 z$
and *right-diff-distrib*: $x \times_7 (y - z) = x \times_7 y - x \times_7 z$
by (*simp-all add*: *cross7-simps*)

hide-fact (**open**) *left-diff-distrib right-diff-distrib*

lemma *cross7-triple1*: $(x \times_7 y) \cdot z = (y \times_7 z) \cdot x$
and *cross7-triple2*: $(x \times_7 y) \cdot z = x \cdot (y \times_7 z)$
by (*simp-all add*: *cross7-def inner-vec-def sum-7 vec-eq-iff algebra-simps*)

lemma *scalar7-triple1*: $x \cdot (y \times_7 z) = y \cdot (z \times_7 x)$
and *scalar7-triple2*: $x \cdot (y \times_7 z) = z \cdot (x \times_7 y)$
by (*simp-all add*: *cross7-def inner-vec-def sum-7 vec-eq-iff algebra-simps*)

lemma *cross7-components*:
$(x \times_7 y)\$1 = x\$2 * y\$4 - x\$4 * y\$2 + x\$3 * y\$7 - x\$7 * y\$3 + x\$5 * y\$6 - x\$6 * y\$5$
$(x \times_7 y)\$2 = x\$4 * y\$1 - x\$1 * y\$4 + x\$3 * y\$5 - x\$5 * y\$3 + x\$6 * y\$7 - x\$7 * y\$6$
$(x \times_7 y)\$3 = x\$5 * y\$2 - x\$2 * y\$5 + x\$4 * y\$6 - x\$6 * y\$4 + x\$7 * y\$1 - x\$1 * y\$7$
$(x \times_7 y)\$4 = x\$1 * y\$2 - x\$2 * y\$1 + x\$6 * y\$3 - x\$3 * y\$6 + x\$5 * y\$7 - x\$7 * y\$5$
$(x \times_7 y)\$5 = x\$6 * y\$1 - x\$1 * y\$6 + x\$2 * y\$3 - x\$3 * y\$2 + x\$7 * y\$4 - x\$4 * y\$7$
$(x \times_7 y)\$6 = x\$1 * y\$5 - x\$5 * y\$1 + x\$7 * y\$2 - x\$2 * y\$7 + x\$3 * y\$4 - x\$4 * y\$3$
$(x \times_7 y)\$7 = x\$1 * y\$3 - x\$3 * y\$1 + x\$4 * y\$5 - x\$5 * y\$4 + x\$2 * y\$6 - x\$6 * y\$2$
by (*simp-all add*: *cross7-def inner-vec-def sum-7 vec-eq-iff algebra-simps*)

Nonassociativity of the 7D cross product showed using a counterexample

lemma *cross7-nonassociative*:
$\neg (\forall (c::real\hat{}7)\ (a::real\hat{}7)\ (b::real\hat{}7).\ c \times_7 (a \times_7 b) = (c \times_7 a) \times_7 b)$
proof−
have $*$: $\exists (c::real\hat{}7)\ (a::real\hat{}7)\ (b::real\hat{}7).\ c \times_7 (a \times_7 b) \neq (c \times_7 a) \times_7 b$
proof−
define $f::real\hat{}7$ **where** $f = vector[0, 0, 0, 0, 0, 1, 1]$
define $g::real\hat{}7$ **where** $g = vector[0, 0, 0, 1, 0, 0, 0]$
define $h::real\hat{}7$ **where** $h = vector[1, 0, 1, 0, 0, 0, 0]$
define u **where** $u = (g \times_7 h)$
define v **where** $v = (f \times_7 g)$

have *1*: $u = vector[0, 1, 0, 0, 0, -1, 0]$
unfolding *cross7-def g-def h-def f-def u-def* **by** *simp*
have *3*: $f \times_7 u = vector[0, 1, 0, 0, 0, 1, -1]$
unfolding *cross7-def f-def* **using** *1* **by** *simp*

have *2*: $v = vector[0, 0, -1, 0, 1, 0, 0]$
unfolding *cross7-def g-def h-def f-def v-def* **by** *simp*
have *4*: $v \times_7 h = vector[0, -1, 0, 0, 0, -1, 1]$
unfolding *cross7-def h-def* **using** *2* **by** *simp*

define $x::real\hat{}7$ **where** $x = vector[0, 1, 0, 0, 0, 1, -1]$

define y::*real*^*7* **where** y= *vector*[$0,\ -1,\ 0,\ 0,\ 0,-1,\ 1$]

have $*$: $x\$2 = 1$ **unfolding** *x-def* **by** *simp*
have $**$: $y\$2 = -1$ **unfolding** *y-def* **by** *simp*

have $***$: $x \neq y$ **using** $*$ $**$ **by** *auto*
have *5*: $f \times_7 u \neq v \times_7 h$
unfolding *x-def y-def*
using $***$
by (*simp add*: *3 4 x-def y-def*)

have *6*: $f \times_7 (g \times_7 h) \neq (f \times_7 g) \times_7 h$ **using** *5* **by** (*simp add*: *u-def v-def*)

show *?thesis* **unfolding** *f-def g-def h-def* **using** *6* **by** *blast*
qed
show *?thesis* **using** $*$ **by** *blast*
qed

The 7D cross product does not satisfy the Jacobi Identity(shown using a counterexample)

lemma *cross7-not-Jacobi*:
$\neg$ ($\forall$ (c::*real*^*7*) (a::*real*^*7*) (b::*real*^*7*) . $(c \times_7 a) \times_7 b + (b \times_7 c) \times_7 a$
$+ (a \times_7 b) \times_7 c = 0$)

proof−
have $*$: $\exists$ (c::*real*^*7*) (a::*real*^*7*) (b::*real*^*7*) . $(c \times_7 a) \times_7 b + (b \times_7 c) \times_7$
a
$+ (a \times_7 b) \times_7 c \neq 0$
proof−

define f::*real*^*7* **where** f= *vector*[$0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1$]
define g::*real*^*7* **where** g= *vector*[$0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0$]
define h::*real*^*7* **where** h= *vector*[$1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0$]
define u **where** $u= (g \times_7 h)$
define v **where** $v= (f \times_7 g)$
define w **where** $w = (h \times_7 f)$

have *1*: u = *vector*[$0,\ 1,\ 0,\ 0,\ 0,\ -1,\ 0$]
unfolding *cross7-def g-def h-def f-def u-def* **by** *simp*
have *3*: $u \times_7 f$ = *vector*[$0,-\ 1,\ 0,\ 0,\ 0,-\ 1,\ 1$]
unfolding *cross7-def f-def* **using** *1* **by** *simp*

have *2*: v = *vector*[$0,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0$]
unfolding *cross7-def g-def h-def f-def v-def* **by** *simp*
have *4*: $v \times_7 h$ = *vector*[$0,\ -1,\ 0,\ 0,\ 0,\ -1,\ 1$]
unfolding *cross7-def h-def* **using** *2* **by** *simp*
have *8*: w = *vector*[$1,\ 0,\ -1,\ -1,\ -1,\ 0,\ 0$]
unfolding *cross7-def h-def f-def w-def* **by** *simp*
have *9*: $w \times_7 g$ = *vector*[$0,\ -1,\ 0,\ 0,\ 0,\ -1,\ 1$]

```
    unfolding cross7-def  h-def f-def w-def g-def apply simp done
  have &: (u ×7 f)$2+( v ×7 h) $2+( w ×7 g) $2 =−3 using 3 4 9 by simp
  have &&:  u ×7 f + v ×7 h +  w ×7 g ≠ 0  using &
    by (metis vector-add-component zero-index zero-neq-neg-numeral)

  show ?thesis  using && u-def v-def w-def by blast
  qed

  show ?thesis using ∗ by blast
qed
```

The vector triple product property fulfilled for the 3D cross product does not hold for the 7D cross product. Shown below with a counterexample.

```
lemma cross7-not-vectortriple:
  ¬(∀ (c::real^7) (a::real^7) ( b::real^7).( c ×7 a ) ×7 b = (c  · b ) ∗R a −  (c  ·
a )∗R b)
proof−
  have ∗: ∃ (c::real^7) (a::real^7) ( b::real^7). (c ×7 a) ×7 b ≠ (c · b ) ∗R a −  (c
· a )∗R b
  proof−
    define g:: real ^ 7 where g = vector[0, 0, 0, 1, 0, 0, 0]
    define h:: real ^ 7 where h = vector[1, 0, 1, 0, 0, 0, 0]
    define f:: real ^ 7 where f = vector[0, 0, 0, 0, 0, 1, 1]
    define u where u = (g ×7 h)
    have 1: u = vector[0, 1, 0, 0, 0, −1, 0]
      unfolding cross7-def g-def h-def f-def u-def by simp
    have 2: u ×7 f = vector[0,− 1, 0, 0, 0,− 1, 1 ]
      unfolding cross7-def f-def using 1 by simp
    have 3: (g · f) ∗R h = 0 unfolding g-def f-def  inner-vec-def
      by (simp add: sum-7)
    have 4: (g · h) ∗R f = 0 unfolding g-def h-def  inner-vec-def
      by (simp add: sum-7)
    have 5: (g · f) ∗R h −(g · h) ∗R f = 0
      using 3 4 by auto
    have 6: u ×7 f ≠ 0 using 2
      by (metis one-neq-zero vector-7(7) zero-index)

    have 7: (g ×7 h) ×7 f ≠ 0 using 2 6 u-def by blast
    have 8: (g ×7 h) ×7 f ≠ (g · f) ∗R h − (g · h ) ∗R f
      using 5 7 by simp
    show ?thesis using 8 by auto
  qed
  show ?thesis using ∗ by auto
qed

lemma axis-nth-neq [simp]: i ≠ j ⟹ axis i x $ j = 0
  by (simp add: axis-def)

lemma cross7-basis:
```

$(axis\ 1\ 1) \times_7 (axis\ 2\ 1) = axis\ 4\ 1\ (axis\ 2\ 1) \times_7 (axis\ 1\ 1) = -(axis\ 4\ 1)$
$(axis\ 2\ 1) \times_7 (axis\ 3\ 1) = axis\ 5\ 1\ (axis\ 3\ 1) \times_7 (axis\ 2\ 1) = -(axis\ 5\ 1)$
$(axis\ 3\ 1) \times_7 (axis\ 4\ 1) = axis\ 6\ 1\ (axis\ 4\ 1) \times_7 (axis\ 3\ 1) = -(axis\ 6\ 1)$
$(axis\ 2\ 1) \times_7 (axis\ 4\ 1) = axis\ 1\ 1\ (axis\ 4\ 1) \times_7 (axis\ 2\ 1) = -(axis\ 1\ 1)$
$(axis\ 4\ 1) \times_7 (axis\ 5\ 1) = axis\ 7\ 1\ (axis\ 5\ 1) \times_7 (axis\ 4\ 1) = -(axis\ 7\ 1)$
$(axis\ 3\ 1) \times_7 (axis\ 5\ 1) = axis\ 2\ 1\ (axis\ 5\ 1) \times_7 (axis\ 3\ 1) = -(axis\ 2\ 1)$
$(axis\ 4\ 1) \times_7 (axis\ 6\ 1) = axis\ 3\ 1\ (axis\ 6\ 1) \times_7 (axis\ 4\ 1) = -(axis\ 3\ 1)$
$(axis\ 5\ 1) \times_7 (axis\ 7\ 1) = axis\ 4\ 1\ (axis\ 7\ 1) \times_7 (axis\ 5\ 1) = -(axis\ 4\ 1)$
$(axis\ 4\ 1) \times_7 (axis\ 1\ 1) = axis\ 2\ 1\ (axis\ 1\ 1) \times_7 (axis\ 4\ 1) = -(axis\ 2\ 1)$
$(axis\ 5\ 1) \times_7 (axis\ 2\ 1) = axis\ 3\ 1\ (axis\ 2\ 1) \times_7 (axis\ 5\ 1) = -(axis\ 3\ 1)$
$(axis\ 6\ 1) \times_7 (axis\ 3\ 1) = axis\ 4\ 1\ (axis\ 3\ 1) \times_7 (axis\ 6\ 1) = -(axis\ 4\ 1)$
$(axis\ 7\ 1) \times_7 (axis\ 4\ 1) = axis\ 5\ 1\ (axis\ 4\ 1) \times_7 (axis\ 7\ 1) = -(axis\ 5\ 1)$
$(axis\ 5\ 1) \times_7 (axis\ 6\ 1) = axis\ 1\ 1\ (axis\ 6\ 1) \times_7 (axis\ 5\ 1) = -(axis\ 1\ 1)$
$(axis\ 6\ 1) \times_7 (axis\ 7\ 1) = axis\ 2\ 1\ (axis\ 7\ 1) \times_7 (axis\ 6\ 1) = -(axis\ 2\ 1)$
$(axis\ 7\ 1) \times_7 (axis\ 1\ 1) = axis\ 3\ 1\ (axis\ 1\ 1) \times_7 (axis\ 7\ 1) = -(axis\ 3\ 1)$
$(axis\ 6\ 1) \times_7 (axis\ 1\ 1) = axis\ 5\ 1\ (axis\ 1\ 1) \times_7 (axis\ 6\ 1) = -(axis\ 5\ 1)$
$(axis\ 7\ 1) \times_7 (axis\ 2\ 1) = axis\ 6\ 1\ (axis\ 2\ 1) \times_7 (axis\ 7\ 1) = -(axis\ 6\ 1)$
$(axis\ 1\ 1) \times_7 (axis\ 3\ 1) = axis\ 7\ 1\ (axis\ 3\ 1) \times_7 (axis\ 1\ 1) = -(axis\ 7\ 1)$
$(axis\ 1\ 1) \times_7 (axis\ 5\ 1) = axis\ 6\ 1\ (axis\ 5\ 1) \times_7 (axis\ 1\ 1) = -(axis\ 6\ 1)$
$(axis\ 2\ 1) \times_7 (axis\ 6\ 1) = axis\ 7\ 1\ (axis\ 6\ 1) \times_7 (axis\ 2\ 1) = -(axis\ 7\ 1)$
$(axis\ 3\ 1) \times_7 (axis\ 7\ 1) = axis\ 1\ 1\ (axis\ 7\ 1) \times_7 (axis\ 3\ 1) = -(axis\ 1\ 1)$
by (*simp-all add*: *vec-eq-iff forall-7 cross7-components*)

lemma *cross7-basis-zero*:
$u=0 \Longrightarrow (u \times_7 axis\ 1\ 1 = 0) \wedge (u \times_7 axis\ 2\ 1 = 0) \wedge (u \times_7 axis\ 3\ 1 = 0)$
$\wedge (u \times_7 axis\ 4\ 1 = 0) \wedge (u \times_7 axis\ 5\ 1 = 0\ ) \wedge (u \times_7 axis\ 6\ 1 = 0\ )$
$\wedge (u \times_7 axis\ 7\ 1 = 0)$
by *auto*

lemma *cross7-basis-nonzero*:
$\neg (u \times_7 axis\ 1\ 1 = 0) \vee \neg (u \times_7 axis\ 2\ 1 = 0) \vee \neg (u \times_7 axis\ 3\ 1 = 0)$
$\vee \neg\ (u \times_7 axis\ 4\ 1 = 0)\ \vee \neg\ (u \times_7 axis\ 5\ 1 = 0\ ) \vee \neg\ (u \times_7 axis\ 6\ 1 = 0\ )$
$\vee \neg\ (u \times_7 axis\ 7\ 1 = 0) \Longrightarrow u \neq 0$
by *auto*

Pythagorean theorem/magnitude

lemma *norm-square-vec-eq*: $norm\ x\ \hat{}\ 2 = (\sum i{\in}UNIV.\ x\ \$\ i\ \hat{}\ 2)$
by (*auto simp*: *norm-vec-def L2-set-def intro*!: *sum-nonneg*)

lemma *norm-cross7-dot-magnitude*: $(norm\ (x \times_7 y))^2 = (norm\ x)^2 * (norm\ y)^2 - (x \cdot y)^2$
unfolding *norm-square-vec-eq sum-7 cross7-components inner-vec-def real-norm-def inner-real-def*
by *algebra*

lemma *cross7-eq-0*: $x \times_7 y = 0 \longleftrightarrow collinear\ \{0,\ x,\ y\}$
proof −
have $x \times_7 y = 0 \longleftrightarrow norm\ (x \times_7 y) = 0$
by *simp*

also have ... ⟷ (*norm x* ∗ *norm y*)^2 = (*x* · *y*)^2
using *norm-cross7-dot-magnitude* [*of x y*] **by** (*auto simp*: *power-mult-distrib*)
also have ... ⟷ |*x* · *y*| = *norm x* ∗ *norm y*
using *power2-eq-iff*
by (*metis* (*mono-tags*, *opaque-lifting*) *abs-minus abs-norm-cancel abs-power2 norm-mult power-abs real-norm-def*)
also have ... ⟷ *collinear* {*0*, *x*, *y*}
by (*rule norm-cauchy-schwarz-equal*)
finally show *?thesis* **.**
qed

lemma *cross7-eq-self*: *x* $\times_7$ *y* = *x* ⟷ *x* = *0 x* $\times_7$ *y* = *y* ⟷ *y* = *0*
apply (*metis cross7-zero-left dot-cross7-self*(*1*) *inner-eq-zero-iff*)
apply (*metis cross7-zero-right dot-cross7-self*(*2*) *inner-eq-zero-iff*)
done

lemma *norm-and-cross7-eq-0*:
x · *y* = *0* ∧ *x* $\times_7$ *y* = *0* ⟷ *x* = *0* ∨ *y* = *0* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
using *cross7-eq-0 norm-cauchy-schwarz-equal* **by** *force*
next
assume *?rhs*
then show *?lhs*
by *auto*
qed

lemma *bilinear-cross7*: *bilinear* ($\times_7$)
apply (*auto simp add*: *bilinear-def linear-def*)
apply *unfold-locales*
apply (*simp-all add*: *cross7-add-right cross7-mult-right cross7-add-left cross7-mult-left*)
done

1.3 Continuity

lemma *continuous-cross7*: ⟦*continuous F f*; *continuous F g*⟧ ⟹ *continuous F* (λ*x*. *f x* $\times_7$ *g x*)
by (*subst continuous-componentwise*)(*auto intro*!: *continuous-intros simp*: *cross7-simps*)

lemma *continuous-on-cross*:
fixes *f* :: ′*a*::*t2-space* ⇒ *real*^*7*
shows ⟦*continuous-on S f*; *continuous-on S g*⟧ ⟹ *continuous-on S* (λ*x*. *f x* $\times_7$ *g x*)
by (*simp add*: *continuous-on-eq-continuous-within continuous-cross7*)

end

2 Theory of Octonions

theory *Octonions*
  imports *Cross-Product-7*
begin

2.1 Basic definitions

As with the complex numbers, coinduction is convenient.

codatatype *octo* =
  Octo (*Ree*: *real*) (*Im1*: *real*) (*Im2*: *real*) (*Im3*: *real*) (*Im4*: *real*)
    (*Im5*: *real*) (*Im6*: *real*) (*Im7*: *real*)

lemma *octo-eqI* [*intro?*]:
  ⟦*Ree x* = *Ree y*; *Im1 x* = *Im1 y*; *Im2 x* = *Im2 y*; *Im3 x* = *Im3 y*;
   Im4 x = *Im4 y*;*Im5 x* = *Im5 y*; *Im6 x* = *Im6 y*; *Im7 x* = *Im7 y*⟧ ⟹ *x* = *y*
  by (*rule octo.expand*) *simp*

lemma *octo-eq-iff*:
  x = *y* ⟷ *Ree x* = *Ree y* ∧ *Im1 x* = *Im1 y* ∧ *Im2 x* = *Im2 y* ∧ *Im3 x* = *Im3 y* ∧
        Im4 x = *Im4 y* ∧ *Im5 x* = *Im5 y* ∧ *Im6 x* = *Im6 y* ∧ *Im7 x* = *Im7 y*
  by (*auto intro*: *octo.expand*)

context
begin

primcorec *octo-e0* :: *octo* (‹*e0*›)
where *Ree e0* = *1* | *Im1 e0* = *0* | *Im2 e0* = *0* | *Im3 e0* = *0*
  | *Im4 e0* = *0* | *Im5 e0* = *0* | *Im6 e0* = *0* | *Im7 e0* = *0*

primcorec *octo-e1* :: *octo* (‹*e1*›)
  where *Ree e1* = *0* | *Im1 e1* = *1* | *Im2 e1* = *0* | *Im3 e1* = *0*
  | *Im4 e1* = *0* | *Im5 e1* = *0* | *Im6 e1* = *0* | *Im7 e1* = *0*

primcorec *octo-e2* :: *octo* (‹*e2*›)
  where *Ree e2* = *0* | *Im1 e2* = *0* | *Im2 e2* = *1* | *Im3 e2* = *0*
  | *Im4 e2* = *0* | *Im5 e2* = *0* | *Im6 e2* = *0* | *Im7 e2* = *0*

primcorec *octo-e3* :: *octo* (‹*e3*›)
  where *Ree e3* = *0* | *Im1 e3* = *0* | *Im2 e3* = *0* | *Im3 e3* = *1*
  | *Im4 e3* = *0* | *Im5 e3* = *0* | *Im6 e3* = *0* | *Im7 e3* = *0*

primcorec *octo-e4* :: *octo* (‹*e4*›)
  where *Ree e4* = *0* | *Im1 e4* = *0* | *Im2 e4* = *0* | *Im3 e4* = *0*
  | *Im4 e4* = *1* | *Im5 e4* = *0* | *Im6 e4* = *0* | *Im7 e4* = *0*

primcorec *octo-e5* :: *octo* (‹*e5*›)
  where *Ree e5* = *0* | *Im1 e5* = *0* | *Im2 e5* = *0* | *Im3 e5* = *0*

| *Im4 e5 = 0* | *Im5 e5 = 1* | *Im6 e5 = 0* | *Im7 e5 = 0*

primcorec *octo-e6* :: *octo* (‹*e6*›)
where *Ree e6 = 0* | *Im1 e6 = 0* | *Im2 e6 = 0* | *Im3 e6 = 0*
| *Im4 e6 = 0* | *Im5 e6 = 0* | *Im6 e6 = 1* | *Im7 e6 = 0*

primcorec *octo-e7* :: *octo* (‹*e7*›)
where *Ree e7 = 0* | *Im1 e7 = 0* | *Im2 e7 = 0* | *Im3 e7 = 0*
| *Im4 e7 = 0* | *Im5 e7 = 0* | *Im6 e7 = 0* | *Im7 e7 = 1*
end

2.2 Addition and Subtraction: An Abelian Group

instantiation *octo* :: *ab-group-add*

begin

primcorec *zero-octo*
where *Ree 0 = 0* |*Im1 0 = 0* | *Im2 0 = 0* | *Im3 0 = 0*
|*Im4 0 = 0* | *Im5 0 = 0* | *Im6 0 = 0* | *Im7 0 = 0*

primcorec *plus-octo*
where
Ree $(x + y)$ = *Ree* x + *Ree* y
| *Im1* $(x + y)$ = *Im1* x + *Im1* y
| *Im2* $(x + y)$ = *Im2* x + *Im2* y
| *Im3* $(x + y)$ = *Im3* x + *Im3* y
| *Im4* $(x + y)$ = *Im4* x + *Im4* y
| *Im5* $(x + y)$ = *Im5* x + *Im5* y
| *Im6* $(x + y)$ = *Im6* x + *Im6* y
| *Im7* $(x + y)$ = *Im7* x + *Im7* y

primcorec *uminus-octo*
where
Ree $(- x)$ = − *Ree* x
| *Im1* $(- x)$ = − *Im1* x
| *Im2* $(- x)$ = − *Im2* x
| *Im3* $(- x)$ = − *Im3* x
| *Im4* $(- x)$ = − *Im4* x
| *Im5* $(- x)$ = − *Im5* x
| *Im6* $(- x)$ = − *Im6* x
| *Im7* $(- x)$ = − *Im7* x

primcorec *minus-octo*
where
Ree $(x - y)$ = *Ree* x − *Ree* y
| *Im1* $(x - y)$ = *Im1* x − *Im1* y
| *Im2* $(x - y)$ = *Im2* x − *Im2* y
| *Im3* $(x - y)$ = *Im3* x − *Im3* y

| *Im4* $(x - y) = Im4\ x - Im4\ y$
| *Im5* $(x - y) = Im5\ x - Im5\ y$
| *Im6* $(x - y) = Im6\ x - Im6\ y$
| *Im7* $(x - y) = Im7\ x - Im7\ y$

instance
by *standard* (*simp-all add*: *octo-eq-iff*)

end

lemma *octo-eq-0-iff*:
$x = 0 \longleftrightarrow Ree\ x\ \hat{}\ 2 + Im1\ x\ \hat{}\ 2 + Im2\ x\ \hat{}\ 2 + Im3\ x\ \hat{}\ 2 +$
$Im4\ x\ \hat{}\ 2 + Im5\ x\ \hat{}\ 2 + Im6\ x\ \hat{}\ 2 + Im7\ x\ \hat{}\ 2 = 0$
proof
assume $(octo.Ree\ x)^2 + (Im1\ x)^2 + (Im2\ x)^2 + (Im3\ x)^2 + (Im4\ x)^2 + (Im5$
$x)^2 + (Im6\ x)^2$
$+ (Im7\ x)^2 = 0$
then have $\forall\, qa.\ qa - x = qa$
by (*simp add*: *add-nonneg-eq-0-iff minus-octo.ctr*)
then show $x = 0$
by *simp*
qed *auto*

2.3 A Normed Vector Space

instantiation *octo* :: *real-vector*

begin

primcorec *scaleR-octo*
where
Ree $(scaleR\ r\ x) = r * Ree\ x$
| *Im1* $(scaleR\ r\ x) = r * Im1\ x$
| *Im2* $(scaleR\ r\ x) = r * Im2\ x$
| *Im3* $(scaleR\ r\ x) = r * Im3\ x$
| *Im4* $(scaleR\ r\ x) = r * Im4\ x$
| *Im5* $(scaleR\ r\ x) = r * Im5\ x$
| *Im6* $(scaleR\ r\ x) = r * Im6\ x$
| *Im7* $(scaleR\ r\ x) = r * Im7\ x$

instance
by *standard* (*auto simp*: *octo-eq-iff distrib-left distrib-right scaleR-add-right*)

end

instantiation *octo*::*one*
begin
primcorec *one-octo*

where
Ree 1 = 1 | *Im1 1 = 0* | *Im2 1 = 0* | *Im3 1 = 0* |
Im4 1 = 0 | *Im5 1 = 0* | *Im6 1 = 0* | *Im7 1 = 0*

instance by *standard*
end

fun *octo-proj*
where
octo-proj x 0 = (Ree (x))
| *octo-proj x (Suc 0) = (Im1(x))*
| *octo-proj x (Suc (Suc 0)) = (Im2 (x))*
| *octo-proj x (Suc (Suc (Suc 0))) = (Im3(x))*
| *octo-proj x (Suc (Suc (Suc (Suc 0)))) = (Im4(x))*
| *octo-proj x (Suc(Suc (Suc (Suc (Suc 0))))) = (Im5(x))*
| *octo-proj x (Suc(Suc (Suc (Suc (Suc (Suc 0)))))) = (Im6(x))*
| *octo-proj x (Suc(Suc(Suc (Suc (Suc (Suc (Suc 0))))))) = (Im7(x))*

lemma *octo-proj-add*:
assumes $i \leq 7$
shows *octo-proj (x+y) i = octo-proj x i + octo-proj y i*

proof −
consider *i = 0* | *i = 1* | *i = 2* | *i = 3* | *i = 4* | *i = 5* | *i = 6* | *i = 7*
using *assms* **by** *force*

then show *?thesis*
by *cases (auto simp: numeral-2-eq-2 numeral-3-eq-3 numeral-4-eq-4 numeral-5-eq-5*

numeral-6-eq-6 numeral-7-eq-7 numeral-7-eq-7)
qed

instantiation *octo ::real-normed-vector*
begin

definition *norm x = sqrt* $((Ree\ x)^2 + (Im1\ x)^2 + (Im2\ x)^2 + (Im3\ x)^2 +$
$(Im4\ x)^2 + (Im5\ x)^2 + (Im6\ x)^2 + (Im7\ x)^2$) **for** *x::octo*

definition *sgn x = x* $/_R$ *norm x* **for** *x :: octo*

definition *dist x y = norm (x − y)* **for** *x y :: octo*

definition [*code del*]:
(*uniformity :: (octo × octo) filter*) = (*INF e∈{0 <..}. principal {(x, y). dist x y < e}*)

definition [*code del*]:
open (U :: octo set) ⟷ (∀ *x∈U. eventually* (λ(*x′, y*). *x′ = x* ⟶ *y ∈ U*) *uniformity*)

lemma *norm-eq-L2*: *norm x* = *L2-set* (*octo-proj x*) {..*7*}
  by (*simp add*: *norm-octo-def L2-set-def eval-nat-numeral*)

instance proof
  fix *r* :: *real* **and** *x y* :: *octo* **and** *S* :: *octo set*
  show (*norm x* = *0*) ⟷ (*x* = *0*)
    by (*simp add*: *norm-octo-def octo-eq-iff add-nonneg-eq-0-iff*)
  have *eq*: *L2-set* (*octo-proj* (*x* + *y*)) {..*7*} = *L2-set* (λ*i*. *octo-proj x i* + *octo-proj y i*) {..*7*}
    by (*rule L2-set-cong*) (*auto simp*: *octo-proj-add*)
  show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
    by (*simp add*: *norm-eq-L2 eq L2-set-triangle-ineq*)
  show *norm* (*scaleR r x*) = |*r*| ∗ *norm x*
    by (*simp add*: *norm-octo-def octo-eq-iff*
          power-mult-distrib distrib-left [*symmetric*] *real-sqrt-mult*)
qed (*rule sgn-octo-def dist-octo-def open-octo-def uniformity-octo-def*)+

end

lemma *norm-octo-squared*:
  norm x ^ *2* = *Ree x* ^ *2* + *Im1 x* ^ *2* + *Im2 x* ^ *2* + *Im3 x* ^ *2* +
    Im4 x ^ *2* + *Im5 x* ^ *2* + *Im6 x* ^ *2* + *Im7 x* ^ *2*
  by (*simp add*: *norm-octo-def*)

instantiation *octo* :: *real-inner*
begin

definition *inner-octo* **where**
  inner-octo x y = *Ree x* ∗ *Ree y* + *Im1 x* ∗ *Im1 y* + *Im2 x* ∗ *Im2 y* + *Im3 x* ∗ *Im3 y*
    + *Im4 x* ∗ *Im4 y* + *Im5 x* ∗ *Im5 y* + *Im6 x* ∗ *Im6 y* + *Im7 x* ∗ *Im7 y* **for** *x y*::*octo*

instance
  by *standard* (*auto simp*: *inner-octo-def algebra-simps norm-octo-def*
          power2-eq-square octo-eq-iff add-nonneg-eq-0-iff)

end

lemma *octo-inner-1* [*simp*]: *inner 1 x* = *Ree x*
 and *octo-inner-1-right* [*simp*]: *inner x 1* = *Ree x*
  unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e1-left* [*simp*]: *inner e1 x* = *Im1 x*
 and *octo-inner-e1-right* [*simp*]: *inner x e1* = *Im1 x*
  unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e2-left* [*simp*]: *inner e2 x* = *Im2 x*

and *octo-inner-e2-right* [*simp*]: *inner x e2* = *Im2 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e3-left* [*simp*]: *inner e3 x* = *Im3 x*
and *octo-inner-e3-right* [*simp*]: *inner x e3* = *Im3 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e4-left* [*simp*]: *inner e4 x* = *Im4 x*
and *octo-inner-e4-right* [*simp*]: *inner x e4* = *Im4 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e5-left* [*simp*]: *inner e5 x* = *Im5 x*
and *octo-inner-e5-right* [*simp*]: *inner x e5* = *Im5 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e6-left* [*simp*]: *inner e6 x* = *Im6 x*
and *octo-inner-e6-right* [*simp*]: *inner x e6* = *Im6 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-inner-e7-left* [*simp*]: *inner e7 x* = *Im7 x*
and *octo-inner-e7-right* [*simp*]: *inner x e7* = *Im7 x*
unfolding *inner-octo-def* **by** *simp-all*

lemma *octo-norm-pow-2-inner*: (*norm x*) ^ *2* = *inner x x* **for** *x*::*octo*
by (*simp add*: *dot-square-norm*)

lemma *octo-norm-property*:
inner x y = (*1*/*2*)∗ ((*norm*(*x*+*y*))^*2* − (*norm*(*x*))^*2* − (*norm*(*y*))^*2*) **for** *x y* ::*octo*
by (*simp add*: *dot-norm norm-octo-def*)

2.4 The Octonionic product and related properties and lemmas

The multiplication is defined following one of the 480 equivalent multiplication tables in an analogy to the definition of the 7D cross product.

instantiation *octo* :: *times*
begin

definition *times-octo* :: [*octo*, *octo*] ⇒ *octo*
where
(*a* ∗ *b*) = (*let*
t0 = *Ree a* ∗ *Ree b* − *Im1 a* ∗ *Im1 b* − *Im2 a* ∗ *Im2 b*− *Im3 a* ∗ *Im3 b*
− *Im4 a* ∗ *Im4 b* − *Im5 a* ∗ *Im5 b* − *Im6 a* ∗ *Im6 b* −*Im7 a* ∗ *Im7 b* ;
t1 = *Ree a* ∗ *Im1 b* + *Im1 a* ∗ *Ree b* + *Im2 a* ∗ *Im4 b* +*Im3 a* ∗ *Im7 b* −
Im4 a ∗ *Im2 b* +*Im5 a* ∗ *Im6 b* − *Im6 a* ∗ *Im5 b* − *Im7 a* ∗ *Im3 b* ;
t2 = *Ree a* ∗ *Im2 b* − *Im1 a* ∗ *Im4 b*+ *Im2 a* ∗ *Ree b* + *Im3 a* ∗ *Im5 b*
+ *Im4 a* ∗ *Im1 b* − *Im5 a* ∗ *Im3 b* + *Im6 a* ∗ *Im7 b* − *Im7 a* ∗*Im6 b* ;
t3 = *Ree a* ∗ *Im3 b* −*Im1 a* ∗ *Im7 b* −*Im2 a* ∗*Im5 b* +*Im3 a* ∗ *Ree b* + *Im4*

```
a * Im6 b
        +  Im5 a *Im2 b  - Im6 a * Im4 b + Im7 a * Im1 b ;
   t4 = Ree a * Im4 b + Im1 a * Im2 b - Im2 a * Im1 b -Im3 a * Im6 b + Im4
a * Ree b
       +Im5 a * Im7 b +Im6 a * Im3 b -Im7 a * Im5 b ;
   t5 = Ree a * Im5 b - Im1 a * Im6 b +Im2 a * Im3 b -Im3 a * Im2 b -Im4
a * Im7 b
        +Im5 a * Ree b  +Im6 a * Im1 b + Im7 a  * Im4 b;
   t6 = Ree a * Im6 b + Im1 a * Im5 b - Im2 a * Im7 b +Im3 a * Im4 b - Im4
a * Im3 b
       -Im5 a * Im1 b + Im6 a * Ree b + Im7 a * Im2 b  ;
   t7 = Ree a * Im7 b + Im1 a * Im3 b +Im2 a * Im6 b - Im3 a * Im1 b + Im4
a * Im5 b
         -Im5 a * Im4 b - Im6 a * Im2 b +Im7 a * Ree b
 in Octo t0 t1 t2 t3 t4 t5 t6 t7)
```

instance by *standard*

end

instantiation *octo* ::*inverse*
begin

```
primcorec inverse-octo
 where
   Ree (inverse x) = Ree x / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x ^ 2
     +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2   )
 | Im1 (inverse x) = - (Im1 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
           +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2)
 | Im2 (inverse x) = - (Im2 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
  +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2 )
 | Im3 (inverse x) = - (Im3 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
    +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2    )
 | Im4 (inverse x) = - (Im4 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
       +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2    )
 | Im5 (inverse x) = - (Im5 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
  +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2)
 | Im6 (inverse x) = - (Im6 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
      +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2    )
 | Im7 (inverse x) = - (Im7 x) / (Ree x ^ 2 + Im1 x ^ 2 + Im2 x ^ 2 + Im3 x
^ 2
    +Im4 x ^ 2 + Im5 x ^ 2 + Im6 x ^ 2 + Im7 x ^2    )
```

definition *x div y* = *x* ∗ (*inverse y*) **for** *x y* :: *octo*

instance by *standard*

end

lemma *octo-mult-components*:
Ree (*x* ∗ *y*) = *Ree x* ∗ *Ree y* − *Im1 x* ∗ *Im1 y* − *Im2 x* ∗ *Im2 y* − *Im3 x* ∗ *Im3 y*
− *Im4 x* ∗ *Im4 y* − *Im5 x* ∗ *Im5 y* − *Im6 x* ∗ *Im6 y*− *Im7 x* ∗ *Im7 y*
Im1 (*x* ∗ *y*) = *Ree x* ∗ *Im1 y* + *Im1 x* ∗ *Ree y* + *Im2 x* ∗ *Im4 y* +*Im3 x* ∗ *Im7 y* −
Im4 x ∗ *Im2 y* +*Im5 x* ∗ *Im6 y* − *Im6 x* ∗ *Im5 y* − *Im7 x* ∗ *Im3 y*
Im2 (*x* ∗ *y*) = *Ree x* ∗ *Im2 y* − *Im1 x* ∗ *Im4 y*+ *Im2 x* ∗ *Ree y* + *Im3 x* ∗ *Im5 y*
+ *Im4 x* ∗ *Im1 y* − *Im5 x* ∗ *Im3 y* + *Im6 x* ∗ *Im7 y* − *Im7 x* ∗*Im6 y*
Im3 (*x* ∗ *y*) = *Ree x* ∗ *Im3 y* −*Im1 x* ∗ *Im7 y* −*Im2 x* ∗*Im5 y* +*Im3 x* ∗ *Ree y* + *Im4 x* ∗ *Im6 y*
+ *Im5 x* ∗*Im2 y* − *Im6 x* ∗ *Im4 y* + *Im7 x* ∗ *Im1 y*
Im4 (*x* ∗*y*) = *Ree x* ∗ *Im4 y* + *Im1 x* ∗ *Im2 y* − *Im2 x* ∗ *Im1 y* −*Im3 x* ∗ *Im6 y* + *Im4 x* ∗ *Ree y*
+*Im5 x* ∗ *Im7 y* +*Im6 x* ∗ *Im3 y* −*Im7 x* ∗ *Im5 y*
Im5 (*x* ∗ *y*) = *Ree x* ∗ *Im5 y* − *Im1 x* ∗ *Im6 y* +*Im2 x* ∗ *Im3 y* −*Im3 x* ∗ *Im2 y* −*Im4 x* ∗ *Im7 y*
+*Im5 x* ∗ *Ree y* +*Im6 x* ∗ *Im1 y* + *Im7 x* ∗ *Im4 y*
Im6 (*x* ∗ *y*) = *Ree x* ∗ *Im6 y* + *Im1 x* ∗ *Im5 y* − *Im2 x* ∗ *Im7 y* +*Im3 x* ∗ *Im4 y* − *Im4 x* ∗ *Im3 y*
−*Im5 x* ∗ *Im1 y* + *Im6 x* ∗ *Ree y* + *Im7 x* ∗ *Im2 y*
Im7 (*x* ∗ *y*) = *Ree x* ∗ *Im7 y* + *Im1 x* ∗ *Im3 y* +*Im2 x* ∗ *Im6 y* − *Im3 x* ∗ *Im1 y* + *Im4 x* ∗ *Im5 y*
−*Im5 x* ∗ *Im4 y* − *Im6 x* ∗ *Im2 y* +*Im7 x* ∗ *Ree y*
unfolding *times-octo-def* **by** *auto*

lemma *octo-distrib-left* :
a ∗ (*b* + *c*) = *a* ∗ *b* + *a* ∗ *c* **for** *a b c* ::*octo*
unfolding *times-octo-def plus-octo-def minus-octo-def uminus-octo-def scaleR-octo-def*
by (*simp add*: *octo-eq-iff octo-mult-components algebra-simps*)

lemma *octo-distrib-right* :
(*b* + *c*) ∗ *a* = *b* ∗ *a* + *c* ∗ *a* **for** *a b c* ::*octo*
unfolding *times-octo-def plus-octo-def minus-octo-def uminus-octo-def scaleR-octo-def*
by (*simp add*: *octo-eq-iff octo-mult-components algebra-simps*)

lemma *multiplicative-norm-octo*: *norm* (*x* ∗ *y*) = *norm x* ∗ *norm y* **for** *x y* ::*octo*
proof −
have *norm* (*x* ∗ *y*) ^ *2* = *norm x* ^ *2* ∗ *norm y* ^ *2*
unfolding *norm-octo-squared octo-mult-components* **by** *algebra*
also have ... = (*norm x* ∗ *norm y*) ^ *2*

```
  by (simp add: power-mult-distrib)
 finally show ?thesis by simp
qed

lemma mult-1-right-octo [simp]: x * 1 = (x :: octo)
 and mult-1-left-octo [simp]: 1 * x = (x :: octo)
 by (simp-all add: times-octo-def)

instance octo :: power ..

lemma power2-eq-square-octo: x ^ 2 = (x * x :: octo)
 by (simp add: numeral-2-eq-2 times-octo-def)

lemma octo-product-alternative-left: x * (x * y) = (x * x :: octo) * y
 unfolding octo-eq-iff octo-mult-components by algebra

lemma octo-product-alternative-right: x * (y * y) = (x * y :: octo) * y
 unfolding octo-eq-iff octo-mult-components by algebra

lemma octo-product-flexible: (x * y) * x = x * (y * x :: octo)
 unfolding octo-eq-iff octo-mult-components by algebra

lemma octo-power-commutes: x ^ y * x = x * (x ^ y :: octo)
 by (induction y) (simp-all add: octo-product-flexible)

lemma octo-product-noncommutative: ¬(∀ x y::octo. (x * y = y * x))
 by (auto simp: times-octo-def)
    (metis (no-types, lifting) Im1-def add-0 mult.commute mult-1 mult-zero-left
octo.case
    octo-e5.simps(2) octo-e5.simps(3) octo-e5.simps(4) octo-e5.simps(5) octo-e5.simps(6)
     octo-e5.simps(8) zero-neq-numeral)

lemma octo-product-nonassociative :
 ¬(∀ x y z::octo. x * (y * z) = (x * y) * z)
proof−
 define x where x = Octo 1 0 0 0 1 0 0 0
 define y where y = Octo 1 3 0 0 0 1 0 0
 define z where z = Octo 0 1 0 1 1 1 0 0
 have x * (y * z) ≠ (x * y) * z
  by (simp add: octo-eq-iff octo-mult-components x-def y-def z-def)
 thus ?thesis by blast
qed
```

2.5 Embedding of the Reals into the Octonions

```
definition octo-of-real :: real ⇒ octo
 where octo-of-real r = scaleR r 1

definition octo-of-nat :: nat ⇒ octo
```

where *octo-of-nat r = scaleR r 1*

definition *octo-of-int* :: *int* ⇒ *octo*
where *octo-of-int r = scaleR r 1*

lemma *octo-of-nat-sel* [*simp*]:
Ree (*octo-of-nat x*) = *of-nat x Im1* (*octo-of-nat x*) = *0 Im2* (*octo-of-nat x*) = *0*
Im3 (*octo-of-nat x*) = *0 Im4* (*octo-of-nat x*) = *0 Im5* (*octo-of-nat x*) = *0*
Im6 (*octo-of-nat x*) = *0 Im7* (*octo-of-nat x*) = *0*
by (*simp-all add*: *octo-of-nat-def*)

lemma *octo-of-real-sel* [*simp*]:
Ree (*octo-of-real x*) = *x Im1* (*octo-of-real x*) = *0 Im2* (*octo-of-real x*) = *0*
Im3 (*octo-of-real x*) = *0 Im4* (*octo-of-real x*) = *0 Im5* (*octo-of-real x*) = *0*
Im6 (*octo-of-real x*) = *0 Im7* (*octo-of-real x*) = *0*
by (*simp-all add*: *octo-of-real-def*)

lemma *octo-of-int-sel* [*simp*]:
Ree (*octo-of-int x*) = *of-int x Im1* (*octo-of-int x*) = *0 Im2* (*octo-of-int x*) = *0*
Im3 (*octo-of-int x*) = *0 Im4* (*octo-of-int x*) = *0 Im5* (*octo-of-int x*) = *0*
Im6 (*octo-of-int x*) = *0 Im7* (*octo-of-int x*) = *0*
by (*simp-all add*: *octo-of-int-def*)

lemma *scaleR-conv-octo-of-real*: *scaleR r x = octo-of-real r ∗ x*
by (*simp add*: *octo-eq-iff octo-mult-components octo-of-real-def*)

lemma *octo-of-real-0* [*simp*]: *octo-of-real 0 = 0*
by (*simp add*: *octo-of-real-def*)

lemma *octo-of-real-1* [*simp*]: *octo-of-real 1 = 1*
by (*simp add*: *octo-of-real-def*)

lemma *octo-of-real-add* [*simp*]: *octo-of-real* (*x + y*) = *octo-of-real x + octo-of-real y*
by (*simp add*: *octo-of-real-def scaleR-left-distrib*)

lemma *octo-of-real-minus* [*simp*]: *octo-of-real* (− *x*) = − *octo-of-real x*
by (*simp add*: *octo-of-real-def*)

lemma *octo-of-real-diff* [*simp*]: *octo-of-real* (*x − y*) = *octo-of-real x − octo-of-real y*
by (*simp add*: *octo-of-real-def scaleR-left-diff-distrib*)

lemma *octo-of-real-mult* [*simp*]: *octo-of-real* (*x ∗ y*) = *octo-of-real x ∗ octo-of-real y*
using *octo-of-real-def*
by (*metis scaleR-conv-octo-of-real scaleR-scaleR*)

lemma *octo-of-real-sum*[*simp*]: *octo-of-real* (*sum f s*) = ($\sum$ *x*∈*s*. *octo-of-real* (*f x*))

by (*induct s rule*: *infinite-finite-induct*) *auto*

lemma *octo-of-real-power* [*simp*]:
octo-of-real $(x \,\widehat{}\, y)$ = (*octo-of-real x* :: *octo*) $\widehat{}\ y$
by (*induct y*)(*simp-all*)

lemma *octo-of-real-eq-iff* [*simp*]: *octo-of-real x* = *octo-of-real y* $\longleftrightarrow$ $x = y$
using *octo-of-real-def*
by (*simp add*: *octo-of-real-def one-octo.code zero-octo.code*)

lemmas *octo-of-real-eq-0-iff* [*simp*] = *octo-of-real-eq-iff* [*of - 0*, *simplified*]
lemmas *octo-of-real-eq-1-iff* [*simp*] = *octo-of-real-eq-iff* [*of - 1*, *simplified*]

lemma *minus-octo-of-real-eq-octo-of-real-iff* [*simp*]: −*octo-of-real x* = *octo-of-real y* $\longleftrightarrow$ $-x = y$
using *octo-of-real-eq-iff*[*of* $-x$ *y*] **by** (*simp only*: *octo-of-real-minus*)

lemma *octo-of-real-eq-minus-of-real-iff* [*simp*]: *octo-of-real x* = −*octo-of-real y* $\longleftrightarrow$ $x = -y$
using *octo-of-real-eq-iff*[*of x* $-y$] **by** (*simp only*: *octo-of-real-minus*)

lemma *octo-of-real-of-nat-eq* [*simp*]: *octo-of-real* (*of-nat x*) = *octo-of-nat x*
unfolding *octo-of-real-def*
by (*simp add*: *octo-of-nat-def*)

lemma *octo-of-real-of-int-eq* [*simp*]: *octo-of-real* (*of-int z*) = *octo-of-int z*
unfolding *octo-of-real-def*
by (*simp add*: *octo-of-int-def*)

lemma *octo-of-int-of-nat*: *octo-of-int* (*of-nat n*) = *octo-of-nat n*
by (*simp add*: *octo-eq-iff*)

lemma *octo-of-nat-add* [*simp*]: *octo-of-nat* $(a + b)$ = *octo-of-nat a* + *octo-of-nat b*
and *octo-of-nat-mult* [*simp*]: *octo-of-nat* $(a * b)$ = *octo-of-nat a* ∗ *octo-of-nat b*
and *octo-of-nat-diff* [*simp*]: $b \le a \Longrightarrow$ *octo-of-nat* $(a - b)$ = *octo-of-nat a* − *octo-of-nat b*
and *octo-of-nat-0* [*simp*]: *octo-of-nat 0* = *0*
and *octo-of-nat-1* [*simp*]: *octo-of-nat 1* = *1*
and *octo-of-nat-Suc-0* [*simp*]: *octo-of-nat* (*Suc 0*) = *1*
by (*simp-all add*: *octo-eq-iff octo-mult-components*)

lemma *octo-of-int-add* [*simp*]: *octo-of-int* $(a + b)$ = *octo-of-int a* + *octo-of-int b*
and *octo-of-int-mult* [*simp*]: *octo-of-int* $(a * b)$ = *octo-of-int a* ∗ *octo-of-int b*
and *octo-of-int-diff* [*simp*]: $b \le a \Longrightarrow$ *octo-of-int* $(a - b)$ = *octo-of-int a* − *octo-of-int b*
and *octo-of-int-0* [*simp*]: *octo-of-int 0* = *0*
and *octo-of-int-1* [*simp*]: *octo-of-int 1* = *1*
by (*simp-all add*: *octo-eq-iff octo-mult-components*)

instance *octo* :: *numeral* **..**

lemma *numeral-octo-conv-of-nat*: *numeral x* = *octo-of-nat* (*numeral x*)
proof (*induction x*)
  case(*Bit0 x*)
  have *numeral x*+ *numeral x* = *octo-of-nat*(*numeral x*+ *numeral x*)
    unfolding *Bit0.IH octo-of-nat-add* **..**
  thus *?case* **by** (*simp add*: *numeral-Bit0*)
next
 case(*Bit1 x*)
  have *numeral x*+ *numeral x* + *numeral num.One*=
octo-of-nat (*numeral x* + *numeral x* + *numeral num.One*)
    unfolding *Bit1.IH octo-of-nat-add* **by** *simp*
  thus *?case* **by** (*simp add*: *numeral-Bit1*)
qed *auto*

lemma *numeral-octo-sel* [*simp*]:
  Ree (*numeral n*) = *numeral n Im1* (*numeral n*) = *0 Im2* (*numeral n*) = *0*
  Im3 (*numeral n*) = *0 Im4* (*numeral n*) = *0 Im5* (*numeral n*) = *0*
  Im6 (*numeral n*) = *0 Im7* (*numeral n*) = *0*
  by (*simp-all add*: *numeral-octo-conv-of-nat*)

lemma *octo-of-real-numeral* [*simp*]: *octo-of-real* (*numeral w*) = *numeral w*
  by (*simp add*: *numeral-octo-conv-of-nat octo-of-real-def octo-of-nat-def*)

lemma *octo-of-real-neg-numeral* [*simp*]: *octo-of-real* (− *numeral w*) = − *numeral w*
  by *simp*

lemma *octo-of-real-times-commute*: *octo-of-real r* ∗ *q* = *q* ∗ *octo-of-real r*
  using *octo-of-real-def times-octo-def* **by** *simp*

lemma *octo-of-real-times-conv-scaleR*: *octo-of-real x* ∗ *y* = *scaleR x y*
  by (*simp add*: *octo-eq-iff octo-mult-components*)

lemma *octo-mult-scaleR-left*: ($r *_R x$) ∗ *y* = $r *_R$ (*x* ∗ *y* :: *octo*)
  by (*simp add*: *octo-eq-iff octo-mult-components algebra-simps*)

lemma *octo-mult-scaleR-right*: *x* ∗ ($r *_R y$) = $r *_R$ (*x* ∗ *y* :: *octo*)
  by (*simp add*: *octo-eq-iff octo-mult-components algebra-simps*)

lemma *scaleR-octo-of-real* [*simp*]: *scaleR r* (*octo-of-real s*) = *octo-of-real* (*r* ∗ *s*)
  by (*simp add*: *octo-of-real-def*)

lemma *octo-of-real-times-left-commute*: *octo-of-real r* ∗ (*x* ∗ *q*) = *x* ∗ (*octo-of-real r* ∗ *q*)
  unfolding *octo-of-real-times-conv-scaleR* **by** (*simp add*: *octo-mult-scaleR-right*)

lemma *nonzero-octo-of-real-inverse*:
  $x \neq 0 \Longrightarrow$ *octo-of-real* (*inverse* x) = *inverse* (*octo-of-real* x :: *octo*)
  by (*simp add*: *octo-eq-iff power2-eq-square divide-simps*)

lemma *octo-of-real-inverse* [*simp*]:
  octo-of-real (*inverse* x) = *inverse* (*octo-of-real* x)
  by (*simp add*: *octo-eq-iff power2-eq-square divide-simps*)

lemma *nonzero-octo-of-real-divide*:
  $y \neq 0 \Longrightarrow$ *octo-of-real* (x / y) = (*octo-of-real* x / *octo-of-real* y ::*octo*)
   by (*simp add*: *divide-inverse divide-octo-def*)

lemma *octo-of-real-divide* [*simp*]:
  octo-of-real (x / y) = (*octo-of-real* x / *octo-of-real* y :: *octo*)
  using *divide-inverse divide-octo-def octo-of-real-def octo-of-real-inverse*
  by (*metis octo-of-real-mult*)

lemma *octo-of-real-inverse-collapse* [*simp*]:
  assumes $c \neq 0$
  shows *octo-of-real* c ∗ *octo-of-real* (*inverse* c) = *1*
        octo-of-real (*inverse* c) ∗ *octo-of-real* c = *1*
 using *assms* **by** (*simp-all add*: *octo-eq-iff octo-mult-components power2-eq-square*)

lemma *octo-divide-numeral*:
  fixes x::*octo* **shows** x / *numeral* y = x $/_R$ *numeral* y
  using *octo-of-real-times-commute*[*of inverse* (*numeral* y)]
  by (*simp add*: *scaleR-conv-octo-of-real divide-octo-def flip*: *octo-of-real-numeral*)

lemma *octo-divide-numeral-sel* [*simp*]:
  Ree (x / *numeral* w) = *Ree* x / *numeral* w
  Im1 (x / *numeral* w) = *Im1* x / *numeral* w
  Im2 (x / *numeral* w) = *Im2* x / *numeral* w
  Im3 (x / *numeral* w) = *Im3* x / *numeral* w
  Im4 (x / *numeral* w) = *Im4* x / *numeral* w
  Im5 (x / *numeral* w) = *Im5* x / *numeral* w
  Im6 (x / *numeral* w) = *Im6* x / *numeral* w
  Im7 (x / *numeral* w) = *Im7* x / *numeral* w
  unfolding *octo-divide-numeral* **by** *simp-all*

lemma *octo-norm-units* [*simp*]:
  norm octo-e1 = *1* *norm* (*e2*::*octo*) = *1* *norm* (*e3*::*octo*) = *1*
  norm (*e4*::*octo*) = *1*   *norm* (*e5*::*octo*) = *1*   *norm* (*e6*::*octo*) = *1* *norm* (*e7*::*octo*) = *1*
  by (*auto simp*: *norm-octo-def*)

lemma *e1-nz* [*simp*]: *e1* $\neq$ *0*
  and *e2-nz* [*simp*]: *e2* $\neq$ *0*
  and *e3-nz* [*simp*]: *e3* $\neq$ *0*
  and *e4-nz* [*simp*]: *e4* $\neq$ *0*

and *e5-nz* [*simp*]: $e5 \neq 0$
and *e6-nz* [*simp*]: $e6 \neq 0$
and *e7-nz* [*simp*]: $e7 \neq 0$
by (*simp-all add*: *octo-eq-iff*)

2.6 "Expansion" into the traditional notation

lemma *octo-unfold*:
$q = (Ree\ q) *_R e0 + (Im1\ q) *_R e1 + (Im2\ q) *_R e2 + (Im3\ q) *_R e3$
$+ (Im4\ q) *_R e4 + (Im5\ q) *_R e5 + (Im6\ q) *_R e6 + (Im7\ q) *_R e7$
by (*simp add*: *octo-eq-iff*)

lemma *octo-trad*: $Octo\ x\ y\ z\ w\ u\ v\ q\ g =$
$x *_R e0 + y *_R e1 + z *_R e2 + w *_R e3 + u *_R e4 + v *_R e5 + q *_R$
$e6 + g *_R e7$
by (*simp add*: *octo-eq-iff*)

lemma *octo-of-real-eq-Octo*: $octo\text{-}of\text{-}real\ a = Octo\ a\ 0\ 0\ 0\ 0\ 0\ 0\ 0$
by (*simp add*: *octo-eq-iff*)

lemma *e1-squared* [*simp*]: $e1 \hat{} 2 = -1$
and *e2-squared* [*simp*]: $e2 \hat{} 2 = -1$
and *e3-squared* [*simp*]: $e3 \hat{} 2 = -1$
and *e4-squared* [*simp*]: $e4 \hat{} 2 = -1$
and *e5-squared* [*simp*]: $e5 \hat{} 2 = -1$
and *e6-squared* [*simp*]: $e6 \hat{} 2 = -1$
and *e7-squared* [*simp*]: $e7 \hat{} 2 = -1$
by (*simp-all add*: *octo-eq-iff power2-eq-square-octo octo-mult-components*)

lemma *inverse-e1* [*simp*]: $inverse\ e1 = -e1$
and *inverse-e2* [*simp*]: $inverse\ e2 = -e2$
and *inverse-e3* [*simp*]: $inverse\ e3 = -e3$
and *inverse-e4* [*simp*]: $inverse\ e4 = -e4$
and *inverse-e5* [*simp*]: $inverse\ e5 = -e5$
and *inverse-e6* [*simp*]: $inverse\ e6 = -e6$
and *inverse-e7* [*simp*]: $inverse\ e7 = -e7$
by (*simp-all add*: *octo-eq-iff*)

2.7 Conjugate of an octonion and related properties.

primcorec *cnj* :: $octo \Rightarrow octo$
where
$Ree\ (cnj\ z) = Ree\ z$
$|\ Im1\ (cnj\ z) = -\ Im1\ z$
$|\ Im2\ (cnj\ z) = -\ Im2\ z$
$|\ Im3\ (cnj\ z) = -\ Im3\ z$
$|\ Im4\ (cnj\ z) = -\ Im4\ z$
$|\ Im5\ (cnj\ z) = -\ Im5\ z$
$|\ Im6\ (cnj\ z) = -\ Im6\ z$
$|\ Im7\ (cnj\ z) = -\ Im7\ z$

lemma *cnj-cancel-iff* [*simp*]: *cnj* x = *cnj* y ⟷ x = y

proof
  show *cnj* x = *cnj* y ⟹ x = y
    by (*simp add*: *octo-eq-iff*)
qed *auto*

lemma *cnj-cnj* [*simp*]:
   cnj(*cnj* q) = q
 by (*simp add*: *octo-eq-iff*)

lemma *cnj-of-real* [*simp*]: *cnj*(*octo-of-real* x) = *octo-of-real* x
  using *octo-eq-iff*
  by (*simp add*: *octo-of-real-eq-Octo*)

lemma *cnj-zero* [*simp*]: *cnj* *0* = *0*
  by (*simp add*: *octo-eq-iff*)

lemma *cnj-zero-iff* [*iff*]: *cnj* z = *0* ⟷ z = *0*
 using *cnj-cnj* **by** (*metis cnj-zero*)

lemma *cnj-one* [*simp*]: *cnj* *1* = *1*
  by (*simp add*: *octo-eq-iff*)

lemma *cnj-one-iff* [*simp*]: *cnj* z = *1* ⟷ z = *1*
  by (*simp add*: *octo-eq-iff*)

lemma *octo-norm-cnj* [*simp*]: *norm*(*cnj* q) = *norm* q
  by (*simp add*: *norm-octo-def*)

lemma *cnj-add* [*simp*]: *cnj* (x + y) = *cnj* x + *cnj* y
  using *octo-eq-iff inner-real-def of-real-0 of-real-inner-1* **by** *simp*

lemma *cnj-sum* [*simp*]: *cnj* (*sum* f S) = ($\sum x \in S$. *cnj* (f x))
  by (*induct* S *rule*: *infinite-finite-induct*) *auto*

lemma *cnj-diff* [*simp*]: *cnj* (x − y) = *cnj* x − *cnj* y
  using *octo-eq-iff*
  by (*metis add.commute add-left-cancel cnj-add diff-add-cancel*)

lemma *cnj-minus* [*simp*]: *cnj* (− x) = − *cnj* x
  using *octo-eq-iff* *cnj-cnj* **by** *auto*

lemma *cnj-inverse* [*simp*]: *cnj* (*inverse* x) = *inverse* (*cnj* x) **for** x y ::*octo*
  using *octo-eq-iff inner-real-def real-inner-1-right* **by** *auto*

lemma *cnj-mult* [*simp*]: *cnj* (x ∗ y) = *cnj* y ∗ *cnj* x **for** x y ::*octo*
  using *octo-eq-iff* *times-octo-def octo-mult-components cnj-cnj*

mult-not-zero nonzero-inverse-mult-distrib **by** *simp*

lemma *cnj-divide* [*simp*]: *cnj* (*x* / *y*) = (*inverse* (*cnj y*)) $*$ *cnj x*
for *x y* ::*octo*
unfolding *divide-octo-def times-octo-def*
using *cnj-inverse cnj-mult octo-mult-components* **by** (*metis times-octo-def*)

lemma *cnj-power* [*simp*]: *cnj* (*x* ^ *y*) = (*cnj x*) ^ *y* **for** *x*::*octo*
by (*induction y*) (*simp-all add*: *octo-power-commutes*)

lemma *cnj-of-nat* [*simp*]: *cnj* (*octo-of-nat x*) = *octo-of-nat*(*x*)
using *cnj-of-real octo-of-real-of-nat-eq* **by** *metis*

lemma *cnj-of-int* [*simp*]: *cnj* (*octo-of-int x*) = *octo-of-nat*(*x*)
using *octo-of-real-def octo-of-real-of-int-eq octo-of-int-def octo-of-nat-def cnj-of-real* **by** *auto*

lemma *cnj-numeral* [*simp*]: *cnj* (*numeral x*) = *numeral x*
by (*simp add*: *numeral-octo-conv-of-nat*)

lemma *cnj-neg-numeral* [*simp*]: *cnj* (− *numeral x*) = − *numeral x*
by (*simp add*: *numeral-octo-conv-of-nat*)

lemma *cnj-scaleR* [*simp*]: *cnj* (*scaleR r x*) = *scaleR r* (*cnj x*)
using *octo-eq-iff inner-real-def ln-one of-real-inner-1* **by** *simp*

lemma *cnj-units* [*simp*]: *cnj e1* = −*e1* *cnj e2* = −*e2* *cnj e3* = −*e3*
cnj e4 = −*e4* *cnj e5* = −*e5* *cnj e6* = −*e6* *cnj e7* = −*e7*
by (*simp-all add*: *octo-eq-iff*)

lemma *cnj-eq-of-real*: *cnj q* = *octo-of-real x* ⟷ *q* = *octo-of-real x*
proof
show *cnj q* = *octo-of-real x* ⟹ *q* = *octo-of-real x*
by (*metis cnj-of-real cnj-cnj*)
qed *auto*

lemma *octo-trad-cnj* : *cnj q* = (*Ree q*) $*_R$ *e0* − (*Im1 q*) $*_R$ *e1* − (*Im2 q*)$*_R$ *e2* − (*Im3 q*) $*_R$ *e3* −
(*Im4 q*) $*_R$ *e4* − (*Im5 q*) $*_R$ *e5* − (*Im6 q*) $*_R$ *e6* − (*Im7 q*)$*_R$ *e7* **for** *q*::*octo*
using *cnj-cnj octo-unfold octo-trad cnj-def Octonions.cnj.code* **by** *auto*

lemma *octonion-conjugate-property*:
cnj x = −(*1/6*) $*_R$ (*x* + (*e1* $*$ *x*) $*$ *e1* + (*e2* $*$ *x*) $*$ *e2* + (*e3* $*$ *x*) $*$ *e3* +
(*e4* $*$ *x*) $*$ *e4* + (*e5* $*$ *x*) $*$ *e5* + (*e6* $*$ *x*) $*$ *e6* + (*e7* $*$ *x*) $*$ *e7*)
by (*simp add*: *octo-eq-iff octo-mult-components*)

lemma *octo-add-cnj*: *q* + *cnj q* = *2* $*_R$ (*Ree q*) $*_R$ *e0* *cnj q* + *q* = (*2*$*_R$ (*Ree q*)$*_R$ *e0*)
by (*simp-all add*: *octo-eq-iff*)

lemma *octo-add-cnj1*: $q + cnj\ q = octo\text{-}of\text{-}real\ (2 *_R\ (Ree\ q))$
$cnj\ q + q = octo\text{-}of\text{-}real\ (2 *_R\ (Ree\ q))$
by (*auto simp*: *octo-eq-iff octo-mult-components*)

lemma *octo-subtract-cnj*:
$q - cnj\ q = 2 *_R (Im1\ q *_R e1 + Im2\ q *_R e2 + Im3\ q *_R e3 +$
$Im4\ q *_R e4 + Im5\ q *_R e5 + Im6\ q *_R e6 + Im7\ q *_R e7)$
by (*simp add*: *octo-eq-iff*)

lemma *octo-mult-cnj-commute*: $cnj\ x * x = x * cnj\ x$
using *times-octo-def* **by** *auto*

lemma *octo-cnj-mult-conv-norm*: $cnj\ x * x = octo\text{-}of\text{-}real\ (norm\ x)\ \hat{}\ 2$
by (*simp add*: *octo-eq-iff octo-mult-components norm-octo-def power2-eq-square*
flip: *octo-of-real-power*)

lemma *octo-mult-cnj-conv-norm*: $x * cnj\ x = octo\text{-}of\text{-}real\ (norm\ x)\ \hat{}\ 2$
by (*simp add*: *octo-eq-iff octo-mult-components norm-octo-def power2-eq-square*
flip: *octo-of-real-power*)

lemma *octo-mult-cnj-conv-norm-aux*: $octo\text{-}of\text{-}real\ (norm\ x\ \hat{}\ 2) = x * cnj\ x$
using *octo-mult-cnj-conv-norm*[*of x*] **by** (*simp add*: *octo-mult-cnj-commute*)

lemma *octo-norm-conj*: $octo\text{-}of\text{-}real\ (\ inner\ x\ y) = (1/2) *_R (x * (cnj\ y) + y * (cnj\ x))$
by (*simp add*: *octo-eq-iff octo-mult-components inner-octo-def*)

lemma *octo-inverse-cnj*: $inverse\ x = cnj\ x\ /_R\ (norm\ x\ \hat{}\ 2)$
by (*auto simp*: *octo-eq-iff norm-octo-def field-simps*)

lemma *inverse-octo-1*: $x \neq 0 \Longrightarrow x * inverse\ x = (1 :: octo)$
by (*simp add*: *octo-mult-scaleR-right octo-mult-cnj-conv-norm-aux* [*symmetric*]
divide-simps octo-inverse-cnj
del: *octo-of-real-power*)

lemma *inverse-octo-1-sym*: $x \neq 0 \Longrightarrow inverse\ x * x = (1 :: octo)$
by (*metis cnj-cnj cnj-inverse cnj-mult cnj-one cnj-zero inverse-octo-1*)

lemma *inverse-0-octo* [*simp*]: $inverse\ 0 = (0 :: octo)$
by (*simp add*: *octo-eq-iff*)

lemma *inverse-octo-commutes*: $inverse\ x * x = x * (inverse\ x :: octo)$
by (*cases* $x = 0$) (*simp-all add*: *inverse-octo-1 inverse-octo-1-sym*)

lemma *octo-inverse-mult*: $inverse\ (x * y) = inverse\ y * inverse\ x$ **for** *x y*::*octo*
proof−
have $inverse\ (x * y) = (cnj\ y * cnj\ x)\ /_R\ (norm\ (x * y)\ \hat{}\ 2)$
by (*simp add*: *octo-inverse-cnj*)

also have ... $=$ (*cnj y* $/_R$ *norm y* ^ *2*) $*$ (*cnj x* $/_R$ *norm x* ^ *2*)
by (*simp add*: *octo-mult-scaleR-left octo-mult-scaleR-right multiplicative-norm-octo power2-eq-square*)
also have ... $=$ *inverse y* $*$ *inverse x*
by (*simp add*: *octo-inverse-cnj*)
finally show *?thesis* **.**
qed

lemma *octo-inverse-eq-cnj*: *norm q* $=$ *1* $\Longrightarrow$ *inverse q* $=$ *cnj q* **for** *q*::*octo*
by (*simp add*: *octo-inverse-cnj*)

lemma *octo-in-Reals-if-Re*: **fixes** *q* ::*real* **shows** *Ree*(*octo-of-real*(*q*)) $=$ *q*
by *simp*

lemma *octo-in-Reals-if-Re-con*: **assumes** *Ree* (*octo-of-real q*) $=$ *q*
shows *q* $\in$ *Reals*
by (*metis Reals-of-real inner-real-def mult.right-neutral of-real-inner-1*)

lemma *octo-in-Reals-if-cnj*: **fixes** *q*:: *real* **shows** *cnj*(*octo-of-real*(*q*)) $=$ *octo-of-real q*
by *simp*

lemma *octo-in-Reals-if-cnj-con*: **assumes** *cnj*(*octo-of-real*(*q*)) $=$ *octo-of-real q*
shows *q* $\in$ *Reals*
by (*metis Reals-of-real inner-real-def mult.right-neutral of-real-inner-1*)

lemma *norm-power2*: *norm q* ^ *2* $=$ *Ree* (*cnj q* $*$ *q*)
by (*simp add*: *octo-mult-components norm-octo-def power2-eq-square*)

lemma *norm-power2-cnj*: *norm q* ^ *2* $=$ *Ree* (*q* $*$ *cnj q*)
by (*simp add*: *octo-mult-components norm-octo-def power2-eq-square*)

lemma *octo-norm-imaginary*: *Ree x* $=$ *0* $\Longrightarrow$ *x* $*$ *x* $= -$(*octo-of-real* (*norm x*))^2
by (*simp add*: *octo-eq-iff octo-mult-components norm-octo-def power2-eq-square flip*: *octo-of-real-power octo-of-real-mult*)

2.8 Linearity and continuity of the components.

lemma *bounded-linear-Ree*: *bounded-linear Ree*
and *bounded-linear-Im1*: *bounded-linear Im1*
and *bounded-linear-Im2*: *bounded-linear Im2*
and *bounded-linear-Im3*: *bounded-linear Im3*
and *bounded-linear-Im4*: *bounded-linear Im4*
and *bounded-linear-Im5*: *bounded-linear Im5*
and *bounded-linear-Im6*: *bounded-linear Im6*
and *bounded-linear-Im7*: *bounded-linear Im7*
by (*simp-all add*: *bounded-linear-intro* [**where** *K=1*] *norm-octo-def real-le-rsqrt add.assoc*)

lemmas *Cauchy-Ree* = *bounded-linear.Cauchy* [*OF bounded-linear-Ree*]
lemmas *Cauchy-Im1* = *bounded-linear.Cauchy* [*OF bounded-linear-Im1*]
lemmas *Cauchy-Im2* = *bounded-linear.Cauchy* [*OF bounded-linear-Im2*]
lemmas *Cauchy-Im3* = *bounded-linear.Cauchy* [*OF bounded-linear-Im3*]
lemmas *Cauchy-Im4* = *bounded-linear.Cauchy* [*OF bounded-linear-Im4*]
lemmas *Cauchy-Im5* = *bounded-linear.Cauchy* [*OF bounded-linear-Im5*]
lemmas *Cauchy-Im6* = *bounded-linear.Cauchy* [*OF bounded-linear-Im6*]
lemmas *Cauchy-Im7* = *bounded-linear.Cauchy* [*OF bounded-linear-Im7*]

lemmas *tendsto-Re* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Ree*]
lemmas *tendsto-Im1* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im1*]
lemmas *tendsto-Im2* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im2*]
lemmas *tendsto-Im3* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im3*]
lemmas *tendsto-Im4* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im4*]
lemmas *tendsto-Im5* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im5*]
lemmas *tendsto-Im6* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im6*]
lemmas *tendsto-Im7* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im7*]

lemmas *isCont-Ree* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Ree*]
lemmas *isCont-Im1* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im1*]
lemmas *isCont-Im2* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im2*]
lemmas *isCont-Im3* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im3*]
lemmas *isCont-Im4* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im4*]
lemmas *isCont-Im5* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im5*]
lemmas *isCont-Im6* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im6*]
lemmas *isCont-Im7* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im7*]

lemmas *continuous-Ree* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Ree*]
lemmas *continuous-Im1* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im1*]
lemmas *continuous-Im2* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im2*]
lemmas *continuous-Im3* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im3*]
lemmas *continuous-Im4* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im4*]
lemmas *continuous-Im5* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im5*]
lemmas *continuous-Im6* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im6*]
lemmas *continuous-Im7* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im7*]

lemmas *continuous-on-Ree* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Ree*]
lemmas *continuous-on-Im1* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im1*]
lemmas *continuous-on-Im2* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im2*]
lemmas *continuous-on-Im3* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im3*]
lemmas *continuous-on-Im4* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im4*]
lemmas *continuous-on-Im5* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im5*]
lemmas *continuous-on-Im6* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF*

bounded-linear-Im6]
lemmas *continuous-on-Im7* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im7*]

lemmas *has-derivative-Ree* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Ree*]
lemmas *has-derivative-Im1* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im1*]
lemmas *has-derivative-Im2* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im2*]
lemmas *has-derivative-Im3* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im3*]
lemmas *has-derivative-Im4* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im4*]
lemmas *has-derivative-Im5* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im5*]
lemmas *has-derivative-Im6* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im6*]
lemmas *has-derivative-Im7* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im7*]

lemmas *sums-Ree* = *bounded-linear.sums* [*OF bounded-linear-Ree*]
lemmas *sums-Im1* = *bounded-linear.sums* [*OF bounded-linear-Im1*]
lemmas *sums-Im2* = *bounded-linear.sums* [*OF bounded-linear-Im2*]
lemmas *sums-Im3* = *bounded-linear.sums* [*OF bounded-linear-Im3*]
lemmas *sums-Im4* = *bounded-linear.sums* [*OF bounded-linear-Im4*]
lemmas *sums-Im5* = *bounded-linear.sums* [*OF bounded-linear-Im5*]
lemmas *sums-Im6* = *bounded-linear.sums* [*OF bounded-linear-Im6*]
lemmas *sums-Im7* = *bounded-linear.sums* [*OF bounded-linear-Im7*]

2.8.1 Octonionic-specific theorems about sums.

lemma *Ree-sum* [*simp*]: *Ree* (*sum f S*) = *sum* (λx. *Ree*(*f x*)) *S*
and *Im1-sum* [*simp*]: *Im1* (*sum f S*) = *sum* (λx. *Im1* (*f x*)) *S*
and *Im2-sum* [*simp*]: *Im2* (*sum f S*) = *sum* (λx. *Im2* (*f x*)) *S*
and *Im3-sum* [*simp*]: *Im3* (*sum f S*) = *sum* (λx. *Im3* (*f x*)) *S*
and *Im4-sum* [*simp*]: *Im4* (*sum f S*) = *sum* (λx. *Im4* (*f x*)) *S*
and *Im5-sum* [*simp*]: *Im5* (*sum f S*) = *sum* (λx. *Im5* (*f x*)) *S*
and *Im6-sum* [*simp*]: *Im6* (*sum f S*) = *sum* (λx. *Im6* (*f x*)) *S*
and *Im7-sum* [*simp*]: *Im7* (*sum f S*) = *sum* (λx. *Im7* (*f x*)) *S*
by (*induct S rule*: *infinite-finite-induct*; *simp*)+

2.8.2 Bound results for real and imaginary components of limits.

lemma *Ree-tendsto-upperbound*:
$[\![$(*f* $\longrightarrow$ *limit*) *net*; $\forall_F$ *x in net*. *octo.Ree* (*f x*) $\leq$ *b*; *net* $\neq$ *bot*$]\!]$ $\Longrightarrow$ *Ree limit* $\leq$ *b*
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Re*])

lemma *Im1-tendsto-upperbound*:

$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im1\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im1\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im1*])

lemma *Im2-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im2\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im2\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im2*])

lemma *Im3-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im3\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im3\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im3*])

lemma *Im4-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im4\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im4\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im4*])

lemma *Im5-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im5\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im5\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im5*])

lemma *Im6-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im6\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im6\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im6*])

lemma *Im7-tendsto-upperbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ Im7\ (f\ x) \leq b;\ net \neq bot]\!] \Longrightarrow Im7\ limit \leq b$
by (*blast intro*: *tendsto-upperbound* [*OF tendsto-Im7*])

lemma *Ree-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq octo.Ree\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Ree\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Re*])

lemma *Im1-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im1\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im1\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im1*])

lemma *Im2-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im2\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im2\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im2*])

lemma *Im3-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im3\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im3\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im3*])

lemma *Im4-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im4\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im4\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im4*])

lemma *Im5-tendsto-lowerbound*:

$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im5\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im5\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im5*])

lemma *Im6-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im6\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im6\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im6*])

lemma *Im7-tendsto-lowerbound*:
$[\![(f \longrightarrow limit)\ net;\ \forall_F\ x\ in\ net.\ b \leq Im7\ (f\ x);\ net \neq bot]\!] \Longrightarrow b \leq Im7\ limit$
by (*blast intro*: *tendsto-lowerbound* [*OF tendsto-Im7*])

lemma *octo-of-real-continuous* [*continuous-intros*]:
continuous net f $\Longrightarrow$ *continuous net* ($\lambda x.$ *octo-of-real* (*f x*))
by (*auto simp*: *octo-of-real-def intro*: *continuous-intros*)

lemma *octo-of-real-continuous-on* [*continuous-intros*]:
continuous-on S f $\Longrightarrow$ *continuous-on S* ($\lambda x.$ *octo-of-real* (*f x*))
by (*auto simp*: *octo-of-real-def intro*: *continuous-intros*)

lemma *of-real-continuous-iff*: *continuous net* ($\lambda x.$ *octo-of-real* (*f x*)) $\longleftrightarrow$ *continuous net f*
proof *safe*
assume *continuous net* ($\lambda x.$ *octo-of-real* (*f x*))
hence *continuous net* ($\lambda x.$ *Ree* (*octo-of-real* (*f x*)))
by (*rule continuous-Ree*)
thus *continuous net f* **by** *simp*
qed (*auto intro*: *continuous-intros*)

lemma *of-real-continuous-on-iff*:
continuous-on S ($\lambda x.$ *octo-of-real*(*f x*)) $\longleftrightarrow$ *continuous-on S f*
proof *safe*
assume *continuous-on S* ($\lambda x.$ *octo-of-real* (*f x*))
hence *continuous-on S* ($\lambda x.$ *Ree* (*octo-of-real* (*f x*)))
by (*rule continuous-on-Ree*)
thus *continuous-on S f* **by** *simp*
qed (*auto intro*: *continuous-intros*)

2.9 Octonions for describing 7D isometries

2.9.1 The *HIm* operator

definition *HIm* :: *octo* $\Rightarrow$ *real*^*7* **where**
HIm q $\equiv$ *vector*[*Im1 q*, *Im2 q*, *Im3 q*, *Im4 q*, *Im5 q*, *Im6 q*, *Im7 q*]

lemma *HIm-Octo*: *HIm* (*Octo w x y z u v q g*) = *vector*[*x*,*y*,*z*, *u*, *v*, *q*, *g*]
by (*simp add*: *HIm-def*)

lemma *him-eq*: *HIm a* = *HIm b* $\longleftrightarrow$ *Im1 a* = *Im1 b* $\wedge$ *Im2 a* = *Im2 b* $\wedge$ *Im3 a* = *Im3 b*
$\wedge$ *Im4 a* = *Im4 b* $\wedge$ *Im5 a* = *Im5 b* $\wedge$ *Im6 a* = *Im6 b* $\wedge$ *Im7 a* = *Im7 b*

by (*metis HIm-def vector-7*)

lemma *him-of-real* [*simp*]: *HIm*(*octo-of-real a*) = *0*
by (*simp add*:*octo-of-real-eq-Octo HIm-Octo vec-eq-iff vector-def*)

lemma *him-0* [*simp*]: *HIm 0* = *0*
by (*metis him-of-real octo-of-real-0*)

lemma *him-1* [*simp*]: *HIm 1* = *0*
using *HIm-def him-0* **by** *auto*

lemma *him-cnj*: *HIm*(*cnj q*) = − *HIm q*
by (*simp add*: *HIm-def vec-eq-iff vector-def*)

lemma *him-mult-left* [*simp*]: *HIm* (*a* $*_R$ *q*) = *a* $*_R$ *HIm q*
by (*simp add*: *HIm-def vec-eq-iff vector-def*)

lemma *him-mult-right* [*simp*]: *HIm* (*q* ∗ *octo-of-real a*) = *HIm q* ∗ *of-real a*
by (*metis Octonions.scaleR-conv-octo-of-real Real-Vector-Spaces.scaleR-conv-of-real him-mult-left octo-of-real-times-commute semiring-normalization-rules*(*7*))

lemma *him-add* [*simp*]: *HIm* (*x* + *y*) = *HIm x* + *HIm y*
and *him-minus* [*simp*]: *HIm* (−*x*) = − *HIm x*
and *him-diff* [*simp*]: *HIm* (*x* − *y*) = *HIm x* − *HIm y*
by (*simp-all add*: *HIm-def vec-eq-iff vector-def*)

lemma *him-sum* [*simp*]: *HIm* (*sum f S*) = ($\sum$ *x*∈*S*. *HIm* (*f x*))
by (*induct S rule*: *infinite-finite-induct*) *auto*

lemma *linear-him*: *linear HIm*
by (*simp add*: *linearI*)

2.9.2 The *Hv* operator

definition *Hv* :: *real*^*7* ⇒ *octo* **where**
Hv v ≡ *Octo 0* (*v*$*1*) (*v*$*2*) (*v*$*3*) (*v*$*4*) (*v*$*5*) (*v*$*6*) (*v*$*7*)

lemma *Hv-sel* [*simp*]:
Ree (*Hv v*) = *0 Im1* (*Hv v*) = *v* $ *1 Im2* (*Hv v*) = *v* $ *2 Im3* (*Hv v*) = *v* $ *3*
Im4 (*Hv v*) = *v* $ *4 Im5* (*Hv v*) = *v* $ *5 Im6* (*Hv v*) = *v* $ *6 Im7* (*Hv v*) = *v* $ *7*
by (*simp-all add*: *Hv-def*)

lemma *hv-vec*: *Hv*(*vec r*) = *Octo 0 r r r r r r r*
by (*simp add*: *Hv-def*)

lemma *hv-eq-zero* [*simp*]: *Hv v* = *0* ⟷ *v* = *0*
by (*simp add*: *octo-eq-iff vec-eq-iff*) (*metis exhaust-7*)

lemma *hv-zero* [*simp*]: *Hv 0 = 0*
by *simp*

lemma *hv-vector* [*simp*]: *Hv*(*vector*[*x*,*y*,*z*,*u*,*v*,*q*,*g*]) = *Octo 0 x y z u v q g*
by (*simp add*: *Hv-def*)

lemma *hv-basis*:
Hv(*axis 1 1*) = *e1 Hv*(*axis 2 1*) = *e2 Hv*(*axis 3 1*) = *e3*
Hv(*axis 4 1*) = *e4 Hv*(*axis 5 1*) = *e5 Hv*(*axis 6 1*) = *e6 Hv*(*axis 7 1*) = *e7*
by (*simp-all add*: *octo-eq-iff*)

lemma *hv-add* [*simp*]: *Hv*(*x* + *y*) = *Hv x* + *Hv y*
by (*simp add*: *Hv-def octo-eq-iff*)

lemma *hv-minus* [*simp*]: *Hv*(−*x*) = −*Hv x*
by (*simp add*: *Hv-def octo-eq-iff*)

lemma *hv-diff* [*simp*]: *Hv*(*x* − *y*) = *Hv x* − *Hv y*
by (*simp add*: *Hv-def octo-eq-iff*)

lemma *hv-cmult* [*simp*]:
Hv(*scaleR a x*) = *scaleR a* (*Hv x*)
by (*simp add*: *Hv-def octo-eq-iff*)

lemma *hv-sum* [*simp*]: *Hv* (*sum f S*) = ($\sum$*x*∈*S*. *Hv* (*f x*))
by (*induct S rule*: *infinite-finite-induct*) *auto*

lemma *hv-inj*: *Hv x* = *Hv y* ⟷ *x* = *y*
by (*simp add*: *Hv-def octo-eq-iff vec-eq-iff*) (*metis* (*full-types*) *exhaust-7*)

lemma *linear-hv*: *linear Hv*
using *octo-of-real-def* **by** (*simp add*: *linearI*)

lemma *him-hv* [*simp*]: *HIm*(*Hv x*) = *x*
using *HIm-def hv-inj octo-eq-iff* **by** *fastforce*

lemma *cnj-hv* [*simp*]: *cnj*(*Hv v*) = −*Hv v*
by (*simp add*: *octo-eq-iff*)

lemma *hv-him*: *Hv*(*HIm q*) = *Octo 0* (*Im1 q*) (*Im2 q*) (*Im3 q*) (*Im4 q*) (*Im5 q*) (*Im6 q*) (*Im7 q*)
by (*simp add*: *HIm-def*)

lemma *hv-him-eq*: *Hv*(*HIm q*) = *q* ⟷ *Ree q* = *0*
by (*simp add*: *hv-him octo-eq-iff*)

lemma *dot-hv* [*simp*]: *Hv u* · *Hv v* = *u* · *v*
by (*simp add*: *Hv-def inner-octo-def inner-vec-def sum-7*)

lemma *norm-hv* [*simp*]: *norm* (*Hv v*) = *norm v*
by (*simp add*: *norm-eq*)

2.9.3 Related basic identities

lemma *mult-hv-eq-cross-dot*: *Hv x* $*$ *Hv y* = *Hv*(*x* $\times_7$ *y*) − *octo-of-real* (*inner x y*)
by (*simp add*: *octo-eq-iff inner-octo-def cross7-components octo-mult-components inner-vec-def sum-7*)

lemma *octonion-identity1-cross7* :
Hv (*x* $\times_7$ *y*) = (*1*/*2*) $*_R$ (*Hv x* $*$ *Hv y* − *Hv y* $*$ *Hv x*)
by (*simp add*: *octo-eq-iff octo-mult-components cross7-components*)

lemma *octonion-identity2-cross7*:
Hv (*x* $\times_7$ (*y* $\times_7$ *z*) + *y* $\times_7$ (*z* $\times_7$ *x*) + *z* $\times_7$ (*x* $\times_7$ *y*)) =
−(*3*/*2*) $*_R$ ((*Hv x* $*$ *Hv y*) $*$ *Hv z* − *Hv x* $*$ (*Hv y* $*$ *Hv z*))
unfolding *octo-eq-iff octo-mult-components cross7-components Hv-sel scaleR-octo.sel vector-add-component minus-octo.sel mult-zero-left mult-zero-right add-0-left add-0-right diff-0 diff-0-right*
by *algebra*

2.10 Representing orthogonal transformations as conjugation or congruence with an octonion.

lemma *HIm-nth* [*simp*]:
HIm x \$ *1* = *Im1 x HIm x* \$ *2* = *Im2 x HIm x* \$ *3* = *Im3 x HIm x* \$ *4* = *Im4 x*
HIm x \$ *5* = *Im5 x HIm x* \$ *6* = *Im6 x HIm x* \$ *7* = *Im7 x*
by (*simp-all add*: *HIm-def*)

lemma *orthogonal-transformation-octo-congruence*:
assumes *norm q* = *1*
shows *orthogonal-transformation* (λx. *HIm*(*cnj q* $*$ *Hv x* $*$ *q*))
proof −
have *nq*: (*Ree q*)^2 + (*Im1 q*)^2 + (*Im2 q*)^2 + (*Im3 q*)^2 + (*Im4 q*)^2 + (*Im5 q*)^2 + (*Im6 q*)^2 + (*Im7 q*)^2 = *1*
using *assms norm-octo-def* **by** *auto*
have *Vector-Spaces.linear* ($*_R$) ($*_R$) (λx. *HIm* (*Octonions.cnj q* $*$ *Hv x* $*$ *q*))
by *unfold-locales* (*simp-all add*: *octo-distrib-left octo-distrib-right octo-mult-scaleR-left octo-mult-scaleR-right*)
moreover have *HIm* (*Octonions.cnj q* $*$ *Hv v* $*$ *q*) · *HIm* (*Octonions.cnj q* $*$ *Hv w* $*$ *q*) =
((*Ree q*)^2 + (*Im1 q*)^2 + (*Im2 q*)^2 + (*Im3 q*)^2+ (*Im4 q*)^2 + (*Im5 q*)^2 + (*Im6 q*)^2 + (*Im7 q*)^2)^*2* $*$ (*v* · *w*) **for** *v w*
unfolding *octo-mult-components cnj.sel Hv-sel inner-vec-def sum-7 HIm-nth inner-real-def*
by *algebra*

ultimately show *?thesis*
  by (*simp add*: *orthogonal-transformation-def linear-def nq*)
qed

lemma *orthogonal-transformation-octo-conjugation*:
  assumes $q \neq 0$
  shows *orthogonal-transformation* (λx. *HIm* (*inverse q* $*$ *Hv x* $*$ *q*))
proof −
  obtain *c d* **where** *eq*: q = *octo-of-real c* $*$ *d* **and** *1*: *norm d* = *1*
  proof
    show *1*: *q* = *octo-of-real* (*norm q*) $*$ (*inverse* (*octo-of-real* (*norm q*)) $*$ *q*)
      using *assms norm-eq-zero right-inverse multiplicative-norm-octo*
      by (*metis Octonions.scaleR-conv-octo-of-real octo-of-real-inverse scaleR-one scaleR-scaleR*)
    show *norm* (*inverse* (*octo-of-real* (*norm q*)) $*$ *q*) = *1*
      using *assms 1 norm-octo-def norm-mult inverse-octo-1 inverse-octo-1-sym nonzero-octo-of-real-inverse octo-inverse-eq-cnj cnj-of-real mult-cancel-left2 multiplicative-norm-octo norm-eq-zero norm-octo-squared norm-power2-cnj octo-mult-cnj-conv-norm power2-eq-square-octo*
      by *metis*
  qed
  have $c \neq 0$
    using *assms eq* **by** (*metis Octonions.scaleR-conv-octo-of-real scale-zero-left*)

  then have *HIm* (*Octonions.cnj d* $*$ *Hv x* $*$ *d*) =
      HIm (*inverse* (*octo-of-real c* $*$ *d*) $*$ *Hv x* $*$ (*octo-of-real c* $*$ *d*)) **for** *x*
  proof(*simp add*: *flip*: *octo-inverse-eq-cnj* [*OF 1*] *of-real-inverse*)
    assume $c \neq 0$
    then have *inverse d* = *inverse d* $*$ *inverse* ($c *_R 1$) $*$ $c *_R 1$
      using *octo-of-real-def octo-of-real-inverse octo-of-real-inverse-collapse*(*1*)
        octo-of-real-times-commute octo-of-real-times-left-commute **by** *force*
    then have *inverse d* $*$ *Hv x* $*$ *d* = *inverse* ($c *_R 1 * d$) $*$ *Hv x* $*$ $c *_R (d * 1)$
      by (*metis* (*no-types*) *mult-1-right-octo octo-inverse-mult octo-mult-scaleR-left octo-mult-scaleR-right*)
    then show
      HIm (*inverse d* $*$ *Hv x* $*$ *d*) = *HIm* (*inverse* (*octo-of-real c* $*$ *d*) $*$ *Hv x* $*$ (*octo-of-real c* $*$ *d*))
      using *octo-mult-scaleR-right octo-of-real-def octo-of-real-times-commute* **by** *presburger*
  qed

  then show *?thesis*
    using *orthogonal-transformation-octo-congruence* [*OF 1*]
    by (*simp add*: *eq*)
qed

bundle *octonion-syntax*
begin

notation *octo-e0* (‹*e0*›)
notation *octo-e1* (‹*e1*›)
notation *octo-e2* (‹*e2*›)
notation *octo-e3* (‹*e3*›)
notation *octo-e4* (‹*e4*›)
notation *octo-e5* (‹*e5*›)
notation *octo-e6* (‹*e6*›)
notation *octo-e7* (‹*e7*›)
end

unbundle *no octonion-syntax*

hide-const (**open**) *Octonions.cnj*

end

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