

# Notes on Gödel’s and Scott’s Variants of the Ontological Argument (Isabelle/HOL dataset)

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## Abstract

Experimental studies with Isabelle/HOL on Kurt Gödel’s modal ontological argument and Dana Scott’s variant of it are presented. They implicitly answer some questions that the authors have received over the last decade(s). In addition, some new results are reported.

Our contribution is explained in full detail in [2]. This document presents the corresponding Isabelle/HOL dataset (which is only slightly modified to meet AFP requirements).

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## 1 Introduction

The Isabelle/HOL dataset associated with [2] is presented. Compared to previous work on Gödel’s modal ontological argument as published in the Archive of Formal Proofs (AFP) [1, 4, 3], our dataset addresses several relevant and in some cases novel aspects, which are combined here in a single publication, including:

1. For the first time, Gödel’s original manuscript [5] has been formalized as closely as possible.
2. The inconsistency of the postulates in Gödel’s original manuscript is explained in detail.
3. Two different ways of eliminating this inconsistency are presented, one of which is novel.
4. Scott’s variant [6] of Gödel’s original manuscript is presented and compared with Gödel’s original variant.
5. In addition to logics S5 and K, the above variants are also studied for logic S4.

6. The above variants are tested for various combinations of of possibilist and actualist quantifiers for individuals (which has not been done systematically in previous publications).
7. Concepts of evil are examined and the derivability of evil is critically questioned.

The purpose of this AFP publication is essentially twofold. One motivation is to make the data sources associated with [2] available in a sustainable, well-maintained way. The other motivation is to support university education in higher-order modal logic by providing a small dataset for reuse that illustrates a systematically explored philosophical argument, emphasizing in particular different notions of quantification.

Compared to [2], the Isabelle sources presented here have been slightly modified to meet some AFP requirements. This concerns the commenting out of calls to sledghemmer (to reduce computational resources) and some minor reformatting (e.g. insertion of new lines). The formalization code itself remains unchanged.

## 2 Interactive and automated theorem proving

### 2.1 SurjectiveCantor.thy (Figure 2 of [2])

The surjective Cantor theorem is used in [2] to illustrate some aspects of interactive and automated theorem proving in Isabelle/HOL as relevant for the paper. To keep the provided data material complete wrt. [2], we include these data sources also here.

```
theory SurjectiveCantor imports Main
begin
```

Surjective Cantor theorem: traditional interactive proof

```
theorem SurjectiveCantor:  $\neg(\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F)$ 
proof
  assume 1:  $\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F$ 
  obtain g::' $a\Rightarrow('a\Rightarrow bool)$  where 2:  $\forall F.\exists X. g X = F$  using 1 by auto
  let ?F =  $\lambda X.\neg g X X$ 
  have 3:  $\exists Y. g Y = ?F$  using 2 by metis
  obtain a::' $a$  where 4:  $g a = ?F$  using 3 by auto
  have 5:  $g a a = ?F a$  using 4 by metis
  have 6:  $g a a = (\neg g a a)$  using 5 by auto
  show False using 6 by auto
qed
```

Avoiding proof by contradiction (Fuenmayor & Benzmüller, 2021)

```
theorem SurjectiveCantor':  $\neg(\exists G.\forall F::'a\Rightarrow bool.\exists X::'a. G X = F)$ 
proof –
```

```

{fix g :: 'a=>('a=>bool)
  have 1:  $\forall X.\exists Y.(\neg g X Y) = (\neg g Y Y)$  by auto
  have 2:  $\forall X.\exists Y.(\neg g X Y) = ((\lambda Z.\neg g Z Z) Y)$  using 1 by auto
  have 3:  $\exists F.\forall X.\exists Y.(\neg g X Y) = (F Y)$  using 2 by auto
  have 4:  $\exists F.\forall X.\neg(\forall Y.(g X Y) = (F Y))$  using 3 by auto
  have  $\exists F.\forall X.\neg(g X = F)$  using 4 by metis
}
hence 5:  $\forall G.\exists F::'a=>bool.\forall X::'a.\neg(G X = F)$  by auto
have 6:  $\neg(\exists G.\forall F::'a=>bool.\exists X::'a. G X = F)$  using 5 by auto
thus ?thesis . — done, avoiding proof by contradiction
qed

```

Surjective Cantor theorem: automated proof by some internal/external theorem provers

```

theorem SurjectiveCantor':  $\neg(\exists G.\forall F::'a=>bool.\exists X::'a. G X = F)$ 
  nitpick[expect=none] — no counterexample found
  — sledgehammer — most internal provers give up
  — sledgehammer[remote_leo2 remote_leo3] — proof found: leo provers succeed
oops

```

Surjective Cantor theorem (wrong formalization attempt): the types are crucial

```

theorem SurjectiveCantor'':  $\neg(\exists G.\forall F::'b.\exists X::'a. G X = F)$ 
  nitpick — counterexample found for card 'a = 1 and card 'b = 1: G=( $\lambda x. (a1 := b1)$ )
  nitpick[satisfy, expect=genuine] — model found for card 'a = 1 and card 'b = 2
  nitpick[card 'a=2, card 'b=3, expect=none] — no counterexample found
oops
end

```

### 3 Mechanization of higher-order modal logic (HOML)

#### 3.1 HOMLinHOL.thy (Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the logic-pluralistic LogiKEY methodology. Here logic S5 is introduced.

```

theory HOMLinHOL imports Main
begin

```

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

```

nitpick-params[user-axioms, expect=genuine, show-all, format=2, max-genuine=3]
declare[[syntax-ambiguity-warning=false]]

```

— Type *i* is associated with possible worlds and type *e* with entities  
**typedecl** *i* — Possible worlds

**typed decl**  $e$  — Individuals/entities  
**type-synonym**  $\sigma = i \Rightarrow bool$  — World-lifted propositions  
**type-synonym**  $\tau = e \Rightarrow \sigma$  — Modal properties

**consts**  $R::i \Rightarrow i \Rightarrow bool$  (**r-**) — Accessibility relation between worlds

**axiomatization where**

$Rrefl: \forall x. \mathbf{r}x$  **and**  
 $Rsymm: \forall x y. \mathbf{r}xy \longrightarrow \mathbf{r}yx$  **and**  
 $Rtrans: \forall x y z. \mathbf{r}xy \wedge \mathbf{r}yz \longrightarrow \mathbf{r}xz$

— Logical connectives (operating on truth-sets)

**abbreviation**  $Mbot::\sigma$  ( $\perp$ ) **where**  $\perp \equiv \lambda w. False$   
**abbreviation**  $Mtop::\sigma$  ( $\top$ ) **where**  $\top \equiv \lambda w. True$   
**abbreviation**  $Mneg::\sigma \Rightarrow \sigma$  ( $\neg$ - [52]53) **where**  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$   
**abbreviation**  $Mand::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\wedge$  50) **where**  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$   
**abbreviation**  $Mor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee$  49) **where**  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$   
**abbreviation**  $Mimp::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $\supset$  48) **where**  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$   
**abbreviation**  $Mequiv::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\leftrightarrow$  47) **where**  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$   
**abbreviation**  $Mbox::\sigma \Rightarrow \sigma$  ( $\Box$ - [54]55) **where**  $\Box\varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$   
**abbreviation**  $Mdia::\sigma \Rightarrow \sigma$  ( $\Diamond$ - [54]55) **where**  $\Diamond\varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$   
**abbreviation**  $Mprimeq::'a \Rightarrow 'a \Rightarrow \sigma$  ( $=$ -) **where**  $x=y \equiv \lambda w. x=w$   
**abbreviation**  $Mprimneg::'a \Rightarrow 'a \Rightarrow \sigma$  ( $\neq$ -) **where**  $x \neq y \equiv \lambda w. x \neq y$   
**abbreviation**  $Mnegpred::\tau \Rightarrow \tau$  ( $\sim$ -) **where**  $\sim\Phi \equiv \lambda x. \lambda w. \neg\Phi x w$   
**abbreviation**  $Mconpred::\tau \Rightarrow \tau \Rightarrow \tau$  (**infixl** . 50) **where**  $\Phi.\Psi \equiv \lambda x. \lambda w. \Phi x w \wedge \Psi x w$   
**abbreviation**  $Mexclor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee^e$  49) **where**  $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation**  $Mallposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall$ ) **where**  $\forall\Phi \equiv \lambda w. \forall x. \Phi x w$   
**abbreviation**  $Mallpossb$  (**binder**  $\forall$  [8]9) **where**  $\forall x. \varphi(x) \equiv \forall\varphi$   
**abbreviation**  $Mexiposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists$ ) **where**  $\exists\Phi \equiv \lambda w. \exists x. \Phi x w$   
**abbreviation**  $Mexipossb$  (**binder**  $\exists$  [8]9) **where**  $\exists x. \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)

**consts**  $existsAt::e \Rightarrow \sigma$  (**-@-**)  
**abbreviation**  $Mallact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall^E$ ) **where**  $\forall^E\Phi \equiv \lambda w. \forall x. x @ w \longrightarrow \Phi x w$   
**abbreviation**  $Mallactb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall^E\varphi$   
**abbreviation**  $Mexiact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists^E$ ) **where**  $\exists^E\Phi \equiv \lambda w. \exists x. x @ w \wedge \Phi x w$   
**abbreviation**  $Mexiactb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)

**abbreviation**  $Mleibeq::'a \Rightarrow 'a \Rightarrow \sigma$  ( $\equiv$ -) **where**  $x \equiv y \equiv \forall P. P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation**  $Mvalid::\sigma \Rightarrow bool$  ( $\lfloor$ -) **where**  $\lfloor\psi \equiv \forall w. \psi w$

**end**

### 3.2 TestsHOML.thy (Figure 4 of [2])

Tests and verifications of properties for the embedding of HOML (S5) in HOL.

**theory** *TestsHOML* **imports** *HOMLinHOL*  
**begin**

— Test for S5 modal logic

**lemma** *axM*:  $[\Box\varphi \supset \varphi]$  **using** *Rrefl* **by** *blast*

**lemma** *axD*:  $[\Box\varphi \supset \Diamond\varphi]$  **using** *Rrefl* **by** *blast*

**lemma** *axB*:  $[\varphi \supset \Box\Diamond\varphi]$  **using** *Rsymm* **by** *blast*

**lemma** *ax4*:  $[\Box\varphi \supset \Box\Box\varphi]$  **using** *Rtrans* **by** *blast*

**lemma** *ax5*:  $[\Diamond\varphi \supset \Box\Diamond\varphi]$  **using** *Rsymm* *Rtrans* **by** *blast*

— Test for Barcan and converse Barcan formula:

**lemma** *BarcanAct*:  $[(\forall^E x. \Box(\varphi x)) \supset \Box(\forall^E x. (\varphi x))]$

**nitpick** $[\text{expect}=\text{genuine}]$  **oops** — Countermodel found

**lemma** *ConvBarcanAct*:  $[\Box(\forall^E x. (\varphi x)) \supset (\forall^E x. \Box(\varphi x))]$

**nitpick** $[\text{expect}=\text{genuine}]$  **oops** — Countermodel found

**lemma** *BarcanPoss*:  $[(\forall x. \Box(\varphi x)) \supset \Box(\forall x. \varphi x)]$  **by** *blast*

**lemma** *ConvBarcanPoss*:  $[\Box(\forall x. (\varphi x)) \supset (\forall x. \Box(\varphi x))]$  **by** *blast*

— A simple Hilbert system for classical propositional logic is derived

**lemma** *Hilbert-A1*:  $[A \supset (B \supset A)]$  **by** *blast*

**lemma** *Hilbert-A2*:  $[(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))]$  **by** *blast*

**lemma** *Hilbert-MP*: **assumes**  $[A]$  **and**  $[A \supset B]$  **shows**  $[B]$  **using** *assms* **by** *blast*

— We have a polymorphic possibilist quantifier for which existential import holds

**lemma** *Quant-1*: **assumes**  $[A]$  **shows**  $[\forall x::'a. A]$  **using** *assms* **by** *auto*

— Existential import holds for possibilist quantifiers

**lemma** *ExImPossibilist1*:  $[\exists x::e. x = x]$  **by** *blast*

**lemma** *ExImPossibilist2*:  $[\exists x::e. x \equiv x]$  **by** *blast*

**lemma** *ExImPossibilist3*:  $[\exists x::e. x = t]$  **by** *blast*

**lemma** *ExImPossibilist4*:  $[\exists x::'a. x \equiv t::'a]$  **by** *blast*

**lemma** *ExImPossibilist*:  $[\exists x::'a. \top]$  **by** *blast*

— We have an actualist quantifier for individuals for which existential import does not hold

**lemma** *Quant-2*: **assumes**  $[A]$  **shows**  $[\forall^E x::e. A]$  **using** *assms* **by** *auto*

— Existential import does not hold for our actualist quantifiers (for individuals)

**lemma** *ExImActualist1*:  $[\exists^E x::e. x = x]$

**nitpick** $[\text{card}=1, \text{expect}=\text{genuine}]$  **oops** — Countermodel found

**lemma** *ExImActualist2*:  $[\exists^E x::e. x \equiv x]$

**nitpick** $[\text{card}=1, \text{expect}=\text{genuine}]$  **oops** — Countermodel found

**lemma** *ExImActualist3*:  $[\exists^E x::e. x = t]$

**nitpick** $[\text{card}=1, \text{expect}=\text{genuine}]$  **oops** — Countermodel found

**lemma** *ExImActualist*:  $[\exists^E x::e. \top]$

**nitpick**[*card=1,expect=genuine*] **oops** — Countermodel found

— Properties of the embedded primitive equality, which coincides with Leibniz equality

**lemma** *EqRef1*:  $[x = x]$  **by** *blast*

**lemma** *EqSym*:  $[x = y \leftrightarrow y = x]$  **by** *blast*

**lemma** *EqTrans*:  $[(x = y) \wedge (y = z) \supset (x = z)]$  **by** *blast*

**lemma** *EQCong*:  $[(x = y) \supset ((\varphi x) = (\varphi y))]$  **by** *blast*

**lemma** *EQFuncExt*:  $[(\varphi = \psi) \supset (\forall x. ((\varphi x) = (\psi x)))]$  **by** *blast*

**lemma** *EQBoolExt1*:  $[(\varphi = \psi) \supset (\varphi \leftrightarrow \psi)]$  **by** *blast*

**lemma** *EQBoolExt2*:  $[(\varphi \leftrightarrow \psi) \supset (\varphi = \psi)]$

**nitpick**[*card=2*] **oops** — Countermodel found

**lemma** *EQBoolExt3*:  $[(\varphi \leftrightarrow \psi)] \longrightarrow [(\varphi = \psi)]$  **by** *blast*

**lemma** *EqPrimLeib*:  $[x = y \leftrightarrow (x \equiv y)]$  **by** *auto*

— Comprehension is natively supported in HOL (due to lambda-abstraction)

**lemma** *Comprehension1*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow A]$  **by** *force*

**lemma** *Comprehension2*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow (A x)]$  **by** *force*

**lemma** *Comprehension3*:  $[\exists \varphi. \forall x y. (\varphi x y) \leftrightarrow (A x y)]$  **by** *force*

— Modal collapse does not hold

**lemma** *ModalCollapse*:  $[\forall \varphi. \varphi \supset \Box \varphi]$

**nitpick**[*card=2,expect=genuine*] **oops** — Countermodel found

— Empty property and self-difference

**lemma** *TruePropertyAndSelfIdentity*:  $[(\lambda x::e. \top) = (\lambda x. x = x)]$  **by** *blast*

**lemma** *EmptyPropertyAndSelfDifference*:  $[(\lambda x::e. \perp) = (\lambda x. x \neq x)]$  **by** *blast*

**lemma** *EmptyProperty2*:  $[\exists x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$  **by** *blast*

**lemma** *EmptyProperty3*:  $[\exists^E x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$  **by** *blast*

**lemma** *EmptyProperty4*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists x. \varphi x]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

**lemma** *EmptyProperty5*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists^E x. \varphi x]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

end

### 3.3 ModalFilter.thy (Figure 5 of [2])

Set filter and ultrafilter formalized for our modal logic setting.

**theory** *ModalFilter* **imports** *HOMLinHOL*

**begin**

**type-synonym**  $\tau = e \Rightarrow \sigma$

**abbreviation** *Element*:: $\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma$  (**infix**  $\in$  90) **where**  $\varphi \in S \equiv S \varphi$

**abbreviation** *EmptySet*:: $\tau$  ( $\emptyset$ ) **where**  $\emptyset \equiv \lambda x. \perp$

**abbreviation** *UniversalSet*:: $\tau$  ( $\mathbf{U}$ ) **where**  $\mathbf{U} \equiv \lambda x. \top$

**abbreviation** *Subset*:: $\tau \Rightarrow \tau \Rightarrow \sigma$  (**infix**  $\subseteq$  80)

**where**  $\varphi \subseteq \psi \equiv \forall x. ((\varphi x) \supset (\psi x))$

**abbreviation** *SubsetE*:: $\tau \Rightarrow \tau \Rightarrow \sigma$  (**infix**  $\subseteq^E$  80)

**where**  $\varphi \subseteq^E \psi \equiv \forall^E x. ((\varphi x) \supset (\psi x))$   
**abbreviation** *Intersection*:: $\tau \Rightarrow \tau \Rightarrow \tau$  (**infix**  $\sqcap$  91)  
**where**  $\varphi \sqcap \psi \equiv \lambda x. ((\varphi x) \wedge (\psi x))$   
**abbreviation** *Inverse*:: $\tau \Rightarrow \tau$  ( $^{-1}$ )  
**where**  $^{-1}\psi \equiv \lambda x. \neg(\psi x)$   
**abbreviation** *Filter*  $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge$   
 $(\forall \varphi \psi. \varphi \in \Phi \wedge \varphi \subseteq^E \psi \supset \psi \in \Phi) \wedge (\forall \varphi \psi. \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$   
**abbreviation** *UFilter*  $\Phi \equiv \text{Filter } \Phi \wedge (\forall \varphi. \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$   
**abbreviation** *FilterP*  $\Phi \equiv \mathbf{U} \in \Phi \wedge \neg(\emptyset \in \Phi) \wedge (\forall \varphi \psi. \varphi \in \Phi \wedge \varphi \subseteq \psi \supset \psi \in \Phi) \wedge$   
 $(\forall \varphi \psi. \varphi \in \Phi \wedge \psi \in \Phi \supset \varphi \sqcap \psi \in \Phi)$   
**abbreviation** *UFilterP*  $\Phi \equiv \text{FilterP } \Phi \wedge (\forall \varphi. \varphi \in \Phi \vee (^{-1}\varphi) \in \Phi)$   
**end**

## 4 Gödel's ontological argument – 1970 manuscript

### 4.1 GoedelVariantHOML1.thy (Figure 6 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *GoedelVariantHOML1* **imports** *HOMLinHOL*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box (\forall^E y. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi \text{Ess. } x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box (\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G \text{Ess. } x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt (verit)*)

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box (\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box (\exists^E y. G y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by**



*smt*

**theorem** *Th3*:  $[\diamond(\exists^E x. G x) \supset \square(\exists^E y. G y)]$   
— sledgehammer(Th2 Rsymm) — Proof found

**proof** —

**have** 1:  $[(\exists^E x. G x) \supset \square(\exists^E y. G y)]$  **using** *Th2* **by** *blast*

**have** 2:  $[\diamond(\exists^E x. G x) \supset \diamond\square(\exists^E y. G y)]$  **using** 1 **by** *blast*

**have** 3:  $[\diamond(\exists^E x. G x) \supset \square(\exists^E y. G y)]$  **using** 2 *Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,expect=unknown*] **oops** — No model found

**lemma** *EmptyEssL*:  $[(\lambda y. \perp) \text{Ess. } x]$  **using** *Essence-def* **by** *auto*

**theorem** *Inconsistency*: *False*

— sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist\_def) — Proof found

**proof** —

**have** 1:  $[\neg(P (\lambda x. \perp))]$  **using** *Ax2a Ax4* **by** *blast*

**have** 2:  $[P (\lambda x. (\lambda y. \perp) \text{Ess. } x) \supset \square(\exists^E z. (\lambda y. \perp) z)]$  **using** *Ax3 Ax4 NecExist-def* **by** *smt*

**have** 3:  $[P (\lambda x. \square(\exists^E z. (\lambda x. \perp) z))]$  **using** 2 *EmptyEssL* **by** *simp*

**have** 4:  $[P (\lambda x. \square \perp)]$  **using** 3 **by** *auto*

**have** 5:  $[P (\lambda x. \perp)]$  **using** 4 *Ax2a Ax4* **by** *smt*

**have** 6:  $[\perp]$  **using** 1 5 **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**end**

## 4.2 GoedelVariantHOML2.thy (Figure 7 of [2])

After an appropriate modification of the definition of essence in Gödel's 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *GoedelVariantHOML2* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* ( $G$ ) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall^E y. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b: [P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a$   $Ax2b$  **by** *blast*

**theorem**  $Th1$ :  $[G x \supset G Ess. x]$  **using**  $Ax2a$   $Ax2b$  *Essence-def* *God-def* **by** (*smt* (*verit*))

**definition** *NecExist* ( $E$ ) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where**  $Ax3: [P E]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists^E y. G y)]$  **using**  $Ax3$   $Th1$  *God-def* *NecExist-def* **by** *smt*

**theorem**  $Th3$ :  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$

— *sledgehammer*( $Th2$  *Rsymm*) — *Proof found*

**proof** —

**have**  $1$ :  $[(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using**  $Th2$  **by** *blast*

**have**  $2$ :  $[\Diamond(\exists^E x. G x) \supset \Diamond\Box(\exists^E y. G y)]$  **using**  $1$  **by** *blast*

**have**  $3$ :  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using**  $2$  *Rsymm* **by** *blast*

**thus** *thesis* **by** *blast*

**qed**

**axiomatization where**  $Ax4: [P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P* ( $\lambda x. \perp$ )]] **oops** — One model found of cardinality one

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen: [(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma**  $L$ :  $[P G]$  **using**  $Ax1Gen$  *God-def* **by** (*smt* (*verit*))

**theorem**  $Th4$ :  $[\Diamond(\exists^E x. G x)]$  **using**  $Ax2a$   $Ax4$   $L$  **by** *blast*

**theorem**  $Th5$ :  $[\Box(\exists^E x. G x)]$  **using**  $Th3$   $Th4$  **by** *blast*

**lemma**  $MC$ :  $[\varphi \supset \Box\varphi]$

— *sledgehammer*( $Ax2a$   $Ax2b$   $Th5$  *God\_def* *Rsymm*) — *Proof found*

**proof** —

```

{fix w fix Q
  have 1:  $\forall x.(G x w \longrightarrow (\forall Z. Z x \supset \Box(\forall^E z. G z \supset Z z))) w$  using Ax2a
Ax2b God-def by smt
  have 2:  $(\exists x. G x w) \longrightarrow ((Q \supset \Box(\forall^E z. G z \supset Q)) w)$  using 1 by force
  have 3:  $(Q \supset \Box Q) w$  using 2 Th5 Rsymm by blast}
thus ?thesis by auto
qed

```

```

lemma PosProps:  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$  using Ax2a Ax4 by blast
lemma NegProps:  $[\neg P (\lambda x. \perp) \wedge \neg P (\lambda x. x \neq x)]$  using Ax2a Ax4 by blast
lemma UniqueEss1:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \leftrightarrow \psi y)]$  using
Essence-def by smt
lemma UniqueEss2:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  nitpick[card i=1] oops
— Countermodel found
lemma UniqueEss3:  $[\varphi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \supset y \equiv x)]$  using Essence-def MC
by auto
lemma Monotheism:  $[G x \wedge G y \supset x \equiv y]$  using Ax2a God-def by smt
lemma Filter:  $[Filter P]$  using Ax1 Ax4 MC NegProps PosProps Rsymm by smt
lemma UltraFilter:  $[UFilter P]$  using Ax2a Filter by smt
lemma True nitpick[satisfy,card=1,eval=[ $P (\lambda x. \perp)$ ]] oops — One model found
of cardinality one

```

end

### 4.3 GoedelVariantHOML3.thy (Figure 8 of [2])

After an appropriate modification of the notion of necessary property inclusion in Gödel's 1970 ontological proof, the inconsistency revealed in Figure 6 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only symmetry of the accessibility relation is actually needed). Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

```

theory GoedelVariantHOML3 imports HOMLinHOL ModalFilter
begin

```

```

consts PositiveProperty:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (P)

```

```

axiomatization where Ax1:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$ 

```

```

axiomatization where Ax2a:  $[P \varphi \vee^e P \sim \varphi]$ 

```

```

definition God (G) where  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$ 

```

```

abbreviation PropertyInclusion ( $-\supset_N-$ ) where  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge$ 
 $(\forall^E y. \varphi y \supset \psi y))$ 

```

```

definition Essence ( $-Ess.-$ ) where  $\varphi \text{ Ess. } x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$ 

```

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a$   $Ax2b$  **by** *blast*

**theorem**  $Th1$ :  $[G x \supset G \text{Ess. } x]$  **using**  $Ax2a$   $Ax2b$  *Essence-def* *God-def* **by** (*smt* (*verit*))

**definition**  $NecExist$  ( $E$ ) **where**  $E x \equiv \forall \varphi. \varphi \text{Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where**  $Ax3$ :  $[P E]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists^E y. G y)]$  **using**  $Ax3$   $Th1$  *God-def* *NecExist-def* **by** *smt*

**theorem**  $Th3$ :  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$

— sledgehammer( $Th2$  *Rsymm*) — Proof found

**proof** —

**have** 1:  $[(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using**  $Th2$  **by** *blast*

**have** 2:  $[\Diamond(\exists^E x. G x) \supset \Diamond\Box(\exists^E y. G y)]$  **using** 1 **by** *blast*

**have** 3:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using** 2 *Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (\lambda x. \perp)]*] **oops** — Two models found of cardinality one

**abbreviation**  $PosProps$   $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom$   $\varphi$   $\Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen$ :  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma**  $L$ :  $[P G]$  **using**  $Ax1Gen$  *God-def* **by** (*smt* (*verit*))

**theorem**  $Th4$ :  $[\Diamond(\exists^E x. G x)]$

— sledgehammer[*timeout=200*]( $Ax2a$   $L$   $Ax1Gen$ ) **oops** — sorry — Proof found

**axiomatization where**  $Th4$ :  $[\Diamond(\exists^E x. G x)]$

**theorem**  $Th5$ :  $[\Box(\exists^E x. G x)]$  **using**  $Th3$   $Th4$  **by** *blast*

**lemma**  $MC$ :  $[\varphi \supset \Box\varphi]$

— sledgehammer( $Ax2a$   $Ax2b$   $Th5$  *God\_def* *Rsymm*) — Proof found

**proof** — {**fix**  $w$  **fix**  $Q$

**have** 1:  $\forall x. (G x w \longrightarrow (\forall Z. Z x \supset \Box(\forall^E z. G z \supset Z z)) w)$  **using**  $Ax2a$   $Ax2b$  *God-def* **by** *smt*

**have** 2:  $(\exists x. G x w) \longrightarrow ((Q \supset \Box(\forall^E z. G z \supset Q)) w)$  **using** 1 **by** *force*

**have** 3:  $(Q \supset \Box Q) w$  **using** 2  $Th5$  *Rsymm* **by** *blast*}

**thus** *?thesis* **by** *auto*

qed

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$  **using** *Ax2a Ax4 L Th4* **by** *smt*  
**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  **using** *Ax2a Ax4 L Th4* **by** *smt*  
**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \leftrightarrow \psi y)]$  **oops** —  
Unclear, open question  
**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  **oops** — Unclear, open  
question  
**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall^E y. \varphi y \supset y \equiv x)]$  **using** *Essence-def MC*  
**by** *auto*  
**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *Ax2a God-def* **by** *smt*  
**lemma** *Filter*:  $[Filter P]$  **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*  
**lemma** *UltraFilter*:  $[UFilter P]$  **using** *Ax2a Filter* **by** *smt*  
**lemma** *True nitpick* $[satisfy, card=1, eval=[P(\lambda x. \top)]]$  **oops** — One model found  
of cardinality one

end

#### 4.4 ThereIsNoEvil1.thy (Figure 10 of [2])

Importing Gödel's modified axioms from Figure 7 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

**theory** *ThereIsNoEvil1* **imports** *GoedelVariantHOML2*  
**begin**

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$

**theorem** *NecNoEvil*:  $[\Box(\neg(\exists^E x. Evil\ x))]$   
— sledgehammer(*Ax2a Ax4 Evil\_def*) — Proof found

**proof** —

**have**  $[\neg P(\lambda y. \perp)]$  **using** *Ax2a Ax4* **by** *blast*  
**hence**  $[(\forall^E x. Evil\ x \supset (\lambda y. \perp)\ x)]$  **using** *Evil-def* **by** *auto*  
**hence**  $[(\forall^E x. Evil\ x \supset \perp)]$  **by** *auto*  
**hence**  $[(\exists^E x. Evil\ x) \supset \perp]$  **by** *auto*  
**hence**  $[\neg(\exists^E x. Evil\ x)]$  **by** *blast*  
**thus** *?thesis* **by** *blast*

qed

end

#### 4.5 ThereIsNoEvil2.thy (Figure 11 of [2])

Importing Gödel's modified axioms from Figure 8 we can prove that necessarily there exists no entity that possesses all non-positive (=negative) properties.

**theory** *ThereIsNoEvil2* **imports** *GoedelVariantHOML3*  
**begin**

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$   
**theorem** *NecNoEvil*:  $\lfloor \Box(\neg(\exists^E x. Evil\ x)) \rfloor$   
— sledgehammer(Ax1Gen Ax2a Evil\_def) **oops** — Proof found

**end**

## 5 Scott's variant

### 5.1 ScottVariantHOML.thy (Figure 12 of [2])

Verification of Scott's variant of Gödel's ontological argument. Actualist quantifiers (avoiding existential import) are used for quantification over entities, otherwise possibilist quantifiers are used.

**theory** *ScottVariantHOML* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *A1*:  $\lfloor \neg P\ \varphi \leftrightarrow P\ \sim \varphi \rfloor$

**axiomatization where** *A2*:  $\lfloor P\ \varphi \wedge \Box(\forall^E y. \varphi\ y \supset \psi\ y) \supset P\ \psi \rfloor$

**theorem** *T1*:  $\lfloor P\ \varphi \supset \Diamond(\exists^E x. \varphi\ x) \rfloor$  **using** *A1 A2* **by** *blast*

**definition** *God* (*G*) **where**  $G\ x \equiv \forall \varphi. P\ \varphi \supset \varphi\ x$

**axiomatization where** *A3*:  $\lfloor P\ G \rfloor$

**theorem** *Coro*:  $\lfloor \Diamond(\exists^E x. G\ x) \rfloor$  **using** *A3 T1* **by** *blast*

**axiomatization where** *A4*:  $\lfloor P\ \varphi \supset \Box P\ \varphi \rfloor$

**definition** *Ess* (*-Ess.-*) **where**  $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi. \psi\ x \supset \Box(\forall^E y. \varphi\ y \supset \psi\ y))$

**theorem** *T2*:  $\lfloor G\ x \supset G\ Ess.\ x \rfloor$  **using** *A1 A4 Ess-def God-def* **by** *fastforce*

**definition** *NecExist* (*NE*) **where**  $NE\ x \equiv \forall \varphi. \varphi\ Ess.\ x \supset \Box(\exists^E x. \varphi\ x)$

**axiomatization where** *A5*:  $\lfloor P\ NE \rfloor$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx. T)]*] **oops** — One model found of cardinality one

**theorem** *T3*:  $\lfloor \Box(\exists^E x. G\ x) \rfloor$   
— sledgehammer(A5 Coro God\_def NecExist\_def Rsymm T2) — Proof found  
**proof** —

**have 1:**  $[(G\ x \supset NE\ x) \wedge (G\ Ess.\ x \supset \Box(\exists^E x.\ G\ x))]$  **using** *A5 Ess-def God-def NecExist-def* **by** *smt*  
**hence 2:**  $[(\exists^E x.\ G\ x) \supset \Box(\exists^E x.\ G\ x)]$  **using** *A5 God-def NecExist-def T2*  
**by** *smt*  
**hence 3:**  $[\Diamond(\exists^E x.\ G\ x) \supset (\Diamond(\Box(\exists^E x.\ G\ x)) \supset \Box(\exists^E x.\ G\ x))]$  **using** *Rsymm*  
**by** *blast*  
**thus** *?thesis using 2 Coro* **by** *blast*  
**qed**

**lemma** *MC:*  $[\varphi \supset \Box\varphi]$   
— *sledgehammer(A1 A4 God\_def Rsymm T3)* — *Proof found*  
**proof** — **{fix w fix Q**  
**have 1:**  $\forall x.(G\ x\ w \longrightarrow (\forall Z. Z\ x \supset \Box(\forall^E z.((G\ z) \supset (Z\ z))))\ w)$  **using** *A1 A4 God-def* **by** *smt*  
**have 2:**  $(\exists x.\ G\ x\ w) \longrightarrow ((Q \supset \Box(\forall^E z.((G\ z) \supset Q)))\ w)$  **using** *1* **by** *force*  
**have 3:**  $(Q \supset \Box Q)\ w$  **using** *2 T3 Rsymm* **by** *blast*  
**thus** *?thesis* **by** *auto*  
**qed**

**lemma** *PosProps:*  $[P(\lambda x.\top) \wedge P(\lambda x.\ x = x)]$  **using** *A1 A2* **by** *blast*  
**lemma** *NegProps:*  $[\neg P(\lambda x.\perp) \wedge \neg P(\lambda x.\ x \neq x)]$  **using** *A1 A2* **by** *blast*  
**lemma** *UniqueEss1:*  $[\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \leftrightarrow \psi\ y)]$  **using** *Ess-def* **by** *smt*  
**lemma** *UniqueEss2:*  $[\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)]$  **nitpick***[card i=1]* **oops**  
— *Countermodel found*  
**lemma** *UniqueEss3:*  $[\varphi\ Ess.\ x \supset \Box(\forall^E y.\ \varphi\ y \supset y \equiv x)]$  **using** *Ess-def MC* **by** *auto*  
**lemma** *Monotheism:*  $[G\ x \wedge G\ y \supset x \equiv y]$  **using** *A1 God-def* **by** *smt*  
**lemma** *Filter:*  $[Filter\ P]$  **using** *A1 God-def Rsymm T1 T3* **by** *(smt (verit, best))*  
**lemma** *UltraFilter:*  $[UFilter\ P]$  **using** *Filter A1* **by** *blast*  
**lemma** *True* **nitpick***[satisfy,card=1,eval=[P(\lambda x.\perp)]]* **oops** — *One model found of cardinality one*

**end**

## 5.2 HOMLinHOLonlyK.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic K is introduced.

**theory** *HOMLinHOLonlyK* **imports** *Main*  
**begin**

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

**nitpick-params***[user-axioms,expect=genuine,show-all,format=2,max-genuine=3]*  
**declare***[[syntax-ambiguity-warning=false]]*

— Type *i* is associated with possible worlds and type *e* with entities

**typedecl** *i* — Possible worlds

**typedecl** *e* — Individuals/entities

**type-synonym**  $\sigma = i \Rightarrow bool$  — World-lifted propositions

**type-synonym**  $\tau = e \Rightarrow \sigma$  — modal properties

**consts**  $R::i \Rightarrow i \Rightarrow bool$  (**-r-**) — Accessibility relation between worlds

— Logical connectives (operating on truth-sets)

**abbreviation**  $Mbot::\sigma$  ( $\perp$ ) **where**  $\perp \equiv \lambda w. False$

**abbreviation**  $Mtop::\sigma$  ( $\top$ ) **where**  $\top \equiv \lambda w. True$

**abbreviation**  $Mneg::\sigma \Rightarrow \sigma$  ( $\neg$ - [52]53) **where**  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$

**abbreviation**  $Mand::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\wedge$  50) **where**  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$

**abbreviation**  $Mor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee$  49) **where**  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$

**abbreviation**  $Mimp::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $\supset$  48) **where**  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$

**abbreviation**  $Mequiv::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\leftrightarrow$  47) **where**  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$

**abbreviation**  $Mbox::\sigma \Rightarrow \sigma$  ( $\Box$ - [54]55) **where**  $\Box\varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$

**abbreviation**  $Mdia::\sigma \Rightarrow \sigma$  ( $\Diamond$ - [54]55) **where**  $\Diamond\varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$

**abbreviation**  $Mprimeq::'a \Rightarrow 'a \Rightarrow \sigma$  (**-=**-) **where**  $x=y \equiv \lambda w. x=y$

**abbreviation**  $Mprimneg::'a \Rightarrow 'a \Rightarrow \sigma$  (**- $\neq$** -) **where**  $x \neq y \equiv \lambda w. x \neq y$

**abbreviation**  $Mnegpred::\tau \Rightarrow \tau$  (**~**-) **where**  $\sim\Phi \equiv \lambda x. \lambda w. \neg\Phi x w$

**abbreviation**  $Mconpred::\tau \Rightarrow \tau \Rightarrow \tau$  (**infixl** . 50) **where**  $\Phi.\Psi \equiv \lambda x. \lambda w. \Phi x w \wedge \Psi x w$

**abbreviation**  $Mexclor::\sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixl**  $\vee^e$  49) **where**  $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation**  $Mallposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall$ ) **where**  $\forall\Phi \equiv \lambda w. \forall x. \Phi x w$

**abbreviation**  $Mallpossb$  (**binder**  $\forall$  [8]9) **where**  $\forall x. \varphi(x) \equiv \forall\varphi$

**abbreviation**  $Mexiposs::('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists$ ) **where**  $\exists\Phi \equiv \lambda w. \exists x. \Phi x w$

**abbreviation**  $Mexipossb$  (**binder**  $\exists$  [8]9) **where**  $\exists x. \varphi(x) \equiv \exists\varphi$

— Actualist quantifiers (for individuals/entities)

**consts**  $existsAt::e \Rightarrow \sigma$  (**-@-**)

**abbreviation**  $Mallact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall^E$ ) **where**  $\forall^E\Phi \equiv \lambda w. \forall x. x @ w \longrightarrow \Phi x w$

**abbreviation**  $Mallactb$  (**binder**  $\forall^E$  [8]9) **where**  $\forall^E x. \varphi(x) \equiv \forall^E\varphi$

**abbreviation**  $Mexiact::(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists^E$ ) **where**  $\exists^E\Phi \equiv \lambda w. \exists x. x @ w \wedge \Phi x w$

**abbreviation**  $Mexiactb$  (**binder**  $\exists^E$  [8]9) **where**  $\exists^E x. \varphi(x) \equiv \exists^E\varphi$

— Leibniz equality (polymorphic)

**abbreviation**  $Mleibeq::'a \Rightarrow 'a \Rightarrow \sigma$  (**- $\equiv$** -) **where**  $x \equiv y \equiv \forall P. P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation**  $Mvalid::\sigma \Rightarrow bool$  (**[**-]) **where**  $[\psi] \equiv \forall w. \psi w$

**end**



### 5.3 ScottVariantHOMLinK.thy (Figure 13 of [2])

Scott's variant of Gödel's argument fails for base logic K (but only in the last step).

**theory** *ScottVariantHOMLinK* **imports** *HOMLinHOLonlyK*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *A1*:  $[\neg P \ \varphi \leftrightarrow P \ \sim \varphi]$

**axiomatization where** *A2*:  $[P \ \varphi \wedge \Box(\forall^E y. \varphi \ y \supset \psi \ y) \supset P \ \psi]$

**theorem** *T1*:  $[P \ \varphi \supset \Diamond(\exists^E x. \varphi \ x)]$  **using** *A1 A2* **by** *blast*

**definition** *God* (*G*) **where**  $G \ x \equiv \forall \varphi. P \ \varphi \supset \varphi \ x$

**axiomatization where** *A3*:  $[P \ G]$

**theorem** *Coro*:  $[\Diamond(\exists^E x. G \ x)]$  **nitpick**[*satisfy,eval=G*] **using** *A3 T1* **by** *blast*

**axiomatization where** *A4*:  $[P \ \varphi \supset \Box P \ \varphi]$

**definition** *Ess* (*-Ess.*) **where**  $\varphi \ \text{Ess.} \ x \equiv \varphi \ x \wedge (\forall \psi. \psi \ x \supset \Box(\forall^E y. \varphi \ y \supset \psi \ y))$

**theorem** *T2*:  $[G \ x \supset G \ \text{Ess.} \ x]$  **using** *A1 A4 Ess-def God-def* **by** *fastforce*

**definition** *NecExist* (*NE*) **where**  $NE \ x \equiv \forall \varphi. \varphi \ \text{Ess.} \ x \supset \Box(\exists^E x. \varphi \ x)$

**axiomatization where** *A5*:  $[P \ NE]$

**lemma** *True* **nitpick**[*satisfy,card=1,eval=[P (\lambda x. \top)]*] **oops** — One model found of cardinality one

**theorem** *T3*:  $[\Box(\exists^E x. G \ x)]$  **nitpick**[*card e=1, card i=2, eval=G*] **oops** — Counterexample

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$  **nitpick**[*card e=1, card i=2, eval=G*] **oops** — Counterexample

**end**

### 5.4 ScottVariantHOMLAndersonQuant.thy (Figure 15 of [2])

Verification of Scott's variant of Gödel's argument with a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *ScottVariantHOMLAndersonQuant* **imports** *HOMLinHOL ModalFilter*

**begin**

**consts** *PositiveProperty*::( $e \Rightarrow \sigma \Rightarrow \sigma$ ) ( $P$ )

**axiomatization where**  $A1$ :  $[\neg P \ \varphi \leftrightarrow P \ \sim \varphi]$

**axiomatization where**  $A2$ :  $[P \ \varphi \wedge \Box(\forall y. \varphi \ y \supset \psi \ y) \supset P \ \psi]$

**theorem**  $T1$ :  $[P \ \varphi \supset \Diamond(\exists x. \varphi \ x)]$  **using**  $A1 \ A2$  **by** *smt*

**definition** *God* ( $G$ ) **where**  $G \ x \equiv \forall \varphi. P \ \varphi \supset \varphi \ x$

**axiomatization where**  $A3$ :  $[P \ G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G \ x)]$  **using**  $A3 \ T1$  **by** *blast*

**axiomatization where**  $A4$ :  $[P \ \varphi \supset \Box P \ \varphi]$

**definition** *Ess* (*-Ess.-*) **where**  $\varphi \ \text{Ess.} \ x \equiv \varphi \ x \wedge (\forall \psi. \psi \ x \supset \Box(\forall y::e. \varphi \ y \supset \psi \ y))$

**theorem**  $T2$ :  $[G \ x \supset G \ \text{Ess.} \ x]$  **using**  $A1 \ A4 \ \text{Ess-def} \ \text{God-def}$  **by** *smt*

**definition** *NecExist* ( $NE$ ) **where**  $NE \ x \equiv \forall \varphi. \varphi \ \text{Ess.} \ x \supset \Box(\exists^E x. \varphi \ x)$

**axiomatization where**  $A5$ :  $[P \ NE]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊤)]*] **oops** — One model found of cardinality one

**theorem**  $T3$ :  $[\Box(\exists^E x. G \ x)]$

— *sledgehammer*( $A5 \ \text{Coro} \ \text{God\_def} \ \text{NecExist\_def} \ \text{Rsymm} \ T2$ ) — Proof found

**proof** —

**have**  $1$ :  $[(G \ x \supset NE \ x) \wedge (G \ \text{Ess.} \ x \supset \Box(\exists^E x. G \ x))]$  **using**  $A5 \ \text{Ess-def} \ \text{God-def} \ \text{NecExist-def}$  **by** *smt*

**hence**  $2$ :  $[(\exists x. G \ x) \supset \Box(\exists^E x. G \ x)]$  **using**  $A5 \ \text{God-def} \ \text{NecExist-def} \ T2$  **by** *smt*

**hence**  $3$ :  $[\Diamond(\exists x. G \ x) \supset (\Diamond(\Box(\exists x. G \ x)) \supset \Box(\exists^E x. G \ x))]$  **using** *Rsymm* **by** *blast*

**thus** *?thesis* **using**  $2 \ \text{Coro}$  **by** *blast*

**qed**

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$

— *sledgehammer*( $A1 \ A4 \ \text{God\_def} \ \text{Rsymm} \ T3$ ) — Proof found

**proof** — **{fix**  $w$  **fix**  $Q$

**have**  $1$ :  $\forall x.(G \ x \ w \longrightarrow (\forall Z. Z \ x \supset \Box(\forall z.((G \ z) \supset (Z \ z)))) \ w)$  **using**  $A1 \ A4 \ \text{God-def}$  **by** *smt*

**have**  $2$ :  $(\exists x. G \ x \ w) \longrightarrow ((Q \supset \Box(\forall z.((G \ z) \supset Q))) \ w)$  **using**  $1$  **by** *force*

**have**  $3$ :  $(Q \supset \Box Q) \ w$  **using**  $2 \ T3 \ \text{Rsymm}$  **by** *blast* **}**

**thus** *?thesis* **by** *auto*  
**qed**

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$  **using** *A1 A2* **by** *blast*  
**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  **using** *A1 A2* **by** *blast*  
**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  **using** *Ess-def*  
**by** *smt*  
**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi = \psi)]$  **nitpick**[*card i=2*] **oops**  
— Countermodel found  
**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  **using** *Ess-def MC* **by**  
*auto*  
**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *A1 God-def* **by** *smt*  
**lemma** *Filter*:  $[Filter P]$  **using** *A1 God-def Rsymm T1 T3* **by** (*smt (verit, best)*)  
**lemma** *UltraFilter*:  $[UFilter P]$  **using** *Filter A1* **by** *blast*  
**lemma** *True* **nitpick**[*satisfy, card=1, eval=[P(\lambda x. \perp)]*] **oops** — One model found  
of cardinality one

**end**

## 6 Appendix

### 6.1 GoedelVariantHOML1poss.thy (Figure 16 of [2])

Gödel's axioms and definitions, as presented in the 1970 manuscript, are inconsistent. In contrast to Figure 6 we here use only possibilist quantifiers and still derive falsity.

**theory** *GoedelVariantHOML1poss* **imports** *HOMLinHOL*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization** **where** *Ax1*:  $[P \varphi \wedge P \psi \supset P(\varphi \cdot \psi)]$

**axiomatization** **where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi \text{ Ess. } x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization** **where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G \text{ Ess. } x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt (verit)*)

```

definition NecExist (E) where  $E\ x \equiv \forall \varphi. \varphi\ Ess.\ x \supset \Box(\exists x. \varphi\ x)$ 

axiomatization where Ax3: [P E]

theorem Th2: [ $G\ x \supset \Box(\exists y. G\ y)$ ] using Ax3 Th1 God-def NecExist-def by smt

theorem Th3: [ $\Diamond(\exists x. G\ x) \supset \Box(\exists y. G\ y)$ ]
  — sledgehammer(Th2 Rsymm) — Proof found
proof —
  have 1: [ $(\exists x. G\ x) \supset \Box(\exists y. G\ y)$ ] using Th2 by blast
  have 2: [ $\Diamond(\exists x. G\ x) \supset \Diamond\Box(\exists y. G\ y)$ ] using 1 by blast
  have 3: [ $\Diamond(\exists x. G\ x) \supset \Box(\exists y. G\ y)$ ] using 2 Rsymm by blast
  thus ?thesis by blast
qed

axiomatization where Ax4: [ $P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi$ ]

lemma True nitpick[satisfy,expect=unknown] oops — No model found

lemma EmptyEssL: [ $(\lambda y. \perp)\ Ess.\ x$ ] using Essence-def by metis

theorem Inconsistency: False
  — sledgehammer(Ax2a Ax3 Ax4 EmptyEssL NecExist_def) — Proof found
proof —
  have 1: [ $\neg(P\ (\lambda x. \perp))$ ] using Ax2a Ax4 by blast
  have 2: [ $P\ (\lambda x. (\lambda y. \perp)\ Ess.\ x \supset \Box(\exists z::e. (\lambda y. \perp)z))$ ] using Ax3 Ax4 NecExist-def by smt
  have 3: [ $P\ (\lambda x. \Box(\exists z. (\lambda x. \perp) z))$ ] using 2 EmptyEssL Ax4 by smt
  have 4: [ $P\ (\lambda x. \Box\perp)$ ] using 3 by auto
  have 5: [ $P\ (\lambda x. \perp)$ ] using 4 Ax2a Ax4 by smt
  have 6: [ $\perp$ ] using 1 5 by blast
  thus ?thesis by blast
qed

end

```

## 6.2 GoedelVariantHOML2poss.thy (Figure 17 of [2])

After an appropriate modification of the definition of essence, the inconsistency revealed in Figure 16 is avoided, and the argument can be successfully verified in modal logic S5 (indeed, as shown, only the modal schema B is actually needed). In contrast to Figure 7 we here use only possibilist quantifiers to obtain these results.

```

theory GoedelVariantHOML2poss imports HOMLinHOL ModalFilter
begin

```

```

consts PositiveProperty::( $e \Rightarrow \sigma$ ) $\Rightarrow \sigma$  (P)

```

**axiomatization where**  $Ax1: [P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where**  $Ax2a: [P \varphi \vee^e P \sim\varphi]$

**definition**  $God (G)$  **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation**  $PropertyInclusion (-\supset_N-)$  **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition**  $Essence (-Ess.-)$  **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b: [P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a Ax2b$  **by** *blast*

**theorem**  $Th1: [G x \supset G Ess. x]$  **using**  $Ax2a Ax2b Essence-def God-def$  **by** (*smt (verit)*)

**definition**  $NecExist (E)$  **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists x. \varphi x)$

**axiomatization where**  $Ax3: [P E]$

**theorem**  $Th2: [G x \supset \Box(\exists y. G y)]$  **using**  $Ax3 Th1 God-def NecExist-def$  **by** *smt*

**theorem**  $Th3: [\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$   
— *sledgehammer(Th2 Rsymm)* — *Proof found*

**proof** —

**have** 1:  $[(\exists x. G x) \supset \Box(\exists y. G y)]$  **using**  $Th2$  **by** *blast*

**have** 2:  $[\Diamond(\exists x. G x) \supset \Diamond\Box(\exists y. G y)]$  **using** 1 **by** *blast*

**have** 3:  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  **using** 2 *Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**axiomatization where**  $Ax4: [P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (\lambda x. \perp)]*] **oops** — One model found of cardinality one

**abbreviation**  $PosProps \Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom \varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen: [(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma**  $L: [P G]$  **using**  $Ax1Gen God-def$  **by** *smt*

**theorem**  $Th4: [\Diamond(\exists x. G x)]$  **using**  $Ax2a Ax4 L$  **by** *blast*

**theorem**  $Th5: [\Box(\exists x. G x)]$  **using**  $Th3 Th4$  **by** *blast*

**lemma**  $MC: [\varphi \supset \Box\varphi]$

— sledgehammer(Ax2a Ax2b Th5 God\_def Rsymm) — Proof found  
**proof** – {**fix**  $w$  **fix**  $Q$   
**have** 1:  $\forall x.(G\ x\ w \longrightarrow (\forall Z. Z\ x \supset \Box(\forall z. G\ z \supset Z\ z))\ w)$  **using**  $Ax2a\ Ax2b$   
*God-def* **by** *smt*  
**have** 2:  $(\exists x. G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z. G\ z \supset Q))\ w)$  **using** 1 **by** *force*  
**have** 3:  $(Q \supset \Box Q)$  **using** 2  $Th5\ Rsymm$  **by** *blast*}  
**thus** *?thesis* **by** *auto*  
**qed**

**lemma** *PosProps*:  $[P\ (\lambda x. \top) \wedge P\ (\lambda x. x = x)]$  **using**  $Ax2a\ Ax4$  **by** *blast*  
**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  **using**  $Ax2a\ Ax4$  **by** *blast*  
**lemma** *UniqueEss1*:  $[\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall y. \varphi\ y \leftrightarrow \psi\ y)]$  **using** *Essence-def*  
**by** *smt*  
**lemma** *UniqueEss2*:  $[\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi \equiv \psi)]$  **nitpick**[*card i=2*] **oops**  
— Countermodel found  
**lemma** *UniqueEss3*:  $[\varphi\ Ess.\ x \supset \Box(\forall y. \varphi\ y \supset y \equiv x)]$  **using** *Essence-def MC*  
**by** *auto*  
**lemma** *Monotheism*:  $[G\ x \wedge G\ y \supset x \equiv y]$  **using**  $Ax2a\ God-def$  **by** *smt*  
**lemma** *Filter*:  $[FilterP\ P]$  **using**  $Ax1\ Ax4\ MC\ NegProps\ PosProps\ Rsymm$  **by**  
*smt*  
**lemma** *UltraFilter*:  $[UFilterP\ P]$  **using**  $Ax2a\ Filter$  **by** *smt*  
**lemma** *True* **nitpick**[*satisfy, card=1, eval=[P (\lambda x. \perp)]*] **oops** — One model found  
of cardinality one

**end**

### 6.3 GoedelVariantHOML3poss.thy (Figure 18 of [2])

After an appropriate modification of the definition of necessary property implication, the inconsistency shown in Figure 16 is avoided, and the argument can be successfully verified. As shown here, this still holds when using possibilist quantifiers only.

**theory** *GoedelVariantHOML3poss* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma\ (P)$

**axiomatization** **where**  $Ax1: [P\ \varphi \wedge P\ \psi \supset P\ (\varphi . \psi)]$

**axiomatization** **where**  $Ax2a: [P\ \varphi \vee^e P \sim \varphi]$

**definition** *God* ( $G$ ) **where**  $G\ x \equiv \forall \varphi. P\ \varphi \supset \varphi\ x$

**abbreviation** *PropertyInclusion* ( $\supset_N$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y::e. \varphi\ y \supset \psi\ y))$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi\ Ess.\ x \equiv \forall \psi. \psi\ x \supset (\varphi \supset_N \psi)$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a$   $Ax2b$  **by** *blast*

**theorem**  $Th1$ :  $[G x \supset G \text{Ess. } x]$  **using**  $Ax2a$   $Ax2b$  *Essence-def* *God-def* **by** (*smt* (*verit*))

**definition**  $NecExist$  ( $E$ ) **where**  $E x \equiv \forall \varphi. (\varphi \text{Ess. } x) \supset \Box(\exists x. \varphi x)$

**axiomatization where**  $Ax3$ :  $[P E]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists y. G y)]$  **using**  $Ax3$   $Th1$  *God-def* *NecExist-def* **by** *smt*

**theorem**  $Th3$ :  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$   
— *sledgehammer*( $Th2$  *Rsymm*) — *Proof found*

**proof** —

**have** 1:  $[(\exists x. G x) \supset \Box(\exists y. G y)]$  **using**  $Th2$  **by** *blast*

**have** 2:  $[\Diamond(\exists x. G x) \supset \Diamond\Box(\exists y. G y)]$  **using** 1 **by** *blast*

**have** 3:  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  **using** 2 *Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy*,*card=1*,*eval*=[ $P(\lambda x. \perp)$ ]] **oops** — One model found of cardinality one

**abbreviation**  $PosProps$   $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom$   $\varphi$   $\Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen$ :  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**lemma**  $L$ :  $[P G]$  **using**  $Ax1Gen$  *God-def* **by** (*smt* (*verit*))

**theorem**  $Th4$ :  $[\Diamond(\exists x. G x)]$

— *sledgehammer*[*timeout=200*]( $Ax2a$   $L$   $Ax1Gen$ ) **oops** — sorry — *Proof found*

**axiomatization where**  $Th4$ :  $[\Diamond(\exists x. G x)]$

**theorem**  $Th5$ :  $[\Box(\exists x. G x)]$  **using**  $Th3$   $Th4$  **by** *blast*

**lemma**  $MC$ :  $[\varphi \supset \Box\varphi]$

— *sledgehammer*( $Ax2a$   $Ax2b$   $Th5$  *God\_def* *Rsymm*) — *Proof found*

**proof** — **{fix**  $w$  **fix**  $Q$

**have** 1:  $\forall x. (G x w \longrightarrow (\forall Z. Z x \supset \Box(\forall z. G z \supset Z z)) w)$  **using**  $Ax2a$   $Ax2b$  *God-def* **by** *smt*

**have** 2:  $(\exists x. G x w) \longrightarrow ((Q \supset \Box(\forall z. G z \supset Q)) w)$  **using** 1 **by** *force*

**have** 3:  $(Q \supset \Box Q)$  **using** 2  $Th5$  *Rsymm* **by** *blast*}

**thus** *?thesis* **by** *auto*

qed

**lemma** *PosProps*:  $[P(\lambda x. \top) \wedge P(\lambda x. x = x)]$  **using** *Ax2a Ax4 L Th4* **by** *smt*  
**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  **using** *Ax2a Ax4 L Th4* **by** *smt*  
**lemma** *UniqueEss1*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  **oops** — Unclear, open question  
**lemma** *UniqueEss2*:  $[\varphi \text{ Ess. } x \wedge \psi \text{ Ess. } x \supset \Box(\varphi \equiv \psi)]$  **oops** — Unclear, open question  
**lemma** *UniqueEss3*:  $[\varphi \text{ Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  **using** *Essence-def MC* **by** *auto*  
**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *Ax2a God-def* **by** *smt*  
**lemma** *Filter*:  $[FilterP P]$  **using** *Ax1 Ax4 MC NegProps PosProps Rsymm* **by** *smt*  
**lemma** *UltraFilter*:  $[UltraFilterP P]$  **using** *Ax2a Filter* **by** *smt*  
**lemma** *True nitpick* $[satisfy, card=1, eval=[P(\lambda x. \top)]]$  **oops** — One model found of cardinality one

end

## 6.4 ScottVariantHOMLposs.thy (Figure 19 of [2])

Scott's variant of Gödel's ontological proof is still valid when using possibilist quantifiers only.

**theory** *ScottVariantHOMLposs* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma (P)$

**axiomatization where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization where** *A2*:  $[P \varphi \wedge \Box(\forall y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists x. \varphi x)]$  **using** *A1 A2* **by** *blast*

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G x)]$  **using** *A3 T1* **by** *blast*

**axiomatization where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (*-Ess.-*) **where**  $\varphi \text{ Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall y::e. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{ Ess. } x]$  **using** *A1 A4 Ess-def God-def* **by** *fastforce*

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{ Ess. } x \supset \Box(\exists x. \varphi x)$



**axiomatization** where  $A5: [P \ NE]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*] **oops** — One model found of cardinality one

**theorem** *T3*:  $[\Box(\exists x. G x)]$

— sledgehammer(A5 Coro God\_def NecExist\_def Rsymm T2) — Proof found

**proof** —

**have** 1:  $[(G x \supset NE x) \wedge (G \text{Ess. } x \supset \Box(\exists x. G x))]$  **using** *A5 Ess-def God-def NecExist-def* **by** *smt*

**hence** 2:  $[(\exists x. G x) \supset \Box(\exists x. G x)]$  **using** *A5 God-def NecExist-def T2* **by** *smt*

**hence** 3:  $[\Diamond(\exists x. G x) \supset (\Diamond(\Box(\exists x. G x)) \supset \Box(\exists x. G x))]$  **using** *Rsymm* **by** *blast*

**thus** *?thesis* **using** 2 *Coro* **by** *blast*

**qed**

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(A1 A4 God\_def Rsymm T3) — Proof found

**proof** — **{fix w fix Q**

**have** 1:  $\forall x.(G x w \longrightarrow (\forall Z. Z x \supset \Box(\forall z.((G z) \supset (Z z)))) w)$  **using** *A1 A4 God-def* **by** *smt*

**have** 2:  $(\exists x. G x w) \longrightarrow ((Q \supset \Box(\forall z.((G z) \supset Q))) w)$  **using** 1 **by** *force*

**have** 3:  $(Q \supset \Box Q) w$  **using** 2 *T3 Rsymm* **by** *blast*

**thus** *?thesis* **by** *auto*

**qed**

**lemma** *PosProps*:  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$  **using** *A1 A2* **by** *blast*

**lemma** *NegProps*:  $[\neg P(\lambda x. \perp) \wedge \neg P(\lambda x. x \neq x)]$  **using** *A1 A2* **by** *blast*

**lemma** *UniqueEss1*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  **using** *Ess-def* **by** *smt*

**lemma** *UniqueEss2*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\varphi \equiv \psi)]$  **nitpick**[*card i=2*] **oops** — Countermodel found

**lemma** *UniqueEss3*:  $[\varphi \text{Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  **using** *Ess-def MC* **by** *auto*

**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *A1 God-def* **by** *smt*

**lemma** *Filter*:  $[FilterP P]$  **using** *A1 God-def Rsymm T1 T3* **by** (*smt (verit, best)*)

**lemma** *UltraFilter*:  $[UFilterP P]$  **using** *Filter A1* **by** *blast*

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (λx.⊥)]*] **oops** — One model found of cardinality one

**end**

## 6.5 EvilDerivable.thy (Figure 20 of [2])

The necessary existence of an Evil-like entity proved from (controversially) modified assumptions. By rejecting Gödel's assumptions and instead postulating corresponding negative versions of them, as shown in the Figure 20, the necessary existence of Evil becomes derivable. The non-positive prop-

erties of this Evil-like entity are however identical to the positive properties of Gödel's God-like entity.

**theory** *EvilDerivable* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**definition** *Evil* (*Evil*) **where**  $Evil\ x \equiv \forall \varphi. \neg P\ \varphi \supset \varphi\ x$

**definition** *Essence* (*-Ess.-*) **where**  $\varphi\ Ess.\ x \equiv \varphi\ x \wedge (\forall \psi. \psi\ x \supset \Box(\forall^E y. \varphi\ y \supset \psi\ y))$

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ Ess.\ x \supset \Box(\exists^E x. \varphi\ x)$

**axiomatization** **where** *A1*:  $\neg P\ \varphi \leftrightarrow P\ \sim\varphi$

**axiomatization** **where** *A2*:  $\neg P\ \varphi \wedge \Box(\forall^E y. \varphi\ y \supset \psi\ y) \supset \neg P\ \psi$

**axiomatization** **where** *A4*:  $\neg P\ Evil$

**axiomatization** **where** *A3*:  $\neg P\ \varphi \supset \Box(\neg P\ \varphi)$

**axiomatization** **where** *A5*:  $\neg P\ E$

**lemma** *True* **nitpick**[*satisfy*,*card i=1*,*eval=[P (\lambda x. \perp)]*,*eval=[P (\lambda x. \top)]*] **oops**  
— Model found

**theorem** *T1*:  $\neg P\ \varphi \supset \Diamond(\exists^E x. \varphi\ x)$  **using** *A1 A2* **by** *blast*

**theorem** *T2*:  $\Diamond(\exists^E x. Evil\ x)$  **using** *A4 T1* **by** *blast*

**theorem** *T3*:  $Evil\ x \supset Evil\ Ess.\ x$  **using** *A1 A3 Essence-def Evil-def* **by** (*smt* (*verit*, *best*))

**theorem** *T4*:  $\Diamond(\exists^E x. Evil\ x) \supset \Box(\exists^E y. Evil\ y)$  **using** *A5 Evil-def NecExist-def Rsymm T3* **by** *smt*

**theorem** *T5*:  $\Box(\exists^E x. Evil\ x)$  **using** *T2 T4* **by** *presburger*

**lemma** *MC*:  $\varphi \supset \Box\varphi$

— *sledgehammer*(*A1 A3 T5 Evil\_def Rsymm*) **oops** — proof found

**lemma** *PosProps*:  $P(\lambda x. \perp) \wedge P(\lambda x. x \neq x)$  **using** *A1 A2* **by** *blast*

**lemma** *NegProps*:  $\neg P(\lambda x. \top) \wedge \neg P(\lambda x. x = x)$  **using** *A1 A2* **by** *blast*

**lemma** *UniqueEss1*:  $\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\forall^E y. \varphi\ y \leftrightarrow \psi\ y)$  **using** *Essence-def* **by** *smt*

**lemma** *UniqueEss2*:  $\varphi\ Ess.\ x \wedge \psi\ Ess.\ x \supset \Box(\varphi = \psi)$  **nitpick**[*card i=2*] **oops**  
— Countermodel found

**lemma** *Monoevilism*:  $Evil\ x \wedge Evil\ y \supset x \equiv y$  **using** *A1 Evil-def* **by** *smt*

**lemma** *Filter*: [*Filter* ( $\lambda \varphi. \neg P \varphi$ )] **using** *A1 Evil-def Rsymm T1 T5* **by** (*smt* (*verit, best*))

**lemma** *UltraFilter*: [*UltraFilter* ( $\lambda \varphi. \neg P \varphi$ )] **using** *Filter A1* **by** *blast*

**end**

## 7 Further Appendices

### 7.1 GoedelVariantHOML1AndersonQuan.thy

The same as GoedelVariantHOML1, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML1AndersonQuant* **imports** *HOMLinHOL*  
**begin**

**consts** *PositiveProperty*::( $e \Rightarrow \sigma$ ) $\Rightarrow \sigma$  (*P*)

**axiomatization** **where** *Ax1*: [ $P \varphi \wedge P \psi \supset P (\varphi . \psi)$ ]

**axiomatization** **where** *Ax2a*: [ $P \varphi \vee^e P \sim \varphi$ ]

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization** **where** *Ax2b*: [ $P \varphi \supset \Box P \varphi$ ]

**lemma** *Ax2b'*: [ $\neg P \varphi \supset \Box(\neg P \varphi)$ ] **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*: [ $G x \supset G Ess. x$ ] **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* (*verit*))

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization** **where** *Ax3*: [ $P E$ ]

**theorem** *Th2*: [ $G x \supset \Box(\exists^E y. G y)$ ] **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**theorem** *Th3*: [ $\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)$ ]

— sledgehammer(Th2 Rsymm) — Proof found

**proof** —

**have** 1: [ $\exists^E x. G x \supset \Box(\exists^E y. G y)$ ] **using** *Th2* **by** *blast*

**have** 2: [ $\Diamond(\exists^E x. G x) \supset \Diamond\Box(\exists^E y. G y)$ ] **using** 1 **by** *blast*

**have** 3: [ $\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)$ ] **using** 2 *Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

qed

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick* $[satisfy, expect=unknown]$  **oops** — No model found

**lemma** *EmptyEssL*:  $[(\lambda y. \perp) \text{Ess. } x]$  **using** *Essence-def* **by** *auto*

**theorem** *Inconsistency*: *False*

— sledgehammer( $Ax2a$   $Ax3$   $Ax4$  *EmptyEssL* *NecExist\_def*) — Proof found

**proof** —

**have** 1:  $[\neg(P (\lambda x. \perp))]$  **using**  $Ax2a$   $Ax4$  **by** *blast*

**have** 2:  $[P (\lambda x. (\lambda y. \perp) \text{Ess. } x \supset \Box(\exists^E z. (\lambda y. \perp) z))]$  **using**  $Ax3$   $Ax4$  *NecExist-def* **by** (*smt* (*verit*))

**have** 3:  $[P (\lambda x. \Box(\exists^E z. (\lambda x. \perp) z))]$  **using** 2 *EmptyEssL* **by** *simp*

**have** 4:  $[P (\lambda x. \Box \perp)]$  **using** 3 **by** *auto*

**have** 5:  $[P (\lambda x. \perp)]$  **using** 4  $Ax2a$   $Ax4$  **by** *smt*

**have** 6:  $[\perp]$  **using** 1 5 **by** *blast*

**thus** *?thesis* **by** *blast*

qed

end

## 7.2 GoedelVariantHOML2AndersonQuan.thy

The same as *GoedelVariantHOML2*, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML2AndersonQuant* **imports** *HOMLinHOL* *ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $P$ )

**axiomatization where**  $Ax1$ :  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**axiomatization where**  $Ax2a$ :  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* ( $G$ ) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi \text{Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a$   $Ax2b$  **by** *blast*

**theorem** *Th1*:  $[G x \supset G \text{Ess. } x]$  **using**  $Ax2a$   $Ax2b$  *Essence-def* *God-def* **by** (*smt* (*verit*))

**definition** *NecExist* (*E*) **where**  $E\ x \equiv \forall \varphi. \varphi\ \text{Ess. } x \supset \Box(\exists^E x. \varphi\ x)$

**axiomatization where** *Ax3*:  $[P\ E]$

**theorem** *Th2*:  $[G\ x \supset \Box(\exists^E y. G\ y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G\ x) \supset \Box(\exists^E y. G\ y)]$   
 — sledgehammer(*Th2 Rsymm*) — Proof found

**proof** —

**have** *1*:  $[(\exists^E x. G\ x) \supset \Box(\exists^E y. G\ y)]$  **using** *Th2* **by** *blast*

**have** *2*:  $[\Diamond(\exists^E x. G\ x) \supset \Diamond\Box(\exists^E y. G\ y)]$  **using** *1* **by** *blast*

**have** *3*:  $[\Diamond(\exists^E x. G\ x) \supset \Box(\exists^E y. G\ y)]$  **using** *2 Rsymm* **by** *blast*

**thus** *?thesis* **by** *blast*

**qed**

**axiomatization where** *Ax4*:  $[P\ \varphi \wedge (\varphi \supset_N \psi) \supset P\ \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (\lambda x. \perp)]*] **oops** — One model found of cardinality one

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi\ \varphi \supset P\ \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi\ \Phi \equiv \Box(\forall z. \varphi\ z \leftrightarrow (\forall \psi. \Phi\ \psi \supset \psi\ z))$

**axiomatization where** *Ax1Gen*:  $[(\text{PosProps } \Phi \wedge \text{ConjOfPropsFrom } \varphi\ \Phi) \supset P\ \varphi]$

**lemma** *L*:  $[P\ G]$  **using** *Ax1Gen God-def* **by** *smt*

**theorem** *Th4*:  $[\Diamond(\exists^E x. G\ x)]$  **using** *Ax2a Ax4 L Rsymm Th2* **by** *metis*

**theorem** *Th5*:  $[\Box(\exists^E x. G\ x)]$  **using** *Th3 Th4* **by** *blast*

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$

— sledgehammer(*Ax2a Ax2b Th5 God\_def Rsymm*) — proof found

**proof** — **{fix** *w* **fix** *Q*

**have** *1*:  $\forall x. (G\ x\ w \longrightarrow (\forall Z. Z\ x \supset \Box(\forall z. G\ z \supset Z\ z))\ w)$  **using** *Ax2a Ax2b God-def* **by** *smt*

**have** *2*:  $(\exists x. G\ x\ w) \longrightarrow ((Q \supset \Box(\forall z. G\ z \supset Q))\ w)$  **using** *1* **by** *force*

**have** *3*:  $(Q \supset \Box Q)\ w$  **using** *2 Th5 Rsymm* **by** *blast*

**thus** *?thesis* **by** *auto*

**qed**

**lemma** *PosProps*:  $[P\ (\lambda x. \top) \wedge P\ (\lambda x. x = x)]$  **using** *Ax2a Ax4* **by** *blast*

**lemma** *NegProps*:  $[\neg P\ (\lambda x. \perp) \wedge \neg P\ (\lambda x. x \neq x)]$  **using** *Ax2a Ax4* **by** *blast*

**lemma** *UniqueEss1*:  $[\varphi\ \text{Ess. } x \wedge \psi\ \text{Ess. } x \supset \Box(\forall y. \varphi\ y \leftrightarrow \psi\ y)]$  **using** *Essence-def* **by** *smt*

**lemma** *UniqueEss2*:  $[\varphi\ \text{Ess. } x \wedge \psi\ \text{Ess. } x \supset \Box(\varphi \equiv \psi)]$  **nitpick**[*card i=2*] **oops**  
 — Countermodel found

**lemma** *UniqueEss3*:  $[\varphi\ \text{Ess. } x \supset \Box(\forall y. \varphi\ y \supset y \equiv x)]$  **using** *Essence-def MC*

**by** *auto*  
**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *Ax2a God-def* **by** *smt*  
**lemma** *Filter*:  $[Filter P]$  **using** *Ax1 Ax4 MC NegProps PosProps Rsymm Ax2a God-def Th5* **by** (*smt (verit, ccfv-threshold)*)  
**lemma** *UltraFilter*:  $[UFilter P]$  **using** *Ax2a Filter* **by** *smt*  
**lemma** *True nitpick* $[satisfy, card=1, eval=[P (\lambda x. \perp)]]$  **oops** — One model found of cardinality one  
**end**

### 7.3 GoedelVariantHOML3AndersonQuan.thy

The same as GoedelVariantHOML3, but now for a mixed use of actualist and possibilist quantifiers for entities; cf. Footnote 20 in [2].

**theory** *GoedelVariantHOML3AndersonQuant* **imports** *HOMLinHOL ModalFilter*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y::e. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt (verit)*)

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$

— sledgehammer(*Th2 Rsymm*) — Proof found

**proof** —

**have** 1:  $[(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using** *Th2* **by** *blast*

**have** 2:  $[\Diamond(\exists^E x. G x) \supset \Diamond\Box(\exists^E y. G y)]$  **using** 1 **by** *blast*

**have** 3:  $[\diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  **using** 2 *Rsymm* **by** *blast*  
**thus** *?thesis* **by** *blast*  
**qed**

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**lemma** *True nitpick*[*satisfy,card=1,eval=[P (\lambda x. \perp)]*] **oops** — Two models found of cardinality one

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$   
**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$   
**axiomatization where** *Ax1Gen*:  $[(\text{PosProps } \Phi \wedge \text{ConjOfPropsFrom } \varphi \Phi) \supset P \varphi]$

**lemma** *L*:  $[P G]$  **using** *Ax1Gen* *God-def* **by** (*smt* (*verit*))

**theorem** *Th4*:  $[\diamond(\exists^E x. G x)]$   
— *sledgehammer*[*timeout=200*](*Ax2a* *L* *Ax1Gen*) **oops** — sorry — Proof found

**axiomatization where** *Th4*:  $[\diamond(\exists^E x. G x)]$

**theorem** *Th5*:  $[\Box(\exists^E x. G x)]$  **using** *Th3* *Th4* **by** *blast*

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$   
— *sledgehammer*(*Ax2a* *Ax2b* *Th5* *God\_def* *Rsymm*) — Proof found  
**proof** — {**fix** *w* **fix** *Q*  
**have** 1:  $\forall x. (G x w \longrightarrow (\forall Z. Z x \supset \Box(\forall z. G z \supset Z z)) w)$  **using** *Ax2a* *Ax2b*  
*God-def* **by** *smt*  
**have** 2:  $(\exists x. G x w) \longrightarrow ((Q \supset \Box(\forall z. G z \supset Q)) w)$  **using** 1 **by** *force*  
**have** 3:  $(Q \supset \Box Q) w$  **using** 2 *Th5* *Rsymm* **by** *blast*}  
**thus** *?thesis* **by** *auto*  
**qed**

**lemma** *PosProps*:  $[P (\lambda x. \top) \wedge P (\lambda x. x = x)]$  **using** *Ax2a* *Ax4* *L* *Th4* **by** *smt*  
**lemma** *NegProps*:  $[\neg P (\lambda x. \perp) \wedge \neg P (\lambda x. x \neq x)]$  **using** *Ax2a* *Ax4* *L* *Th4* **by** *smt*  
**lemma** *UniqueEss1*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\forall y. \varphi y \leftrightarrow \psi y)]$  **oops** — Unclear, open question  
**lemma** *UniqueEss2*:  $[\varphi \text{Ess. } x \wedge \psi \text{Ess. } x \supset \Box(\varphi \equiv \psi)]$  **oops** — Unclear, open question  
**lemma** *UniqueEss3*:  $[\varphi \text{Ess. } x \supset \Box(\forall y. \varphi y \supset y \equiv x)]$  **using** *Essence-def* *MC* **by** *auto*  
**lemma** *Monotheism*:  $[G x \wedge G y \supset x \equiv y]$  **using** *Ax2a* *God-def* **by** *smt*  
**lemma** *Filter*:  $[Filter P]$  **using** *Ax1* *Ax4* *MC* *NegProps* *PosProps* *Rsymm* *Ax2a* *God-def* *Th5* **by** (*smt* (*verit*, *ccfv-threshold*))  
**lemma** *UltraFilter*:  $[UFilter P]$  **using** *Ax2a* *Filter* **by** *blast*  
**lemma** *True nitpick*[*satisfy,card=1,eval=[P (\lambda x. \top)]*] **oops** — One model found of cardinality one

**end**

## 7.4 HOMLinHOLOnlyS4.thy (slight variation of Figure 3 of [2])

Shallow embedding of higher-order modal logic (HOML) in the classical higher-order logic (HOL) of Isabelle/HOL utilizing the LogiKEy methodology. Here logic S4 is introduced.

```
theory HOMLinHOLOnlyS4 imports Main
begin
```

— Global parameters setting for the model finder nitpick and the parser; unimport for the reader

```
nitpick-params[user-axioms,expect=genuine,show-all,format=2,max-genuine=3]
declare[[syntax-ambiguity-warning=false]]
```

— Type *i* is associated with possible worlds and type *e* with entities:

```
typedecl i — Possible worlds
typedecl e — Individuals/entities
type-synonym  $\sigma = i \Rightarrow \text{bool}$  — World-lifted propositions
type-synonym  $\tau = e \Rightarrow \sigma$  — modal properties
```

**consts** *R::i⇒i⇒bool* (**-r-**) — Accessibility relation between worlds  
**axiomatization where**

```
Rrefl:  $\forall x. \mathbf{r}x \mathbf{and}$ 
Rtrans:  $\forall x y z. \mathbf{r}xy \wedge \mathbf{r}yz \longrightarrow \mathbf{r}xz$ 
```

— Logical connectives (operating on truth-sets)

**abbreviation** *Mbot::σ* ( $\perp$ ) **where**  $\perp \equiv \lambda w. \text{False}$

**abbreviation** *Mtop::σ* ( $\top$ ) **where**  $\top \equiv \lambda w. \text{True}$

**abbreviation** *Mneg::σ⇒σ* ( $\neg$ - [52]53) **where**  $\neg\varphi \equiv \lambda w. \neg(\varphi w)$

**abbreviation** *Mand::σ⇒σ⇒σ* (**infixl**  $\wedge$  50) **where**  $\varphi \wedge \psi \equiv \lambda w. \varphi w \wedge \psi w$

**abbreviation** *Mor::σ⇒σ⇒σ* (**infixl**  $\vee$  49) **where**  $\varphi \vee \psi \equiv \lambda w. \varphi w \vee \psi w$

**abbreviation** *Mimp::σ⇒σ⇒σ* (**infixr**  $\supset$  48) **where**  $\varphi \supset \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$

**abbreviation** *Mequiv::σ⇒σ⇒σ* (**infixl**  $\leftrightarrow$  47) **where**  $\varphi \leftrightarrow \psi \equiv \lambda w. \varphi w \longleftrightarrow \psi w$

**abbreviation** *Mbox::σ⇒σ* ( $\Box$ - [54]55) **where**  $\Box\varphi \equiv \lambda w. \forall v. w \mathbf{r} v \longrightarrow \varphi v$

**abbreviation** *Mdia::σ⇒σ* ( $\Diamond$ - [54]55) **where**  $\Diamond\varphi \equiv \lambda w. \exists v. w \mathbf{r} v \wedge \varphi v$

**abbreviation** *Mprimeq::'a⇒'a⇒σ* (**-=-**) **where**  $x=y \equiv \lambda w. x=w$

**abbreviation** *Mprimneg::'a⇒'a⇒σ* (**-≠-**) **where**  $x \neq y \equiv \lambda w. x \neq y$

**abbreviation** *Mnegpred::τ⇒τ* ( $\sim$ -) **where**  $\sim\Phi \equiv \lambda x. \lambda w. \neg\Phi x w$

**abbreviation** *Mconpred::τ⇒τ⇒τ* (**infixl**  $.$  50) **where**  $\Phi.\Psi \equiv \lambda x. \lambda w. \Phi x w \wedge \Psi x w$

**abbreviation** *Mexclor::σ⇒σ⇒σ* (**infixl**  $\vee^e$  49) **where**  $\varphi \vee^e \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$

— Possibilist quantifiers (polymorphic)

**abbreviation** *Mallposs::('a⇒σ)⇒σ* ( $\forall$ ) **where**  $\forall\Phi \equiv \lambda w. \forall x. \Phi x w$

**abbreviation** *Mallpossb* (**binder**  $\forall$  [8]9) **where**  $\forall x. \varphi(x) \equiv \forall\varphi$

**abbreviation** *Mexiposs::('a⇒σ)⇒σ* ( $\exists$ ) **where**  $\exists\Phi \equiv \lambda w. \exists x. \Phi x w$

**abbreviation** *Mexipossb* (**binder**  $\exists$  [8]9) **where**  $\exists x. \varphi(x) \equiv \exists\varphi$



— Actualist quantifiers (for individuals/entities)

**consts** *existsAt*:: $e \Rightarrow \sigma$  ( $-\textcircled{A}$ -)

**abbreviation** *Mallact*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall^E$ ) **where**  $\forall^E \Phi \equiv \lambda w. \forall x. x \textcircled{A} w \longrightarrow \Phi x w$

**abbreviation** *Mallactb* (**binder**  $\forall^E$  [ $\delta$ ]  $g$ ) **where**  $\forall^E x. \varphi(x) \equiv \forall^E \varphi$

**abbreviation** *Mexiact*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists^E$ ) **where**  $\exists^E \Phi \equiv \lambda w. \exists x. x \textcircled{A} w \wedge \Phi x w$

**abbreviation** *Mexiactb* (**binder**  $\exists^E$  [ $\delta$ ]  $g$ ) **where**  $\exists^E x. \varphi(x) \equiv \exists^E \varphi$

— Leibniz equality (polymorphic)

**abbreviation** *Mleibeq*:: $'a \Rightarrow 'a \Rightarrow \sigma$  ( $-\equiv$ -) **where**  $x \equiv y \equiv \forall P. P x \supset P y$

— Meta-logical predicate for global validity

**abbreviation** *Mvalid*:: $\sigma \Rightarrow \text{bool}$  ( $[-]$ ) **where**  $[\psi] \equiv \forall w. \psi w$

**end**

## 7.5 TestsHOMLinS4.thy

Tests and verifications of properties for the embedding of HOML (S4) in HOL.

**theory** *TestsHOMLinS4* **imports** *HOMLinHOLonlyS4*  
**begin**

— Test for S5 modal logic

**lemma** *axM*:  $[\Box \varphi \supset \varphi]$  **using** *Rrefl* **by** *blast*

**lemma** *axD*:  $[\Box \varphi \supset \Diamond \varphi]$  **using** *Rrefl* **by** *blast*

**lemma** *axB*:  $[\varphi \supset \Box \Diamond \varphi]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

**lemma** *ax4*:  $[\Box \varphi \supset \Box \Box \varphi]$  **using** *Rtrans* **by** *blast*

**lemma** *ax5*:  $[\Diamond \varphi \supset \Box \Diamond \varphi]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

— Test for Barcan and converse Barcan formula:

**lemma** *BarcanAct*:  $[(\forall^E x. \Box(\varphi x)) \supset \Box(\forall^E x. (\varphi x))]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

**lemma** *ConvBarcanAct*:  $[\Box(\forall^E x. (\varphi x)) \supset (\forall^E x. \Box(\varphi x))]$

**nitpick**[*expect=genuine*] **oops** — Countermodel found

**lemma** *BarcanPoss*:  $[(\forall x. \Box(\varphi x)) \supset \Box(\forall x. \varphi x)]$  **by** *blast*

**lemma** *ConvBarcanPoss*:  $[\Box(\forall x. (\varphi x)) \supset (\forall x. \Box(\varphi x))]$  **by** *blast*

— A simple Hilbert system for classical propositional logic is derived

**lemma** *Hilbert-A1*:  $[A \supset (B \supset A)]$  **by** *blast*

**lemma** *Hilbert-A2*:  $[(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))]$  **by** *blast*

**lemma** *Hilbert-MP*: **assumes**  $[A]$  **and**  $[A \supset B]$  **shows**  $[B]$  **using** *assms* **by** *blast*

— We have a polymorphic possibilist quantifier for which existential import holds

**lemma** *Quant-1*: **assumes**  $[A]$  **shows**  $[\forall x::'a. A]$  **using** *assms* **by** *auto*

— Existential import holds for possibilist quantifiers

**lemma** *ExImPossibilist1*:  $[\exists x::e. x = x]$  **by** *blast*  
**lemma** *ExImPossibilist2*:  $[\exists x::e. x \equiv x]$  **by** *blast*  
**lemma** *ExImPossibilist3*:  $[\exists x::e. x = t]$  **by** *blast*  
**lemma** *ExImPossibilist4*:  $[\exists x::'a. x \equiv t::'a]$  **by** *blast*  
**lemma** *ExImPossibilist*:  $[\exists x::'a. \top]$  **by** *blast*

— We have an actualist quantifier for individuals for which existential import does not hold

**lemma** *Quant-2*: **assumes**  $[A]$  **shows**  $[\forall^E x::e. A]$  **using** *assms* **by** *auto*

— Existential import does not hold for our actualist quantifiers (for individuals)

**lemma** *ExImActualist1*:  $[\exists^E x::e. x = x]$   
**nitpick** $[card=1, expect=genuine]$  **oops** — Countermodel found  
**lemma** *ExImActualist2*:  $[\exists^E x::e. x \equiv x]$   
**nitpick** $[card=1, expect=genuine]$  **oops** — Countermodel found  
**lemma** *ExImActualist3*:  $[\exists^E x::e. x = t]$   
**nitpick** $[card=1, expect=genuine]$  **oops** — Countermodel found  
**lemma** *ExImActualist*:  $[\exists^E x::e. \top]$   
**nitpick** $[card=1, expect=genuine]$  **oops** — Countermodel found

— Properties of the embedded primitive equality, which coincides with Leibniz equality

**lemma** *EqRef1*:  $[x = x]$  **by** *blast*  
**lemma** *EqSym*:  $[(x = y) \leftrightarrow (y = x)]$  **by** *blast*  
**lemma** *EqTrans*:  $[(x = y) \wedge (y = z) \supset (x = z)]$  **by** *blast*  
**lemma** *EQCong*:  $[(x = y) \supset ((\varphi x) = (\varphi y))]$  **by** *blast*  
**lemma** *EQFuncExt*:  $[(\varphi = \psi) \supset (\forall x. ((\varphi x) = (\psi x)))]$  **by** *blast*  
**lemma** *EQBoolExt1*:  $[(\varphi = \psi) \supset (\varphi \leftrightarrow \psi)]$  **by** *blast*  
**lemma** *EQBoolExt2*:  $[(\varphi \leftrightarrow \psi) \supset (\varphi = \psi)]$   
**nitpick** $[card=2]$  **oops** — Countermodel found  
**lemma** *EQBoolExt3*:  $[(\varphi \leftrightarrow \psi) \longrightarrow [(\varphi = \psi)]]$  **by** *blast*  
**lemma** *EqPrimLeib*:  $[(x = y) \leftrightarrow (x \equiv y)]$  **by** *auto*

— Comprehension is natively supported in HOL (due to lambda-abstraction)

**lemma** *Comprehension1*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow A]$  **by** *force*  
**lemma** *Comprehension2*:  $[\exists \varphi. \forall x. (\varphi x) \leftrightarrow (A x)]$  **by** *force*  
**lemma** *Comprehension3*:  $[\exists \varphi. \forall x y. (\varphi x y) \leftrightarrow (A x y)]$  **by** *force*

— Modal collapse does not hold

**lemma** *ModalCollapse*:  $[\forall \varphi. \varphi \supset \Box \varphi]$   
**nitpick** $[card=2, expect=genuine]$  **oops** — Countermodel found

— Empty property and self-difference

**lemma** *TruePropertyAndSelfIdentity*:  $[(\lambda x::e. \top) = (\lambda x. x = x)]$  **by** *blast*  
**lemma** *EmptyPropertyAndSelfDifference*:  $[(\lambda x::e. \perp) = (\lambda x. x \neq x)]$  **by** *blast*  
**lemma** *EmptyProperty2*:  $[\exists x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$  **by** *blast*  
**lemma** *EmptyProperty3*:  $[\exists^E x. \varphi x] \longrightarrow [\varphi \neq (\lambda x::e. \perp)]$  **by** *blast*  
**lemma** *EmptyProperty4*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists x. \varphi x]$   
**nitpick** $[expect=genuine]$  **oops** — Countermodel found

**lemma** *EmptyProperty5*:  $[\varphi \neq (\lambda x::e. \perp)] \longrightarrow [\exists^E x. \varphi x]$   
**nitpick***[expect=genuine]* **oops** — Countermodel found

**end**

## 7.6 GoedelVariantHOML1inS4.thy

The same as GoedelVariantHOML1, but now in logic S4.

**theory** *GoedelVariantHOML1inS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box (\forall^E y. \varphi y \supset \psi y)$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box (\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt* *(verit)*)

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box (\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box (\exists^E y. G y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem** *Th3*:  $[\Diamond (\exists^E x. G x) \supset \Box (\exists^E y. G y)]$  **using** *Ax3 Essence-def God-def NecExist-def Rrefl* **by** *fastforce*

**end**

## 7.7 GoedelVariantHOML2inS4.thy

The same as GoedelVariantHOML2, but now in logic S4, where the proof of theorem Th3 fails.

```

theory GoedelVariantHOML2inS4 imports HOMLinHOLOnlyS4
begin

consts PositiveProperty::(e⇒σ)⇒σ (P)

axiomatization where Ax1: [P φ ∧ P ψ ⊃ P (φ . ψ)]

abbreviation PosProps Φ ≡ ∀φ. Φ φ ⊃ P φ
abbreviation ConjOfPropsFrom φ Φ ≡ □(∀Ez. φ z ↔ (∀ψ. Φ ψ ⊃ ψ z))
axiomatization where Ax1Gen: [(PosProps Φ ∧ ConjOfPropsFrom φ Φ) ⊃ P φ]

axiomatization where Ax2a: [P φ ∨e P ~φ]

definition God (G) where G x ≡ ∀φ. P φ ⊃ φ x

abbreviation PropertyInclusion (-⊃N-) where φ ⊃N ψ ≡ □(∀Ey. φ y ⊃ ψ y)

definition Essence (-Ess.-) where φ Ess. x ≡ φ x ∧ (∀ψ. ψ x ⊃ (φ ⊃N ψ))

axiomatization where Ax2b: [P φ ⊃ □ P φ]

lemma Ax2b': [¬P φ ⊃ □(¬P φ)] using Ax2a Ax2b by blast

theorem Th1: [G x ⊃ G Ess. x] using Ax2a Ax2b Essence-def God-def by (smt (verit))

definition NecExist (E) where E x ≡ ∀φ. φ Ess. x ⊃ □(∃Ex. φ x)

axiomatization where Ax3: [P E]

axiomatization where Ax4: [P φ ∧ (φ ⊃N ψ) ⊃ P ψ]

theorem Th2: [G x ⊃ □(∃Ey. G y)] using Ax3 Th1 God-def NecExist-def by smt

theorem Th3: [◇(∃Ex. G x) ⊃ □(∃Ey. G y)] — nitpick sledgehammer oops
— Open problem

end

```

## 7.8 GoedelVariantHOML2possInS4.thy

The same as *GoedelVariantHOML2poss*, but now in logic S4, where the proof of theorem *Th3* fails.

```

theory GoedelVariantHOML2possInS4 imports HOMLinHOLOnlyS4
begin

```

```

consts PositiveProperty::(e⇒σ)⇒σ (P)

```

**axiomatization where**  $Ax1$ :  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**abbreviation**  $PosProps \Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation**  $ConjOfPropsFrom \varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where**  $Ax1Gen$ :  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where**  $Ax2a$ :  $[P \varphi \vee^e P \sim\varphi]$

**definition**  $God (G)$  **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation**  $PropertyInclusion (-\supset_N-)$  **where**  $\varphi \supset_N \psi \equiv \Box(\forall y::e. \varphi y \supset \psi y)$

**definition**  $Essence (-Ess.-)$  **where**  $\varphi Ess. x \equiv \varphi x \wedge (\forall \psi. \psi x \supset (\varphi \supset_N \psi))$

**axiomatization where**  $Ax2b$ :  $[P \varphi \supset \Box P \varphi]$

**lemma**  $Ax2b'$ :  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using**  $Ax2a Ax2b$  **by** *blast*

**theorem**  $Th1$ :  $[G x \supset G Ess. x]$  **using**  $Ax2a Ax2b Essence-def God-def$  **by** (*smt (verit)*)

**definition**  $NecExist (E)$  **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists x. \varphi x)$

**axiomatization where**  $Ax3$ :  $[P E]$

**axiomatization where**  $Ax4$ :  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem**  $Th2$ :  $[G x \supset \Box(\exists y. G y)]$  **using**  $Ax3 Th1 God-def NecExist-def$  **by** *smt*

**theorem**  $Th3$ :  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  — nitpick sledgehammer **oops** —  
Open problem

**end**

## 7.9 GoedelVariantHOML3inS4.thy

The same as GoedelVariantHOML3, but now in logic S4, where the proof of theorem Th3 fails.

**theory** *GoedelVariantHOML3inS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma (P)$

**axiomatization where**  $Ax1$ :  $[P \varphi \wedge P \psi \supset P (\varphi \cdot \psi)]$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$   
**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall^E z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$   
**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall^E y. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt (verit)*)

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. \varphi Ess. x \supset \Box(\exists^E x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**theorem** *Th2*:  $[G x \supset \Box(\exists^E y. G y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem** *Th3*:  $[\Diamond(\exists^E x. G x) \supset \Box(\exists^E y. G y)]$  — nitpick sledgehammer **oops**  
— Open problem

**end**

## 7.10 GoedelVariantHOML3possInS4.thy

The same as GoedelVariantHOML3poss, but now in logic S4, where the proof of theorem Th3 fails.

**theory** *GoedelVariantHOML3possInS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:  $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *Ax1*:  $[P \varphi \wedge P \psi \supset P (\varphi . \psi)]$

**abbreviation** *PosProps*  $\Phi \equiv \forall \varphi. \Phi \varphi \supset P \varphi$

**abbreviation** *ConjOfPropsFrom*  $\varphi \Phi \equiv \Box(\forall z. \varphi z \leftrightarrow (\forall \psi. \Phi \psi \supset \psi z))$

**axiomatization where** *Ax1Gen*:  $[(PosProps \Phi \wedge ConjOfPropsFrom \varphi \Phi) \supset P \varphi]$

**axiomatization where** *Ax2a*:  $[P \varphi \vee^e P \sim \varphi]$

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**abbreviation** *PropertyInclusion* ( $-\supset_N-$ ) **where**  $\varphi \supset_N \psi \equiv \Box(\varphi \neq (\lambda x. \perp) \wedge (\forall y. \varphi y \supset \psi y))$

**definition** *Essence* ( $-Ess.-$ ) **where**  $\varphi Ess. x \equiv \forall \psi. \psi x \supset (\varphi \supset_N \psi)$

**axiomatization where** *Ax2b*:  $[P \varphi \supset \Box P \varphi]$

**lemma** *Ax2b'*:  $[\neg P \varphi \supset \Box(\neg P \varphi)]$  **using** *Ax2a Ax2b* **by** *blast*

**theorem** *Th1*:  $[G x \supset G Ess. x]$  **using** *Ax2a Ax2b Essence-def God-def* **by** (*smt (verit)*)

**definition** *NecExist* (*E*) **where**  $E x \equiv \forall \varphi. (\varphi Ess. x) \supset \Box(\exists x. \varphi x)$

**axiomatization where** *Ax3*:  $[P E]$

**axiomatization where** *Ax4*:  $[P \varphi \wedge (\varphi \supset_N \psi) \supset P \psi]$

**theorem** *Th2*:  $[G x \supset \Box(\exists y. G y)]$  **using** *Ax3 Th1 God-def NecExist-def* **by** *smt*

**theorem** *Th3*:  $[\Diamond(\exists x. G x) \supset \Box(\exists y. G y)]$  — nitpick sledgehammer **oops** — Open problem

**end**

## 7.11 ScottVariantHOMLinS4.thy

**theory** *ScottVariantHOMLinS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization where** *A2*:  $[P \varphi \wedge \Box(\forall^E y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists^E x. \varphi x)]$  **using** *A1 A2* **by** *blast*

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists^E x. G x)]$  **using** *A3 T1* **by** *blast*

**axiomatization** **where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (-*Ess*.) **where**  $\varphi \text{ Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall^E y. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{ Ess. } x]$  **using** *A1 A4 Ess-def God-def* **by** *fastforce*

**definition** *NecExist* (*NE*) **where**  $NE x \equiv \forall \varphi. \varphi \text{ Ess. } x \supset \Box(\exists^E x. \varphi x)$

**axiomatization** **where** *A5*:  $[P NE]$

**lemma** *True nitpick* $[satisfy, card=1, eval=[P (\lambda x. \top)]]$  **oops** — One model found of cardinality one

**theorem** *T3*:  $[\Box(\exists^E x. G x)]$  **nitpick** $[card e=1, card i=2]$  **oops** — Countermodel found

**lemma** *MC*:  $[\varphi \supset \Box \varphi]$  **nitpick** $[card e=1, card i=2]$  **oops** — Countermodel found

**end**

## 7.12 ScottVariantHOMLpossInS4.thy

**theory** *ScottVariantHOMLpossInS4* **imports** *HOMLinHOLonlyS4*  
**begin**

**consts** *PositiveProperty*:: $(e \Rightarrow \sigma) \Rightarrow \sigma$  (*P*)

**axiomatization** **where** *A1*:  $[\neg P \varphi \leftrightarrow P \sim \varphi]$

**axiomatization** **where** *A2*:  $[P \varphi \wedge \Box(\forall y. \varphi y \supset \psi y) \supset P \psi]$

**theorem** *T1*:  $[P \varphi \supset \Diamond(\exists x. \varphi x)]$  **using** *A1 A2* **by** *blast*

**definition** *God* (*G*) **where**  $G x \equiv \forall \varphi. P \varphi \supset \varphi x$

**axiomatization** **where** *A3*:  $[P G]$

**theorem** *Coro*:  $[\Diamond(\exists x. G x)]$  **using** *A3 T1* **by** *blast*

**axiomatization** **where** *A4*:  $[P \varphi \supset \Box P \varphi]$

**definition** *Ess* (-*Ess*.) **where**  $\varphi \text{ Ess. } x \equiv \varphi x \wedge (\forall \psi. \psi x \supset \Box(\forall y::e. \varphi y \supset \psi y))$

**theorem** *T2*:  $[G x \supset G \text{ Ess. } x]$  **using** *A1 A4 Ess-def God-def* **by** *fastforce*



**definition** *NecExist* (*NE*) **where**  $NE\ x \equiv \forall \varphi. \varphi\ Ess.\ x \supset \Box(\exists x. \varphi\ x)$

**axiomatization** **where** *A5*:  $[P\ NE]$

**lemma** *True* **nitpick** $[satisfy, card=1, eval=[P\ (\lambda x. \perp)]]$  **oops** — One model found of cardinality one

**theorem** *T3*:  $[\Box(\exists x. G\ x)]$  **nitpick** $[card\ e=1, card\ i=2]$  **oops** — Countermodel found

**lemma** *MC*:  $[\varphi \supset \Box\varphi]$  **nitpick** $[card\ e=1, card\ i=2]$  **oops** — Countermodel found

**end**

## References

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