# The Inductive Unwinding Theorem for CSP Noninterference Security 

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#### Abstract

The necessary and sufficient condition for CSP noninterference security stated by the Ipurge Unwinding Theorem is expressed in terms of a pair of event lists varying over the set of process traces. This does not render it suitable for the subsequent application of rule induction in the case of a process defined inductively, since rule induction may rather be applied to a single variable ranging over an inductively defined set.

Starting from the Ipurge Unwinding Theorem, this paper derives a necessary and sufficient condition for CSP noninterference security that involves a single event list varying over the set of process traces, and is thus suitable for rule induction; hence its name, Inductive Unwinding Theorem. Similarly to the Ipurge Unwinding Theorem, the new theorem only requires to consider individual accepted and refused events for each process trace, and applies to the general case of a possibly intransitive noninterference policy. Specific variants of this theorem are additionally proven for deterministic processes and trace set processes.


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# 1 The Inductive Unwinding Theorem 

theory InductiveUnwinding<br>imports Noninterference-Ipurge-Unwinding.DeterministicProcesses begin

The necessary and sufficient condition for CSP noninterference security [7] stated by the Ipurge Unwinding Theorem [8] is expressed in terms of a pair of event lists varying over the set of process traces. This does not render it suitable for the subsequent application of rule induction in the case of a process defined inductively, since rule induction may rather be applied to a single variable ranging over an inductively defined set (cf. [5]).
However, the formulation of an inductive definition is the standard way of defining a process that admits traces of unbounded length, indeed because it provides rule induction as a powerful method to prove process properties, particularly noninterference security, by considering any indefinitely long trace of the process. Therefore, it is essential to infer some condition equivalent to CSP noninterference security and suitable for being handled by means of rule induction.
Starting from the Ipurge Unwinding Theorem, this paper derives a necessary and sufficient condition for CSP noninterference security that involves a single event list varying over the set of process traces, and is thus suitable for rule induction; hence its name, Inductive Unwinding Theorem. Similarly to the Ipurge Unwinding Theorem, the new theorem only requires to consider individual accepted and refused events for each process trace, and applies to the general case of a possibly intransitive noninterference policy. Specific variants of this theorem are additionally proven for deterministic processes and trace set processes [8].
For details about the theory of Communicating Sequential Processes, to which the notion of process security defined in [7] and applied in this paper refers, cf. [1].
As regards the formal contents of this paper, the salient points of definitions and proofs are commented; for additional information, cf. Isabelle documentation, particularly [5], [4], [3], and [2].

### 1.1 Propaedeutic lemmas

Here below are the proofs of some lemmas on the constants defined in [7] and [8] which are propaedeutic to the demonstration of the Inductive Unwinding Theorem.
Among other things, the lemmas being proven formalize the following statements:

- A set of domains $U$ may affect a set of domains $V$ via an event list $x s$, as expressed through function sinks-aux, just in case $V$ may be affected by $U$ via $x s$, as expressed through function sources-aux.
- The event lists output by function ipurge-tr are not longer than the corresponding input ones.
- Function ipurge-tr-rev is idempotent.


## lemma sources-aux-single-dom:

```
sources-aux I D {u}xs= insert u (sources I Duxs)
```

by (simp add: sources-sinks sources-sinks-aux sinks-aux-single-dom)
lemma sources-interference-eq:
$((D x, u) \in I \vee(\exists v \in$ sources $I D u x s .(D x, v) \in I))=$ ( $D x \in$ sources $I D u(x \# x s)$ )
proof (simp only: sources-sinks rev.simps, subst (1 2) converse-iff [symmetric])
qed (rule sinks-interference-eq)
lemma ex-sinks-sources-aux-1 [rule-format]:
$(\exists u \in$ sinks-aux I D U xs. $\exists v \in V .(u, v) \in I) \longrightarrow$
$(\exists u \in U . \exists v \in$ sources-aux I D Vxs. $(u, v) \in I)$
proof (induction xs arbitrary: V rule: rev-induct, simp, subst sources-aux-append, rule impI)
fix $x$ xs $V$
let
$? V=$ sources-aux $I D V[x]$ and ? $V^{\prime}=\operatorname{insert}(D x) V$
assume

$$
A: \bigwedge V .(\exists u \in \text { sinks-aux I D Uxs. } \exists v \in V .(u, v) \in I) \longrightarrow
$$

$(\exists u \in U . \exists v \in$ sources-aux I D Vxs. $(u, v) \in I)$ and
B: $\exists u \in$ sinks-aux ID $U(x s$ @ $[x]) . \exists v \in V .(u, v) \in I$
show $\exists u \in U . \exists v \in$ sources-aux I $D$ ? $V$ xs. $(u, v) \in I$
proof (cases $\exists u \in$ sinks-aux I D Uxs. $(u, D x) \in I)$
case True
hence $(\exists v \in V .(D x, v) \in I) \vee$
$(\exists u \in$ sinks-aux I D Uxs. $\exists v \in V .(u, v) \in I)$ (is ? $A \vee$ ? $B$ ) using $B$ by simp
moreover \{

```
    assume ?A
    have (\existsu\in sinks-aux I D U xs. \existsv\in? ?V'. (u,v)\inI)\longrightarrow
        (\existsu\inU.\existsv\in sources-aux I D ? V' 'ss. (u,v) \inI)
        using A.
        moreover obtain u where
            C:u\in sinks-aux I D U xs and D: (u,D x) \inI
        using True ..
    have D x ? ? V' by simp
    with D have }\existsv\in? ?V'.(u,v)\inI.
    hence }\existsu\in\mathrm{ sinks-aux I D U xs. }\existsv\in
    ultimately have }\existsu\inU.\existsv\in\mathrm{ sources-aux I D ?V' xs. (u,v) & I ..
    hence ?thesis using <?A〉 by simp
    }
    moreover {
        assume ?B
        have (\existsu\in sinks-aux I D U xs. \existsv\in?V. (u,v)\inI)\longrightarrow
            (\existsu\inU.\existsv\in sources-aux I D ?V Vs. (u,v)\inI)
        using A .
        moreover obtain u}\mathrm{ where
            C:u\in sinks-aux I D U xs and D:\existsv\inV. (u,v)\inI
        using <?B\rangle ..
        have V\subseteq?V by (rule sources-aux-subset)
        hence }\existsv\in? V. (u,v)\inI using D by sim
        hence }\existsu\in\mathrm{ sinks-aux I D U xs. ヨv & ?V. (u,v) <I using C ..
        ultimately have ?thesis ..
    }
    ultimately show ?thesis ..
next
    case False
    have (\existsu\in sinks-aux I D U xs. \existsv\in?V. (u,v)\inI)\longrightarrow
        (\existsu\inU.\existsv\in sources-aux I D ?V \s. ( }u,v)\inI
        using A .
    moreover have \existsu\in sinks-aux I D U xs. \existsv\inV. (u,v)\inI
    using B and False by simp
    then obtain }u\mathrm{ where
        C:u\in sinks-aux I D U xs and D:\existsv\inV. (u,v)\inI ..
    have V\subseteq? V by (rule sources-aux-subset)
    hence }\existsv\in\mathrm{ ?V. (u,v) &I using D by simp
    hence \existsu\in sinks-aux I D U xs. \existsv\in?V. (u,v)\inI using C ..
    ultimately show ?thesis ..
    qed
qed
lemma ex-sinks-sources-aux-2 [rule-format]:
\((\exists u \in U . \exists v \in\) sources-aux I D Vxs. \((u, v) \in I) \longrightarrow\)
    ( \existsu\in sinks-aux I D U xs. \existsv\inV. (u,v) \inI)
proof (induction xs arbitrary: V rule: rev-induct, simp, subst sources-aux-append,
rule impI)
    fix x xs V
```

```
let
    ?V = sources-aux I D V [x] and
    ? V' = insert (Dx)V
    assume
    A: \V. (\existsu\inU.\existsv\in sources-aux I D V xs. (u,v) \inI)\longrightarrow
        (\existsu\in sinks-aux I D U xs. \existsv\inV. (u,v) \inI) and
    B:\existsu\inU.\existsv\in sources-aux I D ?V \s. (u,v)\inI
    show }\existsu\in\mathrm{ sinks-aux IDU(xs@ @ [x]). ヨv GV. (u,v) GI
    proof (cases \existsu\in sinks-aux I D U xs. (u,D x) \inI,
    cases \existsv\inV.(D x,v)\inI, simp-all (no-asm-simp))
    have ( }\existsu\inU.\existsv\in\mathrm{ sources-aux I D V xs. (u,v) fI) }
        (\existsu\in sinks-aux I D U xs. \existsv\inV. (u,v)\inI)
        using A .
    moreover assume }\neg(\existsv\inV.(Dx,v)\inI
    hence }\existsu\inU.\existsv\in\mathrm{ sources-aux I D V xs. (u,v) }\inI\mathrm{ using B by simp
    ultimately show }\existsu\in\mathrm{ sinks-aux I D U xs. }\existsv\inV.(u,v)\inI.
next
    assume C:\neg(\existsu\in sinks-aux I D U xs. (u,D x)\inI)
    have (\existsu\inU.\existsv\insources-aux I D ?V xs. (u,v) \inI)\longrightarrow
        (\existsu\in sinks-aux I D U xs. \existsv\in?V. (u,v)\inI)
    using A .
    hence \existsu\in sinks-aux I D U xs. \existsv\in?V. (u,v)\inI using B ..
    then obtain u where
        D:u\in sinks-aux I D U xs and E:\existsv\in?V. (u,v) \inI ..
    obtain v where F:v\in?V and G:(u,v)\inI using E ..
    have v=D x\veev\inV using F by (cases \existsv\inV.(Dx,v)\inI, simp-all)
    moreover {
        assume v=D x
        hence (u,D x) \inI using G by simp
        hence \existsu\in sinks-aux I D U xs. (u,D x) \inI using D ..
        hence }\existsu\in\mathrm{ sinks-aux I D U xs. }\existsv\inV.(u,v)\in
        using C by contradiction
    }
    moreover {
        assume v}\in
        with G have }\existsv\inV.(u,v)\inI.
        hence \existsu\in sinks-aux I D U xs. \existsv\inV. (u,v)\inI using D ..
    }
    ultimately show \existsu\in sinks-aux I D U xs. \existsv\inV. (u,v)\inI ..
    qed
qed
lemma ex-sinks-sources-aux:
(\existsu\in sinks-aux I D U xs. \existsv\inV. (u,v) \inI)=
    (\existsu\inU.\existsv\in sources-aux I D V xs. (u,v) \inI)
by (rule iffI, erule ex-sinks-sources-aux-1, rule ex-sinks-sources-aux-2)
lemma ipurge-tr-rev-ipurge-tr-sources-aux-1 [rule-format]:
\neg(\existsv\inD`set ys.\existsu\insources-aux I D U zs. (v,u) \inI)\longrightarrow
```

```
    ipurge-tr-rev-aux I D U (xs @ ys @ zs)=
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D'set ys) zs)
proof (induction zs arbitrary:U rule: rev-induct, rule-tac [!] impI,
    simp del: bex-simps)
    fix }
    assume }\neg(\existsv\in\mp@subsup{D}{}{\prime}\mathrm{ set ys. }\existsu\inU.(v,u)\inI
    hence ipurge-tr-rev-aux I D U ys = [] by (simp add: ipurge-tr-rev-aux-nil)
    thus ipurge-tr-rev-aux ID U (xs @ ys)= ipurge-tr-rev-aux I D U xs
    by (simp add: ipurge-tr-rev-aux-append-nil)
next
    fix z zs U
    let
        ?U = sources-aux I D U [z] and
        ? U'}=\operatorname{insert}(Dz)
    assume }\bigwedgeU.\neg(\existsv\inD'set ys. \existsu\in sources-aux I D Uzs. (v,u)\inI)
    ipurge-tr-rev-aux I D U (xs @ ys @ zs)=
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D` set ys)zs)
    hence }\neg(\existsv\inD'set ys. \existsu\in sources-aux I D?U zs. (v,u)\inI)
    ipurge-tr-rev-aux ID?U (xs @ ys @ zs)=
    ipurge-tr-rev-aux I D ?U (xs @ ipurge-tr-aux I D (D'set ys)zs).
moreover assume
    \neg(\existsv\inD'set ys. \existsu\in sources-aux I D U (zs @ [z]). (v,u) \inI)
hence }A\mathrm{ :
    \neg(\existsv\inD'set ys. \existsu\in sources-aux I D ?U zs. (v,u) \inI)
    by (subst (asm) sources-aux-append)
ultimately have B:
    ipurge-tr-rev-aux I D ?U (xs @ ys @ zs)=
    ipurge-tr-rev-aux I D ? U (xs @ ipurge-tr-aux I D (D`set ys)zs)..
have
    ipurge-tr-rev-aux ID U (xs @ ys @ zs @ [z])=
    ipurge-tr-rev-aux ID U((xs @ ys @ zs)@ [z])
    by simp
hence C}C\mathrm{ :
    ipurge-tr-rev-aux I D U (xs@ ys @ zs @ [z])=
    ipurge-tr-rev-aux I D ?U (xs @ ys @ zs) @ ipurge-tr-rev-aux I D U [z]
    (is - = - @ ?ws) by (simp only: ipurge-tr-rev-aux-append)
show
ipurge-tr-rev-aux I D U (xs @ ys @ zs @ [z])=
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D`set ys) (zs @ [z]))
proof (subst C, cases }\existsu\inU.(Dz,u)\inI
    simp-all (no-asm-simp) del: ipurge-tr-aux.simps)
    case True
    have }\neg(\existsv\in\mathrm{ sinks-aux I D(D'set ys) zs. ヨu G?U. (v,u) GI)
    using A by (simp add: ex-sinks-sources-aux)
    hence }\neg(\existsv\in\mathrm{ sinks-aux I D(D'set ys) zs. (v,Dz) GI)
    using True by simp
hence
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D` set ys) (zs @ [z])) =
            ipurge-tr-rev-aux I D U ((xs @ ipurge-tr-aux I D (D'set ys) zs) @ [z])
```

```
    by simp
    also have ... =
    ipurge-tr-rev-aux I D ?U (xs @ ipurge-tr-aux I D (D'set ys) zs)@ ?ws
    by (simp only: ipurge-tr-rev-aux-append)
    also have ... =
    ipurge-tr-rev-aux I D ?U' (xs @ ipurge-tr-aux I D (D'set ys)zs) @ [z]
    using True by simp
finally have
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D' set ys) (zs @ [z])) =
    ipurge-tr-rev-aux I D ? U' (xs @ ipurge-tr-aux I D (D'set ys) zs) @ [z].
    thus
    ipurge-tr-rev-aux I D?U'(xs @ ys @ zs) @ [z]=
        ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux ID (D` set ys) (zs @ [z]))
    using B and True by simp
next
    case False
    have
    ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux ID (D` set ys) (zs @ [z]))=
        ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D'set ys) zs)
    proof (cases \existsv\in sinks-aux I D(D'set ys)zs. (v,Dz)\inI, simp-all)
        have
            ipurge-tr-rev-aux I D U (xs@ ipurge-tr-aux I D (D'set ys)zs @ [z])=
                ipurge-tr-rev-aux I D U ((xs @ ipurge-tr-aux I D (D'set ys) zs)@ [z])
            by simp
            also have ...=
                ipurge-tr-rev-aux I D ?U (xs @ ipurge-tr-aux I D (D` set ys) zs)@ ?ws
            by (simp only: ipurge-tr-rev-aux-append)
                            also have ...=
                ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D(D`set ys)zs)
            using False by simp
    finally show
                            ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D'set ys)zs @ [z])=
                ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D'set ys) zs).
    qed
    thus
    ipurge-tr-rev-aux I D U (xs @ ys @ zs)=
        ipurge-tr-rev-aux I D U (xs @ ipurge-tr-aux I D (D` set ys) (zs @ [z]))
    using B and False by simp
qed
qed
lemma ipurge-tr-rev-ipurge-tr-sources-1:
    assumes A:D y & sources I Du(y#zs)
    shows
    ipurge-tr-rev I Du(xs @ y # zs)=
    ipurge-tr-rev I Du(xs @ ipurge-tr I D (D y) zs)
proof -
    have \neg((Dy,u) \inI\vee (\existsv\in sources I D u zs. (D y,v) \inI))
        using A by (simp only: sources-interference-eq, simp)
```

```
    hence \(\neg\left(\exists v \in D^{\prime}\right.\) set \([y] . \exists u \in\) sources-aux I \(\left.D\{u\} z s .(v, u) \in I\right)\)
    by (simp add: sources-aux-single-dom)
hence
    ipurge-tr-rev-aux I D \(\{u\}(x s @[y] @ z s)=\)
    ipurge-tr-rev-aux I \(D\{u\}\) (xs @ ipurge-tr-aux I \(D(D\) ‘set \([y]) z s)\)
    by (rule ipurge-tr-rev-ipurge-tr-sources-aux-1)
thus ?thesis by (simp add: ipurge-tr-aux-single-dom ipurge-tr-rev-aux-single-dom)
qed
lemma ipurge-tr-length:
    length (ipurge-tr I D u xs) \(\leq\) length \(x s\)
by (induction xs rule: rev-induct, simp-all)
lemma sources-idem:
    sources I D u (ipurge-tr-rev I Duxs) = sources I Duxs
by (induction xs, simp-all)
lemma ipurge-tr-rev-idem:
    ipurge-tr-rev I \(D u\) (ipurge-tr-rev I \(D u x s)=\) ipurge-tr-rev I \(D u x s\)
by (induction xs, simp-all add: sources-idem)
```


### 1.2 Closure of the traces of a secure process under reverse intransitive purge

The derivation of the Inductive Unwinding Theorem from the Ipurge Unwinding Theorem requires to prove that the set of the traces of a secure process is closed under reverse intransitive purge, i.e. function ipurge-tr-rev [8]. This can be expressed formally by means of the following statement:
$\llbracket$ secure P I D; xs traces $P \rrbracket \Longrightarrow$ ipurge-tr-rev I Duxs traces $P$

The reason why such closure property holds is that the reverse intransitive purge of a list $x s$ with regard to a policy $I$, an event-domain map $D$, and a domain $u$ can equivalently be computed as follows: for each item $x$ of $x s$, if $x$ may affect $u$, retain $x$ and go on recursively using as input the sublist of $x s$ following $x$, say $x^{\prime}$; otherwise, discard $x$ and go on recursively using ipurge-tr I $D\left(\begin{array}{l}\text { x }\end{array}\right) x s^{\prime}[7]$ as input.
The result actually matches ipurge-tr-rev I $D u x s$. In fact, for each $x$ not affecting $u$, ipurge-tr I $D\left(\begin{array}{l}D\end{array}\right) x s^{\prime}$ retains any item of $x s^{\prime}$ not affected by $x$, which is the case for any item of $x s^{\prime}$ affecting $u$, since otherwise $x$ would affect $u$.
Furthermore, if $x s$ is a trace of a secure process, the result is still a trace. In fact, for each $x$ not affecting $u$, if $y s$ is the partial output for the sublist of xs preceding $x$, then $y s$ @ ipurge-tr I $D(D x) x s^{\prime}$ is a trace provided such is $y s$ @ $x \# x s^{\prime}$, by virtue of the definition of CSP noninterference security
[7]. Hence, the property of being a trace is conserved upon each recursive call by the concatenation of the partial output and the residual input, until the latter is nil and the former matches the total output.
This argument shows that in order to prove by induction, under the aforesaid assumptions, that the output of such a reverse intransitive purge function is a trace, the partial output has to be passed to the function as an argument, in addition to the residual input, in the recursive calls contained within the definition of the function. Therefore, the output of the function has to be accumulated into one of its parameters, viz. the function needs to be tailrecursive. This suggests to prove the properties of interest of the function by applying the ten-step proof method for theorems on tail-recursive functions described in [6].
The starting point is to formulate a naive definition of the function, which will then be refined as specified by the proof method. The name of the refined function, from which the name of the naive function here below is derived, will be ipurge-tr-rev- $t$, where suffix $t$ stands for tail-recursive.

```
function (sequential) ipurge-tr-rev-t-naive ::
```



```
ipurge-tr-rev-t-naive I D u (x # xs) ys =
    (if D x sources I D u (x# xs)
    then ipurge-tr-rev-t-naive I D u xs (ys @ [x])
    else ipurge-tr-rev-t-naive I D u(ipurge-tr I D (Dx)xs) ys)|
ipurge-tr-rev-t-naive --- ys = ys
oops
```

The parameter into which the output is accumulated is the last one.
As shown by the previous argument, the properties of function ipurge-tr-rev-t-naive that would have to be proven are the following ones:
ipurge-tr-rev-t-naive I D u xs [] = ipurge-tr-rev I D u xs
$\llbracket$ secure $P I D ; x s \in$ traces $P \rrbracket \Longrightarrow$ ipurge-tr-rev-t-naive $I D$ u xs []$\in$ traces $P$
as they clearly entail the above formal statement of the target closure lemma.

### 1.2.1 Step 1

In the definition of the auxiliary tail-recursive function ipurge-tr-rev-t-aux, the Cartesian product of the input types of function ipurge-tr-rev-t-naive will be implemented as a record type.

```
record \(\left({ }^{\prime} a,{ }^{\prime} d\right)\) ipurge-rec \(=\)
    Pol :: \(\left({ }^{\prime} d \times 1\right.\) 'd) set
    Map :: ' \(a \Rightarrow\) 'd
    Dom :: 'd
    In :: ' \(a\) list
    Out :: 'a list
function (sequential) ipurge-tr-rev-t-aux ::
    ( \(\left.{ }^{\prime} a,{ }^{\prime} d\right)\) ipurge-rec \(\Rightarrow\left({ }^{\prime} a,{ }^{\prime} d\right)\) ipurge-rec where
ipurge-tr-rev-t-aux \((\operatorname{Pol}=I, M a p=D, D o m=u, I n=x \# x s\), Out \(=y s \mid)=\)
    (if \(D x \in\) sources I \(D u(x \# x s)\)
    then ipurge-tr-rev-t-aux
        (Pol = I, Map = D, Dom =u, In =xs, Out=ys @ [x])
    else ipurge-tr-rev-t-aux
    (Pol \(=I\), Map \(=D, D o m=u, I n=\) ipurge-tr \(I D(D x) x s\), Out \(=y s)) \mid\)
ipurge-tr-rev-t-aux \(X=X\)
proof (simp-all, atomize-elim)
    fix \(X::\left({ }^{\prime} a,{ }^{\prime} d\right)\) ipurge-rec
    show
        ( \(\exists\) I Dux xs ys.
        \(X=(\) Pol \(=I, M a p=D, D o m=u, I n=x \# x s\), Out \(=y s \mid) \vee\)
        ( \(\exists\) I D u ys.
        \(X=(\) Pol \(=I, M a p=D, D o m=u\), In \(=[]\), Out \(=y s \mid)\)
    proof (cases \(X\), simp-all)
    qed (subst disj-commute, rule spec [OF list.nchotomy])
qed
termination ipurge-tr-rev-t-aux
proof (relation measure \((\lambda X\). length ( \(\operatorname{In} X)\) ), simp-all)
    fix \(D:: ' ~ a \Rightarrow ' d\) and \(I x x s\)
    have length (ipurge-tr I \(D\left(\begin{array}{l}\text { x }\end{array}\right.\) xs) \(\leq\) length \(x s\) by (rule ipurge-tr-length)
    thus length (ipurge-tr I D ( \(D\) x) xs) < Suc (length xs) by simp
qed
```

As shown by this proof, the termination of function ipurge-tr-rev-t-aux is guaranteed by the fact, proven previously, that the event lists output by function ipurge-tr are not longer than the corresponding input ones.

### 1.2.2 Step 2

definition ipurge-tr-rev-t-in ::

$$
\left({ }^{\prime} d \times{ }^{\prime} d\right) \text { set } \Rightarrow\left({ }^{\prime} a \Rightarrow{ }^{\prime} d\right) \Rightarrow^{\prime} d \Rightarrow '^{\prime} a \text { list } \Rightarrow\left({ }^{\prime} a,,^{\prime} d\right) \text { ipurge-rec where }
$$

ipurge-tr-rev-t-in I D u xs 三

$$
\cap \text { Pol }=I, M a p=D, D o m=u, I n=x s, \text { Out }=[] D
$$

definition ipurge-tr-rev-t-out ::
('a, 'd) ipurge-rec $\Rightarrow$ 'a list where
ipurge-tr-rev-t-out $\equiv$ Out

```
definition ipurge-tr-rev-t ::
```



```
ipurge-tr-rev-t I D u xs \equiv
    ipurge-tr-rev-t-out (ipurge-tr-rev-t-aux (ipurge-tr-rev-t-in I D u xs))
```

Since the significant inputs of function ipurge-tr-rev-t-naive match pattern ,,,--- [], those of function ipurge-tr-rev-t-aux, as returned by function ipurge-tr-rev-t-in, match pattern $\cap$ Pol $=-, M a p=-, D o m=-, I n=-$, Out $=[] D$.
In terms of function ipurge-tr-rev-t, the statements to be proven, henceforth respectively named ipurge-tr-rev-t-equiv and ipurge-tr-rev-t-trace, take the following form:

```
ipurge-tr-rev-t I D u xs = ipurge-tr-rev I Duxs
```

$\llbracket$ secure $P I D ; x s \in$ traces $P \rrbracket \Longrightarrow$ ipurge-tr-rev-t $I D u x s \in \operatorname{traces} P$

### 1.2.3 Step 3

inductive-set ipurge-tr-rev-t-set :: (' $\left.a,{ }^{\prime} d\right)$ ipurge-rec $\Rightarrow\left({ }^{\prime} a\right.$, 'd) ipurge-rec set
for $X::\left({ }^{\prime} a,{ }^{\prime} d\right)$ ipurge-rec where
R0: $X \in$ ipurge-tr-rev-t-set $X \mid$
R1: $\llbracket(P$ Pol $=I, M a p=D, D o m=u$, In $=x \# x s$, Out $=y s)$
$\in$ ipurge-tr-rev-t-set $X$;
$D x \in$ sources $I D u(x \# x s) \rrbracket \Longrightarrow$
(Pol $=I, M a p=D, D o m=u, I n=x s$, Out $=y s @[x] D$ $\in$ ipurge-tr-rev-t-set $X \mid$
R2: $\llbracket(0$ Pol $=I, M a p=D, D o m=u$, In $=x \# x s$, Out $=y s)$
$\in$ ipurge-tr-rev-t-set $X$;
$D x \notin$ sources $I D u(x \# x s) \rrbracket \Longrightarrow$
( Pol $=I, M a p=D, D o m=u, I n=$ ipurge-tr $I D(D x) x s$, Out $=y s)$
$\in$ ipurge-tr-rev-t-set $X$

### 1.2.4 Step 4

lemma ipurge-tr-rev-t-subset: assumes $A: Y \in$ ipurge-tr-rev-t-set $X$ shows ipurge-tr-rev-t-set $Y \subseteq$ ipurge-tr-rev-t-set $X$
proof (rule subsetI, erule ipurge-tr-rev-t-set.induct) show $Y \in$ ipurge-tr-rev-t-set $X$ using $A$.
next
fix $I D u x x s$ ys

## assume

(Pol $=I, M a p=D, D o m=u, I n=x \# x s, O u t=y s)$
$\in$ ipurge-tr-rev-t-set $X$ and

```
    \(D x \in\) sources I D \(u(x \# x s)\)
    thus 0 Pol \(=I, M a p=D, D o m=u, I n=x s\), Out \(=y s @[x]\) )
    \(\in\) ipurge-tr-rev-t-set \(X\)
    by (rule R1)
next
    fix \(I D u x\) xs ys
    assume
    \((\) Pol \(=I, M a p=D, D o m=u, I n=x \# x s, O u t=y s)\)
        \(\in\) ipurge-tr-rev-t-set \(X\) and
    \(D x \notin\) sources I D u ( \(x \# x s\) )
    thus \(\|\) Pol \(=I, M a p=D, D o m=u\), In \(=\) ipurge-tr \(I D(D x) x s\), Out \(=y s \|\)
    \(\in\) ipurge-tr-rev-t-set \(X\)
    by (rule R2)
qed
lemma ipurge-tr-rev-t-aux-set:
    ipurge-tr-rev-t-aux \(X \in\) ipurge-tr-rev-t-set \(X\)
proof (induction rule: ipurge-tr-rev-t-aux.induct,
simp-all only: ipurge-tr-rev-t-aux.simps(2) R0)
    fix \(I u x x s\) ys and \(D:: ' a \Rightarrow^{\prime} d\)
    assume
        A: D \(x \in\) sources I \(D u(x \# x s) \Longrightarrow\)
    ipurge-tr-rev-t-aux
        (Pol \(=I\), Map \(=D, D o m=u\), In \(=x s\), Out \(=y s @[x]\) )
        \(\in\) ipurge-tr-rev-t-set
            (Pol \(=I, M a p=D, D o m=u\), In \(=x s\), Out \(=y s @[x]\) )
    (is - \(\Longrightarrow\) ipurge-tr-rev-t-aux ? Y \(\in\)-) and
    B: D x \(\notin\) sources I \(D u(x \# x s) \Longrightarrow\)
    ipurge-tr-rev-t-aux
        (Pol \(=I\), Map \(=D, D o m=u\), In \(=\) ipurge-tr \(I D(D x) x s\), Out \(=y s)\)
        \(\in\) ipurge-tr-rev-t-set
            (Pol \(=I, M a p=D, D o m=u, I n=\) ipurge-tr \(I D(D x) x s\), Out \(=y s)\)
    (is - \(\Longrightarrow\) ipurge-tr-rev-t-aux ? \(Z \in-\) )
show
ipurge-tr-rev-t-aux
    (Pol \(=I, M a p=D, D o m=u, I n=x \# x s\), Out \(=y s)\)
    \(\in\) ipurge-tr-rev-t-set
        (Pol \(=I\), Map \(=D, D o m=u, I n=x \# x s\), Out \(=y s)\)
    (is ipurge-tr-rev-t-aux ? \(X \in-\) )
proof (cases \(D x \in\) sources \(I D u(x \#\) xs \()\), simp-all del: sources.simps \()\)
    case True
    have ? \(X \in\) ipurge-tr-rev-t-set ? \(X\) by (rule R0)
    moreover have ? \(X \in\) ipurge-tr-rev-t-set ? \(X \Longrightarrow\) ? \(Y \in\) ipurge-tr-rev-t-set ? \(X\)
    by (rule R1 [OF - True])
    ultimately have ? \(Y \in\) ipurge-tr-rev-t-set ? \(X\) by simp
    hence ipurge-tr-rev-t-set? \(Y \subseteq\) ipurge-tr-rev-t-set ? \(X\)
    by (rule ipurge-tr-rev-t-subset)
    moreover have ipurge-tr-rev-t-aux ?Y \(\in\) ipurge-tr-rev-t-set?Y
    using True by (rule A)
```

ultimately show ipurge-tr-rev-t-aux ? Y $\in$ ipurge-tr-rev-t-set ? $X$..

## next

case False
have ? $X \in$ ipurge-tr-rev-t-set ? $X$ by (rule R0)
moreover have ? $X \in$ ipurge-tr-rev-t-set ? $X \Longrightarrow$ ? $Z \in$ ipurge-tr-rev-t-set ? $X$
by (rule R2 [OF - False])
ultimately have ? $Z \in$ ipurge-tr-rev-t-set ? $X$ by simp
hence ipurge-tr-rev-t-set? $Z \subseteq$ ipurge-tr-rev-t-set? X
by (rule ipurge-tr-rev-t-subset)
moreover have ipurge-tr-rev-t-aux ? Z $\in$ ipurge-tr-rev-t-set ? $Z$
using False by (rule B)
ultimately show ipurge-tr-rev-t-aux ? $Z \in$ ipurge-tr-rev-t-set ? $X$..
qed
qed

### 1.2.5 Step 5

definition ipurge-tr-rev-t-inv-1 ::
$\left({ }^{\prime} d \times{ }^{\prime} d\right)$ set $\Rightarrow\left({ }^{\prime} a \Rightarrow{ }^{\prime} d\right) \Rightarrow{ }^{\prime} d \Rightarrow{ }^{\prime}$ a list $\Rightarrow\left({ }^{\prime} a,{ }^{\prime} d\right)$ ipurge-rec $\Rightarrow$ bool
where
ipurge-tr-rev-t-inv-1 I D u xs $X \equiv$
Out $X$ @ ipurge-tr-rev I $D u($ In $X)=$ ipurge-tr-rev I $D u$ xs
definition ipurge-tr-rev-t-inv-2 ::
'a process $\Rightarrow\left(' d \times{ }^{\prime} d\right)$ set $\Rightarrow\left({ }^{\prime} a \Rightarrow^{\prime} d\right) \Rightarrow{ }^{\prime}$ a list $\Rightarrow\left({ }^{\prime} a,{ }^{\prime} d\right)$ ipurge-rec $\Rightarrow$ bool where
ipurge-tr-rev-t-inv-2 P I D xs $X \equiv$
secure PID $\longrightarrow x s \in$ traces $P \longrightarrow$ Out $X$ @ In $X \in \operatorname{traces} P$

Two invariants have been defined, one for each of lemmas ipurge-tr-rev-t-equiv, ipurge-tr-rev-t-trace.
More precisely, the invariants are ipurge-tr-rev-t-inv-1 I Duxs and ipurge-tr-rev-t-inv-2 P I Dxs, where the free variables are intended to match those appearing in the aforesaid lemmas.

### 1.2.6 Step 6

lemma ipurge-tr-rev-t-input-1:
ipurge-tr-rev-t-inv-1 I D u xs $\$ Pol $=I, M a p=D, \operatorname{Dom}=u, I n=x s$, Out $=[])$
by (simp add: ipurge-tr-rev-t-inv-1-def)
lemma ipurge-tr-rev-t-input-2:
ipurge-tr-rev-t-inv-2 P I D xs $(P$ Pol $=I, M a p=D, D o m=u, I n=x s$, Out $=[] 1)$ by (simp add: ipurge-tr-rev-t-inv-2-def)

### 1.2.7 Step 7

definition ipurge-tr-rev-t-form :: (' $a,{ }^{\prime} d$ ) ipurge-rec $\Rightarrow$ bool where

$$
\text { ipurge-tr-rev-t-form } X \equiv \text { In } X=[]
$$

lemma ipurge-tr-rev-t-intro-1:
【ipurge-tr-rev-t-inv-1 I D u xs X; ipurge-tr-rev-t-form $X \rrbracket \Longrightarrow$ ipurge-tr-rev-t-out $X=$ ipurge-tr-rev I $D$ u xs
by (simp add: ipurge-tr-rev-t-inv-1-def ipurge-tr-rev-t-form-def ipurge-tr-rev-t-out-def)
lemma ipurge-tr-rev-t-intro-2:
【ipurge-tr-rev-t-inv-2 P I D xs $X$; ipurge-tr-rev-t-form $X \rrbracket \Longrightarrow$
secure P I D $\longrightarrow x s \in$ traces $P \longrightarrow$ ipurge-tr-rev-t-out $X \in$ traces $P$
by (simp add: ipurge-tr-rev-t-inv-2-def ipurge-tr-rev-t-form-def
ipurge-tr-rev-t-out-def)

### 1.2.8 Step 8

lemma ipurge-tr-rev-t-form-aux: ipurge-tr-rev-t-form (ipurge-tr-rev-t-aux X)
by (induction $X$ rule: ipurge-tr-rev-t-aux.induct, simp-all add: ipurge-tr-rev-t-form-def)

### 1.2.9 Step 9

lemma ipurge-tr-rev-t-invariance-aux:
$Y \in$ ipurge-tr-rev-t-set $X \Longrightarrow$
Pol $Y=$ Pol $X \wedge \operatorname{Map} Y=\operatorname{Map} X \wedge \operatorname{Dom} Y=\operatorname{Dom} X$
by (erule ipurge-tr-rev-t-set.induct, simp-all)

The lemma just proven, stating the invariance of the first three record fields over inductive set ipurge-tr-rev-t-set $X$, is used in the following proofs of the invariance of predicates ipurge-tr-rev-t-inv-1 I Duxs and ipurge-tr-rev-t-inv-2 PI D xs.
The equality between the free variables appearing in the predicates and the corresponding fields of the record generating the set, which is required for such invariance properties to hold, is asserted in the enunciation of the properties by means of record updates. In the subsequent proofs of lemmas ipurge-tr-rev-t-equiv, ipurge-tr-rev-t-trace, the enforcement of this equality will be ensured by the identification of both predicate variables and record fields with the related free variables appearing in the lemmas.
lemma ipurge-tr-rev-t-invariance-1:
$\llbracket Y \in$ ipurge-tr-rev-t-set $(X \mid$ Pol $:=I$, Map $:=D$, Dom $:=u$\); ipurge-tr-rev-t-inv-1 I D u ws $(X($ Pol $:=I$, Map $:=D$, Dom $:=u \mid) \rrbracket \Longrightarrow$ ipurge-tr-rev-t-inv-1 I D u ws Y
proof (erule ipurge-tr-rev-t-set.induct, assumption, drule-tac [!] ipurge-tr-rev-t-invariance-aux, simp-all add: ipurge-tr-rev-t-inv-1-def del: sources.simps)
fix $x x s$ ys
assume $A$ : $D x \notin$ sources $I D u(x \# x s)$
hence ipurge-tr-rev I Duxs=ipurge-tr-rev IDu([] @ $x \# x s)$ by simp also have $\ldots$ = ipurge-tr-rev I $D u$ ([] @ ipurge-tr I D (Dx) xs)
using $A$ by (rule ipurge-tr-rev-ipurge-tr-sources-1)
finally have
ipurge-tr-rev I $D u x s=$ ipurge-tr-rev I $D u($ ipurge-tr $I D(D x) x s)$
by $\operatorname{simp}$
moreover assume ys @ ipurge-tr-rev I Duxs = ipurge-tr-rev I D u ws ultimately show
ys @ ipurge-tr-rev I $D u($ ipurge-tr I $D(D x) x s)=$ ipurge-tr-rev I $D u$ ws by $\operatorname{simp}$
qed
lemma ipurge-tr-rev-t-invariance-2:
$\llbracket Y \in$ ipurge-tr-rev-t-set $(X($ Pol $:=I, M a p:=D D)$;
ipurge-tr-rev-t-inv-2 P I D ws $(X($ Pol $:=I$, Map $:=D)) \rrbracket$
ipurge-tr-rev-t-inv-2 P I D ws Y
proof (erule ipurge-tr-rev-t-set.induct, assumption,
drule-tac [!] ipurge-tr-rev-t-invariance-aux,
simp-all add: ipurge-tr-rev-t-inv-2-def, (rule impI)+)
fix $x$ xs ys

## assume

$S$ : secure P I D and
$w s \in$ traces $P$ and
secure P I D $\longrightarrow$ ws $\in$ traces $P \longrightarrow y s @ x \# x s \in \operatorname{traces} P$
hence $y s$ @ $x \# x s \in$ traces $P$ by simp
hence (ys @ $x \# x s,\{ \}) \in$ failures $P$ by (rule traces-failures)
hence $(x \# x s,\{ \}) \in$ futures $P$ ys by (simp add: futures-def)
hence (ipurge-tr $I D(D x)$ xs, ipurge-ref $I D(D x) x s\}) \in$ futures $P$ ys using $S$ by (simp add: secure-def)
hence (ys @ ipurge-tr I D (Dx)xs, ipurge-ref ID (Dx) xs \{\}) failures $P$ by (simp add: futures-def)
thus ys @ ipurge-tr I D ( $D x$ ) xs $\in$ traces $P$ by (rule failures-traces)
qed

### 1.2.10 Step 10

Here below are the proofs of lemmas ipurge-tr-rev-t-equiv, ipurge-tr-rev-t-trace, which are then applied to demonstrate the target closure lemma.

```
lemma ipurge-tr-rev-t-equiv:
    ipurge-tr-rev-t I Duxs=ipurge-tr-rev I D u xs
proof -
    let ?X = (Pol = I, Map = D, Dom = u,In = xs,Out = [])
    have ipurge-tr-rev-t-aux ?X
        \inipurge-tr-rev-t-set (?X(Pol :=I, Map := D, Dom :=u\)
        by (simp add: ipurge-tr-rev-t-aux-set)
    moreover have
```

```
    ipurge-tr-rev-t-inv-1 I D u xs (?X\Pol := I, Map := D, Dom := u\)
    by (simp add: ipurge-tr-rev---input-1)
    ultimately have ipurge-tr-rev-t-inv-1 I D u xs (ipurge-tr-rev-t-aux ?X)
    by (rule ipurge-tr-rev-t-tnvariance-1)
    moreover have ipurge-tr-rev-t-form (ipurge-tr-rev-t-aux ?X)
    by (rule ipurge-tr-rev-t-form-aux)
    ultimately have
        ipurge-tr-rev-t-out (ipurge-tr-rev-t-aux ?X) = ipurge-tr-rev I D u xs
        by (rule ipurge-tr-rev-t-intro-1)
    moreover have ?X = ipurge-tr-rev-t-in I D u xs
    by (simp add: ipurge-tr-rev-t-in-def)
    ultimately show ?thesis by (simp add: ipurge-tr-rev-t-def)
qed
lemma ipurge-tr-rev-t-trace [rule-format]:
    secure P I D \longrightarrowxs traces P\longrightarrow ipurge-tr-rev-t I D u xs \in traces P
proof -
    let ?X = \Pol =I,Map=D,Dom=u,In =xs,Out = []D
    have ipurge-tr-rev-t-aux ?X
        \in ipurge-tr-rev-t-set (?X\Pol := I, Map := D|)
        by (simp add: ipurge-tr-rev-t-aux-set)
    moreover have ipurge-tr-rev-t-inv-2 P I D xs (?X\Pol := I, Map := DD)
    by (simp add: ipurge-tr-rev-t-input-2)
    ultimately have ipurge-tr-rev-t-inv-2 P I D xs (ipurge-tr-rev-t-aux ?X)
    by (rule ipurge-tr-rev-t-invariance-2)
    moreover have ipurge-tr-rev-t-form (ipurge-tr-rev-t-aux ?X)
    by (rule ipurge-tr-rev---form-aux)
    ultimately have secure PID}\longrightarrowxs\in\operatorname{traces P\longrightarrow
        ipurge-tr-rev-t-out (ipurge-tr-rev-t-aux ?X) \in traces P
    by (rule ipurge-tr-rev-t-intro-2)
    moreover have ?X = ipurge-tr-rev-t-in I D u xs
        by (simp add: ipurge-tr-rev-t-in-def)
    ultimately show ?thesis by (simp add: ipurge-tr-rev-t-def)
qed
lemma ipurge-tr-rev-trace:
secure \(P\) I \(D \Longrightarrow\) xs \(\in\) traces \(P \Longrightarrow\) ipurge-tr-rev I D uxs \(\in\) traces \(P\)
by (subst ipurge-tr-rev-t-equiv [symmetric], rule ipurge-tr-rev-t-trace)
```


### 1.3 The Inductive Unwinding Theorem in its general form

In what follows, the Inductive Unwinding Theorem is proven, in the form applying to a generic process. The equivalence of the condition expressed by the theorem to CSP noninterference security, as defined in [7], is demonstrated by showing that it is necessary and sufficient for the verification of the condition expressed by the Ipurge Unwinding Theorem, under the same assumption that the sets of refusals of the process be closed under union (cf. [8]).

Particularly, the closure of the traces of a secure process under function ipurge-tr-rev and the idempotence of this function are used in the proof of condition necessity.
lemma inductive-unwinding-1:
assumes
$R$ : ref-union-closed $P$ and
$S$ : secure P I D
shows $\forall$ xs $\in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s \wedge$
ref-dom-events $P D u($ ipurge-tr-rev $I D u x s)=r e f-d o m-e v e n t s ~ P D u x s$
proof (rule balli) +
fix $x s u$
from $R$ and $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D . \forall x s$ ys.
xs $\in$ traces $P \wedge y s \in$ traces $P \wedge$
ipurge-tr-rev I D uxs = ipurge-tr-rev I Duys $\longrightarrow$
next-dom-events $P D u x s=$ next-dom-events $P D$ u ys $\wedge$
ref-dom-events $P D u x s=$ ref-dom-events $P D u$ ys
by (simp add: ipurge-unwinding weakly-future-consistent-def rel-ipurge-def)
moreover assume $u \in$ range $D \cap(-I)$ " range $D$
ultimately have $\forall x s$ ys.
xs $\in$ traces $P \wedge y s \in$ traces $P \wedge$
ipurge-tr-rev I D u xs = ipurge-tr-rev I Duys $\longrightarrow$
next-dom-events $P D u x s=$ next-dom-events $P D u$ ys $\wedge$
ref-dom-events $P D$ u xs = ref-dom-events $P D u$ ys ..
hence
ipurge-tr-rev I D uxs traces $P \wedge x s \in$ traces $P \wedge$ ipurge-tr-rev IDu(ipurge-tr-rev I Duxs) $=$ ipurge-tr-rev I Duxs $\longrightarrow$ next-dom-events P D u (ipurge-tr-rev I Duxs) = next-dom-events P Duxs $\wedge$ ref-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ ref-dom-events $P D u x s$
by blast
moreover assume $x s: x s \in$ traces $P$
moreover from $S$ and $x s$ have ipurge-tr-rev $I D u x s \in \operatorname{traces} P$
by (rule ipurge-tr-rev-trace)
moreover have
ipurge-tr-rev I Du(ipurge-tr-rev I $D u x s)=$ ipurge-tr-rev I $D u x s$
by (rule ipurge-tr-rev-idem)
ultimately show
next-dom-events $P D u$ (ipurge-tr-rev I $D u x s)=$ next-dom-events $P D u x s \wedge$ ref-dom-events $P D u$ (ipurge-tr-rev I $D u x s)=$ ref-dom-events $P D u x s$ by $\operatorname{simp}$
qed
lemma inductive-unwinding-2:

## assumes

$R$ : ref-union-closed $P$ and
$S: \forall x s \in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s \wedge$
ref-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$
ref-dom-events P D u xs
shows secure PID
proof (simp add: ipurge-unwinding [OF R] weakly-future-consistent-def rel-ipurge-def, rule ballI, (rule allI)+, rule impI, (erule conjE)+)
fix $u x s$ ys
assume $x s \in$ traces $P$
with $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D$. next-dom-events $P D$ u (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s \wedge$ ref-dom-events P $D$ u (ipurge-tr-rev I $D u x s)=$ ref-dom-events $P D u x s .$.
moreover assume $A: u \in$ range $D \cap(-I)$ " range $D$
ultimately have $B$ :
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s \wedge$ ref-dom-events $P D u($ ipurge-tr-rev $I D u x s)=$ ref-dom-events $P D u x s .$.
assume ys $\in$ traces $P$
with $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u$ ys $)=$ next-dom-events $P D u$ ys $\wedge$ ref-dom-events $P D$ u (ipurge-tr-rev I $D$ u ys) $=$ ref-dom-events $P D$ u ys ..

## hence

next-dom-events $P D u$ (ipurge-tr-rev I D u ys) $=$ next-dom-events $P D u$ ys $\wedge$ ref-dom-events $P D u$ (ipurge-tr-rev I $D$ u ys) $=$ ref-dom-events $P D$ u ys
using $A$..
moreover assume ipurge-tr-rev I $D u x s=$ ipurge-tr-rev I $D u$ ys
ultimately show
next-dom-events $P D u x s=$ next-dom-events $P D u$ ys $\wedge$
ref-dom-events $P D$ u xs $=$ ref-dom-events $P D$ u ys
using $B$ by simp
qed
theorem inductive-unwinding:
ref-union-closed $P \Longrightarrow$
secure P I D =
( $\forall$ xs $\in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s \wedge$ ref-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=r e f$-dom-events $P D u x s)$
by (rule iffI, rule inductive-unwinding-1, assumption+, rule inductive-unwinding-2)

Interestingly, this necessary and sufficient condition for the noninterference security of a process resembles the classical definition of noninterference security for a deterministic state machine with outputs formulated in [9], which is formalized in [7] as predicate $c$-secure.
Denoting with (1) the former and with (2) the latter, the differences between them can be summarized as follows:

- The event list appearing in (1) is constrained to vary over process traces, whereas the action list appearing in (2) is unconstrained.
This comes as no surprise, since the state machines used as model of
computation in [9] accept any action list as a trace.
- The definition of function ipurge-tr-rev, used in (1), does not implicitly assume that the noninterference policy be reflexive, even though any policy of practical significance will be such. On the contrary, the definition of the intransitive purge function used in (2), which is formalized in [7] as function c-ipurge, makes this implicit assumption, as shown by the consideration that c-ipurge $I D(D x)[x]=[x]$ regardless of whether $(D x, D x) \in I$ or not.
This is the mathematical reason why the equivalence between CSP noninterference security and classical noninterference security for deterministic state machines with outputs, proven in [7], is subordinated to the assumption that the noninterference policy be reflexive.
- The equality of action outputs appearing in (2) is replaced in (1) by the equality of accepted and refused events.

The binding of the universal quantification over domains contained in (1) does not constitute an actual difference, since in (2) the purge function is only applied to domains in the range of the event-domain map, and its output matches the entire input action list, thus rendering the equation trivial, for domains allowed to be affected by any event domain.

### 1.4 The Inductive Unwinding Theorem for deterministic and trace set processes

Here below are the proofs of specific variants of the Inductive Unwinding Theorem applying to deterministic processes and trace set processes [8]. The variant for deterministic processes is derived, following the above proof of the general form of the theorem, from the Ipurge Unwinding Theorem for deterministic processes [8]. Then, the variant for trace set processes is inferred from the variant for deterministic processes.
Similarly to what happens for the Ipurge Unwinding Theorem, the refusals union closure assumption that characterizes the general form of the Inductive Unwinding Theorem is replaced by the assumption that the process actually be deterministic in the variant for deterministic processes, and by the assumption that the set of traces actually be such in the variant for trace set processes. Moreover, these variants involve accepted events only, in accordance with the fact that in deterministic processes, refused events are completely specified by accepted events (cf. [1], [7]).

```
lemma d-inductive-unwinding-1:
    assumes
    D:deterministic P and
```

S: secure P I D
shows $\forall x s \in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s$
proof (rule ballI)+
fix $x s u$
from $D$ and $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D$. $\forall x s$ ys. xs $\in$ traces $P \wedge y s \in$ traces $P \wedge$
ipurge-tr-rev I $D u x s=$ ipurge-tr-rev $I D u y s \longrightarrow$
next-dom-events $P D u x s=$ next-dom-events $P D u$ ys
by ( simp add: d-ipurge-unwinding d-weakly-future-consistent-def rel-ipurge-def)
moreover assume $u \in$ range $D \cap(-I)$ " range $D$
ultimately have $\forall x s$ ys.
xs $\in$ traces $P \wedge y s \in$ traces $P \wedge$
ipurge-tr-rev $I D$ uxs $=$ ipurge-tr-rev $I D u$ ys $\longrightarrow$ next-dom-events $P D$ u xs = next-dom-events $P D$ u ys ..
hence
ipurge-tr-rev I D u xs $\in$ traces $P \wedge x s \in$ traces $P \wedge$
ipurge-tr-rev I Du(ipurge-tr-rev I $D u x s)=$ ipurge-tr-rev I D uxs $\longrightarrow$ next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s$ by blast
moreover assume $x s: x s \in$ traces $P$
moreover from $S$ and $x s$ have ipurge-tr-rev I $D u x s \in \operatorname{traces} P$
by (rule ipurge-tr-rev-trace)
moreover have
ipurge-tr-rev I $D u($ ipurge-tr-rev I $D u x s)=$ ipurge-tr-rev I $D$ uxs
by (rule ipurge-tr-rev-idem)
ultimately show
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s$ by $\operatorname{simp}$
qed
lemma d-inductive-unwinding-2:
assumes
$D$ : deterministic $P$ and
$S: \forall x s \in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s$
shows secure P I D
proof (simp add: d-ipurge-unwinding [OF D] d-weakly-future-consistent-def rel-ipurge-def, rule ballI, (rule allI)+, rule impI, ( erule conjE)+)
fix $u x s$ ys
assume $x s \in$ traces $P$
with $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events P $D$ u (ipurge-tr-rev I D u xs) = next-dom-events P D uxs ..
moreover assume $A: u \in$ range $D \cap(-I)$ " range $D$
ultimately have $B$ :
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s .$.
assume ys traces $P$
with $S$ have $\forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events P $D$ u (ipurge-tr-rev I D u ys) = next-dom-events P D u ys..

## hence

next-dom-events $P D u$ (ipurge-tr-rev $I D u y s)=$ next-dom-events $P D u$ ys using $A$..
moreover assume ipurge-tr-rev I D uxs =ipurge-tr-rev I D u ys
ultimately show next-dom-events $P D u x s=$ next-dom-events $P D u$ ys
using $B$ by simp

## qed

theorem d-inductive-unwinding:
deterministic $P \Longrightarrow$
secure $P$ I $D=$
$(\forall x s \in$ traces $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
next-dom-events $P D u$ (ipurge-tr-rev $I D u x s)=$ next-dom-events $P D u x s)$
by (rule iffI, rule d-inductive-unwinding-1, assumption+, rule d-inductive-unwinding-2)
theorem ts-inductive-unwinding:
assumes $T$ : trace-set $T$
shows secure (ts-process $T$ ) I $D=$
$(\forall x s \in T . \forall u \in$ range $D \cap(-I)$ " range $D . \forall x \in D-‘\{u\}$.
(ipurge-tr-rev ID uxs @ $[x] \in T)=(x s @[x] \in T)$ )
(is secure ?P I $D=-$ )
proof (subst d-inductive-unwinding, rule ts-process-d [OF T],
simp add: next-dom-events-def ts-process-next-events [OF T] set-eq-iff,
rule iffI, (rule ballI)+, (rule-tac [2] ballI)+, rule-tac [2] allI)
fix $x s u x$
assume $A: \forall x s \in$ traces ? $P . \forall u \in$ range $D \cap(-I)$ " range $D$.
$\forall x .(u=D x \wedge$ ipurge-tr-rev I D uxs @ $[x] \in T)=(u=D x \wedge x s @[x] \in T)$
assume $x s \in T$
moreover have traces ? $P=T$ using $T$ by (rule ts-process-traces)
ultimately have $x s \in$ traces ? P by simp
with $A$ have $\forall u \in$ range $D \cap(-I)$ " range $D$.
$\forall x .(u=D x \wedge$ ipurge-tr-rev I Duxs @ $[x] \in T)=$ $(u=D x \wedge x s @[x] \in T) .$.
moreover assume $u \in$ range $D \cap(-I)$ " range $D$
ultimately have
$\forall x .(u=D x \wedge$ ipurge-tr-rev I $D u$ xs $@[x] \in T)=$ $(u=D x \wedge x s @[x] \in T) .$.
hence $(u=D x \wedge$ ipurge-tr-rev I $D u x s @[x] \in T)=$
$(u=D x \wedge x s @[x] \in T) .$.
moreover assume $x \in D-‘\{u\}$
hence $u=D x$ by simp
ultimately show (ipurge-tr-rev I Duxs @ $[x] \in T)=(x s @[x] \in T)$ by simp
next
fix $x s u x$
assume $A: \forall x s \in T . \forall u \in$ range $D \cap(-I)$ " range $D$.
$\forall x \in D-{ }^{\prime}\{u\}$. (ipurge-tr-rev I D uxs @ $\left.[x] \in T\right)=(x s @[x] \in T)$
assume $x s \in$ traces ? $P$
moreover have traces ? $P=T$ using $T$ by (rule ts-process-traces)
ultimately have $x s \in T$ by $\operatorname{simp}$

```
    with \(A\) have \(\forall u \in\) range \(D \cap(-I)\) " range \(D\).
    \(\forall x \in D-'\{u\}\). (ipurge-tr-rev I D u xs @ \([x] \in T)=(x s @[x] \in T) .\).
    moreover assume \(u \in\) range \(D \cap(-I)\) " range \(D\)
    ultimately have \(B\) :
    \(\forall x \in D-‘\{u\}\). (ipurge-tr-rev I D u xs @ \([x] \in T)=(x s @[x] \in T) .\).
    show \((u=D x \wedge\) ipurge-tr-rev I D u xs @ \([x] \in T)=\)
    \((u=D x \wedge x s @[x] \in T)\)
proof (cases \(D x=u\), simp-all)
    case True
    hence \(x \in D-\) ' \(\{u\}\) by simp
    with \(B\) show (ipurge-tr-rev ID uxs @ \([x] \in T)=(x s @[x] \in T)\)..
    qed
qed
end
```


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