Interval Temporal Logic on Natural Numbers

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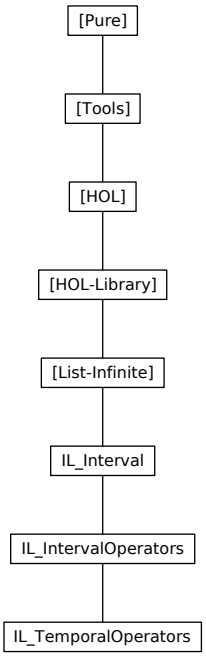
September 13, 2023

Abstract

We introduce a theory of temporal logic operators using sets of natural numbers as time domain, formalized in a shallow embedding manner. The theory comprises special natural intervals (theory IL_Interval: open and closed intervals, continuous and modulo intervals, interval traversing results), operators for shifting intervals to left/right on the number axis as well as expanding/contracting intervals by constant factors (theory IL_IntervalOperators.thy), and ultimately definitions and results for unary and binary temporal operators on arbitrary natural sets (theory IL_TemporalOperators).

Contents


[Pure]
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IL_Interval
IL_IntervalOperators
IL_TemporalOperators

1 Intervals and operations for temporal logic declarations

theory *IL-Interval*
imports
List−Infinite.InfiniteSet2
List−Infinite.SetIntervalStep
begin

1.1 Time intervals – definitions and basic lemmata

1.1.1 Definitions

type-synonym *Time* = *nat*

type-synonym *iT* = *Time set*

Infinite interval starting at some natural *n*.

definition
iFROM :: *Time* ⇒ *iT* ([-...])
where
[*n*...] ≡ {*n*..}

Finite interval starting at $0::'a$ and ending at some natural *n*.

definition
iTILL :: *Time* ⇒ *iT* ([...-])
where
[...*n*] ≡ {..*n*}

Finite bounded interval containing the naturals between *n* and *n* + *d*. *d* denotes the difference between left and right interval bound. The number of elements is $d + (1::'a)$ so that an empty interval cannot be defined.

definition
iIN :: *Time* ⇒ *nat* ⇒ *iT* ([-...,-])
where
[*n*...,*d*] ≡ {*n*..*n*+*d*}

Infinite modulo interval containing all naturals having the same division remainder modulo *m* as *r*, and beginning at *n*.

definition
iMOD :: *Time* ⇒ *nat* ⇒ *iT* ([-, *mod* -])
where
[*r*, *mod m*] ≡ { *x*. *x mod m* = *r mod m* ∧ *r* ≤ *x*}

Finite bounded modulo interval containing all naturals having the same division remainder modulo *m* as *r*, beginning at *n*, and ending after *c* cycles

at $r + m * c$. The number of elements is $c + (1::'a)$ so that an empty interval cannot be defined.

definition
iMODb :: *Time* ⇒ *nat* ⇒ *nat* ⇒ *iT* ([-, *mod* -, -])
where
$[r, \mathit{mod}\ m, c] \equiv \{ x.\ x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c\}$

1.1.2 Membership in an interval

lemmas *iT-defs* = *iFROM-def iTILL-def iIN-def iMOD-def iMODb-def*

lemma *iFROM-iff*: $x \in [n...] = (n \leq x)$
⟨*proof*⟩
lemma *iTILL-iff*: $x \in [...n] = (x \leq n)$
⟨*proof*⟩
lemma *iIN-iff*:$x \in [n...,d] = (n \leq x \wedge x \leq n + d)$
⟨*proof*⟩
lemma *iMOD-iff*: $x \in [r, \mathit{mod}\ m] = (x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x)$
⟨*proof*⟩
lemma *iMODb-iff*: $x \in [r, \mathit{mod}\ m, c] =$
$(x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c)$
⟨*proof*⟩

lemma *iFROM-D*: $x \in [n...] \Longrightarrow (n \leq x)$
⟨*proof*⟩
lemma *iTILL-D*: $x \in [...n] \Longrightarrow (x \leq n)$
⟨*proof*⟩
corollary *iIN-geD*: $x \in [n...,d] \Longrightarrow n \leq x$
⟨*proof*⟩
corollary *iIN-leD*: $x \in [n...,d] \Longrightarrow x \leq n + d$
⟨*proof*⟩
corollary *iMOD-modD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
⟨*proof*⟩
corollary *iMOD-geD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-modD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
⟨*proof*⟩
corollary *iMODb-geD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-leD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x \leq r + m * c$
⟨*proof*⟩

lemmas *iT-iff* = *iFROM-iff iTILL-iff iIN-iff iMOD-iff iMODb-iff*

lemmas *iT-drule* =
iFROM-D
iTILL-D
iIN-geD iIN-leD
iMOD-modD iMOD-geD

iMODb-modD iMODb-geD iMODb-leD

lemma
iFROM-I [*intro*]: $n \leq x \Longrightarrow x \in [n\ldots]$ **and**
iTILL-I [*intro*]: $x \leq n \Longrightarrow x \in [\ldots n]$ **and**
iIN-I [*intro*]: $n \leq x \Longrightarrow x \leq n + d \Longrightarrow x \in [n\ldots,d]$ **and**
iMOD-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \in [r,\ mod\ m]$ **and**
iMODb-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow x \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma
iFROM-E [*elim*]: $x \in [n\ldots] \Longrightarrow (n \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iTILL-E [*elim*]: $x \in [\ldots n] \Longrightarrow (x \leq n \Longrightarrow P) \Longrightarrow P$ **and**
iIN-E [*elim*]: $x \in [n\ldots,d] \Longrightarrow (n \leq x \Longrightarrow x \leq n + d \Longrightarrow P) \Longrightarrow P$ **and**
iMOD-E [*elim*]: $x \in [r,\ mod\ m] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iMODb-E [*elim*]: $x \in [r,\ mod\ m,\ c] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *iIN-Suc-insert-conv*:
$insert\ (Suc\ (n + d))\ [n\ldots,d] = [n\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iTILL-Suc-insert-conv*: $insert\ (Suc\ n)\ [\ldots n] = [\ldots Suc\ n]$
⟨*proof*⟩

lemma *iMODb-Suc-insert-conv*:
$insert\ (r + m * Suc\ c)\ [r,\ mod\ m,\ c] = [r,\ mod\ m,\ Suc\ c]$
⟨*proof*⟩

lemma *iFROM-pred-insert-conv*: $insert\ (n - Suc\ 0)\ [n\ldots] = [n - Suc\ 0\ldots]$
⟨*proof*⟩

lemma *iIN-pred-insert-conv*:
$0 < n \Longrightarrow insert\ (n - Suc\ 0)\ [n\ldots,d] = [n - Suc\ 0\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iMOD-pred-insert-conv*:
$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m] = [r - m,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-pred-insert-conv*:
$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m,\ c] = [r - m,\ mod\ m,\ Suc\ c]$

⟨*proof*⟩

lemma *iFROM-Suc-pred-insert-conv*: *insert n* [*Suc n*...] = [*n*...]
⟨*proof*⟩
lemma *iIN-Suc-pred-insert-conv*: *insert n* [*Suc n*...,*d*] = [*n*...,*Suc d*]
⟨*proof*⟩
lemma *iMOD-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*] = [*r*, *mod m*]
⟨*proof*⟩
lemma *iMODb-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*, *c*] = [*r*, *mod m*, *Suc c*]
⟨*proof*⟩

lemmas *iT-Suc-insert* =
iIN-Suc-insert-conv
iTILL-Suc-insert-conv
iMODb-Suc-insert-conv
lemmas *iT-pred-insert* =
iFROM-pred-insert-conv
iIN-pred-insert-conv
iMOD-pred-insert-conv
iMODb-pred-insert-conv
lemmas *iT-Suc-pred-insert* =
iFROM-Suc-pred-insert-conv
iIN-Suc-pred-insert-conv
iMOD-Suc-pred-insert-conv
iMODb-Suc-pred-insert-conv

lemma *iMOD-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*]; *b* ∈ [*r*, *mod m*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
⟨*proof*⟩
lemma *iMODb-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*, *c*]; *b* ∈ [*r*, *mod m*, *c*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
⟨*proof*⟩

1.1.3 Interval conversions

lemma *iIN-0-iTILL-conv*:[*0*...,*n*] = [...*n*]
⟨*proof*⟩
lemma *iIN-iTILL-iTILL-conv*: *0* < *n* ⟹ [*n*...,*d*] = [...*n*+*d*] − [...*n* − *Suc 0*]
⟨*proof*⟩
lemma *iIN-iFROM-iTILL-conv*: [*n*...,*d*] = [*n*...] ∩ [...*n*+*d*]
⟨*proof*⟩
lemma *iMODb-iMOD-iTILL-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [...*r*+*m*∗*c*]
⟨*proof*⟩
lemma *iMODb-iMOD-iIN-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [*r*...,*m*∗*c*]
⟨*proof*⟩

lemma *iFROM-iTILL-iIN-conv*: *n* ≤ *n*′ ⟹ [*n*...] ∩ [...*n*′] = [*n*...,*n*′−*n*]
⟨*proof*⟩

lemma *iMOD-iTILL-iMODb-conv*:
$r \leq n \Longrightarrow [r,\ mod\ m] \cap [\ldots n] = [r,\ mod\ m,\ (n - r)\ div\ m]$
⟨*proof*⟩

lemma *iMOD-iIN-iMODb-conv*:
$[r,\ mod\ m] \cap [r\ldots,d] = [r,\ mod\ m,\ d\ div\ m]$
⟨*proof*⟩

lemma *iFROM-0*: $[0\ldots] = UNIV$
⟨*proof*⟩

lemma *iTILL-0*: $[\ldots 0] = \{0\}$
⟨*proof*⟩

lemma *iIN-0*: $[n\ldots,0] = \{n\}$
⟨*proof*⟩

lemma *iMOD-0*: $[r,\ mod\ 0] = [r\ldots,0]$
⟨*proof*⟩

lemma *iMODb-mod-0*: $[r,\ mod\ 0,\ c] = [r\ldots,0]$
⟨*proof*⟩

lemma *iMODb-0*: $[r,\ mod\ m,\ 0] = [r\ldots,0]$
⟨*proof*⟩

lemmas *iT-0* =
iFROM-0
iTILL-0
iIN-0
iMOD-0
iMODb-mod-0
iMODb-0

lemma *iMOD-1*: $[r,\ mod\ Suc\ 0] = [r\ldots]$
⟨*proof*⟩

lemma *iMODb-mod-1*: $[r,\ mod\ Suc\ 0,\ c] = [r\ldots,c]$
⟨*proof*⟩

1.1.4 Finiteness and emptiness of intervals

lemma
iFROM-not-empty: $[n\ldots] \neq \{\}$ **and**
iTILL-not-empty: $[\ldots n] \neq \{\}$ **and**
iIN-not-empty: $[n\ldots,d] \neq \{\}$ **and**
iMOD-not-empty: $[r,\ mod\ m] \neq \{\}$ **and**
iMODb-not-empty: $[r,\ mod\ m,\ c] \neq \{\}$
⟨*proof*⟩

lemmas *iT-not-empty* =
iFROM-not-empty
iTILL-not-empty
iIN-not-empty
iMOD-not-empty
iMODb-not-empty

lemma
iTILL-finite: *finite* [...*n*] **and**
iIN-finite: *finite* [*n*...,*d*] **and**
iMODb-finite: *finite* [*r*, *mod m*, *c*] **and**
iMOD-0-finite: *finite* [*r*, *mod 0*]
⟨*proof*⟩

lemma *iFROM-infinite*: *infinite* [*n*...]
⟨*proof*⟩

lemma *iMOD-infinite*: $0 < m \Longrightarrow$ *infinite* [*r*, *mod m*]
⟨*proof*⟩

lemmas *iT-finite* =
iTILL-finite
iIN-finite
iMODb-finite iMOD-0-finite

lemmas *iT-infinite* =
iFROM-infinite
iMOD-infinite

1.1.5 *Min* and *Max* element of an interval

lemma
iTILL-Min: *iMin* [...*n*] = *0* **and**
iFROM-Min: *iMin* [*n*...] = *n* **and**
iIN-Min: *iMin* [*n*...,*d*] = *n* **and**
iMOD-Min: *iMin* [*r*, *mod m*] = *r* **and**
iMODb-Min: *iMin* [*r*, *mod m*, *c*] = *r*
⟨*proof*⟩

lemmas *iT-Min* =
iIN-Min
iTILL-Min
iFROM-Min
iMOD-Min
iMODb-Min

lemma
iTILL-Max: *Max* [...*n*] = *n* **and**

iIN-Max: $Max\ [n...,d] = n+d$ **and**
iMODb-Max: $Max\ [r,\ mod\ m,\ c] = r + m * c$ **and**
iMOD-0-Max: $Max\ [r,\ mod\ 0] = r$
⟨*proof*⟩

lemmas *iT-Max* =
iTILL-Max
iIN-Max
iMODb-Max
iMOD-0-Max

lemma
iTILL-iMax: $iMax\ [...n] = enat\ n$ **and**
iIN-iMax: $iMax\ [n...,d] = enat\ (n+d)$ **and**
iMODb-iMax: $iMax\ [r,\ mod\ m,\ c] = enat\ (r + m * c)$ **and**
iMOD-0-iMax: $iMax\ [r,\ mod\ 0] = enat\ r$ **and**
iFROM-iMax: $iMax\ [n...] = \infty$ **and**
iMOD-iMax: $0 < m \Longrightarrow iMax\ [r,\ mod\ m] = \infty$
⟨*proof*⟩

lemmas *iT-iMax* =
iTILL-iMax
iIN-iMax
iMODb-iMax
iMOD-0-iMax
iFROM-iMax
iMOD-iMax

1.2 Adding and subtracting constants to interval elements

lemma
iFROM-plus: $x \in [n...] \Longrightarrow x + k \in [n...]$ **and**
iFROM-Suc: $x \in [n...] \Longrightarrow Suc\ x \in [n...]$ **and**
iFROM-minus: $[\![\ x \in [n...];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n...]$ **and**
iFROM-pred: $n < x \Longrightarrow x - Suc\ 0 \in [n...]$
⟨*proof*⟩

lemma
iTILL-plus: $[\![\ x \in [...n];\ k \leq n - x\ ]\!] \Longrightarrow x + k \in [...n]$ **and**
iTILL-Suc: $x < n \Longrightarrow Suc\ x \in [...n]$ **and**
iTILL-minus: $x \in [...n] \Longrightarrow x - k \in [...n]$ **and**
iTILL-pred: $x \in [...n] \Longrightarrow x - Suc\ 0 \in [...n]$
⟨*proof*⟩

lemma *iIN-plus*: $[\![\ x \in [n...,d];\ k \leq n + d - x\ ]\!] \Longrightarrow x + k \in [n...,d]$
⟨*proof*⟩

lemma *iIN-Suc*: $[\![\ x \in [n...,d];\ x < n + d\ ]\!] \Longrightarrow Suc\ x \in [n...,d]$
⟨*proof*⟩

lemma *iIN-minus*: $[\![\ x \in [n..,d];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n..,d]$
⟨*proof*⟩

lemma *iIN-pred*: $[\![\ x \in [n..,d];\ n < x\ ]\!] \Longrightarrow x - Suc\ 0 \in [n..,d]$
⟨*proof*⟩

lemma *iMOD-plus-divisor-mult*: $x \in [r,\ mod\ m] \Longrightarrow x + k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-plus-divisor*: $x \in [r,\ mod\ m] \Longrightarrow x + m \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-minus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m];\ k * m \leq x - r\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-minus-divisor-mult2*:
$[\![\ x \in [r,\ mod\ m];\ k \leq (x - r)\ div\ m\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
⟨*proof*⟩

corollary *iMOD-minus-divisor*:
$[\![\ x \in [r,\ mod\ m];\ m + r \leq x\ ]\!] \Longrightarrow x - m \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-plus*:
$x \in [r,\ mod\ m] \Longrightarrow (x + k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩
corollary *iMOD-Suc*:
$x \in [r,\ mod\ m] \Longrightarrow (Suc\ x \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMOD-minus*:
$[\![\ x \in [r,\ mod\ m];\ k \leq x - r\ ]\!] \Longrightarrow (x - k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMOD-pred*:
$[\![\ x \in [r,\ mod\ m];\ r < x\ ]\!] \Longrightarrow (x - Suc\ 0 \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMODb-plus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k * m \leq r + m * c - x\ ]\!] \Longrightarrow x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-plus-divisor-mult2*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq c - (x - r)\ div\ m\ ]\!] \Longrightarrow$
$x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *iMODb-plus-divisor*:

$[\![\ x \in [r,\ mod\ m,\ c];\ x < r + m * c\ ]\!] \Longrightarrow x + m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-minus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m,\ c];\ r + k * m \leq x\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-plus*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq r + m * c - x\ ]\!] \Longrightarrow$
$(x + k \in [r,\ mod\ m,\ c]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMODb-Suc*:
$[\![\ x \in [r,\ mod\ m,\ c];\ x < r + m * c\ ]\!] \Longrightarrow$
$(Suc\ x \in [r,\ mod\ m,\ c]) = (m = Suc\ 0)$
⟨*proof*⟩

lemma *iMODb-minus*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq x - r\ ]\!] \Longrightarrow$
$(x - k \in [r,\ mod\ m,\ c]) = (k\ mod\ m = 0)$
⟨*proof*⟩

corollary *iMODb-pred*:
$[\![\ x \in [r,\ mod\ m,\ c];\ r < x\ ]\!] \Longrightarrow$
$(x - Suc\ 0 \in [r,\ mod\ m,\ c]) = (m = Suc\ 0)$
⟨*proof*⟩

lemmas *iFROM-plus-minus* =
iFROM-plus
iFROM-Suc
iFROM-minus
iFROM-pred

lemmas *iTILL-plus-minus* =
iTILL-plus
iTILL-Suc
iTILL-minus
iTILL-pred

lemmas *iIN-plus-minus* =
iIN-plus
iIN-Suc
iTILL-minus
iIN-pred

lemmas *iMOD-plus-minus-divisor* =
iMOD-plus-divisor-mult
iMOD-plus-divisor
iMOD-minus-divisor-mult

iMOD-minus-divisor-mult2
iMOD-minus-divisor

lemmas *iMOD-plus-minus* =
iMOD-plus
iMOD-Suc
iMOD-minus
iMOD-pred

lemmas *iMODb-plus-minus-divisor* =
iMODb-plus-divisor-mult
iMODb-plus-divisor-mult2
iMODb-plus-divisor
iMODb-minus-divisor-mult

lemmas *iMODb-plus-minus* =
iMODb-plus
iMODb-Suc
iMODb-minus
iMODb-pred

lemmas *iT-plus-minus* =
iFROM-plus-minus
iTILL-plus-minus
iIN-plus-minus
iMOD-plus-minus-divisor
iMOD-plus-minus
iMODb-plus-minus-divisor
iMODb-plus-minus

1.3 Relations between intervals

1.3.1 Auxiliary lemmata

lemma *Suc-in-imp-not-subset-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg S \subseteq [r, mod\ m]$
⟨*proof*⟩

corollary *Suc-in-imp-neq-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m]$
⟨*proof*⟩

lemma *Suc-in-imp-not-subset-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg S \subseteq [r, mod\ m, c]$
⟨*proof*⟩
corollary *Suc-in-imp-neq-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m, c]$
⟨*proof*⟩

1.3.2 Subset relation between intervals

lemma
iIN-iFROM-subset-same: $[n...,d] \subseteq [n...]$ **and**
iIN-iTILL-subset-same: $[n...,d] \subseteq [...n+d]$ **and**
iMOD-iFROM-subset-same: $[r,\ mod\ m] \subseteq [r...]$ **and**
iMODb-iTILL-subset-same: $[r,\ mod\ m,\ c] \subseteq [...r+m*c]$ **and**
iMODb-iIN-subset-same: $[r,\ mod\ m,\ c] \subseteq [r...,m*c]$ **and**
iMODb-iMOD-subset-same: $[r,\ mod\ m,\ c] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemmas *iT-subset-same* =
iIN-iFROM-subset-same
iIN-iTILL-subset-same
iMOD-iFROM-subset-same
iMODb-iTILL-subset-same
iMODb-iIN-subset-same
iMODb-iTILL-subset-same
iMODb-iMOD-subset-same

lemma *iMODb-imp-iMOD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x \in [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-imp-iMODb*:
$[\![\ x \in [r,\ mod\ m];\ x \leq r + m * c\ ]\!] \Longrightarrow x \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMOD-singleton-subset-conv*: $([r,\ mod\ m] \subseteq \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩
lemma *iMOD-singleton-eq-conv*: $([r,\ mod\ m] = \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩

lemma *iMODb-singleton-subset-conv*:
$([r,\ mod\ m,\ c] \subseteq \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩
lemma *iMODb-singleton-eq-conv*:
$([r,\ mod\ m,\ c] = \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩

lemma *iMODb-subset-imp-divisor-mod-0*:
$[\![\ 0 < c';\ [r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow m'\ mod\ m = 0$
⟨*proof*⟩

lemma *iMOD-subset-imp-divisor-mod-0*:
$[r',\ mod\ m'] \subseteq [r,\ mod\ m] \Longrightarrow m'\ mod\ m = 0$
⟨*proof*⟩

lemma *iMOD-subset-imp-iMODb-subset*:
$[\![\ [r',\ mod\ m'] \subseteq [r,\ mod\ m];\ r' + m' * c' \leq r + m * c\ ]\!] \Longrightarrow$
$[r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]$

⟨*proof*⟩

lemma *iMODb-subset-imp-iMOD-subset*:
⟦ [r', *mod* m', c'] ⊆ [r, *mod* m, c]; $0 < c'$ ⟧ ⟹
[r', *mod* m'] ⊆ [r, *mod* m]
⟨*proof*⟩

lemma *iMODb-0-iMOD-subset-conv*:
([r', *mod* m', 0] ⊆ [r, *mod* m]) =
(r' *mod* m = r *mod* m ∧ $r \leq r'$)
⟨*proof*⟩

lemma *iFROM-subset-conv*: ([n'...] ⊆ [n...]) = ($n \leq n'$)
⟨*proof*⟩

lemma *iFROM-iMOD-subset-conv*: ([n'...] ⊆ [r, *mod* m]) = ($r \leq n' \wedge m$ = *Suc* *0*)
⟨*proof*⟩

lemma *iIN-subset-conv*: ([n'...,d'] ⊆ [n...,d]) = ($n \leq n' \wedge n'+d' \leq n+d$)
⟨*proof*⟩

lemma *iIN-iFROM-subset-conv*: ([n'...,d'] ⊆ [n...]) = ($n \leq n'$)
⟨*proof*⟩

lemma *iIN-iTILL-subset-conv*: ([n'...,d'] ⊆ [...n]) = ($n' + d' \leq n$)
⟨*proof*⟩

lemma *iIN-iMOD-subset-conv*:
$0 < d' \Longrightarrow$ ([n'...,d'] ⊆ [r, *mod* m]) = ($r \leq n' \wedge m$ = *Suc* *0*)
⟨*proof*⟩

lemma *iIN-iMODb-subset-conv*:
$0 < d' \Longrightarrow$
([n'...,d'] ⊆ [r, *mod* m, c]) =
($r \leq n' \wedge m$ = *Suc* *0* $\wedge$ $n' + d' \leq r + m * c$)
⟨*proof*⟩

lemma *iTILL-subset-conv*: ([...n'] ⊆ [...n]) = ($n' \leq n$)
⟨*proof*⟩

lemma *iTILL-iFROM-subset-conv*: ([...n'] ⊆ [n...]) = (n = *0*)
⟨*proof*⟩

lemma *iTILL-iIN-subset-conv*: ([...n'] ⊆ [n...,d]) = (n = *0* $\wedge$ $n' \leq d$)
⟨*proof*⟩

lemma *iTILL-iMOD-subset-conv*:

$0 < n' \Longrightarrow ([...n'] \subseteq [r, \, mod \, m]) = (r = 0 \wedge m = Suc \; 0)$
⟨*proof*⟩

lemma *iTILL-iMODb-subset-conv*:
$0 < n' \Longrightarrow ([...n'] \subseteq [r, \, mod \, m, \, c]) = (r = 0 \wedge m = Suc \; 0 \wedge n' \leq r + m * c)$
⟨*proof*⟩

lemma *iMOD-iFROM-subset-conv*: $([r', \, mod \, m']) \subseteq [n...] = (n \leq r')$
⟨*proof*⟩

lemma *iMODb-iFROM-subset-conv*: $([r', \, mod \, m', \, c'] \subseteq [n...]) = (n \leq r')$
⟨*proof*⟩

lemma *iMODb-iIN-subset-conv*:
$([r', \, mod \, m', \, c'] \subseteq [n...,d]) = (n \leq r' \wedge r' + m' * c' \leq n + d)$
⟨*proof*⟩

lemma *iMODb-iTILL-subset-conv*:
$([r', \, mod \, m', \, c'] \subseteq [...n]) = (r' + m' * c' \leq n)$
⟨*proof*⟩

lemma *iMOD-0-subset-conv*: $([r', \, mod \; 0] \subseteq [r, \, mod \, m]) = (r' \; mod \; m = r \; mod \; m$
$\wedge \;\; r \leq r')$
⟨*proof*⟩

lemma *iMOD-subset-conv*: $0 < m \Longrightarrow$
$([r', \, mod \, m'] \subseteq [r, \, mod \, m]) =$
$(r' \; mod \; m = r \; mod \; m \;\; \wedge \;\; r \leq r' \;\; \wedge \;\; m' \; mod \; m = 0)$
⟨*proof*⟩

lemma *iMODb-subset-mod-0-conv*:
$([r', \, mod \, m', \, c'] \subseteq [r, \, mod \; 0, \, c \,]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨*proof*⟩

lemma *iMODb-subset-0-conv*:
$([r', \, mod \, m', \, c'] \subseteq [r, \, mod \, m, \, 0 \,]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨*proof*⟩

lemma *iMODb-0-subset-conv*:
$([r', \, mod \, m', \, 0] \subseteq [r, \, mod \, m, \, c \,]) = (r' \in [r, \, mod \, m, \, c])$
⟨*proof*⟩

lemma *iMODb-mod-0-subset-conv*:
$([r', \, mod \; 0, \, c'] \subseteq [r, \, mod \, m, \, c \,]) = (r' \in [r, \, mod \, m, \, c])$
⟨*proof*⟩

lemma *iMODb-subset-conv'*: $[\![\; 0 < c; \; 0 < c' \;]\!] \Longrightarrow$
$([r', \, mod \, m', \, c'] \subseteq [r, \, mod \, m, \, c]) =$
$(r' \; mod \; m = r \; mod \; m \;\; \wedge \;\; r \leq r' \;\; \wedge \;\; m' \; mod \; m = 0 \;\; \wedge$

$r' + m' * c' \leq r + m * c)$
⟨*proof*⟩

lemma *iMODb-subset-conv*: $[\![\ 0 < m';\ 0 < c'\ ]\!] \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0\ \wedge$
$r' + m' * c' \leq r + m * c)$
⟨*proof*⟩

lemma *iMODb-iMOD-subset-conv*: $0 < c' \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0)$
⟨*proof*⟩

lemmas *iT-subset-conv* =
iFROM-subset-conv
iFROM-iMOD-subset-conv
iTILL-subset-conv
iTILL-iFROM-subset-conv
iTILL-iIN-subset-conv
iTILL-iMOD-subset-conv
iTILL-iMODb-subset-conv
iIN-subset-conv
iIN-iFROM-subset-conv
iIN-iTILL-subset-conv
iIN-iMOD-subset-conv
iIN-iMODb-subset-conv
iMOD-subset-conv
iMOD-iFROM-subset-conv
iMODb-subset-conv'
iMODb-subset-conv
iMODb-iFROM-subset-conv
iMODb-iIN-subset-conv
iMODb-iTILL-subset-conv
iMODb-iMOD-subset-conv

lemma *iFROM-subset*: $n \leq n' \Longrightarrow [n'\ldots] \subseteq [n\ldots]$
⟨*proof*⟩

lemma *not-iFROM-iIN-subset*: $\neg\ [n'\ldots] \subseteq [n\ldots,d]$
⟨*proof*⟩

lemma *not-iFROM-iTILL-subset*: $\neg\ [n'\ldots] \subseteq [\ldots n]$
⟨*proof*⟩

lemma *not-iFROM-iMOD-subset*: $m \neq Suc\ 0 \Longrightarrow \neg\ [n'\ldots] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *not-iFROM-iMODb-subset*: $\neg\ [n'\ldots] \subseteq [r,\ mod\ m,\ c]$

⟨*proof*⟩

lemma *iIN-subset*: ⟦ $n \leq n'$; $n' + d' \leq n + d$ ⟧ ⟹ $[n'..,d'] \subseteq [n..,d]$
⟨*proof*⟩

lemma *iIN-iFROM-subset*: $n \leq n' \Longrightarrow [n'..,d'] \subseteq [n..]$
⟨*proof*⟩

lemma *iIN-iTILL-subset*: $n' + d' \leq n \Longrightarrow [n'..,d'] \subseteq [..n]$
⟨*proof*⟩

lemma *not-iIN-iMODb-subset*: ⟦ $0 < d'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[n'..,d'] \subseteq [r, mod\ m, c]$
⟨*proof*⟩

lemma *not-iIN-iMOD-subset*: ⟦ $0 < d'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[n'..,d'] \subseteq [r, mod\ m]$
⟨*proof*⟩

lemma *iTILL-subset*: $n' \leq n \Longrightarrow [..n'] \subseteq [..n]$
⟨*proof*⟩

lemma *iTILL-iFROM-subset*: $([..n'] \subseteq [0..])$
⟨*proof*⟩

lemma *iTILL-iIN-subset*: $n' \leq d \Longrightarrow ([..n'] \subseteq [0..,d])$
⟨*proof*⟩

lemma *not-iTILL-iMOD-subset*:
⟦ $0 < n'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[..n'] \subseteq [r, mod\ m]$
⟨*proof*⟩

lemma *not-iTILL-iMODb-subset*:
⟦ $0 < n'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[..n'] \subseteq [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMOD-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m'] \subseteq [n..]$
⟨*proof*⟩

lemma *not-iMOD-iIN-subset*: $0 < m' \Longrightarrow$ ¬ $[r', mod\ m'] \subseteq [n..,d]$
⟨*proof*⟩

lemma *not-iMOD-iTILL-subset*: $0 < m' \Longrightarrow$ ¬ $[r', mod\ m'] \subseteq [..n]$
⟨*proof*⟩

lemma *iMOD-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$ ⟧ ⟹ $[r', mod\ m'] \subseteq [r, mod\ m]$
⟨*proof*⟩

lemma *not-iMOD-iMODb-subset*: $0 < m' \Longrightarrow \neg\ [r',\ mod\ m'] \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-iFROM-subset*: $n \leq r' \Longrightarrow [r',\ mod\ m',\ c'] \subseteq [n...]$
⟨*proof*⟩

lemma *iMODb-iTILL-subset*:
$r' + m' * c' \leq n \Longrightarrow [r',\ mod\ m',\ c'] \subseteq [...n]$
⟨*proof*⟩

lemma *iMODb-iIN-subset*:
$[\![\ n \leq r';\ r' + m' * c' \leq n + d\ ]\!] \Longrightarrow [r',\ mod\ m',\ c'] \subseteq [n...,d]$
⟨*proof*⟩

lemma *iMODb-iMOD-subset*:
$[\![\ r \leq r';\ r'\ mod\ m = r\ mod\ m;\ m'\ mod\ m = 0\ ]\!] \Longrightarrow [r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-subset*:
$[\![\ r \leq r';\ r'\ mod\ m = r\ mod\ m;\ m'\ mod\ m = 0;\ r' + m' * c' \leq r + m * c\ ]\!] \Longrightarrow$
$[r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iFROM-trans*: $[\![\ y \in [x...];\ z \in [y...]\ ]\!] \Longrightarrow z \in [x...]$
⟨*proof*⟩

lemma *iTILL-trans*: $[\![\ y \in [...x];\ z \in [...y]\ ]\!] \Longrightarrow z \in [...x]$
⟨*proof*⟩

lemma *iIN-trans*:
$[\![\ y \in [x...,d];\ z \in [y...,d'];\ d' \leq x + d - y\ ]\!] \Longrightarrow z \in [x...,d]$
⟨*proof*⟩

lemma *iMOD-trans*:
$[\![\ y \in [x,\ mod\ m];\ z \in [y,\ mod\ m]\ ]\!] \Longrightarrow z \in [x,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-trans*:
$[\![\ y \in [x,\ mod\ m,\ c];\ z \in [y,\ mod\ m,\ c'];\ m * c' \leq x + m * c - y\ ]\!] \Longrightarrow$
$z \in [x,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMODb-trans'*:
$[\![\ y \in [x,\ mod\ m,\ c];\ z \in [y,\ mod\ m,\ c'];\ c' \leq x\ div\ m + c - y\ div\ m\ ]\!] \Longrightarrow$
$z \in [x,\ mod\ m,\ c]$
⟨*proof*⟩

1.3.3 Equality of intervals

lemma *iFROM-eq-conv*: $([n...] = [n'...]) = (n = n')$
⟨*proof*⟩

lemma *iIN-eq-conv*: $([n...,d] = [n'...,d']) = (n = n' \wedge d = d')$
⟨*proof*⟩

lemma *iTILL-eq-conv*: $([...n] = [...n']) = (n = n')$
⟨*proof*⟩

lemma *iMOD-0-eq-conv*: $([r,\ mod\ 0] = [r',\ mod\ m']) = (r = r' \wedge m' = 0)$
⟨*proof*⟩

lemma *iMOD-eq-conv*: $0 < m \Longrightarrow ([r,\ mod\ m] = [r',\ mod\ m']) = (r = r' \wedge m = m')$
⟨*proof*⟩

lemma *iMODb-mod-0-eq-conv*:
$([r,\ mod\ 0,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
⟨*proof*⟩

lemma *iMODb-0-eq-conv*:
$([r,\ mod\ m,\ 0] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
⟨*proof*⟩

lemma *iMODb-eq-conv*: $[\![\ 0 < m;\ 0 < c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge m = m' \wedge c = c')$
⟨*proof*⟩

lemma *iMOD-iFROM-eq-conv*: $([n...] = [r,\ mod\ m]) = (n = r \wedge m = Suc\ 0)$
⟨*proof*⟩

lemma *iMODb-iIN-0-eq-conv*:
$([n...,0] = [r,\ mod\ m,\ c]) = (n = r \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩

lemma *iMODb-iIN-eq-conv*:
$0 < d \Longrightarrow ([n...,d] = [r,\ mod\ m,\ c]) = (n = r \wedge m = Suc\ 0 \wedge c = d)$
⟨*proof*⟩

1.3.4 Inequality of intervals

lemma *iFROM-iIN-neq*: $[n'...] \neq [n...,d]$
⟨*proof*⟩

corollary *iFROM-iTILL-neq*: $[n'...] \neq [...n]$
⟨*proof*⟩

corollary *iFROM-iMOD-neq*: $m \neq Suc\ 0 \Longrightarrow [n...] \neq [r,\ mod\ m]$

⟨*proof*⟩
corollary *iFROM-iMODb-neq*: $[n...] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

corollary *iIN-iMOD-neq*: $0 < m \Longrightarrow [n...,d] \neq [r,\ mod\ m]$
⟨*proof*⟩

corollary *iIN-iMODb-neq2*: $[\![\ m \neq Suc\ 0;\ 0 < d\ ]\!] \Longrightarrow [n...,d] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iIN-iMODb-neq*: $[\![\ 2 \leq m;\ 0 < c\ ]\!] \Longrightarrow [n...,d] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iTILL-iIN-neq*: $0 < n \Longrightarrow [...n'] \neq [n...,d]$
⟨*proof*⟩

corollary *iTILL-iMOD-neq*: $0 < m \Longrightarrow [...n] \neq [r,\ mod\ m]$
⟨*proof*⟩

corollary *iTILL-iMODb-neq*:
$[\![\ m \neq Suc\ 0;\ 0 < n\ ]\!] \Longrightarrow [...n] \neq [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iMOD-iMODb-neq*: $0 < m \Longrightarrow [r,\ mod\ m] \neq [r',\ mod\ m',\ c']$
⟨*proof*⟩

lemmas *iT-neq* =
iFROM-iTILL-neq iFROM-iIN-neq iFROM-iMOD-neq iFROM-iMODb-neq
iTILL-iIN-neq iTILL-iMOD-neq iTILL-iMODb-neq
iIN-iMOD-neq iIN-iMODb-neq iIN-iMODb-neq2
iMOD-iMODb-neq

1.4 Union and intersection of intervals

lemma *iFROM-union'*: $[n...] \cup [n'...] = [min\ n\ n'...]$
⟨*proof*⟩

corollary *iFROM-union*: $n \leq n' \Longrightarrow [n...] \cup [n'...] = [n...]$
⟨*proof*⟩

lemma *iFROM-inter'*: $[n...] \cap [n'...] = [max\ n\ n'...]$
⟨*proof*⟩

corollary *iFROM-inter*: $n' \leq n \Longrightarrow [n...] \cap [n'...] = [n...]$
⟨*proof*⟩

lemma *iTILL-union'*: $[...n] \cup [...n'] = [...max\ n\ n']$
⟨*proof*⟩

corollary *iTILL-union*: $n' \leq n \Longrightarrow [\ldots n] \cup [\ldots n'] = [\ldots n]$
⟨*proof*⟩

lemma *iTILL-iFROM-union*: $n \leq n' \Longrightarrow [\ldots n'] \cup [n\ldots] = UNIV$
⟨*proof*⟩

lemma *iTILL-inter'*: $[\ldots n] \cap [\ldots n'] = [\ldots min\ n\ n']$
⟨*proof*⟩

corollary *iTILL-inter*: $n \leq n' \Longrightarrow [\ldots n] \cap [\ldots n'] = [\ldots n]$
⟨*proof*⟩

Union and intersection for iIN only when there intersection is not empty and none of them is other's subset, for instance: .. n .. n+d .. n' .. n'+d'

lemma *iIN-union*:
$[\![\ n \leq n';\ n' \leq Suc\ (n + d);\ n + d \leq n' + d'\ ]\!] \Longrightarrow$
$[n\ldots,d] \cup [n'\ldots,d'] = [n\ldots,n' - n + d']$
⟨*proof*⟩

lemma *iIN-append-union*:
$[n\ldots,d] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
⟨*proof*⟩

lemma *iIN-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots,d'] = [n\ldots,Suc\ (d + d')]$
⟨*proof*⟩

lemma *iIN-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
⟨*proof*⟩

lemma *iIN-iFROM-union*:
$n' \leq Suc\ (n + d) \Longrightarrow [n\ldots,d] \cup [n'\ldots] = [min\ n\ n'\ldots]$
⟨*proof*⟩

lemma *iIN-iFROM-append-union*:
$[n\ldots,d] \cup [n + d\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iIN-iFROM-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iIN-iFROM-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iIN-inter*:
$[\![\ n \leq n';\ n' \leq n + d;\ n + d \leq n' + d'\ ]\!] \Longrightarrow$
$[n.\ldots,d] \cap [n'.\ldots,d'] = [n'.\ldots,n + d - n']$
⟨*proof*⟩

lemma *iMOD-union*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [r,\ mod\ m]$
⟨*proof*⟩
lemma *iMOD-union′*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [min\ r\ r',\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inter*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [r',\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inter′*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [max\ r\ r',\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-union*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m;\ r' \leq r + m * c;\ r + m * c \leq r' + m * c'\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [r',\ mod\ m,\ c'] = [r,\ mod\ m,\ r'\ div\ m - r\ div\ m + c']$
⟨*proof*⟩

lemma *iMODb-append-union*:
$[r,\ mod\ m,\ c] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c + c']$
⟨*proof*⟩

lemma *iMODb-iMOD-append-union′*:
$[\![\ r\ mod\ m = r'\ mod\ m;\ r' \leq r + m * Suc\ c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [min\ r\ r',\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-iMOD-append-union*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m;\ r' \leq r + m * Suc\ c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-append-union-Suc*:
$[r,\ mod\ m,\ c] \cup [\ r + m * Suc\ c,\ mod\ m,\ c'] = [r,\ mod\ m,\ Suc\ (c + c')]$
⟨*proof*⟩

lemma *iMODb-append-union-pred*:
$0 < c \Longrightarrow [r,\ mod\ m,\ c - Suc\ 0] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c +$

$c'\rceil$
⟨*proof*⟩

lemma *iMODb-inter*:
$[\![\ r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * c; r + m * c \le r' + m * c'\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cap [r',\ mod\ m,\ c'\rceil = [r',\ mod\ m,\ c - (r'-r)\ div\ m]$
⟨*proof*⟩

lemmas *iT-union'* =
iFROM-union'
iTILL-union'
iMOD-union'
iMODb-iMOD-append-union'
lemmas *iT-union* =
iFROM-union
iTILL-union
iTILL-iFROM-union
iIN-union
iIN-iFROM-union
iMOD-union
iMODb-union
lemmas *iT-union-append* =
iIN-append-union
iIN-append-union-Suc
iIN-append-union-pred
iIN-iFROM-append-union
iIN-iFROM-append-union-Suc
iIN-iFROM-append-union-pred
iMODb-append-union
iMODb-iMOD-append-union
iMODb-append-union-Suc
iMODb-append-union-pred

lemmas *iT-inter'* =
iFROM-inter'
iTILL-inter'
iMOD-inter'
lemmas *iT-inter* =
iFROM-inter
iTILL-inter
iIN-inter
iMOD-inter
iMODb-inter

lemma *mod-partition-Union*:
$0 < m \Longrightarrow (\bigcup k.\ A \cap [k * m \ldots, m - Suc\ 0]) = A$

⟨*proof*⟩

lemma *finite-mod-partition-Union*:
⟦ *0* < *m*; *finite A* ⟧ ⟹
(⋃ *k*≤*Max A div m*. *A* ∩ [*k*∗*m*...,*m* − *Suc 0*]) = *A*
⟨*proof*⟩

lemma *mod-partition-is-disjoint*:
⟦ *0* < (*m*::*nat*); *k* ≠ *k*′ ⟧ ⟹
(*A* ∩ [*k* ∗ *m*...,*m* − *Suc 0*]) ∩ (*A* ∩ [*k*′ ∗ *m*...,*m* − *Suc 0*]) = {}
⟨*proof*⟩

1.5 Cutting intervals

lemma *iTILL-cut-le*: [...*n*] ↓≤ *t* = (*if t* ≤ *n then* [...*t*] *else* [...*n*])
⟨*proof*⟩

corollary *iTILL-cut-le1*: *t* ∈ [...*n*] ⟹ [...*n*] ↓≤ *t* = [...*t*]
⟨*proof*⟩

corollary *iTILL-cut-le2*: *t* ∉ [...*n*] ⟹ [...*n*] ↓≤ *t* = [...*n*]
⟨*proof*⟩

lemma *iFROM-cut-le*:
[*n*...] ↓≤ *t* =
(*if t* < *n then* {} *else* [*n*...,*t*−*n*])
⟨*proof*⟩

corollary *iFROM-cut-le1*: *t* ∈ [*n*...] ⟹ [*n*...] ↓≤ *t* = [*n*...,*t* − *n*]
⟨*proof*⟩

lemma *iIN-cut-le*:
[*n*...,*d*] ↓≤ *t* = (
if t < *n then* {} *else*
if t ≤ *n*+*d then* [*n*...,*t*−*n*]
else [*n*...,*d*])
⟨*proof*⟩

corollary *iIN-cut-le1*:
t ∈ [*n*...,*d*] ⟹ [*n*...,*d*] ↓≤ *t* = [*n*...,*t* − *n*]
⟨*proof*⟩

lemma *iMOD-cut-le*:
[*r*, *mod m*] ↓≤ *t* = (
if t < *r then* {}
else [*r*, *mod m*, (*t* − *r*) *div m*])
⟨*proof*⟩

lemma *iMOD-cut-le1*:
$t \in [r,\ mod\ m] \Longrightarrow$
$[r,\ mod\ m] \downarrow\leq t = [r,\ mod\ m,\ (t - r)\ div\ m]$
⟨*proof*⟩

lemma *iMODb-cut-le*:
$[r,\ mod\ m,\ c] \downarrow\leq t =$ (
if $t < r$ *then* $\{\}$
else if $t < r + m * c$ *then* $[r,\ mod\ m,\ (t - r)\ div\ m]$
else $[r,\ mod\ m,\ c]$)
⟨*proof*⟩

lemma *iMODb-cut-le1*:
$t \in [r,\ mod\ m,\ c] \Longrightarrow$
$[r,\ mod\ m,\ c] \downarrow\leq t = [r,\ mod\ m,\ (t - r)\ div\ m]$
⟨*proof*⟩

lemma *iTILL-cut-less*:
$[\ldots n] \downarrow< t =$ (
if $n < t$ *then* $[\ldots n]$ *else*
if $t = 0$ *then* $\{\}$
else $[\ldots t - Suc\ 0]$)
⟨*proof*⟩

lemma *iTILL-cut-less1*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow [\ldots n] \downarrow< t = [\ldots t - Suc\ 0]$
⟨*proof*⟩

lemma *iFROM-cut-less*:
$[n\ldots] \downarrow< t =$ (
if $t \leq n$ *then* $\{\}$
else $[n\ldots,t - Suc\ n]$)
⟨*proof*⟩

lemma *iFROM-cut-less1*:
$n < t \Longrightarrow [n\ldots] \downarrow< t = [n\ldots,t - Suc\ n]$
⟨*proof*⟩

lemma *iIN-cut-less*:
$[n\ldots,d] \downarrow< t =$ (
if $t \leq n$ *then* $\{\}$ *else*
if $t \leq n + d$ *then* $[n\ldots,\ t - Suc\ n]$
else $[n\ldots,d]$)
⟨*proof*⟩

lemma *iIN-cut-less1*:
⟦ $t \in [n..,d]$; $n < t$ ⟧ ⟹ $[n..,d] \downarrow< t = [n.., t - Suc\ n]$
⟨*proof*⟩

lemma *iMOD-cut-less*:
$[r, mod\ m] \downarrow< t =$ (
if $t \leq r$ *then* {}
else $[r, mod\ m, (t - Suc\ r)\ div\ m]$)
⟨*proof*⟩

lemma *iMOD-cut-less1*:
⟦ $t \in [r, mod\ m]$; $r < t$ ⟧ ⟹
$[r, mod\ m] \downarrow< t = [r, mod\ m, (t - r)\ div\ m - Suc\ 0]$
⟨*proof*⟩

lemma *iMODb-cut-less*:
$[r, mod\ m, c] \downarrow< t =$ (
if $t \leq r$ *then* {} *else*
if $r + m * c < t$ *then* $[r, mod\ m, c]$
else $[r, mod\ m, (t - Suc\ r)\ div\ m]$)
⟨*proof*⟩

lemma *iMODb-cut-less1*: ⟦ $t \in [r, mod\ m, c]$; $r < t$ ⟧ ⟹
$[r, mod\ m, c] \downarrow< t = [r, mod\ m, (t - r)\ div\ m - Suc\ 0]$
⟨*proof*⟩

lemmas *iT-cut-le* =
iTILL-cut-le
iFROM-cut-le
iIN-cut-le
iMOD-cut-le
iMODb-cut-le

lemmas *iT-cut-le1* =
iTILL-cut-le1
iFROM-cut-le1
iIN-cut-le1
iMOD-cut-le1
iMODb-cut-le1

lemmas *iT-cut-less* =
iTILL-cut-less
iFROM-cut-less
iIN-cut-less
iMOD-cut-less
iMODb-cut-less

lemmas *iT-cut-less1* =
iTILL-cut-less1
iFROM-cut-less1
iIN-cut-less1
iMOD-cut-less1
iMODb-cut-less1

lemmas *iT-cut-le-less* =
iTILL-cut-le
iTILL-cut-less
iFROM-cut-le
iFROM-cut-less
iIN-cut-le
iIN-cut-less
iMOD-cut-le
iMOD-cut-less
iMODb-cut-le
iMODb-cut-less

lemmas *iT-cut-le-less1* =
iTILL-cut-le1
iTILL-cut-less1
iFROM-cut-le1
iFROM-cut-less1
iIN-cut-le1
iIN-cut-less1
iMOD-cut-le1
iMOD-cut-less1
iMODb-cut-le1
iMODb-cut-less1

lemma *iTILL-cut-ge*:
$[\ldots n] \downarrow\geq t = (\mathit{if}\ n < t\ \mathit{then}\ \{\}\ \mathit{else}\ [t\ldots,n-t])$
⟨*proof*⟩

corollary *iTILL-cut-ge1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = [t\ldots,n-t]$
⟨*proof*⟩

corollary *iTILL-cut-ge2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = \{\}$
⟨*proof*⟩

lemma *iTILL-cut-greater*:
$[\ldots n] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ \{\}\ \mathit{else}\ [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t])$
⟨*proof*⟩

corollary *iTILL-cut-greater1*:
$t \in [\ldots n] \Longrightarrow t < n \Longrightarrow [\ldots n] \downarrow> t = [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t]$
⟨*proof*⟩

corollary *iTILL-cut-greater2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow> t = \{\}$
⟨*proof*⟩

lemma *iFROM-cut-ge*:
$[n\ldots] \downarrow\geq t = (\mathit{if}\ n \leq t\ \mathit{then}\ [t\ldots]\ \mathit{else}\ [n\ldots])$
⟨*proof*⟩
corollary *iFROM-cut-ge1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\geq t = [t\ldots]$
⟨*proof*⟩

lemma *iFROM-cut-greater*:
$[n\ldots] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ [\mathit{Suc}\ t\ldots]\ \mathit{else}\ [n\ldots])$
⟨*proof*⟩
corollary *iFROM-cut-greater1*:
$t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow> t = [\mathit{Suc}\ t\ldots]$
⟨*proof*⟩

lemma *iIN-cut-ge*:
$[n\ldots,d] \downarrow\geq t =$ (
if $t < n$ *then* $[n\ldots,d]$ *else*
if $t \leq n{+}d$ *then* $[t\ldots,n{+}d{-}t]$
else $\{\}$)
⟨*proof*⟩

corollary *iIN-cut-ge1*: $t \in [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = [t\ldots,n{+}d{-}t]$
⟨*proof*⟩

corollary *iIN-cut-ge2*: $t \notin [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = (\mathit{if}\ t < n\ \mathit{then}\ [n\ldots,d]\ \mathit{else}\ \{\})$
⟨*proof*⟩

lemma *iIN-cut-greater*:
$[n\ldots,d] \downarrow> t =$ (
if $t < n$ *then* $[n\ldots,d]$ *else*
if $t < n{+}d$ *then* $[\mathit{Suc}\ t\ldots,n + d - \mathit{Suc}\ t]$
else $\{\}$)
⟨*proof*⟩

corollary *iIN-cut-greater1*:
$[\![\ t \in [n\ldots,d];\ t < n + d\ ]\!] \Longrightarrow$
$[n\ldots,d] \downarrow> t = [\mathit{Suc}\ t\ldots,n + d - \mathit{Suc}\ t]$
⟨*proof*⟩

lemma *mod-cut-greater-aux-t-less*:

$[\![\ 0 < (m::nat);\ r \leq t\ ]\!] \Longrightarrow$
$t < t + m - (t - r)\ mod\ m$
⟨*proof*⟩

lemma *mod-cut-greater-aux-le-x*:
$[\![\ (r::nat) \leq t;\ t < x;\ x\ mod\ m = r\ mod\ m]\!] \Longrightarrow$
$t + m - (t - r)\ mod\ m \leq x$
⟨*proof*⟩

lemma *iMOD-cut-greater*:
$[r,\ mod\ m] \downarrow> t =$ (
if $t < r$ *then* $[r,\ mod\ m]$ *else*
if $m = 0$ *then* $\{\}$
else $[t + m - (t - r)\ mod\ m,\ mod\ m]$)
⟨*proof*⟩

lemma *iMOD-cut-greater1*:
$t \in [r,\ mod\ m] \Longrightarrow$
$[r,\ mod\ m] \downarrow> t =$ (
if $m = 0$ *then* $\{\}$
else $[t + m,\ mod\ m]$)
⟨*proof*⟩

lemma *iMODb-cut-greater-aux*:
$[\![\ 0 < m;\ t < r + m * c;\ r \leq t]\!] \Longrightarrow$
$(r + m * c - (t + m - (t - r)\ mod\ m))\ div\ m =$
$c - Suc\ ((t - r)\ div\ m)$
⟨*proof*⟩

lemma *iMODb-cut-greater*:
$[r,\ mod\ m,\ c] \downarrow> t =$ (
if $t < r$ *then* $[r,\ mod\ m,\ c]$ *else*
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m - (t - r)\ mod\ m,\ mod\ m,\ c - Suc\ ((t-r)\ div\ m)]$)
⟨*proof*⟩

lemma *iMODb-cut-greater1*:
$t \in [r,\ mod\ m,\ c] \Longrightarrow$
$[r,\ mod\ m,\ c] \downarrow> t =$ (
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m,\ mod\ m,\ c - Suc\ ((t-r)\ div\ m)]$)
⟨*proof*⟩

lemma *iMOD-cut-ge*:
$[r,\ mod\ m] \downarrow\geq t =$ (
if $t \leq r$ *then* $[r,\ mod\ m]$ *else*
if $m = 0$ *then* $\{\}$

else $[t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m]$)
⟨*proof*⟩

lemma *iMOD-cut-ge1*:
$t \in [r,\ mod\ m] \Longrightarrow$
$[r,\ mod\ m] \downarrow\geq t = [t,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-cut-ge*:
$[r,\ mod\ m,\ c] \downarrow\geq t =$ (
if $t \leq r$ *then* $[r,\ mod\ m,\ c]$ *else*
if $r + m * c < t$ *then* $\{\}$
else $[t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m,\ c - (t + m - Suc\ r)\ div\ m]$)
⟨*proof*⟩

lemma *iMODb-cut-ge1*:
$t \in [r,\ mod\ m,\ c] \Longrightarrow$
$[r,\ mod\ m,\ c] \downarrow\geq t =$ (
if $r + m * c < t$ *then* $\{\}$
else $[t,\ mod\ m,\ c - (t - r)\ div\ m]$)
⟨*proof*⟩

lemma *iMOD-0-cut-greater*:
$t \in [r,\ mod\ 0] \Longrightarrow [r,\ mod\ 0] \downarrow> t = \{\}$
⟨*proof*⟩
lemma *iMODb-0-cut-greater*: $t \in [r,\ mod\ 0,\ c] \Longrightarrow$
$[r,\ mod\ 0,\ c] \downarrow> t = \{\}$
⟨*proof*⟩

lemmas *iT-cut-ge* =
iTILL-cut-ge
iFROM-cut-ge
iIN-cut-ge
iMOD-cut-ge
iMODb-cut-ge

lemmas *iT-cut-ge1* =
iTILL-cut-ge1
iFROM-cut-ge1
iIN-cut-ge1
iMOD-cut-ge1
iMODb-cut-ge1

lemmas *iT-cut-greater* =
iTILL-cut-greater
iFROM-cut-greater
iIN-cut-greater

iMOD-cut-greater
iMODb-cut-greater

lemmas *iT-cut-greater1* =
iTILL-cut-greater1
iFROM-cut-greater1
iIN-cut-greater1
iMOD-cut-greater1
iMODb-cut-greater1

lemmas *iT-cut-ge-greater* =
iTILL-cut-ge
iTILL-cut-greater
iFROM-cut-ge
iFROM-cut-greater
iIN-cut-ge
iIN-cut-greater
iMOD-cut-ge
iMOD-cut-greater
iMODb-cut-ge
iMODb-cut-greater
lemmas *iT-cut-ge-greater1* =
iTILL-cut-ge1
iTILL-cut-greater1
iFROM-cut-ge1
iFROM-cut-greater1
iIN-cut-ge1
iIN-cut-greater1
iMOD-cut-ge1
iMOD-cut-greater1
iMODb-cut-ge1
iMODb-cut-greater1

1.6 Cardinality of intervals

lemma *iFROM-card*: *card* $[n...] = 0$
⟨*proof*⟩

lemma *iTILL-card*: *card* $[...n] = Suc\ n$
⟨*proof*⟩

lemma *iIN-card*: *card* $[n...,d] = Suc\ d$
⟨*proof*⟩

lemma *iMOD-0-card*: *card* $[r,\ mod\ 0] = Suc\ 0$
⟨*proof*⟩

lemma *iMOD-card*: $0 < m \Longrightarrow card\ [r,\ mod\ m] = 0$
⟨*proof*⟩

lemma *iMOD-card-if*: *card* [*r*, *mod m*] = (*if m* = *0 then Suc 0 else 0*)
⟨*proof*⟩

lemma *iMODb-mod-0-card*: *card* [*r*, *mod 0*, *c*] = *Suc 0*
⟨*proof*⟩

lemma *iMODb-card*: *0* < *m* ⟹ *card* [*r*, *mod m*, *c*] = *Suc c*
⟨*proof*⟩

lemma *iMODb-card-if*:
card [*r*, *mod m*, *c*] = (*if m* = *0 then Suc 0 else Suc c*)
⟨*proof*⟩

lemmas *iT-card* =
iFROM-card
iTILL-card
iIN-card
iMOD-card-if
iMODb-card-if

Cardinality with *icard*

lemma *iFROM-icard*: *icard* [*n*...] = ∞
⟨*proof*⟩

lemma *iTILL-icard*: *icard* [...*n*] = *enat* (*Suc n*)
⟨*proof*⟩

lemma *iIN-icard*: *icard* [*n*...,*d*] = *enat* (*Suc d*)
⟨*proof*⟩

lemma *iMOD-0-icard*: *icard* [*r*, *mod 0*] = *eSuc 0*
⟨*proof*⟩

lemma *iMOD-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*] = ∞
⟨*proof*⟩

lemma *iMOD-icard-if*: *icard* [*r*, *mod m*] = (*if m* = *0 then eSuc 0 else* ∞)
⟨*proof*⟩

lemma *iMODb-mod-0-icard*: *icard* [*r*, *mod 0*, *c*] = *eSuc 0*
⟨*proof*⟩

lemma *iMODb-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*, *c*] = *enat* (*Suc c*)
⟨*proof*⟩

lemma *iMODb-icard-if*: *icard* [*r*, *mod m*, *c*] = *enat* (*if m* = *0 then Suc 0 else Suc c*)
⟨*proof*⟩

lemmas *iT-icard* =
iFROM-icard
iTILL-icard
iIN-icard
iMOD-icard-if
iMODb-icard-if

1.7 Functions *inext* and *iprev* with intervals

lemma
iFROM-inext: $t \in [n...] \Longrightarrow inext\ t\ [n...] = Suc\ t$ **and**
iTILL-inext: $t < n \Longrightarrow inext\ t\ [...n] = Suc\ t$ **and**
iIN-inext: $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n...,d] = Suc\ t$
⟨*proof*⟩

lemma
iFROM-iprev': $t \in [n...] \Longrightarrow iprev\ (Suc\ t)\ [n...] = t$ **and**
iFROM-iprev: $n < t \Longrightarrow iprev\ t\ [n...] = t - Suc\ 0$ **and**
iTILL-iprev: $t \in [...n] \Longrightarrow iprev\ t\ [...n] = t - Suc\ 0$ **and**
iIN-iprev: $[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow iprev\ t\ [n...,d] = t - Suc\ 0$ **and**
iIN-iprev': $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow iprev\ (Suc\ t)\ [n...,d] = t$
⟨*proof*⟩

lemma *iMOD-inext*: $t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] = t + m$
⟨*proof*⟩
lemma *iMOD-iprev*: $[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow iprev\ t\ [r,\ mod\ m] = t - m$
⟨*proof*⟩

lemma *iMOD-iprev'*: $t \in [r,\ mod\ m] \Longrightarrow iprev\ (t + m)\ [r,\ mod\ m] = t$
⟨*proof*⟩

lemma *iMODb-inext*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$inext\ t\ [r,\ mod\ m,\ c] = t + m$
⟨*proof*⟩

lemma *iMODb-iprev*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow$
$iprev\ t\ [r,\ mod\ m,\ c] = t - m$
⟨*proof*⟩

lemma *iMODb-iprev'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$iprev\ (t + m)\ [r,\ mod\ m,\ c] = t$
⟨*proof*⟩

lemmas *iT-inext* =
iFROM-inext

iTILL-inext
iIN-inext
iMOD-inext
iMODb-inext
lemmas *iT-iprev* =
iFROM-iprev′
iFROM-iprev
iTILL-iprev
iIN-iprev
iIN-iprev′
iMOD-iprev
iMOD-iprev′
iMODb-iprev
iMODb-iprev′

lemma *iFROM-inext-if*:
inext t [*n*...] = (*if t* ∈ [*n*...] *then Suc t else t*)
⟨*proof*⟩

lemma *iTILL-inext-if*:
inext t [...*n*] = (*if t* < *n then Suc t else t*)
⟨*proof*⟩

lemma *iIN-inext-if*:
inext t [*n*...,*d*] = (*if n* ≤ *t* ∧ *t* < *n* + *d then Suc t else t*)
⟨*proof*⟩

lemma *iMOD-inext-if*:
inext t [*r*, *mod m*] = (*if t* ∈ [*r*, *mod m*] *then t* + *m else t*)
⟨*proof*⟩

lemma *iMODb-inext-if*:
inext t [*r*, *mod m*, *c*] =
(*if t* ∈ [*r*, *mod m*, *c*] ∧ *t* < *r* + *m* ∗ *c then t* + *m else t*)
⟨*proof*⟩

lemmas *iT-inext-if* =
iFROM-inext-if
iTILL-inext-if
iIN-inext-if
iMOD-inext-if
iMODb-inext-if

lemma *iFROM-iprev-if*:
iprev t [*n*...] = (*if n* < *t then t* − *Suc 0 else t*)
⟨*proof*⟩
lemma *iTILL-iprev-if*:
iprev t [...*n*] = (*if t* ∈ [...*n*] *then t* − *Suc 0 else t*)
⟨*proof*⟩

lemma *iIN-iprev-if*:
iprev t [n...,d] = (if n < t ∧ t ≤ n + d then t − Suc 0 else t)
⟨*proof*⟩
lemma *iMOD-iprev-if*:
iprev t [r, mod m] =
(if t ∈ [r, mod m] ∧ r < t then t − m else t)
⟨*proof*⟩
lemma *iMODb-iprev-if*:
iprev t [r, mod m, c] =
(if t ∈ [r, mod m, c] ∧ r < t then t − m else t)
⟨*proof*⟩

lemmas *iT-iprev-if =*
iFROM-iprev-if
iTILL-iprev-if
iIN-iprev-if
iMOD-iprev-if
iMODb-iprev-if

The difference between an element and the next/previous element is constant if the element is different from Min/Max of the interval

lemma *iFROM-inext-diff-const*:
t ∈ [n...] ⟹ inext t [n...] − t = Suc 0
⟨*proof*⟩

lemma *iFROM-iprev-diff-const*:
n < t ⟹ t − iprev t [n...] = Suc 0
⟨*proof*⟩

lemma *iFROM-iprev-diff-const′*:
t ∈ [n...] ⟹ Suc t − iprev (Suc t) [n...] = Suc 0
⟨*proof*⟩

lemma *iTILL-inext-diff-const*:
t < n ⟹ inext t [...n] − t = Suc 0
⟨*proof*⟩
lemma *iTILL-iprev-diff-const*:
⟦ t ∈ [...n]; 0 < t ⟧ ⟹ t − iprev t [...n] = Suc 0
⟨*proof*⟩

lemma *iIN-inext-diff-const*:
⟦ n ≤ t; t < n + d ⟧ ⟹ inext t [n...,d] − t = Suc 0
⟨*proof*⟩

lemma *iIN-iprev-diff-const*:
⟦ n < t; t ≤ n + d ⟧ ⟹ t − iprev t [n...,d] = Suc 0
⟨*proof*⟩
lemma *iIN-iprev-diff-const′*:
⟦ n ≤ t; t < n + d ⟧ ⟹ Suc t − iprev (Suc t) [n...,d] = Suc 0

⟨*proof*⟩

lemma *iMOD-inext-diff-const*:
$t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] - t = m$
⟨*proof*⟩

lemma *iMOD-iprev-diff-const'*:
$t \in [r,\ mod\ m] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m] = m$
⟨*proof*⟩

lemma *iMOD-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m] = m$
⟨*proof*⟩

lemma *iMODb-inext-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow inext\ t\ [r,\ mod\ m,\ c] - t = m$
⟨*proof*⟩

lemma *iMODb-iprev-diff-const'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m,\ c] = m$
⟨*proof*⟩

lemma *iMODb-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m,\ c] = m$
⟨*proof*⟩

lemmas *iT-inext-diff-const* =
iFROM-inext-diff-const
iTILL-inext-diff-const
iIN-inext-diff-const
iMOD-inext-diff-const
iMODb-inext-diff-const
lemmas *iT-iprev-diff-const* =
iFROM-iprev-diff-const
iFROM-iprev-diff-const'
iTILL-iprev-diff-const
iIN-iprev-diff-const
iIN-iprev-diff-const'
iMOD-iprev-diff-const'
iMOD-iprev-diff-const
iMODb-iprev-diff-const'
iMODb-iprev-diff-const

1.7.1 Mirroring of intervals

lemma
iIN-mirror-elem: *mirror-elem* $x\ [n\ldots,d] = n + n + d - x$ **and**
iTILL-mirror-elem: *mirror-elem* $x\ [\ldots n] = n - x$ **and**

iMODb-mirror-elem: *mirror-elem x* [*r*, *mod m*, *c*] = *r* + *r* + *m* ∗ *c* − *x*
⟨*proof*⟩

lemma *iMODb-imirror-bounds*:
r′ + *m′* ∗ *c′* ≤ *l* + *r* ⟹
imirror-bounds [*r′*, *mod m′*, *c′*] *l r* = [*l* + *r* − *r′* − *m′* ∗ *c′*, *mod m′*, *c′*]
⟨*proof*⟩

lemma *iIN-imirror-bounds*:
n + *d* ≤ *l* + *r* ⟹ *imirror-bounds* [*n*...,*d*] *l r* = [*l* + *r* − *n* − *d*...,*d*]
⟨*proof*⟩

lemma *iTILL-imirror-bounds*:
n ≤ *l* + *r* ⟹ *imirror-bounds* [...*n*] *l r* = [*l* + *r* − *n*...,*n*]
⟨*proof*⟩

lemmas *iT-imirror-bounds* =
iTILL-imirror-bounds
iIN-imirror-bounds
iMODb-imirror-bounds

lemma *iMODb-imirror-ident*: *imirror* [*r*, *mod m*, *c*] = [*r*, *mod m*, *c*]
⟨*proof*⟩
lemma *iIN-imirror-ident*: *imirror* [*n*...,*d*] = [*n*...,*d*]
⟨*proof*⟩
lemma *iTILL-imirror-ident*: *imirror* [...*n*] = [...*n*]
⟨*proof*⟩

lemmas *iT-imirror-ident* =
iTILL-imirror-ident
iIN-imirror-ident
iMODb-imirror-ident

1.7.2 Functions *inext-nth* and *iprev-nth* on intervals

lemma *iFROM-inext-nth* : [*n*...] → *a* = *n* + *a*
⟨*proof*⟩

lemma *iIN-inext-nth* : *a* ≤ *d* ⟹ [*n*...,*d*] → *a* = *n* + *a*
⟨*proof*⟩

lemma *iIN-iprev-nth*: *a* ≤ *d* ⟹ [*n*...,*d*] ← *a* = *n* + *d* − *a*
⟨*proof*⟩

lemma *iIN-inext-nth-if* :
[*n*...,*d*] → *a* = (*if a* ≤ *d then n* + *a else n* + *d*)
⟨*proof*⟩

lemma *iIN-iprev-nth-if*:
$[n..,d] \leftarrow a = (\mathit{if}\ a \leq d\ \mathit{then}\ n + d - a\ \mathit{else}\ n)$
⟨*proof*⟩

lemma *iTILL-inext-nth* : $a \leq n \Longrightarrow [..n] \rightarrow a = a$
⟨*proof*⟩

lemma *iTILL-inext-nth-if* :
$[..n] \rightarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ a\ \mathit{else}\ n)$
⟨*proof*⟩

lemma *iTILL-iprev-nth*: $a \leq n \Longrightarrow [..n] \leftarrow a = n - a$
⟨*proof*⟩

lemma *iTILL-iprev-nth-if*:
$[..n] \leftarrow a= (\mathit{if}\ a \leq n\ \mathit{then}\ n - a\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *iMOD-inext-nth*: $[r,\ mod\ m] \rightarrow a = r + m * a$
⟨*proof*⟩

lemma *iMODb-inext-nth*: $a \leq c \Longrightarrow [r,\ mod\ m,\ c] \rightarrow a = r + m * a$
⟨*proof*⟩

lemma *iMODb-inext-nth-if*:
$[r,\ mod\ m,\ c] \rightarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * a\ \mathit{else}\ r + m * c)$
⟨*proof*⟩

lemma *iMODb-iprev-nth*:
$a \leq c \Longrightarrow [r,\ mod\ m,\ c] \leftarrow a = r + m * (c - a)$
⟨*proof*⟩

lemma *iMODb-iprev-nth-if*:
$[r,\ mod\ m,\ c] \leftarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * (c - a)\ \mathit{else}\ r)$
⟨*proof*⟩

lemma *iIN-iFROM-inext-nth*:
$a \leq d \Longrightarrow [n..,d] \rightarrow a = [n..] \rightarrow a$
⟨*proof*⟩

lemma *iIN-iFROM-inext*:
$a < n + d \Longrightarrow \mathit{inext}\ a\ [n..,d] = \mathit{inext}\ a\ [n..]$
⟨*proof*⟩

lemma *iMOD-iMODb-inext-nth*:
$a \leq c \Longrightarrow [r,\ mod\ m,\ c] \rightarrow a = [r,\ mod\ m] \rightarrow a$
⟨*proof*⟩

lemma *iMOD-iMODb-inext*:
$a < r + m * c \Longrightarrow inext\ a\ [r,\ mod\ m,\ c] = inext\ a\ [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inext-nth-Suc-diff*:
$([r,\ mod\ m] \rightarrow (Suc\ n)) - ([r,\ mod\ m] \rightarrow n) = m$
⟨*proof*⟩

lemma *iMOD-inext-nth-diff*:
$([r,\ mod\ m] \rightarrow a) - ([r,\ mod\ m] \rightarrow b) = (a - b) * m$
⟨*proof*⟩

lemma *iMODb-inext-nth-diff*: $[\![\ a \leq c;\ b \leq c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] \rightarrow a) - ([r,\ mod\ m,\ c] \rightarrow b) = (a - b) * m$
⟨*proof*⟩

1.8 Induction with intervals

lemma *iFROM-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots];\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iIN-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots,d];\ k \neq n + d;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots,d]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iTILL-induct*:
$[\![\ P\ 0;\ \bigwedge k.\ [\![\ k \in [\ldots n];\ k \neq n;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [\ldots n]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMOD-induct*:
$[\![\ P\ r;\ \bigwedge k.\ [\![\ k \in [r,\ mod\ m];\ P\ k\ ]\!] \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-induct*:
$[\![\ P\ r;\ \bigwedge k.\ [\![\ k \in [r,\ mod\ m,\ c];\ k \neq r + m * c;\ P\ k\ ]\!] \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iIN-rev-induct*:
$[\![\ P\ (n + d);\ \bigwedge k.\ [\![\ k \in [n\ldots,d];\ k \neq n;\ P\ k\ ]\!] \Longrightarrow P\ (k - Suc\ 0);\ a \in [n\ldots,d]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iTILL-rev-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [\ldots n];\ 0 < k;\ P\ k\ ]\!] \Longrightarrow P\ (k - Suc\ 0);\ a \in [\ldots n]\ ]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-rev-induct*:
⟦ $P\ (r + m * c)$; $\bigwedge k.$ ⟦ $k \in [r, mod\ m, c]$; $k \neq r$; $P\ k$ ⟧ $\Longrightarrow P\ (k - m)$; $a \in [r, mod\ m, c]$ ⟧ $\Longrightarrow P\ a$
⟨*proof*⟩

end

2 Arithmetic operators on natural intervals

theory *IL-IntervalOperators*
imports *IL-Interval*
begin

2.1 Arithmetic operations with intervals

2.1.1 Addition of and multiplication by constants

definition *iT-Plus* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus$ 55)
where $I \oplus k \equiv (\lambda n.(n + k))\ `\ I$

definition *iT-Mult* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\otimes$ 55)
where *iT-Mult-def* : $I \otimes k \equiv (\lambda n.(n * k))\ `\ I$

lemma *iT-Plus-image-conv*: $I \oplus k = (\lambda n.(n + k))\ `\ I$
⟨*proof*⟩

lemma *iT-Mult-image-conv*: $I \otimes k = (\lambda n.(n * k))\ `\ I$
⟨*proof*⟩

lemma *iT-Plus-empty*: $\{\} \oplus k = \{\}$
⟨*proof*⟩

lemma *iT-Mult-empty*: $\{\} \otimes k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-not-empty*: $I \neq \{\} \Longrightarrow I \oplus k \neq \{\}$
⟨*proof*⟩

lemma *iT-Mult-not-empty*: $I \neq \{\} \Longrightarrow I \otimes k \neq \{\}$
⟨*proof*⟩

lemma *iT-Plus-empty-iff*: $(I \oplus k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Mult-empty-iff*: $(I \otimes k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Plus-mono*: $A \subseteq B \Longrightarrow A \oplus k \subseteq B \oplus k$
⟨*proof*⟩

lemma *iT-Mult-mono*: $A \subseteq B \Longrightarrow A \otimes k \subseteq B \otimes k$
⟨*proof*⟩

lemma *iT-Mult-0*: $I \neq \{\} \Longrightarrow I \otimes 0 = [\ldots 0]$
⟨*proof*⟩

corollary *iT-Mult-0-if*: $I \otimes 0 = (\mathit{if}\ I = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ [\ldots 0])$
⟨*proof*⟩

lemma *iT-Plus-mem-iff*: $x \in (I \oplus k) = (k \leq x \wedge (x - k) \in I)$
⟨*proof*⟩

lemma *iT-Plus-mem-iff2*: $x + k \in (I \oplus k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff-0*: $x \in (I \otimes 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff*:
$0 < k \Longrightarrow x \in (I \otimes k) = (x\ \mathit{mod}\ k = 0 \wedge x\ \mathit{div}\ k \in I)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff2*: $0 < k \Longrightarrow x * k \in (I \otimes k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-singleton*: $\{a\} \oplus k = \{a + k\}$
⟨*proof*⟩

lemma *iT-Mult-singleton*: $\{a\} \otimes k = \{a * k\}$
⟨*proof*⟩

lemma *iT-Plus-Un*: $(A \cup B) \oplus k = (A \oplus k) \cup (B \oplus k)$
⟨*proof*⟩

lemma *iT-Mult-Un*: $(A \cup B) \otimes k = (A \otimes k) \cup (B \otimes k)$
⟨*proof*⟩

lemma *iT-Plus-Int*: $(A \cap B) \oplus k = (A \oplus k) \cap (B \oplus k)$
⟨*proof*⟩

lemma *iT-Mult-Int*: $0 < k \Longrightarrow (A \cap B) \otimes k = (A \otimes k) \cap (B \otimes k)$
⟨*proof*⟩

lemma *iT-Plus-image*: $f \text{ ` } I \oplus n = (\lambda x.\ f\ x + n) \text{ ` } I$
⟨*proof*⟩

lemma *iT-Mult-image*: $f \text{ ` } I \otimes n = (\lambda x.\ f\ x * n) \text{ ` } I$
⟨*proof*⟩

lemma *iT-Plus-commute*: $I \oplus a \oplus b = I \oplus b \oplus a$
⟨*proof*⟩

lemma *iT-Mult-commute*: $I \otimes a \otimes b = I \otimes b \otimes a$
⟨*proof*⟩

lemma *iT-Plus-assoc*:$I \oplus a \oplus b = I \oplus (a + b)$
⟨*proof*⟩

lemma *iT-Mult-assoc*:$I \otimes a \otimes b = I \otimes (a * b)$
⟨*proof*⟩

lemma *iT-Plus-Mult-distrib*: $I \oplus n \otimes m = I \otimes m \oplus n * m$
⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩
lemma *iT-Plus-finite-iff*: *finite* $(I \oplus k) =$ *finite* I
⟨*proof*⟩

lemma *iT-Mult-0-finite*: *finite* $(I \otimes 0)$
⟨*proof*⟩

lemma *iT-Mult-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \otimes k) =$ *finite* I
⟨*proof*⟩

lemma *iT-Plus-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oplus k) = iMin\ I + k$
⟨*proof*⟩

lemma *iT-Mult-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \otimes k) = iMin\ I * k$
⟨*proof*⟩

lemma *iT-Plus-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \oplus k) = Max\ I + k$
⟨*proof*⟩

lemma *iT-Mult-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \otimes k) = Max\ I * k$
⟨*proof*⟩

corollary
iMOD-mult-0: $[r,\ mod\ m] \otimes 0 = [\ldots 0]$ **and**
iMODb-mult-0: $[r,\ mod\ m,\ c] \otimes 0 = [\ldots 0]$ **and**
iFROM-mult-0: $[n\ldots] \otimes 0 = [\ldots 0]$ **and**
iIN-mult-0: $[n\ldots,d] \otimes 0 = [\ldots 0]$ **and**
iTILL-mult-0: $[\ldots n] \otimes 0 = [\ldots 0]$
⟨*proof*⟩

lemmas *iT-mult-0* =
iTILL-mult-0
iFROM-mult-0
iIN-mult-0
iMOD-mult-0
iMODb-mult-0

lemma *iT-Plus-0*: $I \oplus 0 = I$
⟨*proof*⟩

lemma *iT-Mult-1*: $I \otimes Suc\ 0 = I$
⟨*proof*⟩

corollary
iFROM-add-Min: $iMin\ ([n...] \oplus k) = n + k$ **and**
iIN-add-Min: $iMin\ ([n...,d] \oplus k) = n + k$ **and**
iTILL-add-Min: $iMin\ ([...n] \oplus k) = k$ **and**
iMOD-add-Min: $iMin\ ([r,\ mod\ m] \oplus k) = r + k$ **and**
iMODb-add-Min: $iMin\ ([r,\ mod\ m,\ c] \oplus k) = r + k$
⟨*proof*⟩

corollary
iFROM-mult-Min: $iMin\ ([n...] \otimes k) = n * k$ **and**
iIN-mult-Min: $iMin\ ([n...,d] \otimes k) = n * k$ **and**
iTILL-mult-Min: $iMin\ ([...n] \otimes k) = 0$ **and**
iMOD-mult-Min: $iMin\ ([r,\ mod\ m] \otimes k) = r * k$ **and**
iMODb-mult-Min: $iMin\ ([r,\ mod\ m,\ c] \otimes k) = r * k$
⟨*proof*⟩

lemmas *iT-add-Min* =
iIN-add-Min
iTILL-add-Min
iFROM-add-Min
iMOD-add-Min
iMODb-add-Min

lemmas *iT-mult-Min* =
iIN-mult-Min
iTILL-mult-Min
iFROM-mult-Min
iMOD-mult-Min
iMODb-mult-Min

lemma *iFROM-add*: $[n...] \oplus k = [n+k...]$
⟨*proof*⟩

lemma *iIN-add*: $[n...,d] \oplus k = [n+k...,d]$

⟨*proof*⟩

lemma *iTILL-add*: $[\ldots i] \oplus k = [k\ldots,i]$
⟨*proof*⟩

lemma *iMOD-add*: $[r,\ mod\ m] \oplus k = [r + k,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-add*: $[r,\ mod\ m,\ c] \oplus k = [r + k,\ mod\ m,\ c]$
⟨*proof*⟩

lemmas *iT-add* =
iMOD-add
iMODb-add
iFROM-add
iIN-add
iTILL-add
iT-Plus-singleton

lemma *iFROM-mult*: $[n\ldots] \otimes k = [\ n * k,\ mod\ k\ ]$
⟨*proof*⟩

lemma *iIN-mult*: $[n\ldots,d] \otimes k = [\ n * k,\ mod\ k,\ d\ ]$
⟨*proof*⟩

lemma *iTILL-mult*: $[\ldots n] \otimes k = [\ 0,\ mod\ k,\ n\ ]$
⟨*proof*⟩

lemma *iMOD-mult*: $[r,\ mod\ m\ ] \otimes k = [\ r * k,\ mod\ m * k\ ]$
⟨*proof*⟩

lemma *iMODb-mult*:
$[r,\ mod\ m,\ c\ ] \otimes k = [\ r * k,\ mod\ m * k,\ c\ ]$
⟨*proof*⟩

lemmas *iT-mult* =
iTILL-mult
iFROM-mult
iIN-mult
iMOD-mult
iMODb-mult
iT-Mult-singleton

2.1.2 Some conversions between intervals using constant addition and multiplication

lemma *iFROM-conv*: $[n\ldots] = UNIV \oplus n$
⟨*proof*⟩

lemma *iIN-conv*: $[n...,d] = [...d] \oplus n$
⟨*proof*⟩

lemma *iMOD-conv*: $[r,\ mod\ m] = [0...] \otimes m \oplus r$
⟨*proof*⟩

lemma *iMODb-conv*: $[r,\ mod\ m,\ c] = [...c] \otimes m \oplus r$
⟨*proof*⟩

Some examples showing the utility of iMODb_conv

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
⟨*proof*⟩

lemma $[r,\ mod\ m,\ 4] = \{r,\ r+m,\ r+2*m,\ r+3*m,\ r+4*m\}$
⟨*proof*⟩

lemma $[2,\ mod\ 10,\ 4] = \{2,\ 12,\ 22,\ 32,\ 42\}$
⟨*proof*⟩

2.1.3 Subtraction of constants

definition *iT-Plus-neg* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus -$ 55) **where**
$I \oplus - k \equiv \{x.\ x + k \in I\}$

lemma *iT-Plus-neg-mem-iff*: $(x \in I \oplus - k) = (x + k \in I)$
⟨*proof*⟩

lemma *iT-Plus-neg-mem-iff2*: $k \leq x \Longrightarrow (x - k \in I \oplus - k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-neg-image-conv*: $I \oplus - k = (\lambda n.(n - k))\ `\ (I \downarrow \geq k)$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-eq*: $t \leq k \Longrightarrow (I \downarrow \geq t) \oplus - k = I \oplus - k$
⟨*proof*⟩

lemma *iT-Plus-neg-mono*: $A \subseteq B \Longrightarrow A \oplus - k \subseteq B \oplus - k$
⟨*proof*⟩

lemma *iT-Plus-neg-empty*: $\{\} \oplus - k = \{\}$
⟨*proof*⟩
lemma *iT-Plus-neg-Max-less-empty*:

$[\![\mathit{finite}\ I;\ \mathit{Max}\ I < k\ ]\!] \Longrightarrow I \oplus\!\!- k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-neg-not-empty-iff*: $(I \oplus\!\!- k \neq \{\}) = (\exists\, x{\in}I.\ k \leq x)$
⟨*proof*⟩

lemma *iT-Plus-neg-empty-iff*:
$(I \oplus\!\!- k = \{\}) = (I = \{\} \vee (\mathit{finite}\ I \wedge \mathit{Max}\ I < k))$
⟨*proof*⟩

lemma *iT-Plus-neg-assoc*: $(I \oplus\!\!- a) \oplus\!\!- b = I \oplus\!\!- (a + b)$
⟨*proof*⟩

lemma *iT-Plus-neg-commute*: $I \oplus\!\!- a \oplus\!\!- b = I \oplus\!\!- b \oplus\!\!- a$
⟨*proof*⟩

lemma *iT-Plus-neg-0*: $I \oplus\!\!- 0 = I$
⟨*proof*⟩

lemma *iT-Plus-Plus-neg-assoc*: $b \leq a \Longrightarrow I \oplus a \oplus\!\!- b = I \oplus (a - b)$
⟨*proof*⟩

lemma *iT-Plus-Plus-neg-assoc2*: $a \leq b \Longrightarrow I \oplus a \oplus\!\!- b = I \oplus\!\!- (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-le-cut-eq*:
$a \leq b \Longrightarrow (I \oplus\!\!- a) \oplus b = (I \downarrow\!\geq a) \oplus (b - a)$
⟨*proof*⟩

corollary *iT-Plus-neg-Plus-le-Min-eq*:
$[\![a \leq b;\ a \leq \mathit{iMin}\ I\ ]\!] \Longrightarrow (I \oplus\!\!- a) \oplus b = I \oplus (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-ge-cut-eq*:
$b \leq a \Longrightarrow (I \oplus\!\!- a) \oplus b = (I \downarrow\!\geq a) \oplus\!\!- (a - b)$
⟨*proof*⟩

corollary *iT-Plus-neg-Plus-ge-Min-eq*:
$[\![b \leq a;\ a \leq \mathit{iMin}\ I\ ]\!] \Longrightarrow (I \oplus\!\!- a) \oplus b = I \oplus\!\!- (a - b)$
⟨*proof*⟩

lemma *iT-Plus-neg-Mult-distrib*:
$0 < m \Longrightarrow I \oplus\!\!- n \otimes m = I \otimes m \oplus\!\!- n * m$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-le-inverse*: $k \leq \mathit{iMin}\ I \Longrightarrow I \oplus\!\!- k \oplus k = I$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-inverse*: $I \oplus\!\!- k \oplus k = I \downarrow\!\geq k$

⟨*proof*⟩

lemma *iT-Plus-Plus-neg-inverse*: $I \oplus k \oplus- k = I$
⟨*proof*⟩

lemma *iT-Plus-neg-Un*: $(A \cup B) \oplus- k = (A \oplus- k) \cup (B \oplus- k)$
⟨*proof*⟩

lemma *iT-Plus-neg-Int*: $(A \cap B) \oplus- k = (A \oplus- k) \cap (B \oplus- k)$
⟨*proof*⟩

lemma *iT-Plus-neg-Max-singleton*: $[\![\mathit{finite}\ I;\ I \neq \{\}\ ]\!] \Longrightarrow I \oplus- \mathit{Max}\ I = \{0\}$
⟨*proof*⟩

lemma *iT-Plus-neg-singleton*: $\{a\} \oplus- k = (\mathit{if}\ k \leq a\ \mathit{then}\ \{a - k\}\ \mathit{else}\ \{\})$
⟨*proof*⟩

corollary *iT-Plus-neg-singleton1*: $k \leq a \Longrightarrow \{a\} \oplus- k = \{a-k\}$
⟨*proof*⟩

corollary *iT-Plus-neg-singleton2*: $a < k \Longrightarrow \{a\} \oplus- k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-neg-finite-iff*: $\mathit{finite}\ (I \oplus- k) = \mathit{finite}\ I$
⟨*proof*⟩

lemma *iT-Plus-neg-Min*:
$I \oplus- k \neq \{\} \Longrightarrow \mathit{iMin}\ (I \oplus- k) = \mathit{iMin}\ (I \downarrow\geq k) - k$
⟨*proof*⟩

lemma *iT-Plus-neg-Max*:
$[\![\mathit{finite}\ I;\ I \oplus- k \neq \{\}\ ]\!] \Longrightarrow \mathit{Max}\ (I \oplus- k) = \mathit{Max}\ I - k$
⟨*proof*⟩

Subtractions of constants from intervals

lemma *iFROM-add-neg*: $[n\ldots] \oplus- k = [n - k\ldots]$
⟨*proof*⟩

lemma *iTILL-add-neg*: $[\ldots n] \oplus- k = (\mathit{if}\ k \leq n\ \mathit{then}\ [\ldots n - k]\ \mathit{else}\ \{\})$
⟨*proof*⟩
lemma *iTILL-add-neg1*: $k \leq n \Longrightarrow [\ldots n] \oplus- k = [\ldots n-k]$
⟨*proof*⟩
lemma *iTILL-add-neg2*: $n < k \Longrightarrow [\ldots n] \oplus- k = \{\}$
⟨*proof*⟩

lemma *iIN-add-neg*:
$[n\ldots,d] \oplus- k = ($
if $k \leq n$ *then* $[n - k\ldots,d]$

else if $k \leq n + d$ *then* $[\ldots n + d - k]$ *else* $\{\}$)
⟨*proof*⟩

lemma *iIN-add-neg1*: $k \leq n \Longrightarrow [n\ldots,d] \oplus- k = [n - k\ldots,d]$
⟨*proof*⟩

lemma *iIN-add-neg2*: $[\![\ n \leq k;\ k \leq n + d\ ]\!] \Longrightarrow [n\ldots,d] \oplus- k = [\ldots n + d - k]$
⟨*proof*⟩

lemma *iIN-add-neg3*: $n + d < k \Longrightarrow [n\ldots,d] \oplus- k = \{\}$
⟨*proof*⟩

lemma *iMOD-0-add-neg*: $[r,\ mod\ 0] \oplus- k = \{r\} \oplus- k$
⟨*proof*⟩

lemma *iMOD-gr0-add-neg*:
$0 < m \Longrightarrow$
$[r,\ mod\ m] \oplus- k =$ (
if $k \leq r$ *then* $[r - k,\ mod\ m]$
else $[(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m]$)
⟨*proof*⟩

lemma *iMOD-add-neg*:
$[r,\ mod\ m] \oplus- k =$ (
if $k \leq r$ *then* $[r - k,\ mod\ m]$
else if $0 < m$ *then* $[(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m]$ *else* $\{\}$)
⟨*proof*⟩

corollary *iMOD-add-neg1*:
$k \leq r \Longrightarrow [r,\ mod\ m] \oplus- k = [r - k,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-add-neg2*:
$[\![\ 0 < m;\ r < k\ ]\!] \Longrightarrow [r,\ mod\ m] \oplus- k = [(m + r\ mod\ m - k\ mod\ m)\ mod\ m,$
$mod\ m]$
⟨*proof*⟩

lemma *iMODb-mod-0-add-neg*: $[r,\ mod\ 0,\ c] \oplus- k = \{r\} \oplus- k$
⟨*proof*⟩

lemma *iMODb-add-neg*:
$[r,\ mod\ m,\ c] \oplus- k =$ (
if $k \leq r$ *then* $[r - k,\ mod\ m,\ c]$
else

if $k \leq r + m * c$ *then*
$[(m + r \mathbin{mod} m - k \mathbin{mod} m) \mathbin{mod} m, \mathbin{mod} m, (r + m * c - k) \mathbin{div} m]$
else $\{\}$)
⟨*proof*⟩

lemma *iMODb-add-neg'*:
$[r, \mathbin{mod} m, c] \oplus- k =$ (
if $k \leq r$ *then* $[r - k, \mathbin{mod} m, c]$
else if $k \leq r + m * c$ *then*
if $k \mathbin{mod} m \leq r \mathbin{mod} m$
then $[\ r \mathbin{mod} m - k \mathbin{mod} m, \mathbin{mod} m, c + r \mathbin{div} m - k \mathbin{div} m]$
else $[\ m + r \mathbin{mod} m - k \mathbin{mod} m, \mathbin{mod} m, c + r \mathbin{div} m - Suc\ (k \mathbin{div} m)\ ]$
else $\{\}$)
⟨*proof*⟩

corollary *iMODb-add-neg1*:
$k \leq r \Longrightarrow [r, \mathbin{mod} m, c] \oplus- k = [r - k, \mathbin{mod} m, c]$
⟨*proof*⟩

corollary *iMODb-add-neg2*:
$[\![\ r < k; k \leq r + m * c\ ]\!] \Longrightarrow$
$[r, \mathbin{mod} m, c] \oplus- k =$
$[(m + r \mathbin{mod} m - k \mathbin{mod} m) \mathbin{mod} m, \mathbin{mod} m, (r + m * c - k) \mathbin{div} m]$
⟨*proof*⟩

corollary *iMODb-add-neg2-mod-le*:
$[\![\ r < k; k \leq r + m * c; k \mathbin{mod} m \leq r \mathbin{mod} m\ ]\!] \Longrightarrow$
$[r, \mathbin{mod} m, c] \oplus- k =$
$[\ r \mathbin{mod} m - k \mathbin{mod} m, \mathbin{mod} m, c + r \mathbin{div} m - k \mathbin{div} m]$
⟨*proof*⟩

corollary *iMODb-add-neg2-mod-less*:
$[\![\ r < k; k \leq r + m * c; r \mathbin{mod} m < k \mathbin{mod} m]\!] \Longrightarrow$
$[r, \mathbin{mod} m, c] \oplus- k =$
$[\ m + r \mathbin{mod} m - k \mathbin{mod} m, \mathbin{mod} m, c + r \mathbin{div} m - Suc\ (k \mathbin{div} m)\ ]$
⟨*proof*⟩

lemma *iMODb-add-neg3*: $r + m * c < k \Longrightarrow [r, \mathbin{mod} m, c] \oplus- k = \{\}$
⟨*proof*⟩

lemmas *iT-add-neg* =
iFROM-add-neg
iIN-add-neg
iTILL-add-neg
iMOD-add-neg
iMODb-add-neg
iT-Plus-neg-singleton

2.1.4 Subtraction of intervals from constants

definition *iT-Minus* :: *Time* ⇒ *iT* ⇒ *iT* (**infixl** ⊖ *55*)
where $k \ominus I \equiv \{x.\ x \leq k \wedge (k - x) \in I\}$

lemma *iT-Minus-mem-iff*: $(x \in k \ominus I) = (x \leq k \wedge k - x \in I)$
⟨*proof*⟩

lemma *iT-Minus-mono*: $A \subseteq B \Longrightarrow k \ominus A \subseteq k \ominus B$
⟨*proof*⟩

lemma *iT-Minus-image-conv*: $k \ominus I = (\lambda x.\ k - x)\ `\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-cut-eq*: $k \leq t \Longrightarrow k \ominus (I \downarrow\leq t) = k \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Minus-cut-eq*: $k \ominus (k \ominus (I \downarrow\leq k)) = I \downarrow\leq k$
⟨*proof*⟩

lemma $10 \ominus [\ldots 3] = [7\ldots,3]$
⟨*proof*⟩

lemma *iT-Minus-empty*: $k \ominus \{\} = \{\}$
⟨*proof*⟩

lemma *iT-Minus-0*: $k \ominus \{0\} = \{k\}$
⟨*proof*⟩

lemma *iT-Minus-bound*: $x \in k \ominus I \Longrightarrow x \leq k$
⟨*proof*⟩

lemma *iT-Minus-finite*: *finite* $(k \ominus I)$
⟨*proof*⟩

lemma *iT-Minus-less-Min-empty*: $k < iMin\ I \Longrightarrow k \ominus I = \{\}$
⟨*proof*⟩

lemma *iT-Minus-Min-singleton*: $I \neq \{\} \Longrightarrow (iMin\ I) \ominus I = \{0\}$
⟨*proof*⟩

lemma *iT-Minus-empty-iff*: $(k \ominus I = \{\}) = (I = \{\} \vee k < iMin\ I)$
⟨*proof*⟩

lemma *iT-Minus-imirror-conv*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus\!-\ (iMin\ I + Max\ (I \downarrow\leq k))$
⟨*proof*⟩

lemma *iT-Minus-imirror-conv'*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ (I \downarrow\leq k) + Max\ (I \downarrow\leq k))$
⟨*proof*⟩

lemma *iT-Minus-Max*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow Max\ (k \ominus I) = k - (iMin\ I)$
⟨*proof*⟩

lemma *iT-Minus-Min*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow iMin\ (k \ominus I) = k - (Max\ (I \downarrow\leq k))$
⟨*proof*⟩

lemma *iT-Minus-Minus-eq*: $[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow k \ominus (k \ominus I) = I$
⟨*proof*⟩

lemma *iT-Minus-Minus-eq2*: $I \subseteq [\ldots k] \Longrightarrow k \ominus (k \ominus I) = I$
⟨*proof*⟩

lemma *iT-Minus-Minus*: $a \ominus (b \ominus I) = (I \downarrow\leq b) \oplus a \oplus- b$
⟨*proof*⟩

lemma *iT-Minus-Plus-empty*: $k < n \Longrightarrow k \ominus (I \oplus n) = \{\}$
⟨*proof*⟩
lemma *iT-Minus-Plus-commute*: $n \leq k \Longrightarrow k \ominus (I \oplus n) = (k - n) \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Plus-cut-assoc*: $(k \ominus I) \oplus n = (k + n) \ominus (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-Plus-assoc*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩
lemma *iT-Minus-Plus-assoc2*:
$I \subseteq [\ldots k] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Un*: $k \ominus (A \cup B) = (k \ominus A) \cup (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-Int*: $k \ominus (A \cap B) = (k \ominus A) \cap (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-singleton*: $k \ominus \{a\} = (if\ a \leq k\ then\ \{k - a\}\ else\ \{\})$
⟨*proof*⟩
corollary *iT-Minus-singleton1*: $a \leq k \Longrightarrow k \ominus \{a\} = \{k-a\}$
⟨*proof*⟩
corollary *iT-Minus-singleton2*: $k < a \Longrightarrow k \ominus \{a\} = \{\}$

⟨*proof*⟩

lemma *iMOD-sub*:
$k \ominus [r,\ mod\ m] =$
$(if\ r \leq k\ then\ [(k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]\ else\ \{\})$
⟨*proof*⟩

corollary *iMOD-sub1*:
$r \leq k \Longrightarrow k \ominus [r,\ mod\ m] = [(k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]$
⟨*proof*⟩

corollary *iMOD-sub2*: $k < r \Longrightarrow k \ominus [r,\ mod\ m] = \{\}$
⟨*proof*⟩

lemma *iTILL-sub*: $k \ominus [\ldots n] = (if\ n \leq k\ then\ [k - n\ldots,n]\ else\ [\ldots k])$
⟨*proof*⟩

corollary *iTILL-sub1*: $n \leq k \Longrightarrow k \ominus [\ldots n] = [k - n\ldots,n]$
⟨*proof*⟩

corollary *iTILL-sub2*: $k \leq n \Longrightarrow k \ominus [\ldots n] = [\ldots k]$
⟨*proof*⟩

lemma *iMODb-sub*:
$k \ominus [r,\ mod\ m,\ c] = ($
$if\ r + m * c \leq k\ then\ [\ k - r - m * c,\ mod\ m,\ c]\ else$
$if\ r \leq k\ then\ [\ (k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]\ else\ \{\})$
⟨*proof*⟩

corollary *iMODb-sub1*:
$[\![\ r \leq k;\ k \leq r + m * c\ ]\!] \Longrightarrow$
$k \ominus [r,\ mod\ m,\ c] = [(k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]$
⟨*proof*⟩

corollary *iMODb-sub2*: $k < r \Longrightarrow k \ominus [r,\ mod\ m,\ c] = \{\}$
⟨*proof*⟩

corollary *iMODb-sub3*:
$r + m * c \leq k \Longrightarrow k \ominus [r,\ mod\ m,\ c] = [\ k - r - m * c,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *iFROM-sub*: $k \ominus [n\ldots] = (if\ n \leq k\ then\ [\ldots k - n]\ else\ \{\})$
⟨*proof*⟩

corollary *iFROM-sub1*: $n \leq k \Longrightarrow k \ominus [n\ldots] = [\ldots k-n]$
⟨*proof*⟩

corollary *iFROM-sub-empty*: $k < n \Longrightarrow k \ominus [n..] = \{\}$
⟨*proof*⟩

lemma *iIN-sub*:
$k \ominus [n..,d] = ($
if $n + d \leq k$ *then* $[k - (n + d)..,d]$
else if $n \leq k$ *then* $[..k - n]$ *else* $\{\})$
⟨*proof*⟩

lemma *iIN-sub1*: $n + d \leq k \Longrightarrow k \ominus [n..,d] = [k - (n + d)..,d]$
⟨*proof*⟩

lemma *iIN-sub2*: $[\![n \leq k; k \leq n + d]\!] \Longrightarrow k \ominus [n..,d] = [..k - n]$
⟨*proof*⟩

lemma *iIN-sub3*: $k < n \Longrightarrow k \ominus [n..,d] = \{\}$
⟨*proof*⟩

lemmas *iT-sub* =
iFROM-sub
iIN-sub
iTILL-sub
iMOD-sub
iMODb-sub
iT-Minus-singleton

2.1.5 Division of intervals by constants

Monotonicity and injectivity of artithmetic operators

lemma *iMOD-div-right-strict-mono-on*:
$[\![0 < k; k \leq m]\!] \Longrightarrow$ *strict-mono-on* $(\lambda x.\ x\ div\ k)$ $[r, mod\ m]$
⟨*proof*⟩

corollary *iMOD-div-right-inj-on*:
$[\![0 < k; k \leq m]\!] \Longrightarrow$ *inj-on* $(\lambda x.\ x\ div\ k)$ $[r, mod\ m]$
⟨*proof*⟩

lemma *iMOD-mult-div-right-inj-on*:
inj-on $(\lambda x.\ x\ div\ (k::nat))$ $[r, mod\ (k * m)]$
⟨*proof*⟩

lemma *iMOD-mult-div-right-inj-on2*:
$m\ mod\ k = 0 \Longrightarrow$ *inj-on* $(\lambda x.\ x\ div\ k)$ $[r, mod\ m]$
⟨*proof*⟩

lemma *iMODb-div-right-strict-mono-on*:

⟦ $0 < k$; $k \leq m$ ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
⟨*proof*⟩

corollary *iMODb-div-right-inj-on*:
⟦ $0 < k$; $k \leq m$ ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
⟨*proof*⟩

lemma *iMODb-mult-div-right-inj-on*:
inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*), *c*]
⟨*proof*⟩

corollary *iMODb-mult-div-right-inj-on2*:
m mod k = *0* ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
⟨*proof*⟩

definition *iT-Div* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ⊘ *55*)
where *I* ⊘ *k* ≡ (λ*n*.(*n div k*)) ‘ *I*

lemma *iT-Div-image-conv*: *I* ⊘ *k* = (λ*n*.(*n div k*)) ‘ *I*
⟨*proof*⟩

lemma *iT-Div-mono*: *A* ⊆ *B* ⟹ *A* ⊘ *k* ⊆ *B* ⊘ *k*
⟨*proof*⟩

lemma *iT-Div-empty*: {} ⊘ *k* = {}
⟨*proof*⟩
lemma *iT-Div-not-empty*: *I* ≠ {} ⟹ *I* ⊘ *k* ≠ {}
⟨*proof*⟩

lemma *iT-Div-empty-iff*: (*I* ⊘ *k* = {}) = (*I* = {})
⟨*proof*⟩

lemma *iT-Div-0*: *I* ≠ {} ⟹ *I* ⊘ *0* = [*...0*]
⟨*proof*⟩
corollary *iT-Div-0-if*: *I* ⊘ *0* = (*if I* = {} *then* {} *else* [*...0*])
⟨*proof*⟩
corollary
iFROM-div-0: [*n...*] ⊘ *0* = [*...0*] **and**
iTILL-div-0: [*...n*] ⊘ *0* = [*...0*] **and**
iIN-div-0: [*n...,d*] ⊘ *0* = [*...0*] **and**
iMOD-div-0: [*r*, *mod m*] ⊘ *0* = [*...0*] **and**
iMODb-div-0: [*r*, *mod m*, *c*] ⊘ *0* = [*...0*]
⟨*proof*⟩

lemmas *iT-div-0* =
iTILL-div-0
iFROM-div-0

iIN-div-0
iMOD-div-0
iMODb-div-0

lemma *iT-Div-1*: $I \oslash Suc\ 0 = I$
⟨*proof*⟩

lemma *iT-Div-mem-iff-0*: $x \in (I \oslash 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩

lemma *iT-Div-mem-iff*:
$0 < k \Longrightarrow x \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x)$
⟨*proof*⟩

lemma *iT-Div-mem-iff2*:
$0 < k \Longrightarrow x\ div\ k \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x\ div\ k)$
⟨*proof*⟩

lemma *iT-Div-mem-iff-Int*:
$0 < k \Longrightarrow x \in (I \oslash k) = (I \cap [x * k..,k - Suc\ 0] \neq \{\})$
⟨*proof*⟩

lemma *iT-Div-imp-mem*:
$0 < k \Longrightarrow x \in I \Longrightarrow x\ div\ k \in (I \oslash k)$
⟨*proof*⟩

lemma *iT-Div-singleton*: $\{a\} \oslash k = \{a\ div\ k\}$
⟨*proof*⟩

lemma *iT-Div-Un*: $(A \cup B) \oslash k = (A \oslash k) \cup (B \oslash k)$
⟨*proof*⟩

lemma *iT-Div-insert*: $(insert\ n\ I) \oslash k = insert\ (n\ div\ k)\ (I \oslash k)$
⟨*proof*⟩

lemma *not-iT-Div-Int*: $\neg\ (\forall\ k\ A\ B.\ (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k))$
⟨*proof*⟩

lemma *subset-iT-Div-Int*: $A \subseteq B \Longrightarrow (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k)$
⟨*proof*⟩

lemma *iFROM-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A\ ]\!] \Longrightarrow (A \cap [n..]) \oslash k = (A \oslash k) \cap ([n..] \oslash k)$
⟨*proof*⟩

lemma *iIN-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A;\ \forall\, x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d\ ]\!] \Longrightarrow$

$(A \cap [n...,d]) \oslash k = (A \oslash k) \cap ([n...,d] \oslash k)$
⟨*proof*⟩
corollary *iTILL-iT-Div-Int*:
$[\![\ 0 < k;\ \forall x{\in}A.\ x\ div\ k \leq n\ div\ k \longrightarrow x \leq n\ ]\!] \Longrightarrow$
$(A \cap [...n]) \oslash k = (A \oslash k) \cap ([...n] \oslash k)$
⟨*proof*⟩
lemma *iIN-iT-Div-Int-mod-0*:
$[\![\ 0 < k;\ n\ mod\ k = 0;\ \forall x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d\ ]\!] \Longrightarrow$
$(A \cap [n...,d]) \oslash k = (A \oslash k) \cap ([n...,d] \oslash k)$
⟨*proof*⟩

lemma *mod-partition-iT-Div-Int*:
$[\![\ 0 < k;\ 0 < d\ ]\!] \Longrightarrow$
$(A \cap [n * k...,d * k - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n * k...,d * k - Suc\ 0] \oslash k)$
⟨*proof*⟩⟨*proof*⟩
corollary *mod-partition-iT-Div-Int2*:
$[\![\ 0 < k;\ 0 < d;\ n\ mod\ k = 0;\ d\ mod\ k = 0\ ]\!] \Longrightarrow$
$(A \cap [n...,d - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n...,d - Suc\ 0] \oslash k)$
⟨*proof*⟩

corollary *mod-partition-iT-Div-Int-one-segment*:
$0 < k \Longrightarrow$
$(A \cap [n * k...,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n * k...,k - Suc\ 0] \oslash k)$
⟨*proof*⟩

corollary *mod-partition-iT-Div-Int-one-segment2*:
$[\![\ 0 < k;\ n\ mod\ k = 0\ ]\!] \Longrightarrow$
$(A \cap [n...,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n...,k - Suc\ 0] \oslash k)$
⟨*proof*⟩

lemma *iT-Div-assoc*:$I \oslash a \oslash b = I \oslash (a * b)$
⟨*proof*⟩

lemma *iT-Div-commute*: $I \oslash a \oslash b = I \oslash b \oslash a$
⟨*proof*⟩

lemma *iT-Mult-Div-self*: $0 < k \Longrightarrow I \otimes k \oslash k = I$
⟨*proof*⟩
lemma *iT-Mult-Div*:
$[\![\ 0 < d;\ \ k\ mod\ d = 0\ ]\!] \Longrightarrow I \otimes k \oslash d = I \otimes (k\ div\ d)$
⟨*proof*⟩

lemma *iT-Div-Mult-self*:
$0 < k \Longrightarrow I \oslash k \otimes k = \{y.\ \exists x \in I.\ y = x - x\ mod\ k\}$
⟨*proof*⟩

lemma *iT-Plus-Div-distrib-mod-less*:
$\forall x{\in}I.\ x\ mod\ m + n\ mod\ m < m \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩
corollary *iT-Plus-Div-distrib-mod-0*:
$n\ mod\ m = 0 \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩

lemma *iT-Div-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oslash k) = iMin\ I\ div\ k$
⟨*proof*⟩

corollary
iFROM-div-Min: $iMin\ ([n...] \oslash k) = n\ div\ k$ **and**
iIN-div-Min: $iMin\ ([n...,d] \oslash k) = n\ div\ k$ **and**
iTILL-div-Min: $iMin\ ([...n] \oslash k) = 0$ **and**
iMOD-div-Min: $iMin\ ([r,\ mod\ m] \oslash k) = r\ div\ k$ **and**
iMODb-div-Min: $iMin\ ([r,\ mod\ m,\ c] \oslash k) = r\ div\ k$
⟨*proof*⟩

lemmas *iT-div-Min* =
iFROM-div-Min
iIN-div-Min
iTILL-div-Min
iMOD-div-Min
iMODb-div-Min

lemma *iT-Div-Max*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oslash k) = Max\ I\ div\ k$
⟨*proof*⟩

corollary
iIN-div-Max: $Max\ ([n...,d] \oslash k) = (n + d)\ div\ k$ **and**
iTILL-div-Max: $Max\ ([...n] \oslash k) = n\ div\ k$ **and**
iMODb-div-Max: $Max\ ([r,\ mod\ m,\ c] \oslash k) = (r + m * c)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-0-finite*: $finite\ (I \oslash 0)$
⟨*proof*⟩

lemma *iT-Div-infinite-iff*: $0 < k \Longrightarrow infinite\ (I \oslash k) = infinite\ I$
⟨*proof*⟩
lemma *iT-Div-finite-iff*: $0 < k \Longrightarrow finite\ (I \oslash k) = finite\ I$
⟨*proof*⟩

lemma *iFROM-div*: $0 < k \Longrightarrow [n...] \oslash k = [n\ div\ k...]$
⟨*proof*⟩

lemma *iIN-div*:
$0 < k \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k...,\ d\ div\ k + (n\ mod\ k + d\ mod\ k)\ div\ k\ ]$

⟨*proof*⟩

corollary *iIN-div-if*:
$0 < k \Longrightarrow [n...,d] \oslash k =$
$[n\ div\ k..., d\ div\ k + (if\ n\ mod\ k + d\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
⟨*proof*⟩

corollary *iIN-div-eq1*:
$[\![\ 0 < k;\ n\ mod\ k + d\ mod\ k < k\ ]\!] \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k...,d\ div\ k]$
⟨*proof*⟩

corollary *iIN-div-eq2*:
$[\![\ 0 < k;\ k \leq n\ mod\ k + d\ mod\ k\ ]\!] \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k..., Suc\ (d\ div\ k)]$
⟨*proof*⟩

corollary *iIN-div-mod-eq-0*:
$[\![\ 0 < k;\ n\ mod\ k = 0\ ]\!] \Longrightarrow [n...,d] \oslash k = [n\ div\ k...,d\ div\ k]$
⟨*proof*⟩

lemma *iTILL-div*:
$0 < k \Longrightarrow [...n] \oslash k = [...n\ div\ k]$
⟨*proof*⟩

lemma *iMOD-div-ge*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow [r,\ mod\ m] \oslash k = [r\ div\ k...]$
⟨*proof*⟩
corollary *iMOD-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m] \oslash m = [r\ div\ m...]$
⟨*proof*⟩

lemma *iMOD-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k)\ ]$
⟨*proof*⟩

lemma *iMODb-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m,\ c] \oslash m = [r\ div\ m...,c]$
⟨*proof*⟩

lemma *iMODb-div-ge*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k...,(r + m * c)\ div\ k - r\ div\ k]$
⟨*proof*⟩

corollary *iMODb-div-ge-if*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow$

$[r,\ mod\ m,\ c] \oslash k =$
$[r\ div\ k \ldots,\ m * c\ div\ k + (if\ r\ mod\ k + m * c\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
⟨*proof*⟩

lemma *iMODb-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k),\ c\ ]$
⟨*proof*⟩

lemmas *iT-div* =
iTILL-div
iFROM-div
iIN-div
iMOD-div
iMODb-div
iT-Div-singleton

This lemma is valid for all $k \leq m$,i. e., also for k with $m\ mod\ k \neq (0::'a)$.

lemma *iMODb-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ k \leq c;\ [r',\ mod\ m',\ c'] = [r,\ mod\ m,\ c] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ div\ k \wedge m' = m\ div\ k \wedge c' = c$
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-is-0-not-ex0*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k;\ k \leq c;\ r\ mod\ k = 0\ ]\!] \Longrightarrow$
$\neg(\exists\, r'\ m'\ c'.\ [r',\ mod\ m',\ c'] = [r,\ mod\ m,\ c] \oslash k)$
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-not-ex--arith-aux1*:
$[\![\ (0::nat) < k;\ k < m;\ 0 < x1\ ]\!] \Longrightarrow$
$x1 * m + x2 - x\ mod\ k + x3 + x\ mod\ k = x1 * m + x2 + x3$
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-not-ex*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k;\ k \leq c\ ]\!] \Longrightarrow$
$\neg(\exists\, r'\ m'\ c'.\ [r',\ mod\ m',\ c'] = [r,\ mod\ m,\ c] \oslash k)$
⟨*proof*⟩

lemma *iMOD-div-eq-imp-iMODb-div-eq*:
$[\![\ 0 < k;\ k \leq m;\ [r',\ mod\ m'] = [r,\ mod\ m] \oslash k\ ]\!] \Longrightarrow$
$[r',\ mod\ m',\ c] = [r,\ mod\ m,\ c] \oslash k$
⟨*proof*⟩

lemma *iMOD-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ [r',\ mod\ m'] = [r,\ mod\ m] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ div\ k \wedge m' = m\ div\ k$

⟨*proof*⟩

lemma *iMOD-div-mod-gr0-not-ex*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$ ⟧ ⟹
¬ $(\exists\, r'\ m'.\ [r',\ mod\ m'] = [r,\ mod\ m] \oslash k)$
⟨*proof*⟩

2.2 Interval cut operators with arithmetic interval operators

lemma
iT-Plus-cut-le2: $(I \oplus k) \downarrow\leq (t + k) = (I \downarrow\leq t) \oplus k$ **and**
iT-Plus-cut-less2: $(I \oplus k) \downarrow< (t + k) = (I \downarrow< t) \oplus k$ **and**
iT-Plus-cut-ge2: $(I \oplus k) \downarrow\geq (t + k) = (I \downarrow\geq t) \oplus k$ **and**
iT-Plus-cut-greater2: $(I \oplus k) \downarrow> (t + k) = (I \downarrow> t) \oplus k$
⟨*proof*⟩

lemma *iT-Plus-cut-le*:
$(I \oplus k) \downarrow\leq t = (\mathit{if}\ t < k\ \mathit{then}\ \{\}\ \mathit{else}\ I \downarrow\leq (t - k) \oplus k)$
⟨*proof*⟩

lemma *iT-Plus-cut-less*: $(I \oplus k) \downarrow< t = I \downarrow< (t - k) \oplus k$
⟨*proof*⟩

lemma *iT-Plus-cut-ge*: $(I \oplus k) \downarrow\geq t = I \downarrow\geq (t - k) \oplus k$
⟨*proof*⟩

lemma *iT-Plus-cut-greater*:
$(I \oplus k) \downarrow> t = (\mathit{if}\ t < k\ \mathit{then}\ I \oplus k\ \mathit{else}\ I \downarrow> (t - k) \oplus k)$
⟨*proof*⟩

lemma
iT-Mult-cut-le2: $0 < k \Longrightarrow (I \otimes k) \downarrow\leq (t * k) = (I \downarrow\leq t) \otimes k$ **and**
iT-Mult-cut-less2: $0 < k \Longrightarrow (I \otimes k) \downarrow< (t * k) = (I \downarrow< t) \otimes k$ **and**
iT-Mult-cut-ge2: $0 < k \Longrightarrow (I \otimes k) \downarrow\geq (t * k) = (I \downarrow\geq t) \otimes k$ **and**
iT-Mult-cut-greater2: $0 < k \Longrightarrow (I \otimes k) \downarrow> (t * k) = (I \downarrow> t) \otimes k$
⟨*proof*⟩

lemma *iT-Mult-cut-le*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\leq t = (I \downarrow\leq (t\ div\ k)) \otimes k$
⟨*proof*⟩

lemma *iT-Mult-cut-less*:
$0 < k \Longrightarrow (I \otimes k) \downarrow< t =$
$(\mathit{if}\ t \bmod k = 0\ \mathit{then}\ (I \downarrow< (t\ div\ k))\ \mathit{else}\ I \downarrow< Suc\ (t\ div\ k)) \otimes k$
⟨*proof*⟩

lemma *iT-Mult-cut-greater*:
$0 < k \Longrightarrow (I \otimes k) \downarrow> t = (I \downarrow> (t\ div\ k)) \otimes k$

⟨*proof*⟩

lemma *iT-Mult-cut-ge*:
0 < *k* ⟹ (*I* ⊗ *k*) ↓≥ *t* =
(*if t mod k* = *0 then* (*I* ↓≥ (*t div k*)) *else I* ↓≥ *Suc* (*t div k*)) ⊗ *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le2*: *k* ≤ *t* ⟹ (*I* ⊕− *k*) ↓≤ (*t* − *k*) = (*I* ↓≤ *t*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-less2*: (*I* ⊕− *k*) ↓< (*t* − *k*) = (*I* ↓< *t*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-ge2*: (*I* ⊕− *k*) ↓≥ (*t* − *k*) = (*I* ↓≥ *t*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-greater2*: *k* ≤ *t* ⟹ (*I* ⊕− *k*) ↓> (*t* − *k*) = (*I* ↓> *t*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le*: (*I* ⊕− *k*) ↓≤ *t* = *I* ↓≤ (*t* + *k*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-less*: (*I* ⊕− *k*) ↓< *t* = *I* ↓< (*t* + *k*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-ge*: (*I* ⊕− *k*) ↓≥ *t* = *I* ↓≥ (*t* + *k*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-greater*: (*I* ⊕− *k*) ↓> *t* = *I* ↓> (*t* + *k*) ⊕− *k*
⟨*proof*⟩

lemma *iT-Minus-cut-le2*: *t* ≤ *k* ⟹ (*k* ⊖ *I*) ↓≤ (*k* − *t*) = *k* ⊖ (*I* ↓≥ *t*)
⟨*proof*⟩

lemma *iT-Minus-cut-less2*: (*k* ⊖ *I*) ↓< (*k* − *t*) = *k* ⊖ (*I* ↓> *t*)
⟨*proof*⟩

lemma *iT-Minus-cut-ge2*: (*k* ⊖ *I*) ↓≥ (*k* − *t*) = *k* ⊖ (*I* ↓≤ *t*)
⟨*proof*⟩

lemma *iT-Minus-cut-greater2*: *t* ≤ *k* ⟹ (*k* ⊖ *I*) ↓> (*k* − *t*) = *k* ⊖ (*I* ↓< *t*)
⟨*proof*⟩

lemma *iT-Minus-cut-le*: (*k* ⊖ *I*) ↓≤ *t* = *k* ⊖ (*I* ↓≥ (*k* − *t*))
⟨*proof*⟩

lemma *iT-Minus-cut-less*:

$(k \ominus I) \downarrow< t = (\mathit{if}\ t \leq k\ \mathit{then}\ k \ominus (I \downarrow> (k - t))\ \mathit{else}\ k \ominus I)$
⟨*proof*⟩

lemma *iT-Minus-cut-ge*:
$(k \ominus I) \downarrow\geq t = (\mathit{if}\ t \leq k\ \mathit{then}\ k \ominus (I \downarrow\leq (k - t))\ \mathit{else}\ \{\})$
⟨*proof*⟩

lemma *iT-Minus-cut-greater*: $(k \ominus I) \downarrow> t = k \ominus (I \downarrow< (k - t))$
⟨*proof*⟩

lemma *iT-Div-cut-le*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq t = I \downarrow\leq (t * k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-less*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< t = I \downarrow< (t * k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-ge*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq t = I \downarrow\geq (t * k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-greater*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> t = I \downarrow> (t * k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-le2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq (t\ \mathit{div}\ k) = I \downarrow\leq (t - t\ \mathit{mod}\ k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-less2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< (t\ \mathit{div}\ k) = I \downarrow< (t - t\ \mathit{mod}\ k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-ge2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq (t\ \mathit{div}\ k) = I \downarrow\geq (t - t\ \mathit{mod}\ k) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-greater2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> (t\ \mathit{div}\ k) = I \downarrow> (t - t\ \mathit{mod}\ k + (k - \mathit{Suc}\ 0)) \oslash k$
⟨*proof*⟩

2.3 *inext* and *iprev* with interval operators

lemma *iT-Plus-inext*: $\mathit{inext}\ (n + k)\ (I \oplus k) = (\mathit{inext}\ n\ I) + k$
⟨*proof*⟩

lemma *iT-Plus-iprev*: $iprev\ (n + k)\ (I \oplus k) = (iprev\ n\ I) + k$
⟨*proof*⟩

lemma *iT-Plus-inext2*: $k \leq n \Longrightarrow inext\ n\ (I \oplus k) = (inext\ (n - k)\ I) + k$
⟨*proof*⟩
lemma *iT-Plus-prev2*: $k \leq n \Longrightarrow iprev\ n\ (I \oplus k) = (iprev\ (n - k)\ I) + k$
⟨*proof*⟩

lemma *iT-Mult-inext*: $inext\ (n * k)\ (I \otimes k) = (inext\ n\ I) * k$
⟨*proof*⟩

lemma *iT-Mult-iprev*: $iprev\ (n * k)\ (I \otimes k) = (iprev\ n\ I) * k$
⟨*proof*⟩

lemma *iT-Mult-inext2-if*:
$inext\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (inext\ (n\ div\ k)\ I) * k\ else\ n)$
⟨*proof*⟩

lemma *iT-Mult-iprev2-if*:
$iprev\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (iprev\ (n\ div\ k)\ I) * k\ else\ n)$
⟨*proof*⟩

corollary *iT-Mult-inext2*:
$n\ mod\ k = 0 \Longrightarrow inext\ n\ (I \otimes k) = (inext\ (n\ div\ k)\ I) * k$
⟨*proof*⟩

corollary *iT-Mult-iprev2*:
$n\ mod\ k = 0 \Longrightarrow iprev\ n\ (I \otimes k) = (iprev\ (n\ div\ k)\ I) * k$
⟨*proof*⟩

lemma *iT-Plus-neg-inext*:
$k \leq n \Longrightarrow inext\ (n - k)\ (I \oplus - k) = inext\ n\ I - k$
⟨*proof*⟩

lemma *iT-Plus-neg-iprev*:
$iprev\ (n - k)\ (I \oplus - k) = iprev\ n\ (I \downarrow \geq k) - k$
⟨*proof*⟩

corollary *iT-Plus-neg-inext2*: $inext\ n\ (I \oplus - k) = inext\ (n + k)\ I - k$
⟨*proof*⟩

corollary *iT-Plus-neg-iprev2*: $iprev\ n\ (I \oplus - k) = iprev\ (n + k)\ (I \downarrow \geq k) - k$
⟨*proof*⟩

lemma *iT-Minus-inext*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow inext\ (k - n)\ (k \ominus I) = k - iprev\ n\ I$
⟨*proof*⟩

corollary *iT-Minus-inext2*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow inext\ n\ (k \ominus I) = k - iprev\ (k - n)\ I$
⟨*proof*⟩

lemma *iT-Minus-iprev*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow iprev\ (k - n)\ (k \ominus I) = k - inext\ n\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-iprev2*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow iprev\ n\ (k \ominus I) = k - inext\ (k - n)\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Plus-inext-nth*: $I \neq \{\} \Longrightarrow (I \oplus k) \rightarrow n = (I \rightarrow n) + k$
⟨*proof*⟩

lemma *iT-Plus-iprev-nth*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow (I \oplus k) \leftarrow n = (I \leftarrow n) + k$
⟨*proof*⟩

lemma *iT-Mult-inext-nth*: $I \neq \{\} \Longrightarrow (I \otimes k) \rightarrow n = (I \rightarrow n) * k$
⟨*proof*⟩

lemma *iT-Mult-iprev-nth*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow (I \otimes k) \leftarrow n = (I \leftarrow n) * k$
⟨*proof*⟩

lemma *iT-Plus-neg-inext-nth*:
$I \oplus- k \neq \{\} \Longrightarrow (I \oplus- k) \rightarrow n = (I \downarrow\geq k \rightarrow n) - k$
⟨*proof*⟩

lemma *iT-Plus-neg-iprev-nth*:
$[\![\ finite\ I;\ I \oplus- k \neq \{\}\ ]\!] \Longrightarrow (I \oplus- k) \leftarrow n = (I \downarrow\geq k \leftarrow n) - k$
⟨*proof*⟩

lemma *iT-Minus-inext-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \rightarrow n = k - ((I \downarrow\leq k) \leftarrow n)$
⟨*proof*⟩

lemma *iT-Minus-iprev-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \leftarrow n = k - ((I \downarrow\leq k) \rightarrow n)$
⟨*proof*⟩

lemma *iT-Div-ge-inext-nth*:
$[\![\ I \neq \{\};\ \forall\, x{\in}I.\ \forall\, y{\in}I.\ x < y \longrightarrow x + k \leq y\ ]\!] \Longrightarrow$
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-mod-inext-nth*:
$[\![\ I \neq \{\};\ \forall\, x{\in}I.\ \forall\, y{\in}I.\ x\ mod\ k = y\ mod\ k\ ]\!] \Longrightarrow$

$(I \oslash k) \to n = (I \to n)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-ge-iprev-nth*:
$[\![\ finite\ I;\ I \neq \{\};\ \forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y\ ]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-mod-iprev-nth*:
$[\![\ finite\ I;\ I \neq \{\};\ \forall x{\in}I.\ \forall y{\in}I.\ x\ mod\ k = y\ mod\ k\ ]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
⟨*proof*⟩

2.4 Cardinality of intervals with interval operators

lemma *iT-Plus-card*: $card\ (I \oplus k) = card\ I$
⟨*proof*⟩
lemma *iT-Mult-card*: $0 < k \Longrightarrow card\ (I \otimes k) = card\ I$
⟨*proof*⟩

lemma *iT-Plus-neg-card*: $card\ (I \oplus{-}\ k) = card\ (I \downarrow{\geq}\ k)$
⟨*proof*⟩

lemma *iT-Plus-neg-card-le*: $card\ (I \oplus{-}\ k) \leq card\ I$
⟨*proof*⟩

lemma *iT-Minus-card*: $card\ (k \ominus I) = card\ (I \downarrow{\leq}\ k)$
⟨*proof*⟩

lemma *iT-Minus-card-le*: $finite\ I \Longrightarrow card\ (k \ominus I) \leq card\ I$
⟨*proof*⟩

lemma *iT-Div-0-card-if*:
$card\ (I \oslash 0) = (if\ I = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *Int-empty-sum*:
$(\sum k{\leq}(n{::}nat).\ if\ \{\} \cap (I\ k) = \{\}\ then\ 0\ else\ Suc\ 0) = 0$
⟨*proof*⟩

lemma *iT-Div-mod-partition-card*:
$card\ (I \cap [n * d\ldots,d - Suc\ 0] \oslash d) =$
$(if\ I \cap [n * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Div-conv-count*:
$0 < d \Longrightarrow I \oslash d = \{k.\ I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\}$
⟨*proof*⟩

lemma *iT-Div-conv-count2*:
$[\![\ 0 < d;\ \textit{finite}\ I;\ \textit{Max}\ I\ \textit{div}\ d \leq n\ ]\!] \Longrightarrow$
$I \oslash d = \{k.\ k \leq n \wedge I \cap [k * d\ldots,d - \textit{Suc}\ 0] \neq \{\}\}$
⟨*proof*⟩

lemma *mod-partition-count-Suc*:
$\{k.\ k \leq \textit{Suc}\ n \wedge I \cap [k * d\ldots,d - \textit{Suc}\ 0] \neq \{\}\} =$
$\{k.\ k \leq n \wedge I \cap [k * d\ldots,d - \textit{Suc}\ 0] \neq \{\}\} \cup$
$(\textit{if}\ I \cap [\textit{Suc}\ n * d\ldots,d - \textit{Suc}\ 0] \neq \{\}\ \textit{then}\ \{\textit{Suc}\ n\}\ \textit{else}\ \{\})$
⟨*proof*⟩

lemma *iT-Div-card*:
$\bigwedge I.[\![\ 0 < d;\ \textit{finite}\ I;\ \textit{Max}\ I\ \textit{div}\ d \leq n]\!] \Longrightarrow$
$\textit{card}\ (I \oslash d) = (\sum k{\leq}n.$
$\textit{if}\ I \cap [k * d\ldots,d - \textit{Suc}\ 0] = \{\}\ \textit{then}\ 0\ \textit{else}\ \textit{Suc}\ 0)$
⟨*proof*⟩

corollary *iT-Div-card-Suc*:
$\bigwedge I.[\![\ 0 < d;\ \textit{finite}\ I;\ \textit{Max}\ I\ \textit{div}\ d \leq n]\!] \Longrightarrow$
$\textit{card}\ (I \oslash d) = (\sum k{<}\textit{Suc}\ n.$
$\textit{if}\ I \cap [k * d\ldots,d - \textit{Suc}\ 0] = \{\}\ \textit{then}\ 0\ \textit{else}\ \textit{Suc}\ 0)$
⟨*proof*⟩
corollary *iT-Div-Max-card*: $[\![\ 0 < d;\ \textit{finite}\ I\ ]\!] \Longrightarrow$
$\textit{card}\ (I \oslash d) = (\sum k{\leq}\textit{Max}\ I\ \textit{div}\ d.$
$\textit{if}\ I \cap [k * d\ldots,d - \textit{Suc}\ 0] = \{\}\ \textit{then}\ 0\ \textit{else}\ \textit{Suc}\ 0)$
⟨*proof*⟩

lemma *iT-Div-card-le*: $0 < k \Longrightarrow \textit{card}\ (I \oslash k) \leq \textit{card}\ I$
⟨*proof*⟩

lemma *iT-Div-card-inj-on*:
$\textit{inj-on}\ (\lambda n.\ n\ \textit{div}\ k)\ I \Longrightarrow \textit{card}\ (I \oslash k) = \textit{card}\ I$
⟨*proof*⟩

lemma *mod-Suc'*:
$0 < n \Longrightarrow \textit{Suc}\ m\ \textit{mod}\ n = (\textit{if}\ m\ \textit{mod}\ n < n - \textit{Suc}\ 0\ \textit{then}\ \textit{Suc}\ (m\ \textit{mod}\ n)\ \textit{else}\ 0)$
⟨*proof*⟩

lemma *div-Suc*:
$0 < n \Longrightarrow \textit{Suc}\ m\ \textit{div}\ n = (\textit{if}\ \textit{Suc}\ (m\ \textit{mod}\ n) = n\ \textit{then}\ \textit{Suc}\ (m\ \textit{div}\ n)\ \textit{else}\ m\ \textit{div}\ n)$
⟨*proof*⟩

lemma *div-Suc'*:
$0 < n \Longrightarrow Suc\ m\ div\ n = (if\ m\ mod\ n < n - Suc\ 0\ then\ m\ div\ n\ else\ Suc\ (m\ div\ n))$
⟨*proof*⟩

lemma *iT-Div-card-ge-aux*:
$\bigwedge I.\ [\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \leq n\ ]\!] \Longrightarrow$
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \leq card\ (I \oslash d)$
⟨*proof*⟩

lemma *iT-Div-card-ge*:
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \leq card\ (I \oslash d)$
⟨*proof*⟩

corollary *iT-Div-card-ge-div*: $card\ I\ div\ d \leq card\ (I \oslash d)$
⟨*proof*⟩

There is no better lower bound function f for $i \oslash d$ with *card* i and d as arguments.

lemma *iT-Div-card-ge--is-maximal-lower-bound*:
$\forall I\ d.\ card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \leq f\ (card\ I)\ d \wedge$
$f\ (card\ I)\ d \leq card\ (I \oslash d) \Longrightarrow$
$f\ (card\ (I::nat\ set))\ d = card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Plus-icard*: $icard\ (I \oplus k) = icard\ I$
⟨*proof*⟩

lemma *iT-Mult-icard*: $0 < k \Longrightarrow icard\ (I \otimes k) = icard\ I$
⟨*proof*⟩

lemma *iT-Plus-neg-icard*: $icard\ (I \oplus- k) = icard\ (I \downarrow\geq k)$
⟨*proof*⟩

lemma *iT-Plus-neg-icard-le*: $icard\ (I \oplus- k) \leq icard\ I$
⟨*proof*⟩

lemma *iT-Minus-icard*: $icard\ (k \ominus I) = icard\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-icard-le*: $icard\ (k \ominus I) \leq icard\ I$
⟨*proof*⟩

lemma *iT-Div-0-icard-if*: $icard\ (I \oslash 0) = enat\ (if\ I = \{\}\ then\ 0\ else\ Suc\ 0)$
⟨*proof*⟩

lemma *iT-Div-mod-partition-icard*:
$icard\ (I \cap [n * d\ldots,d - Suc\ 0] \oslash d) =$

enat (*if* $I \cap [n * d..,d - Suc\ 0] = \{\}$ *then 0 else Suc 0*)
⟨*proof*⟩

lemma *iT-Div-icard*:
$[\![0 < d; finite\ I \Longrightarrow Max\ I\ div\ d \leq n]\!] \Longrightarrow$
icard $(I \oslash d) =$
(*if finite I then enat* $(\sum k{\leq}n.$ *if* $I \cap [k * d..,d - Suc\ 0] = \{\}$ *then 0 else Suc 0*) *else* ∞)
⟨*proof*⟩

corollary *iT-Div-Max-icard*: $0 < d \Longrightarrow$
icard $(I \oslash d) =$ (*if finite I*
then enat $(\sum k{\leq}Max\ I\ div\ d.$ *if* $I \cap [k * d..,d - Suc\ 0] = \{\}$ *then 0 else Suc 0*) *else* ∞)
⟨*proof*⟩

lemma *iT-Div-icard-le*: $0 < k \Longrightarrow$ *icard* $(I \oslash k) \leq$ *icard I*
⟨*proof*⟩

lemma *iT-Div-icard-inj-on*: *inj-on* $(\lambda n.\ n\ div\ k)\ I \Longrightarrow$ *icard* $(I \oslash k) =$ *icard I*
⟨*proof*⟩

lemma *iT-Div-icard-ge*: *icard I div* (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*) ≤ *icard* $(I \oslash d)$
⟨*proof*⟩

corollary *iT-Div-icard-ge-div*: *icard I div* (*enat d*) ≤ *icard* $(I \oslash d)$
⟨*proof*⟩

lemma *iT-Div-icard-ge--is-maximal-lower-bound*:
$\forall I\ d.$ *icard I div* (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*)
$\leq f$ (*icard I*) $d \wedge$
f (*icard I*) $d \leq$ *icard* $(I \oslash d) \Longrightarrow$
f (*icard* (*I*::*nat set*)) $d =$
icard I div (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*)
⟨*proof*⟩

2.5 Results about sets of intervals

2.5.1 Set of intervals without and with empty interval

definition *iFROM-UN-set* :: (*nat set*) *set*
where *iFROM-UN-set* $\equiv \bigcup n.\ \{[n..]\}$

definition *iTILL-UN-set* :: (*nat set*) *set*
where *iTILL-UN-set* $\equiv \bigcup n.\ \{[..n]\}$

definition *iIN-UN-set* :: (*nat set*) *set*
where *iIN-UN-set* $\equiv \bigcup n\ d.\ \{[n..,d]\}$

definition *iMOD-UN-set* :: (*nat set*) *set*
  where *iMOD-UN-set* $\equiv \bigcup r\ m.\ \{[r,\ mod\ m]\}$

definition *iMODb-UN-set* :: (*nat set*) *set*
  where *iMODb-UN-set* $\equiv \bigcup r\ m\ c.\ \{[r,\ mod\ m,\ c]\}$

definition *iFROM-set* :: (*nat set*) *set*
  where *iFROM-set* $\equiv \{[n...]\ |n.\ True\}$

definition *iTILL-set* :: (*nat set*) *set*
  where *iTILL-set* $\equiv \{[...n]\ |n.\ True\}$

definition *iIN-set* :: (*nat set*) *set*
  where *iIN-set* $\equiv \{[n...,d]\ |n\ d.\ True\}$

definition *iMOD-set* :: (*nat set*) *set*
  where *iMOD-set* $\equiv \{[r,\ mod\ m]\ |r\ m.\ True\}$

definition *iMODb-set* :: (*nat set*) *set*
  where *iMODb-set* $\equiv \{[r,\ mod\ m,\ c]\ |r\ m\ c.\ True\}$

definition *iMOD2-set* :: (*nat set*) *set*
  where *iMOD2-set* $\equiv \{[r,\ mod\ m]\ |r\ m.\ 2 \leq m\}$

definition *iMODb2-set* :: (*nat set*) *set*
  where *iMODb2-set* $\equiv \{[r,\ mod\ m,\ c]\ |r\ m\ c.\ 2 \leq m \wedge 1 \leq c\}$

definition *iMOD2-UN-set* :: (*nat set*) *set*
  where *iMOD2-UN-set* $\equiv \bigcup r.\ \bigcup m{\in}\{2..\}.\ \{[r,\ mod\ m]\}$

definition *iMODb2-UN-set* :: (*nat set*) *set*
  where *iMODb2-UN-set* $\equiv \bigcup r.\ \bigcup m{\in}\{2..\}.\ \bigcup c{\in}\{1..\}.\ \{[r,\ mod\ m,\ c]\}$

lemmas *i-set-defs* =
  iFROM-set-def iTILL-set-def iIN-set-def
  iMOD-set-def iMODb-set-def
  iMOD2-set-def iMODb2-set-def
lemmas *i-UN-set-defs* =
  iFROM-UN-set-def iTILL-UN-set-def iIN-UN-set-def
  iMOD-UN-set-def iMODb-UN-set-def
  iMOD2-UN-set-def iMODb2-UN-set-def

lemma *iFROM-set-UN-set-eq*: *iFROM-set* = *iFROM-UN-set*
⟨*proof*⟩

lemma
iTILL-set-UN-set-eq: *iTILL-set* = *iTILL-UN-set* **and**
iIN-set-UN-set-eq: *iIN-set* = *iIN-UN-set* **and**
iMOD-set-UN-set-eq: *iMOD-set* = *iMOD-UN-set* **and**
iMODb-set-UN-set-eq: *iMODb-set* = *iMODb-UN-set*
⟨*proof*⟩

lemma *iMOD2-set-UN-set-eq*: *iMOD2-set* = *iMOD2-UN-set*
⟨*proof*⟩

lemma *iMODb2-set-UN-set-eq*: *iMODb2-set* = *iMODb2-UN-set*
⟨*proof*⟩

lemmas *i-set-i-UN-set-sets-eq* =
iFROM-set-UN-set-eq
iTILL-set-UN-set-eq
iIN-set-UN-set-eq
iMOD-set-UN-set-eq
iMODb-set-UN-set-eq
iMOD2-set-UN-set-eq
iMODb2-set-UN-set-eq

lemma *iMOD2-set-iMOD-set-subset*: *iMOD2-set* ⊆ *iMOD-set*
⟨*proof*⟩

lemma *iMODb2-set-iMODb-set-subset*: *iMODb2-set* ⊆ *iMODb-set*
⟨*proof*⟩

definition *i-set* :: (*nat set*) *set*
where *i-set* ≡ *iFROM-set* ∪ *iTILL-set* ∪ *iIN-set* ∪ *iMOD-set* ∪ *iMODb-set*

definition *i-UN-set* :: (*nat set*) *set*
where *i-UN-set* ≡ *iFROM-UN-set* ∪ *iTILL-UN-set* ∪ *iIN-UN-set* ∪ *iMOD-UN-set*
∪ *iMODb-UN-set*

Minimal definitions for *i-set* and *i-set*

definition *i-set-min* :: (*nat set*) *set*
where *i-set-min* ≡ *iFROM-set* ∪ *iIN-set* ∪ *iMOD2-set* ∪ *iMODb2-set*

definition *i-UN-set-min* :: (*nat set*) *set*
where *i-UN-set-min* ≡ *iFROM-UN-set* ∪ *iIN-UN-set* ∪ *iMOD2-UN-set* ∪ *iMODb2-UN-set*

definition *i-set0* :: (*nat set*) *set*
where *i-set0* ≡ *insert* {} *i-set*

lemma *i-set-i-UN-set-eq*: *i-set* = *i-UN-set*
⟨*proof*⟩

lemma *i-set-min-i-UN-set-min-eq*: *i-set-min* = *i-UN-set-min*
⟨*proof*⟩

lemma *i-set-min-eq*: *i-set* = *i-set-min*
⟨*proof*⟩

corollary *i-UN-set-i-UN-min-set-eq*: *i-UN-set* = *i-UN-set-min*
⟨*proof*⟩

lemma *i-set-min-is-minimal-let*:
let s1 = *iFROM-set*; *s2*= *iIN-set*; *s3*= *iMOD2-set*; *s4*= *iMODb2-set in*
s1 ∩ *s2* = {} ∧ *s1* ∩ *s3* = {} ∧ *s1* ∩ *s4* = {} ∧
s2 ∩ *s3* = {} ∧ *s2* ∩ *s4* = {} ∧ *s3* ∩ *s4* = {}
⟨*proof*⟩

lemmas *i-set-min-is-minimal* = *i-set-min-is-minimal-let*[*simplified*]

inductive-set *i-set-ind*:: (*nat set*) *set*
where
i-set-ind-FROM[*intro*!]: [*n*...] ∈ *i-set-ind*
| *i-set-ind-TILL*[*intro*!]: [...*n*] ∈ *i-set-ind*
| *i-set-ind-IN*[*intro*!]: [*n*...,*d*] ∈ *i-set-ind*
| *i-set-ind-MOD*[*intro*!]: [*r*, *mod m*] ∈ *i-set-ind*
| *i-set-ind-MODb*[*intro*!]: [*r*, *mod m*, *c*] ∈ *i-set-ind*

inductive-set *i-set0-ind* :: (*nat set*) *set*
where
i-set0-ind-empty[*intro*!] : {} ∈ *i-set0-ind*
| *i-set0-ind-i-set*[*intro*]: *I* ∈ *i-set-ind* ⟹ *I* ∈ *i-set0-ind*

The introduction rule *i-set0-ind-i-set* is not declared a safe introduction rule, because it would disturb the correct usage of the *safe* method.

lemma *i-set-ind-subset-i-set0-ind*: *i-set-ind* ⊆ *i-set0-ind*
⟨*proof*⟩

lemma
i-set0-ind-FROM[*intro*!] : [*n*...] ∈ *i-set0-ind* **and**
i-set0-ind-TILL[*intro*!] : [...*n*] ∈ *i-set0-ind* **and**
i-set0-ind-IN[*intro*!] : [*n*...,*d*] ∈ *i-set0-ind* **and**
i-set0-ind-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set0-ind* **and**
i-set0-ind-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set0-ind*
⟨*proof*⟩

lemmas *i-set0-ind-intros2* =
i-set0-ind-empty
i-set0-ind-FROM
i-set0-ind-TILL

i-set0-ind-IN
i-set0-ind-MOD
i-set0-ind-MODb

lemma *i-set-i-set-ind-eq*: $i\text{-}set = i\text{-}set\text{-}ind$
⟨*proof*⟩

lemma *i-set0-i-set0-ind-eq*: $i\text{-}set0 = i\text{-}set0\text{-}ind$
⟨*proof*⟩

lemma *i-set-imp-not-empty*: $I \in i\text{-}set \Longrightarrow I \neq \{\}$
⟨*proof*⟩

lemma *i-set0-i-set-mem-conv*: $(I \in i\text{-}set0) = (I \in i\text{-}set \vee I = \{\})$
⟨*proof*⟩

lemma *i-set-i-set0-mem-conv*: $(I \in i\text{-}set) = (I \in i\text{-}set0 \wedge I \neq \{\})$
⟨*proof*⟩

lemma *i-set0-i-set-conv*: $i\text{-}set0 - \{\{\}\} = i\text{-}set$
⟨*proof*⟩

corollary *i-set-subset-i-set0*: $i\text{-}set \subseteq i\text{-}set0$
⟨*proof*⟩

lemma *i-set-singleton*: $\{a\} \in i\text{-}set$
⟨*proof*⟩

lemma *i-set0-singleton*: $\{a\} \in i\text{-}set0$
⟨*proof*⟩

corollary
i-set-FROM[*intro*!] : $[n...] \in i\text{-}set$ **and**
i-set-TILL[*intro*!] : $[...n] \in i\text{-}set$ **and**
i-set-IN[*intro*!] : $[n...,d] \in i\text{-}set$ **and**
i-set-MOD[*intro*!] : $[r,\ mod\ m] \in i\text{-}set$ **and**
i-set-MODb[*intro*!] : $[r,\ mod\ m,\ c] \in i\text{-}set$
⟨*proof*⟩

lemmas *i-set-intros* =
i-set-FROM
i-set-TILL
i-set-IN
i-set-MOD
i-set-MODb

lemma
i-set0-empty[*intro*!]: $\{\} \in i\text{-}set0$ **and**
i-set0-FROM[*intro*!] : $[n...] \in i\text{-}set0$ **and**

i-set0-TILL[*intro*!] : $[\ldots n] \in$ *i-set0* **and**
i-set0-IN[*intro*!] : $[n\ldots,d] \in$ *i-set0* **and**
i-set0-MOD[*intro*!] : $[r,\ mod\ m] \in$ *i-set0* **and**
i-set0-MODb[*intro*!] : $[r,\ mod\ m,\ c] \in$ *i-set0*
⟨*proof*⟩

lemmas *i-set0-intros* =
i-set0-empty
i-set0-FROM
i-set0-TILL
i-set0-IN
i-set0-MOD
i-set0-MODb

lemma *i-set-infinite-as-iMOD*:
$[\![\ I \in$ *i-set*; *infinite* $I\ ]\!] \Longrightarrow \exists\, r\ m.\ I = [r,\ mod\ m]$
⟨*proof*⟩

lemma *i-set-finite-as-iMODb*:
$[\![\ I \in$ *i-set*; *finite* $I\ ]\!] \Longrightarrow \exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *i-set-as-iMOD-iMODb*:
$I \in$ *i-set* $\Longrightarrow (\exists\, r\ m.\ I = [r,\ mod\ m]) \vee (\exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c])$
⟨*proof*⟩

2.5.2 Interval sets are closed under cutting

lemma *i-set-cut-le-ge-closed-disj*:
$[\![\ I \in$ *i-set*; $t \in I$; *cut-op* $= (\downarrow\leq) \vee$ *cut-op* $= (\downarrow\geq)\ ]\!] \Longrightarrow$
cut-op $I\ t \in$ *i-set*
⟨*proof*⟩

corollary
i-set-cut-le-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\leq t \in$ *i-set* **and**
i-set-cut-ge-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\geq t \in$ *i-set*
⟨*proof*⟩

lemmas *i-set-cut-le-ge-closed* = *i-set-cut-le-closed i-set-cut-ge-closed*

lemma *i-set0-cut-closed-disj*:
$[\![\ I \in$ *i-set0*;
cut-op $= (\downarrow\leq) \vee$ *cut-op* $= (\downarrow\geq) \vee$
cut-op $= (\downarrow<) \vee$ *cut-op* $= (\downarrow>)\ ]\!] \Longrightarrow$
cut-op $I\ t \in$ *i-set0*
⟨*proof*⟩

corollary

i-set0-cut-le-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\leq t \in$ *i-set0* **and**
i-set0-cut-less-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow< t \in$ *i-set0* **and**
i-set0-cut-ge-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\geq t \in$ *i-set0* **and**
i-set0-cut-greater-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow> t \in$ *i-set0*
⟨*proof*⟩

lemmas *i-set0-cut-closed* =
i-set0-cut-le-closed
i-set0-cut-less-closed
i-set0-cut-ge-closed
i-set0-cut-greater-closed

2.5.3 Interval sets are closed under addition and multiplication

lemma *i-set-Plus-closed*: $I \in$ *i-set* $\Longrightarrow I \oplus k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-Mult-closed*: $I \in$ *i-set* $\Longrightarrow I \otimes k \in$ *i-set*
⟨*proof*⟩

lemma *i-set0-Plus-closed*: $I \in$ *i-set0* $\Longrightarrow I \oplus k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-Mult-closed*: $I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0*
⟨*proof*⟩

2.5.4 Interval sets are closed with certain conditions under subtraction

lemma *i-set-Plus-neg-closed*:
$[\![I \in$ *i-set*; $\exists x{\in}I.\ k \leq x]\!] \Longrightarrow I \oplus- k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-Minus-closed*:
$[\![I \in$ *i-set*; *iMin* $I \leq k]\!] \Longrightarrow k \ominus I \in$ *i-set*
⟨*proof*⟩

lemma *i-set0-Plus-neg-closed*: $I \in$ *i-set0* $\Longrightarrow I \oplus- k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-Minus-closed*: $I \in$ *i-set0* $\Longrightarrow k \ominus I \in$ *i-set0*
⟨*proof*⟩

lemmas *i-set-IntOp-closed* =
i-set-Plus-closed
i-set-Mult-closed
i-set-Plus-neg-closed
i-set-Minus-closed

lemmas *i-set0-IntOp-closed* =
i-set0-Plus-closed
i-set0-Mult-closed
i-set0-Plus-neg-closed
i-set0-Minus-closed

2.5.5 Interval sets are not closed under division

lemma *iMOD-div-mod-gr0-not-in-i-set*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k\ ]\!] \Longrightarrow [r,\ mod\ m] \oslash k \notin i\text{-}set$
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-not-in-i-set*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k;\ k \leq c\ ]\!] \Longrightarrow [r,\ mod\ m,\ c] \oslash k \notin i\text{-}set$
⟨*proof*⟩

lemma $[0,\ mod\ 3] \oslash 2 \notin i\text{-}set$
⟨*proof*⟩

lemma *i-set-Div-not-closed*: $Suc\ 0 < k \Longrightarrow \exists\, I{\in}i\text{-}set.\ I \oslash k \notin i\text{-}set$
⟨*proof*⟩
lemma *i-set0-Div-not-closed*: $Suc\ 0 < k \Longrightarrow \exists\, I{\in}i\text{-}set0.\ I \oslash k \notin i\text{-}set0$
⟨*proof*⟩

2.5.6 Sets of intervals closed under division

inductive-set *NatMultiples* :: *nat set* ⇒ *nat set*
for *F* :: *nat set*
where
NatMultiples-Factor: $k \in F \Longrightarrow k \in NatMultiples\ F$
| *NatMultiples-Product*: $[\![\ k \in F;\ m \in NatMultiples\ F\ ]\!] \Longrightarrow k * m \in NatMultiples\ F$

lemma *NatMultiples-ex-divisor*: $m \in NatMultiples\ F \Longrightarrow \exists\, k{\in}F.\ m\ mod\ k = 0$
⟨*proof*⟩

lemma *NatMultiples-product-factor*: $[\![\ a \in F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩

lemma *NatMultiples-product-factor-multiple*:
$[\![\ a \in F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩

lemma *NatMultiples-product-multiple-factor*:
$[\![\ a \in NatMultiples\ F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩

lemma *NatMultiples-product-multiple*:
$[\![\ a \in NatMultiples\ F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩

Subset of *i-set* containing only continuous intervals, i. e., without *iMOD* and *iMODb*.

inductive-set *i-set-cont* :: (*nat set*) *set*
where
  i-set-cont-FROM[*intro*]: [*n*...] ∈ *i-set-cont*
| *i-set-cont-TILL*[*intro*]: [...*n*] ∈ *i-set-cont*
| *i-set-cont-IN*[*intro*]: [*n*...,*d*] ∈ *i-set-cont*

definition *i-set0-cont* :: (*nat set*) *set*
  where *i-set0-cont* ≡ *insert* {} *i-set-cont*

lemma *i-set-cont-subset-i-set*: *i-set-cont* ⊆ *i-set*
⟨*proof*⟩

lemma *i-set0-cont-subset-i-set0*: *i-set0-cont* ⊆ *i-set0*
⟨*proof*⟩

Minimal definition of *i-set-cont*

inductive-set *i-set-cont-min* :: (*nat set*) *set*
where
  i-set-cont-FROM[*intro*]: [*n*...] ∈ *i-set-cont-min*
| *i-set-cont-IN*[*intro*]: [*n*...,*d*] ∈ *i-set-cont-min*

definition *i-set0-cont-min* :: (*nat set*) *set*
  where *i-set0-cont-min* ≡ *insert* {} *i-set-cont-min*

lemma *i-set-cont-min-eq*: *i-set-cont* = *i-set-cont-min*
⟨*proof*⟩

inext and *iprev* with continuous intervals

lemma *i-set-cont-inext*:
  ⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *finite I* ⟹ *n* < *Max I* ⟧ ⟹ *inext n I* = *Suc n*
⟨*proof*⟩

lemma *i-set-cont-iprev*:
  ⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *iMin I* < *n* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
⟨*proof*⟩

lemma *i-set-cont-inext--less*:
  ⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n* < *n0*; *n0* ∈ *I* ⟧ ⟹ *inext n I* = *Suc n*
⟨*proof*⟩

lemma *i-set-cont-iprev--greater*:
  ⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n0* < *n*; *n0* ∈ *I* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
⟨*proof*⟩

Sets of modulo intervals

inductive-set *i-set-mult* :: *nat* ⇒ (*nat set*) *set*

for $k :: nat$
where
i-set-mult-FROM[*intro*!]: $[n...] \in$ *i-set-mult* k
| *i-set-mult-TILL*[*intro*!]: $[...n] \in$ *i-set-mult* k
| *i-set-mult-IN*[*intro*!]: $[n...,d] \in$ *i-set-mult* k
| *i-set-mult-MOD*[*intro*!]: $[r,\ mod\ m * k] \in$ *i-set-mult* k
| *i-set-mult-MODb*[*intro*!]: $[r,\ mod\ m * k,\ c] \in$ *i-set-mult* k

definition *i-set0-mult* :: $nat \Rightarrow (nat\ set)\ set$
where *i-set0-mult* $k \equiv$ *insert* $\{\}$ (*i-set-mult* k)

lemma
i-set0-mult-empty[*intro*!]: $\{\} \in$ *i-set0-mult* k **and**
i-set0-mult-FROM[*intro*!] : $[n...] \in$ *i-set0-mult* k **and**
i-set0-mult-TILL[*intro*!] : $[...n] \in$ *i-set0-mult* k **and**
i-set0-mult-IN[*intro*!] : $[n...,d] \in$ *i-set0-mult* k **and**
i-set0-mult-MOD[*intro*!] : $[r,\ mod\ m * k] \in$ *i-set0-mult* k **and**
i-set0-mult-MODb[*intro*!] : $[r,\ mod\ m * k,\ c] \in$ *i-set0-mult* k
⟨*proof*⟩

lemmas *i-set0-mult-intros* =
i-set0-mult-empty
i-set0-mult-FROM
i-set0-mult-TILL
i-set0-mult-IN
i-set0-mult-MOD
i-set0-mult-MODb

lemma *i-set-mult-subset-i-set0-mult*: *i-set-mult* $k \subseteq$ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set-mult-subset-i-set*: *i-set-mult* $k \subseteq$ *i-set*
⟨*proof*⟩

lemma *i-set0-mult-subset-i-set0*: *i-set0-mult* $k \subseteq$ *i-set0*
⟨*proof*⟩

lemma *i-set-mult-0-eq-i-set-cont*: *i-set-mult* $0 =$ *i-set-cont*
⟨*proof*⟩

lemma *i-set0-mult-0-eq-i-set0-cont*: *i-set0-mult* $0 =$ *i-set0-cont*
⟨*proof*⟩

lemma *i-set-mult-1-eq-i-set*: *i-set-mult* $(Suc\ 0) =$ *i-set*
⟨*proof*⟩

lemma *i-set0-mult-1-eq-i-set0*: *i-set0-mult* $(Suc\ 0) =$ *i-set0*
⟨*proof*⟩

lemma *i-set-mult-imp-not-empty*: $I \in$ *i-set-mult* $k \Longrightarrow I \neq \{\}$
⟨*proof*⟩

lemma *iMOD-in-i-set-mult-imp-divisor-mod-0*:
$[\![m \neq Suc\ 0; [r, mod\ m] \in$ *i-set-mult* $k\ ]\!] \Longrightarrow m\ mod\ k = 0$
⟨*proof*⟩

lemma
divisor-mod-0-imp-iMOD-in-i-set-mult: $m\ mod\ k = 0 \Longrightarrow [r, mod\ m] \in$ *i-set-mult* k **and**
divisor-mod-0-imp-iMODb-in-i-set-mult: $m\ mod\ k = 0 \Longrightarrow [r, mod\ m, c] \in$ *i-set-mult* k
⟨*proof*⟩

lemma *iMOD-in-i-set-mult--divisor-mod-0-conv*:
$m \neq Suc\ 0 \Longrightarrow ([r, mod\ m] \in$ *i-set-mult* $k) = (m\ mod\ k = 0)$
⟨*proof*⟩

lemma *i-set-mult-neq1-subset-i-set*: $k \neq Suc\ 0 \Longrightarrow$ *i-set-mult* $k \subset$ *i-set*
⟨*proof*⟩

lemma *mod-0-imp-i-set-mult-subset*:
$a\ mod\ b = 0 \Longrightarrow$ *i-set-mult* $a \subseteq$ *i-set-mult* b
⟨*proof*⟩

lemma *i-set-mult-subset-imp-mod-0*:
$[\![a \neq Suc\ 0; (i\text{-}set\text{-}mult\ a \subseteq i\text{-}set\text{-}mult\ b)\ ]\!] \Longrightarrow a\ mod\ b = 0$
⟨*proof*⟩

lemma *i-set-mult-subset-conv*:
$a \neq Suc\ 0 \Longrightarrow (i\text{-}set\text{-}mult\ a \subseteq i\text{-}set\text{-}mult\ b) = (a\ mod\ b = 0)$
⟨*proof*⟩

lemma *i-set-mult-mod-0-div*:
$[\![I \in$ *i-set-mult* $k; k\ mod\ d = 0\ ]\!] \Longrightarrow I \oslash d \in$ *i-set-mult* $(k\ div\ d)$
⟨*proof*⟩

Intervals from *i-set-mult* k remain in *i-set* after division by d a divisor of k.

corollary *i-set-mult-mod-0-div-i-set*:
$[\![I \in$ *i-set-mult* $k; k\ mod\ d = 0\ ]\!] \Longrightarrow I \oslash d \in$ *i-set*
⟨*proof*⟩
corollary *i-set-mult-div-self-i-set*:
$I \in$ *i-set-mult* $k \Longrightarrow I \oslash k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-mod-0-mult-in-i-set-mult*:
$[\![I \in$ *i-set*; $m\ mod\ k = 0\ ]\!] \Longrightarrow I \otimes m \in$ *i-set-mult* k

⟨*proof*⟩

lemma *i-set-self-mult-in-i-set-mult*:
$I \in$ *i-set* $\Longrightarrow I \otimes k \in$ *i-set-mult* k
⟨*proof*⟩

lemma *i-set-mult-mod-gr0-div-not-in-i-set*:
$[\![\ 0 < k;\ 0 < d;\ 0 < k \bmod d\ ]\!] \Longrightarrow \exists I \in$ *i-set-mult* $k.\ I \oslash d \notin$ *i-set*
⟨*proof*⟩

lemma *i-set0-mult-mod-0-div*:
$[\![\ I \in$ *i-set0-mult* $k;\ k \bmod d = 0\ ]\!] \Longrightarrow I \oslash d \in$ *i-set0-mult* $(k \operatorname{div} d)$
⟨*proof*⟩

corollary *i-set0-mult-mod-0-div-i-set0*:
$[\![\ I \in$ *i-set0-mult* $k;\ k \bmod d = 0\ ]\!] \Longrightarrow I \oslash d \in$ *i-set0*
⟨*proof*⟩

corollary *i-set0-mult-div-self-i-set0*:
$I \in$ *i-set0-mult* $k \Longrightarrow I \oslash k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-mod-0-mult-in-i-set0-mult*:
$[\![\ I \in$ *i-set0*; $m \bmod k = 0\ ]\!] \Longrightarrow I \otimes m \in$ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set0-self-mult-in-i-set0-mult*:
$I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set0-mult-mod-gr0-div-not-in-i-set0*:
$[\![\ 0 < k;\ 0 < d;\ 0 < k \bmod d\ ]\!] \Longrightarrow \exists I \in$ *i-set0-mult* $k.\ I \oslash d \notin$ *i-set0*
⟨*proof*⟩

end

3 Temporal logic operators on natural intervals

theory *IL-TemporalOperators*
imports *IL-IntervalOperators*
begin

Bool : some additional properties

instantiation *bool* :: {*ord*, *zero*, *one*, *plus*, *times*, *order*}
begin

definition *Zero-bool-def* [*simp*]: $0 \equiv$ *False*
definition *One-bool-def* [*simp*]: $1 \equiv$ *True*
definition *add-bool-def*: $a + b \equiv a \vee b$

definition *mult-bool-def*: $a * b \equiv a \wedge b$

instance ⟨*proof*⟩

end

value *False* < *True*
value *True* < *True*
value *True* ≤ *True*

lemmas *bool-op-rel-defs* =
add-bool-def
mult-bool-def
less-bool-def
le-bool-def

instance *bool* :: *wellorder*
⟨*proof*⟩

instance *bool* :: *comm-semiring-1*
⟨*proof*⟩

3.1 Basic definitions

lemma *UNIV-nat*: $\mathbb{N}$ = (*UNIV*::*nat set*)
⟨*proof*⟩

Universal temporal operator: Always/Globally

definition *iAll* :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Always
where *iAll I P* ≡ ∀ *t*∈*I*. *P t*

Existential temporal operator: Eventually/Finally

definition *iEx* :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Eventually
where *iEx I P* ≡ ∃ *t*∈*I*. *P t*

syntax
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*□ - -./ -) [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◇ - -./ -) [*0*, *0*, *10*] *10*)
translations
□ *t I*. *P* ⇌ *CONST iAll I* (λ*t*. *P*)
◇ *t I*. *P* ⇌ *CONST iEx I* (λ*t*. *P*)

Future temporal operator: Next

definition *iNext* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Next
where *iNext t0 I P* ≡ *P* (*inext t0 I*)

Past temporal operator: Last/Previous

definition *iLast* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Last

where *iLast t0 I P* ≡ *P* (*iprev t0 I*)

syntax
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*○ - - -./ -) [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖ - - -./ -) [*0*, *0*, *10*] *10*)
translations
○ *t t0 I*. *P* ⇌ *CONST iNext t0 I* (λ*t*. *P*)
⊖ *t t0 I*. *P* ⇌ *CONST iLast t0 I* (λ*t*. *P*)

lemma ○ *t 10* [*0*...]. (*t* + *10* > *10*)
⟨*proof*⟩

The following versions of Next and Last operator differ in the cases where no next/previous element exists or specified time point is not in interval: the weak versions return *True* and the strong versions return *False*.

definition *iNextWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Next
where *iNextWeak t0 I P* ≡ (□ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iNextStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Next
where *iNextStrong t0 I P* ≡ (◇ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iLastWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Last
where *iLastWeak t0 I P* ≡ (□ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

definition *iLastStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Last
where *iLastStrong t0 I P* ≡ (◇ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

syntax
-iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*○$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*○$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
translations
○$_W$ *t t0 I*. *P* ⇌ *CONST iNextWeak t0 I* (λ*t*. *P*)
○$_S$ *t t0 I*. *P* ⇌ *CONST iNextStrong t0 I* (λ*t*. *P*)
⊖$_W$ *t t0 I*. *P* ⇌ *CONST iLastWeak t0 I* (λ*t*. *P*)
⊖$_S$ *t t0 I*. *P* ⇌ *CONST iLastStrong t0 I* (λ*t*. *P*)

Some examples for Next and Last operator

lemma ○ *t 5* [*0*...,*10*]. ([*0*::*int*,*10*,*20*,*30*,*40*,*50*,*60*,*70*,*80*,*90*] ! *t* < *80*)
⟨*proof*⟩

lemma $\ominus\ t\ 5\ [0...,10].\ ([0::int,10,20,30,40,50,60,70,80,90]\ !\ t < 80)$
⟨*proof*⟩

Temporal Until operator

definition *iUntil* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Until
where $iUntil\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t')$

Temporal Since operator (past operator corresponding to Until)

definition *iSince* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Since
where $iSince\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t')$

syntax
-iUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{U}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{S}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{U}\ t'\ I.\ Q \rightleftharpoons CONST\ iUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{S}\ t'\ I.\ Q \rightleftharpoons CONST\ iSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition *iWeakUntil* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ —
Weak Until/Wating-for/Unless
where $iWeakUntil\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t'))$

definition *iWeakSince* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ —
Weak Since/Back-to
where $iWeakSince\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t'))$

syntax
-iWeakUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{W}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iWeakSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{B}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{W}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{B}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition *iRelease* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ —
Release
where $iRelease\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \vee (\Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow\leq t).\ Q\ t'))$

definition *iTrigger* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ —
Trigger
where $iTrigger\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \vee (\Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow\geq t).\ Q\ t'))$

syntax

-iRelease :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
((-./ - (3$\mathcal{R}$ - -)./ -) [10, 0, 0, 0, 10] 10)
-iTrigger :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
((-./ - (3$\mathcal{T}$ - -)./ -) [10, 0, 0, 0, 10] 10)
translations
$P.\ t\ \mathcal{R}\ t'\ I.\ Q \rightleftharpoons CONST\ iRelease\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{T}\ t'\ I.\ Q \rightleftharpoons CONST\ iTrigger\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

lemmas *iTL-Next-defs* =
iNext-def iLast-def
iNextWeak-def iLastWeak-def
iNextStrong-def iLastStrong-def
lemmas *iTL-defs* =
iAll-def iEx-def
iUntil-def iSince-def
iWeakUntil-def iWeakSince-def
iRelease-def iTrigger-def
iTL-Next-defs

⟨*ML*⟩

3.2 Basic lemmata for temporal operators

3.2.1 Intro/elim rules

lemma
iexI[*intro*]: $[\![\ P\ t;\ t \in I\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
rev-iexI[*intro?*]: $[\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
iexE[*elim!*]: $[\![\ \Diamond\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma
iallI[*intro!*]: $(\bigwedge t.\ t \in I \Longrightarrow P\ t) \Longrightarrow \Box\ t\ I.\ P\ t$ **and**
ispec[*dest?*]: $[\![\ \Box\ t\ I.\ P\ t;\ t \in I\ ]\!] \Longrightarrow P\ t$ **and**
iallE[*elim*]: $[\![\ \Box\ t\ I.\ P\ t;\ P\ t \Longrightarrow Q;\ t \notin I \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma
iuntilI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\,\mathcal{U}\ t\ I.\ Q\ t$ **and**
rev-iuntilI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\,\mathcal{U}\ t\ I.\ Q\ t$
⟨*proof*⟩

lemma
iuntilE[*elim*]:
$[\![\ P'\ t'.\ t'\,\mathcal{U}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma
isinceI[*intro*]:
⟦ $Q\ t$; $(\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')$; $t \in I$ ⟧ $\Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$ **and**
rev-isinceI[*intro?*]:
⟦ $t \in I$; $Q\ t$; $(\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')$ ⟧ $\Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
⟨*proof*⟩
lemma
isinceE[*elim*]:
⟦ $P'\ t'.\ t'\ \mathcal{S}\ t\ I.\ P\ t$; $\bigwedge t.$ ⟦ $t \in I$; $P\ t$ ⟧ $\Longrightarrow Q$ ⟧ $\Longrightarrow Q$
⟨*proof*⟩

3.2.2 Rewrite rules for trivial simplification

lemma *iall-triv*[*simp*]: $(\Box\ t\ I.\ P) = ((\exists t.\ t \in I) \longrightarrow P)$
⟨*proof*⟩

lemma *iex-triv*[*simp*]: $(\Diamond\ t\ I.\ P) = ((\exists t.\ t \in I) \wedge P)$
⟨*proof*⟩

lemma *iex-conjL1*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iex-conjR1*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iex-conjL2*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iex-conjR2*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iex-commute*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P\ t1\ t2) =$
$(\Diamond\ t2\ I2.\ \Diamond\ t1\ I1.\ P\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conjL1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$

⟨*proof*⟩

lemma *iall-conjR1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iall-conjL2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conjR2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
⟨*proof*⟩

lemma *iall-commute*:
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P\ t1\ t2) =$
$(\Box\ t2\ I2.\ \Box\ t1\ I1.\ P\ t1\ t2)$
⟨*proof*⟩

lemma *iall-conj-distrib*:
$(\Box\ t\ I.\ P\ t \wedge Q\ t) = ((\Box\ t\ I.\ P\ t) \wedge (\Box\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iex-disj-distrib*:
$(\Diamond\ t\ I.\ P\ t \vee Q\ t) = ((\Diamond\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iall-conj-distrib2*:
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2 \wedge Q\ t1\ t2) =$
$((\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2) \wedge (\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩

lemma *iex-disj-distrib2*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2 \vee Q\ t1\ t2) =$
$((\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2) \vee (\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩

lemma *iUntil-disj-distrib*:
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (Q1\ t2 \vee Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q1\ t2) \vee (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩

lemma *iSince-disj-distrib*:

$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ (Q1\ t2\ \vee\ Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q1\ t2)\ \vee\ (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩

lemma
iNext-iff: $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iff: $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩

lemma
iNext-iEx-iff: $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iEx-iff: $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩

lemma *inext-singleton-cut-greater-not-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 \neq \{\}) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *iprev-singleton-cut-less-not-empty-iff*:
$(\{iprev\ t0\ I\} \downarrow< t0 \neq \{\}) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *inext-singleton-cut-greater-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 = \{\}) = (I \downarrow> t0 = \{\} \vee t0 \notin I)$
⟨*proof*⟩

lemma *iprev-singleton-cut-less-empty-iff*:
$(\{iprev\ t0\ I\} \downarrow< t0 = \{\}) = (I \downarrow< t0 = \{\} \vee t0 \notin I)$
⟨*proof*⟩

lemma *iNextWeak-iff* : $(\bigcirc_W\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \vee (I \downarrow> t0 = \{\}) \vee t0 \notin I)$
⟨*proof*⟩

lemma *iNextStrong-iff* : $(\bigcirc_S\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \wedge (I \downarrow> t0 \neq \{\}) \wedge t0 \in I)$
⟨*proof*⟩

lemma *iLastWeak-iff* : $(\ominus_W\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \vee (I \downarrow< t0 = \{\}) \vee t0 \notin I)$
⟨*proof*⟩

lemma *iLastStrong-iff* : $(\ominus_S\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \wedge (I \downarrow< t0 \neq \{\}) \wedge t0 \in I)$
⟨*proof*⟩

lemmas *iTL-Next-iff* =
iNext-iff iLast-iff

iNextWeak-iff iNextStrong-iff
iLastWeak-iff iLastStrong-iff

lemma
iNext-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{iprev\ t0\ I\}.\ P\ t)$
⟨*proof*⟩
lemmas *iNextLast-iff-singleton = iNext-iff-singleton iLast-iff-singleton*

lemma
iNext-iEx-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iEx-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{iprev\ t0\ I\}.\ P\ t)$
⟨*proof*⟩

lemma
iNextWeak-iAll-conv: $(\bigcirc_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iNextStrong-iEx-conv: $(\bigcirc_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iLastWeak-iAll-conv: $(\ominus_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$ **and**
iLastStrong-iEx-conv: $(\ominus_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$
⟨*proof*⟩

lemma
iAll-True[*simp*]: $\Box\ t\ I.\ True$ **and**
iAll-False[*simp*]: $(\Box\ t\ I.\ False) = (I = \{\})$ **and**
iEx-True[*simp*]: $(\Diamond\ t\ I.\ True) = (I \neq \{\})$ **and**
iEx-False[*simp*]: $\neg\ (\Diamond\ t\ I.\ False)$
⟨*proof*⟩

lemma *empty-iff-iAll-False*: $(I = \{\}) = (\Box\ t\ I.\ False)$ ⟨*proof*⟩
lemma *not-empty-iff-iEx-True*: $(I \neq \{\}) = (\Diamond\ t\ I.\ True)$ ⟨*proof*⟩

lemma
iNext-True: $\bigcirc\ t\ t0\ I.\ True$ **and**
iNextWeak-True: $(\bigcirc_W\ t\ t0\ I.\ True)$ **and**
iNext-False: $\neg\ (\bigcirc\ t\ t0\ I.\ False)$ **and**
iNextStrong-False: $\neg\ (\bigcirc_S\ t\ t0\ I.\ False)$
⟨*proof*⟩

lemma
iNextStrong-True: $(\bigcirc_S\ t\ t0\ I.\ True) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$ **and**
iNextWeak-False: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ False)) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma
iLast-True: $\ominus\ t\ t0\ I.\ True$ **and**

iLastWeak-True: $(\ominus_W\ t\ t0\ I.\ True)$ **and**
iLast-False: $\neg\ (\ominus\ t\ t0\ I.\ False)$ **and**
iLastStrong-False: $\neg\ (\ominus_S\ t\ t0\ I.\ False)$
⟨*proof*⟩

lemma
iLastStrong-True: $(\ominus_S\ t\ t0\ I.\ True) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$ **and**
iLastWeak-False: $(\neg\ (\ominus_W\ t\ t0\ I.\ False)) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *iUntil-True-left*[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iUntil-True*[*simp*]: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$
⟨*proof*⟩

lemma *iUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (I \neq \{\} \wedge Q\ (iMin\ I))$
⟨*proof*⟩

lemma *iUntil-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ False)$
⟨*proof*⟩

lemma *iSince-True-left*[*simp*]: $(True.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iSince-True-if*:
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) =$
$(if\ finite\ I\ then\ I \neq \{\}\ else\ \Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow> t1).\ P\ t2)$
⟨*proof*⟩

corollary *iSince-True-finite*[*simp*]: $finite\ I \Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) = (I \neq \{\})$
⟨*proof*⟩

lemma *iSince-False-left*[*simp*]: $(False.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (finite\ I \wedge I \neq \{\} \wedge Q\ (Max\ I))$
⟨*proof*⟩

lemma *iSince-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ False)$
⟨*proof*⟩

lemma *iWeakUntil-True-left*[*simp*]: $True.\ t'\ \mathcal{W}\ t\ I.\ Q\ t$
⟨*proof*⟩

lemma *iWeakUntil-True*[*simp*]: $P\ t'.\ t'\ \mathcal{W}\ t\ I.\ True$
⟨*proof*⟩

lemma *iWeakUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = (I = \{\} \vee Q\ (iMin\ I))$
⟨*proof*⟩

lemma *iWeakUntil-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ False) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iWeakSince-True-left*[*simp*]: $True.\ t'\ \mathcal{B}\ t\ I.\ Q\ t$
⟨*proof*⟩

lemma *iWeakSince-True-disj*:
$(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ True) =$
$(I = \{\} \vee (\Diamond\ t1\ I.\ \Box\ t2\ (I\ \downarrow{>}\ t1).\ P\ t2))$
⟨*proof*⟩

lemma *iWeakSince-True-finite*[*simp*]: $finite\ I \Longrightarrow (P\ t'.\ t'\ \mathcal{B}\ t\ I.\ True)$
⟨*proof*⟩

lemma *iWeakSince-False-left*[*simp*]: $(False.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = (I = \{\} \vee (finite\ I \wedge Q\ (Max\ I)))$
⟨*proof*⟩

lemma *iWeakSince-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ False) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iRelease-True-left*[*simp*]: $(True.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (I = \{\} \vee Q\ (iMin\ I))$
⟨*proof*⟩

lemma *iRelease-True*[*simp*]: $P\ t'.\ t'\ \mathcal{R}\ t\ I.\ True$
⟨*proof*⟩

lemma *iRelease-False-left*[*simp*]: $(False.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iRelease-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ False) = (I = \{\})$
⟨*proof*⟩

lemma *iTrigger-True-left*[*simp*]: $(True.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (I = \{\} \vee (\Diamond\ t1\ I.\ \Box\ t2\ (I\ \downarrow{\geq}\ t1).\ Q\ t2))$
⟨*proof*⟩

lemma *iTrigger-True*[*simp*]: $P\ t'.\ t'\ \mathcal{T}\ t\ I.\ True$
⟨*proof*⟩

lemma *iTrigger-False-left*[*simp*]: $(False.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
⟨*proof*⟩

lemma *iTrigger-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ False) = (I = \{\})$
⟨*proof*⟩

lemma
iUntil-TrueTrue[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$ **and**

iSince-TrueTrue[*simp*]: (*True*. t' $\mathcal{S}$ t I. *True*) = ($I \neq \{\}$) **and**
iWeakUntil-TrueTrue[*simp*]: *True*. t' $\mathcal{W}$ t I. *True* **and**
iWeakSince-TrueTrue[*simp*]: *True*. t' $\mathcal{B}$ t I. *True* **and**
iRelease-TrueTrue[*simp*]: *True*. t' $\mathcal{R}$ t I. *True* **and**
iTrigger-TrueTrue[*simp*]: *True*. t' $\mathcal{T}$ t I. *True*
⟨*proof*⟩

3.2.3 Empty sets and singletons

lemma *iAll-empty*[*simp*]: $\Box$ t {}. P t ⟨*proof*⟩
lemma *iEx-empty*[*simp*]: $\neg$ ($\Diamond$ t {}. P t) ⟨*proof*⟩

lemma *iUntil-empty*[*simp*]: $\neg$ (P *t0*. *t0* $\mathcal{U}$ *t1* {}. Q *t1*) ⟨*proof*⟩
lemma *iSince-empty*[*simp*]: $\neg$ (P *t0*. *t0* $\mathcal{S}$ *t1* {}. Q *t1*) ⟨*proof*⟩
lemma *iWeakUntil-empty*[*simp*]: P *t0*. *t0* $\mathcal{W}$ *t1* {}. Q *t1* ⟨*proof*⟩
lemma *iWeakSince-empty*[*simp*]: P *t0*. *t0* $\mathcal{B}$ *t1* {}. Q *t1* ⟨*proof*⟩

lemma *iRelease-empty*[*simp*]: P *t0*. *t0* $\mathcal{R}$ *t1* {}. Q *t1* ⟨*proof*⟩
lemma *iTrigger-empty*[*simp*]: P *t0*. *t0* $\mathcal{T}$ *t1* {}. Q *t1* ⟨*proof*⟩

lemmas *iTL-empty* =
iAll-empty iEx-empty
iUntil-empty iSince-empty
iWeakUntil-empty iWeakSince-empty
iRelease-empty iTrigger-empty

lemma *iAll-singleton*[*simp*]: ($\Box$ t' {t}. P t') = P t ⟨*proof*⟩
lemma *iEx-singleton*[*simp*]: ($\Diamond$ t' {t}. P t') = P t ⟨*proof*⟩

lemma *iUntil-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{U}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩

lemma *iSince-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{S}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩

lemma *iWeakUntil-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{W}$ *t1* {t}. Q *t1*) = (P t $\vee$ Q t)
⟨*proof*⟩

lemma *iWeakSince-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{B}$ *t1* {t}. Q *t1*) = (P t $\vee$ Q t)
⟨*proof*⟩

lemma *iRelease-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{R}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩

lemma *iTrigger-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{T}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩

lemmas *iTL-singleton* =
iAll-singleton iEx-singleton

iUntil-singleton iSince-singleton
iWeakUntil-singleton iWeakSince-singleton
iRelease-singleton iTrigger-singleton

3.2.4 Conversions between temporal operators

lemma *iAll-iEx-conv*: $(\Box\ t\ I.\ P\ t) = (\neg\ (\Diamond\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩
lemma *iEx-iAll-conv*: $(\Diamond\ t\ I.\ P\ t) = (\neg\ (\Box\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩

lemma *not-iAll*[*simp*]: $(\neg\ (\Box\ t\ I.\ P\ t)) = (\Diamond\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩
lemma *not-iEx*[*simp*]: $(\neg\ (\Diamond\ t\ I.\ P\ t)) = (\Box\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩

lemma *iUntil-iEx-conv*: $(\mathit{True}.\ t'\ \mathcal{U}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩
lemma *iSince-iEx-conv*: $(\mathit{True}.\ t'\ \mathcal{S}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩

lemma *iRelease-iAll-conv*: $(\mathit{False}.\ t'\ \mathcal{R}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iTrigger-iAll-conv*: $(\mathit{False}.\ t'\ \mathcal{T}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iWeakUntil-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩

lemma *iWeakSince-iSince-conv*: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩

lemma *iUntil-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iSince-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iRelease-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{W}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩

lemma *iRelease-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{U}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iTrigger-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{B}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩

lemma *iTrigger-iSince-conv*: $(P\ t'.\ t' \mathcal{T}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t' \mathcal{S}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iRelease-not-iUntil-conv*: $(P\ t'.\ t' \mathcal{R}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{U}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩
lemma *iUntil-not-iRelease-conv*: $(P\ t'.\ t' \mathcal{U}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{R}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

The Trigger operator $\mathcal{T}$is a past operator, so that it is used for time intervals, that are bounded by a current time point, and thus are finite. For an infinite interval the stated relation to the Since operator $\mathcal{S}$would not be fulfilled.

lemma *iTrigger-not-iSince-conv*: *finite* $I \Longrightarrow (P\ t'.\ t' \mathcal{T}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{S}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

lemma *iSince-not-iTrigger-conv*: *finite* $I \Longrightarrow (P\ t'.\ t' \mathcal{S}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{T}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

lemma *not-iUntil*:
$(\neg\ (P\ t1.\ t1 \mathcal{U}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I \downarrow< t).\ \neg\ P\ t')))$
⟨*proof*⟩

lemma *not-iSince*:
$(\neg\ (P\ t1.\ t1 \mathcal{S}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I \downarrow> t).\ \neg\ P\ t')))$
⟨*proof*⟩

lemma *iWeakUntil-conj-iUntil-conv*:
$(P\ t1.\ t1 \mathcal{W}\ t2\ I.\ (P\ t2 \wedge Q\ t2)) = (\neg\ (\neg\ Q\ t1.\ t1 \mathcal{U}\ t2\ I.\ \neg\ P\ t2))$
⟨*proof*⟩

lemma *iUntil-disj-iUntil-conv*:
$(P\ t1 \vee Q\ t1.\ t1 \mathcal{U}\ t2\ I.\ Q\ t2) =$
$(P\ t1.\ t1 \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-disj-iWeakUntil-conv*:
$(P\ t1 \vee Q\ t1.\ t1 \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(P\ t1.\ t1 \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-iUntil-conj-conv*:
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (\neg\ P\ t2\ \wedge\ \neg\ Q\ t2)))$
⟨*proof*⟩

Negation and temporal operators

lemma
not-iNext[*simp*]: $(\neg\ (\bigcirc\ t\ t0\ I.\ P\ t)) = (\bigcirc\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextWeak[*simp*]: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ P\ t)) = (\bigcirc_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextStrong[*simp*]: $(\neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)) = (\bigcirc_W\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLast[*simp*]: $(\neg\ (\ominus\ t\ t0\ I.\ P\ t)) = (\ominus\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastWeak[*simp*]: $(\neg\ (\ominus_W\ t\ t0\ I.\ P\ t)) = (\ominus_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastStrong[*simp*]: $(\neg\ (\ominus_S\ t\ t0\ I.\ P\ t)) = (\ominus_W\ t\ t0\ I.\ \neg\ P\ t)$
⟨*proof*⟩

lemma *not-iWeakUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\downarrow< t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩
lemma *not-iWeakSince*:
$(\neg\ (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\downarrow> t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩

lemma *not-iRelease*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ P\ t \longrightarrow (\Diamond\ t\ I\downarrow\leq t.\ \neg\ Q\ t)))$
⟨*proof*⟩

lemma *not-iRelease-iUntil-conv*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t)$
⟨*proof*⟩

lemma *not-iTrigger*:
$(\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ \neg\ P\ t \vee (\Diamond\ t\ I\downarrow\geq t.\ \neg\ Q\ t)))$
⟨*proof*⟩

lemma *not-iTrigger-iSince-conv*:
finite $I \Longrightarrow (\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t)$
⟨*proof*⟩

3.2.5 Some implication results

lemma *all-imp-iall*: $\forall x.\ P\ x \Longrightarrow \Box\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *bex-imp-lex*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \exists x.\ P\ x$ ⟨*proof*⟩

lemma *iAll-imp-iEx*: $I \neq \{\} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *i-set-iAll-imp-iEx*: $I \in$ *i-set* $\Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$

⟨*proof*⟩

lemmas *iT-iAll-imp-iEx* = *iT-not-empty*[*THEN iAll-imp-iEx*]

lemma *iUntil-imp-iEx*: *P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2* $\Longrightarrow$ $\Diamond$ *t I*. *Q t*
⟨*proof*⟩

lemma *iSince-imp-iEx*: *P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2* $\Longrightarrow$ $\Diamond$ *t I*. *Q t*
⟨*proof*⟩

lemma *iall-subset-imp-iall*: ⟦ $\Box$ *t B*. *P t*; *A* $\subseteq$ *B* ⟧ $\Longrightarrow$ $\Box$ *t A*. *P t*
⟨*proof*⟩

lemma *iex-subset-imp-iex*: ⟦ $\Diamond$ *t A*. *P t*; *A* $\subseteq$ *B* ⟧ $\Longrightarrow$ $\Diamond$ *t B*. *P t*
⟨*proof*⟩

lemma *iall-mp*: ⟦ $\Box$ *t I*. *P t* $\longrightarrow$ *Q t*; $\Box$ *t I*. *P t* ⟧ $\Longrightarrow$ $\Box$ *t I*. *Q t* ⟨*proof*⟩
lemma *iex-mp*: ⟦ $\Box$ *t I*. *P t* $\longrightarrow$ *Q t*; $\Diamond$ *t I*. *P t* ⟧ $\Longrightarrow$ $\Diamond$ *t I*. *Q t* ⟨*proof*⟩

lemma *iUntil-imp*:
⟦ *P1 t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iSince-imp*:
⟦ *P1 t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iWeakUntil-imp*:
⟦ *P1 t1*. *t1* $\mathcal{W}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{W}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iWeakSince-imp*:
⟦ *P1 t1*. *t1* $\mathcal{B}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{B}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iRelease-imp*:
⟦ *P1 t1*. *t1* $\mathcal{R}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{R}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iTrigger-imp*:
⟦ *P1 t1*. *t1* $\mathcal{T}$ *t2 I*. *Q t2*; $\Box$ *t I*. *P1 t* $\longrightarrow$ *P2 t* ⟧ $\Longrightarrow$ *P2 t1*. *t1* $\mathcal{T}$ *t2 I*. *Q t2*
⟨*proof*⟩

lemma *iMin-imp-iUntil*:
⟦ *I* $\neq$ {}; *Q* (*iMin I*) ⟧ $\Longrightarrow$ *P t'*. *t'* $\mathcal{U}$ *t I*. *Q t*
⟨*proof*⟩

lemma *Max-imp-iSince*:
$[\![\mathit{finite}\ I;\ I \neq \{\};\ Q\ (\mathit{Max}\ I)]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
⟨*proof*⟩

3.2.6 Congruence rules for temporal operators' predicates

lemma *iAll-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Box\ t\ I.\ P\ (f\ t)\ t) = (\Box\ t\ I.\ P\ (g\ t)\ t)$
⟨*proof*⟩

lemma *iEx-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Diamond\ t\ I.\ P\ (f\ t)\ t) = (\Diamond\ t\ I.\ P\ (g\ t)\ t)$
⟨*proof*⟩

lemma *iUntil-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iUntil-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (g\ t2)\ t2)$
⟨*proof*⟩

lemma *iSince-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iSince-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (g\ t2)\ t2)$
⟨*proof*⟩

lemma *bex-subst*:
$\forall\, x{\in}A.\ P\ x \longrightarrow (Q\ x = Q'\ x) \Longrightarrow$
$(\exists\, x{\in}A.\ P\ x \wedge Q\ x) = (\exists\, x{\in}A.\ P\ x \wedge Q'\ x)$
⟨*proof*⟩

lemma *iEx-subst*:
$\Box\ t\ I.\ P\ t \longrightarrow (Q\ t = Q'\ t) \Longrightarrow$
$(\Diamond\ t\ I.\ P\ t \wedge Q\ t) = (\Diamond\ t\ I.\ P\ t \wedge Q'\ t)$
⟨*proof*⟩

3.2.7 Temporal operators with set unions/intersections and subsets

lemma *iAll-subset*: $[\![A \subseteq B;\ \Box\ t\ B.\ P\ t]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
⟨*proof*⟩

lemma *iEx-subset*: $[\![A \subseteq B;\ \Diamond\ t\ A.\ P\ t]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$

⟨*proof*⟩

lemma *iUntil-append*:
⟦ *finite A*; *Max A* ≤ *iMin B* ⟧ ⟹
P t1. *t1* $\mathcal{U}$ *t2 A*. *Q t2* ⟹ *P t1*. *t1* $\mathcal{U}$ *t2* (*A* ∪ *B*). *Q t2*
⟨*proof*⟩

lemma *iSince-prepend*:
⟦ *finite A*; *Max A* ≤ *iMin B* ⟧ ⟹
P t1. *t1* $\mathcal{S}$ *t2 B*. *Q t2* ⟹ *P t1*. *t1* $\mathcal{S}$ *t2* (*A* ∪ *B*). *Q t2*
⟨*proof*⟩

lemma
iAll-union: ⟦ □ *t A*. *P t*; □ *t B*. *P t* ⟧ ⟹ □ *t* (*A* ∪ *B*). *P t* **and**
iAll-union-conv: (□ *t A* ∪ *B*. *P t*) = ((□ *t A*. *P t*) ∧ (□ *t B*. *P t*))
⟨*proof*⟩

lemma
iEx-union: (◇ *t A*. *P t*) ∨ (◇ *t B*. *P t*) ⟹ ◇ *t* (*A* ∪ *B*). *P t* **and**
iEx-union-conv: (◇ *t* (*A* ∪ *B*). *P t*) = ((◇ *t A*. *P t*) ∨ (◇ *t B*. *P t*))
⟨*proof*⟩

lemma *iAll-inter*: (□ *t A*. *P t*) ∨ (□ *t B*. *P t*) ⟹ □ *t* (*A* ∩ *B*). *P t* ⟨*proof*⟩

lemma *not-iEx-inter*:
∃ *A B P*. (◇ *t A*. *P t*) ∧ (◇ *t B*. *P t*) ∧ ¬ (◇ *t* (*A* ∩ *B*). *P t*)
⟨*proof*⟩

lemma
iAll-insert: ⟦ *P t*; □ *t I*. *P t* ⟧ ⟹ □ *t'* (*insert t I*). *P t'* **and**
iAll-insert-conv: (□ *t'* (*insert t I*). *P t'*) = (*P t* ∧ (□ *t' I*. *P t'*))
⟨*proof*⟩

lemma
iEx-insert: ⟦ *P t* ∨ (◇ *t I*. *P t*) ⟧ ⟹ ◇ *t'* (*insert t I*). *P t'* **and**
iEx-insert-conv: (◇ *t'* (*insert t I*). *P t'*) = (*P t* ∨ (◇ *t' I*. *P t'*))
⟨*proof*⟩

3.3 Further results for temporal operators

lemma *Collect-minI-iEx*: ◇ *t I*. *P t* ⟹ ◇ *t I*. *P t* ∧ (□ *t'* (*I* ↓< *t*). ¬ *P t'*)
⟨*proof*⟩

lemma *iUntil-disj-conv1*:
I ≠ {} ⟹
(*P t'*. *t'* $\mathcal{U}$ *t I*. *Q t*) = (*Q* (*iMin I*) ∨ (*P t'*. *t'* $\mathcal{U}$ *t I*. *Q t* ∧ *iMin I* < *t*))
⟨*proof*⟩

lemma *iSince-disj-conv1*:

$[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (Q\ (Max\ I) \vee (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t \wedge t < Max\ I))$
⟨*proof*⟩

lemma *iUntil-next*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) =$
$(Q\ (iMin\ I) \vee (P\ (iMin\ I) \wedge (P\ t'.\ t'\ \mathcal{U}\ t\ (I \downarrow> (iMin\ I)).\ Q\ t)))$
⟨*proof*⟩

lemma *iSince-prev*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) =$
$(Q\ (Max\ I) \vee (P\ (Max\ I) \wedge (P\ t'.\ t'\ \mathcal{S}\ t\ (I \downarrow< Max\ I).\ Q\ t)))$
⟨*proof*⟩

lemma *iNext-induct-rule*:
$[\![\ P\ (iMin\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t')); t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩

lemma *iNext-induct*:
$[\![\ P\ (iMin\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'))\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$
⟨*proof*⟩

lemma *iLast-induct-rule*:
$[\![\ P\ (Max\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t')); finite\ I; t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩

lemma *iLast-induct*:
$[\![\ P\ (Max\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t')); finite\ I\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$
⟨*proof*⟩

lemma *iUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iSince-conj-not*: *finite* $I \Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iWeakSince-conj-not*: *finite* $I \Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iNextStrong-imp-iNextWeak*: $(\bigcirc_S\ t\ t0\ I.\ P\ t) \longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *iLastStrong-imp-iLastWeak*: $(\ominus_S\ t\ t0\ I.\ P\ t) \longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma *infin-imp-iNextWeak-iNextStrong-eq-iNext*:
$[\![\ \mathit{infinite}\ I;\ t0 \in I\ ]\!] \Longrightarrow$
$((\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)) \wedge ((\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t))$
⟨*proof*⟩

lemma *infin-imp-iNextWeak-eq-iNext*: $[\![\ \mathit{infinite}\ I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNext*: $[\![\ \mathit{infinite}\ I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNextWeak*: $[\![\ \mathit{infinite}\ I;\ t0 \in I\ ]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
not-in-iNext-eq: $t0 \notin I \Longrightarrow (\bigcirc\ t\ t0\ I.\ P\ t) = (P\ t0)$ **and**
not-in-iLast-eq: $t0 \notin I \Longrightarrow (\ominus\ t\ t0\ I.\ P\ t) = (P\ t0)$
⟨*proof*⟩

lemma
not-in-iNextWeak-eq: $t0 \notin I \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastWeak-eq: $t0 \notin I \Longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
not-in-iNextStrong-eq: $t0 \notin I \Longrightarrow \neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastStrong-eq: $t0 \notin I \Longrightarrow \neg\ (\ominus_S\ t\ t0\ I.\ P\ t)$
⟨*proof*⟩

lemma
iNext-UNIV: $(\bigcirc\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextWeak-UNIV: $(\bigcirc_W\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextStrong-UNIV: $(\bigcirc_S\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$
⟨*proof*⟩

lemma
iLast-UNIV: $(\ominus\ t\ t0\ UNIV.\ P\ t) = P\ (t0 - Suc\ 0)$ **and**
iLastWeak-UNIV: $(\ominus_W\ t\ t0\ UNIV.\ P\ t) = (\mathit{if}\ 0 < t0\ \mathit{then}\ P\ (t0 - Suc\ 0)\ \mathit{else}\ True)$ **and**
iLastStrong-UNIV: $(\ominus_S\ t\ t0\ UNIV.\ P\ t) = (\mathit{if}\ 0 < t0\ \mathit{then}\ P\ (t0 - Suc\ 0)\ \mathit{else}\ False)$
⟨*proof*⟩

lemmas *iTL-Next-UNIV* =
iNext-UNIV iNextWeak-UNIV iNextStrong-UNIV
iLast-UNIV iLastWeak-UNIV iLastStrong-UNIV

lemma *inext-nth-iNext-Suc*: $(\bigcirc\ t\ (I \rightarrow n)\ I.\ P\ t) = P\ (I \rightarrow Suc\ n)$
⟨*proof*⟩

lemma *iprev-nth-iLast-Suc*: $(\ominus\ t\ (I \leftarrow n)\ I.\ P\ t) = P\ (I \leftarrow Suc\ n)$
⟨*proof*⟩

3.4 Temporal operators and arithmetic interval operators

Shifting intervals through addition and subtraction of constants. Mirroring intervals through subtraction of intervals from constants. Expanding and compressing intervals through multiplication and division by constants.

Always operator

lemma *iT-Plus-iAll-conv*: $(\Box\ t\ I \oplus k.\ P\ t) = (\Box\ t\ I.\ P\ (t + k))$
⟨*proof*⟩

lemma *iT-Mult-iAll-conv*: $(\Box\ t\ I \otimes k.\ P\ t) = (\Box\ t\ I.\ P\ (t * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iAll-conv*: $(\Box\ t\ I \oplus\!- k.\ P\ t) = (\Box\ t\ (I \downarrow\geq k).\ P\ (t - k))$
⟨*proof*⟩

lemma *iT-Minus-iAll-conv*: $(\Box\ t\ k \ominus I.\ P\ t) = (\Box\ t\ (I \downarrow\leq k).\ P\ (k - t))$
⟨*proof*⟩

lemma *iT-Div-iAll-conv*: $(\Box\ t\ I \oslash k.\ P\ t) = (\Box\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

lemmas *iT-arith-iAll-conv* =
iT-Plus-iAll-conv
iT-Mult-iAll-conv
iT-Plus-neg-iAll-conv
iT-Minus-iAll-conv
iT-Div-iAll-conv

Eventually operator

lemma
iT-Plus-iEx-conv: $(\Diamond\ t\ I \oplus k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t + k))$ **and**
iT-Mult-iEx-conv: $(\Diamond\ t\ I \otimes k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t * k))$ **and**
iT-Plus-neg-iEx-conv: $(\Diamond\ t\ I \oplus\!- k.\ P\ t) = (\Diamond\ t\ (I \downarrow\geq k).\ P\ (t - k))$ **and**
iT-Minus-iEx-conv: $(\Diamond\ t\ k \ominus I.\ P\ t) = (\Diamond\ t\ (I \downarrow\leq k).\ P\ (k - t))$ **and**
iT-Div-iEx-conv: $(\Diamond\ t\ I \oslash k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

Until and Since operators

lemma *iT-Plus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{U}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Until and Since operators can be converted into each other through substraction of intervals from constants

lemma *iUntil-iSince-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iSince-iUntil-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Plus-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{S}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iSince-conv*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iSince-conv*:

$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Weak Until and Weak Since operators

lemma *iT-Plus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{W}\ t2\ (I \downarrow\!\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{B}\ t2\ (I \downarrow\!\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

lemma *iT-Plus-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩

lemma *iT-Mult-iWeakSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩

lemma *iT-Plus-neg-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{B}\ t2\ (I \downarrow\!\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩

lemma *iT-Minus-iWeakSince-conv*:
$(P\ t1.\ t1\ \mathcal{B}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{W}\ t2\ (I \downarrow\!\leq k).\ Q\ (k - t2))$
⟨*proof*⟩

lemma *iT-Div-iWeakSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Release and Trigger operators

lemma *iT-Plus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1$

$\mathcal{R}$ *t2 I*. *Q* (*t2* + *k*))
⟨*proof*⟩

lemma *iT-Mult-iRelease-conv*: (*P t1*. *t1* $\mathcal{R}$ *t2* (*I* ⊗ *k*). *Q t2*) = (*P* (*t1* ∗ *k*). *t1* $\mathcal{R}$ *t2 I*. *Q* (*t2* ∗ *k*))
⟨*proof*⟩

lemma *iT-Plus-neg-iRelease-conv*: (*P t1*. *t1* $\mathcal{R}$ *t2* (*I* ⊕− *k*). *Q t2*) = (*P* (*t1* − *k*). *t1* $\mathcal{R}$ *t2* (*I* ↓≥ *k*). *Q* (*t2* − *k*))
⟨*proof*⟩

lemma *iT-Minus-iRelease-conv*: (*P t1*. *t1* $\mathcal{R}$ *t2* (*k* ⊖ *I*). *Q t2*) = (*P* (*k* − *t1*). *t1* $\mathcal{T}$ *t2* (*I* ↓≤ *k*). *Q* (*k* − *t2*))
⟨*proof*⟩

lemma *iT-Div-iRelease-conv*: (*P t1*. *t1* $\mathcal{R}$ *t2* (*I* ⊘ *k*). *Q t2*) = (*P* (*t1 div k*). *t1* $\mathcal{R}$ *t2 I*. *Q* (*t2 div k*))
⟨*proof*⟩

lemma *iT-Plus-iTrigger-conv*: (*P t1*. *t1* $\mathcal{T}$ *t2* (*I* ⊕ *k*). *Q t2*) = (*P* (*t1* + *k*). *t1* $\mathcal{T}$ *t2 I*. *Q* (*t2* + *k*))
⟨*proof*⟩

lemma *iT-Mult-iTrigger-conv*: *0* < *k* ⟹ (*P t1*. *t1* $\mathcal{T}$ *t2* (*I* ⊗ *k*). *Q t2*) = (*P* (*t1* ∗ *k*). *t1* $\mathcal{T}$ *t2 I*. *Q* (*t2* ∗ *k*))
⟨*proof*⟩

lemma *iT-Plus-neg-iTrigger-conv*: (*P t1*. *t1* $\mathcal{T}$ *t2* (*I* ⊕− *k*). *Q t2*) = (*P* (*t1* − *k*). *t1* $\mathcal{T}$ *t2* (*I* ↓≥ *k*). *Q* (*t2* − *k*))
⟨*proof*⟩

lemma *iT-Minus-iTrigger-conv*:
(*P t1*. *t1* $\mathcal{T}$ *t2* (*k* ⊖ *I*). *Q t2*) = (*P* (*k* − *t1*). *t1* $\mathcal{R}$ *t2* (*I* ↓≤ *k*). *Q* (*k* − *t2*))
⟨*proof*⟩

lemma *iT-Div-iTrigger-conv*:
0 < *k* ⟹ (*P t1*. *t1* $\mathcal{T}$ *t2* (*I* ⊘ *k*). *Q t2*) = (*P* (*t1 div k*). *t1* $\mathcal{T}$ *t2 I*. *Q* (*t2 div k*))
⟨*proof*⟩

end