

Nominal 2

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Abstract

Dealing with binders, renaming of bound variables, capture-avoiding substitution, etc., is very often a major problem in formal proofs, especially in proofs by structural and rule induction. Nominal Isabelle is designed to make such proofs easy to formalise: it provides an infrastructure for declaring nominal datatypes (that is alpha-equivalence classes) and for defining functions over them by structural recursion. It also provides induction principles that have Barendregts variable convention already built in.

This entry can be used as a more advanced replacement for HOL/Nominal in the Isabelle distribution.

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```

theory Nominal2-Base
imports HOL-Library.Infinite-Set
         HOL-Library.Multiset
         HOL-Library.FSet
         FinFun.FinFun
keywords
  atom-decl equivariance :: thy-decl
begin

declare [[typedef-overloaded]]

```

1 Atoms and Sorts

A simple implementation for *atom-sorts* is strings.

To deal with Church-like binding we use trees of strings as sorts.

```
datatype atom-sort = Sort string atom-sort list
```

```
datatype atom = Atom atom-sort nat
```

Basic projection function.

```
primrec
  sort-of :: atom  $\Rightarrow$  atom-sort
where
  sort-of (Atom s n) = s
```

```
primrec
  nat-of :: atom  $\Rightarrow$  nat
where
  nat-of (Atom s n) = n
```

There are infinitely many atoms of each sort.

```
lemma INFM-sort-of-eq:
  shows INFM a. sort-of a = s
  <proof>
```

```
lemma infinite-sort-of-eq:
  shows infinite {a. sort-of a = s}
  <proof>
```

```
lemma atom-infinite [simp]:
  shows infinite (UNIV :: atom set)
```

<proof>

lemma *obtain-atom*:

fixes $X :: \text{atom set}$

assumes $X: \text{finite } X$

obtains a **where** $a \notin X$ $\text{sort-of } a = s$

<proof>

lemma *atom-components-eq-iff*:

fixes $a b :: \text{atom}$

shows $a = b \iff \text{sort-of } a = \text{sort-of } b \wedge \text{nat-of } a = \text{nat-of } b$

<proof>

2 Sort-Respecting Permutations

definition

$\text{perm} \equiv \{f. \text{bij } f \wedge \text{finite } \{a. f a \neq a\} \wedge (\forall a. \text{sort-of } (f a) = \text{sort-of } a)\}$

typedef $\text{perm} = \text{perm}$

<proof>

lemma *permI*:

assumes $\text{bij } f$ **and** $\text{MOST } x. f x = x$ **and** $\bigwedge a. \text{sort-of } (f a) = \text{sort-of } a$

shows $f \in \text{perm}$

<proof>

lemma *perm-is-bij*: $f \in \text{perm} \implies \text{bij } f$

<proof>

lemma *perm-is-finite*: $f \in \text{perm} \implies \text{finite } \{a. f a \neq a\}$

<proof>

lemma *perm-is-sort-respecting*: $f \in \text{perm} \implies \text{sort-of } (f a) = \text{sort-of } a$

<proof>

lemma *perm-MOST*: $f \in \text{perm} \implies \text{MOST } x. f x = x$

<proof>

lemma *perm-id*: $\text{id} \in \text{perm}$

<proof>

lemma *perm-comp*:

assumes $f: f \in \text{perm}$ **and** $g: g \in \text{perm}$

shows $(f \circ g) \in \text{perm}$

<proof>

lemma *perm-inv*:

assumes $f: f \in \text{perm}$

shows $(\text{inv } f) \in \text{perm}$

<proof>

lemma *bij-Rep-perm*: *bij (Rep-perm p)*
<proof>

lemma *finite-Rep-perm*: *finite {a. Rep-perm p a ≠ a}*
<proof>

lemma *sort-of-Rep-perm*: *sort-of (Rep-perm p a) = sort-of a*
<proof>

lemma *Rep-perm-ext*:
Rep-perm p1 = Rep-perm p2 ⇒ p1 = p2
<proof>

instance *perm :: size <proof>*

2.1 Permutations form a (multiplicative) group

instantiation *perm :: group-add*
begin

definition
0 = Abs-perm id

definition
- p = Abs-perm (inv (Rep-perm p))

definition
p + q = Abs-perm (Rep-perm p ∘ Rep-perm q)

definition
(p1::perm) - p2 = p1 + - p2

lemma *Rep-perm-0*: *Rep-perm 0 = id*
<proof>

lemma *Rep-perm-add*:
Rep-perm (p1 + p2) = Rep-perm p1 ∘ Rep-perm p2
<proof>

lemma *Rep-perm-uminus*:
Rep-perm (- p) = inv (Rep-perm p)
<proof>

instance
<proof>

end

3 Implementation of swappings

definition

$swap :: atom \Rightarrow atom \Rightarrow perm (('(- \rightleftharpoons -'))$

where

$(a \rightleftharpoons b) =$
 Abs-perm (if sort-of a = sort-of b
 then ($\lambda c.$ if a = c then b else if b = c then a else c)
 else id)

lemma *Rep-perm-swap:*

$Rep-perm (a \rightleftharpoons b) =$
 (if sort-of a = sort-of b
 then ($\lambda c.$ if a = c then b else if b = c then a else c)
 else id)

<proof>

lemmas *Rep-perm-simps =*

Rep-perm-0
Rep-perm-add
Rep-perm-uminus
Rep-perm-swap

lemma *swap-different-sorts [simp]:*

$sort-of\ a \neq\ sort-of\ b \implies (a \rightleftharpoons b) = 0$
<proof>

lemma *swap-cancel:*

shows $(a \rightleftharpoons b) + (a \rightleftharpoons b) = 0$
and $(a \rightleftharpoons b) + (b \rightleftharpoons a) = 0$
<proof>

lemma *swap-self [simp]:*

$(a \rightleftharpoons a) = 0$
<proof>

lemma *minus-swap [simp]:*

$-(a \rightleftharpoons b) = (a \rightleftharpoons b)$
<proof>

lemma *swap-commute:*

$(a \rightleftharpoons b) = (b \rightleftharpoons a)$
<proof>

lemma *swap-triple:*

assumes $a \neq b$ **and** $c \neq b$
assumes $sort-of\ a = sort-of\ b$ $sort-of\ b = sort-of\ c$
shows $(a \rightleftharpoons c) + (b \rightleftharpoons c) + (a \rightleftharpoons c) = (a \rightleftharpoons b)$
<proof>

4 Permutation Types

Infix syntax for *permute* has higher precedence than addition, but lower than unary minus.

```
class pt =  
  fixes permute :: perm  $\Rightarrow$  'a  $\Rightarrow$  'a (- · - [76, 75] 75)  
  assumes permute-zero [simp]: 0 · x = x  
  assumes permute-plus [simp]: (p + q) · x = p · (q · x)  
begin
```

```
lemma permute-diff [simp]:  
  shows (p - q) · x = p · - q · x  
   $\langle$ proof $\rangle$ 
```

```
lemma permute-minus-cancel [simp]:  
  shows p · - p · x = x  
  and - p · p · x = x  
   $\langle$ proof $\rangle$ 
```

```
lemma permute-swap-cancel [simp]:  
  shows (a  $\rightleftharpoons$  b) · (a  $\rightleftharpoons$  b) · x = x  
   $\langle$ proof $\rangle$ 
```

```
lemma permute-swap-cancel2 [simp]:  
  shows (a  $\rightleftharpoons$  b) · (b  $\rightleftharpoons$  a) · x = x  
   $\langle$ proof $\rangle$ 
```

```
lemma inj-permute [simp]:  
  shows inj (permute p)  
   $\langle$ proof $\rangle$ 
```

```
lemma surj-permute [simp]:  
  shows surj (permute p)  
   $\langle$ proof $\rangle$ 
```

```
lemma bij-permute [simp]:  
  shows bij (permute p)  
   $\langle$ proof $\rangle$ 
```

```
lemma inv-permute:  
  shows inv (permute p) = permute (- p)  
   $\langle$ proof $\rangle$ 
```

```
lemma permute-minus:  
  shows permute (- p) = inv (permute p)  
   $\langle$ proof $\rangle$ 
```

```
lemma permute-eq-iff [simp]:  
  shows p · x = p · y  $\longleftrightarrow$  x = y
```


<proof>

end

4.1 Permutations for atoms

instantiation *atom* :: *pt*

begin

definition

$$p \cdot a = (\text{Rep-perm } p) a$$

instance

<proof>

end

lemma *sort-of-permute* [*simp*]:

shows *sort-of* ($p \cdot a$) = *sort-of* a

<proof>

lemma *swap-atom*:

shows ($a \rightleftharpoons b$) \cdot $c =$

(*if* *sort-of* $a =$ *sort-of* b

then (*if* $c = a$ *then* b *else* *if* $c = b$ *then* a *else* c) *else* c)

<proof>

lemma *swap-atom-simps* [*simp*]:

sort-of $a =$ *sort-of* $b \implies (a \rightleftharpoons b) \cdot a = b$

sort-of $a =$ *sort-of* $b \implies (a \rightleftharpoons b) \cdot b = a$

$c \neq a \implies c \neq b \implies (a \rightleftharpoons b) \cdot c = c$

<proof>

lemma *perm-eq-iff*:

fixes p q :: *perm*

shows $p = q \iff (\forall a::\text{atom}. p \cdot a = q \cdot a)$

<proof>

4.2 Permutations for permutations

instantiation *perm* :: *pt*

begin

definition

$$p \cdot q = p + q - p$$

instance

<proof>

end

lemma *permute-self*:

shows $p \cdot p = p$

\langle *proof* \rangle

lemma *permute-minus-self*:

shows $- p \cdot p = p$

\langle *proof* \rangle

4.3 Permutations for functions

instantiation *fun* :: (*pt*, *pt*) *pt*

begin

definition

$p \cdot f = (\lambda x. p \cdot (f (- p \cdot x)))$

instance

\langle *proof* \rangle

end

lemma *permute-fun-app-eq*:

shows $p \cdot (f x) = (p \cdot f) (p \cdot x)$

\langle *proof* \rangle

lemma *permute-fun-comp*:

shows $p \cdot f = (\text{permute } p) \circ f \circ (\text{permute } (-p))$

\langle *proof* \rangle

4.4 Permutations for booleans

instantiation *bool* :: *pt*

begin

definition $p \cdot (b::\text{bool}) = b$

instance

\langle *proof* \rangle

end

lemma *permute-boolE*:

fixes $P::\text{bool}$

shows $p \cdot P \implies P$

\langle *proof* \rangle

lemma *permute-boolI*:

fixes $P::\text{bool}$

shows $P \implies p \cdot P$

<proof>

4.5 Permutations for sets

instantiation *set* :: (*pt*) *pt*
begin

definition

$p \cdot X = \{p \cdot x \mid x. x \in X\}$

instance

<proof>

end

lemma *permute-set-eq*:

shows $p \cdot X = \{x. \neg p \cdot x \in X\}$

<proof>

lemma *permute-set-eq-image*:

shows $p \cdot X = \text{permute } p \text{ ` } X$

<proof>

lemma *permute-set-eq-vimage*:

shows $p \cdot X = \text{permute } (\neg p) \text{ - ` } X$

<proof>

lemma *permute-finite* [*simp*]:

shows $\text{finite } (p \cdot X) = \text{finite } X$

<proof>

lemma *swap-set-not-in*:

assumes $a \notin S \ b \notin S$

shows $(a \rightleftharpoons b) \cdot S = S$

<proof>

lemma *swap-set-in*:

assumes $a \in S \ b \notin S \ \text{sort-of } a = \text{sort-of } b$

shows $(a \rightleftharpoons b) \cdot S \neq S$

<proof>

lemma *swap-set-in-eq*:

assumes $a \in S \ b \notin S \ \text{sort-of } a = \text{sort-of } b$

shows $(a \rightleftharpoons b) \cdot S = (S - \{a\}) \cup \{b\}$

<proof>

lemma *swap-set-both-in*:

assumes $a \in S \ b \in S$

shows $(a \rightleftharpoons b) \cdot S = S$

<proof>

lemma *mem-permute-iff*:

shows $(p \cdot x) \in (p \cdot X) \longleftrightarrow x \in X$

<proof>

lemma *empty-eqt*:

shows $p \cdot \{\} = \{\}$

<proof>

lemma *insert-eqt*:

shows $p \cdot (\text{insert } x \ A) = \text{insert } (p \cdot x) \ (p \cdot A)$

<proof>

4.6 Permutations for *unit*

instantiation *unit* :: *pt*

begin

definition $p \cdot (u::\text{unit}) = u$

instance

<proof>

end

4.7 Permutations for products

instantiation *prod* :: (*pt*, *pt*) *pt*

begin

primrec

permute-prod

where

Pair-eqt: $p \cdot (x, y) = (p \cdot x, p \cdot y)$

instance

<proof>

end

4.8 Permutations for sums

instantiation *sum* :: (*pt*, *pt*) *pt*

begin

primrec

permute-sum

where

Inl-eqt: $p \cdot (\text{Inl } x) = \text{Inl } (p \cdot x)$

| *Inr-eqvt*: $p \cdot (\text{Inr } y) = \text{Inr } (p \cdot y)$

instance

<proof>

end

4.9 Permutations for 'a list

instantiation *list* :: (pt) pt

begin

primrec

permute-list

where

Nil-eqvt: $p \cdot [] = []$

| *Cons-eqvt*: $p \cdot (x \# xs) = p \cdot x \# p \cdot xs$

instance

<proof>

end

lemma *set-eqvt*:

shows $p \cdot (\text{set } xs) = \text{set } (p \cdot xs)$

<proof>

4.10 Permutations for 'a option

instantiation *option* :: (pt) pt

begin

primrec

permute-option

where

None-eqvt: $p \cdot \text{None} = \text{None}$

| *Some-eqvt*: $p \cdot (\text{Some } x) = \text{Some } (p \cdot x)$

instance

<proof>

end

4.11 Permutations for 'a multiset

instantiation *multiset* :: (pt) pt

begin

definition

$p \cdot M = \{\# p \cdot x. x \# M \#\}$

```

instance
  ⟨proof⟩

end

lemma permutate-multiset [simp]:
  fixes  $M\ N::('a::pt)\ \text{multiset}$ 
  shows  $(p \cdot \{\#\}) = (\{\#\} ::('a::pt)\ \text{multiset})$ 
  and  $(p \cdot \text{add-mset } x\ M) = \text{add-mset } (p \cdot x)\ (p \cdot M)$ 
  and  $(p \cdot (M + N)) = (p \cdot M) + (p \cdot N)$ 
  ⟨proof⟩

```

4.12 Permutations for $'a$ fset

```

instantiation fset :: (pt) pt
begin

context includes fset.lifting begin
lift-definition
  permutate-fset ::  $\text{perm} \Rightarrow 'a\ \text{fset} \Rightarrow 'a\ \text{fset}$ 
is permutate ::  $\text{perm} \Rightarrow 'a\ \text{set} \Rightarrow 'a\ \text{set}$  ⟨proof⟩
end

```

```

context includes fset.lifting begin
instance
  ⟨proof⟩
end

```

end

```

context includes fset.lifting
begin
lemma permutate-fset [simp]:
  fixes  $S::('a::pt)\ \text{fset}$ 
  shows  $(p \cdot \{\|\}) = (\{\|\} ::('a::pt)\ \text{fset})$ 
  and  $(p \cdot \text{finsert } x\ S) = \text{finsert } (p \cdot x)\ (p \cdot S)$ 
  ⟨proof⟩

```

```

lemma fset-eqvt:
  shows  $p \cdot (\text{fset } S) = \text{fset } (p \cdot S)$ 
  ⟨proof⟩
end

```

4.13 Permutations for $('a, 'b)$ finfun

```

instantiation finfun :: (pt, pt) pt
begin

```

```

lift-definition

```

```

    permute-finfun :: perm ⇒ ('a, 'b) finfun ⇒ ('a, 'b) finfun
  is
    permute :: perm ⇒ ('a ⇒ 'b) ⇒ ('a ⇒ 'b)
    ⟨proof⟩

  instance
    ⟨proof⟩

  end

```

4.14 Permutations for *char*, *nat*, and *int*

```

  instantiation char :: pt
  begin

  definition p · (c::char) = c

  instance
    ⟨proof⟩

  end

  instantiation nat :: pt
  begin

  definition p · (n::nat) = n

  instance
    ⟨proof⟩

  end

  instantiation int :: pt
  begin

  definition p · (i::int) = i

  instance
    ⟨proof⟩

  end

```

5 Pure types

Pure types will have always empty support.

```

class pure = pt +
  assumes permute-pure: p · x = x

  Types unit and bool are pure.

```

instance *unit* :: *pure*
⟨*proof*⟩

instance *bool* :: *pure*
⟨*proof*⟩

Other type constructors preserve purity.

instance *fun* :: (*pure*, *pure*) *pure*
⟨*proof*⟩

instance *set* :: (*pure*) *pure*
⟨*proof*⟩

instance *prod* :: (*pure*, *pure*) *pure*
⟨*proof*⟩

instance *sum* :: (*pure*, *pure*) *pure*
⟨*proof*⟩

instance *list* :: (*pure*) *pure*
⟨*proof*⟩

instance *option* :: (*pure*) *pure*
⟨*proof*⟩

5.1 Types *char*, *nat*, and *int*

instance *char* :: *pure*
⟨*proof*⟩

instance *nat* :: *pure*
⟨*proof*⟩

instance *int* :: *pure*
⟨*proof*⟩

6 Infrastructure for Equivariance and *Perm-simp*

6.1 Basic functions about permutations

⟨*ML*⟩

6.2 Eqvt infrastructure

Setup of the theorem attributes *eqvt* and *eqvt-raw*.

⟨*ML*⟩

lemmas [*eqvt*] =

permute-prod.simps
permute-list.simps
permute-option.simps
permute-sum.simps

empty-eqvt insert-eqvt set-eqvt

permute-fset fset-eqvt

permute-multiset

6.3 *perm-simp* infrastructure

definition

unpermute $p = \text{permute } (- \ p)$

lemma *eqvt-apply*:

fixes $f :: 'a::pt \Rightarrow 'b::pt$
and $x :: 'a::pt$
shows $p \cdot (f \ x) \equiv (p \cdot f) \ (p \cdot x)$
<proof>

lemma *eqvt-lambda*:

fixes $f :: 'a::pt \Rightarrow 'b::pt$
shows $p \cdot f \equiv (\lambda x. \ p \cdot (f \ (\text{unpermute } \ p \ x)))$
<proof>

lemma *eqvt-bound*:

shows $p \cdot \text{unpermute } \ p \ x \equiv x$
<proof>

provides *perm-simp* methods

<ML>

6.3.1 Equivariance for permutations and swapping

lemma *permute-eqvt*:

shows $p \cdot (q \cdot x) = (p \cdot q) \cdot (p \cdot x)$
<proof>

lemma *permute-raw* [*eqvt-raw*]:

shows $p \cdot \text{permute} \equiv \text{permute}$
<proof>

lemma *zero-raw-raw* [*eqvt*]:

shows $p \cdot (0::perm) = 0$
<proof>

lemma *add-perm-eqvt* [eqvt]:
fixes $p\ p1\ p2 :: perm$
shows $p \cdot (p1 + p2) = p \cdot p1 + p \cdot p2$
<proof>

lemma *swap-eqvt* [eqvt]:
shows $p \cdot (a \rightleftharpoons b) = (p \cdot a \rightleftharpoons p \cdot b)$
<proof>

lemma *uminus-eqvt* [eqvt]:
fixes $p\ q::perm$
shows $p \cdot (-\ q) = -(p \cdot q)$
<proof>

6.3.2 Equivariance of Logical Operators

lemma *eq-eqvt* [eqvt]:
shows $p \cdot (x = y) \longleftrightarrow (p \cdot x) = (p \cdot y)$
<proof>

lemma *Not-eqvt* [eqvt]:
shows $p \cdot (\neg\ A) \longleftrightarrow \neg\ (p \cdot A)$
<proof>

lemma *conj-eqvt* [eqvt]:
shows $p \cdot (A \wedge B) \longleftrightarrow (p \cdot A) \wedge (p \cdot B)$
<proof>

lemma *imp-eqvt* [eqvt]:
shows $p \cdot (A \longrightarrow B) \longleftrightarrow (p \cdot A) \longrightarrow (p \cdot B)$
<proof>

declare *imp-eqvt*[folded *HOL.induct-implies-def*, eqvt]

lemma *all-eqvt* [eqvt]:
shows $p \cdot (\forall x. P\ x) = (\forall x. (p \cdot P)\ x)$
<proof>

declare *all-eqvt*[folded *HOL.induct-forall-def*, eqvt]

lemma *ex-eqvt* [eqvt]:
shows $p \cdot (\exists x. P\ x) = (\exists x. (p \cdot P)\ x)$
<proof>

lemma *ex1-eqvt* [eqvt]:
shows $p \cdot (\exists!x. P\ x) = (\exists!x. (p \cdot P)\ x)$

$\langle proof \rangle$

lemma *if-eqvt* [eqvt]:

shows $p \cdot (\text{if } b \text{ then } x \text{ else } y) = (\text{if } p \cdot b \text{ then } p \cdot x \text{ else } p \cdot y)$
 $\langle proof \rangle$

lemma *True-eqvt* [eqvt]:

shows $p \cdot \text{True} = \text{True}$
 $\langle proof \rangle$

lemma *False-eqvt* [eqvt]:

shows $p \cdot \text{False} = \text{False}$
 $\langle proof \rangle$

lemma *disj-eqvt* [eqvt]:

shows $p \cdot (A \vee B) \longleftrightarrow (p \cdot A) \vee (p \cdot B)$
 $\langle proof \rangle$

lemma *all-eqvt2*:

shows $p \cdot (\forall x. P x) = (\forall x. p \cdot P (- p \cdot x))$
 $\langle proof \rangle$

lemma *ex-eqvt2*:

shows $p \cdot (\exists x. P x) = (\exists x. p \cdot P (- p \cdot x))$
 $\langle proof \rangle$

lemma *ex1-eqvt2*:

shows $p \cdot (\exists!x. P x) = (\exists!x. p \cdot P (- p \cdot x))$
 $\langle proof \rangle$

lemma *the-eqvt*:

assumes *unique*: $\exists!x. P x$
shows $(p \cdot (\text{THE } x. P x)) = (\text{THE } x. (p \cdot P) x)$
 $\langle proof \rangle$

lemma *the-eqvt2*:

assumes *unique*: $\exists!x. P x$
shows $(p \cdot (\text{THE } x. P x)) = (\text{THE } x. p \cdot P (- p \cdot x))$
 $\langle proof \rangle$

6.3.3 Equivariance of Set operators

lemma *mem-eqvt* [eqvt]:

shows $p \cdot (x \in A) \longleftrightarrow (p \cdot x) \in (p \cdot A)$
 $\langle proof \rangle$

lemma *Collect-eqvt* [eqvt]:

shows $p \cdot \{x. P x\} = \{x. (p \cdot P) x\}$
 $\langle proof \rangle$

lemma *Bex-eqvt* [eqvt]:
shows $p \cdot (\exists x \in S. P x) = (\exists x \in (p \cdot S). (p \cdot P) x)$
 ⟨proof⟩

lemma *Ball-eqvt* [eqvt]:
shows $p \cdot (\forall x \in S. P x) = (\forall x \in (p \cdot S). (p \cdot P) x)$
 ⟨proof⟩

lemma *image-eqvt* [eqvt]:
shows $p \cdot (f \cdot A) = (p \cdot f) \cdot (p \cdot A)$
 ⟨proof⟩

lemma *Image-eqvt* [eqvt]:
shows $p \cdot (R \cdot A) = (p \cdot R) \cdot (p \cdot A)$
 ⟨proof⟩

lemma *UNIV-eqvt* [eqvt]:
shows $p \cdot UNIV = UNIV$
 ⟨proof⟩

lemma *inter-eqvt* [eqvt]:
shows $p \cdot (A \cap B) = (p \cdot A) \cap (p \cdot B)$
 ⟨proof⟩

lemma *Inter-eqvt* [eqvt]:
shows $p \cdot \bigcap S = \bigcap (p \cdot S)$
 ⟨proof⟩

lemma *union-eqvt* [eqvt]:
shows $p \cdot (A \cup B) = (p \cdot A) \cup (p \cdot B)$
 ⟨proof⟩

lemma *Union-eqvt* [eqvt]:
shows $p \cdot \bigcup A = \bigcup (p \cdot A)$
 ⟨proof⟩

lemma *Diff-eqvt* [eqvt]:
fixes $A B :: 'a::pt\ set$
shows $p \cdot (A - B) = (p \cdot A) - (p \cdot B)$
 ⟨proof⟩

lemma *Compl-eqvt* [eqvt]:
fixes $A :: 'a::pt\ set$
shows $p \cdot (- A) = - (p \cdot A)$
 ⟨proof⟩

lemma *subset-eqvt* [eqvt]:
shows $p \cdot (S \subseteq T) \longleftrightarrow (p \cdot S) \subseteq (p \cdot T)$

<proof>

lemma *psubset-eqvt* [*eqvt*]:
shows $p \cdot (S \subset T) \longleftrightarrow (p \cdot S) \subset (p \cdot T)$
<proof>

lemma *vimage-eqvt* [*eqvt*]:
shows $p \cdot (f -' A) = (p \cdot f) -' (p \cdot A)$
<proof>

lemma *foldr-eqvt*[*eqvt*]:
 $p \cdot \text{foldr } f \text{ } xs = \text{foldr } (p \cdot f) (p \cdot xs)$
<proof>

lemma *Sigma-eqvt*:
shows $(p \cdot (X \times Y)) = (p \cdot X) \times (p \cdot Y)$
<proof>

In order to prove that *lfp* is equivariant we need two auxiliary classes which specify that (*<=>*) and *Inf* are equivariant. Instances for *bool* and *fun* are given.

class *le-eqvt* = *pt* +
assumes *le-eqvt* [*eqvt*]: $p \cdot (x \leq y) = ((p \cdot x) \leq (p \cdot (y :: 'a :: \{order, pt\})))$

class *inf-eqvt* = *pt* +
assumes *inf-eqvt* [*eqvt*]: $p \cdot (\text{Inf } X) = \text{Inf } (p \cdot (X :: 'a :: \{complete-lattice, pt\} \text{ set}))$

instantiation *bool* :: *le-eqvt*
begin

instance
<proof>

end

instantiation *fun* :: (*pt*, *le-eqvt*) *le-eqvt*
begin

instance
<proof>

end

instantiation *bool* :: *inf-eqvt*
begin

instance

<proof>

end

instantiation *fun* :: (*pt*, *inf-eqvt*) *inf-eqvt*
begin

instance

<proof>

end

lemma *lfp-eqvt* [*eqvt*]:

fixes $F :: ('a \Rightarrow 'b) \Rightarrow ('a :: pt \Rightarrow 'b :: \{inf-eqvt, le-eqvt\})$

shows $p \cdot (lfp\ F) = lfp\ (p \cdot F)$

<proof>

lemma *finite-eqvt* [*eqvt*]:

shows $p \cdot finite\ A = finite\ (p \cdot A)$

<proof>

lemma *fun-upd-eqvt* [*eqvt*]:

shows $p \cdot (f(x := y)) = (p \cdot f)((p \cdot x) := (p \cdot y))$

<proof>

lemma *comp-eqvt* [*eqvt*]:

shows $p \cdot (f \circ g) = (p \cdot f) \circ (p \cdot g)$

<proof>

6.3.4 Equivariance for product operations

lemma *fst-eqvt* [*eqvt*]:

shows $p \cdot (fst\ x) = fst\ (p \cdot x)$

<proof>

lemma *snd-eqvt* [*eqvt*]:

shows $p \cdot (snd\ x) = snd\ (p \cdot x)$

<proof>

lemma *split-eqvt* [*eqvt*]:

shows $p \cdot (case-prod\ P\ x) = case-prod\ (p \cdot P)\ (p \cdot x)$

<proof>

6.3.5 Equivariance for list operations

lemma *append-eqvt* [*eqvt*]:

shows $p \cdot (xs\ @\ ys) = (p \cdot xs)\ @\ (p \cdot ys)$

<proof>

lemma *rev-eqvt* [*eqvt*]:

shows $p \cdot (\text{rev } xs) = \text{rev } (p \cdot xs)$
<proof>

lemma *map-eqvt* [eqvt]:
shows $p \cdot (\text{map } f \text{ } xs) = \text{map } (p \cdot f) (p \cdot xs)$
<proof>

lemma *removeAll-eqvt* [eqvt]:
shows $p \cdot (\text{removeAll } x \text{ } xs) = \text{removeAll } (p \cdot x) (p \cdot xs)$
<proof>

lemma *filter-eqvt* [eqvt]:
shows $p \cdot (\text{filter } f \text{ } xs) = \text{filter } (p \cdot f) (p \cdot xs)$
<proof>

lemma *distinct-eqvt* [eqvt]:
shows $p \cdot (\text{distinct } xs) = \text{distinct } (p \cdot xs)$
<proof>

lemma *length-eqvt* [eqvt]:
shows $p \cdot (\text{length } xs) = \text{length } (p \cdot xs)$
<proof>

6.3.6 Equivariance for 'a option

lemma *map-option-eqvt*[eqvt]:
shows $p \cdot (\text{map-option } f \text{ } x) = \text{map-option } (p \cdot f) (p \cdot x)$
<proof>

6.3.7 Equivariance for 'a fset

context includes *fset.lifting* **begin**

lemma *in-fset-eqvt* [eqvt]:
shows $(p \cdot (x \mid \in \mid S)) = ((p \cdot x) \mid \in \mid (p \cdot S))$
<proof>

lemma *union-fset-eqvt* [eqvt]:
shows $(p \cdot (S \mid \cup \mid T)) = ((p \cdot S) \mid \cup \mid (p \cdot T))$
<proof>

lemma *inter-fset-eqvt* [eqvt]:
shows $(p \cdot (S \mid \cap \mid T)) = ((p \cdot S) \mid \cap \mid (p \cdot T))$
<proof>

lemma *subset-fset-eqvt* [eqvt]:
shows $(p \cdot (S \mid \subseteq \mid T)) = ((p \cdot S) \mid \subseteq \mid (p \cdot T))$
<proof>

lemma *map-fset-eqvt* [eqvt]:
shows $p \cdot (f \mid \uparrow \mid S) = (p \cdot f) \mid \uparrow \mid (p \cdot S)$

$\langle proof \rangle$
end

6.3.8 Equivariance for (\cdot, \cdot) *finfun*

lemma *finfun-update-eqv* [eqvt]:
shows $(p \cdot (\text{finfun-update } f \ a \ b)) = \text{finfun-update } (p \cdot f) \ (p \cdot a) \ (p \cdot b)$
 $\langle proof \rangle$

lemma *finfun-const-eqv* [eqvt]:
shows $(p \cdot (\text{finfun-const } b)) = \text{finfun-const } (p \cdot b)$
 $\langle proof \rangle$

lemma *finfun-apply-eqv* [eqvt]:
shows $(p \cdot (\text{finfun-apply } f \ b)) = \text{finfun-apply } (p \cdot f) \ (p \cdot b)$
 $\langle proof \rangle$

7 Supp, Freshness and Supports

context *pt*
begin

definition
supp :: $'a \Rightarrow \text{atom set}$
where
 $\text{supp } x = \{a. \text{infinite } \{b. (a \rightleftharpoons b) \cdot x \neq x\}\}$

definition
fresh :: $\text{atom} \Rightarrow 'a \Rightarrow \text{bool}$ (- $\#$ - [55, 55] 55)
where
 $a \# x \equiv a \notin \text{supp } x$

end

lemma *supp-conv-fresh*:
shows $\text{supp } x = \{a. \neg a \# x\}$
 $\langle proof \rangle$

lemma *swap-rel-trans*:
assumes *sort-of* $a = \text{sort-of } b$
assumes *sort-of* $b = \text{sort-of } c$
assumes $(a \rightleftharpoons c) \cdot x = x$
assumes $(b \rightleftharpoons c) \cdot x = x$
shows $(a \rightleftharpoons b) \cdot x = x$
 $\langle proof \rangle$

lemma *swap-fresh-fresh*:
assumes $a: a \# x$
and $b: b \# x$

shows $(a \rightleftharpoons b) \cdot x = x$
<proof>

7.1 supp and fresh are equivariant

lemma *supp-eqvt* [*eqvt*]:
shows $p \cdot (\text{supp } x) = \text{supp } (p \cdot x)$
<proof>

lemma *fresh-eqvt* [*eqvt*]:
shows $p \cdot (a \# x) = (p \cdot a) \# (p \cdot x)$
<proof>

lemma *fresh-permute-iff*:
shows $(p \cdot a) \# (p \cdot x) \longleftrightarrow a \# x$
<proof>

lemma *fresh-permute-left*:
shows $a \# p \cdot x \longleftrightarrow - p \cdot a \# x$
<proof>

8 supports

definition

supports :: *atom set* \Rightarrow *'a::pt* \Rightarrow *bool* (**infixl** *supports* 80)

where

$S \text{ supports } x \equiv \forall a b. (a \notin S \wedge b \notin S \longrightarrow (a \rightleftharpoons b) \cdot x = x)$

lemma *supp-is-subset*:
fixes $S :: \text{atom set}$
and $x :: \text{'a::pt}$
assumes $a1: S \text{ supports } x$
and $a2: \text{finite } S$
shows $(\text{supp } x) \subseteq S$
<proof>

lemma *supports-finite*:
fixes $S :: \text{atom set}$
and $x :: \text{'a::pt}$
assumes $a1: S \text{ supports } x$
and $a2: \text{finite } S$
shows $\text{finite } (\text{supp } x)$
<proof>

lemma *supp-supports*:
fixes $x :: \text{'a::pt}$
shows $(\text{supp } x) \text{ supports } x$
<proof>

lemma *supports-fresh*:
fixes $x :: 'a::pt$
assumes $a1: S \text{ supports } x$
and $a2: \text{finite } S$
and $a3: a \notin S$
shows $a \# x$
 $\langle \text{proof} \rangle$

lemma *supp-is-least-supports*:
fixes $S :: \text{atom set}$
and $x :: 'a::pt$
assumes $a1: S \text{ supports } x$
and $a2: \text{finite } S$
and $a3: \bigwedge S'. \text{finite } S' \implies (S' \text{ supports } x) \implies S \subseteq S'$
shows $(\text{supp } x) = S$
 $\langle \text{proof} \rangle$

lemma *subsetCI*:
shows $(\bigwedge x. x \in A \implies x \notin B \implies \text{False}) \implies A \subseteq B$
 $\langle \text{proof} \rangle$

lemma *finite-supp-unique*:
assumes $a1: S \text{ supports } x$
assumes $a2: \text{finite } S$
assumes $a3: \bigwedge a b. \llbracket a \in S; b \notin S; \text{sort-of } a = \text{sort-of } b \rrbracket \implies (a \rightleftharpoons b) \cdot x \neq x$
shows $(\text{supp } x) = S$
 $\langle \text{proof} \rangle$

9 Support w.r.t. relations

This definition is used for unquotient types, where alpha-equivalence does not coincide with equality.

definition
 $\text{supp-rel } R \ x = \{a. \text{infinite } \{b. \neg(R ((a \rightleftharpoons b) \cdot x) x)\}\}$

10 Finitely-supported types

class $fs = pt +$
assumes $\text{finite-supp}: \text{finite } (\text{supp } x)$

lemma *pure-supp*:
fixes $x::'a::\text{pure}$
shows $\text{supp } x = \{\}$
 $\langle \text{proof} \rangle$

lemma *pure-fresh*:
fixes $x::'a::\text{pure}$

shows $a \# x$
 $\langle proof \rangle$

instance $pure < fs$
 $\langle proof \rangle$

10.1 Type $atom$ is finitely-supported.

lemma $supp-atom$:
shows $supp\ a = \{a\}$
 $\langle proof \rangle$

lemma $fresh-atom$:
shows $a \# b \longleftrightarrow a \neq b$
 $\langle proof \rangle$

instance $atom :: fs$
 $\langle proof \rangle$

11 Type $perm$ is finitely-supported.

lemma $perm-swap-eq$:
shows $(a \rightleftharpoons b) \cdot p = p \longleftrightarrow (p \cdot (a \rightleftharpoons b)) = (a \rightleftharpoons b)$
 $\langle proof \rangle$

lemma $supports-perm$:
shows $\{a. p \cdot a \neq a\}$ supports p
 $\langle proof \rangle$

lemma $finite-perm-lemma$:
shows $finite\ \{a::atom. p \cdot a \neq a\}$
 $\langle proof \rangle$

lemma $supp-perm$:
shows $supp\ p = \{a. p \cdot a \neq a\}$
 $\langle proof \rangle$

lemma $fresh-perm$:
shows $a \# p \longleftrightarrow p \cdot a = a$
 $\langle proof \rangle$

lemma $supp-swap$:
shows $supp\ (a \rightleftharpoons b) = (if\ a = b \vee sort-of\ a \neq sort-of\ b\ then\ \{\}\ else\ \{a, b\})$
 $\langle proof \rangle$

lemma $fresh-swap$:
shows $a \# (b \rightleftharpoons c) \longleftrightarrow (sort-of\ b \neq sort-of\ c) \vee b = c \vee (a \# b \wedge a \# c)$
 $\langle proof \rangle$

lemma *fresh-zero-perm*:

shows $a \# (0::perm)$

$\langle proof \rangle$

lemma *supp-zero-perm*:

shows $supp (0::perm) = \{\}$

$\langle proof \rangle$

lemma *fresh-plus-perm*:

fixes $p q::perm$

assumes $a \# p \ a \# q$

shows $a \# (p + q)$

$\langle proof \rangle$

lemma *supp-plus-perm*:

fixes $p q::perm$

shows $supp (p + q) \subseteq supp p \cup supp q$

$\langle proof \rangle$

lemma *fresh-minus-perm*:

fixes $p::perm$

shows $a \# (- p) \longleftrightarrow a \# p$

$\langle proof \rangle$

lemma *supp-minus-perm*:

fixes $p::perm$

shows $supp (- p) = supp p$

$\langle proof \rangle$

lemma *plus-perm-eq*:

fixes $p q::perm$

assumes $asm: supp p \cap supp q = \{\}$

shows $p + q = q + p$

$\langle proof \rangle$

lemma *supp-plus-perm-eq*:

fixes $p q::perm$

assumes $asm: supp p \cap supp q = \{\}$

shows $supp (p + q) = supp p \cup supp q$

$\langle proof \rangle$

lemma *perm-eq-iff2*:

fixes $p q :: perm$

shows $p = q \longleftrightarrow (\forall a::atom \in supp p \cup supp q. p \cdot a = q \cdot a)$

$\langle proof \rangle$

instance *perm* :: *fs*

$\langle proof \rangle$

12 Finite Support instances for other types

12.1 Type $'a \times 'b$ is finitely-supported.

lemma *supp-Pair*:

shows $\text{supp } (x, y) = \text{supp } x \cup \text{supp } y$
<proof>

lemma *fresh-Pair*:

shows $a \# (x, y) \longleftrightarrow a \# x \wedge a \# y$
<proof>

lemma *supp-Unit*:

shows $\text{supp } () = \{\}$
<proof>

lemma *fresh-Unit*:

shows $a \# ()$
<proof>

instance *prod* :: $(fs, fs) fs$

<proof>

12.2 Type $'a + 'b$ is finitely supported

lemma *supp-Inl*:

shows $\text{supp } (\text{Inl } x) = \text{supp } x$
<proof>

lemma *supp-Inr*:

shows $\text{supp } (\text{Inr } x) = \text{supp } x$
<proof>

lemma *fresh-Inl*:

shows $a \# \text{Inl } x \longleftrightarrow a \# x$
<proof>

lemma *fresh-Inr*:

shows $a \# \text{Inr } y \longleftrightarrow a \# y$
<proof>

instance *sum* :: $(fs, fs) fs$

<proof>

12.3 Type $'a \text{ option}$ is finitely supported

lemma *supp-None*:

shows $\text{supp } \text{None} = \{\}$
<proof>

lemma *supp-Some*:
shows $\text{supp } (\text{Some } x) = \text{supp } x$
<proof>

lemma *fresh-None*:
shows $a \# \text{None}$
<proof>

lemma *fresh-Some*:
shows $a \# \text{Some } x \longleftrightarrow a \# x$
<proof>

instance *option* :: (*fs*) *fs*
<proof>

12.3.1 Type 'a list is finitely supported

lemma *supp-Nil*:
shows $\text{supp } [] = \{\}$
<proof>

lemma *fresh-Nil*:
shows $a \# []$
<proof>

lemma *supp-Cons*:
shows $\text{supp } (x \# xs) = \text{supp } x \cup \text{supp } xs$
<proof>

lemma *fresh-Cons*:
shows $a \# (x \# xs) \longleftrightarrow a \# x \wedge a \# xs$
<proof>

lemma *supp-append*:
shows $\text{supp } (xs @ ys) = \text{supp } xs \cup \text{supp } ys$
<proof>

lemma *fresh-append*:
shows $a \# (xs @ ys) \longleftrightarrow a \# xs \wedge a \# ys$
<proof>

lemma *supp-rev*:
shows $\text{supp } (\text{rev } xs) = \text{supp } xs$
<proof>

lemma *fresh-rev*:
shows $a \# \text{rev } xs \longleftrightarrow a \# xs$
<proof>

lemma *supp-removeAll*:
fixes $x::atom$
shows $supp (removeAll\ x\ xs) = supp\ xs - \{x\}$
 $\langle proof \rangle$

lemma *supp-of-atom-list*:
fixes $as::atom\ list$
shows $supp\ as = set\ as$
 $\langle proof \rangle$

instance $list :: (fs)\ fs$
 $\langle proof \rangle$

13 Support and Freshness for Applications

lemma *fresh-conv-MOST*:
shows $a \# x \longleftrightarrow (MOST\ b. (a \rightleftharpoons b) \cdot x = x)$
 $\langle proof \rangle$

lemma *fresh-fun-app*:
assumes $a \# f$ **and** $a \# x$
shows $a \# f\ x$
 $\langle proof \rangle$

lemma *supp-fun-app*:
shows $supp (f\ x) \subseteq (supp\ f) \cup (supp\ x)$
 $\langle proof \rangle$

13.1 Equivariance Predicate *eqvt* and *eqvt-at*

definition
 $eqvt\ f \equiv \forall p. p \cdot f = f$

lemma *eqvt-boolI*:
fixes $f::bool$
shows $eqvt\ f$
 $\langle proof \rangle$

equivariance of a function at a given argument

definition
 $eqvt-at\ f\ x \equiv \forall p. p \cdot (f\ x) = f (p \cdot x)$

lemma *eqvtI*:
shows $(\bigwedge p. p \cdot f \equiv f) \implies eqvt\ f$
 $\langle proof \rangle$

lemma *eqvt-at-perm*:
assumes $eqvt-at\ f\ x$
shows $eqvt-at\ f (q \cdot x)$

<proof>

lemma *supp-fun-eqvt*:
 assumes *a*: *eqvt f*
 shows $\text{supp } f = \{\}$
<proof>

lemma *fresh-fun-eqvt*:
 assumes *a*: *eqvt f*
 shows $a \# f$
<proof>

lemma *fresh-fun-eqvt-app*:
 assumes *a*: *eqvt f*
 shows $a \# x \implies a \# f x$
<proof>

lemma *supp-fun-app-eqvt*:
 assumes *a*: *eqvt f*
 shows $\text{supp } (f x) \subseteq \text{supp } x$
<proof>

lemma *supp-eqvt-at*:
 assumes *asm*: *eqvt-at f x*
 and *fin*: *finite (supp x)*
 shows $\text{supp } (f x) \subseteq \text{supp } x$
<proof>

lemma *finite-supp-eqvt-at*:
 assumes *asm*: *eqvt-at f x*
 and *fin*: *finite (supp x)*
 shows *finite (supp (f x))*
<proof>

lemma *fresh-eqvt-at*:
 assumes *asm*: *eqvt-at f x*
 and *fin*: *finite (supp x)*
 and *fresh*: $a \# x$
 shows $a \# f x$
<proof>

for handling of freshness of functions

<ML>

13.2 helper functions for *nominal-functions*

lemma *THE-defaultI2*:
 assumes $\exists!x. P x \wedge x. P x \implies Q x$
 shows $Q (\text{THE-default } d P)$
<proof>

lemma *the-default-eqvt*:
assumes *unique*: $\exists!x. P x$
shows $(p \cdot (\text{THE-default } d P)) = (\text{THE-default } (p \cdot d) (p \cdot P))$
 $\langle \text{proof} \rangle$

lemma *fundef-ex1-eqvt*:
fixes $x::'a::pt$
assumes *f-def*: $f == (\lambda x::'a. \text{THE-default } (d x) (G x))$
assumes *eqvt*: *eqvt* G
assumes *ex1*: $\exists!y. G x y$
shows $(p \cdot (f x)) = f (p \cdot x)$
 $\langle \text{proof} \rangle$

lemma *fundef-ex1-eqvt-at*:
fixes $x::'a::pt$
assumes *f-def*: $f == (\lambda x::'a. \text{THE-default } (d x) (G x))$
assumes *eqvt*: *eqvt* G
assumes *ex1*: $\exists!y. G x y$
shows *eqvt-at* $f x$
 $\langle \text{proof} \rangle$

lemma *fundef-ex1-prop*:
fixes $x::'a::pt$
assumes *f-def*: $f == (\lambda x::'a. \text{THE-default } (d x) (G x))$
assumes *P-all*: $\bigwedge x y. G x y \implies P x y$
assumes *ex1*: $\exists!y. G x y$
shows $P x (f x)$
 $\langle \text{proof} \rangle$

14 Support of Finite Sets of Finitely Supported Elements

support and freshness for atom sets

lemma *supp-finite-atom-set*:
fixes $S::\text{atom set}$
assumes *finite* S
shows *supp* $S = S$
 $\langle \text{proof} \rangle$

lemma *supp-cofinite-atom-set*:
fixes $S::\text{atom set}$
assumes *finite* $(UNIV - S)$
shows *supp* $S = (UNIV - S)$
 $\langle \text{proof} \rangle$

lemma *fresh-finite-atom-set*:
fixes $S::\text{atom set}$

assumes *finite S*
shows $a \# S \longleftrightarrow a \notin S$
<proof>

lemma *fresh-minus-atom-set*:
fixes $S::\text{atom set}$
assumes *finite S*
shows $a \# S - T \longleftrightarrow (a \notin T \longrightarrow a \# S)$
<proof>

lemma *Union-supports-set*:
shows $(\bigcup x \in S. \text{supp } x) \text{ supports } S$
<proof>

lemma *Union-of-finite-supp-sets*:
fixes $S::('a::\text{fs set})$
assumes $\text{fin}: \text{finite } S$
shows *finite* $(\bigcup x \in S. \text{supp } x)$
<proof>

lemma *Union-included-in-supp*:
fixes $S::('a::\text{fs set})$
assumes $\text{fin}: \text{finite } S$
shows $(\bigcup x \in S. \text{supp } x) \subseteq \text{supp } S$
<proof>

lemma *supp-of-finite-sets*:
fixes $S::('a::\text{fs set})$
assumes $\text{fin}: \text{finite } S$
shows $(\text{supp } S) = (\bigcup x \in S. \text{supp } x)$
<proof>

lemma *finite-sets-supp*:
fixes $S::('a::\text{fs set})$
assumes *finite S*
shows *finite* $(\text{supp } S)$
<proof>

lemma *supp-of-finite-union*:
fixes $S T::('a::\text{fs set})$
assumes $\text{fin1}: \text{finite } S$
and $\text{fin2}: \text{finite } T$
shows $\text{supp } (S \cup T) = \text{supp } S \cup \text{supp } T$
<proof>

lemma *fresh-finite-union*:
fixes $S T::('a::\text{fs set})$
assumes $\text{fin1}: \text{finite } S$
and $\text{fin2}: \text{finite } T$

shows $a \# (S \cup T) \longleftrightarrow a \# S \wedge a \# T$
<proof>

lemma *supp-of-finite-insert*:
fixes $S::('a::fs) \text{ set}$
assumes $fin: \text{finite } S$
shows $\text{supp } (\text{insert } x \ S) = \text{supp } x \cup \text{supp } S$
<proof>

lemma *fresh-finite-insert*:
fixes $S::('a::fs) \text{ set}$
assumes $fin: \text{finite } S$
shows $a \# (\text{insert } x \ S) \longleftrightarrow a \# x \wedge a \# S$
<proof>

lemma *supp-set-empty*:
shows $\text{supp } \{\} = \{\}$
<proof>

lemma *fresh-set-empty*:
shows $a \# \{\}$
<proof>

lemma *supp-set*:
fixes $xs :: ('a::fs) \text{ list}$
shows $\text{supp } (\text{set } xs) = \text{supp } xs$
<proof>

lemma *fresh-set*:
fixes $xs :: ('a::fs) \text{ list}$
shows $a \# (\text{set } xs) \longleftrightarrow a \# xs$
<proof>

14.1 Type 'a multiset is finitely supported

lemma *set-mset-eqvt* [eqvt]:
shows $p \cdot (\text{set-mset } M) = \text{set-mset } (p \cdot M)$
<proof>

lemma *supp-set-mset*:
shows $\text{supp } (\text{set-mset } M) \subseteq \text{supp } M$
<proof>

lemma *Union-finite-multiset*:
fixes $M::'a::fs \text{ multiset}$
shows $\text{finite } (\bigcup \{\text{supp } x \mid x. x \in\# M\})$
<proof>

lemma *Union-supports-multiset*:

shows $\bigcup \{ \text{supp } x \mid x. x \in \# M \}$ *supports* M
<proof>

lemma *Union-included-multiset*:
fixes $M :: ('a :: fs) \text{ multiset}$
shows $(\bigcup \{ \text{supp } x \mid x. x \in \# M \}) \subseteq \text{supp } M$
<proof>

lemma *supp-of-multisets*:
fixes $M :: ('a :: fs) \text{ multiset}$
shows $(\text{supp } M) = (\bigcup \{ \text{supp } x \mid x. x \in \# M \})$
<proof>

lemma *multisets-supp-finite*:
fixes $M :: ('a :: fs) \text{ multiset}$
shows *finite* $(\text{supp } M)$
<proof>

lemma *supp-of-multiset-union*:
fixes $M N :: ('a :: fs) \text{ multiset}$
shows $\text{supp } (M + N) = \text{supp } M \cup \text{supp } N$
<proof>

lemma *supp-empty-mset* [*simp*]:
shows $\text{supp } \{ \# \} = \{ \}$
<proof>

instance *multiset* :: $(fs) fs$
<proof>

14.2 Type $'a$ *fset* is finitely supported

lemma *supp-fset* [*simp*]:
shows $\text{supp } (\text{fset } S) = \text{supp } S$
<proof>

lemma *supp-empty-fset* [*simp*]:
shows $\text{supp } \{ \{ \} \} = \{ \}$
<proof>

lemma *fresh-empty-fset*:
shows $a \# \{ \{ \} \}$
<proof>

lemma *supp-finsert* [*simp*]:
fixes $x :: 'a :: fs$
and $S :: 'a \text{ fset}$
shows $\text{supp } (\text{finsert } x S) = \text{supp } x \cup \text{supp } S$
<proof>

lemma *fresh-finsert*:
fixes $x::'a::fs$
and $S::'a\ fset$
shows $a \# \text{finsert } x\ S \longleftrightarrow a \# x \wedge a \# S$
 $\langle \text{proof} \rangle$

lemma *fset-finite-supp*:
fixes $S::('a::fs)\ fset$
shows $\text{finite } (\text{supp } S)$
 $\langle \text{proof} \rangle$

lemma *supp-union-fset*:
fixes $S\ T::'a::fs\ fset$
shows $\text{supp } (S \mid\cup\mid T) = \text{supp } S \cup \text{supp } T$
 $\langle \text{proof} \rangle$

lemma *fresh-union-fset*:
fixes $S\ T::'a::fs\ fset$
shows $a \# S \mid\cup\mid T \longleftrightarrow a \# S \wedge a \# T$
 $\langle \text{proof} \rangle$

instance *fset* :: $(fs)\ fs$
 $\langle \text{proof} \rangle$

14.3 Type $('a, 'b)$ *finfun* is finitely supported

lemma *fresh-finfun-const*:
shows $a \# (\text{finfun-const } b) \longleftrightarrow a \# b$
 $\langle \text{proof} \rangle$

lemma *fresh-finfun-update*:
shows $\llbracket a \# f; a \# x; a \# y \rrbracket \implies a \# \text{finfun-update } f\ x\ y$
 $\langle \text{proof} \rangle$

lemma *supp-finfun-const*:
shows $\text{supp } (\text{finfun-const } b) = \text{supp}(b)$
 $\langle \text{proof} \rangle$

lemma *supp-finfun-update*:
shows $\text{supp } (\text{finfun-update } f\ x\ y) \subseteq \text{supp}(f, x, y)$
 $\langle \text{proof} \rangle$

instance *finfun* :: $(fs, fs)\ fs$
 $\langle \text{proof} \rangle$

15 Freshness and Fresh-Star

lemma *fresh-Unit-elim*:

shows $(a \# () \implies PROP C) \equiv PROP C$
 ⟨proof⟩

lemma *fresh-Pair-elim*:

shows $(a \# (x, y) \implies PROP C) \equiv (a \# x \implies a \# y \implies PROP C)$
 ⟨proof⟩

lemma [*simp*]:

shows $a \# x1 \implies a \# x2 \implies a \# (x1, x2)$
 ⟨proof⟩

lemma *fresh-PairD*:

shows $a \# (x, y) \implies a \# x$
and $a \# (x, y) \implies a \# y$
 ⟨proof⟩

⟨ML⟩

The fresh-star generalisation of fresh is used in strong induction principles.

definition

fresh-star :: *atom set* \Rightarrow 'a::pt \Rightarrow bool (- #* - [80,80] 80)

where

$as \#* x \equiv \forall a \in as. a \# x$

lemma *fresh-star-supp-conv*:

shows $supp x \#* y \implies supp y \#* x$
 ⟨proof⟩

lemma *fresh-star-perm-set-conv*:

fixes *p*::perm
assumes *fresh*: $as \#* p$
and *fn*: *finite as*
shows $supp p \#* as$
 ⟨proof⟩

lemma *fresh-star-atom-set-conv*:

assumes *fresh*: $as \#* bs$
and *fn*: *finite as finite bs*
shows $bs \#* as$
 ⟨proof⟩

lemma *atom-fresh-star-disjoint*:

assumes *fn*: *finite bs*
shows $as \#* bs \longleftrightarrow (as \cap bs = \{\})$

⟨proof⟩

lemma *fresh-star-Pair*:

shows $as \#* (x, y) = (as \#* x \wedge as \#* y)$
<proof>

lemma *fresh-star-list*:

shows $as \#* (xs @ ys) \longleftrightarrow as \#* xs \wedge as \#* ys$
and $as \#* (x \# xs) \longleftrightarrow as \#* x \wedge as \#* xs$
and $as \#* []$
<proof>

lemma *fresh-star-set*:

fixes $xs::('a::fs) list$
shows $as \#* set xs \longleftrightarrow as \#* xs$
<proof>

lemma *fresh-star-singleton*:

fixes $a::atom$
shows $as \#* \{a\} \longleftrightarrow as \#* a$
<proof>

lemma *fresh-star-fset*:

fixes $xs::('a::fs) list$
shows $as \#* fset S \longleftrightarrow as \#* S$
<proof>

lemma *fresh-star-Un*:

shows $(as \cup bs) \#* x = (as \#* x \wedge bs \#* x)$
<proof>

lemma *fresh-star-insert*:

shows $(insert a as) \#* x = (a \# x \wedge as \#* x)$
<proof>

lemma *fresh-star-Un-elim*:

$((as \cup bs) \#* x \Longrightarrow PROP C) \equiv (as \#* x \Longrightarrow bs \#* x \Longrightarrow PROP C)$
<proof>

lemma *fresh-star-insert-elim*:

$(insert a as \#* x \Longrightarrow PROP C) \equiv (a \# x \Longrightarrow as \#* x \Longrightarrow PROP C)$
<proof>

lemma *fresh-star-empty-elim*:

$(\{\} \#* x \Longrightarrow PROP C) \equiv PROP C$
<proof>

lemma *fresh-star-Unit-elim*:

shows $(a \#* ()) \Longrightarrow PROP C \equiv PROP C$

$\langle proof \rangle$

lemma *fresh-star-Pair-elim*:

shows $(a \#* (x, y) \implies PROP C) \equiv (a \#* x \implies a \#* y \implies PROP C)$
 $\langle proof \rangle$

lemma *fresh-star-zero*:

shows $as \#* (0::perm)$
 $\langle proof \rangle$

lemma *fresh-star-plus*:

fixes $p q::perm$
shows $\llbracket a \#* p; a \#* q \rrbracket \implies a \#* (p + q)$
 $\langle proof \rangle$

lemma *fresh-star-permute-iff*:

shows $(p \cdot a) \#* (p \cdot x) \longleftrightarrow a \#* x$
 $\langle proof \rangle$

lemma *fresh-star-eqvt* [*eqvt*]:

shows $p \cdot (as \#* x) \longleftrightarrow (p \cdot as) \#* (p \cdot x)$
 $\langle proof \rangle$

16 Induction principle for permutations

lemma *smaller-supp*:

assumes $a: a \in supp p$
shows $supp ((p \cdot a \rightleftharpoons a) + p) \subset supp p$
 $\langle proof \rangle$

lemma *perm-struct-induct*[*consumes 1, case-names zero swap*]:

assumes $S: supp p \subseteq S$
and *zero*: $P 0$
and *swap*: $\bigwedge p a b. \llbracket P p; supp p \subseteq S; a \in S; b \in S; a \neq b; sort-of a = sort-of b \rrbracket$
 $\implies P ((a \rightleftharpoons b) + p)$
shows $P p$
 $\langle proof \rangle$

lemma *perm-simple-struct-induct*[*case-names zero swap*]:

assumes *zero*: $P 0$
and *swap*: $\bigwedge p a b. \llbracket P p; a \neq b; sort-of a = sort-of b \rrbracket \implies P ((a \rightleftharpoons b) + p)$
shows $P p$
 $\langle proof \rangle$

lemma *perm-struct-induct2*[*consumes 1, case-names zero swap plus*]:

assumes $S: supp p \subseteq S$
assumes *zero*: $P 0$
assumes *swap*: $\bigwedge a b. \llbracket sort-of a = sort-of b; a \neq b; a \in S; b \in S \rrbracket \implies P (a \rightleftharpoons$

b)
assumes *plus*: $\bigwedge p1\ p2. \llbracket P\ p1; P\ p2; \text{supp}\ p1 \subseteq S; \text{supp}\ p2 \subseteq S \rrbracket \implies P\ (p1 + p2)$
shows $P\ p$
 $\langle \text{proof} \rangle$

lemma *perm-simple-struct-induct2*[*case-names zero swap plus*]:
assumes *zero*: $P\ 0$
assumes *swap*: $\bigwedge a\ b. \llbracket \text{sort-of}\ a = \text{sort-of}\ b; a \neq b \rrbracket \implies P\ (a \rightleftharpoons b)$
assumes *plus*: $\bigwedge p1\ p2. \llbracket P\ p1; P\ p2 \rrbracket \implies P\ (p1 + p2)$
shows $P\ p$
 $\langle \text{proof} \rangle$

lemma *supp-perm-singleton*:
fixes $p::\text{perm}$
shows $\text{supp}\ p \subseteq \{b\} \longleftrightarrow p = 0$
 $\langle \text{proof} \rangle$

lemma *supp-perm-pair*:
fixes $p::\text{perm}$
shows $\text{supp}\ p \subseteq \{a, b\} \longleftrightarrow p = 0 \vee p = (b \rightleftharpoons a)$
 $\langle \text{proof} \rangle$

lemma *supp-perm-eq*:
assumes $(\text{supp}\ x) \#* p$
shows $p \cdot x = x$
 $\langle \text{proof} \rangle$

same lemma as above, but proved with a different induction principle

lemma *supp-perm-eq-test*:
assumes $(\text{supp}\ x) \#* p$
shows $p \cdot x = x$
 $\langle \text{proof} \rangle$

lemma *perm-supp-eq*:
assumes $a: (\text{supp}\ p) \#* x$
shows $p \cdot x = x$
 $\langle \text{proof} \rangle$

lemma *supp-perm-perm-eq*:
assumes $a: \forall a \in \text{supp}\ x. p \cdot a = q \cdot a$
shows $p \cdot x = q \cdot x$
 $\langle \text{proof} \rangle$

disagreement set

definition
 $dset :: \text{perm} \Rightarrow \text{perm} \Rightarrow \text{atom}\ \text{set}$
where
 $dset\ p\ q = \{a::\text{atom}. p \cdot a \neq q \cdot a\}$

lemma *ds-fresh*:
assumes $dset\ p\ q\ \#* x$
shows $p \cdot x = q \cdot x$
 $\langle proof \rangle$

lemma *atom-set-perm-eq*:
assumes $a: as\ \#* p$
shows $p \cdot as = as$
 $\langle proof \rangle$

17 Avoiding of atom sets

For every set of atoms, there is another set of atoms avoiding a finitely supported c and there is a permutation which 'translates' between both sets.

lemma *at-set-avoiding-aux*:
fixes $Xs::atom\ set$
and $As::atom\ set$
assumes $b: Xs \subseteq As$
and $c: finite\ As$
shows $\exists p. (p \cdot Xs) \cap As = \{\} \wedge (supp\ p) = (Xs \cup (p \cdot Xs))$
 $\langle proof \rangle$

lemma *at-set-avoiding*:
assumes $a: finite\ Xs$
and $b: finite\ (supp\ c)$
obtains $p::perm$ **where** $(p \cdot Xs)\ \#* c$ **and** $(supp\ p) = (Xs \cup (p \cdot Xs))$
 $\langle proof \rangle$

lemma *at-set-avoiding1*:
assumes $finite\ xs$
and $finite\ (supp\ c)$
shows $\exists p. (p \cdot xs)\ \#* c$
 $\langle proof \rangle$

lemma *at-set-avoiding2*:
assumes $finite\ xs$
and $finite\ (supp\ c)\ finite\ (supp\ x)$
and $xs\ \#* x$
shows $\exists p. (p \cdot xs)\ \#* c \wedge supp\ x\ \#* p$
 $\langle proof \rangle$

lemma *at-set-avoiding3*:
assumes $finite\ xs$
and $finite\ (supp\ c)\ finite\ (supp\ x)$
and $xs\ \#* x$
shows $\exists p. (p \cdot xs)\ \#* c \wedge supp\ x\ \#* p \wedge supp\ p = xs \cup (p \cdot xs)$

$\langle proof \rangle$

lemma *at-set-avoiding2-atom*:

assumes *finite* (*supp* *c*) *finite* (*supp* *x*)

and $b: a \# x$

shows $\exists p. (p \cdot a) \# c \wedge \text{supp } x \#* p$

$\langle proof \rangle$

18 Renaming permutations

lemma *set-renaming-perm*:

assumes *b*: *finite* *bs*

shows $\exists q. (\forall b \in bs. q \cdot b = p \cdot b) \wedge \text{supp } q \subseteq bs \cup (p \cdot bs)$

$\langle proof \rangle$

lemma *set-renaming-perm2*:

shows $\exists q. (\forall b \in bs. q \cdot b = p \cdot b) \wedge \text{supp } q \subseteq bs \cup (p \cdot bs)$

$\langle proof \rangle$

lemma *list-renaming-perm*:

shows $\exists q. (\forall b \in \text{set } bs. q \cdot b = p \cdot b) \wedge \text{supp } q \subseteq \text{set } bs \cup (p \cdot \text{set } bs)$

$\langle proof \rangle$

19 Concrete Atoms Types

Class *at-base* allows types containing multiple sorts of atoms. Class *at* only allows types with a single sort.

class *at-base* = *pt* +

fixes *atom* :: 'a \Rightarrow *atom*

assumes *atom-eq-iff* [*simp*]: *atom* *a* = *atom* *b* \longleftrightarrow *a* = *b*

assumes *atom-eqvt*: $p \cdot (\text{atom } a) = \text{atom } (p \cdot a)$

declare *atom-eqvt* [*eqvt*]

class *at* = *at-base* +

assumes *sort-of-atom-eq* [*simp*]: *sort-of* (*atom* *a*) = *sort-of* (*atom* *b*)

lemma *sort-ineq* [*simp*]:

assumes *sort-of* (*atom* *a*) \neq *sort-of* (*atom* *b*)

shows *atom* *a* \neq *atom* *b*

$\langle proof \rangle$

lemma *supp-at-base*:

fixes *a*::'a::*at-base*

shows *supp* *a* = {*atom* *a*}

$\langle proof \rangle$

lemma *fresh-at-base*:

shows $\text{sort-of } a \neq \text{sort-of } (\text{atom } b) \implies a \# b$
and $a \# b \longleftrightarrow a \neq \text{atom } b$
 ⟨proof⟩

lemma *fresh-ineq-at-base* [simp]:
shows $a \neq \text{atom } b \implies a \# b$
 ⟨proof⟩

lemma *fresh-atom-at-base* [simp]:
fixes $b::'a::\text{at-base}$
shows $a \# \text{atom } b \longleftrightarrow a \# b$
 ⟨proof⟩

lemma *fresh-star-atom-at-base*:
fixes $b::'a::\text{at-base}$
shows $as \#* \text{atom } b \longleftrightarrow as \#* b$
 ⟨proof⟩

lemma *if-fresh-at-base* [simp]:
shows $\text{atom } a \# x \implies P (\text{if } a = x \text{ then } t \text{ else } s) = P s$
and $\text{atom } a \# x \implies P (\text{if } x = a \text{ then } t \text{ else } s) = P s$
 ⟨proof⟩

⟨ML⟩

instance *at-base* < *fs*
 ⟨proof⟩

lemma *at-base-infinite* [simp]:
shows $\text{infinite } (\text{UNIV} :: 'a::\text{at-base set}) \text{ (is infinite ?U)}$
 ⟨proof⟩

lemma *swap-at-base-simps* [simp]:
fixes $x y::'a::\text{at-base}$
shows $\text{sort-of } (\text{atom } x) = \text{sort-of } (\text{atom } y) \implies (\text{atom } x \rightleftharpoons \text{atom } y) \cdot x = y$
and $\text{sort-of } (\text{atom } x) = \text{sort-of } (\text{atom } y) \implies (\text{atom } x \rightleftharpoons \text{atom } y) \cdot y = x$
and $\text{atom } x \neq a \implies \text{atom } x \neq b \implies (a \rightleftharpoons b) \cdot x = x$
 ⟨proof⟩

lemma *obtain-at-base*:
assumes $X: \text{finite } X$
obtains $a::'a::\text{at-base}$ **where** $\text{atom } a \notin X$
 ⟨proof⟩

lemma *obtain-fresh'*:

assumes $fin: finite (supp\ x)$
obtains $a::'a::at-base$ **where** $atom\ a \# x$
 $\langle proof \rangle$

lemma *obtain-fresh*:
fixes $x::'b::fs$
obtains $a::'a::at-base$ **where** $atom\ a \# x$
 $\langle proof \rangle$

lemma *supp-finite-set-at-base*:
assumes $a: finite\ S$
shows $supp\ S = atom\ 'S$
 $\langle proof \rangle$

lemma *fresh-finite-set-at-base*:
fixes $a::'a::at-base$
assumes $a: finite\ S$
shows $atom\ a \# S \longleftrightarrow a \notin S$
 $\langle proof \rangle$

lemma *fresh-at-base-permute-iff* [*simp*]:
fixes $a::'a::at-base$
shows $atom\ (p \cdot a) \# p \cdot x \longleftrightarrow atom\ a \# x$
 $\langle proof \rangle$

lemma *fresh-at-base-permI*:
shows $atom\ a \# p \implies p \cdot a = a$
 $\langle proof \rangle$

20 Infrastructure for concrete atom types

definition
 $flip :: 'a::at-base \Rightarrow 'a \Rightarrow perm\ ('(- \leftrightarrow -))$
where
 $(a \leftrightarrow b) = (atom\ a \iff atom\ b)$

lemma *flip-fresh-fresh*:
assumes $atom\ a \# x\ atom\ b \# x$
shows $(a \leftrightarrow b) \cdot x = x$
 $\langle proof \rangle$

lemma *flip-self* [*simp*]: $(a \leftrightarrow a) = 0$
 $\langle proof \rangle$

lemma *flip-commute*: $(a \leftrightarrow b) = (b \leftrightarrow a)$
 $\langle proof \rangle$

lemma *minus-flip* [simp]: $-(a \leftrightarrow b) = (a \leftrightarrow b)$
<proof>

lemma *add-flip-cancel*: $(a \leftrightarrow b) + (a \leftrightarrow b) = 0$
<proof>

lemma *permute-flip-cancel* [simp]: $(a \leftrightarrow b) \cdot (a \leftrightarrow b) \cdot x = x$
<proof>

lemma *permute-flip-cancel2* [simp]: $(a \leftrightarrow b) \cdot (b \leftrightarrow a) \cdot x = x$
<proof>

lemma *flip-eqvt* [eqvt]:
shows $p \cdot (a \leftrightarrow b) = (p \cdot a \leftrightarrow p \cdot b)$
<proof>

lemma *flip-at-base-simps* [simp]:
shows $\text{sort-of } (atom\ a) = \text{sort-of } (atom\ b) \implies (a \leftrightarrow b) \cdot a = b$
and $\text{sort-of } (atom\ a) = \text{sort-of } (atom\ b) \implies (a \leftrightarrow b) \cdot b = a$
and $\llbracket a \neq c; b \neq c \rrbracket \implies (a \leftrightarrow b) \cdot c = c$
and $\text{sort-of } (atom\ a) \neq \text{sort-of } (atom\ b) \implies (a \leftrightarrow b) \cdot x = x$
<proof>

the following two lemmas do not hold for *at-base*, only for single sort atoms from *at*

lemma *flip-triple*:
fixes $a\ b\ c :: 'a :: at$
assumes $a \neq b$ **and** $c \neq b$
shows $(a \leftrightarrow c) + (b \leftrightarrow c) + (a \leftrightarrow c) = (a \leftrightarrow b)$
<proof>

lemma *permute-flip-at*:
fixes $a\ b\ c :: 'a :: at$
shows $(a \leftrightarrow b) \cdot c = (\text{if } c = a \text{ then } b \text{ else if } c = b \text{ then } a \text{ else } c)$
<proof>

lemma *flip-at-simps* [simp]:
fixes $a\ b :: 'a :: at$
shows $(a \leftrightarrow b) \cdot a = b$
and $(a \leftrightarrow b) \cdot b = a$
<proof>

20.1 Syntax for coercing at-elements to the atom-type

syntax
-atom-constrain :: $logic \Rightarrow type \Rightarrow logic$ ($-\dots-$ [4, 0] 3)

translations
-atom-constrain $a\ t \Rightarrow CONST\ atom\ (-constrain\ a\ t)$

20.2 A lemma for proving instances of class *at*.

<ML>

New atom types are defined as subtypes of *atom*.

lemma *exists-eq-simple-sort*:

shows $\exists a. a \in \{a. \text{sort-of } a = s\}$

<proof>

lemma *exists-eq-sort*:

shows $\exists a. a \in \{a. \text{sort-of } a \in \text{range sort-fun}\}$

<proof>

lemma *at-base-class*:

fixes *sort-fun* :: $'b \Rightarrow \text{atom-sort}$

fixes *Rep* :: $'a \Rightarrow \text{atom}$ **and** *Abs* :: $\text{atom} \Rightarrow 'a$

assumes *type*: *type-definition Rep Abs* $\{a. \text{sort-of } a \in \text{range sort-fun}\}$

assumes *atom-def*: $\bigwedge a. \text{atom } a = \text{Rep } a$

assumes *permute-def*: $\bigwedge p a. p \cdot a = \text{Abs } (p \cdot \text{Rep } a)$

shows *OFCLASS*('a, *at-base-class*)

<proof>

lemma *at-class*:

fixes *s* :: *atom-sort*

fixes *Rep* :: $'a \Rightarrow \text{atom}$ **and** *Abs* :: $\text{atom} \Rightarrow 'a$

assumes *type*: *type-definition Rep Abs* $\{a. \text{sort-of } a = s\}$

assumes *atom-def*: $\bigwedge a. \text{atom } a = \text{Rep } a$

assumes *permute-def*: $\bigwedge p a. p \cdot a = \text{Abs } (p \cdot \text{Rep } a)$

shows *OFCLASS*('a, *at-class*)

<proof>

lemma *at-class-sort*:

fixes *s* :: *atom-sort*

fixes *Rep* :: $'a \Rightarrow \text{atom}$ **and** *Abs* :: $\text{atom} \Rightarrow 'a$

fixes *a*:: $'a$

assumes *type*: *type-definition Rep Abs* $\{a. \text{sort-of } a = s\}$

assumes *atom-def*: $\bigwedge a. \text{atom } a = \text{Rep } a$

shows *sort-of* (*atom a*) = *s*

<proof>

<ML>

21 Library functions for the nominal infrastructure

<ML>

22 The freshness lemma according to Andy Pitts

lemma *freshness-lemma*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \exists a. \text{atom } a \# (h, h \ a)$
shows $\exists x. \forall a. \text{atom } a \# h \longrightarrow h \ a = x$
 $\langle \text{proof} \rangle$

lemma *freshness-lemma-unique*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \exists a. \text{atom } a \# (h, h \ a)$
shows $\exists! x. \forall a. \text{atom } a \# h \longrightarrow h \ a = x$
 $\langle \text{proof} \rangle$

packaging the freshness lemma into a function

definition

$\text{Fresh} :: ('a::at \Rightarrow 'b::pt) \Rightarrow 'b$

where

$\text{Fresh } h = (\text{THE } x. \forall a. \text{atom } a \# h \longrightarrow h \ a = x)$

lemma *Fresh-apply*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \exists a. \text{atom } a \# (h, h \ a)$
assumes $b: \text{atom } a \# h$
shows $\text{Fresh } h = h \ a$
 $\langle \text{proof} \rangle$

lemma *Fresh-apply'*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \text{atom } a \# h \ \text{atom } a \# h \ a$
shows $\text{Fresh } h = h \ a$
 $\langle \text{proof} \rangle$

$\langle \text{ML} \rangle$

lemma *Fresh-eqvt*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \exists a. \text{atom } a \# (h, h \ a)$
shows $p \cdot (\text{Fresh } h) = \text{Fresh } (p \cdot h)$
 $\langle \text{proof} \rangle$

lemma *Fresh-supports*:

fixes $h :: 'a::at \Rightarrow 'b::pt$
assumes $a: \exists a. \text{atom } a \# (h, h \ a)$
shows $(\text{supp } h) \ \text{supports } (\text{Fresh } h)$
 $\langle \text{proof} \rangle$

notation Fresh (**binder** *FRESH* 10)


```

lemma FRESH-f-iff:
  fixes  $P :: 'a::at \Rightarrow 'b::pure$ 
  fixes  $f :: 'b \Rightarrow 'c::pure$ 
  assumes  $P: finite (supp P)$ 
  shows  $(FRESH\ x.\ f\ (P\ x)) = f\ (FRESH\ x.\ P\ x)$ 
   $\langle proof \rangle$ 

```

```

lemma FRESH-binop-iff:
  fixes  $P :: 'a::at \Rightarrow 'b::pure$ 
  fixes  $Q :: 'a::at \Rightarrow 'c::pure$ 
  fixes  $binop :: 'b \Rightarrow 'c \Rightarrow 'd::pure$ 
  assumes  $P: finite (supp P)$ 
  and  $Q: finite (supp Q)$ 
  shows  $(FRESH\ x.\ binop\ (P\ x)\ (Q\ x)) = binop\ (FRESH\ x.\ P\ x)\ (FRESH\ x.\ Q\ x)$ 
   $\langle proof \rangle$ 

```

```

lemma FRESH-conj-iff:
  fixes  $P\ Q :: 'a::at \Rightarrow bool$ 
  assumes  $P: finite (supp P)$  and  $Q: finite (supp Q)$ 
  shows  $(FRESH\ x.\ P\ x \wedge Q\ x) \longleftrightarrow (FRESH\ x.\ P\ x) \wedge (FRESH\ x.\ Q\ x)$ 
   $\langle proof \rangle$ 

```

```

lemma FRESH-disj-iff:
  fixes  $P\ Q :: 'a::at \Rightarrow bool$ 
  assumes  $P: finite (supp P)$  and  $Q: finite (supp Q)$ 
  shows  $(FRESH\ x.\ P\ x \vee Q\ x) \longleftrightarrow (FRESH\ x.\ P\ x) \vee (FRESH\ x.\ Q\ x)$ 
   $\langle proof \rangle$ 

```

23 Automation for creating concrete atom types

At the moment only single-sort concrete atoms are supported.

$\langle ML \rangle$

24 Automatic equivariance procedure for inductive definitions

$\langle ML \rangle$

```

end
theory Nominal2-Abs
imports Nominal2-Base
         HOL-Library.Quotient-List
         HOL-Library.Quotient-Product
begin

```

25 Abstractions

fun

alpha-set

where

alpha-set[*simp del*]:

alpha-set (*bs*, *x*) *R f p* (*cs*, *y*) \longleftrightarrow

$f\ x - bs = f\ y - cs \wedge$

$(f\ x - bs) \#* p \wedge$

$R\ (p \cdot x)\ y \wedge$

$p \cdot bs = cs$

fun

alpha-res

where

alpha-res[*simp del*]:

alpha-res (*bs*, *x*) *R f p* (*cs*, *y*) \longleftrightarrow

$f\ x - bs = f\ y - cs \wedge$

$(f\ x - bs) \#* p \wedge$

$R\ (p \cdot x)\ y$

fun

alpha-lst

where

alpha-lst[*simp del*]:

alpha-lst (*bs*, *x*) *R f p* (*cs*, *y*) \longleftrightarrow

$f\ x - set\ bs = f\ y - set\ cs \wedge$

$(f\ x - set\ bs) \#* p \wedge$

$R\ (p \cdot x)\ y \wedge$

$p \cdot bs = cs$

lemmas *alphas* = *alpha-set.simps alpha-res.simps alpha-lst.simps*

notation

alpha-set ($- \approx_{set} - - - - [100, 100, 100, 100, 100] 100$) **and**

alpha-res ($- \approx_{res} - - - - [100, 100, 100, 100, 100] 100$) **and**

alpha-lst ($- \approx_{lst} - - - - [100, 100, 100, 100, 100] 100$)

26 Mono

lemma [*mono*]:

shows $R1 \leq R2 \implies \text{alpha-set } bs\ R1 \leq \text{alpha-set } bs\ R2$

and $R1 \leq R2 \implies \text{alpha-res } bs\ R1 \leq \text{alpha-res } bs\ R2$

and $R1 \leq R2 \implies \text{alpha-lst } cs\ R1 \leq \text{alpha-lst } cs\ R2$

<proof>

27 Equivariance

lemma *alpha-eqvt*[*eqvt*]:

shows $(bs, x) \approx_{set} R f q (cs, y) \implies (p \cdot bs, p \cdot x) \approx_{set} (p \cdot R) (p \cdot f) (p \cdot q)$
 $(p \cdot cs, p \cdot y)$
and $(bs, x) \approx_{res} R f q (cs, y) \implies (p \cdot bs, p \cdot x) \approx_{res} (p \cdot R) (p \cdot f) (p \cdot q)$
 $(p \cdot cs, p \cdot y)$
and $(ds, x) \approx_{lst} R f q (es, y) \implies (p \cdot ds, p \cdot x) \approx_{lst} (p \cdot R) (p \cdot f) (p \cdot q) (p$
 $\cdot es, p \cdot y)$
<proof>

28 Equivalence

lemma *alpha-refl*:

assumes $a: R x x$
shows $(bs, x) \approx_{set} R f 0 (bs, x)$
and $(bs, x) \approx_{res} R f 0 (bs, x)$
and $(cs, x) \approx_{lst} R f 0 (cs, x)$
<proof>

lemma *alpha-sym*:

assumes $a: R (p \cdot x) y \implies R (- p \cdot y) x$
shows $(bs, x) \approx_{set} R f p (cs, y) \implies (cs, y) \approx_{set} R f (- p) (bs, x)$
and $(bs, x) \approx_{res} R f p (cs, y) \implies (cs, y) \approx_{res} R f (- p) (bs, x)$
and $(ds, x) \approx_{lst} R f p (es, y) \implies (es, y) \approx_{lst} R f (- p) (ds, x)$
<proof>

lemma *alpha-trans*:

assumes $a: \llbracket R (p \cdot x) y; R (q \cdot y) z \rrbracket \implies R ((q + p) \cdot x) z$
shows $\llbracket (bs, x) \approx_{set} R f p (cs, y); (cs, y) \approx_{set} R f q (ds, z) \rrbracket \implies (bs, x) \approx_{set} R$
 $f (q + p) (ds, z)$
and $\llbracket (bs, x) \approx_{res} R f p (cs, y); (cs, y) \approx_{res} R f q (ds, z) \rrbracket \implies (bs, x) \approx_{res} R$
 $f (q + p) (ds, z)$
and $\llbracket (es, x) \approx_{lst} R f p (gs, y); (gs, y) \approx_{lst} R f q (hs, z) \rrbracket \implies (es, x) \approx_{lst} R f$
 $(q + p) (hs, z)$
<proof>

lemma *alpha-sym-eqvt*:

assumes $a: R (p \cdot x) y \implies R y (p \cdot x)$
and $b: p \cdot R = R$
shows $(bs, x) \approx_{set} R f p (cs, y) \implies (cs, y) \approx_{set} R f (- p) (bs, x)$
and $(bs, x) \approx_{res} R f p (cs, y) \implies (cs, y) \approx_{res} R f (- p) (bs, x)$
and $(ds, x) \approx_{lst} R f p (es, y) \implies (es, y) \approx_{lst} R f (- p) (ds, x)$
<proof>

lemma *alpha-set-trans-eqvt*:

assumes $b: (cs, y) \approx_{set} R f q (ds, z)$
and $a: (bs, x) \approx_{set} R f p (cs, y)$
and $d: q \cdot R = R$

and $c: \llbracket R (p \cdot x) y; R y (- q \cdot z) \rrbracket \Longrightarrow R (p \cdot x) (- q \cdot z)$
shows $(bs, x) \approx_{set} R f (q + p) (ds, z)$
 $\langle proof \rangle$

lemma *alpha-res-trans-eqv*:

assumes $b: (cs, y) \approx_{res} R f q (ds, z)$
and $a: (bs, x) \approx_{res} R f p (cs, y)$
and $d: q \cdot R = R$
and $c: \llbracket R (p \cdot x) y; R y (- q \cdot z) \rrbracket \Longrightarrow R (p \cdot x) (- q \cdot z)$
shows $(bs, x) \approx_{res} R f (q + p) (ds, z)$
 $\langle proof \rangle$

lemma *alpha-lst-trans-eqv*:

assumes $b: (cs, y) \approx_{lst} R f q (ds, z)$
and $a: (bs, x) \approx_{lst} R f p (cs, y)$
and $d: q \cdot R = R$
and $c: \llbracket R (p \cdot x) y; R y (- q \cdot z) \rrbracket \Longrightarrow R (p \cdot x) (- q \cdot z)$
shows $(bs, x) \approx_{lst} R f (q + p) (ds, z)$
 $\langle proof \rangle$

lemmas *alpha-trans-eqv = alpha-set-trans-eqv alpha-res-trans-eqv alpha-lst-trans-eqv*

29 General Abstractions

fun

alpha-abs-set

where

$[simp\ del]:$

$alpha-abs-set (bs, x) (cs, y) \longleftrightarrow (\exists p. (bs, x) \approx_{set} ((=))\ supp\ p (cs, y))$

fun

alpha-abs-lst

where

$[simp\ del]:$

$alpha-abs-lst (bs, x) (cs, y) \longleftrightarrow (\exists p. (bs, x) \approx_{lst} ((=))\ supp\ p (cs, y))$

fun

alpha-abs-res

where

$[simp\ del]:$

$alpha-abs-res (bs, x) (cs, y) \longleftrightarrow (\exists p. (bs, x) \approx_{res} ((=))\ supp\ p (cs, y))$

notation

alpha-abs-set (**infix** $\approx_{abs'-set}$ 50) **and**

alpha-abs-lst (**infix** $\approx_{abs'-lst}$ 50) **and**

alpha-abs-res (**infix** $\approx_{abs'-res}$ 50)

lemmas *alphas-abs = alpha-abs-set.simps alpha-abs-res.simps alpha-abs-lst.simps*

lemma *alphas-abs-refl*:
shows $(bs, x) \approx_{abs-set} (bs, x)$
and $(bs, x) \approx_{abs-res} (bs, x)$
and $(cs, x) \approx_{abs-lst} (cs, x)$
 $\langle proof \rangle$

lemma *alphas-abs-sym*:
shows $(bs, x) \approx_{abs-set} (cs, y) \implies (cs, y) \approx_{abs-set} (bs, x)$
and $(bs, x) \approx_{abs-res} (cs, y) \implies (cs, y) \approx_{abs-res} (bs, x)$
and $(ds, x) \approx_{abs-lst} (es, y) \implies (es, y) \approx_{abs-lst} (ds, x)$
 $\langle proof \rangle$

lemma *alphas-abs-trans*:
shows $\llbracket (bs, x) \approx_{abs-set} (cs, y); (cs, y) \approx_{abs-set} (ds, z) \rrbracket \implies (bs, x) \approx_{abs-set} (ds, z)$
and $\llbracket (bs, x) \approx_{abs-res} (cs, y); (cs, y) \approx_{abs-res} (ds, z) \rrbracket \implies (bs, x) \approx_{abs-res} (ds, z)$
and $\llbracket (es, x) \approx_{abs-lst} (gs, y); (gs, y) \approx_{abs-lst} (hs, z) \rrbracket \implies (es, x) \approx_{abs-lst} (hs, z)$
 $\langle proof \rangle$

lemma *alphas-abs-evt*:
shows $(bs, x) \approx_{abs-set} (cs, y) \implies (p \cdot bs, p \cdot x) \approx_{abs-set} (p \cdot cs, p \cdot y)$
and $(bs, x) \approx_{abs-res} (cs, y) \implies (p \cdot bs, p \cdot x) \approx_{abs-res} (p \cdot cs, p \cdot y)$
and $(ds, x) \approx_{abs-lst} (es, y) \implies (p \cdot ds, p \cdot x) \approx_{abs-lst} (p \cdot es, p \cdot y)$
 $\langle proof \rangle$

30 Strengthening the equivalence

lemma *disjoint-right-eq*:
assumes $a: A \cup B1 = A \cup B2$
and $b: A \cap B1 = \{\} \ A \cap B2 = \{\}$
shows $B1 = B2$
 $\langle proof \rangle$

lemma *supp-property-res*:
assumes $a: (as, x) \approx_{res} (=) \text{supp } p (as', x')$
shows $p \cdot (\text{supp } x \cap as) = \text{supp } x' \cap as'$
 $\langle proof \rangle$

lemma *alpha-abs-res-stronger1-aux*:
assumes $asm: (as, x) \approx_{res} (=) \text{supp } p' (as', x')$
shows $\exists p. (as, x) \approx_{res} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq (\text{supp } x \cap as) \cup (\text{supp } x' \cap as')$
 $\langle proof \rangle$

lemma *alpha-abs-res-minimal*:
assumes $asm: (as, x) \approx_{res} (=) \text{supp } p (as', x')$

shows $(as \cap \text{supp } x, x) \approx_{\text{res}} (=) \text{supp } p (as' \cap \text{supp } x', x')$
 ⟨proof⟩

lemma *alpha-abs-res-abs-set*:

assumes *asm*: $(as, x) \approx_{\text{res}} (=) \text{supp } p (as', x')$
shows $(as \cap \text{supp } x, x) \approx_{\text{set}} (=) \text{supp } p (as' \cap \text{supp } x', x')$
 ⟨proof⟩

lemma *alpha-abs-set-abs-res*:

assumes *asm*: $(as \cap \text{supp } x, x) \approx_{\text{set}} (=) \text{supp } p (as' \cap \text{supp } x', x')$
shows $(as, x) \approx_{\text{res}} (=) \text{supp } p (as', x')$
 ⟨proof⟩

lemma *alpha-abs-res-stronger1*:

assumes *asm*: $(as, x) \approx_{\text{res}} (=) \text{supp } p' (as', x')$
shows $\exists p. (as, x) \approx_{\text{res}} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq as \cup as'$
 ⟨proof⟩

lemma *alpha-abs-set-stronger1*:

assumes *asm*: $(as, x) \approx_{\text{set}} (=) \text{supp } p' (as', x')$
shows $\exists p. (as, x) \approx_{\text{set}} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq as \cup as'$
 ⟨proof⟩

lemma *alpha-abs-lst-stronger1*:

assumes *asm*: $(as, x) \approx_{\text{lst}} (=) \text{supp } p' (as', x')$
shows $\exists p. (as, x) \approx_{\text{lst}} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq \text{set } as \cup \text{set } as'$
 ⟨proof⟩

lemma *alphas-abs-stronger*:

shows $(as, x) \approx_{\text{abs-set}} (as', x') \iff (\exists p. (as, x) \approx_{\text{set}} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq as \cup as')$
and $(as, x) \approx_{\text{abs-res}} (as', x') \iff (\exists p. (as, x) \approx_{\text{res}} (=) \text{supp } p (as', x') \wedge \text{supp } p \subseteq as \cup as')$
and $(bs, x) \approx_{\text{abs-lst}} (bs', x') \iff (\exists p. (bs, x) \approx_{\text{lst}} (=) \text{supp } p (bs', x') \wedge \text{supp } p \subseteq \text{set } bs \cup \text{set } bs')$
 ⟨proof⟩

lemma *alpha-res-alpha-set*:

$(bs, x) \approx_{\text{res}} (=) \text{supp } p (cs, y) \iff (bs \cap \text{supp } x, x) \approx_{\text{set}} (=) \text{supp } p (cs \cap \text{supp } y, y)$
 ⟨proof⟩

31 Quotient types

quotient-type

'a abs-set = $(\text{atom set} \times 'a::\text{pt}) / \text{alpha-abs-set}$
 ⟨proof⟩

quotient-type

'b abs-res = (atom set × 'b::pt) / alpha-abs-res
 ⟨proof⟩

quotient-type

'c abs-lst = (atom list × 'c::pt) / alpha-abs-lst
 ⟨proof⟩

quotient-definition

Abs-set ([-]set. - [60, 60] 60)

where

Abs-set::atom set ⇒ ('a::pt) ⇒ 'a abs-set

is

Pair::atom set ⇒ ('a::pt) ⇒ (atom set × 'a) ⟨proof⟩

quotient-definition

Abs-res ([-]res. - [60, 60] 60)

where

Abs-res::atom set ⇒ ('a::pt) ⇒ 'a abs-res

is

Pair::atom set ⇒ ('a::pt) ⇒ (atom set × 'a) ⟨proof⟩

quotient-definition

Abs-lst ([-]lst. - [60, 60] 60)

where

Abs-lst::atom list ⇒ ('a::pt) ⇒ 'a abs-lst

is

Pair::atom list ⇒ ('a::pt) ⇒ (atom list × 'a) ⟨proof⟩

lemma [quot-respect]:

shows ((=) ==> (=) ==> alpha-abs-set) Pair Pair

and ((=) ==> (=) ==> alpha-abs-res) Pair Pair

and ((=) ==> (=) ==> alpha-abs-lst) Pair Pair

⟨proof⟩

lemma [quot-respect]:

shows ((=) ==> alpha-abs-set ==> alpha-abs-set) permute permute

and ((=) ==> alpha-abs-res ==> alpha-abs-res) permute permute

and ((=) ==> alpha-abs-lst ==> alpha-abs-lst) permute permute

⟨proof⟩

lemma Abs-eq-iff:

shows [bs]set. x = [bs']set. y ⟷ (∃ p. (bs, x) ≈set (=) supp p (bs', y))

and [bs]res. x = [bs']res. y ⟷ (∃ p. (bs, x) ≈res (=) supp p (bs', y))

and [cs]lst. x = [cs']lst. y ⟷ (∃ p. (cs, x) ≈lst (=) supp p (cs', y))

⟨proof⟩

lemma Abs-eq-iff2:

shows [bs]set. x = [bs']set. y ⟷ (∃ p. (bs, x) ≈set ((=)) supp p (bs', y) ∧ supp p ⊆ bs ∪ bs')

and $[bs]res. x = [bs']res. y \longleftrightarrow (\exists p. (bs, x) \approx_{res} ((=)) supp p (bs', y) \wedge supp p \subseteq bs \cup bs')$
and $[cs]lst. x = [cs']lst. y \longleftrightarrow (\exists p. (cs, x) \approx_{lst} ((=)) supp p (cs', y) \wedge supp p \subseteq set cs \cup set cs')$
 $\langle proof \rangle$

lemma *Abs-eq-res-set*:

shows $[bs]res. x = [cs]res. y \longleftrightarrow [bs \cap supp x]set. x = [cs \cap supp y]set. y$
 $\langle proof \rangle$

lemma *Abs-eq-res-supp*:

assumes *asm*: $supp x \subseteq bs$
shows $[as]res. x = [as \cap bs]res. x$
 $\langle proof \rangle$

lemma *Abs-exhausts*[*cases type*]:

shows $(\bigwedge as (x::'a::pt). y1 = [as]set. x \implies P1) \implies P1$
and $(\bigwedge as (x::'a::pt). y2 = [as]res. x \implies P2) \implies P2$
and $(\bigwedge bs (x::'a::pt). y3 = [bs]lst. x \implies P3) \implies P3$
 $\langle proof \rangle$

instantiation *abs-set* :: (*pt*) *pt*

begin

quotient-definition

permute-abs-set::*perm* $\Rightarrow ('a::pt \text{ abs-set}) \Rightarrow 'a \text{ abs-set}$

is

permute:: *perm* $\Rightarrow (atom \text{ set} \times 'a::pt) \Rightarrow (atom \text{ set} \times 'a::pt)$
 $\langle proof \rangle$

lemma *permute-Abs-set*[*simp*]:

fixes *x*::'*a*::*pt*
shows $(p \cdot ([as]set. x)) = [p \cdot as]set. (p \cdot x)$
 $\langle proof \rangle$

instance

$\langle proof \rangle$

end

instantiation *abs-res* :: (*pt*) *pt*

begin

quotient-definition

permute-abs-res::*perm* $\Rightarrow ('a::pt \text{ abs-res}) \Rightarrow 'a \text{ abs-res}$

is

permute:: *perm* $\Rightarrow (atom \text{ set} \times 'a::pt) \Rightarrow (atom \text{ set} \times 'a::pt)$
 $\langle proof \rangle$

lemma *permute-Abs-res[simp]*:

fixes $x::'a::pt$

shows $(p \cdot ([as]res. x)) = [p \cdot as]res. (p \cdot x)$

<proof>

instance

<proof>

end

instantiation *abs-lst* :: (pt) *pt*

begin

quotient-definition

permute-abs-lst::perm \Rightarrow $('a::pt \text{ abs-lst}) \Rightarrow 'a \text{ abs-lst}$

is

permute::perm \Rightarrow $(atom \text{ list} \times 'a::pt) \Rightarrow (atom \text{ list} \times 'a::pt)$

<proof>

lemma *permute-Abs-lst[simp]*:

fixes $x::'a::pt$

shows $(p \cdot ([as]lst. x)) = [p \cdot as]lst. (p \cdot x)$

<proof>

instance

<proof>

end

lemmas *permute-Abs[eqvt]* = *permute-Abs-set permute-Abs-res permute-Abs-lst*

lemma *Abs-swap1*:

assumes $a1: a \notin (supp\ x) - bs$

and $a2: b \notin (supp\ x) - bs$

shows $[bs]set. x = [(a \rightleftharpoons b) \cdot bs]set. ((a \rightleftharpoons b) \cdot x)$

and $[bs]res. x = [(a \rightleftharpoons b) \cdot bs]res. ((a \rightleftharpoons b) \cdot x)$

<proof>

lemma *Abs-swap2*:

assumes $a1: a \notin (supp\ x) - (set\ bs)$

and $a2: b \notin (supp\ x) - (set\ bs)$

shows $[bs]lst. x = [(a \rightleftharpoons b) \cdot bs]lst. ((a \rightleftharpoons b) \cdot x)$

<proof>

lemma *Abs-supports*:

shows $((supp\ x) - as) \text{ supports } ([as]set. x)$

and $((supp\ x) - as) \text{ supports } ([as]res. x)$

and $((\text{supp } x) - \text{set } bs) \text{ supports } ([bs]lst. x)$
 $\langle \text{proof} \rangle$

function

$\text{supp-set} :: ('a::pt) \text{ abs-set} \Rightarrow \text{atom set}$ **and**
 $\text{supp-res} :: ('a::pt) \text{ abs-res} \Rightarrow \text{atom set}$ **and**
 $\text{supp-lst} :: ('a::pt) \text{ abs-lst} \Rightarrow \text{atom set}$

where

$\text{supp-set } ([as]set. x) = \text{supp } x - as$
 $| \text{supp-res } ([as]res. x) = \text{supp } x - as$
 $| \text{supp-lst } (Abs-lst cs x) = (\text{supp } x) - (\text{set } cs)$
 $\langle \text{proof} \rangle$

termination

$\langle \text{proof} \rangle$

lemma *supp-funs-eqvt*[*eqvt*]:

shows $(p \cdot \text{supp-set } x) = \text{supp-set } (p \cdot x)$
and $(p \cdot \text{supp-res } y) = \text{supp-res } (p \cdot y)$
and $(p \cdot \text{supp-lst } z) = \text{supp-lst } (p \cdot z)$
 $\langle \text{proof} \rangle$

lemma *Abs-fresh-aux*:

shows $a \# [bs]set. x \Longrightarrow a \# \text{supp-set } ([bs]set. x)$
and $a \# [bs]res. x \Longrightarrow a \# \text{supp-res } ([bs]res. x)$
and $a \# [cs]lst. x \Longrightarrow a \# \text{supp-lst } ([cs]lst. x)$
 $\langle \text{proof} \rangle$

lemma *Abs-supp-subset1*:

assumes $a: \text{finite } (\text{supp } x)$
shows $(\text{supp } x) - as \subseteq \text{supp } ([as]set. x)$
and $(\text{supp } x) - as \subseteq \text{supp } ([as]res. x)$
and $(\text{supp } x) - (\text{set } bs) \subseteq \text{supp } ([bs]lst. x)$
 $\langle \text{proof} \rangle$

lemma *Abs-supp-subset2*:

assumes $a: \text{finite } (\text{supp } x)$
shows $\text{supp } ([as]set. x) \subseteq (\text{supp } x) - as$
and $\text{supp } ([as]res. x) \subseteq (\text{supp } x) - as$
and $\text{supp } ([bs]lst. x) \subseteq (\text{supp } x) - (\text{set } bs)$
 $\langle \text{proof} \rangle$

lemma *Abs-finite-supp*:

assumes $a: \text{finite } (\text{supp } x)$
shows $\text{supp } ([as]set. x) = (\text{supp } x) - as$
and $\text{supp } ([as]res. x) = (\text{supp } x) - as$
and $\text{supp } ([bs]lst. x) = (\text{supp } x) - (\text{set } bs)$
 $\langle \text{proof} \rangle$

lemma *supp-Abs*:

fixes $x::'a::fs$

shows $supp ([as]set. x) = (supp x) - as$

and $supp ([as]res. x) = (supp x) - as$

and $supp ([bs]lst. x) = (supp x) - (set bs)$

<proof>

instance *abs-set* :: $(fs) fs$

<proof>

instance *abs-res* :: $(fs) fs$

<proof>

instance *abs-lst* :: $(fs) fs$

<proof>

lemma *Abs-fresh-iff*:

fixes $x::'a::fs$

shows $a \# [bs]set. x \longleftrightarrow a \in bs \vee (a \notin bs \wedge a \# x)$

and $a \# [bs]res. x \longleftrightarrow a \in bs \vee (a \notin bs \wedge a \# x)$

and $a \# [cs]lst. x \longleftrightarrow a \in (set cs) \vee (a \notin (set cs) \wedge a \# x)$

<proof>

lemma *Abs-fresh-star-iff*:

fixes $x::'a::fs$

shows $as \#* ([bs]set. x) \longleftrightarrow (as - bs) \#* x$

and $as \#* ([bs]res. x) \longleftrightarrow (as - bs) \#* x$

and $as \#* ([cs]lst. x) \longleftrightarrow (as - set cs) \#* x$

<proof>

lemma *Abs-fresh-star*:

fixes $x::'a::fs$

shows $as \subseteq as' \Longrightarrow as \#* ([as']set. x)$

and $as \subseteq as' \Longrightarrow as \#* ([as']res. x)$

and $bs \subseteq set bs' \Longrightarrow bs \#* ([bs']lst. x)$

<proof>

lemma *Abs-fresh-star2*:

fixes $x::'a::fs$

shows $as \cap bs = \{\} \Longrightarrow as \#* ([bs]set. x) \longleftrightarrow as \#* x$

and $as \cap bs = \{\} \Longrightarrow as \#* ([bs]res. x) \longleftrightarrow as \#* x$

and $cs \cap set ds = \{\} \Longrightarrow cs \#* ([ds]lst. x) \longleftrightarrow cs \#* x$

<proof>

32 Abstractions of single atoms

lemma *Abs1-eq*:

fixes $x y::'a::fs$

shows $[\{atom a\}]set. x = [\{atom a\}]set. y \longleftrightarrow x = y$

and $[\{atom\ a\}]res. x = [\{atom\ a\}]res. y \longleftrightarrow x = y$
and $[[atom\ a]]lst. x = [[atom\ a]]lst. y \longleftrightarrow x = y$
 $\langle proof \rangle$

lemma *Abs1-eq-iff-fresh*:

fixes $x\ y::'a::fs$
and $a\ b\ c::'b::at$
assumes $atom\ c \# (a, b, x, y)$
shows $[\{atom\ a\}]set. x = [\{atom\ b\}]set. y \longleftrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y$
and $[\{atom\ a\}]res. x = [\{atom\ b\}]res. y \longleftrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y$
and $[[atom\ a]]lst. x = [[atom\ b]]lst. y \longleftrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y$
 $\langle proof \rangle$

lemma *Abs1-eq-iff-all*:

fixes $x\ y::'a::fs$
and $z::'c::fs$
and $a\ b::'b::at$
shows $[\{atom\ a\}]set. x = [\{atom\ b\}]set. y \longleftrightarrow (\forall c. atom\ c \# z \longrightarrow atom\ c \# (a, b, x, y) \longrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y)$
and $[\{atom\ a\}]res. x = [\{atom\ b\}]res. y \longleftrightarrow (\forall c. atom\ c \# z \longrightarrow atom\ c \# (a, b, x, y) \longrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y)$
and $[[atom\ a]]lst. x = [[atom\ b]]lst. y \longleftrightarrow (\forall c. atom\ c \# z \longrightarrow atom\ c \# (a, b, x, y) \longrightarrow (a \leftrightarrow c) \cdot x = (b \leftrightarrow c) \cdot y)$
 $\langle proof \rangle$

lemma *Abs1-eq-iff*:

fixes $x\ y::'a::fs$
and $a\ b::'b::at$
shows $[\{atom\ a\}]set. x = [\{atom\ b\}]set. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge x = (a \leftrightarrow b) \cdot y \wedge atom\ a \# y)$
and $[\{atom\ a\}]res. x = [\{atom\ b\}]res. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge x = (a \leftrightarrow b) \cdot y \wedge atom\ a \# y)$
and $[[atom\ a]]lst. x = [[atom\ b]]lst. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge x = (a \leftrightarrow b) \cdot y \wedge atom\ a \# y)$
 $\langle proof \rangle$

lemma *Abs1-eq-iff'*:

fixes $x::'a::fs$
and $a\ b::'b::at$
shows $[\{atom\ a\}]set. x = [\{atom\ b\}]set. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge (b \leftrightarrow a) \cdot x = y \wedge atom\ b \# x)$
and $[\{atom\ a\}]res. x = [\{atom\ b\}]res. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge (b \leftrightarrow a) \cdot x = y \wedge atom\ b \# x)$
and $[[atom\ a]]lst. x = [[atom\ b]]lst. y \longleftrightarrow (a = b \wedge x = y) \vee (a \neq b \wedge (b \leftrightarrow a) \cdot x = y \wedge atom\ b \# x)$
 $\langle proof \rangle$

$\langle ML \rangle$

32.1 Renaming of bodies of abstractions

lemma *Abs-rename-set*:

fixes $x::'a::fs$

assumes $a: (p \cdot bs) \#* x$

shows $\exists q. [bs]set. x = [p \cdot bs]set. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

lemma *Abs-rename-res*:

fixes $x::'a::fs$

assumes $a: (p \cdot bs) \#* x$

shows $\exists q. [bs]res. x = [p \cdot bs]res. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

lemma *Abs-rename-lst*:

fixes $x::'a::fs$

assumes $a: (p \cdot (set\ bs)) \#* x$

shows $\exists q. [bs]lst. x = [p \cdot bs]lst. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

for deep recursive binders

lemma *Abs-rename-set'*:

fixes $x::'a::fs$

assumes $a: (p \cdot bs) \#* x$

shows $\exists q. [bs]set. x = [q \cdot bs]set. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

lemma *Abs-rename-res'*:

fixes $x::'a::fs$

assumes $a: (p \cdot bs) \#* x$

shows $\exists q. [bs]res. x = [q \cdot bs]res. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

lemma *Abs-rename-lst'*:

fixes $x::'a::fs$

assumes $a: (p \cdot (set\ bs)) \#* x$

shows $\exists q. [bs]lst. x = [q \cdot bs]lst. (q \cdot x) \wedge q \cdot bs = p \cdot bs$
 $\langle proof \rangle$

33 Infrastructure for building tuples of relations and functions

fun

$prod-fv :: ('a \Rightarrow atom\ set) \Rightarrow ('b \Rightarrow atom\ set) \Rightarrow ('a \times 'b) \Rightarrow atom\ set$

where

$prod-fv\ fv1\ fv2\ (x, y) = fv1\ x \cup fv2\ y$

definition

$prod-alpha :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \times 'b \Rightarrow 'a \times 'b \Rightarrow bool)$

where

$prod-alpha = rel-prod$

lemma [quot-respect]:

shows $((R1 \implies (=)) \implies (R2 \implies (=)) \implies rel-prod\ R1\ R2 \implies (=))\ prod-fv\ prod-fv$
 $\langle proof \rangle$

lemma [quot-preserve]:

assumes $q1: Quotient3\ R1\ abs1\ rep1$

and $q2: Quotient3\ R2\ abs2\ rep2$

shows $((abs1 \dashrightarrow id) \dashrightarrow (abs2 \dashrightarrow id) \dashrightarrow map-prod\ rep1\ rep2 \dashrightarrow id)\ prod-fv = prod-fv$
 $\langle proof \rangle$

lemma [mono]:

shows $A \leq B \implies C \leq D \implies prod-alpha\ A\ C \leq prod-alpha\ B\ D$

$\langle proof \rangle$

lemma [eqvt]:

shows $p \cdot prod-alpha\ A\ B\ x\ y = prod-alpha\ (p \cdot A)\ (p \cdot B)\ (p \cdot x)\ (p \cdot y)$

$\langle proof \rangle$

lemma [eqvt]:

shows $p \cdot prod-fv\ A\ B\ (x, y) = prod-fv\ (p \cdot A)\ (p \cdot B)\ (p \cdot x, p \cdot y)$

$\langle proof \rangle$

lemma *prod-fv-supp*:

shows $prod-fv\ supp\ supp = supp$

$\langle proof \rangle$

lemma *prod-alpha-eq*:

shows $prod-alpha\ ((=))\ ((=)) = ((=))$

$\langle proof \rangle$

end

theory *Nominal2-FCB*

imports *Nominal2-Abs*

begin

A tactic which solves all trivial cases in function definitions, and leaves the others unchanged.

$\langle ML \rangle$

lemma *Abs-lst1-fcb*:

fixes $x\ y :: 'a :: at$

and $S\ T :: 'b :: fs$

assumes $e: [[atom\ x]]lst.\ T = [[atom\ y]]lst.\ S$

and $f1: \llbracket x \neq y; atom\ y \# T; atom\ x \# (y \leftrightarrow x) \cdot T \rrbracket \Longrightarrow atom\ x \# f\ x\ T$

and $f2: \llbracket x \neq y; atom\ y \# T; atom\ x \# (y \leftrightarrow x) \cdot T \rrbracket \Longrightarrow atom\ y \# f\ x\ T$

and $p: \llbracket S = (x \leftrightarrow y) \cdot T; x \neq y; atom\ y \# T; atom\ x \# S \rrbracket$

$\Longrightarrow (x \leftrightarrow y) \cdot (f\ x\ T) = f\ y\ S$

shows $f\ x\ T = f\ y\ S$

<proof>

lemma *Abs-lst-fcb*:

fixes $xs\ ys :: 'a :: fs$

and $S\ T :: 'b :: fs$

assumes $e: (Abs-lst\ (ba\ xs)\ T) = (Abs-lst\ (ba\ ys)\ S)$

and $f1: \bigwedge x. x \in set\ (ba\ xs) \Longrightarrow x \# f\ xs\ T$

and $f2: \bigwedge x. \llbracket supp\ T - set\ (ba\ xs) = supp\ S - set\ (ba\ ys); x \in set\ (ba\ ys) \rrbracket$
 $\Longrightarrow x \# f\ xs\ T$

and $eqv: \bigwedge p. \llbracket p \cdot T = S; p \cdot ba\ xs = ba\ ys; supp\ p \subseteq set\ (ba\ xs) \cup set\ (ba\ ys) \rrbracket$

$\Longrightarrow p \cdot (f\ xs\ T) = f\ ys\ S$

shows $f\ xs\ T = f\ ys\ S$

<proof>

lemma *Abs-set-fcb*:

fixes $xs\ ys :: 'a :: fs$

and $S\ T :: 'b :: fs$

assumes $e: (Abs-set\ (ba\ xs)\ T) = (Abs-set\ (ba\ ys)\ S)$

and $f1: \bigwedge x. x \in ba\ xs \Longrightarrow x \# f\ xs\ T$

and $f2: \bigwedge x. \llbracket supp\ T - ba\ xs = supp\ S - ba\ ys; x \in ba\ ys \rrbracket \Longrightarrow x \# f\ xs\ T$

and $eqv: \bigwedge p. \llbracket p \cdot T = S; p \cdot ba\ xs = ba\ ys; supp\ p \subseteq ba\ xs \cup ba\ ys \rrbracket \Longrightarrow p \cdot (f\ xs\ T) = f\ ys\ S$

shows $f\ xs\ T = f\ ys\ S$

<proof>

lemma *Abs-res-fcb*:

fixes $xs\ ys :: ('a :: at-base)\ set$

and $S\ T :: 'b :: fs$

assumes $e: (Abs-res\ (atom\ 'xs)\ T) = (Abs-res\ (atom\ 'ys)\ S)$

and $f1: \bigwedge x. x \in atom\ 'xs \Longrightarrow x \in supp\ T \Longrightarrow x \# f\ xs\ T$

and $f2: \bigwedge x. \llbracket supp\ T - atom\ 'xs = supp\ S - atom\ 'ys; x \in atom\ 'ys; x \in supp\ S \rrbracket \Longrightarrow x \# f\ xs\ T$

and $eqv: \bigwedge p. \llbracket p \cdot T = S; supp\ p \subseteq atom\ 'xs \cap supp\ T \cup atom\ 'ys \cap supp\ S;$

$p \cdot (atom\ 'xs \cap supp\ T) = atom\ 'ys \cap supp\ S \rrbracket \Longrightarrow p \cdot (f\ xs\ T) = f\ ys\ S$

shows $f\ xs\ T = f\ ys\ S$

<proof>

lemma *Abs-set-fcb2*:

fixes *as bs* :: *atom set*

and *x y* :: '*b* :: *fs*

and *c*::'*c*::*fs*

assumes *eq*: [*as*]set. *x* = [*bs*]set. *y*

and *fin*: *finite as finite bs*

and *fcb1*: *as* #* *f as x c*

and *fresh1*: *as* #* *c*

and *fresh2*: *bs* #* *c*

and *perm1*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f \text{ as } x c) = f (p \cdot \text{as}) (p \cdot x) c$

and *perm2*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f \text{ bs } y c) = f (p \cdot \text{bs}) (p \cdot y) c$

shows *f as x c* = *f bs y c*

<proof>

lemma *Abs-res-fcb2*:

fixes *as bs* :: *atom set*

and *x y* :: '*b* :: *fs*

and *c*::'*c*::*fs*

assumes *eq*: [*as*]res. *x* = [*bs*]res. *y*

and *fin*: *finite as finite bs*

and *fcb1*: (*as* ∩ *supp x*) #* *f (as* ∩ *supp x) x c*

and *fresh1*: *as* #* *c*

and *fresh2*: *bs* #* *c*

and *perm1*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f (as \cap \text{supp } x) x c) = f (p \cdot (as \cap \text{supp } x)) (p \cdot x) c$

and *perm2*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f (bs \cap \text{supp } y) y c) = f (p \cdot (bs \cap \text{supp } y)) (p \cdot y) c$

shows *f (as* ∩ *supp x) x c* = *f (bs* ∩ *supp y) y c*

<proof>

lemma *Abs-lst-fcb2*:

fixes *as bs* :: *atom list*

and *x y* :: '*b* :: *fs*

and *c*::'*c*::*fs*

assumes *eq*: [*as*]lst. *x* = [*bs*]lst. *y*

and *fcb1*: (*set as*) #* *f as x c*

and *fresh1*: *set as* #* *c*

and *fresh2*: *set bs* #* *c*

and *perm1*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f \text{ as } x c) = f (p \cdot \text{as}) (p \cdot x) c$

and *perm2*: $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f \text{ bs } y c) = f (p \cdot \text{bs}) (p \cdot y) c$

shows *f as x c* = *f bs y c*

<proof>

lemma *Abs-lst1-fcb2*:

fixes *a b* :: *atom*

and *x y* :: '*b* :: *fs*

and *c*::'*c* :: *fs*

assumes *e*: [[*a*]]lst. *x* = [[*b*]]lst. *y*


```

and fc1:  $a \# f a x c$ 
and fresh:  $\{a, b\} \#* c$ 
and perm1:  $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f a x c) = f (p \cdot a) (p \cdot x) c$ 
and perm2:  $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f b y c) = f (p \cdot b) (p \cdot y) c$ 
shows  $f a x c = f b y c$ 
<proof>

```

```

lemma Abs-lst1-fcb2':
fixes  $a b :: 'a::\text{at-base}$ 
and  $x y :: 'b :: fs$ 
and  $c::'c :: fs$ 
assumes  $e: [[\text{atom } a]]\text{lst. } x = [[\text{atom } b]]\text{lst. } y$ 
and fc1:  $\text{atom } a \# f a x c$ 
and fresh:  $\{\text{atom } a, \text{atom } b\} \#* c$ 
and perm1:  $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f a x c) = f (p \cdot a) (p \cdot x) c$ 
and perm2:  $\bigwedge p. \text{supp } p \#* c \implies p \cdot (f b y c) = f (p \cdot b) (p \cdot y) c$ 
shows  $f a x c = f b y c$ 
<proof>

```

```

end
theory Nominal2
imports
  Nominal2-Base Nominal2-Abs Nominal2-FCB
keywords
  nominal-datatype :: thy-defn and
  nominal-function nominal-inductive nominal-termination :: thy-goal-defn and
  avoids binds
begin

```

<ML>

34 Interface for *nominal-datatype*

<ML>

Infrastructure for adding *-raw* to types and terms

<ML>

35 Preparing and parsing of the specification

<ML>

associates every *SOME* with the index in the list; drops *NONEs*

<ML>

adds an empty binding clause for every argument that is not already part of a binding clause

<ML>

end

theory *Atoms*
imports *Nominal2-Base*
begin

36 *nat-of* is an example of a function without finite support

lemma *not-fresh-nat-of*:
 shows $\neg a \# \text{nat-of}$
 $\langle \text{proof} \rangle$

lemma *supp-nat-of*:
 shows $\text{supp nat-of} = \text{UNIV}$
 $\langle \text{proof} \rangle$

37 Manual instantiation of class *at*.

typedef *name* = {*a*. *sort-of a* = *Sort "name" []*}
 $\langle \text{proof} \rangle$

instantiation *name* :: *at*
begin

definition
 $p \cdot a = \text{Abs-name } (p \cdot \text{Rep-name } a)$

definition
 $\text{atom } a = \text{Rep-name } a$

instance
 $\langle \text{proof} \rangle$

end

lemma *sort-of-atom-name*:
 shows $\text{sort-of } (\text{atom } (a::\text{name})) = \text{Sort "name" []}$
 $\langle \text{proof} \rangle$

 Custom syntax for concrete atoms of type *at*

term *a::name*

38 Automatic instantiation of class *at*.

atom-decl *name2*

lemma
 $sort-of (atom (a::name2)) \neq sort-of (atom (b::name))$
 $\langle proof \rangle$

example swappings

lemma
fixes $a b::atom$
assumes $sort-of a = sort-of b$
shows $(a \rightleftharpoons b) \cdot (a, b) = (b, a)$
 $\langle proof \rangle$

lemma
fixes $a b::name2$
shows $(a \leftrightarrow b) \cdot (a, b) = (b, a)$
 $\langle proof \rangle$

39 An example for multiple-sort atoms

datatype $ty =$
 $TVar\ string$
 $| Fun\ ty\ ty\ (- \rightarrow -)$

primrec
 $sort-of-ty::ty \Rightarrow atom-sort$
where
 $sort-of-ty (TVar\ s) = Sort\ "TVar"\ [Sort\ s\ []]$
 $| sort-of-ty (Fun\ ty1\ ty2) = Sort\ "Fun"\ [sort-of-ty\ ty1,\ sort-of-ty\ ty2]$

lemma $sort-of-ty-eq-iff:$
shows $sort-of-ty\ x = sort-of-ty\ y \longleftrightarrow x = y$
 $\langle proof \rangle$

declare $sort-of-ty.simps\ [simp\ del]$

typedef $var = \{a.\ sort-of\ a \in range\ sort-of-ty\}$
 $\langle proof \rangle$

instantiation $var :: at-base$

begin

definition
 $p \cdot a = Abs-var\ (p \cdot Rep-var\ a)$

definition
 $atom\ a = Rep-var\ a$

instance
 $\langle proof \rangle$

end

Constructor for variables.

definition

$Var :: nat \Rightarrow ty \Rightarrow var$

where

$Var\ x\ t = Abs-var\ (Atom\ (sort-of-ty\ t)\ x)$

lemma *Var-eq-iff* [simp]:

shows $Var\ x\ s = Var\ y\ t \iff x = y \wedge s = t$
<proof>

lemma *sort-of-atom-var* [simp]:

$sort-of\ (atom\ (Var\ n\ ty)) = sort-of-ty\ ty$
<proof>

lemma

assumes $\alpha \neq \beta$

shows $(Var\ x\ \alpha \leftrightarrow Var\ y\ \alpha) \cdot (Var\ x\ \alpha, Var\ x\ \beta) = (Var\ y\ \alpha, Var\ x\ \beta)$
<proof>

Projecting out the type component of a variable.

definition

$ty-of :: var \Rightarrow ty$

where

$ty-of\ x = inv\ sort-of-ty\ (sort-of\ (atom\ x))$

Functions *Var/ty-of* satisfy many of the same properties as *Atom/sort-of*.

lemma *ty-of-Var* [simp]:

shows $ty-of\ (Var\ x\ t) = t$
<proof>

lemma *ty-of-permute* [simp]:

shows $ty-of\ (p \cdot x) = ty-of\ x$
<proof>

40 Tests with subtyping and automatic coercions

declare [[*coercion-enabled*]]

atom-decl *var1*

atom-decl *var2*

declare [[*coercion atom::var1* \Rightarrow *atom*]]

declare [[*coercion atom::var2* \Rightarrow *atom*]]

lemma

```

fixes a::var1 and b::var2
shows  $a \# t \wedge b \# t$ 
⟨proof⟩

```

```

lemma
fixes as::var1 set
shows  $atom \ 'as \ \#* \ t$ 

⟨proof⟩

```

end

```

theory Eqvt
imports Nominal2-Base
begin

```

```

declare [[trace-eqvt = false]]

```

```

lemma
fixes B::'a::pt
shows  $p \cdot (B = C)$ 
⟨proof⟩

```

```

lemma
fixes B::bool
shows  $p \cdot (B = C)$ 
⟨proof⟩

```

```

lemma
fixes B::bool
shows  $p \cdot (A \longrightarrow B = C)$ 
⟨proof⟩

```

```

lemma
shows  $p \cdot (\lambda(x::'a::pt). A \longrightarrow (B::'a \Rightarrow bool) \ x = C) = foo$ 
⟨proof⟩

```

```

lemma
shows  $p \cdot (\lambda(B::bool. A \longrightarrow (B = C)) = foo$ 
⟨proof⟩

```

```

lemma
shows  $p \cdot (\lambda x y. \exists z. x = z \wedge x = y \longrightarrow z \neq x) = foo$ 
⟨proof⟩

```

```

lemma
  shows  $p \cdot (\lambda f x. f (g (f x))) = foo$ 
  <proof>

lemma
  fixes  $p q :: perm$ 
  and  $x :: 'a :: pt$ 
  shows  $p \cdot (q \cdot x) = foo$ 
  <proof>

lemma
  fixes  $p q r :: perm$ 
  and  $x :: 'a :: pt$ 
  shows  $p \cdot (q \cdot r \cdot x) = foo$ 
  <proof>

lemma
  fixes  $p r :: perm$ 
  shows  $p \cdot (\lambda q :: perm. q \cdot (r \cdot x)) = foo$ 
  <proof>

lemma
  fixes  $C D :: bool$ 
  shows  $B (p \cdot (C = D))$ 
  <proof>

declare [[trace-eqvt = false]]

  there is no raw eqvt-rule for The

lemma  $p \cdot (THE x. P x) = foo$ 
  <proof>

lemma
  fixes  $P :: ('b \Rightarrow bool) \Rightarrow ('b :: pt) \Rightarrow ('a :: pt)$ 
  shows  $p \cdot (P The) = foo$ 
  <proof>

lemma
  fixes  $P :: ('a :: pt) \Rightarrow ('b :: pt) \Rightarrow bool$ 
  shows  $p \cdot (\lambda(a, b). P a b) = (\lambda(a, b). (p \cdot P) a b)$ 
  <proof>

thm eqvts
thm eqvts-raw

  <ML>

```

end