Interval Temporal Logic on Natural Numbers

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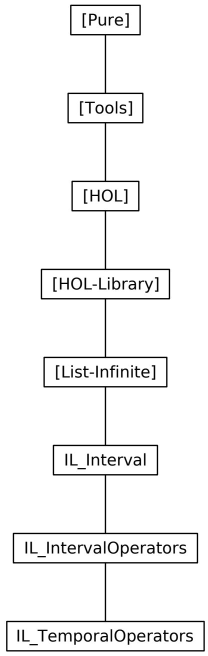
Abstract

We introduce a theory of temporal logic operators using sets of natural numbers as time domain, formalized in a shallow embedding manner. The theory comprises special natural intervals (theory IL_Interval: open and closed intervals, continuous and modulo intervals, interval traversing results), operators for shifting intervals to left/right on the number axis as well as expanding/contracting intervals by constant factors (theory IL_IntervalOperators.thy), and ultimately definitions and results for unary and binary temporal operators on arbitrary natural sets (theory IL_TemporalOperators).

Contents


[Pure]
[Tools]
[HOL]
[HOL-Library]
[List-Infinite]
IL_Interval
IL_IntervalOperators
IL_TemporalOperators

1 Intervals and operations for temporal logic declarations

theory *IL-Interval*
imports
List−Infinite.InfiniteSet2
List−Infinite.SetIntervalStep
begin

1.1 Time intervals – definitions and basic lemmata

1.1.1 Definitions

type-synonym *Time* = *nat*

type-synonym *iT* = *Time set*

Infinite interval starting at some natural *n*.

definition
iFROM :: *Time* ⇒ *iT* (‹[-. . .]›)
where
[*n*. . .] ≡ {*n*..}

Finite interval starting at *0* and ending at some natural *n*.

definition
iTILL :: *Time* ⇒ *iT* (‹[. . .-]›)
where
[. . .*n*] ≡ {..*n*}

Finite bounded interval containing the naturals between *n* and *n* + *d*. *d* denotes the difference between left and right interval bound. The number of elements is *d* + *1* so that an empty interval cannot be defined.

definition
iIN :: *Time* ⇒ *nat* ⇒ *iT* (‹[-. . .,-]›)
where
[*n*. . .,*d*] ≡ {*n*..*n*+*d*}

Infinite modulo interval containing all naturals having the same division remainder modulo *m* as *r*, and beginning at *n*.

definition
iMOD :: *Time* ⇒ *nat* ⇒ *iT* (‹[-, *mod* -]›)
where
[*r*, *mod m*] ≡ { *x*. *x mod m* = *r mod m* ∧ *r* ≤ *x*}

Finite bounded modulo interval containing all naturals having the same division remainder modulo *m* as *r*, beginning at *n*, and ending after *c* cycles at $r + m * c$. The number of elements is $c + 1$ so that an empty interval cannot be defined.

definition
iMODb :: *Time* ⇒ *nat* ⇒ *nat* ⇒ *iT* (‹[-, *mod* -, -]›)
where
$[r,\ mod\ m,\ c] \equiv \{\ x.\ x\ mod\ m = r\ mod\ m \wedge r \leq x \wedge x \leq r + m * c\}$

1.1.2 Membership in an interval

lemmas *iT-defs* = *iFROM-def iTILL-def iIN-def iMOD-def iMODb-def*

lemma *iFROM-iff*: $x \in [n\ldots] = (n \leq x)$
by (*simp add*: *iFROM-def*)
lemma *iTILL-iff*: $x \in [\ldots n] = (x \leq n)$
by (*simp add*: *iTILL-def*)
lemma *iIN-iff*:$x \in [n\ldots,d] = (n \leq x \wedge x \leq n + d)$
by (*simp add*: *iIN-def*)
lemma *iMOD-iff*: $x \in [r,\ mod\ m] = (x\ mod\ m = r\ mod\ m \wedge r \leq x)$
by (*simp add*: *iMOD-def*)
lemma *iMODb-iff*: $x \in [r,\ mod\ m,\ c] =$
$(x\ mod\ m = r\ mod\ m \wedge r \leq x \wedge x \leq r + m * c)$
by (*simp add*: *iMODb-def*)

lemma *iFROM-D*: $x \in [n\ldots] \Longrightarrow (n \leq x)$
by (*rule iFROM-iff*[*THEN iffD1*])
lemma *iTILL-D*: $x \in [\ldots n] \Longrightarrow (x \leq n)$
by (*rule iTILL-iff*[*THEN iffD1*])
corollary *iIN-geD*: $x \in [n\ldots,d] \Longrightarrow n \leq x$
by (*simp add*: *iIN-iff*)
corollary *iIN-leD*: $x \in [n\ldots,d] \Longrightarrow x \leq n + d$
by (*simp add*: *iIN-iff*)
corollary *iMOD-modD*: $x \in [r,\ mod\ m] \Longrightarrow x\ mod\ m = r\ mod\ m$
by (*simp add*: *iMOD-iff*)
corollary *iMOD-geD*: $x \in [r,\ mod\ m] \Longrightarrow r \leq x$
by (*simp add*: *iMOD-iff*)
corollary *iMODb-modD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x\ mod\ m = r\ mod\ m$
by (*simp add*: *iMODb-iff*)
corollary *iMODb-geD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow r \leq x$
by (*simp add*: *iMODb-iff*)
corollary *iMODb-leD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x \leq r + m * c$
by (*simp add*: *iMODb-iff*)

lemmas *iT-iff* = *iFROM-iff iTILL-iff iIN-iff iMOD-iff iMODb-iff*

lemmas *iT-drule* =
iFROM-D
iTILL-D
iIN-geD iIN-leD
iMOD-modD iMOD-geD

iMODb-modD iMODb-geD iMODb-leD

lemma
iFROM-I [*intro*]: $n \le x \Longrightarrow x \in [n\ldots]$ **and**
iTILL-I [*intro*]: $x \le n \Longrightarrow x \in [\ldots n]$ **and**
iIN-I [*intro*]: $n \le x \Longrightarrow x \le n + d \Longrightarrow x \in [n\ldots,d]$ **and**
iMOD-I [*intro*]: $x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \in [r, \bmod m]$ **and**
iMODb-I [*intro*]: $x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \le r + m * c \Longrightarrow x \in [r, \bmod m, c]$
by (*simp add*: *iT-iff*)+

lemma
iFROM-E [*elim*]: $x \in [n\ldots] \Longrightarrow (n \le x \Longrightarrow P) \Longrightarrow P$ **and**
iTILL-E [*elim*]: $x \in [\ldots n] \Longrightarrow (x \le n \Longrightarrow P) \Longrightarrow P$ **and**
iIN-E [*elim*]: $x \in [n\ldots,d] \Longrightarrow (n \le x \Longrightarrow x \le n + d \Longrightarrow P) \Longrightarrow P$ **and**
iMOD-E [*elim*]: $x \in [r, \bmod m] \Longrightarrow (x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow P) \Longrightarrow P$ **and**
iMODb-E [*elim*]: $x \in [r, \bmod m, c] \Longrightarrow (x \bmod m = r \bmod m \Longrightarrow r \le x \Longrightarrow x \le r + m * c \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *iT-iff*)+

lemma *iIN-Suc-insert-conv*:
insert (*Suc* (*n* + *d*)) [*n*...,*d*] = [*n*...,*Suc d*]
by (*fastforce simp*: *iIN-iff*)

lemma *iTILL-Suc-insert-conv*: *insert* (*Suc n*) [...*n*] = [...*Suc n*]
by (*fastforce simp*: *iIN-Suc-insert-conv*[*of 0 n*])

lemma *iMODb-Suc-insert-conv*:
insert (*r* + *m* ∗ *Suc c*) [*r*, *mod m*, *c*] = [*r*, *mod m*, *Suc c*]
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff add.commute*[*of - r*])
apply (*simp add*: *add.commute*[*of m*])
apply (*simp add*: *add.assoc*[*symmetric*])
apply (*rule iffI*)
 apply *fastforce*
apply (*elim conjE*)
apply (*drule-tac x*=*x* **in** *order-le-less*[*THEN iffD1*, *rule-format*])
apply (*erule disjE*)
 apply (*frule less-mod-eq-imp-add-divisor-le*[**where** *m*=*m*], *simp*)
 apply (*drule add-le-imp-le-right*)
 apply *simp*
apply *simp*
done

lemma *iFROM-pred-insert-conv*: *insert* (*n* − *Suc 0*) [*n*. . .] = [*n* − *Suc 0*. . .]
by (*fastforce simp*: *iFROM-iff*)

lemma *iIN-pred-insert-conv*:
  0 < *n* ⟹ *insert* (*n* − *Suc 0*) [*n*. . .,*d*] = [*n* − *Suc 0*. . .,*Suc d*]
by (*fastforce simp*: *iIN-iff*)

lemma *iMOD-pred-insert-conv*:
  m ≤ *r* ⟹ *insert* (*r* − *m*) [*r*, *mod m*] = [*r* − *m*, *mod m*]
apply (*case-tac m* = *0*)
 apply (*simp add*: *iMOD-iff insert-absorb*)
apply *simp*
apply (*rule set-eqI*)
apply (*simp add*: *iMOD-iff mod-diff-self2*)
apply (*rule iffI*)
 apply (*erule disjE*)
  apply (*simp add*: *mod-diff-self2*)
 apply (*simp add*: *le-imp-diff-le*)
apply (*erule conjE*)
apply (*drule order-le-less*[*THEN iffD1*, *of r*−*m*], *erule disjE*)
 prefer *2*
 apply *simp*
apply (*frule order-less-le-trans*[*of - m r*], *assumption*)
apply (*drule less-mod-eq-imp-add-divisor-le*[*of r*−*m - m*])
 apply (*simp add*: *mod-diff-self2*)
apply *simp*
done

lemma *iMODb-pred-insert-conv*:
  m ≤ *r* ⟹ *insert* (*r* − *m*) [*r*, *mod m*, *c*] = [*r* − *m*, *mod m*, *Suc c*]
apply (*rule set-eqI*)
apply (*frule iMOD-pred-insert-conv*)
apply (*drule-tac f*=*λs*. *x* ∈ *s* **in** *arg-cong*)
apply (*force simp*: *iMOD-iff iMODb-iff*)
done

lemma *iFROM-Suc-pred-insert-conv*: *insert n* [*Suc n*. . .] = [*n*. . .]
by (*insert iFROM-pred-insert-conv*[*of Suc n*], *simp*)
lemma *iIN-Suc-pred-insert-conv*: *insert n* [*Suc n*. . .,*d*] = [*n*. . .,*Suc d*]
by (*insert iIN-pred-insert-conv*[*of Suc n*], *simp*)
lemma *iMOD-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*] = [*r*, *mod m*]
by (*insert iMOD-pred-insert-conv*[*of m r* + *m*], *simp*)
lemma *iMODb-Suc-pred-insert-conv*: *insert r* [*r* + *m*, *mod m*, *c*] = [*r*, *mod m*, *Suc c*]
by (*insert iMODb-pred-insert-conv*[*of m r* + *m*], *simp*)

lemmas *iT-Suc-insert* =
  iIN-Suc-insert-conv
  iTILL-Suc-insert-conv

iMODb-Suc-insert-conv
lemmas *iT-pred-insert* =
iFROM-pred-insert-conv
iIN-pred-insert-conv
iMOD-pred-insert-conv
iMODb-pred-insert-conv
lemmas *iT-Suc-pred-insert* =
iFROM-Suc-pred-insert-conv
iIN-Suc-pred-insert-conv
iMOD-Suc-pred-insert-conv
iMODb-Suc-pred-insert-conv

lemma *iMOD-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*]; *b* ∈ [*r*, *mod m*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
by (*simp add*: *iMOD-iff mod-eq-imp-diff-mod-0*)
lemma *iMODb-mem-diff*: ⟦ *a* ∈ [*r*, *mod m*, *c*]; *b* ∈ [*r*, *mod m*, *c*] ⟧ ⟹ (*a* − *b*) *mod m* = *0*
by (*simp add*: *iMODb-iff mod-eq-imp-diff-mod-0*)

1.1.3 Interval conversions

lemma *iIN-0-iTILL-conv*:[*0*. . .,*n*] = [. . .*n*]
by (*simp add*: *iTILL-def iIN-def atMost-atLeastAtMost-0-conv*)
lemma *iIN-iTILL-iTILL-conv*: *0* < *n* ⟹ [*n*. . .,*d*] = [. . .*n*+*d*] − [. . .*n* − *Suc 0*]
by (*fastforce simp*: *iTILL-iff iIN-iff*)
lemma *iIN-iFROM-iTILL-conv*: [*n*. . .,*d*] = [*n*. . .] ∩ [. . .*n*+*d*]
by (*simp add*: *iT-defs atLeastAtMost-def*)
lemma *iMODb-iMOD-iTILL-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [. . .*r*+*m*∗*c*]
by (*force simp*: *iT-defs set-interval-defs*)
lemma *iMODb-iMOD-iIN-conv*: [*r*, *mod m*, *c*] = [*r*, *mod m*] ∩ [*r*. . .,*m*∗*c*]
by (*force simp*: *iT-defs set-interval-defs*)

lemma *iFROM-iTILL-iIN-conv*: *n* ≤ *n′* ⟹ [*n*. . .] ∩ [. . .*n′*] = [*n*. . .,*n′*−*n*]
by (*simp add*: *iT-defs atLeastAtMost-def*)
lemma *iMOD-iTILL-iMODb-conv*:
r ≤ *n* ⟹ [*r*, *mod m*] ∩ [. . .*n*] = [*r*, *mod m*, (*n* − *r*) *div m*]
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff minus-mod-eq-mult-div* [*symmetric*])
apply (*rule iffI*)
apply *clarify*
apply (*frule-tac x*=*x* **and** *y*=*n* **and** *m*=*m* **in** *le-imp-sub-mod-le*)
apply (*simp add*: *mod-diff-right-eq*)
apply *fastforce*
done

lemma *iMOD-iIN-iMODb-conv*:
[*r*, *mod m*] ∩ [*r*. . .,*d*] = [*r*, *mod m*, *d div m*]
apply (*case-tac r* = *0*)
apply (*simp add*: *iIN-0-iTILL-conv iMOD-iTILL-iMODb-conv*)

apply (*simp add*: *iIN-iTILL-iTILL-conv Diff-Int-distrib iMOD-iTILL-iMODb-conv diff-add-inverse*)
apply (*rule subst*[*of* {} - $\lambda t. \forall x.(x - t) = x$, *THEN spec*])
prefer *2*
apply *simp*
apply (*rule sym*)
apply (*fastforce simp*: *disjoint-iff-not-equal iMOD-iff iTILL-iff*)
done

lemma *iFROM-0*: [*0*. . .] = *UNIV*
by (*simp add*: *iFROM-def*)

lemma *iTILL-0*: [. . .*0*] = {*0*}
by (*simp add*: *iTILL-def*)

lemma *iIN-0*: [*n*. . .,*0*] = {*n*}
by (*simp add*: *iIN-def*)

lemma *iMOD-0*: [*r*, *mod 0*] = [*r*. . .,*0*]
by (*fastforce simp*: *iIN-0 iMOD-def*)

lemma *iMODb-mod-0*: [*r*, *mod 0*, *c*] = [*r*. . .,*0*]
by (*fastforce simp*: *iMODb-def iIN-0*)

lemma *iMODb-0*: [*r*, *mod m*, *0*] = [*r*. . .,*0*]
by (*fastforce simp*: *iMODb-def iIN-0 set-eq-iff*)

lemmas *iT-0* =
iFROM-0
iTILL-0
iIN-0
iMOD-0
iMODb-mod-0
iMODb-0

lemma *iMOD-1*: [*r*, *mod Suc 0*] = [*r*. . .]
by (*fastforce simp*: *iFROM-iff*)

lemma *iMODb-mod-1*: [*r*, *mod Suc 0*, *c*] = [*r*. . .,*c*]
by (*fastforce simp*: *iT-iff*)

1.1.4 Finiteness and emptiness of intervals

lemma
iFROM-not-empty: [*n*. . .] ≠ {} **and**
iTILL-not-empty: [. . .*n*] ≠ {} **and**
iIN-not-empty: [*n*. . .,*d*] ≠ {} **and**
iMOD-not-empty: [*r*, *mod m*] ≠ {} **and**
iMODb-not-empty: [*r*, *mod m*, *c*] ≠ {}

by (*fastforce simp*: *iT-iff*)+

lemmas *iT-not-empty* =
iFROM-not-empty
iTILL-not-empty
iIN-not-empty
iMOD-not-empty
iMODb-not-empty

lemma
iTILL-finite: *finite* [...*n*] **and**
iIN-finite: *finite* [*n*...,*d*] **and**
iMODb-finite: *finite* [*r*, *mod m*, *c*] **and**
iMOD-0-finite: *finite* [*r*, *mod 0*]
by (*simp add*: *iT-defs*)+

lemma *iFROM-infinite*: *infinite* [*n*...]
by (*simp add*: *iT-defs infinite-atLeast*)

lemma *iMOD-infinite*: $0 < m \Longrightarrow$ *infinite* [*r*, *mod m*]
apply (*rule infinite-nat-iff-asc-chain*[*THEN iffD2*])
apply (*rule iT-not-empty*)
apply (*rule ballI*, *rename-tac n*)
apply (*rule-tac x*=*n*+*m* **in** *bexI*, *simp*)
apply (*simp add*: *iMOD-iff*)
done

lemmas *iT-finite* =
iTILL-finite
iIN-finite
iMODb-finite iMOD-0-finite

lemmas *iT-infinite* =
iFROM-infinite
iMOD-infinite

1.1.5 *Min* and *Max* element of an interval

lemma
iTILL-Min: *iMin* [...*n*] = *0* **and**
iFROM-Min: *iMin* [*n*...] = *n* **and**
iIN-Min: *iMin* [*n*...,*d*] = *n* **and**
iMOD-Min: *iMin* [*r*, *mod m*] = *r* **and**
iMODb-Min: *iMin* [*r*, *mod m*, *c*] = *r*
by (*rule iMin-equality*, (*simp add*: *iT-iff*)+)+

lemmas *iT-Min* =
iIN-Min
iTILL-Min

iFROM-Min
iMOD-Min
iMODb-Min

lemma
iTILL-Max: *Max* [. . .*n*] = *n* **and**
iIN-Max: *Max* [*n*. . .,*d*] = *n*+*d* **and**
iMODb-Max: *Max* [*r*, *mod m*, *c*] = *r* + *m* ∗ *c* **and**
iMOD-0-Max: *Max* [*r*, *mod 0*] = *r*
by (*rule Max-equality*, (*simp add*: *iT-iff iT-finite*)+)+

lemmas *iT-Max* =
iTILL-Max
iIN-Max
iMODb-Max
iMOD-0-Max

lemma
iTILL-iMax: *iMax* [. . .*n*] = *enat n* **and**
iIN-iMax: *iMax* [*n*. . .,*d*] = *enat* (*n*+*d*) **and**
iMODb-iMax: *iMax* [*r*, *mod m*, *c*] = *enat* (*r* + *m* ∗ *c*) **and**
iMOD-0-iMax: *iMax* [*r*, *mod 0*] = *enat r* **and**
iFROM-iMax: *iMax* [*n*. . .] = ∞ **and**
iMOD-iMax: *0* < *m* =⇒ *iMax* [*r*, *mod m*] = ∞
by (*simp add*: *iMax-def iT-finite iT-infinite iT-Max*)+

lemmas *iT-iMax* =
iTILL-iMax
iIN-iMax
iMODb-iMax
iMOD-0-iMax
iFROM-iMax
iMOD-iMax

1.2 Adding and subtracting constants to interval elements

lemma
iFROM-plus: *x* ∈ [*n*. . .] =⇒ *x* + *k* ∈ [*n*. . .] **and**
iFROM-Suc: *x* ∈ [*n*. . .] =⇒ *Suc x* ∈ [*n*. . .] **and**
iFROM-minus: ⟦ *x* ∈ [*n*. . .]; *k* ≤ *x* − *n* ⟧ =⇒ *x* − *k* ∈ [*n*. . .] **and**
iFROM-pred: *n* < *x* =⇒ *x* − *Suc 0* ∈ [*n*. . .]
by (*simp add*: *iFROM-iff*)+

lemma
iTILL-plus: ⟦ *x* ∈ [. . .*n*]; *k* ≤ *n* − *x* ⟧ =⇒ *x* + *k* ∈ [. . .*n*] **and**
iTILL-Suc: *x* < *n* =⇒ *Suc x* ∈ [. . .*n*] **and**
iTILL-minus: *x* ∈ [. . .*n*] =⇒ *x* − *k* ∈ [. . .*n*] **and**
iTILL-pred: *x* ∈ [. . .*n*] =⇒ *x* − *Suc 0* ∈ [. . .*n*]
by (*simp add*: *iTILL-iff*)+

lemma *iIN-plus*: ⟦ $x \in [n\ldots,d]$; $k \le n + d - x$ ⟧ $\Longrightarrow x + k \in [n\ldots,d]$
by (*fastforce simp*: *iIN-iff*)

lemma *iIN-Suc*: ⟦ $x \in [n\ldots,d]$; $x < n + d$ ⟧ $\Longrightarrow$ *Suc* $x \in [n\ldots,d]$
by (*simp add*: *iIN-iff*)

lemma *iIN-minus*: ⟦ $x \in [n\ldots,d]$; $k \le x - n$ ⟧ $\Longrightarrow x - k \in [n\ldots,d]$
by (*fastforce simp*: *iIN-iff*)

lemma *iIN-pred*: ⟦ $x \in [n\ldots,d]$; $n < x$ ⟧ $\Longrightarrow x -$ *Suc 0* $\in [n\ldots,d]$
by (*fastforce simp*: *iIN-iff*)

lemma *iMOD-plus-divisor-mult*: $x \in [r,\ mod\ m] \Longrightarrow x + k * m \in [r,\ mod\ m]$
by (*simp add*: *iMOD-def*)

corollary *iMOD-plus-divisor*: $x \in [r,\ mod\ m] \Longrightarrow x + m \in [r,\ mod\ m]$
by (*simp add*: *iMOD-def*)

lemma *iMOD-minus-divisor-mult*:
⟦ $x \in [r,\ mod\ m]$; $k * m \le x - r$ ⟧ $\Longrightarrow x - k * m \in [r,\ mod\ m]$
by (*fastforce simp*: *iMOD-def mod-diff-mult-self1*)

corollary *iMOD-minus-divisor-mult2*:
⟦ $x \in [r,\ mod\ m]$; $k \le (x - r)\ div\ m$ ⟧ $\Longrightarrow x - k * m \in [r,\ mod\ m]$
apply (*rule iMOD-minus-divisor-mult*, *assumption*)
apply (*clarsimp simp*: *iMOD-iff*)
apply (*drule mult-le-mono1*[*of - - m*])
apply (*simp add*: *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*])
done

corollary *iMOD-minus-divisor*:
⟦ $x \in [r,\ mod\ m]$; $m + r \le x$ ⟧ $\Longrightarrow x - m \in [r,\ mod\ m]$
apply (*frule iMOD-geD*)
apply (*insert iMOD-minus-divisor-mult*[*of x r m 1*])
apply *simp*
done

lemma *iMOD-plus*:
$x \in [r,\ mod\ m] \Longrightarrow (x + k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
apply *safe*
 apply (*drule iMOD-modD*)+
 apply (*rule mod-add-eq-imp-mod-0*[*of x*, *THEN iffD1*])
  apply *simp*
 apply (*erule dvdE*)
apply (*simp add*: *mult.commute iMOD-plus-divisor-mult*)
done
corollary *iMOD-Suc*:
$x \in [r,\ mod\ m] \Longrightarrow$ (*Suc* $x \in [r,\ mod\ m]) = (m =$ *Suc 0*$)$

apply (*simp add*: *iMOD-iff*, *safe*)
apply (*simp add*: *mod-Suc*, *split if-split-asm*)
apply *simp*+
done

lemma *iMOD-minus*:
$[\![x \in [r, mod\ m]; k \le x - r]\!] \Longrightarrow (x - k \in [r, mod\ m]) = (k\ mod\ m = 0)$
apply *safe*
apply (*clarsimp simp*: *iMOD-iff*)
apply (*rule mod-add-eq-imp-mod-0*[*of x* − *k k*, *THEN iffD1*])
apply *simp*
apply (*erule dvdE*)
apply (*simp add*: *mult.commute iMOD-minus-divisor-mult*)
done

corollary *iMOD-pred*:
$[\![x \in [r, mod\ m]; r < x]\!] \Longrightarrow (x - Suc\ 0 \in [r, mod\ m]) = (m = Suc\ 0)$
apply *safe*
apply (*simp add*: *iMOD-Suc*[*of x* − *Suc 0 r*, *THEN iffD1*])
apply (*simp add*: *iMOD-iff*)
done

lemma *iMODb-plus-divisor-mult*:
$[\![x \in [r, mod\ m, c]; k * m \le r + m * c - x]\!] \Longrightarrow x + k * m \in [r, mod\ m, c]$
by (*fastforce simp*: *iMODb-def*)

lemma *iMODb-plus-divisor-mult2*:
$[\![x \in [r, mod\ m, c]; k \le c - (x - r)\ div\ m]\!] \Longrightarrow$
$x + k * m \in [r, mod\ m, c]$
apply (*rule iMODb-plus-divisor-mult*, *assumption*)
apply (*clarsimp simp*: *iMODb-iff*)
apply (*drule mult-le-mono1*[*of - - m*])
apply (*simp add*: *diff-mult-distrib*
mod-0-div-mult-cancel[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
add.commute[*of r*] *mult.commute*[*of c*])
done
lemma *iMODb-plus-divisor*:
$[\![x \in [r, mod\ m, c]; x < r + m * c]\!] \Longrightarrow x + m \in [r, mod\ m, c]$
by (*simp add*: *iMODb-iff less-mod-eq-imp-add-divisor-le*)

lemma *iMODb-minus-divisor-mult*:
$[\![x \in [r, mod\ m, c]; r + k * m \le x]\!] \Longrightarrow x - k * m \in [r, mod\ m, c]$
by (*fastforce simp*: *iMODb-def mod-diff-mult-self1*)

lemma *iMODb-plus*:
$[\![x \in [r, mod\ m, c]; k \le r + m * c - x]\!] \Longrightarrow$
$(x + k \in [r, mod\ m, c]) = (k\ mod\ m = 0)$
apply *safe*
apply (*rule mod-add-eq-imp-mod-0*[*of x*, *THEN iffD1*])

apply (*simp add*: *iT-iff*)
apply *fastforce*
done

corollary *iMODb-Suc*:
$[\![x \in [r, mod\ m, c]; x < r + m * c]\!] \Longrightarrow$
$(Suc\ x \in [r, mod\ m, c]) = (m = Suc\ 0)$
apply (*rule iffI*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-Suc*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-1 iFROM-Suc iTILL-Suc*)
done

lemma *iMODb-minus*:
$[\![x \in [r, mod\ m, c]; k \le x - r]\!] \Longrightarrow$
$(x - k \in [r, mod\ m, c]) = (k\ mod\ m = 0)$
apply (*rule iffI*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-minus*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-minus iTILL-minus*)
done

corollary *iMODb-pred*:
$[\![x \in [r, mod\ m, c]; r < x]\!] \Longrightarrow$
$(x - Suc\ 0 \in [r, mod\ m, c]) = (m = Suc\ 0)$
apply (*rule iffI*)
apply (*subgoal-tac* $x \in [r, mod\ m] \wedge x - Suc\ 0 \in [r, mod\ m]$)
prefer *2*
apply (*simp add*: *iT-iff*)
apply (*clarsimp simp*: *iMOD-pred*)
apply (*fastforce simp add*: *iMODb-iff*)
done

lemmas *iFROM-plus-minus* =
iFROM-plus
iFROM-Suc
iFROM-minus
iFROM-pred

lemmas *iTILL-plus-minus* =
iTILL-plus
iTILL-Suc
iTILL-minus
iTILL-pred

lemmas *iIN-plus-minus* =
iIN-plus
iIN-Suc
iTILL-minus
iIN-pred

lemmas *iMOD-plus-minus-divisor* =
iMOD-plus-divisor-mult
iMOD-plus-divisor
iMOD-minus-divisor-mult
iMOD-minus-divisor-mult2
iMOD-minus-divisor

lemmas *iMOD-plus-minus* =
iMOD-plus
iMOD-Suc
iMOD-minus
iMOD-pred

lemmas *iMODb-plus-minus-divisor* =
iMODb-plus-divisor-mult
iMODb-plus-divisor-mult2
iMODb-plus-divisor
iMODb-minus-divisor-mult

lemmas *iMODb-plus-minus* =
iMODb-plus
iMODb-Suc
iMODb-minus
iMODb-pred

lemmas *iT-plus-minus* =
iFROM-plus-minus
iTILL-plus-minus
iIN-plus-minus
iMOD-plus-minus-divisor
iMOD-plus-minus
iMODb-plus-minus-divisor
iMODb-plus-minus

1.3 Relations between intervals

1.3.1 Auxiliary lemmata

lemma *Suc-in-imp-not-subset-iMOD*:
⟦ $n \in S$; $Suc\ n \in S$; $m \neq Suc\ 0$ ⟧ $\Longrightarrow \neg\ S \subseteq [r,\ mod\ m]$
by (*blast intro*: *iMOD-Suc*[*THEN iffD1*])

corollary *Suc-in-imp-neq-iMOD*:
⟦ $n \in S$; $Suc\ n \in S$; $m \neq Suc\ 0$ ⟧ $\Longrightarrow S \neq [r,\ mod\ m]$
by (*blast dest*: *Suc-in-imp-not-subset-iMOD*)

lemma *Suc-in-imp-not-subset-iMODb*:
⟦ $n \in S$; $Suc\ n \in S$; $m \neq Suc\ 0$ ⟧ $\Longrightarrow \neg\ S \subseteq [r,\ mod\ m,\ c]$
apply (*rule ccontr*, *simp*)
apply (*frule subsetD*[*of - - n*], *assumption*)

apply (*drule subsetD*[*of - - Suc n*], *assumption*)
apply (*frule iMODb-Suc*[*THEN iffD1*])
 apply (*drule iMODb-leD*[*of Suc n*])
 apply *simp*
apply *blast+*
done
corollary *Suc-in-imp-neq-iMODb*:
 $[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r,\ mod\ m,\ c]$
by (*blast dest*: *Suc-in-imp-not-subset-iMODb*)

1.3.2 Subset relation between intervals

lemma
 iIN-iFROM-subset-same: $[n\ldots,d] \subseteq [n\ldots]$ **and**
 iIN-iTILL-subset-same: $[n\ldots,d] \subseteq [\ldots n+d]$ **and**
 iMOD-iFROM-subset-same: $[r,\ mod\ m] \subseteq [r\ldots]$ **and**
 iMODb-iTILL-subset-same: $[r,\ mod\ m,\ c] \subseteq [\ldots r+m*c]$ **and**
 iMODb-iIN-subset-same: $[r,\ mod\ m,\ c] \subseteq [r\ldots,m*c]$ **and**
 iMODb-iMOD-subset-same: $[r,\ mod\ m,\ c] \subseteq [r,\ mod\ m]$
by (*simp add*: *subset-iff iT-iff*)+

lemmas *iT-subset-same* =
 iIN-iFROM-subset-same
 iIN-iTILL-subset-same
 iMOD-iFROM-subset-same
 iMODb-iTILL-subset-same
 iMODb-iIN-subset-same
 iMODb-iTILL-subset-same
 iMODb-iMOD-subset-same

lemma *iMODb-imp-iMOD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x \in [r,\ mod\ m]$
by (*blast intro*: *iMODb-iMOD-subset-same*)

lemma *iMOD-imp-iMODb*:
 $[\![x \in [r,\ mod\ m]; x \le r + m * c]\!] \Longrightarrow x \in [r,\ mod\ m,\ c]$
by (*simp add*: *iT-iff*)

lemma *iMOD-singleton-subset-conv*: $([r,\ mod\ m] \subseteq \{a\}) = (r = a \wedge m = 0)$
apply (*rule iffI*)
 apply (*simp add*: *subset-singleton-conv iT-not-empty*)
 apply (*simp add*: *set-eq-iff iT-iff*)
 apply (*frule-tac x=r* **in** *spec*, *drule-tac x=r+m* **in** *spec*)
 apply *simp*
apply (*simp add*: *iMOD-0 iIN-0*)
done
lemma *iMOD-singleton-eq-conv*: $([r,\ mod\ m] = \{a\}) = (r = a \wedge m = 0)$
apply (*rule-tac* $t=[r,\ mod\ m] = \{a\}$ **and** $s=[r,\ mod\ m] \subseteq \{a\}$ **in** *subst*)
 apply (*simp add*: *subset-singleton-conv iMOD-not-empty*)
apply (*simp add*: *iMOD-singleton-subset-conv*)

done

lemma *iMODb-singleton-subset-conv*:
  $([r, mod\ m, c] \subseteq \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
apply (*rule iffI*)
 apply (*simp add*: *subset-singleton-conv iT-not-empty*)
 apply (*simp add*: *set-eq-iff iT-iff*)
 apply (*frule-tac x=r* **in** *spec*, *drule-tac x=r+m* **in** *spec*)
 apply *clarsimp*
apply (*fastforce simp*: *iMODb-0 iMODb-mod-0 iIN-0*)
done
lemma *iMODb-singleton-eq-conv*:
  $([r, mod\ m, c] = \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
apply (*rule-tac t*=$[r, mod\ m, c] = \{a\}$ **and** *s*=$[r, mod\ m, c] \subseteq \{a\}$ **in** *subst*)
 apply (*simp add*: *subset-singleton-conv iMODb-not-empty*)
apply (*simp add*: *iMODb-singleton-subset-conv*)
done

lemma *iMODb-subset-imp-divisor-mod-0*:
  $[\![\ 0 < c'; [r', mod\ m', c'] \subseteq [r, mod\ m, c]\ ]\!] \Longrightarrow m'\ mod\ m = 0$
apply (*simp add*: *subset-iff iMODb-iff*)
apply (*drule gr0-imp-self-le-mult1*[*of - m'*])
apply (*rule mod-add-eq-imp-mod-0*[*of r' m' m*, *THEN iffD1*])
apply (*frule-tac x=r'* **in** *spec*, *drule-tac x=r'+m'* **in** *spec*)
apply *simp*
done

lemma *iMOD-subset-imp-divisor-mod-0*:
  $[r', mod\ m'] \subseteq [r, mod\ m] \Longrightarrow m'\ mod\ m = 0$
apply (*simp add*: *subset-iff iMOD-iff*)
apply (*rule mod-add-eq-imp-mod-0*[*of r' m' m*, *THEN iffD1*])
apply *simp*
done

lemma *iMOD-subset-imp-iMODb-subset*:
  $[\![\ [r', mod\ m'] \subseteq [r, mod\ m]; r' + m' * c' \le r + m * c\ ]\!] \Longrightarrow$
  $[r', mod\ m', c'] \subseteq [r, mod\ m, c]$
by (*simp add*: *subset-iff iT-iff*)

lemma *iMODb-subset-imp-iMOD-subset*:
  $[\![\ [r', mod\ m', c'] \subseteq [r, mod\ m, c]; 0 < c'\ ]\!] \Longrightarrow$
  $[r', mod\ m'] \subseteq [r, mod\ m]$
apply (*frule subsetD*[*of - - r'*])
 apply (*simp add*: *iMODb-iff*)
apply (*rule subsetI*)
apply (*simp add*: *iMOD-iff iMODb-iff*, *clarify*)
apply (*drule mod-eq-mod-0-imp-mod-eq*[**where** *m=m* **and** *m'=m'*])
 apply (*simp add*: *iMODb-subset-imp-divisor-mod-0*)
apply *simp*

done

lemma *iMODb-0-iMOD-subset-conv*:
  $([r', \, mod \; m', \, 0] \subseteq [r, \, mod \; m]) =$
  $(r' \; mod \; m = r \; mod \; m \wedge r \leq r')$
by (*simp add*: *iMODb-0 iIN-0 singleton-subset-conv iMOD-iff*)

lemma *iFROM-subset-conv*: $([n'\ldots] \subseteq [n\ldots]) = (n \leq n')$
by (*simp add*: *iFROM-def*)

lemma *iFROM-iMOD-subset-conv*: $([n'\ldots] \subseteq [r, \, mod \; m]) = (r \leq n' \wedge m = Suc \; 0)$
apply (*rule iffI*)
 apply (*rule conjI*)
  apply (*drule iMin-subset*[*OF iFROM-not-empty*])
  apply (*simp add*: *iT-Min*)
 apply (*rule ccontr*)
 apply (*cut-tac Suc-in-imp-not-subset-iMOD*[*of* n' $[n'\ldots]$ *m r*])
 apply (*simp add*: *iT-iff*)+
apply (*simp add*: *subset-iff iT-iff*)
done

lemma *iIN-subset-conv*: $([n'\ldots,d'] \subseteq [n\ldots,d]) = (n \leq n' \wedge n'+d' \leq n+d)$
apply (*rule iffI*)
 apply (*frule iMin-subset*[*OF iIN-not-empty*])
 apply (*drule Max-subset*[*OF iIN-not-empty - iIN-finite*])
 apply (*simp add*: *iIN-Min iIN-Max*)
apply (*simp add*: *subset-iff iIN-iff*)
done

lemma *iIN-iFROM-subset-conv*: $([n'\ldots,d'] \subseteq [n\ldots]) = (n \leq n')$
by (*fastforce simp*: *subset-iff iFROM-iff iIN-iff*)

lemma *iIN-iTILL-subset-conv*: $([n'\ldots,d'] \subseteq [\ldots n]) = (n' + d' \leq n)$
by (*fastforce simp*: *subset-iff iT-iff*)

lemma *iIN-iMOD-subset-conv*:
  $0 < d' \Longrightarrow ([n'\ldots,d'] \subseteq [r, \, mod \; m]) = (r \leq n' \wedge m = Suc \; 0)$
apply (*rule iffI*)
 apply (*frule iMin-subset*[*OF iIN-not-empty*])
 apply (*simp add*: *iT-Min*)
 apply (*subgoal-tac* $n' \in [n'\ldots,d']$)
  prefer *2*
  apply (*simp add*: *iIN-iff*)
 apply (*rule ccontr*)
 apply (*frule Suc-in-imp-not-subset-iMOD*[**where** *r*=*r* **and** *m*=*m*])
 apply (*simp add*: *iIN-Suc*)+
apply (*simp add*: *iMOD-1 iIN-iFROM-subset-conv*)
done

lemma *iIN-iMODb-subset-conv*:
  $0 < d' \Longrightarrow$
  $([n' \ldots, d'] \subseteq [r,\ mod\ m,\ c]) =$
  $(r \le n' \wedge m = Suc\ 0 \wedge n' + d' \le r + m * c)$
apply (*rule iffI*)
 apply (*frule subset-trans*[*OF - iMODb-iMOD-subset-same*])
 apply (*simp add*: *iIN-iMOD-subset-conv iMODb-mod-1 iIN-subset-conv*)
apply (*clarsimp simp*: *iMODb-mod-1 iIN-subset-conv*)
done

lemma *iTILL-subset-conv*: $([\ldots n'] \subseteq [\ldots n]) = (n' \le n)$
by (*simp add*: *iTILL-def*)

lemma *iTILL-iFROM-subset-conv*: $([\ldots n'] \subseteq [n \ldots]) = (n = 0)$
apply (*rule iffI*)
 apply (*drule subsetD*[*of - - 0*])
 apply (*simp add*: *iT-iff*)+
apply (*simp add*: *iFROM-0*)
done

lemma *iTILL-iIN-subset-conv*: $([\ldots n'] \subseteq [n \ldots, d]) = (n = 0 \wedge n' \le d)$
apply (*rule iffI*)
 apply (*frule iMin-subset*[*OF iTILL-not-empty*])
 apply (*drule Max-subset*[*OF iTILL-not-empty - iIN-finite*])
 apply (*simp add*: *iT-Min iT-Max*)
apply (*simp add*: *iIN-0-iTILL-conv iTILL-subset-conv*)
done

lemma *iTILL-iMOD-subset-conv*:
  $0 < n' \Longrightarrow ([\ldots n'] \subseteq [r,\ mod\ m]) = (r = 0 \wedge m = Suc\ 0)$
apply (*drule iIN-iMOD-subset-conv*[*of n' 0 r m*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done

lemma *iTILL-iMODb-subset-conv*:
  $0 < n' \Longrightarrow ([\ldots n'] \subseteq [r,\ mod\ m,\ c]) = (r = 0 \wedge m = Suc\ 0 \wedge n' \le r + m * c)$
apply (*drule iIN-iMODb-subset-conv*[*of n' 0 r m c*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done

lemma *iMOD-iFROM-subset-conv*: $([r',\ mod\ m']) \subseteq [n \ldots] = (n \le r')$
by (*fastforce simp*: *subset-iff iT-iff*)

lemma *iMODb-iFROM-subset-conv*: $([r',\ mod\ m',\ c'] \subseteq [n \ldots]) = (n \le r')$
by (*fastforce simp*: *subset-iff iT-iff*)

lemma *iMODb-iIN-subset-conv*:

$([r', mod\ m', c'] \subseteq [n\ldots,d]) = (n \le r' \wedge r' + m' * c' \le n + d)$
by (*fastforce simp*: *subset-iff iT-iff*)

lemma *iMODb-iTILL-subset-conv*:
$([r', mod\ m', c'] \subseteq [\ldots n]) = (r' + m' * c' \le n)$
by (*fastforce simp*: *subset-iff iT-iff*)

lemma *iMOD-0-subset-conv*: $([r', mod\ 0] \subseteq [r, mod\ m]) = (r'\ mod\ m = r\ mod\ m \wedge r \le r')$
by (*fastforce simp*: *iMOD-0 iIN-0 singleton-subset-conv iMOD-iff*)

lemma *iMOD-subset-conv*: $0 < m \Longrightarrow$
$([r', mod\ m'] \subseteq [r, mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m \wedge r \le r' \wedge m'\ mod\ m = 0)$
apply (*rule iffI*)
apply (*frule subsetD*[*of - - r'*])
apply (*simp add*: *iMOD-iff*)
apply (*drule iMOD-subset-imp-divisor-mod-0*)
apply (*simp add*: *iMOD-iff*)
apply (*rule subsetI*)
apply (*simp add*: *iMOD-iff*, *elim conjE*)
apply (*drule mod-eq-mod-0-imp-mod-eq*[**where** $m'=m'$ **and** $m=m$])
apply *simp+*
done

lemma *iMODb-subset-mod-0-conv*:
$([r', mod\ m', c'] \subseteq [r, mod\ 0, c]) = (r'=r \wedge (m'=0 \vee c'=0))$
by (*simp add*: *iMODb-mod-0 iIN-0 iMODb-singleton-subset-conv*)

lemma *iMODb-subset-0-conv*:
$([r', mod\ m', c'] \subseteq [r, mod\ m, 0]) = (r'=r \wedge (m'=0 \vee c'=0))$
by (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-subset-conv*)

lemma *iMODb-0-subset-conv*:
$([r', mod\ m', 0] \subseteq [r, mod\ m, c]) = (r' \in [r, mod\ m, c])$
by (*simp add*: *iMODb-0 iIN-0*)

lemma *iMODb-mod-0-subset-conv*:
$([r', mod\ 0, c'] \subseteq [r, mod\ m, c]) = (r' \in [r, mod\ m, c])$
by (*simp add*: *iMODb-mod-0 iIN-0*)

lemma *iMODb-subset-conv'*: $[\![0 < c; 0 < c']\!] \Longrightarrow$
$([r', mod\ m', c'] \subseteq [r, mod\ m, c]) =$
$(r'\ mod\ m = r\ mod\ m \wedge r \le r' \wedge m'\ mod\ m = 0 \wedge$
$r' + m' * c' \le r + m * c)$
apply (*rule iffI*)
apply (*frule iMODb-subset-imp-iMOD-subset*, *assumption*)
apply (*drule iMOD-subset-imp-divisor-mod-0*)
apply (*frule subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]])

apply (*drule Max-subset*[*OF iMODb-not-empty - iMODb-finite*])
apply (*simp add*: *iMODb-iff iMODb-Min iMODb-Max*)
apply (*elim conjE*)
apply (*case-tac m = 0*, *simp add*: *iMODb-mod-0*)
apply (*simp add*: *iMOD-subset-imp-iMODb-subset iMOD-subset-conv*)
done

lemma *iMODb-subset-conv*: ⟦ $0 < m'$; $0 < c'$ ⟧ ⟹
([r', *mod* m', c'] ⊆ [r, *mod* m, c]) =
(r' *mod* m = r *mod* m ∧ $r \le r'$ ∧ m' *mod* $m = 0$ ∧
$r' + m' * c' \le r + m * c$)
apply (*case-tac c = 0*)
apply (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-subset-conv linorder-not-le*, *intro impI*)
apply (*case-tac* $r' < r$, *simp*)
apply (*simp add*: *linorder-not-less*)
apply (*insert add-less-le-mono*[*of 0* $m' * c'$ *r* r'])
apply *simp*
apply (*simp add*: *iMODb-subset-conv′*)
done

lemma *iMODb-iMOD-subset-conv*: $0 < c'$ ⟹
([r', *mod* m', c'] ⊆ [r, *mod* m]) =
(r' *mod* m = r *mod* m ∧ $r \le r'$ ∧ m' *mod* $m = 0$)
apply (*rule iffI*)
apply (*frule subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]])
apply (*simp add*: *iMODb-Min iMOD-iff*, *elim conjE*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv*)
apply (*subgoal-tac* [r', *mod* m', c'] ⊆ [r, *mod* m] ∩ [...$r' + m' * c'$])
prefer *2*
apply (*simp add*: *iMODb-iMOD-iTILL-conv*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv iMODb-subset-imp-divisor-mod-0*)
apply (*rule subset-trans*[*OF iMODb-iMOD-subset-same*])
apply (*case-tac m = 0*, *simp*)
apply (*simp add*: *iMOD-subset-conv*)
done

lemmas *iT-subset-conv* =
iFROM-subset-conv
iFROM-iMOD-subset-conv
iTILL-subset-conv
iTILL-iFROM-subset-conv
iTILL-iIN-subset-conv
iTILL-iMOD-subset-conv
iTILL-iMODb-subset-conv
iIN-subset-conv
iIN-iFROM-subset-conv
iIN-iTILL-subset-conv
iIN-iMOD-subset-conv

iIN-iMODb-subset-conv
iMOD-subset-conv
iMOD-iFROM-subset-conv
iMODb-subset-conv′
iMODb-subset-conv
iMODb-iFROM-subset-conv
iMODb-iIN-subset-conv
iMODb-iTILL-subset-conv
iMODb-iMOD-subset-conv

lemma *iFROM-subset*: $n \le n' \Longrightarrow [n'\ldots] \subseteq [n\ldots]$
by (*simp add*: *iFROM-subset-conv*)

lemma *not-iFROM-iIN-subset*: $\neg\ [n'\ldots] \subseteq [n\ldots,d]$
apply (*rule ccontr*, *simp*)
apply (*drule subsetD*[*of - - max n′* (*Suc* (*n* + *d*))])
apply (*simp add*: *iFROM-iff*)
apply (*simp add*: *iIN-iff*)
done

lemma *not-iFROM-iTILL-subset*: $\neg\ [n'\ldots] \subseteq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv* [*symmetric*] *not-iFROM-iIN-subset*)

lemma *not-iFROM-iMOD-subset*: $m \ne Suc\ 0 \Longrightarrow \neg\ [n'\ldots] \subseteq [r,\ mod\ m]$
apply (*rule Suc-in-imp-not-subset-iMOD*[*of n′*])
apply (*simp add*: *iT-iff*)+
done

lemma *not-iFROM-iMODb-subset*: $\neg\ [n'\ldots] \subseteq [r,\ mod\ m,\ c]$
by (*rule infinite-not-subset-finite*[*OF iFROM-infinite iMODb-finite*])

lemma *iIN-subset*: $[\![\ n \le n';\ n' + d' \le n + d\ ]\!] \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots,d]$
by (*simp add*: *iIN-subset-conv*)

lemma *iIN-iFROM-subset*: $n \le n' \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots]$
by (*simp add*: *subset-iff iT-iff*)

lemma *iIN-iTILL-subset*: $n' + d' \le n \Longrightarrow [n'\ldots,d'] \subseteq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-subset*)

lemma *not-iIN-iMODb-subset*: $[\![\ 0 < d';\ m \ne Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m,\ c]$
apply (*rule Suc-in-imp-not-subset-iMODb*[*of n′*])
apply (*simp add*: *iIN-iff*)+
done

lemma *not-iIN-iMOD-subset*: $[\![\ 0 < d';\ m \ne Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m]$
apply (*rule ccontr*, *simp*)

apply (*case-tac* $r \leq n' + d'$)
 apply (*drule Int-greatest*[*OF - iIN-iTILL-subset*[*OF order-refl*]])
 apply (*simp add*: *iMOD-iTILL-iMODb-conv not-iIN-iMODb-subset*)
apply (*drule subsetD*[*of - -* $n'+d'$])
apply (*simp add*: *iT-iff*)+
done

lemma *iTILL-subset*: $n' \leq n \Longrightarrow [\ldots n'] \subseteq [\ldots n]$
by (*rule iTILL-subset-conv*[*THEN iffD2*])

lemma *iTILL-iFROM-subset*: $([\ldots n'] \subseteq [0\ldots])$
by (*simp add*: *iFROM-0*)

lemma *iTILL-iIN-subset*: $n' \leq d \Longrightarrow ([\ldots n'] \subseteq [0\ldots,d])$
by (*simp add*: *iIN-0-iTILL-conv iTILL-subset*)

lemma *not-iTILL-iMOD-subset*:
 $[\![0 < n'; m \neq Suc\ 0]\!] \Longrightarrow \neg [\ldots n'] \subseteq [r, mod\ m]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *not-iIN-iMOD-subset*)

lemma *not-iTILL-iMODb-subset*:
 $[\![0 < n'; m \neq Suc\ 0]\!] \Longrightarrow \neg [\ldots n'] \subseteq [r, mod\ m, c]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *not-iIN-iMODb-subset*)

lemma *iMOD-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m'] \subseteq [n\ldots]$
by (*rule iMOD-iFROM-subset-conv*[*THEN iffD2*])

lemma *not-iMOD-iIN-subset*: $0 < m' \Longrightarrow \neg [r', mod\ m'] \subseteq [n\ldots,d]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iIN-finite*])

lemma *not-iMOD-iTILL-subset*: $0 < m' \Longrightarrow \neg [r', mod\ m'] \subseteq [\ldots n]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iTILL-finite*])

lemma *iMOD-subset*:
 $[\![r \leq r'; r'\ mod\ m = r\ mod\ m; m'\ mod\ m = 0]\!] \Longrightarrow [r', mod\ m'] \subseteq [r, mod\ m]$
apply (*case-tac* $m = 0$, *simp*)
apply (*simp add*: *iMOD-subset-conv*)
done

lemma *not-iMOD-iMODb-subset*: $0 < m' \Longrightarrow \neg [r', mod\ m'] \subseteq [r, mod\ m, c]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iMODb-finite*])

lemma *iMODb-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m', c'] \subseteq [n\ldots]$
by (*rule iMODb-iFROM-subset-conv*[*THEN iffD2*])

lemma *iMODb-iTILL-subset*:
 $r' + m' * c' \leq n \Longrightarrow [r', mod\ m', c'] \subseteq [\ldots n]$
by (*rule iMODb-iTILL-subset-conv*[*THEN iffD2*])

lemma *iMODb-iIN-subset*:
$\llbracket\ n \le r'; r' + m' * c' \le n + d\ \rrbracket \Longrightarrow [r', mod\ m', c'] \subseteq [n\ldots,d]$
by (*simp add*: *iMODb-iIN-subset-conv*)

lemma *iMODb-iMOD-subset*:
$\llbracket\ r \le r'; r'\ mod\ m = r\ mod\ m; m'\ mod\ m = 0\ \rrbracket \Longrightarrow [r', mod\ m', c'] \subseteq [r, mod\ m]$
apply (*case-tac* $c' = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMOD-iff*)
apply (*simp add*: *iMODb-iMOD-subset-conv*)
done

lemma *iMODb-subset*:
$\llbracket\ r \le r'; r'\ mod\ m = r\ mod\ m; m'\ mod\ m = 0; r' + m' * c' \le r + m * c\ \rrbracket \Longrightarrow$
$[r', mod\ m', c'] \subseteq [r, mod\ m, c]$
apply (*case-tac* $m' = 0$)
apply (*simp add*: *iMODb-mod-0 iIN-0 iMODb-iff*)
apply (*case-tac* $c' = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMODb-iff*)
apply (*simp add*: *iMODb-subset-conv*)
done

lemma *iFROM-trans*: $\llbracket\ y \in [x\ldots]; z \in [y\ldots]\ \rrbracket \Longrightarrow z \in [x\ldots]$
by (*rule subsetD*[*OF iFROM-subset*[*OF iFROM-D*]])

lemma *iTILL-trans*: $\llbracket\ y \in [\ldots x]; z \in [\ldots y]\ \rrbracket \Longrightarrow z \in [\ldots x]$
by (*rule subsetD*[*OF iTILL-subset*[*OF iTILL-D*]])

lemma *iIN-trans*:
$\llbracket\ y \in [x\ldots,d]; z \in [y\ldots,d']; d' \le x + d - y\ \rrbracket \Longrightarrow z \in [x\ldots,d]$
by *fastforce*

lemma *iMOD-trans*:
$\llbracket\ y \in [x, mod\ m]; z \in [y, mod\ m]\ \rrbracket \Longrightarrow z \in [x, mod\ m]$
by (*rule subsetD*[*OF iMOD-subset*[*OF iMOD-geD iMOD-modD mod-self*]])

lemma *iMODb-trans*:
$\llbracket\ y \in [x, mod\ m, c]; z \in [y, mod\ m, c']; m * c' \le x + m * c - y\ \rrbracket \Longrightarrow$
$z \in [x, mod\ m, c]$
by *fastforce*

lemma *iMODb-trans'*:
$\llbracket\ y \in [x, mod\ m, c]; z \in [y, mod\ m, c']; c' \le x\ div\ m + c - y\ div\ m\ \rrbracket \Longrightarrow$
$z \in [x, mod\ m, c]$
apply (*rule iMODb-trans*[**where** c'=c'], *assumption*+)
apply (*frule iMODb-geD*, *frule div-le-mono*[*of x y m*])
apply (*simp add*: *add.commute*[*of - c*] *add.commute*[*of - m*∗*c*])
apply (*drule mult-le-mono*[*OF le-refl*, *of - - m*])
apply (*simp add*: *add-mult-distrib2 diff-mult-distrib2 minus-mod-eq-mult-div* [*symmetric*])

apply (*simp add*: *iMODb-iff*)
done

1.3.3 Equality of intervals

lemma *iFROM-eq-conv*: $([n\ldots] = [n'\ldots]) = (n = n')$
apply (*rule iffI*)
 apply (*drule set-eq-subset*[*THEN iffD1*])
 apply (*simp add*: *iFROM-subset-conv*)
apply *simp*
done

lemma *iIN-eq-conv*: $([n\ldots,d] = [n'\ldots,d']) = (n = n' \wedge d = d')$
apply (*rule iffI*)
 apply (*drule set-eq-subset*[*THEN iffD1*])
 apply (*simp add*: *iIN-subset-conv*)
apply *simp*
done

lemma *iTILL-eq-conv*: $([\ldots n] = [\ldots n']) = (n = n')$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-eq-conv*)

lemma *iMOD-0-eq-conv*: $([r,\ mod\ 0] = [r',\ mod\ m']) = (r = r' \wedge m' = 0)$
apply (*simp add*: *iMOD-0 iIN-0*)
apply (*simp add*: *iMOD-singleton-eq-conv eq-sym-conv*[*of* {*r*}] *eq-sym-conv*[*of r*])
done

lemma *iMOD-eq-conv*: $0 < m \Longrightarrow ([r,\ mod\ m] = [r',\ mod\ m']) = (r = r' \wedge m = m')$
apply (*case-tac* $m' = 0$)
 apply (*simp add*: *eq-sym-conv*[*of* [*r*, *mod m*]] *iMOD-0-eq-conv*)
apply (*rule iffI*)
 apply (*fastforce simp add*: *set-eq-subset iMOD-subset-conv*)
apply *simp*
done

lemma *iMODb-mod-0-eq-conv*:
  $([r,\ mod\ 0,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
apply (*simp add*: *iMODb-mod-0 iIN-0*)
apply (*fastforce simp*: *iMODb-singleton-eq-conv eq-sym-conv*[*of* {*r*}])
done

lemma *iMODb-0-eq-conv*:
  $([r,\ mod\ m,\ 0] = [r',\ mod\ m',\ c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
apply (*simp add*: *iMODb-0 iIN-0*)
apply (*fastforce simp*: *iMODb-singleton-eq-conv eq-sym-conv*[*of* {*r*}])
done

lemma *iMODb-eq-conv*: $[\![\ 0 < m;\ 0 < c\ ]\!] \Longrightarrow$

$([r,\ mod\ m,\ c] = [r',\ mod\ m',\ c']) = (r = r' \wedge m = m' \wedge c = c')$
apply (*case-tac* $c' = 0$)
 apply (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-eq-conv*)
apply (*rule iffI*)
 apply (*fastforce simp*: *set-eq-subset iMODb-subset-conv'*)
apply *simp*
done

lemma *iMOD-iFROM-eq-conv*: $([n\ldots] = [r,\ mod\ m]) = (n = r \wedge m = Suc\ 0)$
by (*fastforce simp*: *iMOD-1*[*symmetric*] *iMOD-eq-conv*)

lemma *iMODb-iIN-0-eq-conv*:
$([n\ldots,0] = [r,\ mod\ m,\ c]) = (n = r \wedge (m = 0 \vee c = 0))$
by (*simp add*: *iIN-0 eq-commute*[*of* $\{n\}$] *eq-commute*[*of n*] *iMODb-singleton-eq-conv*)

lemma *iMODb-iIN-eq-conv*:
$0 < d \Longrightarrow ([n\ldots,d] = [r,\ mod\ m,\ c]) = (n = r \wedge m = Suc\ 0 \wedge c = d)$
by (*fastforce simp*: *iMODb-mod-1*[*symmetric*] *iMODb-eq-conv*)

1.3.4 Inequality of intervals

lemma *iFROM-iIN-neq*: $[n'\ldots] \neq [n\ldots,d]$
apply (*rule ccontr*)
apply (*insert iFROM-infinite*[*of n'*], *insert iIN-finite*[*of n d*])
apply *simp*
done

corollary *iFROM-iTILL-neq*: $[n'\ldots] \neq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iFROM-iIN-neq*)

corollary *iFROM-iMOD-neq*: $m \neq Suc\ 0 \Longrightarrow [n\ldots] \neq [r,\ mod\ m]$
apply (*subgoal-tac* $n \in [n\ldots]$)
 prefer *2*
 apply (*simp add*: *iFROM-iff*)
apply (*simp add*: *Suc-in-imp-neq-iMOD iFROM-Suc*)
done
corollary *iFROM-iMODb-neq*: $[n\ldots] \neq [r,\ mod\ m,\ c]$
apply (*rule ccontr*)
apply (*insert iMODb-finite*[*of r m c*], *insert iFROM-infinite*[*of n*])
apply *simp*
done

corollary *iIN-iMOD-neq*: $0 < m \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m]$
apply (*rule ccontr*)
apply (*insert iMOD-infinite*[*of m r*], *insert iIN-finite*[*of n d*])
apply *simp*
done

corollary *iIN-iMODb-neq2*: $[\![\ m \neq Suc\ 0;\ 0 < d\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$

apply (*subgoal-tac* $n \in [n\ldots,d]$)
 prefer *2*
 apply (*simp add*: *iIN-iff*)
apply (*simp add*: *Suc-in-imp-neq-iMODb iIN-Suc*)
done

lemma *iIN-iMODb-neq*: $[\![\ 2 \le m;\ 0 < c\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$
apply (*simp add*: *nat-ge2-conv*, *elim conjE*)
apply (*case-tac d=0*)
 apply (*rule not-sym*)
 apply (*simp add*: *iIN-0 iMODb-singleton-eq-conv*)
apply (*simp add*: *iIN-iMODb-neq2*)
done

lemma *iTILL-iIN-neq*: $0 < n \Longrightarrow [\ldots n'] \neq [n\ldots,d]$
by (*fastforce simp*: *set-eq-iff iT-iff*)

corollary *iTILL-iMOD-neq*: $0 < m \Longrightarrow [\ldots n] \neq [r,\ mod\ m]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iMOD-neq*)

corollary *iTILL-iMODb-neq*:
  $[\![\ m \neq Suc\ 0;\ 0 < n\ ]\!] \Longrightarrow [\ldots n] \neq [r,\ mod\ m,\ c]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iMODb-neq2*)

lemma *iMOD-iMODb-neq*: $0 < m \Longrightarrow [r,\ mod\ m] \neq [r',\ mod\ m',\ c']$
apply (*rule ccontr*)
apply (*insert iMODb-finite*[*of r' m' c'*], *insert iMOD-infinite*[*of m r*])
apply *simp*
done

lemmas *iT-neq* =
  iFROM-iTILL-neq iFROM-iIN-neq iFROM-iMOD-neq iFROM-iMODb-neq
  iTILL-iIN-neq iTILL-iMOD-neq iTILL-iMODb-neq
  iIN-iMOD-neq iIN-iMODb-neq iIN-iMODb-neq2
  iMOD-iMODb-neq

1.4 Union and intersection of intervals

lemma *iFROM-union'*: $[n\ldots] \cup [n'\ldots] = [min\ n\ n'\ldots]$
by (*fastforce simp*: *iFROM-iff*)

corollary *iFROM-union*: $n \le n' \Longrightarrow [n\ldots] \cup [n'\ldots] = [n\ldots]$
by (*simp add*: *iFROM-union' min-eqL*)

lemma *iFROM-inter'*: $[n\ldots] \cap [n'\ldots] = [max\ n\ n'\ldots]$
by (*fastforce simp*: *iFROM-iff*)

corollary *iFROM-inter*: $n' \le n \Longrightarrow [n\ldots] \cap [n'\ldots] = [n\ldots]$
by (*simp add*: *iFROM-inter' max-eqL*)

lemma *iTILL-union'*: $[\ldots n] \cup [\ldots n'] = [\ldots max\ n\ n']$
by (*fastforce simp*: *iTILL-iff*)

corollary *iTILL-union*: $n' \le n \Longrightarrow [\ldots n] \cup [\ldots n'] = [\ldots n]$
by (*simp add*: *iTILL-union' max-eqL*)

lemma *iTILL-iFROM-union*: $n \le n' \Longrightarrow [\ldots n'] \cup [n\ldots] = UNIV$
by (*fastforce simp*: *iT-iff*)

lemma *iTILL-inter'*: $[\ldots n] \cap [\ldots n'] = [\ldots min\ n\ n']$
by (*fastforce simp*: *iT-iff*)

corollary *iTILL-inter*: $n \le n' \Longrightarrow [\ldots n] \cap [\ldots n'] = [\ldots n]$
by (*simp add*: *iTILL-inter' min-eqL*)

Union and intersection for iIN only when there intersection is not empty and none of them is other's subset, for instance: .. n .. n+d .. n' .. n'+d'

lemma *iIN-union*:
$[\![\ n \le n';\ n' \le Suc\ (n + d);\ n + d \le n' + d'\ ]\!] \Longrightarrow$
$[n\ldots,d] \cup [n'\ldots,d'] = [n\ldots,n' - n + d']$
by (*fastforce simp add*: *iIN-iff*)

lemma *iIN-append-union*:
$[n\ldots,d] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
by (*simp add*: *iIN-union*)

lemma *iIN-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots,d'] = [n\ldots,Suc\ (d + d')]$
by (*simp add*: *iIN-union*)

lemma *iIN-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
by (*simp add*: *iIN-union*)

lemma *iIN-iFROM-union*:
$n' \le Suc\ (n + d) \Longrightarrow [n\ldots,d] \cup [n'\ldots] = [min\ n\ n'\ldots]$
by (*fastforce simp*: *iIN-iff*)

lemma *iIN-iFROM-append-union*:
$[n\ldots,d] \cup [n + d\ldots] = [n\ldots]$
by (*simp add*: *iIN-iFROM-union min-eqL*)

lemma *iIN-iFROM-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots] = [n\ldots]$
by (*simp add*: *iIN-iFROM-union min-eqL*)

lemma *iIN-iFROM-append-union-pred*:

$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots] = [n\ldots]$
by (*simp add*: *iIN-iFROM-union min-eqL*)

lemma *iIN-inter*:
$[\![n \le n'; n' \le n + d; n + d \le n' + d']\!] \Longrightarrow$
$[n\ldots,d] \cap [n'\ldots,d'] = [n'\ldots,n + d - n']$
by (*fastforce simp*: *iIN-iff*)

lemma *iMOD-union*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [r,\ mod\ m]$
by (*fastforce simp*: *iT-iff*)
lemma *iMOD-union'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [min\ r\ r',\ mod\ m]$
apply (*case-tac* $r \le r'$)
apply (*fastforce simp*: *iMOD-union min-eq*)+
done

lemma *iMOD-inter*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [r',\ mod\ m]$
by (*fastforce simp*: *iT-iff*)

lemma *iMOD-inter'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [max\ r\ r',\ mod\ m]$
apply (*case-tac* $r \le r'$)
apply (*fastforce simp*: *iMOD-inter max-eq*)+
done

lemma *iMODb-union*:
$[\![r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * c; r + m * c \le r' + m * c']\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [r',\ mod\ m,\ c'] = [r,\ mod\ m,\ r'\ div\ m - r\ div\ m + c']$
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff*)
apply (*drule sym*[*of r mod m*], *simp*)
apply (*fastforce simp*: *add-mult-distrib2 diff-mult-distrib2 minus-mod-eq-mult-div* [*symmetric*])
done

lemma *iMODb-append-union*:
$[r,\ mod\ m,\ c] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c + c']$
apply (*insert iMODb-union*[*of r r + m * c m c c'*])
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMODb-mod-0*)
apply *simp*
done

lemma *iMODb-iMOD-append-union′*:
$[\![\ r\ mod\ m = r'\ mod\ m;\ r' \le r + m * Suc\ c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [min\ r\ r',\ mod\ m]$
apply (*subgoal-tac* ($min\ r\ r'$) $mod\ m = r'\ mod\ m$)
prefer *2*
apply (*simp add*: *min-def*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff*)
apply (*drule sym*[*of r mod m*], *simp*)
apply (*rule iffI*)
apply *fastforce*
apply (*clarsimp simp*: *linorder-not-le*)
apply (*case-tac* $r \le r'$)
apply (*simp add*: *min-eqL*)
apply (*rule add-le-imp-le-right*[*of - m*])
apply (*rule less-mod-eq-imp-add-divisor-le*)
apply *simp+*
done

lemma *iMODb-iMOD-append-union*:
$[\![\ r \le r';\ r\ mod\ m = r'\ mod\ m;\ r' \le r + m * Suc\ c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [r,\ mod\ m]$
by (*simp add*: *iMODb-iMOD-append-union′ min-eqL*)

lemma *iMODb-append-union-Suc*:
$[r,\ mod\ m,\ c] \cup [\ r + m * Suc\ c,\ mod\ m,\ c'] = [r,\ mod\ m,\ Suc\ (c + c')]$
apply (*subst insert-absorb*[*of* $r + m * c$ $[r,\ mod\ m,\ c] \cup [\ r + m * Suc\ c,\ mod\ m,\ c']$, *symmetric*])
apply (*simp add*: *iT-iff*)
apply (*simp del*: *Un-insert-right add*: *Un-insert-right*[*symmetric*] *add.commute*[*of m*] *add.assoc*[*symmetric*] *iMODb-Suc-pred-insert-conv*)
apply (*simp add*: *iMODb-append-union*)
done

lemma *iMODb-append-union-pred*:
$0 < c \Longrightarrow [r,\ mod\ m,\ c - Suc\ 0] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c + c']$
by (*insert iMODb-append-union-Suc*[*of r m* $c - Suc\ 0$ *c′*], *simp*)

lemma *iMODb-inter*:
$[\![\ r \le r';\ r\ mod\ m = r'\ mod\ m;\ r' \le r + m * c;\ r + m * c \le r' + m * c'\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cap [r',\ mod\ m,\ c'] = [r',\ mod\ m,\ c - (r'-r)\ div\ m]$
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff*)
apply (*simp add*: *diff-mult-distrib2*)
apply (*simp add*: *mult.commute*[*of -* $(r' - r)\ div\ m$])
apply (*simp add*: *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*])

apply (*simp add*: *add.commute*[*of* - *r*])
apply *fastforce*
done

lemmas *iT-union'* =
iFROM-union'
iTILL-union'
iMOD-union'
iMODb-iMOD-append-union'
lemmas *iT-union* =
iFROM-union
iTILL-union
iTILL-iFROM-union
iIN-union
iIN-iFROM-union
iMOD-union
iMODb-union
lemmas *iT-union-append* =
iIN-append-union
iIN-append-union-Suc
iIN-append-union-pred
iIN-iFROM-append-union
iIN-iFROM-append-union-Suc
iIN-iFROM-append-union-pred
iMODb-append-union
iMODb-iMOD-append-union
iMODb-append-union-Suc
iMODb-append-union-pred

lemmas *iT-inter'* =
iFROM-inter'
iTILL-inter'
iMOD-inter'
lemmas *iT-inter* =
iFROM-inter
iTILL-inter
iIN-inter
iMOD-inter
iMODb-inter

lemma *mod-partition-Union*:
$0 < m \Longrightarrow (\bigcup k.\ A \cap [k * m\ldots,m - Suc\ 0]) = A$
apply *simp*
apply (*rule subst*[**where** *s*=*UNIV* **and** *P*=λ*x*. *A* ∩ *x* = *A*])
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff*)
apply (*rule-tac x*=*x div m* **in** *exI*)

apply (*simp add*: *div-mult-cancel*)
apply (*subst add.commute*)
apply (*rule le-add-diff*)
apply (*simp add*: *Suc-mod-le-divisor*)
apply *simp*
done

lemma *finite-mod-partition-Union*:
$[\![0 < m; finite\ A]\!] \Longrightarrow$
$(\bigcup k \le Max\ A\ div\ m.\ A \cap [k*m\ldots,m - Suc\ 0]) = A$
apply (*rule subst*[*OF mod-partition-Union*[*of m*], **where**
$P = \lambda x.\ (\bigcup k \le Max\ A\ div\ m.\ A \cap [k*m\ldots,m - Suc\ 0]) = x$])
apply *assumption*
apply (*rule set-eqI*)
apply (*simp add*: *iIN-iff*)
apply (*rule iffI*, *blast*)
apply *clarsimp*
apply (*rename-tac x x1*)
apply (*rule-tac x=x div m* **in** *bexI*)
apply (*frule in-imp-not-empty*[**where** *A=A*])
apply (*frule-tac Max-ge*, *assumption*)
apply (*cut-tac n=x* **and** *k=x div m* **and** *m=m* **in** *div-imp-le-less*)
apply *clarsimp+*
apply (*drule-tac m=x* **in** *less-imp-le-pred*)
apply (*simp add*: *add.commute*[*of m*])
apply (*simp add*: *div-le-mono*)
done

lemma *mod-partition-is-disjoint*:
$[\![0 < (m::nat); k \ne k']\!] \Longrightarrow$
$(A \cap [k * m\ldots,m - Suc\ 0]) \cap (A \cap [k' * m\ldots,m - Suc\ 0]) = \{\}$
apply (*clarsimp simp add*: *all-not-in-conv*[*symmetric*] *iT-iff*)
apply (*subgoal-tac* $\bigwedge k.\ [\![k * m \le x; x \le k * m + m - Suc\ 0]\!] \Longrightarrow x\ div\ m = k$)
apply *blast*
apply (*rule le-less-imp-div*, *assumption*)
apply *simp*
done

1.5 Cutting intervals

lemma *iTILL-cut-le*: $[\ldots n] \downarrow\le t = (if\ t \le n\ then\ [\ldots t]\ else\ [\ldots n])$
unfolding *i-cut-defs iT-defs atMost-def*
by *force*

corollary *iTILL-cut-le1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\le t = [\ldots t]$
by (*simp add*: *iTILL-cut-le iT-iff*)

corollary *iTILL-cut-le2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\le t = [\ldots n]$
by (*simp add*: *iTILL-cut-le iT-iff*)

lemma *iFROM-cut-le*:
  $[n\ldots] \downarrow\leq t =$
  $(\mathit{if}\ t < n\ \mathit{then}\ \{\}\ \mathit{else}\ [n\ldots,t-n])$
by (*simp add*: *set-eq-iff i-cut-mem-iff iT-iff*)

corollary *iFROM-cut-le1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\leq t = [n\ldots,t - n]$
by (*simp add*: *iFROM-cut-le iT-iff*)

lemma *iIN-cut-le*:
  $[n\ldots,d] \downarrow\leq t = ($
  $\mathit{if}\ t < n\ \mathit{then}\ \{\}\ \mathit{else}$
  $\mathit{if}\ t \leq n+d\ \mathit{then}\ [n\ldots,t-n]$
  $\mathit{else}\ [n\ldots,d])$
by (*force simp*: *set-eq-iff i-cut-mem-iff iT-iff*)

corollary *iIN-cut-le1*:
  $t \in [n\ldots,d] \Longrightarrow [n\ldots,d] \downarrow\leq t = [n\ldots,t - n]$
by (*simp add*: *iIN-cut-le iT-iff*)

lemma *iMOD-cut-le*:
  $[r,\ \mathit{mod}\ m] \downarrow\leq t = ($
  $\mathit{if}\ t < r\ \mathit{then}\ \{\}$
  $\mathit{else}\ [r,\ \mathit{mod}\ m,\ (t - r)\ \mathit{div}\ m])$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMOD-0 iMODb-0 iIN-0 i-cut-empty i-cut-singleton*)
apply (*case-tac* $t < r$)
 apply (*simp add*: *cut-le-Min-empty iMOD-Min*)
apply (*clarsimp simp*: *linorder-not-less set-eq-iff i-cut-mem-iff iT-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*clarsimp simp*: *minus-mod-eq-mult-div* [*symmetric*])
apply (*drule-tac* $x=r$ **and** $y=x$ **in** *le-imp-less-or-eq*, *erule disjE*)
 prefer *2*
 apply *simp*
apply (*drule-tac* $x=r$ **and** $y=x$ **and** $m=m$ **in** *less-mod-eq-imp-add-divisor-le*, *simp*)
apply (*rule iffI*)
 apply (*rule-tac* $x=x$ **in** *subst*[*OF mod-eq-imp-diff-mod-eq*[*of - m r t*], *rule-format*], *simp*+)
 apply (*subgoal-tac* $r + (t - x)\ \mathit{mod}\ m \leq t$)
  prefer *2*
  apply (*simp add*: *order-trans*[*OF add-le-mono2*[*OF mod-le-divisor*]])
 apply *simp*
 apply (*simp add*: *le-imp-sub-mod-le*)
apply (*subgoal-tac* $r + (t - r)\ \mathit{mod}\ m \leq t$)
 prefer *2*
 apply (*rule ccontr*)
 apply *simp*

apply *simp*
done

lemma *iMOD-cut-le1*:
  $t \in [r, \, mod \; m] \Longrightarrow$
  $[r, \, mod \; m] \downarrow\leq t = [r, \, mod \; m, \, (t - r) \; div \; m]$
by (*simp add*: *iMOD-cut-le iT-iff*)

lemma *iMODb-cut-le*:
  $[r, \, mod \; m, \, c] \downarrow\leq t = ($
    if $t < r$ *then* $\{\}$
    else if $t < r + m * c$ *then* $[r, \, mod \; m, \, (t - r) \; div \; m]$
    else $[r, \, mod \; m, \, c])$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMODb-mod-0 iIN-0 cut-le-singleton*)
apply (*case-tac* $t < r$)
 apply (*simp add*: *cut-le-Min-empty iT-Min*)
apply (*case-tac* $r + m * c \leq t$)
 apply (*simp add*: *cut-le-Max-all iT-Max iT-finite*)
apply (*simp add*: *linorder-not-le linorder-not-less*)
apply (*rule-tac* t=c **and** s=$(r + m * c - r) \; div \; m$ **in** *subst*, *simp*)
apply (*subst iMOD-iTILL-iMODb-conv*[*symmetric*], *simp*)
apply (*simp add*: *cut-le-Int-right iTILL-cut-le*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
done

lemma *iMODb-cut-le1*:
  $t \in [r, \, mod \; m, \, c] \Longrightarrow$
  $[r, \, mod \; m, \, c] \downarrow\leq t = [r, \, mod \; m, \, (t - r) \; div \; m]$
by (*clarsimp simp*: *iMODb-cut-le iT-iff iMODb-mod-0*)

lemma *iTILL-cut-less*:
  $[\ldots n] \downarrow< t = ($
    if $n < t$ *then* $[\ldots n]$ *else*
    if $t = 0$ *then* $\{\}$
    else $[\ldots t - Suc \; 0])$
apply (*case-tac* $n < t$)
 apply (*simp add*: *cut-less-Max-all iT-Max iT-finite*)
apply (*case-tac* $t = 0$)
 apply (*simp add*: *cut-less-0-empty*)
apply (*fastforce simp*: *nat-cut-less-le-conv iTILL-cut-le*)
done

lemma *iTILL-cut-less1*:
  $[\![\; t \in [\ldots n]; \; 0 < t \;]\!] \Longrightarrow [\ldots n] \downarrow< t = [\ldots t - Suc \; 0]$
by (*simp add*: *iTILL-cut-less iT-iff*)

lemma *iFROM-cut-less*:
  $[n\ldots] \downarrow< t = ($
    if $t \le n$ *then* $\{\}$
    else $[n\ldots,t - Suc\ n])$
apply (*case-tac* $t \le n$)
 apply (*simp add*: *cut-less-Min-empty iT-Min*)
apply (*fastforce simp*: *nat-cut-less-le-conv iFROM-cut-le*)
done

lemma *iFROM-cut-less1*:
  $n < t \Longrightarrow [n\ldots] \downarrow< t = [n\ldots,t - Suc\ n]$
by (*simp add*: *iFROM-cut-less*)

lemma *iIN-cut-less*:
  $[n\ldots,d] \downarrow< t = ($
    if $t \le n$ *then* $\{\}$ *else*
    if $t \le n + d$ *then* $[n\ldots, t - Suc\ n]$
    else $[n\ldots,d])$
apply (*case-tac* $t \le n$)
 apply (*simp add*: *cut-less-Min-empty iT-Min*)
apply (*case-tac* $n + d < t$)
 apply (*simp add*: *cut-less-Max-all iT-Max iT-finite*)
apply (*fastforce simp*: *nat-cut-less-le-conv iIN-cut-le*)
done

lemma *iIN-cut-less1*:
  $[\![\ t \in [n\ldots,d];\ n < t\ ]\!] \Longrightarrow [n\ldots,d] \downarrow< t = [n\ldots, t - Suc\ n]$
by (*simp add*: *iIN-cut-less iT-iff*)

lemma *iMOD-cut-less*:
  $[r, mod\ m] \downarrow< t = ($
    if $t \le r$ *then* $\{\}$
    else $[r, mod\ m, (t - Suc\ r)\ div\ m])$
apply (*case-tac* $t = 0$)
 apply (*simp add*: *cut-less-0-empty*)
apply (*simp add*: *nat-cut-less-le-conv iMOD-cut-le*)
apply *fastforce*
done

lemma *iMOD-cut-less1*:
  $[\![\ t \in [r, mod\ m];\ r < t\ ]\!] \Longrightarrow$
  $[r, mod\ m] \downarrow< t = [r, mod\ m, (t - r)\ div\ m - Suc\ 0]$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMOD-0 iMODb-mod-0 iIN-0*)
apply (*simp add*: *iMOD-cut-less*)
apply (*simp add*: *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0 iT-iff*)
done

lemma *iMODb-cut-less*:
  [*r*, *mod m*, *c*] ↓< *t* = (
    if t ≤ *r then* {} *else*
    if r + *m* ∗ *c* < *t then* [*r*, *mod m*, *c*]
    else [*r*, *mod m*, (*t* − *Suc r*) *div m*])
apply (*case-tac t* = *0*)
 apply (*simp add*: *cut-less-0-empty*)
apply (*simp add*: *nat-cut-less-le-conv iMODb-cut-le*)
apply *fastforce*
done

lemma *iMODb-cut-less1*: ⟦ *t* ∈ [*r*, *mod m*, *c*]; *r* < *t* ⟧ ⟹
  [*r*, *mod m*, *c*] ↓< *t* = [*r*, *mod m*, (*t* − *r*) *div m* − *Suc 0*]
apply (*case-tac m* = *0*)
 apply (*simp add*: *iMODb-mod-0 iIN-0*)
apply (*simp add*: *iMODb-cut-less*)
apply (*simp add*: *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0 iT-iff*)
done

lemmas *iT-cut-le* =
  iTILL-cut-le
  iFROM-cut-le
  iIN-cut-le
  iMOD-cut-le
  iMODb-cut-le

lemmas *iT-cut-le1* =
  iTILL-cut-le1
  iFROM-cut-le1
  iIN-cut-le1
  iMOD-cut-le1
  iMODb-cut-le1

lemmas *iT-cut-less* =
  iTILL-cut-less
  iFROM-cut-less
  iIN-cut-less
  iMOD-cut-less
  iMODb-cut-less

lemmas *iT-cut-less1* =
  iTILL-cut-less1
  iFROM-cut-less1
  iIN-cut-less1
  iMOD-cut-less1
  iMODb-cut-less1

lemmas *iT-cut-le-less* =
iTILL-cut-le
iTILL-cut-less
iFROM-cut-le
iFROM-cut-less
iIN-cut-le
iIN-cut-less
iMOD-cut-le
iMOD-cut-less
iMODb-cut-le
iMODb-cut-less

lemmas *iT-cut-le-less1* =
iTILL-cut-le1
iTILL-cut-less1
iFROM-cut-le1
iFROM-cut-less1
iIN-cut-le1
iIN-cut-less1
iMOD-cut-le1
iMOD-cut-less1
iMODb-cut-le1
iMODb-cut-less1

lemma *iTILL-cut-ge*:
$[\ldots n] \downarrow\geq t = (\mathit{if}\ n < t\ \mathit{then}\ \{\}\ \mathit{else}\ [t\ldots,n-t])$
by (*force simp*: *i-cut-mem-iff iT-iff*)

corollary *iTILL-cut-ge1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = [t\ldots,n-t]$
by (*simp add*: *iTILL-cut-ge iT-iff*)

corollary *iTILL-cut-ge2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = \{\}$
by (*simp add*: *iTILL-cut-ge iT-iff*)

lemma *iTILL-cut-greater*:
$[\ldots n] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ \{\}\ \mathit{else}\ [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t])$
by (*force simp*: *i-cut-mem-iff iT-iff*)

corollary *iTILL-cut-greater1*:
$t \in [\ldots n] \Longrightarrow t < n \Longrightarrow [\ldots n] \downarrow> t = [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t]$
by (*simp add*: *iTILL-cut-greater iT-iff*)

corollary *iTILL-cut-greater2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow> t = \{\}$
by (*simp add*: *iTILL-cut-greater iT-iff*)

lemma *iFROM-cut-ge*:
$[n\ldots] \downarrow\geq t = (\mathit{if}\ n \leq t\ \mathit{then}\ [t\ldots]\ \mathit{else}\ [n\ldots])$
by (*force simp*: *i-cut-mem-iff iT-iff*)

corollary *iFROM-cut-ge1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\geq t = [t\ldots]$
by (*simp add*: *iFROM-cut-ge iT-iff*)

lemma *iFROM-cut-greater*:
$[n\ldots] \downarrow> t = (\textit{if } n \leq t \textit{ then } [\textit{Suc } t\ldots] \textit{ else } [n\ldots])$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iFROM-cut-greater1*:
$t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow> t = [\textit{Suc } t\ldots]$
by (*simp add*: *iFROM-cut-greater iT-iff*)

lemma *iIN-cut-ge*:
$[n\ldots,d] \downarrow\geq t = ($
$\textit{if } t < n \textit{ then } [n\ldots,d] \textit{ else}$
$\textit{if } t \leq n+d \textit{ then } [t\ldots,n+d-t]$
$\textit{else } \{\})$
by (*force simp*: *i-cut-mem-iff iT-iff*)

corollary *iIN-cut-ge1*: $t \in [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = [t\ldots,n+d-t]$
by (*simp add*: *iIN-cut-ge iT-iff*)

corollary *iIN-cut-ge2*: $t \notin [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = (\textit{if } t < n \textit{ then } [n\ldots,d] \textit{ else } \{\})$
by (*simp add*: *iIN-cut-ge iT-iff*)

lemma *iIN-cut-greater*:
$[n\ldots,d] \downarrow> t = ($
$\textit{if } t < n \textit{ then } [n\ldots,d] \textit{ else}$
$\textit{if } t < n+d \textit{ then } [\textit{Suc } t\ldots,n + d - \textit{Suc } t]$
$\textit{else } \{\})$
by (*force simp*: *i-cut-mem-iff iT-iff*)

corollary *iIN-cut-greater1*:
$[\![t \in [n\ldots,d]; t < n + d]\!] \Longrightarrow$
$[n\ldots,d] \downarrow> t = [\textit{Suc } t\ldots,n + d - \textit{Suc } t]$
by (*simp add*: *iIN-cut-greater iT-iff*)

lemma *mod-cut-greater-aux-t-less*:
$[\![0 < (m::nat); r \leq t]\!] \Longrightarrow$
$t < t + m - (t - r) \textit{ mod } m$
by (*simp add*: *less-add-diff add.commute*)

lemma *mod-cut-greater-aux-le-x*:
$[\![(r::nat) \leq t; t < x; x \textit{ mod } m = r \textit{ mod } m]\!] \Longrightarrow$
$t + m - (t - r) \textit{ mod } m \leq x$

apply (*insert diff-mod-le*[*of t r m*])
apply (*subst diff-add-assoc2*, *simp*)
apply (*rule less-mod-eq-imp-add-divisor-le*, *simp*)
apply (*simp add*: *sub-diff-mod-eq*)
done

lemma *iMOD-cut-greater*:
[*r*, *mod m*] ↓> *t* = (
if t < *r then* [*r*, *mod m*] *else*
if m = *0 then* {}
else [*t* + *m* − (*t* − *r*) *mod m*, *mod m*])
apply (*case-tac m* = *0*)
apply (*simp add*: *iMOD-0 iIN-0 i-cut-singleton*)
apply (*case-tac t* < *r*)
apply (*simp add*: *iT-Min cut-greater-Min-all*)
apply (*simp add*: *linorder-not-less*)
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-iff*, *clarify*)
apply (*subgoal-tac* (*t* − *r*) *mod m* ≤ *t*)
prefer *2*
apply (*rule order-trans*[*OF mod-le-dividend*], *simp*)
apply (*simp add*: *diff-add-assoc2 del*: *add-diff-assoc2*)
apply (*simp add*: *sub-diff-mod-eq del*: *add-diff-assoc2*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply *clarify*
apply (*rule less-mod-eq-imp-add-divisor-le*)
apply *simp*
apply (*simp add*: *sub-diff-mod-eq*)
apply (*subgoal-tac t* < *x*)
prefer *2*
apply (*rule-tac y*=*t* − (*t* − *r*) *mod m* + *m* **in** *order-less-le-trans*)
apply (*simp add*: *mod-cut-greater-aux-t-less*)
apply *simp*+
done

lemma *iMOD-cut-greater1*:
t ∈ [*r*, *mod m*] ⟹
[*r*, *mod m*] ↓> *t* = (
if m = *0 then* {}
else [*t* + *m*, *mod m*])
by (*simp add*: *iMOD-cut-greater iT-iff mod-eq-imp-diff-mod-0*)

lemma *iMODb-cut-greater-aux*:
⟦ *0* < *m*; *t* < *r* + *m* ∗ *c*; *r* ≤ *t*⟧ ⟹
(*r* + *m* ∗ *c* − (*t* + *m* − (*t* − *r*) *mod m*)) *div m* =
c − *Suc* ((*t* − *r*) *div m*)
apply (*subgoal-tac r* ≤ *t* + *m* − (*t* − *r*) *mod m*)
prefer *2*
apply (*rule order-trans*[*of - t*], *simp*)

apply (*simp add*: *mod-cut-greater-aux-t-less less-imp-le*)
apply (*rule-tac t*=(*r* + *m* ∗ *c* − (*t* + *m* − (*t* − *r*) *mod m*)) **and** *s*=*m* ∗ (*c* − *Suc* ((*t* − *r*) *div m*)) **in** *subst*)
apply (*simp add*: *diff-mult-distrib2 minus-mod-eq-mult-div* [*symmetric*] *del*: *diff-diff-left*)
apply *simp*
done

lemma *iMODb-cut-greater*:
[*r*, *mod m*, *c*] ↓> *t* = (
if t < *r then* [*r*, *mod m*, *c*] *else*
if r + *m* ∗ *c* ≤ *t then* {}
else [*t* + *m* − (*t* − *r*) *mod m*, *mod m*, *c* − *Suc* ((*t*−*r*) *div m*)])
apply (*case-tac m* = *0*)
apply (*simp add*: *iMODb-mod-0 iIN-0 i-cut-singleton*)
apply (*case-tac r* + *m* ∗ *c* ≤ *t*)
apply (*simp add*: *cut-greater-Max-empty iT-Max iT-finite*)
apply (*case-tac t* < *r*)
apply (*simp add*: *cut-greater-Min-all iT-Min*)
apply (*simp add*: *linorder-not-less linorder-not-le*)
apply (*rule-tac t*=[*r*, *mod m*, *c*] **and** *s*=[*r*, *mod m*] ∩ [. . .*r* + *m* ∗ *c*] **in** *subst*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
apply (*simp add*: *i-cut-Int-left iMOD-cut-greater*)
apply (*subst iMOD-iTILL-iMODb-conv*)
apply (*rule mod-cut-greater-aux-le-x*, *simp*+)
apply (*simp add*: *iMODb-cut-greater-aux*)
done

lemma *iMODb-cut-greater1*:
t ∈ [*r*, *mod m*, *c*] ⟹
[*r*, *mod m*, *c*] ↓> *t* = (
if r + *m* ∗ *c* ≤ *t then* {}
else [*t* + *m*, *mod m*, *c* − *Suc* ((*t*−*r*) *div m*)])
by (*simp add*: *iMODb-cut-greater iT-iff mod-eq-imp-diff-mod-0*)

lemma *iMOD-cut-ge*:
[*r*, *mod m*] ↓≥ *t* = (
if t ≤ *r then* [*r*, *mod m*] *else*
if m = *0 then* {}
else [*t* + *m* − *Suc* ((*t* − *Suc r*) *mod m*), *mod m*])
apply (*case-tac t* = *0*)
apply (*simp add*: *cut-ge-0-all*)
apply (*force simp*: *nat-cut-greater-ge-conv*[*symmetric*] *iMOD-cut-greater*)
done

lemma *iMOD-cut-ge1*:
t ∈ [*r*, *mod m*] ⟹
[*r*, *mod m*] ↓≥ *t* = [*t*, *mod m*]

by (*fastforce simp*: *iMOD-cut-ge*)

lemma *iMODb-cut-ge*:
 [*r*, *mod m*, *c*] ↓≥ *t* = (
  if t ≤ *r then* [*r*, *mod m*, *c*] *else*
  if r + *m* ∗ *c* < *t then* {}
  else [*t* + *m* − *Suc* ((*t* − *Suc r*) *mod m*), *mod m*, *c* − (*t* + *m* − *Suc r*) *div m*])
apply (*case-tac m* = *0*)
 apply (*simp add*: *iMODb-mod-0 iIN-0 i-cut-singleton*)
apply (*case-tac r* + *m* ∗ *c* < *t*)
 apply (*simp add*: *cut-ge-Max-empty iT-Max iT-finite*)
apply (*case-tac t* ≤ *r*)
 apply (*simp add*: *cut-ge-Min-all iT-Min*)
apply (*simp add*: *linorder-not-less linorder-not-le*)
apply (*case-tac r mod m* = *t mod m*)
 apply (*simp add*: *diff-mod-pred*)
 apply (*simp add*: *mod-0-imp-diff-Suc-div-conv mod-eq-diff-mod-0-conv diff-add-assoc2*
del: *add-diff-assoc2*)
 apply (*subgoal-tac 0* < (*t* − *r*) *div m*)
  prefer *2*
  apply (*frule-tac x*=*r* **in** *less-mod-eq-imp-add-divisor-le*)
   apply (*simp add*: *mod-eq-diff-mod-0-conv*)
  apply (*drule add-le-imp-le-diff2*)
  apply (*drule-tac m*=*m* **and** *k*=*m* **in** *div-le-mono*)
  apply *simp*
 apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-iff*, *intro allI*)
 apply (*simp add*: *mod-eq-diff-mod-0-conv*[*symmetric*])
 apply (*rule conj-cong*, *simp*)
 apply (*case-tac t* ≤ *x*)
  prefer *2*
  apply *simp*
 apply (*simp add*: *diff-mult-distrib2 minus-mod-eq-mult-div* [*symmetric*] *mod-eq-diff-mod-0-conv*
add.commute[*of r*])
apply (*subgoal-tac Suc* ((*t* − *Suc r*) *mod m*) = (*t* − *r*) *mod m*)
 prefer *2*
 apply (*clarsimp simp add*: *diff-mod-pred mod-eq-diff-mod-0-conv*)
apply (*rule-tac t*=[*r*, *mod m*, *c*] **and** *s*=[*r*, *mod m*] ∩ [. . .*r* + *m* ∗ *c*] **in** *subst*)
 apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
apply (*simp add*: *i-cut-Int-left iMOD-cut-ge*)
apply (*subst iMOD-iTILL-iMODb-conv*)
 apply (*drule-tac x*=*t* **in** *le-imp-less-or-eq*, *erule disjE*)
 apply (*rule mod-cut-greater-aux-le-x*, *simp*+)
apply (*rule arg-cong* [**where** *y*=*c* − (*t* + *m* − *Suc r*) *div m*])
apply (*drule-tac x*=*t* **in** *le-imp-less-or-eq*, *erule disjE*)
 prefer *2*
 apply *simp*
apply (*simp add*: *iMODb-cut-greater-aux*)

```
apply (rule arg-cong[where f=(−) c])
apply (simp add: diff-add-assoc2 del: add-diff-assoc2)
apply (rule-tac t=t − Suc r and s=t − r − Suc 0 in subst, simp)
apply (subst div-diff1-eq[of - Suc 0])
apply (case-tac m = Suc 0, simp)
apply simp
done

lemma iMODb-cut-ge1:
  t ∈ [r, mod m, c] =⇒
  [r, mod m, c] ↓≥ t = (
    if r + m ∗ c < t then {}
    else [t, mod m, c − (t − r) div m])
apply (case-tac m = 0)
 apply (simp add: iMODb-mod-0 iT-iff iIN-0 i-cut-singleton)
apply (clarsimp simp: iMODb-cut-ge iT-iff)
apply (simp add: mod-eq-imp-diff-mod-eq-divisor)
apply (rule-tac t=t + m − Suc r and s=t − r + (m − Suc 0) in subst, simp)
apply (subst div-add1-eq)
apply (simp add: mod-eq-imp-diff-mod-0)
done

lemma iMOD-0-cut-greater:
  t ∈ [r, mod 0] =⇒ [r, mod 0] ↓> t = {}
by (simp add: iT-iff iMOD-0 iIN-0 i-cut-singleton)
lemma iMODb-0-cut-greater: t ∈ [r, mod 0, c] =⇒
  [r, mod 0, c] ↓> t = {}
by (simp add: iT-iff iMODb-mod-0 iIN-0 i-cut-singleton)

lemmas iT-cut-ge =
  iTILL-cut-ge
  iFROM-cut-ge
  iIN-cut-ge
  iMOD-cut-ge
  iMODb-cut-ge

lemmas iT-cut-ge1 =
  iTILL-cut-ge1
  iFROM-cut-ge1
  iIN-cut-ge1
  iMOD-cut-ge1
  iMODb-cut-ge1

lemmas iT-cut-greater =
  iTILL-cut-greater
  iFROM-cut-greater
  iIN-cut-greater
  iMOD-cut-greater
  iMODb-cut-greater
```

lemmas *iT-cut-greater1* =
iTILL-cut-greater1
iFROM-cut-greater1
iIN-cut-greater1
iMOD-cut-greater1
iMODb-cut-greater1

lemmas *iT-cut-ge-greater* =
iTILL-cut-ge
iTILL-cut-greater
iFROM-cut-ge
iFROM-cut-greater
iIN-cut-ge
iIN-cut-greater
iMOD-cut-ge
iMOD-cut-greater
iMODb-cut-ge
iMODb-cut-greater
lemmas *iT-cut-ge-greater1* =
iTILL-cut-ge1
iTILL-cut-greater1
iFROM-cut-ge1
iFROM-cut-greater1
iIN-cut-ge1
iIN-cut-greater1
iMOD-cut-ge1
iMOD-cut-greater1
iMODb-cut-ge1
iMODb-cut-greater1

1.6 Cardinality of intervals

lemma *iFROM-card*: *card* [*n*...] = *0*
by (*simp add*: *iFROM-infinite*)

lemma *iTILL-card*: *card* [...*n*] = *Suc n*
by (*simp add*: *iTILL-def*)

lemma *iIN-card*: *card* [*n*...,*d*] = *Suc d*
by (*simp add*: *iIN-def*)

lemma *iMOD-0-card*: *card* [*r*, *mod 0*] = *Suc 0*
by (*simp add*: *iMOD-0 iIN-card*)

lemma *iMOD-card*: *0* < *m* ⟹ *card* [*r*, *mod m*] = *0*
by (*simp add*: *iMOD-infinite*)

lemma *iMOD-card-if*: *card* [*r*, *mod m*] = (*if m* = *0 then Suc 0 else 0*)

by (*simp add*: *iMOD-0-card iMOD-card*)

lemma *iMODb-mod-0-card*: *card* [*r*, *mod 0*, *c*] = *Suc 0*
by (*simp add*: *iMODb-mod-0 iIN-card*)

lemma *iMODb-card*: *0* < *m* ⟹ *card* [*r*, *mod m*, *c*] = *Suc c*
apply (*induct c*)
 apply (*simp add*: *iMODb-0 iIN-card*)
apply (*subst iMODb-Suc-insert-conv*[*symmetric*])
apply (*subst card-insert-disjoint*)
apply (*simp add*: *iT-finite iT-iff*)+
done

lemma *iMODb-card-if*:
  card [*r*, *mod m*, *c*] = (*if m* = *0 then Suc 0 else Suc c*)
by (*simp add*: *iMODb-mod-0-card iMODb-card*)

lemmas *iT-card* =
  iFROM-card
  iTILL-card
  iIN-card
  iMOD-card-if
  iMODb-card-if

Cardinality with *icard*

lemma *iFROM-icard*: *icard* [*n*. . .] = ∞
by (*simp add*: *iFROM-infinite*)

lemma *iTILL-icard*: *icard* [. . .*n*] = *enat* (*Suc n*)
by (*simp add*: *icard-finite iT-finite iT-card*)

lemma *iIN-icard*: *icard* [*n*. . .,*d*] = *enat* (*Suc d*)
by (*simp add*: *icard-finite iT-finite iT-card*)

lemma *iMOD-0-icard*: *icard* [*r*, *mod 0*] = *eSuc 0*
by (*simp add*: *icard-finite iT-finite iT-card eSuc-enat*)

lemma *iMOD-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*] = ∞
by (*simp add*: *iMOD-infinite*)

lemma *iMOD-icard-if*: *icard* [*r*, *mod m*] = (*if m* = *0 then eSuc 0 else* ∞)
by (*simp add*: *icard-finite iT-finite iT-infinite eSuc-enat iT-card*)

lemma *iMODb-mod-0-icard*: *icard* [*r*, *mod 0*, *c*] = *eSuc 0*
by (*simp add*: *icard-finite iT-finite eSuc-enat iT-card*)

lemma *iMODb-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*, *c*] = *enat* (*Suc c*)
by (*simp add*: *icard-finite iT-finite iMODb-card*)

lemma *iMODb-icard-if*: *icard* [*r*, *mod m*, *c*] = *enat* (*if m* = *0 then Suc 0 else Suc c*)
by (*simp add*: *icard-finite iT-finite iMODb-card-if*)

lemmas *iT-icard* =
iFROM-icard
iTILL-icard
iIN-icard
iMOD-icard-if
iMODb-icard-if

1.7 Functions *inext* and *iprev* with intervals

lemma
iFROM-inext: *t* ∈ [*n*. . .] ⟹ *inext t* [*n*. . .] = *Suc t* **and**
iTILL-inext: *t* < *n* ⟹ *inext t* [. . .*n*] = *Suc t* **and**
iIN-inext: ⟦ *n* ≤ *t*; *t* < *n* + *d* ⟧ ⟹ *inext t* [*n*. . .,*d*] = *Suc t*
by (*simp add*: *iT-defs inext-atLeast inext-atMost inext-atLeastAtMost*)+

lemma
iFROM-iprev': *t* ∈ [*n*. . .] ⟹ *iprev* (*Suc t*) [*n*. . .] = *t* **and**
iFROM-iprev: *n* < *t* ⟹ *iprev t* [*n*. . .] = *t* − *Suc 0* **and**
iTILL-iprev: *t* ∈ [. . .*n*] ⟹ *iprev t* [. . .*n*] = *t* − *Suc 0* **and**
iIN-iprev: ⟦ *n* < *t*; *t* ≤ *n* + *d* ⟧ ⟹ *iprev t* [*n*. . .,*d*] = *t* − *Suc 0* **and**
iIN-iprev': ⟦ *n* ≤ *t*; *t* < *n* + *d* ⟧ ⟹ *iprev* (*Suc t*) [*n*. . .,*d*] = *t*
by (*simp add*: *iT-defs iprev-atLeast iprev-atMost iprev-atLeastAtMost*)+

lemma *iMOD-inext*: *t* ∈ [*r*, *mod m*] ⟹ *inext t* [*r*, *mod m*] = *t* + *m*
by (*clarsimp simp add*: *inext-def iMOD-cut-greater iT-iff iT-Min iT-not-empty mod-eq-imp-diff-mod-0*)
lemma *iMOD-iprev*: ⟦ *t* ∈ [*r*, *mod m*]; *r* < *t* ⟧ ⟹ *iprev t* [*r*, *mod m*] = *t* − *m*
apply (*case-tac m* = *0*, *simp add*: *iMOD-iff*)
apply (*clarsimp simp add*: *iprev-def iMOD-cut-less iT-iff iT-Max iT-not-empty minus-mod-eq-mult-div* [*symmetric*])
apply (*simp del*: *add-Suc-right add*: *add-Suc-right*[*symmetric*] *mod-eq-imp-diff-mod-eq-divisor*)
apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done

lemma *iMOD-iprev'*: *t* ∈ [*r*, *mod m*] ⟹ *iprev* (*t* + *m*) [*r*, *mod m*] = *t*
apply (*case-tac m* = *0*)
apply (*simp add*: *iMOD-0 iIN-0 iprev-singleton*)
apply (*simp add*: *iMOD-iprev iT-iff*)
done

lemma *iMODb-inext*:
⟦ *t* ∈ [*r*, *mod m*, *c*]; *t* < *r* + *m* ∗ *c* ⟧ ⟹
inext t [*r*, *mod m*, *c*] = *t* + *m*
by (*clarsimp simp add*: *inext-def iMODb-cut-greater iT-iff iT-Min iT-not-empty mod-eq-imp-diff-mod-0*)

lemma *iMODb-iprev*:
  ⟦ $t \in [r, mod\ m, c]$; $r < t$ ⟧ $\Longrightarrow$
  iprev $t\ [r, mod\ m, c] = t - m$
apply (*case-tac* $m = 0$, *simp add*: *iMODb-iff*)
apply (*clarsimp simp add*: *iprev-def iMODb-cut-less iT-iff iT-Max iT-not-empty minus-mod-eq-mult-div* [*symmetric*])
apply (*simp del*: *add-Suc-right add*: *add-Suc-right*[*symmetric*] *mod-eq-imp-diff-mod-eq-divisor*)
apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done

lemma *iMODb-iprev′*:
  ⟦ $t \in [r, mod\ m, c]$; $t < r + m * c$ ⟧ $\Longrightarrow$
  iprev $(t + m)\ [r, mod\ m, c] = t$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMODb-mod-0 iIN-0 iprev-singleton*)
apply (*simp add*: *iMODb-iprev iT-iff less-mod-eq-imp-add-divisor-le*)
done

lemmas *iT-inext* =
  iFROM-inext
  iTILL-inext
  iIN-inext
  iMOD-inext
  iMODb-inext
lemmas *iT-iprev* =
  iFROM-iprev′
  iFROM-iprev
  iTILL-iprev
  iIN-iprev
  iIN-iprev′
  iMOD-iprev
  iMOD-iprev′
  iMODb-iprev
  iMODb-iprev′

lemma *iFROM-inext-if*:
  inext $t\ [n\ldots] = ($*if* $t \in [n\ldots]$ *then Suc* t *else* $t)$
by (*simp add*: *iFROM-inext not-in-inext-fix*)

lemma *iTILL-inext-if*:
  inext $t\ [\ldots n] = ($*if* $t < n$ *then Suc* t *else* $t)$
by (*simp add*: *iTILL-inext iT-finite iT-Max inext-ge-Max*)

lemma *iIN-inext-if*:
  inext $t\ [n\ldots,d] = ($*if* $n \le t \wedge t < n + d$ *then Suc* t *else* $t)$
by (*fastforce simp*: *iIN-inext iT-iff not-in-inext-fix iT-finite iT-Max inext-ge-Max*)

lemma *iMOD-inext-if*:

inext t [*r*, *mod m*] = (*if t* ∈ [*r*, *mod m*] *then t* + *m else t*)
by (*simp add*: *iMOD-inext not-in-inext-fix*)

lemma *iMODb-inext-if*:
inext t [*r*, *mod m*, *c*] =
(*if t* ∈ [*r*, *mod m*, *c*] ∧ *t* < *r* + *m* ∗ *c then t* + *m else t*)
by (*fastforce simp*: *iMODb-inext iT-iff not-in-inext-fix iT-finite iT-Max inext-ge-Max*)

lemmas *iT-inext-if* =
iFROM-inext-if
iTILL-inext-if
iIN-inext-if
iMOD-inext-if
iMODb-inext-if

lemma *iFROM-iprev-if*:
iprev t [*n*. . .] = (*if n* < *t then t* − *Suc 0 else t*)
by (*simp add*: *iFROM-iprev iT-Min iprev-le-iMin*)
lemma *iTILL-iprev-if*:
iprev t [. . .*n*] = (*if t* ∈ [. . .*n*] *then t* − *Suc 0 else t*)
by (*simp add*: *iTILL-iprev not-in-iprev-fix*)
lemma *iIN-iprev-if*:
iprev t [*n*. . .,*d*] = (*if n* < *t* ∧ *t* ≤ *n* + *d then t* − *Suc 0 else t*)
by (*fastforce simp*: *iIN-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)
lemma *iMOD-iprev-if*:
iprev t [*r*, *mod m*] =
(*if t* ∈ [*r*, *mod m*] ∧ *r* < *t then t* − *m else t*)
by (*fastforce simp add*: *iMOD-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)
lemma *iMODb-iprev-if*:
iprev t [*r*, *mod m*, *c*] =
(*if t* ∈ [*r*, *mod m*, *c*] ∧ *r* < *t then t* − *m else t*)
by (*fastforce simp add*: *iMODb-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)

lemmas *iT-iprev-if* =
iFROM-iprev-if
iTILL-iprev-if
iIN-iprev-if
iMOD-iprev-if
iMODb-iprev-if

The difference between an element and the next/previous element is constant if the element is different from Min/Max of the interval

lemma *iFROM-inext-diff-const*:
t ∈ [*n*. . .] =⇒ *inext t* [*n*. . .] − *t* = *Suc 0*
by (*simp add*: *iFROM-inext*)

lemma *iFROM-iprev-diff-const*:
n < *t* =⇒ *t* − *iprev t* [*n*. . .] = *Suc 0*
by (*simp add*: *iFROM-iprev*)

lemma *iFROM-iprev-diff-const'*:
$t \in [n\ldots] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots] = Suc\ 0$
by (*simp add*: *iFROM-iprev'*)

lemma *iTILL-inext-diff-const*:
$t < n \Longrightarrow inext\ t\ [\ldots n] - t = Suc\ 0$
by (*simp add*: *iTILL-inext*)
lemma *iTILL-iprev-diff-const*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow t - iprev\ t\ [\ldots n] = Suc\ 0$
by (*simp add*: *iTILL-iprev*)

lemma *iIN-inext-diff-const*:
$[\![\ n \le t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] - t = Suc\ 0$
by (*simp add*: *iIN-inext*)

lemma *iIN-iprev-diff-const*:
$[\![\ n < t;\ t \le n + d\ ]\!] \Longrightarrow t - iprev\ t\ [n\ldots,d] = Suc\ 0$
by (*simp add*: *iIN-iprev*)
lemma *iIN-iprev-diff-const'*:
$[\![\ n \le t;\ t < n + d\ ]\!] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots,d] = Suc\ 0$
by (*simp add*: *iIN-iprev*)

lemma *iMOD-inext-diff-const*:
$t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] - t = m$
by (*simp add*: *iMOD-inext*)

lemma *iMOD-iprev-diff-const'*:
$t \in [r,\ mod\ m] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m] = m$
by (*simp add*: *iMOD-iprev'*)

lemma *iMOD-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m] = m$
apply (*simp add*: *iMOD-iprev iT-iff*)
apply (*drule less-mod-eq-imp-add-divisor-le*[**where** *m*=*m*], *simp*+)
done

lemma *iMODb-inext-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow inext\ t\ [r,\ mod\ m,\ c] - t = m$
by (*simp add*: *iMODb-inext*)

lemma *iMODb-iprev-diff-const'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m,\ c] = m$
by (*simp add*: *iMODb-iprev'*)

lemma *iMODb-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m,\ c] = m$
apply (*simp add*: *iMODb-iprev iT-iff*)

apply (*drule less-mod-eq-imp-add-divisor-le*[**where** *m*=*m*], *simp*+)
done

lemmas *iT-inext-diff-const* =
iFROM-inext-diff-const
iTILL-inext-diff-const
iIN-inext-diff-const
iMOD-inext-diff-const
iMODb-inext-diff-const
lemmas *iT-iprev-diff-const* =
iFROM-iprev-diff-const
iFROM-iprev-diff-const'
iTILL-iprev-diff-const
iIN-iprev-diff-const
iIN-iprev-diff-const'
iMOD-iprev-diff-const'
iMOD-iprev-diff-const
iMODb-iprev-diff-const'
iMODb-iprev-diff-const

1.7.1 Mirroring of intervals

lemma
iIN-mirror-elem: *mirror-elem* x [n...,d] = $n + n + d - x$ **and**
iTILL-mirror-elem: *mirror-elem* x [...n] = $n - x$ **and**
iMODb-mirror-elem: *mirror-elem* x [r, *mod* m, c] = $r + r + m * c - x$
by (*simp add*: *mirror-elem-def nat-mirror-def iT-Min iT-Max*)+

lemma *iMODb-imirror-bounds*:
$r' + m' * c' \le l + r \Longrightarrow$
imirror-bounds [r', *mod* m', c'] l r = [$l + r - r' - m' * c'$, *mod* m', c']
apply (*clarsimp simp*: *set-eq-iff Bex-def imirror-bounds-iff iT-iff*)
apply (*frule diff-le-mono*[*of - - r'*], *simp*)
apply (*simp add*: *mod-diff-right-eq*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac x'*)
apply (*rule-tac a*=*x'* **in** *ssubst*[*OF mod-diff-right-eq*, *rule-format*], *simp*+)
apply (*simp add*: *diff-le-mono2*)
apply *clarsimp*
apply (*rule-tac x*=*l*+*r*−*x* **in** *exI*)
apply (*simp add*: *le-diff-swap*)
apply (*simp add*: *le-diff-conv2*)
apply (*subst mod-sub-eq-mod-swap*, *simp*+)
apply (*simp add*: *mod-diff-right-eq*)
done

lemma *iIN-imirror-bounds*:
$n + d \le l + r \Longrightarrow$ *imirror-bounds* [n...,d] l r = [$l + r - n - d$...,d]
apply (*insert iMODb-imirror-bounds*[*of n Suc 0 d l r*])

apply (*simp add*: *iMODb-mod-1*)
done

lemma *iTILL-imirror-bounds*:
  $n \le l + r \Longrightarrow$ *imirror-bounds* $[\ldots n]$ l r $= [l + r - n\ldots,n]$
apply (*insert iIN-imirror-bounds*[*of 0 n l r*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done

lemmas *iT-imirror-bounds* =
  iTILL-imirror-bounds
  iIN-imirror-bounds
  iMODb-imirror-bounds

lemma *iMODb-imirror-ident*: *imirror* [r, *mod* m, c] = [r, *mod* m, c]
by (*simp add*: *imirror-eq-imirror-bounds iMODb-Min iMODb-Max iMODb-imirror-bounds*)
lemma *iIN-imirror-ident*: *imirror* $[n\ldots,d] = [n\ldots,d]$
by (*simp add*: *iMODb-mod-1*[*symmetric*] *iMODb-imirror-ident*)
lemma *iTILL-imirror-ident*: *imirror* $[\ldots n] = [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-imirror-ident*)

lemmas *iT-imirror-ident* =
  iTILL-imirror-ident
  iIN-imirror-ident
  iMODb-imirror-ident

1.7.2 Functions *inext-nth* and *iprev-nth* on intervals

lemma *iFROM-inext-nth* : $[n\ldots] \rightarrow a = n + a$
by (*simp add*: *iT-defs inext-nth-atLeast*)

lemma *iIN-inext-nth* : $a \le d \Longrightarrow [n\ldots,d] \rightarrow a = n + a$
by (*simp add*: *iT-defs inext-nth-atLeastAtMost*)

lemma *iIN-iprev-nth*: $a \le d \Longrightarrow [n\ldots,d] \leftarrow a = n + d - a$
by (*simp add*: *iT-defs iprev-nth-atLeastAtMost*)

lemma *iIN-inext-nth-if* :
  $[n\ldots,d] \rightarrow a =$ (*if* $a \le d$ *then* $n + a$ *else* $n + d$)
by (*simp add*: *iIN-inext-nth inext-nth-card-Max iT-finite iT-not-empty iT-Max iT-card*)

lemma *iIN-iprev-nth-if*:
  $[n\ldots,d] \leftarrow a =$ (*if* $a \le d$ *then* $n + d - a$ *else* n)
by (*simp add*: *iIN-iprev-nth iprev-nth-card-iMin iT-finite iT-not-empty iT-Min iT-card*)

lemma *iTILL-inext-nth* : $a \le n \Longrightarrow [\ldots n] \rightarrow a = a$
by (*simp add*: *iTILL-def inext-nth-atMost*)

lemma *iTILL-inext-nth-if* :
$[\ldots n] \rightarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ a\ \mathit{else}\ n)$
by (*insert iIN-inext-nth-if*[*of 0 n a*], *simp add*: *iIN-0-iTILL-conv*)

lemma *iTILL-iprev-nth*: $a \leq n \Longrightarrow [\ldots n] \leftarrow a = n - a$
by (*simp add*: *iTILL-def iprev-nth-atMost*)

lemma *iTILL-iprev-nth-if*:
$[\ldots n] \leftarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ n - a\ \mathit{else}\ 0)$
by (*insert iIN-iprev-nth-if*[*of 0 n a*], *simp add*: *iIN-0-iTILL-conv*)

lemma *iMOD-inext-nth*: $[r,\ \mathit{mod}\ m] \rightarrow a = r + m * a$
apply (*induct a*)
apply (*simp add*: *iT-Min*)
apply (*simp add*: *iMOD-inext-if iT-iff*)
done

lemma *iMODb-inext-nth*: $a \leq c \Longrightarrow [r,\ \mathit{mod}\ m,\ c] \rightarrow a = r + m * a$
apply (*case-tac m = 0*)
apply (*simp add*: *iMODb-mod-0 iIN-0 inext-nth-singleton*)
apply (*induct a*)
apply (*simp add*: *iMODb-Min*)
apply (*simp add*: *iMODb-inext-if iT-iff*)
done

lemma *iMODb-inext-nth-if*:
$[r,\ \mathit{mod}\ m,\ c] \rightarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * a\ \mathit{else}\ r + m * c)$
by (*simp add*: *iMODb-inext-nth inext-nth-card-Max iT-finite iT-not-empty iT-Max iT-card*)

lemma *iMODb-iprev-nth*:
$a \leq c \Longrightarrow [r,\ \mathit{mod}\ m,\ c] \leftarrow a = r + m * (c - a)$
apply (*case-tac m = 0*)
apply (*simp add*: *iMODb-mod-0 iIN-0 iprev-nth-singleton*)
apply (*induct a*)
apply (*simp add*: *iMODb-Max*)
apply (*simp add*: *iMODb-iprev-if iT-iff*)
apply (*frule mult-left-mono*[*of - - m*], *simp*)
apply (*simp add*: *diff-mult-distrib2*)
done

lemma *iMODb-iprev-nth-if*:
$[r,\ \mathit{mod}\ m,\ c] \leftarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * (c - a)\ \mathit{else}\ r)$
by (*simp add*: *iMODb-iprev-nth iprev-nth-card-iMin iT-finite iT-not-empty iT-Min iT-card*)

lemma *iIN-iFROM-inext-nth*:
$a \le d \Longrightarrow [n\ldots,d] \to a = [n\ldots] \to a$
by (*simp add*: *iIN-inext-nth iFROM-inext-nth*)

lemma *iIN-iFROM-inext*:
$a < n + d \Longrightarrow inext\ a\ [n\ldots,d] = inext\ a\ [n\ldots]$
by (*simp add*: *iT-inext-if iT-iff*)

lemma *iMOD-iMODb-inext-nth*:
$a \le c \Longrightarrow [r,\ mod\ m,\ c] \to a = [r,\ mod\ m] \to a$
by (*simp add*: *iMOD-inext-nth iMODb-inext-nth*)

lemma *iMOD-iMODb-inext*:
$a < r + m * c \Longrightarrow inext\ a\ [r,\ mod\ m,\ c] = inext\ a\ [r,\ mod\ m]$
by (*simp add*: *iT-inext-if iT-iff*)

lemma *iMOD-inext-nth-Suc-diff*:
$([r,\ mod\ m] \to (Suc\ n)) - ([r,\ mod\ m] \to n) = m$
by (*simp add*: *iMOD-inext-nth del*: *inext-nth.simps*)

lemma *iMOD-inext-nth-diff*:
$([r,\ mod\ m] \to a) - ([r,\ mod\ m] \to b) = (a - b) * m$
by (*simp add*: *iMOD-inext-nth diff-mult-distrib mult.commute*[*of m*])

lemma *iMODb-inext-nth-diff*: $[\![\ a \le c;\ b \le c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] \to a) - ([r,\ mod\ m,\ c] \to b) = (a - b) * m$
by (*simp add*: *iMODb-inext-nth diff-mult-distrib mult.commute*[*of m*])

1.8 Induction with intervals

lemma *iFROM-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots];\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots]\ ]\!] \Longrightarrow P\ a$
apply (*rule inext-induct*[*of - [n...]*])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iIN-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n\ldots,d];\ k \ne n + d;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n\ldots,d]\ ]\!] \Longrightarrow P\ a$
apply (*rule inext-induct*[*of - [n...,d]*])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iTILL-induct*:
$[\![\ P\ 0;\ \bigwedge k.\ [\![\ k \in [\ldots n];\ k \ne n;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [\ldots n]\ ]\!] \Longrightarrow P\ a$
apply (*rule inext-induct*[*of - [...n]*])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iMOD-induct*:
$\llbracket\ P\ r;\ \bigwedge k.\ \llbracket\ k \in [r,\ mod\ m];\ P\ k\ \rrbracket \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m]\ \rrbracket \Longrightarrow P\ a$
apply (*rule inext-induct*[*of - *[*r*, *mod m*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iMODb-induct*:
$\llbracket\ P\ r;\ \bigwedge k.\ \llbracket\ k \in [r,\ mod\ m,\ c];\ k \neq r + m * c;\ P\ k\ \rrbracket \Longrightarrow P\ (k + m);\ a \in [r,\ mod\ m,\ c]\ \rrbracket \Longrightarrow P\ a$
apply (*rule inext-induct*[*of - *[*r*, *mod m*, *c*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iIN-rev-induct*:
$\llbracket\ P\ (n + d);\ \bigwedge k.\ \llbracket\ k \in [n\ldots,d];\ k \neq n;\ P\ k\ \rrbracket \Longrightarrow P\ (k - Suc\ 0);\ a \in [n\ldots,d]\ \rrbracket \Longrightarrow P\ a$
apply (*rule iprev-induct*[*of - *[*n*. . .,*d*]])
apply (*simp add*: *iT-Max iT-finite iT-iprev-if*)+
done

lemma *iTILL-rev-induct*:
$\llbracket\ P\ n;\ \bigwedge k.\ \llbracket\ k \in [\ldots n];\ 0 < k;\ P\ k\ \rrbracket \Longrightarrow P\ (k - Suc\ 0);\ a \in [\ldots n]\ \rrbracket \Longrightarrow P\ a$
apply (*rule iprev-induct*[*of - *[. . .*n*]])
apply (*fastforce simp*: *iT-Max iT-finite iT-iprev-if*)+
done

lemma *iMODb-rev-induct*:
$\llbracket\ P\ (r + m * c);\ \bigwedge k.\ \llbracket\ k \in [r,\ mod\ m,\ c];\ k \neq r;\ P\ k\ \rrbracket \Longrightarrow P\ (k - m);\ a \in [r,\ mod\ m,\ c]\ \rrbracket \Longrightarrow P\ a$
apply (*rule iprev-induct*[*of - *[*r*, *mod m*, *c*]])
apply (*simp add*: *iT-Max iT-finite iT-iprev-if*)+
done

end

2 Arithmetic operators on natural intervals

theory *IL-IntervalOperators*
imports *IL-Interval*
begin

2.1 Arithmetic operations with intervals

2.1.1 Addition of and multiplication by constants

definition *iT-Plus* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ‹⊕› *55*)
where $I \oplus k \equiv (\lambda n.(n + k))$ ‘ I

definition *iT-Mult* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ‹⊗› *55*)

where *iT-Mult-def* : $I \otimes k \equiv (\lambda n.(n * k))$ ‘ I

lemma *iT-Plus-image-conv*: $I \oplus k = (\lambda n.(n + k))$ ‘ I
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-image-conv*: $I \otimes k = (\lambda n.(n * k))$ ‘ I
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-empty*: $\{\} \oplus k = \{\}$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-empty*: $\{\} \otimes k = \{\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-not-empty*: $I \neq \{\} \Longrightarrow I \oplus k \neq \{\}$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-not-empty*: $I \neq \{\} \Longrightarrow I \otimes k \neq \{\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-empty-iff*: $(I \oplus k = \{\}) = (I = \{\})$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-empty-iff*: $(I \otimes k = \{\}) = (I = \{\})$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-mono*: $A \subseteq B \Longrightarrow A \oplus k \subseteq B \oplus k$
by (*simp add*: *iT-Plus-def image-mono*)

lemma *iT-Mult-mono*: $A \subseteq B \Longrightarrow A \otimes k \subseteq B \otimes k$
by (*simp add*: *iT-Mult-def image-mono*)

lemma *iT-Mult-0*: $I \neq \{\} \Longrightarrow I \otimes 0 = [\ldots 0]$
by (*fastforce simp add*: *iTILL-def iT-Mult-def*)

corollary *iT-Mult-0-if*: $I \otimes 0 = (\textit{if } I = \{\} \textit{ then } \{\} \textit{ else } [\ldots 0])$
by (*simp add*: *iT-Mult-empty iT-Mult-0*)

lemma *iT-Plus-mem-iff*: $x \in (I \oplus k) = (k \le x \wedge (x - k) \in I)$
apply (*simp add*: *iT-Plus-def image-iff*)
apply (*rule iffI*)
apply *fastforce*
apply (*rule-tac x=x − k* **in** *bexI*, *simp+*)
done

lemma *iT-Plus-mem-iff2*: $x + k \in (I \oplus k) = (x \in I)$

by (*simp add*: *iT-Plus-def image-iff*)

lemma *iT-Mult-mem-iff-0*: $x \in (I \otimes 0) = (I \neq \{\} \wedge x = 0)$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Mult-empty*)
apply (*simp add*: *iT-Mult-0 iT-iff*)
done

lemma *iT-Mult-mem-iff*:
 $0 < k \Longrightarrow x \in (I \otimes k) = (x \bmod k = 0 \wedge x \operatorname{div} k \in I)$
by (*fastforce simp*: *iT-Mult-def image-iff*)

lemma *iT-Mult-mem-iff2*: $0 < k \Longrightarrow x * k \in (I \otimes k) = (x \in I)$
by (*simp add*: *iT-Mult-def image-iff*)

lemma *iT-Plus-singleton*: $\{a\} \oplus k = \{a + k\}$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-singleton*: $\{a\} \otimes k = \{a * k\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-Un*: $(A \cup B) \oplus k = (A \oplus k) \cup (B \oplus k)$
by (*simp add*: *iT-Plus-def image-Un*)

lemma *iT-Mult-Un*: $(A \cup B) \otimes k = (A \otimes k) \cup (B \otimes k)$
by (*simp add*: *iT-Mult-def image-Un*)

lemma *iT-Plus-Int*: $(A \cap B) \oplus k = (A \oplus k) \cap (B \oplus k)$
by (*simp add*: *iT-Plus-def image-Int*)

lemma *iT-Mult-Int*: $0 < k \Longrightarrow (A \cap B) \otimes k = (A \otimes k) \cap (B \otimes k)$
by (*simp add*: *iT-Mult-def image-Int mult-right-inj*)

lemma *iT-Plus-image*: $f \text{ ` } I \oplus n = (\lambda x.\ f\ x + n) \text{ ` } I$
by (*fastforce simp*: *iT-Plus-def*)

lemma *iT-Mult-image*: $f \text{ ` } I \otimes n = (\lambda x.\ f\ x * n) \text{ ` } I$
by (*fastforce simp*: *iT-Mult-def*)

lemma *iT-Plus-commute*: $I \oplus a \oplus b = I \oplus b \oplus a$
by (*fastforce simp*: *iT-Plus-def*)

lemma *iT-Mult-commute*: $I \otimes a \otimes b = I \otimes b \otimes a$
by (*fastforce simp*: *iT-Mult-def*)

lemma *iT-Plus-assoc*:$I \oplus a \oplus b = I \oplus (a + b)$
by (*fastforce simp*: *iT-Plus-def*)

lemma *iT-Mult-assoc*:$I \otimes a \otimes b = I \otimes (a * b)$
by (*fastforce simp*: *iT-Mult-def*)

lemma *iT-Plus-Mult-distrib*: $I \oplus n \otimes m = I \otimes m \oplus n * m$
by (*simp add*: *iT-Plus-def iT-Mult-def image-image add-mult-distrib*)

lemma *iT-Plus-finite-iff*: *finite* $(I \oplus k)$ = *finite* I
by (*simp add*: *iT-Plus-def inj-on-finite-image-iff*)

lemma *iT-Mult-0-finite*: *finite* $(I \otimes 0)$
by (*simp add*: *iT-Mult-0-if iTILL-0*)

lemma *iT-Mult-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \otimes k)$ = *finite* I
by (*simp add*: *iT-Mult-def inj-on-finite-image-iff* [*OF inj-imp-inj-on*] *mult-right-inj*)

lemma *iT-Plus-Min*: $I \neq \{\} \Longrightarrow$ *iMin* $(I \oplus k)$ = *iMin* $I + k$
by (*simp add*: *iT-Plus-def iMin-mono2 mono-def*)

lemma *iT-Mult-Min*: $I \neq \{\} \Longrightarrow$ *iMin* $(I \otimes k)$ = *iMin* $I * k$
by (*simp add*: *iT-Mult-def iMin-mono2 mono-def*)

lemma *iT-Plus-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow$ *Max* $(I \oplus k)$ = *Max* $I + k$
by (*simp add*: *iT-Plus-def Max-mono2 mono-def*)

lemma *iT-Mult-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow$ *Max* $(I \otimes k)$ = *Max* $I * k$
by (*simp add*: *iT-Mult-def Max-mono2 mono-def*)

corollary
  iMOD-mult-0: $[r,$ *mod* $m] \otimes 0 = [\ldots 0]$ **and**
  iMODb-mult-0: $[r,$ *mod* $m,\ c] \otimes 0 = [\ldots 0]$ **and**
  iFROM-mult-0: $[n\ldots] \otimes 0 = [\ldots 0]$ **and**
  iIN-mult-0: $[n\ldots,d] \otimes 0 = [\ldots 0]$ **and**
  iTILL-mult-0: $[\ldots n] \otimes 0 = [\ldots 0]$
by (*simp add*: *iT-not-empty iT-Mult-0*)+

lemmas *iT-mult-0* =
  iTILL-mult-0
  iFROM-mult-0
  iIN-mult-0
  iMOD-mult-0
  iMODb-mult-0

lemma *iT-Plus-0*: $I \oplus 0 = I$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-1*: $I \otimes$ *Suc* $0 = I$
by (*simp add*: *iT-Mult-def*)

corollary
iFROM-add-Min: *iMin* ([*n*. . .] ⊕ *k*) = *n* + *k* **and**
iIN-add-Min: *iMin* ([*n*. . .,*d*] ⊕ *k*) = *n* + *k* **and**
iTILL-add-Min: *iMin* ([. . .*n*] ⊕ *k*) = *k* **and**
iMOD-add-Min: *iMin* ([*r*, *mod m*] ⊕ *k*) = *r* + *k* **and**
iMODb-add-Min: *iMin* ([*r*, *mod m*, *c*] ⊕ *k*) = *r* + *k*
by (*simp add*: *iT-not-empty iT-Plus-Min iT-Min*)+

corollary
iFROM-mult-Min: *iMin* ([*n*. . .] ⊗ *k*) = *n* ∗ *k* **and**
iIN-mult-Min: *iMin* ([*n*. . .,*d*] ⊗ *k*) = *n* ∗ *k* **and**
iTILL-mult-Min: *iMin* ([. . .*n*] ⊗ *k*) = *0* **and**
iMOD-mult-Min: *iMin* ([*r*, *mod m*] ⊗ *k*) = *r* ∗ *k* **and**
iMODb-mult-Min: *iMin* ([*r*, *mod m*, *c*] ⊗ *k*) = *r* ∗ *k*
by (*simp add*: *iT-not-empty iT-Mult-Min iT-Min*)+

lemmas *iT-add-Min* =
iIN-add-Min
iTILL-add-Min
iFROM-add-Min
iMOD-add-Min
iMODb-add-Min

lemmas *iT-mult-Min* =
iIN-mult-Min
iTILL-mult-Min
iFROM-mult-Min
iMOD-mult-Min
iMODb-mult-Min

lemma *iFROM-add*: [*n*. . .] ⊕ *k* = [*n*+*k*. . .]
by (*simp add*: *iFROM-def iT-Plus-def image-add-atLeast*)

lemma *iIN-add*: [*n*. . .,*d*] ⊕ *k* = [*n*+*k*. . .,*d*]
by (*fastforce simp add*: *iIN-def iT-Plus-def*)

lemma *iTILL-add*: [. . .*i*] ⊕ *k* = [*k*. . .,*i*]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-add*)

lemma *iMOD-add*: [*r*, *mod m*] ⊕ *k* = [*r* + *k*, *mod m*]
apply (*clarsimp simp*: *set-eq-iff iMOD-def iT-Plus-def image-iff*)
apply (*rule iffI*)
apply (*clarsimp simp*: *mod-add*)
apply (*rule-tac x*=*x* − *k* **in** *exI*)
apply *clarsimp*
apply (*simp add*: *mod-sub-add*)
done

lemma *iMODb-add*: [*r*, *mod m*, *c*] ⊕ *k* = [*r* + *k*, *mod m*, *c*]
by (*simp add*: *iMODb-iMOD-iIN-conv iT-Plus-Int iMOD-add iIN-add*)

lemmas *iT-add* =
iMOD-add
iMODb-add
iFROM-add
iIN-add
iTILL-add
iT-Plus-singleton

lemma *iFROM-mult*: [*n*. . .] ⊗ *k* = [*n* ∗ *k*, *mod k*]
apply (*case-tac k* = *0*)
apply (*simp add*: *iMOD-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply (*drule mult-le-mono1*[*of - - k*])
apply (*rule order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*drule div-le-mono*[*of - - k*])
apply *simp*
done

lemma *iIN-mult*: [*n*. . .,*d*] ⊗ *k* = [*n* ∗ *k*, *mod k*, *d*]
apply (*case-tac k* = *0*)
apply (*simp add*: *iMODb-mod-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply (*elim conjE*)
apply (*drule mult-le-mono1*[*of - - k*], *drule mult-le-mono1*[*of - - k*])
apply (*rule conjI*)
apply (*rule order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*simp add*: *div-mult-cancel add-mult-distrib mult.commute*[*of k*])
apply (*erule conjE*)
apply (*drule div-le-mono*[*of - - k*], *drule div-le-mono*[*of - - k*])
apply *simp*
done

lemma *iTILL-mult*: [. . .*n*] ⊗ *k* = [*0*, *mod k*, *n*]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-mult*)

lemma *iMOD-mult*: [*r*, *mod m*] ⊗ *k* = [*r* ∗ *k*, *mod m* ∗ *k*]
apply (*case-tac k* = *0*)
apply (*simp add*: *iMOD-0 iT-mult-0 iIN-0 iTILL-0*)

apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*subst mult.commute*[*of m k*])
apply (*simp add*: *mod-mult2-eq*)
apply (*rule iffI*)
 apply (*elim conjE*)
 apply (*drule mult-le-mono1*[*of - - k*])
 apply (*simp add*: *div-mult-cancel*)
apply (*elim conjE*)
apply (*subgoal-tac x mod k = 0*)
 prefer *2*
 apply (*drule-tac arg-cong*[**where** *f=λx. x mod k*])
 apply (*simp add*: *mult.commute*[*of k*])
apply (*drule div-le-mono*[*of - - k*])
apply *simp*
done

lemma *iMODb-mult*:
  [*r*, *mod m*, *c*] ⊗ *k* = [*r* ∗ *k*, *mod m* ∗ *k*, *c*]
apply (*case-tac k = 0*)
 apply (*simp add*: *iMODb-mod-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*subst iMODb-iMOD-iTILL-conv*)
apply (*simp add*: *iT-Mult-Int iMOD-mult iTILL-mult iMODb-iMOD-iTILL-conv*)
apply (*subst Int-assoc*[*symmetric*])
apply (*subst Int-absorb2*)
 apply (*simp add*: *iMOD-subset*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv add-mult-distrib2*)
done

lemmas *iT-mult* =
  iTILL-mult
  iFROM-mult
  iIN-mult
  iMOD-mult
  iMODb-mult
  iT-Mult-singleton

2.1.2 Some conversions between intervals using constant addition and multiplication

lemma *iFROM-conv*: [*n*. . .] = *UNIV* ⊕ *n*
by (*simp add*: *iFROM-0*[*symmetric*] *iFROM-add*)

lemma *iIN-conv*: [*n*. . .,*d*] = [. . .*d*] ⊕ *n*
by (*simp add*: *iTILL-add*)

lemma *iMOD-conv*: [*r*, *mod m*] = [*0*. . .] ⊗ *m* ⊕ *r*
apply (*case-tac m = 0*)
 apply (*simp add*: *iMOD-0 iT-mult-0 iTILL-add*)
apply (*simp add*: *iFROM-mult iMOD-add*)

done

lemma *iMODb-conv*: $[r,\ mod\ m,\ c] = [\ldots c] \otimes m \oplus r$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMODb-mod-0 iT-mult-0 iTILL-add*)
apply (*simp add*: *iTILL-mult iMODb-add*)
done

Some examples showing the utility of iMODb_conv

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp add*: *iT-defs*)
apply *safe*
defer *1*
apply *simp+*

The direct proof without iMODb_conv fails

oops

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def*)
apply *safe*
apply *simp+*
done

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def*)
apply (*simp add*: *atMost-def*)
apply *safe*
apply *simp+*
done

lemma $[r,\ mod\ m,\ 4] = \{r,\ r+m,\ r+2*m,\ r+3*m,\ r+4*m\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def atMost-def*)
apply (*simp add*: *image-Collect*)
apply *safe*
apply *fastforce+*
done

lemma $[2,\ mod\ 10,\ 4] = \{2,\ 12,\ 22,\ 32,\ 42\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Plus-def iT-Mult-def*)
apply *fastforce*
done

2.1.3 Subtraction of constants

definition *iT-Plus-neg* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ‹⊕−› *55*) **where**

$I \oplus- k \equiv \{x.\ x + k \in I\}$

lemma *iT-Plus-neg-mem-iff*: $(x \in I \oplus- k) = (x + k \in I)$
by (*simp add*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-mem-iff2*: $k \le x \Longrightarrow (x - k \in I \oplus- k) = (x \in I)$
by (*simp add*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-image-conv*: $I \oplus- k = (\lambda n.(n - k))$ ‘ $(I \downarrow\ge k)$
apply (*simp add*: *iT-Plus-neg-def cut-ge-def*, *safe*)
apply (*rule-tac x=x+k* **in** *image-eqI*)
apply *simp+*
done

lemma *iT-Plus-neg-cut-eq*: $t \le k \Longrightarrow (I \downarrow\ge t) \oplus- k = I \oplus- k$
by (*simp add*: *set-eq-iff iT-Plus-neg-mem-iff cut-ge-mem-iff*)

lemma *iT-Plus-neg-mono*: $A \subseteq B \Longrightarrow A \oplus- k \subseteq B \oplus- k$
by (*simp add*: *iT-Plus-neg-def subset-iff*)

lemma *iT-Plus-neg-empty*: $\{\} \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-def*)
lemma *iT-Plus-neg-Max-less-empty*:
$[\![$ *finite* I; *Max* $I < k$ $]\!] \Longrightarrow I \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-Max-empty*)

lemma *iT-Plus-neg-not-empty-iff*: $(I \oplus- k \ne \{\}) = (\exists x{\in}I.\ k \le x)$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-not-empty-iff*)

lemma *iT-Plus-neg-empty-iff*:
$(I \oplus- k = \{\}) = (I = \{\} \lor ($*finite* $I \land$ *Max* $I < k))$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-neg-empty*)
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*case-tac infinite I*)
 apply (*simp add*: *nat-cut-ge-infinite-not-empty*)
apply (*simp add*: *cut-ge-empty-iff*)
done

lemma *iT-Plus-neg-assoc*: $(I \oplus- a) \oplus- b = I \oplus- (a + b)$
apply (*simp add*: *iT-Plus-neg-def*)
apply (*simp add*: *add.assoc add.commute*[*of b*])
done

lemma *iT-Plus-neg-commute*: $I \oplus- a \oplus- b = I \oplus- b \oplus- a$
by (*simp add*: *iT-Plus-neg-assoc add.commute*[*of b*])

lemma *iT-Plus-neg-0*: $I \oplus- 0 = I$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-0-all*)

lemma *iT-Plus-Plus-neg-assoc*: $b \le a \Longrightarrow I \oplus a \oplus- b = I \oplus (a - b)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
 apply *fastforce*
apply (*rule-tac x=x + b* **in** *exI*)
apply (*simp add*: *le-diff-conv*)
done

lemma *iT-Plus-Plus-neg-assoc2*: $a \le b \Longrightarrow I \oplus a \oplus- b = I \oplus- (b - a)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
 apply *fastforce*
apply (*clarify*, *rename-tac x'*)
apply (*rule-tac x=x' + a* **in** *exI*)
apply *simp*
done

lemma *iT-Plus-neg-Plus-le-cut-eq*:
  $a \le b \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\ge a) \oplus (b - a)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
 apply (*clarify*, *rename-tac x'*)
 apply (*subgoal-tac x' = x + a - b*)
  prefer *2*
  apply *simp*
 apply (*simp add*: *le-imp-diff-le le-add-diff*)
apply *fastforce*
done

corollary *iT-Plus-neg-Plus-le-Min-eq*:
  $[\![a \le b; a \le iMin\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus (b - a)$
by (*simp add*: *iT-Plus-neg-Plus-le-cut-eq cut-ge-Min-all*)

lemma *iT-Plus-neg-Plus-ge-cut-eq*:
  $b \le a \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\ge a) \oplus- (a - b)$
apply (*simp add*: *iT-Plus-neg-image-conv iT-Plus-def cut-cut-ge max-eqL*)
apply (*subst image-comp*)
apply (*rule image-cong*, *simp*)
apply (*simp add*: *cut-ge-mem-iff*)
done

corollary *iT-Plus-neg-Plus-ge-Min-eq*:
  $[\![b \le a; a \le iMin\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus- (a - b)$
by (*simp add*: *iT-Plus-neg-Plus-ge-cut-eq cut-ge-Min-all*)

lemma *iT-Plus-neg-Mult-distrib*:
$0 < m \Longrightarrow I \oplus- n \otimes m = I \otimes m \oplus- n * m$
apply (*clarsimp simp*: *set-eq-iff iT-Plus-neg-image-conv image-iff iT-Plus-def iT-Mult-def Bex-def cut-ge-mem-iff*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac* x')
apply (*rule-tac* $x{=}x' * m$ **in** *exI*)
apply (*simp add*: *diff-mult-distrib*)
apply (*clarsimp*, *rename-tac* x')
apply (*rule-tac* $x{=}x' - n$ **in** *exI*)
apply (*simp add*: *diff-mult-distrib*)
apply *fastforce*
done

lemma *iT-Plus-neg-Plus-le-inverse*: $k \le iMin\ I \Longrightarrow I \oplus- k \oplus k = I$
by (*simp add*: *iT-Plus-neg-Plus-le-Min-eq iT-Plus-0*)

lemma *iT-Plus-neg-Plus-inverse*: $I \oplus- k \oplus k = I \downarrow\ge k$
by (*simp add*: *iT-Plus-neg-Plus-ge-cut-eq iT-Plus-neg-0*)

lemma *iT-Plus-Plus-neg-inverse*: $I \oplus k \oplus- k = I$
by (*simp add*: *iT-Plus-Plus-neg-assoc iT-Plus-0*)

lemma *iT-Plus-neg-Un*: $(A \cup B) \oplus- k = (A \oplus- k) \cup (B \oplus- k)$
by (*fastforce simp*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-Int*: $(A \cap B) \oplus- k = (A \oplus- k) \cap (B \oplus- k)$
by (*fastforce simp*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-Max-singleton*: $[\![$ *finite* I; $I \ne \{\}$ $]\!] \Longrightarrow I \oplus- Max\ I = \{0\}$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Plus-neg-def*)
apply (*case-tac* $x = 0$)
apply *simp*
apply *fastforce*
done

lemma *iT-Plus-neg-singleton*: $\{a\} \oplus- k = ($*if* $k \le a$ *then* $\{a - k\}$ *else* $\{\})$
by (*force simp add*: *set-eq-iff iT-Plus-neg-mem-iff singleton-iff*)

corollary *iT-Plus-neg-singleton1*: $k \le a \Longrightarrow \{a\} \oplus- k = \{a{-}k\}$
by (*simp add*: *iT-Plus-neg-singleton*)

corollary *iT-Plus-neg-singleton2*: $a < k \Longrightarrow \{a\} \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-singleton*)

lemma *iT-Plus-neg-finite-iff*: *finite* $(I \oplus- k) =$ *finite* I
apply (*simp add*: *iT-Plus-neg-image-conv*)

apply (*subst inj-on-finite-image-iff*)
apply (*metis cut-geE inj-on-diff-nat*)
apply (*simp add*: *nat-cut-ge-finite-iff*)
done

lemma *iT-Plus-neg-Min*:
$I \oplus- k \neq \{\} \Longrightarrow iMin\ (I \oplus- k) = iMin\ (I \downarrow\geq k) - k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *iMin-mono2 monoI*)
done

lemma *iT-Plus-neg-Max*:
$[\![\ finite\ I;\ I \oplus- k \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oplus- k) = Max\ I - k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *Max-mono2 monoI cut-ge-finite cut-ge-Max-eq*)
done

Subtractions of constants from intervals

lemma *iFROM-add-neg*: $[n\ldots] \oplus- k = [n - k\ldots]$
by (*fastforce simp*: *set-eq-iff iT-Plus-neg-mem-iff*)

lemma *iTILL-add-neg*: $[\ldots n] \oplus- k = (if\ k \leq n\ then\ [\ldots n - k]\ else\ \{\})$
by (*force simp add*: *set-eq-iff iT-Plus-neg-mem-iff iT-iff*)
lemma *iTILL-add-neg1*: $k \leq n \Longrightarrow [\ldots n] \oplus- k = [\ldots n{-}k]$
by (*simp add*: *iTILL-add-neg*)
lemma *iTILL-add-neg2*: $n < k \Longrightarrow [\ldots n] \oplus- k = \{\}$
by (*simp add*: *iTILL-add-neg*)

lemma *iIN-add-neg*:
$[n\ldots,d] \oplus- k = ($
$if\ k \leq n\ then\ [n - k\ldots,d]$
$else\ if\ k \leq n + d\ then\ [\ldots n + d - k]\ else\ \{\})$
by (*simp add*: *iIN-iFROM-iTILL-conv iT-Plus-neg-Int iFROM-add-neg iTILL-add-neg iFROM-0*)

lemma *iIN-add-neg1*: $k \leq n \Longrightarrow [n\ldots,d] \oplus- k = [n - k\ldots,d]$
by (*simp add*: *iIN-add-neg*)

lemma *iIN-add-neg2*: $[\![\ n \leq k;\ k \leq n + d\ ]\!] \Longrightarrow [n\ldots,d] \oplus- k = [\ldots n + d - k]$
by (*simp add*: *iIN-add-neg iIN-0-iTILL-conv*)

lemma *iIN-add-neg3*: $n + d < k \Longrightarrow [n\ldots,d] \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-Max-less-empty iT-finite iT-Max*)

lemma *iMOD-0-add-neg*: $[r,\ mod\ 0] \oplus- k = \{r\} \oplus- k$
by (*simp add*: *iMOD-0 iIN-0*)

lemma *iMOD-gr0-add-neg*:
  $0 < m \Longrightarrow$
  $[r, mod\ m] \oplus- k = ($
    $if\ k \le r\ then\ [r - k, mod\ m]$
    $else\ [(m + r\ mod\ m - k\ mod\ m)\ mod\ m, mod\ m])$
apply (*rule set-eqI*)
apply (*simp add*: *iMOD-def iT-Plus-neg-def*)
apply (*simp add*: *eq-sym-conv*[*of* - *r mod m*])
apply (*intro conjI impI*)
apply (*simp add*: *eq-sym-conv*[*of* - (*r* − *k*) *mod m*] *mod-sub-add le-diff-conv*)
apply (*simp add*: *eq-commute*[*of r mod m*] *mod-add-eq-mod-conv*)
apply *safe*
apply (*drule sym*)
apply *simp*
done

lemma *iMOD-add-neg*:
  $[r, mod\ m] \oplus- k = ($
    $if\ k \le r\ then\ [r - k, mod\ m]$
    $else\ if\ 0 < m\ then\ [(m + r\ mod\ m - k\ mod\ m)\ mod\ m, mod\ m]\ else\ \{\})$
apply (*case-tac* $0 < m$)
apply (*simp add*: *iMOD-gr0-add-neg*)
apply (*simp add*: *iMOD-0 iIN-0 iT-Plus-neg-singleton*)
done

corollary *iMOD-add-neg1*:
  $k \le r \Longrightarrow [r, mod\ m] \oplus- k = [r - k, mod\ m]$
by (*simp add*: *iMOD-add-neg*)

lemma *iMOD-add-neg2*:
  $[\![\ 0 < m; r < k\ ]\!] \Longrightarrow [r, mod\ m] \oplus- k = [(m + r\ mod\ m - k\ mod\ m)\ mod\ m,$
$mod\ m]$
by (*simp add*: *iMOD-add-neg*)

lemma *iMODb-mod-0-add-neg*: $[r, mod\ 0, c] \oplus- k = \{r\} \oplus- k$
by (*simp add*: *iMODb-mod-0 iIN-0*)

lemma *iMODb-add-neg*:
  $[r, mod\ m, c] \oplus- k = ($
    $if\ k \le r\ then\ [r - k, mod\ m, c]$
    $else$
      $if\ k \le r + m * c\ then$
      $[(m + r\ mod\ m - k\ mod\ m)\ mod\ m, mod\ m, (r + m * c - k)\ div\ m]$
      $else\ \{\})$
apply (*clarsimp simp add*: *iMODb-iMOD-iIN-conv iT-Plus-neg-Int iMOD-add-neg*
iIN-add-neg)

apply (*simp add*: *iMOD-iIN-iMODb-conv*)
apply (*rule-tac t*=(m + r *mod* m − k *mod* m) *mod* m **and** *s*=(r + m ∗ c − k) *mod* m **in** *subst*)
apply (*simp add*: *mod-diff1-eq*[*of k*])
apply (*subst iMOD-iTILL-iMODb-conv*, *simp*)
apply (*subst sub-mod-div-eq-div*, *simp*)
done

lemma *iMODb-add-neg′*:
[*r*, *mod m*, *c*] ⊕− *k* = (
if k ≤ *r then* [*r* − *k*, *mod m*, *c*]
else if k ≤ *r* + *m* ∗ *c then*
if k mod m ≤ *r mod m*
then [*r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *k div m*]
else [*m* + *r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *Suc* (*k div m*)]
else {})
apply (*clarsimp simp add*: *iMODb-add-neg*)
apply (*case-tac m* = *0*, *simp+*)
apply (*case-tac k mod m* ≤ *r mod m*)
apply (*clarsimp simp*: *linorder-not-le*)
apply (*simp add*: *divisor-add-diff-mod-if*)
apply (*simp add*: *div-diff1-eq-if*)
apply (*clarsimp simp*: *linorder-not-le*)
apply (*simp add*: *div-diff1-eq-if*)
done

corollary *iMODb-add-neg1*:
k ≤ *r* ⟹ [*r*, *mod m*, *c*] ⊕− *k* = [*r* − *k*, *mod m*, *c*]
by (*simp add*: *iMODb-add-neg*)

corollary *iMODb-add-neg2*:
⟦ *r* < *k*; *k* ≤ *r* + *m* ∗ *c* ⟧ ⟹
[*r*, *mod m*, *c*] ⊕− *k* =
[(*m* + *r mod m* − *k mod m*) *mod m*, *mod m*, (*r* + *m* ∗ *c* − *k*) *div m*]
by (*simp add*: *iMODb-add-neg*)

corollary *iMODb-add-neg2-mod-le*:
⟦ *r* < *k*; *k* ≤ *r* + *m* ∗ *c*; *k mod m* ≤ *r mod m* ⟧ ⟹
[*r*, *mod m*, *c*] ⊕− *k* =
[*r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *k div m*]
by (*simp add*: *iMODb-add-neg′*)

corollary *iMODb-add-neg2-mod-less*:
⟦ *r* < *k*; *k* ≤ *r* + *m* ∗ *c*; *r mod m* < *k mod m*⟧ ⟹
[*r*, *mod m*, *c*] ⊕− *k* =
[*m* + *r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *Suc* (*k div m*)]
by (*simp add*: *iMODb-add-neg′*)

lemma *iMODb-add-neg3*: *r* + *m* ∗ *c* < *k* ⟹ [*r*, *mod m*, *c*] ⊕− *k* = {}

by (*simp add*: *iMODb-add-neg*)

lemmas *iT-add-neg* =
iFROM-add-neg
iIN-add-neg
iTILL-add-neg
iMOD-add-neg
iMODb-add-neg
iT-Plus-neg-singleton

2.1.4 Subtraction of intervals from constants

definition *iT-Minus* :: *Time* ⇒ *iT* ⇒ *iT* (**infixl** ‹⊖› *55*)
where $k \ominus I \equiv \{x.\ x \le k \wedge (k - x) \in I\}$

lemma *iT-Minus-mem-iff*: $(x \in k \ominus I) = (x \le k \wedge k - x \in I)$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-mono*: $A \subseteq B \Longrightarrow k \ominus A \subseteq k \ominus B$
by (*simp add*: *subset-iff iT-Minus-mem-iff*)

lemma *iT-Minus-image-conv*: $k \ominus I = (\lambda x.\ k - x)$ ‘ $(I \downarrow\le k)$
by (*fastforce simp*: *iT-Minus-def cut-le-def image-iff*)

lemma *iT-Minus-cut-eq*: $k \le t \Longrightarrow k \ominus (I \downarrow\le t) = k \ominus I$
by (*fastforce simp*: *set-eq-iff iT-Minus-mem-iff*)

lemma *iT-Minus-Minus-cut-eq*: $k \ominus (k \ominus (I \downarrow\le k)) = I \downarrow\le k$
by (*fastforce simp*: *iT-Minus-def*)

lemma $10 \ominus [\ldots 3] = [7\ldots,3]$
by (*fastforce simp*: *iT-Minus-def*)

lemma *iT-Minus-empty*: $k \ominus \{\} = \{\}$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-0*: $k \ominus \{0\} = \{k\}$
by (*simp add*: *iT-Minus-image-conv cut-le-def image-Collect*)

lemma *iT-Minus-bound*: $x \in k \ominus I \Longrightarrow x \le k$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-finite*: *finite* $(k \ominus I)$
apply (*rule finite-nat-iff-bounded-le2*[*THEN iffD2*])
apply (*rule-tac x=k* **in** *exI*)
apply (*simp add*: *iT-Minus-bound*)
done

lemma *iT-Minus-less-Min-empty*: $k < iMin\ I \Longrightarrow k \ominus I = \{\}$

by (*simp add*: *iT-Minus-image-conv cut-le-Min-empty*)

lemma *iT-Minus-Min-singleton*: $I \neq \{\} \Longrightarrow (iMin\ I) \ominus I = \{0\}$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-mem-iff*)
apply (*fastforce intro*: *iMinI-ex2*)
done

lemma *iT-Minus-empty-iff*: $(k \ominus I = \{\}) = (I = \{\} \vee k < iMin\ I)$
apply (*case-tac* $I = \{\}$, *simp add*: *iT-Minus-empty*)
apply (*simp add*: *iT-Minus-image-conv cut-le-empty-iff iMin-gr-iff*)
done

lemma *iT-Minus-imirror-conv*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ I + Max\ (I \downarrow\leq k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty cut-le-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*case-tac* $k < iMin\ I$)
apply (*simp add*: *iT-Minus-less-Min-empty cut-le-Min-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*simp add*: *linorder-not-less*)
apply (*frule cut-le-Min-not-empty*[*of - k*], *assumption*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-neg-image-conv iT-Plus-neg-mem-iff iT-Plus-mem-iff imirror-iff image-iff Bex-def i-cut-mem-iff cut-le-Min-eq*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac* x')
apply (*rule-tac* $x=k - x' + iMin\ I + Max\ (I \downarrow\leq k)$ **in** *exI*, *simp*)
apply (*simp add*: *add.assoc le-add-diff*)
apply (*simp add*: *add.commute*[*of k*] *le-add-diff nat-cut-le-finite cut-leI trans-le-add2*)
apply (*rule-tac* $x=x'$ **in** *exI*, *simp*)
apply (*clarsimp*, *rename-tac x1 x2*)
apply (*rule-tac* $x=x2$ **in** *exI*)
apply *simp*
apply (*drule add-right-cancel*[*THEN iffD2*, *of - - k*], *simp*)
apply (*simp add*: *trans-le-add2 nat-cut-le-finite cut-le-mem-iff*)
done

lemma *iT-Minus-imirror-conv'*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ (I \downarrow\leq k) + Max\ (I \downarrow\leq k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty cut-le-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*case-tac* $k < iMin\ I$)
apply (*simp add*: *iT-Minus-less-Min-empty cut-le-Min-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*simp add*: *cut-le-Min-not-empty cut-le-Min-eq iT-Minus-imirror-conv*)
done

lemma *iT-Minus-Max*:
 $[\![I \neq \{\}; iMin\ I \le k]\!] \Longrightarrow Max\ (k \ominus I) = k - (iMin\ I)$
apply (*rule Max-equality*)
 apply (*simp add*: *iT-Minus-mem-iff iMinI-ex2*)
 apply (*simp add*: *iT-Minus-finite*)
apply (*fastforce simp*: *iT-Minus-def*)
done

lemma *iT-Minus-Min*:
 $[\![I \neq \{\}; iMin\ I \le k]\!] \Longrightarrow iMin\ (k \ominus I) = k - (Max\ (I \downarrow\le k))$
apply (*insert nat-cut-le-finite*[*of I k*])
apply (*frule cut-le-Min-not-empty*[*of - k*], *assumption*)
apply (*rule iMin-equality*)
 apply (*simp add*: *iT-Minus-mem-iff nat-cut-le-Max-le del*: *Max-le-iff*)
 apply (*simp add*: *subsetD*[*OF cut-le-subset*, *OF Max-in*])
apply (*clarsimp simp add*: *iT-Minus-image-conv image-iff*, *rename-tac x'*)
apply (*rule diff-le-mono2*)
apply (*simp add*: *Max-ge-iff cut-le-mem-iff*)
done

lemma *iT-Minus-Minus-eq*: $[\![finite\ I; Max\ I \le k]\!] \Longrightarrow k \ominus (k \ominus I) = I$
apply (*simp add*: *iT-Minus-cut-eq*[*of k k I*, *symmetric*] *iT-Minus-Minus-cut-eq*)
apply (*simp add*: *cut-le-Max-all*)
done

lemma *iT-Minus-Minus-eq2*: $I \subseteq [\ldots k] \Longrightarrow k \ominus (k \ominus I) = I$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Minus-empty*)
apply (*rule iT-Minus-Minus-eq*)
 apply (*simp add*: *finite-subset iTILL-finite*)
apply (*frule Max-subset*)
apply (*simp add*: *iTILL-finite iTILL-Max*)+
done

lemma *iT-Minus-Minus*: $a \ominus (b \ominus I) = (I \downarrow\le b) \oplus a \oplus- b$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-image-conv iT-Plus-neg-image-conv image-iff Bex-def i-cut-mem-iff*)
apply *fastforce*
done

lemma *iT-Minus-Plus-empty*: $k < n \Longrightarrow k \ominus (I \oplus n) = \{\}$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-empty iT-Minus-empty*)
apply (*simp add*: *iT-Minus-empty-iff iT-Plus-empty-iff iT-Plus-Min*)
done
lemma *iT-Minus-Plus-commute*: $n \le k \Longrightarrow k \ominus (I \oplus n) = (k - n) \ominus I$

apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-image-conv image-iff Bex-def i-cut-mem-iff*)
apply *fastforce*
done

lemma *iT-Minus-Plus-cut-assoc*: $(k \ominus I) \oplus n = (k + n) \ominus (I \downarrow\leq k)$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Plus-mem-iff iT-Minus-mem-iff cut-le-mem-iff*)
apply *fastforce*
done

lemma *iT-Minus-Plus-assoc*:
$[\![$ *finite I*; *Max I* $\leq k$ $]\!] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
by (*insert iT-Minus-Plus-cut-assoc*[*of k I n*], *simp add*: *cut-le-Max-all*)
lemma *iT-Minus-Plus-assoc2*:
$I \subseteq [\ldots k] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty iT-Plus-empty*)
apply (*rule iT-Minus-Plus-assoc*)
apply (*simp add*: *finite-subset iTILL-finite*)
apply (*frule Max-subset*)
apply (*simp add*: *iTILL-finite iTILL-Max*)+
done

lemma *iT-Minus-Un*: $k \ominus (A \cup B) = (k \ominus A) \cup (k \ominus B)$
by (*fastforce simp*: *iT-Minus-def*)

lemma *iT-Minus-Int*: $k \ominus (A \cap B) = (k \ominus A) \cap (k \ominus B)$
by (*fastforce simp*: *set-eq-iff iT-Minus-mem-iff*)

lemma *iT-Minus-singleton*: $k \ominus \{a\} = ($*if* $a \leq k$ *then* $\{k - a\}$ *else* $\{\})$
by (*simp add*: *iT-Minus-image-conv cut-le-singleton*)
corollary *iT-Minus-singleton1*: $a \leq k \Longrightarrow k \ominus \{a\} = \{k{-}a\}$
by (*simp add*: *iT-Minus-singleton*)
corollary *iT-Minus-singleton2*: $k < a \Longrightarrow k \ominus \{a\} = \{\}$
by (*simp add*: *iT-Minus-singleton*)

lemma *iMOD-sub*:
$k \ominus [r,$ *mod* $m] =$
$($*if* $r \leq k$ *then* $[(k - r)$ *mod* $m,$ *mod* $m, (k - r)$ *div* $m]$ *else* $\{\})$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-mem-iff iT-iff*)
apply (*fastforce simp add*: *mod-sub-eq-mod-swap*[*of r*, *symmetric*])
done

corollary *iMOD-sub1*:
$r \leq k \Longrightarrow k \ominus [r,$ *mod* $m] = [(k - r)$ *mod* $m,$ *mod* $m, (k - r)$ *div* $m]$

by (*simp add*: *iMOD-sub*)

corollary *iMOD-sub2*: $k < r \Longrightarrow k \ominus [r,\ mod\ m] = \{\}$
apply (*rule iT-Minus-less-Min-empty*)
apply (*simp add*: *iMOD-Min*)
done

lemma *iTILL-sub*: $k \ominus [\ldots n] = (if\ n \leq k\ then\ [k - n\ldots,n]\ else\ [\ldots k])$
by (*force simp add*: *set-eq-iff iT-Minus-mem-iff iT-iff*)

corollary *iTILL-sub1*: $n \leq k \Longrightarrow k \ominus [\ldots n] = [k - n\ldots,n]$
by (*simp add*: *iTILL-sub*)

corollary *iTILL-sub2*: $k \leq n \Longrightarrow k \ominus [\ldots n] = [\ldots k]$
by (*simp add*: *iTILL-sub iIN-0-iTILL-conv*)

lemma *iMODb-sub*:
  $k \ominus [r,\ mod\ m,\ c] = ($
    $if\ r + m * c \leq k\ then\ [\ k - r - m * c,\ mod\ m,\ c]\ else$
      $if\ r \leq k\ then\ [\ (k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]\ else\ \{\})$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMODb-mod-0 iIN-0 iT-Minus-singleton*)
apply (*subst iMODb-iMOD-iTILL-conv*)
apply (*subst iT-Minus-Int*)
apply (*simp add*: *iMOD-sub iTILL-sub*)
apply (*intro conjI impI*)
 apply *simp*
 apply (*subgoal-tac* $(k - r)\ mod\ m \leq k - (r + m * c)$)
  prefer *2*
  apply (*subgoal-tac* $m * c \leq k - r - (k - r)\ mod\ m$)
   prefer *2*
   apply (*drule add-le-imp-le-diff2*)
   apply (*drule div-le-mono*[*of - - m*], *simp*)
   apply (*drule mult-le-mono2*[*of - - m*])
   apply (*simp add*: *minus-mod-eq-mult-div* [*symmetric*])
  apply (*simp add*: *le-diff-conv2*[*OF mod-le-dividend*] *del*: *diff-diff-left*)
 apply (*subst iMODb-iMOD-iIN-conv*)
 apply (*simp add*: *Int-assoc minus-mod-eq-mult-div* [*symmetric*])
 apply (*subst iIN-inter*, *simp*+)
 apply (*rule set-eqI*)
 apply (*fastforce simp add*: *iT-iff mod-diff-mult-self2 diff-diff-left*[*symmetric*] *simp del*: *diff-diff-left*)
apply (*simp add*: *Int-absorb2 iMODb-iTILL-subset*)
done

corollary *iMODb-sub1*:
  $[\![\ r \leq k;\ k \leq r + m * c\ ]\!] \Longrightarrow$

$k \ominus [r, mod\ m, c] = [(k - r)\ mod\ m, mod\ m, (k - r)\ div\ m]$
by (*clarsimp simp*: *iMODb-sub iMODb-mod-0*)

corollary *iMODb-sub2*: $k < r \Longrightarrow k \ominus [r, mod\ m, c] = \{\}$
apply (*rule iT-Minus-less-Min-empty*)
apply (*simp add*: *iMODb-Min*)
done

corollary *iMODb-sub3*:
$r + m * c \le k \Longrightarrow k \ominus [r, mod\ m, c] = [\ k - r - m * c, mod\ m, c]$
by (*simp add*: *iMODb-sub*)

lemma *iFROM-sub*: $k \ominus [n\ldots] = (if\ n \le k\ then\ [\ldots k - n]\ else\ \{\})$
by (*simp add*: *iMOD-1*[*symmetric*] *iMOD-sub iMODb-mod-1 iIN-0-iTILL-conv*)

corollary *iFROM-sub1*: $n \le k \Longrightarrow k \ominus [n\ldots] = [\ldots k{-}n]$
by (*simp add*: *iFROM-sub*)

corollary *iFROM-sub-empty*: $k < n \Longrightarrow k \ominus [n\ldots] = \{\}$
by (*simp add*: *iFROM-sub*)

lemma *iIN-sub*:
$k \ominus [n\ldots,d] = ($
$if\ n + d \le k\ then\ [k - (n + d)\ldots,d]$
$else\ if\ n \le k\ then\ [\ldots k - n]\ else\ \{\})$
apply (*simp add*: *iMODb-mod-1*[*symmetric*] *iMODb-sub*)
apply (*simp add*: *iMODb-mod-1 iIN-0-iTILL-conv*)
done

lemma *iIN-sub1*: $n + d \le k \Longrightarrow k \ominus [n\ldots,d] = [k - (n + d)\ldots,d]$
by (*simp add*: *iIN-sub*)

lemma *iIN-sub2*: $[\![\ n \le k; k \le n + d\ ]\!] \Longrightarrow k \ominus [n\ldots,d] = [\ldots k - n]$
by (*clarsimp simp*: *iIN-sub iIN-0-iTILL-conv*)

lemma *iIN-sub3*: $k < n \Longrightarrow k \ominus [n\ldots,d] = \{\}$
by (*simp add*: *iIN-sub*)

lemmas *iT-sub* =
iFROM-sub
iIN-sub
iTILL-sub
iMOD-sub
iMODb-sub
iT-Minus-singleton

2.1.5 Division of intervals by constants

Monotonicity and injectivity of artithmetic operators

lemma *iMOD-div-right-strict-mono-on*:
  ⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*]
apply (*rule div-right-strict-mono-on*, *assumption*)
apply (*clarsimp simp*: *iT-iff*)
apply (*drule-tac s*=*y mod m* **in** *sym*, *simp*)
apply (*rule-tac y*=*x* + *m* **in** *order-trans*, *simp*)
apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done

corollary *iMOD-div-right-inj-on*:
  ⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]
by (*rule strict-mono-on-imp-inj-on*[*OF iMOD-div-right-strict-mono-on*])

lemma *iMOD-mult-div-right-inj-on*:
  inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*)]
apply (*case-tac k* ∗ *m* = *0*)
 apply (*simp del*: *mult-is-0 mult-eq-0-iff add*: *iMOD-0 iIN-0*)
apply (*simp add*: *iMOD-div-right-inj-on*)
done

lemma *iMOD-mult-div-right-inj-on2*:
  m mod k = *0* ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]
  by (*auto simp add*: *iMOD-mult-div-right-inj-on*)

lemma *iMODb-div-right-strict-mono-on*:
  ⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
by (*rule strict-mono-on-subset*[*OF iMOD-div-right-strict-mono-on iMODb-iMOD-subset-same*])

corollary *iMODb-div-right-inj-on*:
  ⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
by (*rule strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*])

lemma *iMODb-mult-div-right-inj-on*:
  inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*), *c*]
by (*rule subset-inj-on*[*OF iMOD-mult-div-right-inj-on iMODb-iMOD-subset-same*])

corollary *iMODb-mult-div-right-inj-on2*:
  m mod k = *0* ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
  by (*auto simp add*: *iMODb-mult-div-right-inj-on*)

definition *iT-Div* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ‹⊘› *55*)
  where *I* ⊘ *k* ≡ (λ*n*.(*n div k*)) ‘ *I*

lemma *iT-Div-image-conv*: *I* ⊘ *k* = (λ*n*.(*n div k*)) ‘ *I*
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-mono*: $A \subseteq B \Longrightarrow A \oslash k \subseteq B \oslash k$
by (*simp add*: *iT-Div-def image-mono*)

lemma *iT-Div-empty*: $\{\} \oslash k = \{\}$
by (*simp add*: *iT-Div-def*)
lemma *iT-Div-not-empty*: $I \neq \{\} \Longrightarrow I \oslash k \neq \{\}$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-empty-iff*: $(I \oslash k = \{\}) = (I = \{\})$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-0*: $I \neq \{\} \Longrightarrow I \oslash 0 = [\ldots 0]$
by (*force simp*: *iT-Div-def*)
corollary *iT-Div-0-if*: $I \oslash 0 = (\mathit{if}\ I = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ [\ldots 0])$
by (*force simp*: *iT-Div-def*)
corollary
iFROM-div-0: $[n\ldots] \oslash 0 = [\ldots 0]$ **and**
iTILL-div-0: $[\ldots n] \oslash 0 = [\ldots 0]$ **and**
iIN-div-0: $[n\ldots,d] \oslash 0 = [\ldots 0]$ **and**
iMOD-div-0: $[r,\ \mathit{mod}\ m] \oslash 0 = [\ldots 0]$ **and**
iMODb-div-0: $[r,\ \mathit{mod}\ m,\ c] \oslash 0 = [\ldots 0]$
by (*simp add*: *iT-Div-0 iT-not-empty*)+

lemmas *iT-div-0* =
iTILL-div-0
iFROM-div-0
iIN-div-0
iMOD-div-0
iMODb-div-0

lemma *iT-Div-1*: $I \oslash \mathit{Suc}\ 0 = I$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-mem-iff-0*: $x \in (I \oslash 0) = (I \neq \{\} \wedge x = 0)$
by (*force simp*: *iT-Div-0-if*)

lemma *iT-Div-mem-iff*:
$0 < k \Longrightarrow x \in (I \oslash k) = (\exists y \in I.\ y\ \mathit{div}\ k = x)$
by (*force simp*: *iT-Div-def*)

lemma *iT-Div-mem-iff2*:
$0 < k \Longrightarrow x\ \mathit{div}\ k \in (I \oslash k) = (\exists y \in I.\ y\ \mathit{div}\ k = x\ \mathit{div}\ k)$
by (*rule iT-Div-mem-iff*)

lemma *iT-Div-mem-iff-Int*:
$0 < k \Longrightarrow x \in (I \oslash k) = (I \cap [x * k\ldots,k - \mathit{Suc}\ 0] \neq \{\})$
apply (*simp add*: *ex-in-conv*[*symmetric*] *iT-Div-mem-iff iT-iff*)

apply (*simp add*: *le-less-div-conv*[*symmetric*] *add.commute*[*of k*])
apply (*subst less-eq-le-pred*, *simp*)
apply *blast*
done

lemma *iT-Div-imp-mem*:
$0 < k \Longrightarrow x \in I \Longrightarrow x\ div\ k \in (I \oslash k)$
by (*force simp*: *iT-Div-mem-iff2*)

lemma *iT-Div-singleton*: $\{a\} \oslash k = \{a\ div\ k\}$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-Un*: $(A \cup B) \oslash k = (A \oslash k) \cup (B \oslash k)$
by (*fastforce simp*: *iT-Div-def*)

lemma *iT-Div-insert*: $(insert\ n\ I) \oslash k = insert\ (n\ div\ k)\ (I \oslash k)$
by (*fastforce simp*: *iT-Div-def*)

lemma *not-iT-Div-Int*: $\neg\ (\forall\ k\ A\ B.\ (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k))$
apply *simp*
apply (
rule-tac x=3 **in** *exI*,
rule-tac x={*0*} **in** *exI*,
rule-tac x={*1*} **in** *exI*)
by (*simp add*: *iT-Div-def*)

lemma *subset-iT-Div-Int*: $A \subseteq B \Longrightarrow (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k)$
by (*simp add*: *iT-Div-def subset-image-Int*)

lemma *iFROM-iT-Div-Int*:
$[\![\ 0 < k;\ n \le iMin\ A\ ]\!] \Longrightarrow (A \cap [n\ldots]) \oslash k = (A \oslash k) \cap ([n\ldots] \oslash k)$
apply (*rule subset-iT-Div-Int*)
apply (*blast intro*: *order-trans iMin-le*)
done

lemma *iIN-iT-Div-Int*:
$[\![\ 0 < k;\ n \le iMin\ A;\ \forall x{\in}A.\ x\ div\ k \le (n + d)\ div\ k \longrightarrow x \le n + d\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
 apply *blast*
apply (*clarsimp*, *rename-tac x1 x2*)
apply (*frule iMin-le*)
apply (*rule-tac x=x1* **in** *exI*, *simp*)
apply (*drule-tac x=x1* **in** *bspec*, *simp*)
apply (*drule div-le-mono*[*of - n + d k*])

apply *simp*
done
corollary *iTILL-iT-Div-Int*:
$[\![0 < k; \forall x \in A.\ x\ div\ k \le n\ div\ k \longrightarrow x \le n]\!] \Longrightarrow$
$(A \cap [\ldots n]) \oslash k = (A \oslash k) \cap ([\ldots n] \oslash k)$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iT-Div-Int*)
lemma *iIN-iT-Div-Int-mod-0*:
$[\![0 < k; n\ mod\ k = 0; \forall x \in A.\ x\ div\ k \le (n + d)\ div\ k \longrightarrow x \le n + d]\!] \Longrightarrow$
$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
apply *blast*
apply (*elim conjE exE*, *rename-tac x1 x2*)
apply (*rule-tac x=x1* **in** *exI*, *simp*)
apply (*rule conjI*)
apply (*rule ccontr*, *simp add*: *linorder-not-le*)
apply (*drule-tac m=n* **and** *n=x2* **and** *k=k* **in** *div-le-mono*)
apply (*drule-tac a=x1* **and** *m=k* **in** *less-mod-0-imp-div-less*)
apply *simp+*
apply (*drule-tac x=x1* **in** *bspec*, *simp*)
apply (*drule div-le-mono*[*of - n + d k*])
apply *simp*
done

lemma *mod-partition-iT-Div-Int*:
$[\![0 < k; 0 < d]\!] \Longrightarrow$
$(A \cap [n * k\ldots,d * k - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n * k\ldots,d * k - Suc\ 0] \oslash k)$
apply (*rule iIN-iT-Div-Int-mod-0*, *simp+*)
apply (*clarify*, *rename-tac x*)
apply (*simp add*: *mod-0-imp-sub-1-div-conv*)
apply (*rule ccontr*, *simp add*: *linorder-not-le pred-less-eq-le*)
apply (*drule-tac n=x* **and** *k=k* **in** *div-le-mono*)
apply *simp*
done

corollary *mod-partition-iT-Div-Int2*:
$[\![0 < k; 0 < d; n\ mod\ k = 0; d\ mod\ k = 0]\!] \Longrightarrow$
$(A \cap [n\ldots,d - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n\ldots,d - Suc\ 0] \oslash k)$
by (*auto simp add*: *ac-simps mod-partition-iT-Div-Int elim*!: *dvdE*)

corollary *mod-partition-iT-Div-Int-one-segment*:
$0 < k \Longrightarrow$
$(A \cap [n * k\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n * k\ldots,k - Suc\ 0] \oslash k)$
by (*insert mod-partition-iT-Div-Int*[**where** *d=1*], *simp*)

corollary *mod-partition-iT-Div-Int-one-segment2*:

$[\![\ 0 < k;\ n\ mod\ k = 0\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n\ldots,k - Suc\ 0] \oslash k)$
using *mod-partition-iT-Div-Int2*[**where** *k*=*k* **and** *d*=*k* **and** *n*=*n*]
by (*insert mod-partition-iT-Div-Int2*[**where** *k*=*k* **and** *d*=*k* **and** *n*=*n*], *simp*)

lemma *iT-Div-assoc*:$I \oslash a \oslash b = I \oslash (a * b)$
by (*simp add*: *iT-Div-def image-image div-mult2-eq*)

lemma *iT-Div-commute*: $I \oslash a \oslash b = I \oslash b \oslash a$
by (*simp add*: *iT-Div-assoc mult.commute*[*of a*])

lemma *iT-Mult-Div-self*: $0 < k \Longrightarrow I \otimes k \oslash k = I$
by (*simp add*: *iT-Mult-def iT-Div-def image-image*)
lemma *iT-Mult-Div*:
$[\![\ 0 < d;\ \ k\ mod\ d = 0\ ]\!] \Longrightarrow I \otimes k \oslash d = I \otimes (k\ div\ d)$
by (*auto simp add*: *ac-simps iT-Mult-assoc*[*symmetric*] *iT-Mult-Div-self*)

lemma *iT-Div-Mult-self*:
$0 < k \Longrightarrow I \oslash k \otimes k = \{y.\ \exists x \in I.\ y = x - x\ mod\ k\}$
by (*simp add*: *set-eq-iff iT-Mult-def iT-Div-def image-image image-iff div-mult-cancel*)

lemma *iT-Plus-Div-distrib-mod-less*:
$\forall x{\in}I.\ x\ mod\ m + n\ mod\ m < m \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
by (*simp add*: *set-eq-iff iT-Div-def iT-Plus-def image-image image-iff div-add1-eq1*)
corollary *iT-Plus-Div-distrib-mod-0*:
$n\ mod\ m = 0 \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
apply (*case-tac* $m = 0$, *simp add*: *iT-Plus-0 iT-Div-0*)
apply (*simp add*: *iT-Plus-Div-distrib-mod-less*)
done

lemma *iT-Div-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oslash k) = iMin\ I\ div\ k$
by (*simp add*: *iT-Div-def iMin-mono2 mono-def div-le-mono*)

corollary
iFROM-div-Min: $iMin\ ([n\ldots] \oslash k) = n\ div\ k$ **and**
iIN-div-Min: $iMin\ ([n\ldots,d] \oslash k) = n\ div\ k$ **and**
iTILL-div-Min: $iMin\ ([\ldots n] \oslash k) = 0$ **and**
iMOD-div-Min: $iMin\ ([r,\ mod\ m] \oslash k) = r\ div\ k$ **and**
iMODb-div-Min: $iMin\ ([r,\ mod\ m,\ c] \oslash k) = r\ div\ k$
by (*simp add*: *iT-not-empty iT-Div-Min iT-Min*)+

lemmas *iT-div-Min* =
iFROM-div-Min
iIN-div-Min
iTILL-div-Min
iMOD-div-Min
iMODb-div-Min

lemma *iT-Div-Max*: ⟦ *finite I*; *I* ≠ {} ⟧ ⟹ *Max* (*I* ⊘ *k*) = *Max I div k*
by (*simp add*: *iT-Div-def Max-mono2 mono-def div-le-mono*)

corollary
  iIN-div-Max: *Max* ([*n*. . .,*d*] ⊘ *k*) = (*n* + *d*) *div k* **and**
  iTILL-div-Max: *Max* ([. . .*n*] ⊘ *k*) = *n div k* **and**
  iMODb-div-Max: *Max* ([*r*, *mod m*, *c*] ⊘ *k*) = (*r* + *m* ∗ *c*) *div k*
by (*simp add*: *iT-not-empty iT-finite iT-Div-Max iT-Max*)+

lemma *iT-Div-0-finite*: *finite* (*I* ⊘ *0*)
by (*simp add*: *iT-Div-0-if iTILL-0*)

lemma *iT-Div-infinite-iff*: *0* < *k* ⟹ *infinite* (*I* ⊘ *k*) = *infinite I*
apply (*unfold iT-Div-def*)
apply (*rule iffI*)
 apply (*rule infinite-image-imp-infinite*, *assumption*)
apply (*clarsimp simp*: *infinite-nat-iff-unbounded-le image-iff*, *rename-tac x1*)
apply (*drule-tac x*=*x1* ∗ *k* **in** *spec*, *clarsimp*, *rename-tac x2*)
apply (*drule div-le-mono*[*of - - k*], *simp*)
apply (*rule-tac x*=*x2 div k* **in** *exI*)
apply *fastforce*
done
lemma *iT-Div-finite-iff*: *0* < *k* ⟹ *finite* (*I* ⊘ *k*) = *finite I*
by (*insert iT-Div-infinite-iff*, *simp*)

lemma *iFROM-div*: *0* < *k* ⟹ [*n*. . .] ⊘ *k* = [*n div k*. . .]
apply (*clarsimp simp*: *set-eq-iff iT-Div-def image-iff Bex-def iFROM-iff*, *rename-tac x*)
apply (*rule iffI*)
 apply (*clarsimp simp*: *div-le-mono*)
apply (*rule-tac x*=*n mod k* + *k* ∗ *x* **in** *exI*)
apply *simp*
apply (*subst add.commute*, *subst le-diff-conv*[*symmetric*])
apply (*subst minus-mod-eq-mult-div*)
apply *simp*
done

lemma *iIN-div*:
  0 < *k* ⟹
  [*n*. . .,*d*] ⊘ *k* = [*n div k*. . ., *d div k* + (*n mod k* + *d mod k*) *div k*]
apply (*clarsimp simp*: *set-eq-iff iT-Div-def image-iff Bex-def iIN-iff*, *rename-tac x*)
apply (*rule iffI*)
 apply *clarify*
 apply (*drule div-le-mono*[*of n - k*])
 apply (*drule div-le-mono*[*of - n* + *d k*])
 apply (*simp add*: *div-add1-eq*[*of n d*])
apply (*clarify*, *rename-tac x*)

apply (*simp add*: *add.assoc*[*symmetric*] *div-add1-eq*[*symmetric*])
apply (*frule mult-le-mono1*[*of n div k - k*])
apply (*frule mult-le-mono1*[*of - (n + d) div k k*])
apply (*simp add*: *mult.commute*[*of - k*] *minus-mod-eq-mult-div* [*symmetric*])
apply (*simp add*: *le-diff-conv le-diff-conv2*[*OF mod-le-dividend*])
apply (*drule order-le-less*[*of - (n + d) div k*, *THEN iffD1*], *erule disjE*)
 apply (*rule-tac x=k ∗ x + n mod k* **in** *exI*)
 apply (*simp add*: *add.commute*[*of - n mod k*])
 apply (*case-tac n mod k* ≤ (*n* + *d*) *mod k*, *simp*)
 apply (*simp add*: *linorder-not-le*)
 apply (*drule-tac m=x* **in** *less-imp-le-pred*)
 apply (*drule-tac i=x* **and** *k=k* **in** *mult-le-mono2*)
 apply (*simp add*: *diff-mult-distrib2 minus-mod-eq-mult-div* [*symmetric*])
 apply (*subst add.commute*[*of n mod k*])
 apply (*subst le-diff-conv2*[*symmetric*])
  apply (*simp add*: *trans-le-add1*)
 apply (*rule order-trans*, *assumption*)
 apply (*rule diff-le-mono2*)
 apply (*simp add*: *trans-le-add2*)
apply (*rule-tac x=n + d* **in** *exI*, *simp*)
done

corollary *iIN-div-if*:
  0 < *k* ⟹ [*n*. . .,*d*] ⊘ *k* =
  [*n div k*. . ., *d div k* + (*if n mod k* + *d mod k* < *k then 0 else Suc 0*)]
apply (*simp add*: *iIN-div*)
apply (*simp add*: *iIN-def add.assoc*[*symmetric*] *div-add1-eq*[*symmetric*] *div-add1-eq2*[**where** *a=n*])
done

corollary *iIN-div-eq1*:
  ⟦ *0* < *k*; *n mod k* + *d mod k* < *k* ⟧ ⟹
  [*n*. . .,*d*] ⊘ *k* = [*n div k*. . .,*d div k*]
by (*simp add*: *iIN-div-if*)

corollary *iIN-div-eq2*:
  ⟦ *0* < *k*; *k* ≤ *n mod k* + *d mod k* ⟧ ⟹
  [*n*. . .,*d*] ⊘ *k* = [*n div k*. . ., *Suc* (*d div k*)]
by (*simp add*: *iIN-div-if*)

corollary *iIN-div-mod-eq-0*:
  ⟦ *0* < *k*; *n mod k* = *0* ⟧ ⟹ [*n*. . .,*d*] ⊘ *k* = [*n div k*. . .,*d div k*]
by (*simp add*: *iIN-div-eq1*)

lemma *iTILL-div*:
  0 < *k* ⟹ [. . .*n*] ⊘ *k* = [. . .*n div k*]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-div-if*)

lemma *iMOD-div-ge*:
⟦ *0* < *m*; *m* ≤ *k* ⟧ ⟹ [*r*, *mod m*] ⊘ *k* = [*r div k*. . .]
apply (*frule less-le-trans*[*of - - k*], *assumption*)
apply (*clarsimp simp*: *set-eq-iff iT-Div-mem-iff Bex-def iT-iff*, *rename-tac x*)
apply (*rule iffI*)
apply (*fastforce simp*: *div-le-mono*)
apply (*rule-tac x*=
if x ∗ *k* < *r then r else*
((*if x* ∗ *k mod m* ≤ *r mod m then 0 else m*) + *r mod m* + (*x* ∗ *k* − *x* ∗ *k mod m*))
in *exI*)
apply (*case-tac x* ∗ *k* < *r*)
apply *simp*
apply (*drule less-imp-le*[*of - r*], *drule div-le-mono*[*of - r k*], *simp*)
apply (*simp add*: *linorder-not-less linorder-not-le*)
apply (*simp add*: *div-le-conv add.commute*[*of k*])
apply (*subst diff-add-assoc*, *simp*)+
apply (*simp add*: *div-mult-cancel*[*symmetric*] *del*: *add-diff-assoc*)
apply (*case-tac x* ∗ *k mod m* = *0*)
apply (*clarsimp elim*!: *dvdE*)
apply (*drule sym*)
apply (*simp add*: *mult.commute*[*of m*])
apply (*blast intro*: *div-less order-less-le-trans mod-less-divisor*)
apply *simp*
apply (*intro conjI impI*)
apply (*simp add*: *div-mult-cancel*)
apply (*simp add*: *div-mult-cancel*)
apply (*subst add.commute*, *subst diff-add-assoc*, *simp*)
apply (*subst add.commute*, *subst div-mult-self1*, *simp*)
apply (*subst div-less*)
apply (*rule order-less-le-trans*[*of - m*], *simp add*: *less-imp-diff-less*)
apply *simp*
apply *simp*
apply (*rule-tac y*=*x* ∗ *k* **in** *order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*rule le-add-diff*)
apply (*simp add*: *trans-le-add1*)
apply (*simp add*: *div-mult-cancel*)
apply (*subst diff-add-assoc2*, *simp add*: *trans-le-add1*)
apply *simp*
done
corollary *iMOD-div-self*:
0 < *m* ⟹ [*r*, *mod m*] ⊘ *m* = [*r div m*. . .]
by (*simp add*: *iMOD-div-ge*)

lemma *iMOD-div*:
⟦ *0* < *k*; *m mod k* = *0* ⟧ ⟹
[*r*, *mod m*] ⊘ *k* = [*r div k*, *mod* (*m div k*)]
apply (*case-tac m* = *0*)

apply (*simp add*: *iMOD-0 iIN-0 iT-Div-singleton*)
apply (*clarsimp elim*!: *dvdE*)
apply (*rename-tac q*)
apply *hypsubst-thin*
apply (*cut-tac r*=*r div k* **and** *k*=*k* **and** *m*=*q* **in** *iMOD-mult*)
apply (*drule arg-cong*[**where** *f*=$\lambda x.\ x \oplus (r\ mod\ k)$])
apply (*drule sym*)
apply (*simp add*: *iMOD-add mult.commute*[*of k*])
apply (*cut-tac I*=[*r div k*, *mod q*] ⊗ *k* **and** *m*=*k* **and** *n*=*r mod k* **in** *iT-Plus-Div-distrib-mod-less*)
apply (*rule ballI*)
apply (*simp only*: *iMOD-mult iMOD-iff* , *elim conjE*)
apply (*drule mod-factor-imp-mod-0*)
apply *simp*
apply (*simp add*: *iT-Plus-0*)
apply (*simp add*: *iT-Mult-Div*[*OF - mod-self*] *iT-Mult-1*)
done

lemma *iMODb-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m,\ c] \oslash m = [r\ div\ m\ldots,c]$
apply (*subst iMODb-iMOD-iTILL-conv*)
apply (*subst iTILL-iT-Div-Int*)
apply *simp*
apply (*clarsimp simp*: *iT-iff simp del*: *div-mult-self1 div-mult-self2*, *rename-tac x*)
apply (*drule div-le-mod-le-imp-le*)
apply *simp*+
apply (*simp add*: *iMOD-div-self iTILL-div iFROM-iTILL-iIN-conv*)
done

lemma *iMODb-div-ge*:
$[\![\ 0 < m;\ m \le k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k\ldots,(r + m * c)\ div\ k - r\ div\ k]$
apply (*case-tac* $m = k$)
apply (*simp add*: *iMODb-div-self*)
apply (*drule le-neq-trans*, *simp*+)
apply (*induct c*)
apply (*simp add*: *iMODb-0 iIN-0 iT-Div-singleton*)
apply (*rule-tac t*=[*r*, *mod m*, *Suc c*] **and** *s*=[*r*, *mod m*, *c*] ∪ {*r* + *m* ∗ *c* + *m*} **in** *subst*)
apply (*cut-tac c*=*c* **and** *c′*=*0* **and** *r*=*r* **and** *m*=*m* **in** *iMODb-append-union-Suc*[*symmetric*])
apply (*simp add*: *iMODb-0 iIN-0 add.commute*[*of m*] *add.assoc*)
apply (*subst iT-Div-Un*)
apply (*simp add*: *iT-Div-singleton*)
apply (*simp add*: *add.commute*[*of m*] *add.assoc*[*symmetric*])
apply (*case-tac* $(r + m * c)\ mod\ k + m\ mod\ k < k$)
apply (*simp add*: *div-add1-eq1*)
apply (*rule insert-absorb*)
apply (*simp add*: *iIN-iff div-le-mono*)
apply (*simp add*: *linorder-not-less*)

apply (*simp add*: *div-add1-eq2*)
apply (*rule-tac t=Suc* ((*r* + *m* ∗ *c*) *div k*) **and** *s=Suc* (*r div k* + ((*r* + *m* ∗ *c*) *div k* − *r div k*)) **in** *subst*)
 apply (*simp add*: *div-le-mono*)
apply (*simp add*: *iIN-Suc-insert-conv*)
done

corollary *iMODb-div-ge-if*:
  ⟦ *0* < *m*; *m* ≤ *k* ⟧ =⇒
  [*r*, *mod m*, *c*] ⊘ *k* =
  [*r div k*. . ., *m* ∗ *c div k* + (*if r mod k* + *m* ∗ *c mod k* < *k then 0 else Suc 0*)]
by (*simp add*: *iMODb-div-ge div-add1-eq-if*[*of - r*])

lemma *iMODb-div*:
  ⟦ *0* < *k*; *m mod k* = *0* ⟧ =⇒
  [*r*, *mod m*, *c*] ⊘ *k* = [*r div k*, *mod* (*m div k*), *c*]
apply (*subst iMODb-iMOD-iTILL-conv*)
apply (*subst iTILL-iT-Div-Int*)
  apply *simp*
 apply (*simp add*: *Ball-def iMOD-iff* , *intro allI impI*, *elim conjE*, *rename-tac x*)
 apply (*drule div-le-mod-le-imp-le*)
  apply (*subst mod-add1-eq-if*)
  apply (*simp add*: *mod-0-imp-mod-mult-right-0*)
  apply (*drule mod-eq-mod-0-imp-mod-eq*, *simp*+)
apply (*simp add*: *iMOD-div iTILL-div*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv div-le-mono*)
apply (*clarsimp simp*: *mult.assoc iMODb-mod-0 iMOD-0 elim*!: *dvdE*)
done

lemmas *iT-div* =
  iTILL-div
  iFROM-div
  iIN-div
  iMOD-div
  iMODb-div
  iT-Div-singleton

This lemma is valid for all $k \leq m$,i. e., also for k with $m\ mod\ k \neq 0$.

lemma *iMODb-div-unique*:
  ⟦ *0* < *k*; *k* ≤ *m*; *k* ≤ *c*; [*r′*, *mod m′*, *c′*] = [*r*, *mod m*, *c*] ⊘ *k* ⟧ =⇒
  r′ = *r div k* ∧ *m′* = *m div k* ∧ *c′* = *c*
apply (*case-tac r′* ≠ *r div k*)
 apply (*drule arg-cong*[**where** *f=iMin*])
 apply (*simp add*: *iT-Min iT-not-empty iT-Div-Min*)
apply *simp*
apply (*case-tac m′* = *0* ∨ *c′* = *0*)
 apply (*subgoal-tac* [*r div k*, *mod m′*, *c′*] = {*r div k*})
  prefer *2*
  apply (*rule iMODb-singleton-eq-conv*[*THEN iffD2*], *simp*)

apply *simp*
apply (*drule arg-cong*[**where** *f=Max*])
apply (*simp add*: *iMODb-mod-0 iIN-0 iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite iT-not-empty*)
apply (*subgoal-tac r div k* < (*r* + *m* ∗ *c*) *div k*, *simp*)
apply (*subst div-add1-eq-if*, *simp*)
apply *clarsimp*
apply (*rule order-less-le-trans*[*of - k* ∗ *k div k*], *simp*)
apply (*rule div-le-mono*)
apply (*simp add*: *mult-mono*)
apply (*subgoal-tac c′* = *c*)
prefer *2*
apply (*drule arg-cong*[**where** *f=λA. card A*])
apply (*simp add*: *iT-Div-def card-image*[*OF iMODb-div-right-inj-on*] *iMODb-card*)
apply *clarsimp*
apply (*frule iMODb-div-right-strict-mono-on*[*of k m r c*], *assumption*)
apply (*frule-tac a=k* **and** *b=0* **and** *m=m′* **and** *r=r div k* **and** *c=c* **in** *iMODb-inext-nth-diff*, *simp*)
apply (*simp add*: *iT-Div-Min iT-not-empty iT-Min*)
apply (*simp add*: *iT-Div-def inext-nth-image*[*OF iMODb-not-empty*])
apply (*simp add*: *iMODb-inext-nth*)
done

lemma *iMODb-div-mod-gr0-is-0-not-ex0*:
⟦ *0* < *k*; *k* < *m*; *0* < *m mod k*; *k* ≤ *c*; *r mod k* = *0* ⟧ =⇒
¬(∃ *r′ m′ c′*. [*r′*, *mod m′*, *c′*] = [*r*, *mod m*, *c*] ⊘ *k*)
apply (*rule ccontr*, *simp*, *elim exE conjE*)
apply (*frule-tac r′=r′* **and** *m′=m′* **and** *c′=c′* **and** *r=r* **and** *k=k* **and** *m=m* **and** *c=c*
in *iMODb-div-unique*[*OF - less-imp-le*], *simp+*)
apply (*drule arg-cong*[**where** *f=Max*])
apply (*simp add*: *iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite iT-not-empty*)
apply (*simp add*: *div-add1-eq1*)
apply (*simp add*: *mult.commute*[*of m*])
apply (*simp add*: *div-mult1-eq*[*of c m*] *div-eq-0-conv*)
apply (*subgoal-tac c* ≤ *c* ∗ (*m mod k*))
apply *simp+*
done

lemma *iMODb-div-mod-gr0-not-ex--arith-aux1*:
⟦ (*0*::*nat*) < *k*; *k* < *m*; *0* < *x1* ⟧ =⇒
x1 ∗ *m* + *x2* − *x mod k* + *x3* + *x mod k* = *x1* ∗ *m* + *x2* + *x3*
apply (*drule Suc-leI*[*of - x1*])
apply (*drule mult-le-mono1*[*of Suc 0 - m*])
apply (*subgoal-tac x mod k* ≤ *x1* ∗ *m*)
prefer *2*
apply (*rule order-trans*[*OF mod-le-divisor*], *assumption*)

apply (*rule order-less-imp-le*)
apply (*rule order-less-le-trans*)
apply *simp+*
done

lemma *iMODb-div-mod-gr0-not-ex*:
⟦ *0* < *k*; *k* < *m*; *0* < *m mod k*; *k* ≤ *c* ⟧ ⟹
¬(∃ *r′ m′ c′*. [*r′*, *mod m′*, *c′*] = [*r*, *mod m*, *c*] ⊘ *k*)
apply (*case-tac r mod k* = *0*)
apply (*simp add*: *iMODb-div-mod-gr0-is-0-not-ex0*)
apply (*rule ccontr*, *simp*, *elim exE conjE*)
apply (*frule-tac r′*=*r′* **and** *m′*=*m′* **and** *c′*=*c′* **and** *r*=*r* **and** *k*=*k* **and** *m*=*m* **and** *c*=*c*
in *iMODb-div-unique*[*OF - less-imp-le*], *simp+*)
apply *clarsimp*
apply (*drule arg-cong*[**where** *f*=*Max*])
apply (*simp add*: *iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite iT-not-empty*)
apply (*simp add*: *div-add1-eq*[*of r m* ∗ *c*])
apply (*simp add*: *mult.commute*[*of - c*])
apply (*clarsimp simp add*: *div-mult1-eq*[*of c m k*])
apply (*subgoal-tac Suc 0* ≤ *c* ∗ (*m mod k*) *div k*, *simp*)
apply (*thin-tac -* = *0*)+
apply (*drule div-le-mono*[*of k c k*], *simp*)
apply (*rule order-trans*[*of - c div k*], *simp*)
apply (*rule div-le-mono*, *simp*)
done

lemma *iMOD-div-eq-imp-iMODb-div-eq*:
⟦ *0* < *k*; *k* ≤ *m*; [*r′*, *mod m′*] = [*r*, *mod m*] ⊘ *k* ⟧ ⟹
[*r′*, *mod m′*, *c*] = [*r*, *mod m*, *c*] ⊘ *k*
apply (*subgoal-tac r′* = *r div k*)
prefer *2*
apply (*drule arg-cong*[**where** *f*=*iMin*])
apply (*simp add*: *iT-Div-Min iMOD-not-empty iMOD-Min*)
apply *clarsimp*
apply (*frule iMOD-div-right-strict-mono-on*[*of - m r*], *assumption*)
apply (*frule card-image*[*OF strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*[*of k m r c*]]], *assumption*)
apply (*simp add*: *iMODb-card*)
apply (*subgoal-tac r* + *m* ∗ *c* ∈ [*r*, *mod m*])
prefer *2*
apply (*simp add*: *iMOD-iff*)
apply (*subgoal-tac* [*r*, *mod m*, *c*] = [*r*, *mod m*] ↓≤ (*r* + *m* ∗ *c*))
prefer *2*
apply (*simp add*: *iMOD-cut-le1*)
apply (*simp add*: *iT-Div-def*)
apply (*simp add*: *cut-le-image*[*symmetric*])

apply (*drule sym*)
apply (*simp add*: *iMOD-cut-le*)
apply (*simp add*: *linorder-not-le*[*of r div k*, *symmetric*])
apply (*simp add*: *div-le-mono*)
apply (*case-tac m′ = 0*)
 apply (*simp add*: *iMODb-mod-0-card*)
apply (*rule arg-cong*[**where** *f*=λ*c*. [*r div k*, *mod m′*, *c*]])
apply (*simp add*: *iMODb-card*)
done

lemma *iMOD-div-unique*:
  ⟦ *0* < *k*; *k* ≤ *m*; [*r′*, *mod m′*] = [*r*, *mod m*] ⊘ *k* ⟧ ⟹
  r′ = *r div k* ∧ *m′* = *m div k*
apply (*frule iMOD-div-eq-imp-iMODb-div-eq*[*of k m r′ m′ r k*], *assumption*+)
apply (*simp add*: *iMODb-div-unique*[*of k - k*])
done

lemma *iMOD-div-mod-gr0-not-ex*:
  ⟦ *0* < *k*; *k* < *m*; *0* < *m mod k* ⟧ ⟹
  ¬ (∃ *r′ m′*. [*r′*, *mod m′*] = [*r*, *mod m*] ⊘ *k*)
apply (*rule ccontr*, *clarsimp*)
apply (*frule-tac k*=*k* **and** *m*=*m* **and** *r′*=*r′* **and** *m′*=*m′* **and** *c*=*k*
  in *iMOD-div-eq-imp-iMODb-div-eq*[*OF* - *less-imp-le*], *assumption*+)
apply (*frule iMODb-div-mod-gr0-not-ex*[*of k m k r*], *simp*+)
done

2.2 Interval cut operators with arithmetic interval operators

lemma
  iT-Plus-cut-le2: (*I* ⊕ *k*) ↓≤ (*t* + *k*) = (*I* ↓≤ *t*) ⊕ *k* **and**
  iT-Plus-cut-less2: (*I* ⊕ *k*) ↓< (*t* + *k*) = (*I* ↓< *t*) ⊕ *k* **and**
  iT-Plus-cut-ge2: (*I* ⊕ *k*) ↓≥ (*t* + *k*) = (*I* ↓≥ *t*) ⊕ *k* **and**
  iT-Plus-cut-greater2: (*I* ⊕ *k*) ↓> (*t* + *k*) = (*I* ↓> *t*) ⊕ *k*
unfolding *iT-Plus-def* **by** *fastforce*+

lemma *iT-Plus-cut-le*:
  (*I* ⊕ *k*) ↓≤ *t* = (*if t* < *k then* {} *else I* ↓≤ (*t* − *k*) ⊕ *k*)
apply (*case-tac t* < *k*)
 apply (*simp add*: *cut-le-empty-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-le2*[*of I k t* − *k*], *simp*)
done

lemma *iT-Plus-cut-less*: (*I* ⊕ *k*) ↓< *t* = *I* ↓< (*t* − *k*) ⊕ *k*
apply (*case-tac t* < *k*)
 apply (*simp add*: *cut-less-0-empty iT-Plus-empty cut-less-empty-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-less2*[*of I k t* − *k*], *simp*)
done

lemma *iT-Plus-cut-ge*: $(I \oplus k) \downarrow\geq t = I \downarrow\geq (t - k) \oplus k$
apply (*case-tac* $t < k$)
 apply (*simp add*: *cut-ge-0-all cut-ge-all-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-ge2*[*of* I k $t - k$], *simp*)
done

lemma *iT-Plus-cut-greater*:
 $(I \oplus k) \downarrow> t = (\mathit{if}\ t < k\ \mathit{then}\ I \oplus k\ \mathit{else}\ I \downarrow> (t - k) \oplus k)$
apply (*case-tac* $t < k$)
 apply (*simp add*: *cut-greater-all-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-greater2*[*of* I k $t - k$], *simp*)
done

lemma
 iT-Mult-cut-le2: $0 < k \Longrightarrow (I \otimes k) \downarrow\leq (t * k) = (I \downarrow\leq t) \otimes k$ **and**
 iT-Mult-cut-less2: $0 < k \Longrightarrow (I \otimes k) \downarrow< (t * k) = (I \downarrow< t) \otimes k$ **and**
 iT-Mult-cut-ge2: $0 < k \Longrightarrow (I \otimes k) \downarrow\geq (t * k) = (I \downarrow\geq t) \otimes k$ **and**
 iT-Mult-cut-greater2: $0 < k \Longrightarrow (I \otimes k) \downarrow> (t * k) = (I \downarrow> t) \otimes k$
unfolding *iT-Mult-def* **by** *fastforce+*

lemma *iT-Mult-cut-le*:
 $0 < k \Longrightarrow (I \otimes k) \downarrow\leq t = (I \downarrow\leq (t\ \mathit{div}\ k)) \otimes k$
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-le-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
 apply (*simp add*: *div-le-mono*)
apply (*rule div-le-mod-le-imp-le*, *simp+*)
done

lemma *iT-Mult-cut-less*:
 $0 < k \Longrightarrow (I \otimes k) \downarrow< t =$
  $(\mathit{if}\ t\ \mathit{mod}\ k = 0\ \mathit{then}\ (I \downarrow< (t\ \mathit{div}\ k))\ \mathit{else}\ I \downarrow< \mathit{Suc}\ (t\ \mathit{div}\ k)) \otimes k$
apply (*case-tac* $t\ \mathit{mod}\ k = 0$)
 apply (*clarsimp simp add*: *mult.commute*[*of* k] *iT-Mult-cut-less2 elim*!: *dvdE*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-less-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*subst less-Suc-eq-le*)
apply (*rule iffI*)
 apply (*rule div-le-mono*, *simp*)
apply (*rule ccontr*, *simp add*: *linorder-not-less*)
apply (*drule le-imp-less-or-eq*[*of t*], *erule disjE*)
 apply (*fastforce dest*: *less-mod-0-imp-div-less*[*of t - k*])
apply *simp*
done

lemma *iT-Mult-cut-greater*:
 $0 < k \Longrightarrow (I \otimes k) \downarrow> t = (I \downarrow> (t\ \mathit{div}\ k)) \otimes k$
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-greater-mem-iff*)

apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
 apply (*simp add*: *less-mod-ge-imp-div-less*)
apply (*rule ccontr*, *simp add*: *linorder-not-less*)
apply (*fastforce dest*: *div-le-mono*[*of - - k*])
done

lemma *iT-Mult-cut-ge*:
  0 < *k* ⟹ (*I* ⊗ *k*) ↓≥ *t* =
    (*if t mod k* = *0 then* (*I* ↓≥ (*t div k*)) *else I* ↓≥ *Suc* (*t div k*)) ⊗ *k*
apply (*case-tac t mod k* = *0*)
 apply (*clarsimp simp add*: *mult.commute*[*of k*] *iT-Mult-cut-ge2 elim*!: *dvdE*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-ge-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
 apply (*rule Suc-leI*)
 apply (*simp add*: *le-mod-greater-imp-div-less*)
apply (*rule ccontr*)
apply (*drule Suc-le-lessD*)
apply (*simp add*: *linorder-not-le*)
apply (*fastforce dest*: *div-le-mono*[*OF order-less-imp-le*, *of - t k*])
done

lemma *iT-Plus-neg-cut-le2*: *k* ≤ *t* ⟹ (*I* ⊕− *k*) ↓≤ (*t* − *k*) = (*I* ↓≤ *t*) ⊕− *k*
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of* (↓≤) (↓≥)])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done

lemma *iT-Plus-neg-cut-less2*: (*I* ⊕− *k*) ↓< (*t* − *k*) = (*I* ↓< *t*) ⊕− *k*
apply (*case-tac t* ≤ *k*)
 apply (*simp add*: *cut-less-0-empty*)
 apply (*case-tac I* ↓< *t* = {})
  apply (*simp add*: *iT-Plus-neg-empty*)
 apply (*rule sym*, *rule iT-Plus-neg-Max-less-empty*[*OF nat-cut-less-finite*])
 apply (*rule order-less-le-trans*[*OF cut-less-Max-less*[*OF nat-cut-less-finite*]], *assumption*+)
apply (*simp add*: *linorder-not-le iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of* (↓<) (↓≥)])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done

lemma *iT-Plus-neg-cut-ge2*: (*I* ⊕− *k*) ↓≥ (*t* − *k*) = (*I* ↓≥ *t*) ⊕− *k*
apply (*case-tac t* ≤ *k*)
 apply (*simp add*: *cut-ge-0-all iT-Plus-neg-cut-eq*)
apply (*simp add*: *linorder-not-le iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of* (↓≥) (↓≥)])

apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done

lemma *iT-Plus-neg-cut-greater2*: $k \le t \Longrightarrow (I \oplus- k) \downarrow> (t - k) = (I \downarrow> t) \oplus- k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of* (↓>) (↓≥)])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done

lemma *iT-Plus-neg-cut-le*: $(I \oplus- k) \downarrow\le t = I \downarrow\le (t + k) \oplus- k$
by (*insert iT-Plus-neg-cut-le2*[*of k t + k I*, *OF le-add2*], *simp*)

lemma *iT-Plus-neg-cut-less*: $(I \oplus- k) \downarrow< t = I \downarrow< (t + k) \oplus- k$
by (*insert iT-Plus-neg-cut-less2*[*of I k t + k*], *simp*)

lemma *iT-Plus-neg-cut-ge*: $(I \oplus- k) \downarrow\ge t = I \downarrow\ge (t + k) \oplus- k$
by (*insert iT-Plus-neg-cut-ge2*[*of I k t + k*], *simp*)

lemma *iT-Plus-neg-cut-greater*: $(I \oplus- k) \downarrow> t = I \downarrow> (t + k) \oplus- k$
by (*insert iT-Plus-neg-cut-greater2*[*of k t + k I*], *simp*)

lemma *iT-Minus-cut-le2*: $t \le k \Longrightarrow (k \ominus I) \downarrow\le (k - t) = k \ominus (I \downarrow\ge t)$
by (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-less2*: $(k \ominus I) \downarrow< (k - t) = k \ominus (I \downarrow> t)$
by (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-ge2*: $(k \ominus I) \downarrow\ge (k - t) = k \ominus (I \downarrow\le t)$
by (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-greater2*: $t \le k \Longrightarrow (k \ominus I) \downarrow> (k - t) = k \ominus (I \downarrow< t)$
by (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-le*: $(k \ominus I) \downarrow\le t = k \ominus (I \downarrow\ge (k - t))$
by (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-less*:
$(k \ominus I) \downarrow< t = (\textit{if } t \le k \textit{ then } k \ominus (I \downarrow> (k - t)) \textit{ else } k \ominus I)$
apply (*case-tac* $t \le k$)
 apply (*cut-tac iT-Minus-cut-less2*[*of k I k − t*], *simp*)
apply (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done

lemma *iT-Minus-cut-ge*:
$(k \ominus I) \downarrow\ge t = (\textit{if } t \le k \textit{ then } k \ominus (I \downarrow\le (k - t)) \textit{ else } \{\})$

apply (*case-tac* $t \le k$)
apply (*cut-tac iT-Minus-cut-ge2*[*of* k I $k - t$], *simp*)
apply (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done

lemma *iT-Minus-cut-greater*: $(k \ominus I) \downarrow> t = k \ominus (I \downarrow< (k - t))$
apply (*case-tac* $t \le k$)
apply (*cut-tac iT-Minus-cut-greater2*[*of* $k - t$ k I], *simp+*)
apply (*fastforce simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done

lemma *iT-Div-cut-le*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\le t = I \downarrow\le (t * k + (k - Suc\ 0)) \oslash k$
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-Div-mem-iff Bex-def*)
apply (*fastforce simp*: *div-le-conv*)
done

lemma *iT-Div-cut-less*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< t = I \downarrow< (t * k) \oslash k$
apply (*case-tac* $t = 0$)
apply (*simp add*: *cut-less-0-empty iT-Div-empty*)
apply (*simp add*: *nat-cut-less-le-conv iT-Div-cut-le diff-mult-distrib*)
done

lemma *iT-Div-cut-ge*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\ge t = I \downarrow\ge (t * k) \oslash k$
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-Div-mem-iff Bex-def*)
apply (*fastforce simp*: *le-div-conv*)
done

lemma *iT-Div-cut-greater*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> t = I \downarrow> (t * k + (k - Suc\ 0)) \oslash k$
by (*simp add*: *nat-cut-ge-greater-conv*[*symmetric*] *iT-Div-cut-ge add.commute*[*of* k])

lemma *iT-Div-cut-le2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\le (t\ div\ k) = I \downarrow\le (t - t\ mod\ k + (k - Suc\ 0)) \oslash k$
by (*frule iT-Div-cut-le*[*of* k I $t\ div\ k$], *simp add*: *div-mult-cancel*)

lemma *iT-Div-cut-less2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< (t\ div\ k) = I \downarrow< (t - t\ mod\ k) \oslash k$
by (*frule iT-Div-cut-less*[*of* k I $t\ div\ k$], *simp add*: *div-mult-cancel*)

lemma *iT-Div-cut-ge2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\ge (t\ div\ k) = I \downarrow\ge (t - t\ mod\ k) \oslash k$
by (*frule iT-Div-cut-ge*[*of* k I $t\ div\ k$], *simp add*: *div-mult-cancel*)

lemma *iT-Div-cut-greater2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> (t\ div\ k) = I \downarrow> (t - t\ mod\ k + (k - Suc\ 0)) \oslash k$
by (*frule iT-Div-cut-greater*[*of k I t div k*], *simp add*: *div-mult-cancel*)

2.3 *inext* and *iprev* with interval operators

lemma *iT-Plus-inext*: $inext\ (n + k)\ (I \oplus k) = (inext\ n\ I) + k$
by (*unfold iT-Plus-def*, *rule inext-image2*[*OF add-right-strict-mono*])

lemma *iT-Plus-iprev*: $iprev\ (n + k)\ (I \oplus k) = (iprev\ n\ I) + k$
by (*unfold iT-Plus-def*, *rule iprev-image2*[*OF add-right-strict-mono*])

lemma *iT-Plus-inext2*: $k \le n \Longrightarrow inext\ n\ (I \oplus k) = (inext\ (n - k)\ I) + k$
by (*insert iT-Plus-inext*[*of n − k k I*], *simp*)
lemma *iT-Plus-prev2*: $k \le n \Longrightarrow iprev\ n\ (I \oplus k) = (iprev\ (n - k)\ I) + k$
by (*insert iT-Plus-iprev*[*of n − k k I*], *simp*)

lemma *iT-Mult-inext*: $inext\ (n * k)\ (I \otimes k) = (inext\ n\ I) * k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Mult-empty inext-empty*)
apply (*case-tac* $k = 0$)
 apply (*simp add*: *iT-Mult-0 iTILL-0 inext-singleton*)
apply (*simp add*: *iT-Mult-def inext-image2*[*OF mult-right-strict-mono*])
done

lemma *iT-Mult-iprev*: $iprev\ (n * k)\ (I \otimes k) = (iprev\ n\ I) * k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Mult-empty iprev-empty*)
apply (*case-tac* $k = 0$)
 apply (*simp add*: *iT-Mult-0 iTILL-0 iprev-singleton*)
apply (*simp add*: *iT-Mult-def iprev-image2*[*OF mult-right-strict-mono*])
done

lemma *iT-Mult-inext2-if*:
$inext\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (inext\ (n\ div\ k)\ I) * k\ else\ n)$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Mult-empty inext-empty div-mult-cancel*)
apply (*case-tac* $k = 0$)
 apply (*simp add*: *iT-Mult-0 iTILL-0 inext-singleton*)
apply (*case-tac* $n\ mod\ k = 0$)
 apply (*clarsimp simp*: *mult.commute*[*of k*] *iT-Mult-inext elim*!: *dvdE*)
apply (*simp add*: *not-in-inext-fix iT-Mult-mem-iff*)
done

lemma *iT-Mult-iprev2-if*:
$iprev\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (iprev\ (n\ div\ k)\ I) * k\ else\ n)$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Mult-empty iprev-empty div-mult-cancel*)

apply (*case-tac* $k = 0$)
 apply (*simp add*: *iT-Mult-0 iTILL-0 iprev-singleton*)
apply (*case-tac* n *mod* $k = 0$)
 apply (*clarsimp simp*: *mult.commute*[*of k*] *iT-Mult-iprev elim*!: *dvdE*)
apply (*simp add*: *not-in-iprev-fix iT-Mult-mem-iff*)
done

corollary *iT-Mult-inext2*:
  $n \ mod \ k = 0 \Longrightarrow inext \ n \ (I \otimes k) = (inext \ (n \ div \ k) \ I) * k$
by (*simp add*: *iT-Mult-inext2-if*)

corollary *iT-Mult-iprev2*:
  $n \ mod \ k = 0 \Longrightarrow iprev \ n \ (I \otimes k) = (iprev \ (n \ div \ k) \ I) * k$
by (*simp add*: *iT-Mult-iprev2-if*)

lemma *iT-Plus-neg-inext*:
  $k \le n \Longrightarrow inext \ (n - k) \ (I \oplus- \ k) = inext \ n \ I - k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-neg-empty inext-empty*)
apply (*case-tac* $n \in I$)
 apply (*simp add*: *iT-Plus-neg-image-conv*)
 apply (*rule subst*[*OF inext-cut-ge-conv*, *of k*], *simp*)
 apply (*rule inext-image*)
  apply (*simp add*: *cut-ge-mem-iff*)
 apply (*subst cut-ge-Int-conv*)
 apply (*rule strict-mono-on-subset*[*OF - Int-lower2*])
 apply (*rule sub-left-strict-mono-on*)
apply (*subgoal-tac* $n - k \notin I \oplus- \ k$)
 prefer *2*
 apply (*simp add*: *iT-Plus-neg-mem-iff*)
apply (*simp add*: *not-in-inext-fix*)
done

lemma *iT-Plus-neg-iprev*:
  $iprev \ (n - k) \ (I \oplus- \ k) = iprev \ n \ (I \downarrow\ge k) - k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-neg-empty i-cut-empty iprev-empty*)
apply (*case-tac* $n < k$)
 apply (*simp add*: *iprev-le-iMin*)
 apply (*simp add*: *order-trans*[*OF iprev-mono*])
apply (*simp add*: *linorder-not-less*)
apply (*case-tac* $n \in I$)
 apply (*frule iT-Plus-neg-mem-iff2*[*THEN iffD2*, *of - - I*], *assumption*)
 apply (*simp add*: *iT-Plus-neg-image-conv*)
 apply (*rule iprev-image*)
  apply (*simp add*: *cut-ge-mem-iff*)
 apply (*subst cut-ge-Int-conv*)
 apply (*rule strict-mono-on-subset*[*OF - Int-lower2*])
 apply (*rule sub-left-strict-mono-on*)

apply (*frule cut-ge-not-in-imp*[*of - - k*])
apply (*subgoal-tac* $n - k \notin I \oplus- k$)
 prefer *2*
 apply (*simp add*: *iT-Plus-neg-mem-iff*)
apply (*simp add*: *not-in-iprev-fix*)
done

corollary *iT-Plus-neg-inext2*: *inext* n $(I \oplus- k)$ = *inext* $(n + k)$ $I - k$
by (*insert iT-Plus-neg-inext*[*of k n + k I*, *OF le-add2*], *simp*)

corollary *iT-Plus-neg-iprev2*: *iprev* n $(I \oplus- k)$ = *iprev* $(n + k)$ $(I \downarrow\geq k) - k$
by (*insert iT-Plus-neg-iprev*[*of n + k k I*], *simp*)

lemma *iT-Minus-inext*:
 $[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow$ *inext* $(k - n)$ $(k \ominus I) = k -$ *iprev* n I
apply (*subgoal-tac iMin* $I \leq k$)
 prefer *2*
 apply (*simp add*: *iT-Minus-empty-iff*)
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\}$)
 prefer *2*
 apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*case-tac* $n \in I$)
 apply (*simp add*: *iT-Minus-imirror-conv*)
 apply (*simp add*: *iT-Plus-neg-inext2*)
 apply (*subgoal-tac* $n \leq$ *iMin* I + *Max* $(I \downarrow\leq k)$)
  prefer *2*
  apply (*rule trans-le-add2*)
  apply (*rule Max-ge*[*OF nat-cut-le-finite*])
  apply (*simp add*: *cut-le-mem-iff*)
 apply (*simp add*: *diff-add-assoc del*: *add-diff-assoc*)
 apply (*subst add.commute*[*of k*], *subst iT-Plus-inext*)
 apply (*simp add*: *cut-le-Min-eq*[*of I*, *symmetric*])
 apply (*fold nat-mirror-def mirror-elem-def*)
 apply (*simp add*: *inext-imirror-iprev-conv*[*OF nat-cut-le-finite*])
 apply (*simp add*: *iprev-cut-le-conv*)
 apply (*simp add*: *mirror-elem-def nat-mirror-def*)
 apply (*frule iprev-mono*[*THEN order-trans*, *of n iMin* $(I \downarrow\leq k)$ + *Max* $(I \downarrow\leq k)$
I])
 apply *simp*
apply (*subgoal-tac* $k - n \notin k \ominus I$)
 prefer *2*
 apply (*simp add*: *iT-Minus-mem-iff*)
apply (*simp add*: *not-in-inext-fix not-in-iprev-fix*)
done

corollary *iT-Minus-inext2*:
 $[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow$ *inext* n $(k \ominus I) = k -$ *iprev* $(k - n)$ I
by (*insert iT-Minus-inext*[*of k I k* $-$ *n*], *simp*)

lemma *iT-Minus-iprev*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow iprev\ (k - n)\ (k \ominus I) = k - inext\ n\ (I \downarrow\leq k)$
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\}$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*subst iT-Minus-cut-eq*[*OF le-refl*, *of - I*, *symmetric*])
apply (*insert iT-Minus-inext2*[*of k k* $\ominus$ ($I \downarrow\leq k$) *n*])
apply (*simp add*: *iT-Minus-Minus-cut-eq*)
apply (*rule diff-diff-cancel*[*symmetric*])
apply (*rule order-trans*[*OF iprev-mono*])
apply *simp*
done

lemma *iT-Minus-iprev2*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow iprev\ n\ (k \ominus I) = k - inext\ (k - n)\ (I \downarrow\leq k)$
by (*insert iT-Minus-iprev*[*of k I k* $-$ *n*], *simp*)

lemma *iT-Plus-inext-nth*: $I \neq \{\} \Longrightarrow (I \oplus k) \to n = (I \to n) + k$
apply (*induct n*)
apply (*simp add*: *iT-Plus-Min*)
apply (*simp add*: *iT-Plus-inext*)
done

lemma *iT-Plus-iprev-nth*: $[\![finite\ I; I \neq \{\}]\!] \Longrightarrow (I \oplus k) \leftarrow n = (I \leftarrow n) + k$
apply (*induct n*)
apply (*simp add*: *iT-Plus-Max*)
apply (*simp add*: *iT-Plus-iprev*)
done

lemma *iT-Mult-inext-nth*: $I \neq \{\} \Longrightarrow (I \otimes k) \to n = (I \to n) * k$
apply (*induct n*)
apply (*simp add*: *iT-Mult-Min*)
apply (*simp add*: *iT-Mult-inext*)
done

lemma *iT-Mult-iprev-nth*: $[\![finite\ I; I \neq \{\}]\!] \Longrightarrow (I \otimes k) \leftarrow n = (I \leftarrow n) * k$
apply (*induct n*)
apply (*simp add*: *iT-Mult-Max*)
apply (*simp add*: *iT-Mult-iprev*)
done

lemma *iT-Plus-neg-inext-nth*:
$I \oplus- k \neq \{\} \Longrightarrow (I \oplus- k) \to n = (I \downarrow\geq k \to n) - k$
apply (*subgoal-tac* $I \downarrow\geq k \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-ge-not-empty-iff iT-Plus-neg-not-empty-iff*)
apply (*induct n*)

apply (*simp add*: *iT-Plus-neg-Min*)
apply (*simp add*: *iT-Plus-neg-cut-eq*[*of k k I*, *symmetric*])
apply (*rule iT-Plus-neg-inext*)
apply (*rule cut-ge-bound*[*of - I*])
apply (*simp add*: *inext-nth-closed*)
done

lemma *iT-Plus-neg-iprev-nth*:
$[\![$ *finite I*; $I \oplus- k \neq \{\}$ $]\!] \Longrightarrow (I \oplus- k) \leftarrow n = (I \downarrow\geq k \leftarrow n) - k$
apply (*subgoal-tac* $I \downarrow\geq k \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-ge-not-empty-iff iT-Plus-neg-not-empty-iff*)
apply (*induct n*)
apply (*simp add*: *iT-Plus-neg-Max cut-ge-Max-eq*)
apply (*simp add*: *iT-Plus-neg-iprev*)
done

lemma *iT-Minus-inext-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \rightarrow n = k - ((I \downarrow\leq k) \leftarrow n)$
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\} \wedge I \neq \{\} \wedge$ *iMin* $I \leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*elim conjE*)
apply (*induct n*)
apply(*simp add*: *iT-Minus-Min*)
apply (*simp add*: *iT-Minus-cut-eq*[*OF order-refl*, *of - I*, *symmetric*])
apply (*rule iT-Minus-inext*)
apply *simp*
apply (*rule cut-le-bound*, *rule iprev-nth-closed*[*OF nat-cut-le-finite*])
apply *assumption*
done

lemma *iT-Minus-iprev-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \leftarrow n = k - ((I \downarrow\leq k) \rightarrow n)$
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\} \wedge I \neq \{\} \wedge$ *iMin* $I \leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*elim conjE*)
apply (*induct n*)
apply(*simp add*: *iT-Minus-Max cut-le-Min-eq*)
apply *simp*
apply (*rule iT-Minus-iprev*)
apply *simp*
apply (*rule cut-le-bound*, *rule inext-nth-closed*)
apply *assumption*
done

lemma *iT-Div-ge-inext-nth*:
$[\![I \neq \{\}; \forall x \in I. \forall y \in I. x < y \longrightarrow x + k \leq y]\!] \Longrightarrow$

$(I \oslash k) \rightarrow n = (I \rightarrow n)$ *div k*
apply (*case-tac k = 0*)
apply (*simp add*: *iT-Div-0 iTILL-0 inext-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule inext-nth-image*[*OF - div-right-strict-mono-on*])

lemma *iT-Div-mod-inext-nth*:
$[\![I \neq \{\}; \forall x \in I.\ \forall y \in I.\ x \bmod k = y \bmod k]\!] \Longrightarrow$
$(I \oslash k) \rightarrow n = (I \rightarrow n)$ *div k*
apply (*case-tac k = 0*)
apply (*simp add*: *iT-Div-0 iTILL-0 inext-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule inext-nth-image*[*OF - mod-eq-div-right-strict-mono-on*])

lemma *iT-Div-ge-iprev-nth*:
$[\![$ *finite I*; $I \neq \{\}; \forall x \in I.\ \forall y \in I.\ x < y \longrightarrow x + k \leq y]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)$ *div k*
apply (*case-tac k = 0*)
apply (*simp add*: *iT-Div-0 iTILL-0 iprev-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule iprev-nth-image*[*OF - - div-right-strict-mono-on*])

lemma *iT-Div-mod-iprev-nth*:
$[\![$ *finite I*; $I \neq \{\}; \forall x \in I.\ \forall y \in I.\ x \bmod k = y \bmod k]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)$ *div k*
apply (*case-tac k = 0*)
apply (*simp add*: *iT-Div-0 iTILL-0 iprev-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule iprev-nth-image*[*OF - - mod-eq-div-right-strict-mono-on*])

2.4 Cardinality of intervals with interval operators

lemma *iT-Plus-card*: *card* $(I \oplus k) =$ *card I*
apply (*unfold iT-Plus-def*)
apply (*rule card-image*)
apply (*rule inj-imp-inj-on*)
apply (*rule add-right-inj*)
done
lemma *iT-Mult-card*: $0 < k \Longrightarrow$ *card* $(I \otimes k) =$ *card I*
apply (*unfold iT-Mult-def*)
apply (*rule card-image*)
apply (*rule inj-imp-inj-on*)
apply (*rule mult-right-inj*)
apply *assumption*
done

lemma *iT-Plus-neg-card*: *card* $(I \oplus- k) =$ *card* $(I \downarrow\geq k)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*rule card-image*)

apply (*subst cut-ge-Int-conv*)
apply (*rule subset-inj-on*[*OF - Int-lower2*])
apply (*rule sub-left-inj-on*)
done

lemma *iT-Plus-neg-card-le*: *card* (*I* ⊕− *k*) ≤ *card I*
apply (*simp add*: *iT-Plus-neg-card*)
apply (*case-tac finite I*)
apply (*rule cut-ge-card*, *assumption*)
apply (*simp add*: *nat-cut-ge-finite-iff*)
done

lemma *iT-Minus-card*: *card* (*k* ⊖ *I*) = *card* (*I* ↓≤ *k*)
apply (*simp add*: *iT-Minus-image-conv*)
apply (*rule card-image*)
apply (*subst cut-le-Int-conv*)
apply (*rule subset-inj-on*[*OF - Int-lower2*])
apply (*rule sub-right-inj-on*)
done

lemma *iT-Minus-card-le*: *finite I* ⟹ *card* (*k* ⊖ *I*) ≤ *card I*
by (*subst iT-Minus-card*, *rule cut-le-card*)

lemma *iT-Div-0-card-if*:
card (*I* ⊘ *0*) = (*if I* ={} *then 0 else Suc 0*)
by (*fastforce simp*: *iT-Div-empty iT-Div-0 iTILL-0*)

lemma *Int-empty-sum*:
(∑ *k*≤(*n*::*nat*). *if* {} ∩ (*I k*) = {} *then 0 else Suc 0*) = *0*
apply (*rule sum-eq-0-iff*[*THEN iffD2*])
apply (*rule finite-atMost*)
apply *simp*
done

lemma *iT-Div-mod-partition-card*:
card (*I* ∩ [*n* ∗ *d*. . .,*d* − *Suc 0*] ⊘ *d*) =
(*if I* ∩ [*n* ∗ *d*. . .,*d* − *Suc 0*] = {} *then 0 else Suc 0*)
apply (*case-tac d* = *0*)
apply (*simp add*: *iIN-0 iTILL-0 iT-Div-0-if*)
apply (*split if-split*, *rule conjI*)
apply (*simp add*: *iT-Div-empty*)
apply *clarsimp*
apply (*subgoal-tac I* ∩ [*n* ∗ *d*. . .,*d* − *Suc 0*] ⊘ *d* = {*n*}, *simp*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
apply (*clarsimp simp*: *le-less-imp-div*)
apply (*drule ex-in-conv*[*THEN iffD2*], *clarsimp simp*: *iIN-iff*, *rename-tac x′*)
apply (*rule-tac x*=*x′* **in** *exI*)

apply (*simp add*: *le-less-imp-div*)
done

lemma *iT-Div-conv-count*:
  $0 < d \Longrightarrow I \oslash d = \{k.\ I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\}$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Div-empty*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff-Int*)
done

lemma *iT-Div-conv-count2*:
  $[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n\ ]\!] \Longrightarrow$
  $I \oslash d = \{k.\ k \le n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\}$
apply (*simp add*: *iT-Div-conv-count*)
apply (*rule set-eqI*, *simp*)
apply (*rule iffI*)
 apply *simp*
 apply (*rule ccontr*)
 apply (*drule ex-in-conv*[*THEN iffD2*], *clarify*, *rename-tac* x')
 apply (*clarsimp simp*: *linorder-not-le iIN-iff*)
 apply (*drule order-le-less-trans*, *simp*)
 apply (*drule div-less-conv*[*THEN iffD1*, *of - Max I*], *simp*)
 apply (*drule-tac* $x=x'$ **in** *Max-ge*, *simp*)
 apply *simp+*
done

lemma *mod-partition-count-Suc*:
  $\{k.\ k \le Suc\ n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\} =$
  $\{k.\ k \le n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\} \cup$
    $(if\ I \cap [Suc\ n * d\ldots,d - Suc\ 0] \neq \{\}\ then\ \{Suc\ n\}\ else\ \{\})$
apply (*rule set-eqI*, *rename-tac x*)
apply (*simp add*: *le-less*[*of - Suc n*] *less-Suc-eq-le*)
apply (*simp add*: *conj-disj-distribR*)
apply (*intro conjI impI*)
 apply *fastforce*
apply (*rule iffI*, *clarsimp+*)
done

lemma *iT-Div-card*:
  $\bigwedge I.[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \le n]\!] \Longrightarrow$
  $card\ (I \oslash d) = (\sum k{\le}n.$
    $if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Div-empty*)
apply (*simp add*: *iT-Div-conv-count2*)
apply (*induct n*)
 apply (*simp add*: *div-eq-0-conv iIN-0-iTILL-conv*)
 apply (*subgoal-tac* $I \cap [\ldots d - Suc\ 0] \neq \{\}$)

prefer *2*
apply (*simp add*: *ex-in-conv*[*symmetric*], *fastforce*)
apply (*simp add*: *card-1-singleton-conv*)
apply (*rule-tac x=0* **in** *exI*)
apply (*rule set-eqI*)
apply (*simp add*: *ex-in-conv*[*symmetric*], *fastforce*)
apply *simp*
apply (*simp add*: *mod-partition-count-Suc*)
apply (*drule-tac* $x{=}I \cap [\ldots n * d + d - Suc\ 0]$ **in** *meta-spec*)
apply *simp*
apply (*case-tac* $I \cap [\ldots n * d + d - Suc\ 0] = \{\}$)
apply *simp*
apply (*subgoal-tac* $\{k.\ k \leq n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\} = \{\}$, *simp*)
apply (*clarsimp*, *rename-tac x*)
apply (*subgoal-tac* $I \cap [x * d\ldots,d - Suc\ 0] \subseteq I \cap [\ldots n * d + d - Suc\ 0]$, *simp*)
apply (*rule Int-mono*[*OF order-refl*])
apply (*simp add*: *iIN-iTILL-subset-conv*)
apply (*simp add*: *diff-le-mono*)
apply (*subgoal-tac* $Max\ (I \cap [\ldots n * d + d - Suc\ 0])\ div\ d \leq n$)
prefer *2*
apply (*simp add*: *div-le-conv add.commute*[*of d*] *iTILL-iff*)
apply (*subgoal-tac* $\bigwedge k.\ k \leq n \Longrightarrow [\ldots n * d + d - Suc\ 0] \cap [k * d\ldots,d - Suc\ 0] = [k * d\ldots,d - Suc\ 0]$)
prefer *2*
apply (*subst Int-commute*)
apply (*simp add*: *iTILL-def cut-le-Int-conv*[*symmetric*])
apply (*rule cut-le-Max-all*[*OF iIN-finite*])
apply (*simp add*: *iIN-Max diff-le-mono*)
apply (*simp add*: *Int-assoc*)
apply (*subgoal-tac*
$\{k.\ k \leq n \wedge I \cap ([\ldots n * d + d - Suc\ 0] \cap [k * d\ldots,d - Suc\ 0]) \neq \{\}\} =$
$\{k.\ k \leq n \wedge I \cap [k * d\ldots,d - Suc\ 0] \neq \{\}\}$)
prefer *2*
apply (*rule set-eqI*, *rename-tac x*)
apply *simp*
apply (*rule conj-cong*, *simp*)
apply *simp*
apply *simp*
done

corollary *iT-Div-card-Suc*:
$\bigwedge I.[\![\ 0 < d;\ finite\ I;\ Max\ I\ div\ d \leq n]\!] \Longrightarrow$
$card\ (I \oslash d) = (\sum k{<}Suc\ n.$
$if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
by (*simp add*: *lessThan-Suc-atMost iT-Div-card*)
corollary *iT-Div-Max-card*: $[\![\ 0 < d;\ finite\ I\ ]\!] \Longrightarrow$
$card\ (I \oslash d) = (\sum k{\leq}Max\ I\ div\ d.$
$if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
by (*simp add*: *iT-Div-card*)

lemma *iT-Div-card-le*: *0* < *k* ⟹ *card* (*I* ⊘ *k*) ≤ *card I*
apply (*case-tac finite I*)
 apply (*simp add*: *iT-Div-def card-image-le*)
apply (*simp add*: *iT-Div-finite-iff*)
done

lemma *iT-Div-card-inj-on*:
  inj-on (λ*n*. *n div k*) *I* ⟹ *card* (*I* ⊘ *k*) = *card I*
by (*unfold iT-Div-def*, *rule card-image*)

lemma *mod-Suc′*:
  0 < *n* ⟹ *Suc m mod n* = (*if m mod n* < *n* − *Suc 0 then Suc* (*m mod n*) *else 0*)
apply (*simp add*: *mod-Suc*)
apply (*intro conjI impI*)
 apply *simp*
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
done

lemma *div-Suc*:
  0 < *n* ⟹ *Suc m div n* = (*if Suc* (*m mod n*) = *n then Suc* (*m div n*) *else m div n*)
apply (*drule Suc-leI*, *drule le-imp-less-or-eq*)
apply (*case-tac n* = *Suc 0*, *simp*)
apply (*split if-split*, *intro conjI impI*)
 apply (*rule-tac t*=*Suc m* **and** *s*=*m* + *1* **in** *subst*, *simp*)
 apply (*subst div-add1-eq2*, *simp*+)
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
apply (*rule-tac t*=*Suc m* **and** *s*=*m* + *1* **in** *subst*, *simp*)
apply (*subst div-add1-eq1*, *simp*+)
done

lemma *div-Suc′*:
  0 < *n* ⟹ *Suc m div n* = (*if m mod n* < *n* − *Suc 0 then m div n else Suc* (*m div n*))
apply (*simp add*: *div-Suc*)
apply (*intro conjI impI*)
 apply *simp*
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
done

lemma *iT-Div-card-ge-aux*:
  ⋀*I*. ⟦ *0* < *d*; *finite I*; *Max I div d* ≤ *n* ⟧ ⟹

card I div d + (*if card I mod d* = *0 then 0 else Suc 0*) ≤ *card* (*I* ⊘ *d*)
apply (*induct n*)
apply (*case-tac I* = {}, *simp*)
apply (*frule-tac m=d* **and** *n=Max I* **and** *k=0* **in** *div-le-conv*[*THEN iffD1*, *rule-format*], *assumption*)
apply (*simp del*: *Max-le-iff*)
apply (*subgoal-tac I* ⊘ *d* = {*0*})
prefer *2*
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff*)
apply (*rule iffI*)
apply (*fastforce simp*: *div-eq-0-conv′*)
apply *fastforce*
apply (*simp add*: *iT-Div-singleton card-singleton del*: *Max-le-iff*)
apply (*drule Suc-le-mono*[*THEN iffD2*, *of - d* − *Suc 0*])
apply (*drule order-trans*[*OF nat-card-le-Max*])
apply (*simp*, *intro conjI impI*)
apply (*drule div-le-mono*[*of - d d*])
apply *simp*
apply (*subgoal-tac card I* ≠ *d*, *simp*)
apply *clarsimp*
apply (*drule order-le-less*[*THEN iffD1*], *erule disjE*)
apply *simp*
apply (*rule-tac t=I* **and** *s=I* ∩ [. . .*n* ∗ *d* + *d* − *Suc 0*] ∪ *I* ∩ [*Suc n* ∗ *d*. . .,*d* − *Suc 0*] **in** *subst*)
apply (*simp add*: *Int-Un-distrib*[*symmetric*] *add.commute*[*of d*])
apply (*subst iIN-0-iTILL-conv*[*symmetric*])
apply (*simp add*: *iIN-union*)
apply (*rule Int-absorb2*)
apply (*simp add*: *iIN-0-iTILL-conv iTILL-def*)
apply (*case-tac I* = {}, *simp*)
apply (*simp add*: *subset-atMost-Max-le-conv le-less-div-conv*[*symmetric*] *less-eq-le-pred*[*symmetric*] *add.commute*[*of d*])
apply (*cut-tac A=I* ∩ [. . .*n* ∗ *d* + *d* − *Suc 0*] **and** *B=I* ∩ [*Suc n* ∗ *d*. . .,*d* − *Suc 0*] **in** *card-Un-disjoint*)
apply *simp*
apply *simp*
apply (*clarsimp simp*: *disjoint-iff-in-not-in1 iT-iff*)
apply (*case-tac I* ∩ [. . .*n* ∗ *d* + *d* − *Suc 0*] = {})
apply (*simp add*: *iT-Div-mod-partition-card del*: *mult-Suc*)
apply (*intro conjI impI*)
apply (*rule div-le-conv*[*THEN iffD2*], *assumption*)
apply *simp*
apply (*rule order-trans*[*OF Int-card2*[*OF iIN-finite*]])
apply (*simp add*: *iIN-card*)
apply (*cut-tac A=I* **and** *n=Suc n* ∗ *d* **and** *d=d* − *Suc 0* **in** *Int-card2*[*OF iIN-finite*, *rule-format*])
apply (*simp add*: *iIN-card*)
apply (*drule order-le-less*[*THEN iffD1*], *erule disjE*)

apply *simp*
apply *simp*
apply (*subgoal-tac Max* ($I \cap [\ldots n * d + d - Suc\ 0]$) *div d* $\leq$ *n*)
prefer *2*
apply (*rule div-le-conv*[*THEN iffD2*], *assumption*)
apply (*rule order-trans*[*OF Max-Int-le2*[*OF - iTILL-finite*]], *assumption*)
apply (*simp add*: *iTILL-Max*)
apply (*simp only*: *iT-Div-Un*)
apply (*cut-tac* $A=I \cap [\ldots n * d + d - Suc\ 0] \oslash d$ **and** $B=I \cap [Suc\ n * d\ldots,d - Suc\ 0] \oslash d$ **in** *card-Un-disjoint*)
apply (*simp add*: *iT-Div-finite-iff*)
apply (*simp add*: *iT-Div-finite-iff*)
apply (*subst iIN-0-iTILL-conv*[*symmetric*])
apply (*subst mod-partition-iT-Div-Int-one-segment*, *simp*)
apply (*cut-tac* $n=0$ **and** $d=n * d+d$ **and** $k=d$ **and** $A=I$ **in** *mod-partition-iT-Div-Int2*, *simp+*)
apply (*rule disjoint-iff-in-not-in1*[*THEN iffD2*])
apply *clarsimp*
apply (*simp add*: *iIN-div-mod-eq-0*)
apply (*simp add*: *mod-0-imp-sub-1-div-conv iIN-0-iTILL-conv iIN-0 iTILL-iff*)
apply (*simp only*: *iT-Div-mod-partition-card*)
apply (*subgoal-tac finite* ($I \cap [\ldots n * d + d - Suc\ 0]$))
prefer *2*
apply *simp*
apply *simp*
apply (*intro conjI impI*)
apply (*rule add-le-divisor-imp-le-Suc-div*)
apply (*rule add-leD1*, *blast*)
apply (*rule Int-card2*[*OF iIN-finite*, *THEN le-trans*])
apply (*simp add*: *iIN-card*)
apply (*cut-tac* $A=I$ **and** $n=Suc\ n * d$ **and** $d=d - Suc\ 0$ **in** *Int-card2*[*OF iIN-finite*, *rule-format*])
apply (*simp add*: *iIN-card*)
apply (*rule-tac y=let* $I=I \cap [\ldots n * d + d - Suc\ 0]$ *in*
card I div d + (*if card I mod d* = *0 then 0 else Suc 0*) **in** *order-trans*)
prefer *2*
apply (*simp add*: *Let-def*)
apply (*unfold Let-def*)
apply (*split if-split*, *intro conjI impI*)
apply (*subgoal-tac card* ($I \cap [Suc\ n * d\ldots,d - Suc\ 0]$) $\neq$ *d*)
prefer *2*
apply (*rule ccontr*, *simp*)
apply (*simp add*: *div-add1-eq1-mod-0-left*)
apply (*simp add*: *add-le-divisor-imp-le-Suc-div*)
done

lemma *iT-Div-card-ge*:
card I div d + (*if card I mod d* = *0 then 0 else Suc 0*) $\leq$ *card* ($I \oslash d$)
apply (*case-tac* $I = \{\}$, *simp*)

apply (*case-tac finite I*)
 prefer *2*
 apply *simp*
apply (*case-tac d = 0*)
 apply (*simp add*: *iT-Div-0 iTILL-0*)
apply (*simp add*: *iT-Div-card-ge-aux*[*OF - - order-refl*])
done

corollary *iT-Div-card-ge-div*: *card I div d* ≤ *card* (*I* ⊘ *d*)
by (*rule iT-Div-card-ge*[*THEN add-leD1*])

There is no better lower bound function *f* for *i* ⊘ *d* with *card i* and *d* as arguments.

lemma *iT-Div-card-ge--is-maximal-lower-bound*:
 ∀ *I d*. *card I div d* + (*if card I mod d* = *0 then 0 else Suc 0*) ≤ *f* (*card I*) *d* ∧
 f (*card I*) *d* ≤ *card* (*I* ⊘ *d*) ⟹
 f (*card* (*I*::*nat set*)) *d* = *card I div d* + (*if card I mod d* = *0 then 0 else Suc 0*)
apply (*case-tac I* = {})
 apply (*erule-tac x*=*I* **in** *allE*, *erule-tac x*=*d* **in** *allE*)
 apply (*simp add*: *iT-Div-empty*)
apply (*case-tac d* = *0*)
 apply (*frule-tac x*={} **in** *spec*, *erule-tac x*=*I* **in** *allE*)
 apply (*erule-tac x*=*d* **in** *allE*, *erule-tac x*=*d* **in** *allE*)
 apply (*clarsimp simp*: *iT-Div-0 iTILL-card iT-Div-empty*)
apply (*rule order-antisym*)
 prefer *2*
 apply *simp*
apply (*case-tac finite I*)
 prefer *2*
 apply (*erule-tac x*=*I* **in** *allE*, *erule-tac x*=*d* **in** *allE*)
 apply (*simp add*: *iT-Div-finite-iff*)
apply (*erule-tac x*=[. . .*card I* − *Suc 0*] **in** *allE*, *erule-tac x*=*d* **in** *allE*, *elim conjE*)
apply (*frule not-empty-card-gr0-conv*[*THEN iffD1*], *assumption*)
apply (*simp add*: *iTILL-card iTILL-div*)
apply (*intro conjI impI*)
 apply (*simp add*: *mod-0-imp-sub-1-div-conv*)
 apply (*subgoal-tac d* ≤ *card I*)
 prefer *2*
  apply (*clarsimp elim*!: *dvdE*)
 apply (*drule div-le-mono*[*of d - d*])
 apply *simp*
apply (*case-tac d* = *Suc 0*, *simp*)
apply (*simp add*: *div-diff1-eq1*)
done

lemma *iT-Plus-icard*: *icard* (*I* ⊕ *k*) = *icard I*
by (*simp add*: *iT-Plus-def icard-image*)

lemma *iT-Mult-icard*: $0 < k \Longrightarrow icard\ (I \otimes k) = icard\ I$
apply (*unfold iT-Mult-def*)
apply (*rule icard-image*)
apply (*rule inj-imp-inj-on*)
apply (*simp add*: *mult-right-inj*)
done

lemma *iT-Plus-neg-icard*: $icard\ (I \oplus- k) = icard\ (I \downarrow\geq k)$
apply (*case-tac finite I*)
 apply (*simp add*: *iT-Plus-neg-finite-iff cut-ge-finite icard-finite iT-Plus-neg-card*)
apply (*simp add*: *iT-Plus-neg-finite-iff nat-cut-ge-finite-iff*)
done

lemma *iT-Plus-neg-icard-le*: $icard\ (I \oplus- k) \leq icard\ I$
apply (*case-tac finite I*)
 apply (*simp add*: *iT-Plus-neg-finite-iff icard-finite iT-Plus-neg-card-le*)
apply *simp*
done

lemma *iT-Minus-icard*: $icard\ (k \ominus I) = icard\ (I \downarrow\leq k)$
by (*simp add*: *icard-finite iT-Minus-finite nat-cut-le-finite iT-Minus-card*)

lemma *iT-Minus-icard-le*: $icard\ (k \ominus I) \leq icard\ I$
apply (*case-tac finite I*)
 apply (*simp add*: *icard-finite iT-Minus-finite iT-Minus-card-le*)
apply *simp*
done

lemma *iT-Div-0-icard-if*: $icard\ (I \oslash 0) = enat\ (if\ I = \{\}\ then\ 0\ else\ Suc\ 0)$
by (*simp add*: *icard-finite iT-Div-0-finite iT-Div-0-card-if*)

lemma *iT-Div-mod-partition-icard*:
  $icard\ (I \cap [n * d\ldots,d - Suc\ 0] \oslash d) =$
  $enat\ (if\ I \cap [n * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)$
apply (*subgoal-tac finite* $(I \cap [n * d\ldots,d - Suc\ 0] \oslash d)$)
 prefer *2*
 apply (*case-tac* $d = 0$, *simp add*: *iT-Div-0-finite*)
 apply (*simp add*: *iT-Div-finite-iff iIN-finite*)
apply (*simp add*: *icard-finite iT-Div-mod-partition-card*)
done

lemma *iT-Div-icard*:
  $[\![\ 0 < d;\ finite\ I \Longrightarrow Max\ I\ div\ d \leq n]\!] \Longrightarrow$
  $icard\ (I \oslash d) =$
  $(if\ finite\ I\ then\ enat\ (\sum k{\leq}n.\ if\ I \cap [k * d\ldots,d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc$
$0)\ else\ \infty)$
by (*simp add*: *icard-finite iT-Div-finite-iff iT-Div-card*)

corollary *iT-Div-Max-icard*: $0 < d \Longrightarrow$

icard (*I* ⊘ *d*) = (*if finite I*
then enat (∑ *k*≤*Max I div d*. *if I* ∩ [*k* ∗ *d*. . .,*d* − *Suc 0*] = {} *then 0 else Suc 0*) *else* ∞)
by (*simp add*: *iT-Div-icard*)

lemma *iT-Div-icard-le*: *0* < *k* ⟹ *icard* (*I* ⊘ *k*) ≤ *icard I*
apply (*case-tac finite I*)
apply (*simp add*: *iT-Div-finite-iff icard-finite iT-Div-card-le*)
apply *simp*
done

lemma *iT-Div-icard-inj-on*: *inj-on* (λ*n*. *n div k*) *I* ⟹ *icard* (*I* ⊘ *k*) = *icard I*
by (*simp add*: *iT-Div-def icard-image*)

lemma *iT-Div-icard-ge*: *icard I div* (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*) ≤ *icard* (*I* ⊘ *d*)
apply (*case-tac d* = *0*)
apply (*simp add*: *icard-finite iT-Div-0-finite*)
apply (*case-tac icard I*)
apply (*fastforce simp*: *iT-Div-0-card-if*)
apply (*simp add*: *iT-Div-0-card-if icard-infinite-conv infinite-imp-nonempty*)
apply (*case-tac finite I*)
apply (*simp add*: *iT-Div-finite-iff icard-finite iT-Div-card-ge*)
apply (*simp add*: *iT-Div-finite-iff*)
done

corollary *iT-Div-icard-ge-div*: *icard I div* (*enat d*) ≤ *icard* (*I* ⊘ *d*)
by (*rule iT-Div-icard-ge*[*THEN iadd-ileD1*])

lemma *iT-Div-icard-ge--is-maximal-lower-bound*:
∀ *I d*. *icard I div* (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*)
≤ *f* (*icard I*) *d* ∧
f (*icard I*) *d* ≤ *icard* (*I* ⊘ *d*) ⟹
f (*icard* (*I*::*nat set*)) *d* =
icard I div (*enat d*) + *enat* (*if icard I mod* (*enat d*) = *0 then 0 else Suc 0*)
apply (*case-tac d* = *0*)
apply (*drule-tac x*=*I* **in** *spec*, *drule-tac x*=*d* **in** *spec*, *erule conjE*)
apply (*simp add*: *iT-Div-0-icard-if icard-0-eq*[*unfolded zero-enat-def*])
apply (*case-tac finite I*)
prefer *2*
apply (*drule-tac x*=*I* **in** *spec*, *drule-tac x*=*d* **in** *spec*)
apply *simp*
apply *simp*
apply (*frule-tac iT-Div-finite-iff*[*THEN iffD2*], *assumption*)
apply (*cut-tac f*=λ*c d*. *the-enat* (*f* (*enat c*) *d*) **and** *I*=*I* **and** *d*=*d* **in** *iT-Div-card-ge--is-maximal-lower-bound*)
apply (*intro allI*, *rename-tac I′ d′*)
apply (*subgoal-tac* ⋀*k*. *f 0 k* = *0*)
prefer *2*

apply (*drule-tac x={}* **in** *spec*, *drule-tac x=k* **in** *spec*, *erule conjE*)
apply (*simp add*: *iT-Div-empty*)
apply (*drule-tac x=I′* **in** *spec*, *drule-tac x=d′* **in** *spec*, *erule conjE*)
apply (*case-tac d′ = 0*)
apply (*simp add*: *idiv-by-0 imod-by-0 iT-Div-0-card-if iT-Div-0-icard-if*)
apply (*case-tac I′ = {}*, *simp*)
apply (*case-tac finite I′*)
apply (*simp add*: *icard-finite*)
apply *simp*
apply *simp*
apply (*case-tac finite I′*)
apply (*frule-tac I=I′* **and** *k=d′* **in** *iT-Div-finite-iff*[*THEN iffD2*, *rule-format*], *assumption*)
apply (*simp add*: *icard-finite*)
apply (*subgoal-tac ∃ n. f* (*enat* (*card I′*)) *d′ = enat n*)
prefer *2*
apply (*rule enat-ile*, *assumption*)
apply *clarsimp*
apply (*subgoal-tac infinite* (*I′ ⊘ d′*))
prefer *2*
apply (*simp add*: *iT-Div-finite-iff*)
apply *simp*
apply (*drule-tac x=I* **in** *spec*, *drule-tac x=d* **in** *spec*, *erule conjE*)
apply (*simp add*: *icard-finite*)
apply (*subgoal-tac ∃ n. f* (*enat* (*card I*)) *d = enat n*)
prefer *2*
apply (*rule enat-ile*, *assumption*)
apply *clarsimp*
done

2.5 Results about sets of intervals

2.5.1 Set of intervals without and with empty interval

definition *iFROM-UN-set* :: (*nat set*) *set*
where *iFROM-UN-set* ≡ ⋃ *n*. {[*n*. . .]}

definition *iTILL-UN-set* :: (*nat set*) *set*
where *iTILL-UN-set* ≡ ⋃ *n*. {[. . .*n*]}

definition *iIN-UN-set* :: (*nat set*) *set*
where *iIN-UN-set* ≡ ⋃ *n d*. {[*n*. . .,*d*]}

definition *iMOD-UN-set* :: (*nat set*) *set*
where *iMOD-UN-set* ≡ ⋃ *r m*. {[*r*, *mod m*]}

definition *iMODb-UN-set* :: (*nat set*) *set*
where *iMODb-UN-set* ≡ ⋃ *r m c*. {[*r*, *mod m*, *c*]}

definition *iFROM-set* :: (*nat set*) *set*
where *iFROM-set* ≡ {[*n*. . .] |*n*. *True*}

definition *iTILL-set* :: (*nat set*) *set*
where *iTILL-set* ≡ {[. . .*n*] |*n*. *True*}

definition *iIN-set* :: (*nat set*) *set*
where *iIN-set* ≡ {[*n*. . .,*d*] |*n d*. *True*}

definition *iMOD-set* :: (*nat set*) *set*
where *iMOD-set* ≡ {[*r*, *mod m*] |*r m*. *True*}

definition *iMODb-set* :: (*nat set*) *set*
where *iMODb-set* ≡ {[*r*, *mod m*, *c*] |*r m c*. *True*}

definition *iMOD2-set* :: (*nat set*) *set*
where *iMOD2-set* ≡ {[*r*, *mod m*] |*r m*. *2* ≤ *m*}

definition *iMODb2-set* :: (*nat set*) *set*
where *iMODb2-set* ≡ {[*r*, *mod m*, *c*] |*r m c*. *2* ≤ *m* ∧ *1* ≤ *c*}

definition *iMOD2-UN-set* :: (*nat set*) *set*
where *iMOD2-UN-set* ≡ ⋃ *r*. ⋃ *m*∈{*2*..}. {[*r*, *mod m*]}

definition *iMODb2-UN-set* :: (*nat set*) *set*
where *iMODb2-UN-set* ≡ ⋃ *r*. ⋃ *m*∈{*2*..}. ⋃ *c*∈{*1*..}. {[*r*, *mod m*, *c*]}

lemmas *i-set-defs* =
iFROM-set-def iTILL-set-def iIN-set-def
iMOD-set-def iMODb-set-def
iMOD2-set-def iMODb2-set-def
lemmas *i-UN-set-defs* =
iFROM-UN-set-def iTILL-UN-set-def iIN-UN-set-def
iMOD-UN-set-def iMODb-UN-set-def
iMOD2-UN-set-def iMODb2-UN-set-def

lemma *iFROM-set-UN-set-eq*: *iFROM-set* = *iFROM-UN-set*
by (*fastforce simp*: *iFROM-set-def iFROM-UN-set-def*)

lemma
iTILL-set-UN-set-eq: *iTILL-set* = *iTILL-UN-set* **and**
iIN-set-UN-set-eq: *iIN-set* = *iIN-UN-set* **and**
iMOD-set-UN-set-eq: *iMOD-set* = *iMOD-UN-set* **and**
iMODb-set-UN-set-eq: *iMODb-set* = *iMODb-UN-set*
by (*fastforce simp*: *i-set-defs i-UN-set-defs*)+

lemma *iMOD2-set-UN-set-eq*: *iMOD2-set* = *iMOD2-UN-set*

by (*fastforce simp*: *i-set-defs i-UN-set-defs*)

lemma *iMODb2-set-UN-set-eq*: *iMODb2-set* = *iMODb2-UN-set*
by (*fastforce simp*: *i-set-defs i-UN-set-defs*)

lemmas *i-set-i-UN-set-sets-eq* =
iFROM-set-UN-set-eq
iTILL-set-UN-set-eq
iIN-set-UN-set-eq
iMOD-set-UN-set-eq
iMODb-set-UN-set-eq
iMOD2-set-UN-set-eq
iMODb2-set-UN-set-eq

lemma *iMOD2-set-iMOD-set-subset*: *iMOD2-set* ⊆ *iMOD-set*
by (*fastforce simp*: *i-set-defs*)

lemma *iMODb2-set-iMODb-set-subset*: *iMODb2-set* ⊆ *iMODb-set*
by (*fastforce simp*: *i-set-defs*)

definition *i-set* :: (*nat set*) *set*
where *i-set* ≡ *iFROM-set* ∪ *iTILL-set* ∪ *iIN-set* ∪ *iMOD-set* ∪ *iMODb-set*

definition *i-UN-set* :: (*nat set*) *set*
where *i-UN-set* ≡ *iFROM-UN-set* ∪ *iTILL-UN-set* ∪ *iIN-UN-set* ∪ *iMOD-UN-set* ∪ *iMODb-UN-set*

Minimal definitions for *i-set* and *i-set*

definition *i-set-min* :: (*nat set*) *set*
where *i-set-min* ≡ *iFROM-set* ∪ *iIN-set* ∪ *iMOD2-set* ∪ *iMODb2-set*

definition *i-UN-set-min* :: (*nat set*) *set*
where *i-UN-set-min* ≡ *iFROM-UN-set* ∪ *iIN-UN-set* ∪ *iMOD2-UN-set* ∪ *iMODb2-UN-set*

definition *i-set0* :: (*nat set*) *set*
where *i-set0* ≡ *insert* {} *i-set*

lemma *i-set-i-UN-set-eq*: *i-set* = *i-UN-set*
by (*simp add*: *i-set-def i-UN-set-def i-set-i-UN-set-sets-eq*)

lemma *i-set-min-i-UN-set-min-eq*: *i-set-min* = *i-UN-set-min*
by (*simp add*: *i-set-min-def i-UN-set-min-def i-set-i-UN-set-sets-eq*)

lemma *i-set-min-eq*: *i-set* = *i-set-min*
apply (*unfold i-set-def i-set-min-def*)
apply (*rule equalityI*)
apply (*rule subsetI*)

apply (*simp add*: *i-set-defs*)
apply (*elim disjE*)
apply *blast*
apply (*fastforce simp*: *iIN-0-iTILL-conv*[*symmetric*])
apply *blast*
apply (*elim exE*)
apply (*case-tac* $2 \le m$, *blast*)
apply (*simp add*: *nat-ge2-conv*)
apply (*fastforce simp*: *iMOD-0 iMOD-1*)
apply (*elim exE*)
apply (*case-tac* $1 \le c$)
apply (*case-tac* $2 \le m$, *fastforce*)
apply (*simp add*: *nat-ge2-conv*)
apply (*fastforce simp*: *iMODb-mod-0 iMODb-mod-1*)
apply (*fastforce simp*: *linorder-not-le less-Suc-eq-le iMODb-0*)
apply (*rule Un-mono*)+
apply (*simp-all add*: *iMOD2-set-iMOD-set-subset iMODb2-set-iMODb-set-subset*)
done

corollary *i-UN-set-i-UN-min-set-eq*: *i-UN-set* = *i-UN-set-min*
by (*simp add*: *i-set-min-eq i-set-i-UN-set-eq*[*symmetric*] *i-set-min-i-UN-set-min-eq*[*symmetric*])

lemma *i-set-min-is-minimal-let*:
let s1 = *iFROM-set*; *s2*= *iIN-set*; *s3*= *iMOD2-set*; *s4*= *iMODb2-set in*
$s1 \cap s2 = \{\} \wedge s1 \cap s3 = \{\} \wedge s1 \cap s4 = \{\} \wedge$
$s2 \cap s3 = \{\} \wedge s2 \cap s4 = \{\} \wedge s3 \cap s4 = \{\}$
apply (*unfold Let-def i-set-defs*, *intro conjI*)
apply (*rule disjoint-iff-in-not-in1*[*THEN iffD2*], *clarsimp simp*: *iT-neq*)+
done

lemmas *i-set-min-is-minimal* = *i-set-min-is-minimal-let*[*simplified*]

inductive-set *i-set-ind*:: (*nat set*) *set*
where
i-set-ind-FROM[*intro*!]: [*n*. . .] ∈ *i-set-ind*
| *i-set-ind-TILL*[*intro*!]: [. . .*n*] ∈ *i-set-ind*
| *i-set-ind-IN*[*intro*!]: [*n*. . .,*d*] ∈ *i-set-ind*
| *i-set-ind-MOD*[*intro*!]: [*r*, *mod m*] ∈ *i-set-ind*
| *i-set-ind-MODb*[*intro*!]: [*r*, *mod m*, *c*] ∈ *i-set-ind*

inductive-set *i-set0-ind* :: (*nat set*) *set*
where
i-set0-ind-empty[*intro*!] : {} ∈ *i-set0-ind*
| *i-set0-ind-i-set*[*intro*]: *I* ∈ *i-set-ind* ⟹ *I* ∈ *i-set0-ind*

The introduction rule *i-set0-ind-i-set* is not declared a safe introduction rule, because it would disturb the correct usage of the *safe* method.

lemma *i-set-ind-subset-i-set0-ind*: *i-set-ind* ⊆ *i-set0-ind*

by (*rule subsetI*, *rule i-set0-ind-i-set*)

lemma
i-set0-ind-FROM[*intro*!] : [*n*. . .] ∈ *i-set0-ind* **and**
i-set0-ind-TILL[*intro*!] : [. . .*n*] ∈ *i-set0-ind* **and**
i-set0-ind-IN[*intro*!] : [*n*. . .,*d*] ∈ *i-set0-ind* **and**
i-set0-ind-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set0-ind* **and**
i-set0-ind-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set0-ind*
by (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*], *rule i-set-ind.intros*)+

lemmas *i-set0-ind-intros2* =
i-set0-ind-empty
i-set0-ind-FROM
i-set0-ind-TILL
i-set0-ind-IN
i-set0-ind-MOD
i-set0-ind-MODb

lemma *i-set-i-set-ind-eq*: *i-set* = *i-set-ind*
apply (*rule set-eqI*, *unfold i-set-def i-set-defs*)
apply (*rule iffI*, *blast*)
apply (*induct-tac x rule*: *i-set-ind.induct*)
apply *blast*+
done

lemma *i-set0-i-set0-ind-eq*: *i-set0* = *i-set0-ind*
apply (*rule set-eqI*, *unfold i-set0-def*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rule iffI*)
apply *blast*
apply (*rule-tac a*=*x* **in** *i-set0-ind.cases*)
apply *blast*+
done

lemma *i-set-imp-not-empty*: *I* ∈ *i-set* =⇒ *I* ≠ {}
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*rule iT-not-empty*)+
done

lemma *i-set0-i-set-mem-conv*: (*I* ∈ *i-set0*) = (*I* ∈ *i-set* ∨ *I* = {})
apply (*simp add*: *i-set-i-set-ind-eq i-set0-i-set0-ind-eq*)
apply (*rule iffI*)
apply (*rule i-set0-ind.cases*[*of I*])
apply *blast*+
done

lemma *i-set-i-set0-mem-conv*: (*I* ∈ *i-set*) = (*I* ∈ *i-set0* ∧ *I* ≠ {})
apply (*insert i-set-imp-not-empty*[*of I*])

apply (*fastforce simp*: *i-set0-i-set-mem-conv*)
done

lemma *i-set0-i-set-conv*: *i-set0* − {{}} = *i-set*
by (*fastforce simp*: *i-set-i-set0-mem-conv*)

corollary *i-set-subset-i-set0*: *i-set* ⊆ *i-set0*
by (*simp add*: *i-set0-i-set-conv*[*symmetric*])

lemma *i-set-singleton*: {*a*} ∈ *i-set*
by (*fastforce simp*: *i-set-def iIN-set-def iIN-0*[*symmetric*])

lemma *i-set0-singleton*: {*a*} ∈ *i-set0*
apply (*rule subsetD*[*OF i-set-subset-i-set0*])
apply (*simp add*: *iIN-0*[*symmetric*] *i-set-i-set-ind-eq i-set-ind.intros*)
done

corollary
i-set-FROM[*intro*!] : [*n*. . .] ∈ *i-set* **and**
i-set-TILL[*intro*!] : [. . .*n*] ∈ *i-set* **and**
i-set-IN[*intro*!] : [*n*. . .,*d*] ∈ *i-set* **and**
i-set-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set* **and**
i-set-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set*
by (*rule ssubst*[*OF i-set-i-set-ind-eq*], *rule i-set-ind.intros*)+

lemmas *i-set-intros* =
i-set-FROM
i-set-TILL
i-set-IN
i-set-MOD
i-set-MODb

lemma
i-set0-empty[*intro*!]: {} ∈ *i-set0* **and**
i-set0-FROM[*intro*!] : [*n*. . .] ∈ *i-set0* **and**
i-set0-TILL[*intro*!] : [. . .*n*] ∈ *i-set0* **and**
i-set0-IN[*intro*!] : [*n*. . .,*d*] ∈ *i-set0* **and**
i-set0-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set0* **and**
i-set0-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set0*
by (*rule ssubst*[*OF i-set0-i-set0-ind-eq*], *rule i-set0-ind-intros2*)+

lemmas *i-set0-intros* =
i-set0-empty
i-set0-FROM
i-set0-TILL
i-set0-IN
i-set0-MOD
i-set0-MODb

lemma *i-set-infinite-as-iMOD*:
⟦ $I \in$ *i-set*; *infinite I* ⟧ $\Longrightarrow \exists\, r\ m.\ I = [r,\ mod\ m]$
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-finite*)
apply (*rule-tac x=n* **in** *exI*, *rule-tac x=Suc 0* **in** *exI*, *simp add*: *iMOD-1*)
apply *blast*
done

lemma *i-set-finite-as-iMODb*:
⟦ $I \in$ *i-set*; *finite I* ⟧ $\Longrightarrow \exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c]$
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*simp add*: *iT-infinite*)
apply (*rule-tac x=0* **in** *exI*, *rule-tac x=Suc 0* **in** *exI*, *rule-tac x=n* **in** *exI*)
 apply (*simp add*: *iMODb-mod-1 iIN-0-iTILL-conv*)
apply (*rule-tac x=n* **in** *exI*, *rule-tac x=Suc 0* **in** *exI*, *rule-tac x=d* **in** *exI*)
 apply (*simp add*: *iMODb-mod-1*)
apply (*case-tac m = 0*)
 apply (*rule-tac x=r* **in** *exI*, *rule-tac x=Suc 0* **in** *exI*, *rule-tac x=0* **in** *exI*)
 apply (*simp add*: *iMOD-0 iIN-0 iMODb-0*)
apply (*simp add*: *iT-infinite*)
apply *blast*
done

lemma *i-set-as-iMOD-iMODb*:
$I \in$ *i-set* $\Longrightarrow (\exists\, r\ m.\ I = [r,\ mod\ m]) \lor (\exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c])$
by (*blast intro*: *i-set-finite-as-iMODb i-set-infinite-as-iMOD*)

2.5.2 Interval sets are closed under cutting

lemma *i-set-cut-le-ge-closed-disj*:
⟦ $I \in$ *i-set*; $t \in I$; *cut-op* = (↓≤) ∨ *cut-op* = (↓≥) ⟧ $\Longrightarrow$
cut-op I t $\in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply *safe*
apply (*simp-all add*: *iT-cut-le1 iT-cut-ge1 i-set-ind.intros iMODb-iff*)
done

corollary
i-set-cut-le-closed: ⟦ $I \in$ *i-set*; $t \in I$ ⟧ $\Longrightarrow$ I ↓≤ t $\in$ *i-set* **and**
i-set-cut-ge-closed: ⟦ $I \in$ *i-set*; $t \in I$ ⟧ $\Longrightarrow$ I ↓≥ t $\in$ *i-set*
by (*simp-all add*: *i-set-cut-le-ge-closed-disj*)

lemmas *i-set-cut-le-ge-closed* = *i-set-cut-le-closed i-set-cut-ge-closed*

lemma *i-set0-cut-closed-disj*:

⟦ $I \in$ *i-set0*;
 cut-op = (↓≤) ∨ *cut-op* = (↓≥) ∨
 cut-op = (↓<) ∨ *cut-op* = (↓>) ⟧ ⟹
cut-op I t ∈ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
 apply (*rule ssubst*[*OF set-restriction-empty*, **where** *P*=λ*x*. *x* ∈ *i-set0-ind*])
  apply (*rule i-cut-set-restriction-disj*[*of cut-op*], *blast*)
  apply *blast*
 apply *blast*
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply *safe*
apply (*simp-all add*: *iT-cut-le iT-cut-ge iT-cut-less iT-cut-greater i-set0-ind-intros2*)
done

corollary
 i-set0-cut-le-closed: *I* ∈ *i-set0* ⟹ *I* ↓≤ *t* ∈ *i-set0* **and**
 i-set0-cut-less-closed: *I* ∈ *i-set0* ⟹ *I* ↓< *t* ∈ *i-set0* **and**
 i-set0-cut-ge-closed: *I* ∈ *i-set0* ⟹ *I* ↓≥ *t* ∈ *i-set0* **and**
 i-set0-cut-greater-closed: *I* ∈ *i-set0* ⟹ *I* ↓> *t* ∈ *i-set0*
by (*simp-all add*: *i-set0-cut-closed-disj*)

lemmas *i-set0-cut-closed* =
 i-set0-cut-le-closed
 i-set0-cut-less-closed
 i-set0-cut-ge-closed
 i-set0-cut-greater-closed

2.5.3 Interval sets are closed under addition and multiplication

lemma *i-set-Plus-closed*: *I* ∈ *i-set* ⟹ *I* ⊕ *k* ∈ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-add i-set-ind.intros*)
done

lemma *i-set-Mult-closed*: *I* ∈ *i-set* ⟹ *I* ⊗ *k* ∈ *i-set*
apply (*case-tac k* = *0*)
 apply (*simp add*: *i-set-imp-not-empty iT-Mult-0-if i-set-intros*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-mult i-set-ind.intros*)
done

lemma *i-set0-Plus-closed*: *I* ∈ *i-set0* ⟹ *I* ⊕ *k* ∈ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct I rule*: *i-set0-ind.induct*)
 apply (*simp add*: *iT-Plus-empty i-set0-ind-empty*)

apply (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*])
apply (*simp add*: *i-set-i-set-ind-eq*[*symmetric*] *i-set-Plus-closed*)
done

lemma *i-set0-Mult-closed*: $I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct I rule*: *i-set0-ind.induct*)
apply (*simp add*: *iT-Mult-empty i-set0-ind-empty*)
apply (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*])
apply (*simp add*: *i-set-i-set-ind-eq*[*symmetric*] *i-set-Mult-closed*)
done

2.5.4 Interval sets are closed with certain conditions under subtraction

lemma *i-set-Plus-neg-closed*:
$[\![I \in$ *i-set*; $\exists x{\in}I.\ k \leq x\ ]\!] \Longrightarrow I \oplus- k \in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*fastforce simp*: *iT-iff iT-add-neg*)+
done

lemma *i-set-Minus-closed*:
$[\![I \in$ *i-set*; *iMin* $I \leq k\ ]\!] \Longrightarrow k \ominus I \in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*fastforce simp*: *iT-Min iT-sub*)+
done

lemma *i-set0-Plus-neg-closed*: $I \in$ *i-set0* $\Longrightarrow I \oplus- k \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
apply (*fastforce simp*: *iT-Plus-neg-empty*)
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply (*fastforce simp*: *iT-add-neg*)+
done

lemma *i-set0-Minus-closed*: $I \in$ *i-set0* $\Longrightarrow k \ominus I \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
apply (*simp add*: *iT-Minus-empty i-set0-ind-empty*)
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply (*fastforce simp*: *iT-sub*)+
done

lemmas *i-set-IntOp-closed* =
i-set-Plus-closed
i-set-Mult-closed

i-set-Plus-neg-closed
i-set-Minus-closed
lemmas *i-set0-IntOp-closed* =
i-set0-Plus-closed
i-set0-Mult-closed
i-set0-Plus-neg-closed
i-set0-Minus-closed

2.5.5 Interval sets are not closed under division

lemma *iMOD-div-mod-gr0-not-in-i-set*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k\ ]\!] \Longrightarrow [r,\ mod\ m] \oslash k \notin i\text{-}set$
apply (*rule ccontr*, *simp*)
apply (*drule i-set-infinite-as-iMOD*)
apply (*simp add*: *iT-Div-finite-iff iMOD-infinite*)
apply (*elim exE*, *rename-tac r′ m′*)
apply (*frule iMOD-div-mod-gr0-not-ex*[*of k m r*], *assumption*+)
apply *fastforce*
done

lemma *iMODb-div-mod-gr0-not-in-i-set*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k;\ k \le c\ ]\!] \Longrightarrow [r,\ mod\ m,\ c] \oslash k \notin i\text{-}set$
apply (*rule ccontr*, *simp*)
apply (*drule i-set-finite-as-iMODb*)
apply (*simp add*: *iT-Div-finite-iff iMODb-finite*)
apply (*elim exE*, *rename-tac r′ m′ c′*)
apply (*frule iMODb-div-mod-gr0-not-ex*[*of k m c r*], *assumption*+)
apply *fastforce*
done

lemma $[0,\ mod\ 3] \oslash 2 \notin i\text{-}set$
by (*rule iMOD-div-mod-gr0-not-in-i-set*, *simp*+)

lemma *i-set-Div-not-closed*: $Suc\ 0 < k \Longrightarrow \exists I{\in}i\text{-}set.\ I \oslash k \notin i\text{-}set$
apply (*rule-tac x*=[*0*, *mod* (*Suc k*)] **in** *bexI*)
apply (*rule iMOD-div-mod-gr0-not-in-i-set*)
apply (*simp-all add*: *mod-Suc i-set-MOD*)
done
lemma *i-set0-Div-not-closed*: $Suc\ 0 < k \Longrightarrow \exists I{\in}i\text{-}set0.\ I \oslash k \notin i\text{-}set0$
apply (*frule i-set-Div-not-closed*, *erule bexE*)
apply (*rule-tac x*=*I* **in** *bexI*)
apply (*simp add*: *i-set0-def iT-Div-not-empty i-set-imp-not-empty*)
apply (*rule subsetD*[*OF i-set-subset-i-set0*], *assumption*)
done

2.5.6 Sets of intervals closed under division

inductive-set *NatMultiples* :: *nat set* ⇒ *nat set*
for *F* :: *nat set*
where

NatMultiples-Factor: $k \in F \Longrightarrow k \in$ *NatMultiples F*
| *NatMultiples-Product*: ⟦ $k \in F$; $m \in$ *NatMultiples F* ⟧ $\Longrightarrow k * m \in$ *NatMultiples F*

lemma *NatMultiples-ex-divisor*: $m \in$ *NatMultiples F* $\Longrightarrow \exists k{\in}F.\ m \ mod\ k = 0$
apply (*induct m rule*: *NatMultiples.induct*)
apply (*rule-tac x=k* **in** *bexI*, *simp+*)+
done

lemma *NatMultiples-product-factor*: ⟦ $a \in F$; $b \in F$ ⟧ $\Longrightarrow a * b \in$ *NatMultiples F*
by (*drule NatMultiples-Factor*[*of b*], *rule NatMultiples-Product*)

lemma *NatMultiples-product-factor-multiple*:
⟦ $a \in F$; $b \in$ *NatMultiples F* ⟧ $\Longrightarrow a * b \in$ *NatMultiples F*
by (*rule NatMultiples-Product*)

lemma *NatMultiples-product-multiple-factor*:
⟦ $a \in$ *NatMultiples F*; $b \in F$ ⟧ $\Longrightarrow a * b \in$ *NatMultiples F*
by (*simp add*: *mult.commute*[*of a*] *NatMultiples-Product*)

lemma *NatMultiples-product-multiple*:
⟦ $a \in$ *NatMultiples F*; $b \in$ *NatMultiples F* ⟧ $\Longrightarrow a * b \in$ *NatMultiples F*
apply (*induct a rule*: *NatMultiples.induct*)
apply (*simp add*: *NatMultiples-Product*)
apply (*simp add*: *mult.assoc*[*of - - b*] *NatMultiples-Product*)
done

Subset of *i-set* containing only continuous intervals, i. e., without *iMOD* and *iMODb*.

inductive-set *i-set-cont* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: [*n*...] $\in$ *i-set-cont*
| *i-set-cont-TILL*[*intro*]: [...*n*] $\in$ *i-set-cont*
| *i-set-cont-IN*[*intro*]: [*n*...,*d*] $\in$ *i-set-cont*

definition *i-set0-cont* :: (*nat set*) *set*
where *i-set0-cont* $\equiv$ *insert* {} *i-set-cont*

lemma *i-set-cont-subset-i-set*: *i-set-cont* $\subseteq$ *i-set*
apply (*unfold subset-eq*, *rule ballI*, *rename-tac x*)
apply (*rule-tac a=x* **in** *i-set-cont.cases*)
apply *blast+*
done

lemma *i-set0-cont-subset-i-set0*: *i-set0-cont* $\subseteq$ *i-set0*
apply (*unfold i-set0-cont-def i-set0-def*)
apply (*rule insert-mono*)
apply (*rule i-set-cont-subset-i-set*)
done

Minimal definition of *i-set-cont*

inductive-set *i-set-cont-min* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: [*n*...] ∈ *i-set-cont-min*
| *i-set-cont-IN*[*intro*]: [*n*...,*d*] ∈ *i-set-cont-min*

definition *i-set0-cont-min* :: (*nat set*) *set*
where *i-set0-cont-min* ≡ *insert* {} *i-set-cont-min*

lemma *i-set-cont-min-eq*: *i-set-cont* = *i-set-cont-min*
apply (*rule set-eqI*, *rule iffI*)
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-cont.cases*)
apply (*fastforce simp*: *iIN-0-iTILL-conv*[*symmetric*])+
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-cont-min.cases*)
apply *blast*+
done

inext and *iprev* with continuous intervals

lemma *i-set-cont-inext*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *finite I* ⟹ *n* < *Max I* ⟧ ⟹ *inext n I* = *Suc n*
apply (*simp add*: *i-set-cont-min-eq*)
apply (*rule i-set-cont-min.cases*, *assumption*)
apply (*simp-all add*: *iT-finite iT-Max iT-inext-if iT-iff*)
done

lemma *i-set-cont-iprev*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *iMin I* < *n* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
apply (*simp add*: *i-set-cont-min-eq*)
apply (*rule i-set-cont-min.cases*, *assumption*)
apply (*simp-all add*: *iT-iprev-if iT-Min iT-iff*)
done

lemma *i-set-cont-inext--less*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n* < *n0*; *n0* ∈ *I* ⟧ ⟹ *inext n I* = *Suc n*
apply (*case-tac finite I*)
apply (*rule i-set-cont-inext*, *assumption*+)
apply (*rule order-less-le-trans*[*OF - Max-ge*], *assumption*+)
apply (*rule i-set-cont.cases*, *assumption*)
apply (*simp-all add*: *iT-finite iT-inext*)
done

lemma *i-set-cont-iprev--greater*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n0* < *n*; *n0* ∈ *I* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
apply (*rule i-set-cont-iprev*, *assumption*+)
apply (*rule order-le-less-trans*[*OF iMin-le*, *of n0*], *assumption*+)
done

Sets of modulo intervals

inductive-set *i-set-mult* :: *nat* ⇒ (*nat set*) *set*

for *k* :: *nat*
where
i-set-mult-FROM[*intro*!]: [*n*. . .] ∈ *i-set-mult k*
| *i-set-mult-TILL*[*intro*!]: [. . .*n*] ∈ *i-set-mult k*
| *i-set-mult-IN*[*intro*!]: [*n*. . .,*d*] ∈ *i-set-mult k*
| *i-set-mult-MOD*[*intro*!]: [*r*, *mod m* ∗ *k*] ∈ *i-set-mult k*
| *i-set-mult-MODb*[*intro*!]: [*r*, *mod m* ∗ *k*, *c*] ∈ *i-set-mult k*

definition *i-set0-mult* :: *nat* ⇒ (*nat set*) *set*
where *i-set0-mult k* ≡ *insert* {} (*i-set-mult k*)

lemma
i-set0-mult-empty[*intro*!]: {} ∈ *i-set0-mult k* **and**
i-set0-mult-FROM[*intro*!] : [*n*. . .] ∈ *i-set0-mult k* **and**
i-set0-mult-TILL[*intro*!] : [. . .*n*] ∈ *i-set0-mult k* **and**
i-set0-mult-IN[*intro*!] : [*n*. . .,*d*] ∈ *i-set0-mult k* **and**
i-set0-mult-MOD[*intro*!] : [*r*, *mod m* ∗ *k*] ∈ *i-set0-mult k* **and**
i-set0-mult-MODb[*intro*!] : [*r*, *mod m* ∗ *k*, *c*] ∈ *i-set0-mult k*
by (*simp-all add*: *i-set0-mult-def i-set-mult.intros*)

lemmas *i-set0-mult-intros* =
i-set0-mult-empty
i-set0-mult-FROM
i-set0-mult-TILL
i-set0-mult-IN
i-set0-mult-MOD
i-set0-mult-MODb

lemma *i-set-mult-subset-i-set0-mult*: *i-set-mult k* ⊆ *i-set0-mult k*
by (*unfold i-set0-mult-def*, *rule subset-insertI*)

lemma *i-set-mult-subset-i-set*: *i-set-mult k* ⊆ *i-set*
apply (*clarsimp simp*: *subset-iff*)
apply (*rule-tac a*=*t* **in** *i-set-mult.cases*[*of* - *k*])
apply (*simp-all add*: *i-set-intros*)
done

lemma *i-set0-mult-subset-i-set0*: *i-set0-mult k* ⊆ *i-set0*
apply (*simp add*: *i-set0-mult-def i-set0-empty*)
apply (*rule order-trans*[*OF* - *i-set-subset-i-set0*, *OF i-set-mult-subset-i-set*])
done

lemma *i-set-mult-0-eq-i-set-cont*: *i-set-mult 0* = *i-set-cont*
apply (*rule set-eqI*, *rule iffI*)
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-mult.cases*[*of* - *0*])
apply (*simp-all add*: *i-set-cont.intros iMOD-0 iMODb-mod-0*)
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-cont.cases*)
apply (*simp-all add*: *i-set-mult.intros*)
done

lemma *i-set0-mult-0-eq-i-set0-cont*: *i-set0-mult 0 = i-set0-cont*
by (*simp add*: *i-set0-mult-def i-set0-cont-def i-set-mult-0-eq-i-set-cont*)

lemma *i-set-mult-1-eq-i-set*: *i-set-mult* (*Suc 0*) = *i-set*
apply (*rule set-eqI*, *rule iffI*)
 apply (*rename-tac x*, *induct-tac x rule*: *i-set-mult.induct*[*of* - *1*])
 apply (*simp-all add*: *i-set-intros*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rename-tac x*, *induct-tac x rule*: *i-set-ind.induct*)
apply (*simp-all add*: *i-set-mult.intros*)
apply (*rule-tac t=m* **and** *s=m* ∗ *Suc 0* **in** *subst*, *simp*, *rule i-set-mult.intros*)+
done

lemma *i-set0-mult-1-eq-i-set0*: *i-set0-mult* (*Suc 0*) = *i-set0*
by (*simp add*: *i-set0-mult-def i-set0-def i-set-mult-1-eq-i-set*)

lemma *i-set-mult-imp-not-empty*: *I* ∈ *i-set-mult k* ⟹ *I* ≠ {}
by (*induct I rule*: *i-set-mult.induct*, *simp-all add*: *iT-not-empty*)

lemma *iMOD-in-i-set-mult-imp-divisor-mod-0*:
 ⟦ *m* ≠ *Suc 0*; [*r*, *mod m*] ∈ *i-set-mult k* ⟧ ⟹ *m mod k = 0*
apply (*case-tac m = 0*, *simp*)
apply (*rule i-set-mult.cases*[*of* [*r*, *mod m*] *k*], *assumption*)
apply (*blast dest*: *iFROM-iMOD-neq*)
apply (*blast dest*: *iTILL-iMOD-neq*)
apply (*blast dest*: *iIN-iMOD-neq*)
apply (*simp add*: *iMOD-eq-conv*)
apply (*blast dest*: *iMOD-iMODb-neq*)
done

lemma
 divisor-mod-0-imp-iMOD-in-i-set-mult: *m mod k = 0* ⟹ [*r*, *mod m*] ∈ *i-set-mult k* **and**
 divisor-mod-0-imp-iMODb-in-i-set-mult: *m mod k = 0* ⟹ [*r*, *mod m*, *c*] ∈ *i-set-mult k*
 by (*auto simp add*: *ac-simps*)

lemma *iMOD-in-i-set-mult--divisor-mod-0-conv*:
 m ≠ *Suc 0* ⟹ ([*r*, *mod m*] ∈ *i-set-mult k*) = (*m mod k = 0*)
apply (*rule iffI*)
 apply (*simp add*: *iMOD-in-i-set-mult-imp-divisor-mod-0*)
apply (*simp add*: *divisor-mod-0-imp-iMOD-in-i-set-mult*)
done

lemma *i-set-mult-neq1-subset-i-set*: *k* ≠ *Suc 0* ⟹ *i-set-mult k* ⊂ *i-set*
apply (*rule psubsetI*, *rule i-set-mult-subset-i-set*)
apply (*simp add*: *set-eq-iff*)
apply (*drule neq-iff*[*THEN iffD1*], *erule disjE*)

```
 apply (simp add: i-set-mult-0-eq-i-set-cont)
 apply (thin-tac k = 0)
 apply (rule-tac x=[0, mod 2] in exI)
 apply (rule ccontr)
 apply (simp add: i-set-intros)
 apply (drule i-set-cont.cases[where P=False])
  apply (drule sym, simp add: iT-neq)+
 apply simp
apply (rule-tac x=[0, mod Suc k] in exI)
apply (rule ccontr)
apply (simp add: i-set-intros)
apply (insert iMOD-in-i-set-mult-imp-divisor-mod-0[of Suc k 0 k])
apply (simp add: mod-Suc)
done

lemma mod-0-imp-i-set-mult-subset:
  a mod b = 0 ⟹ i-set-mult a ⊆ i-set-mult b
  apply (auto simp add: ac-simps elim!: dvdE)
apply (rule-tac a=x and k=k * b in i-set-mult.cases)
apply (simp-all add: i-set-mult.intros mult.assoc[symmetric])
done

lemma i-set-mult-subset-imp-mod-0:
  ⟦ a ≠ Suc 0; (i-set-mult a ⊆ i-set-mult b) ⟧ ⟹ a mod b = 0
apply (simp add: subset-iff)
apply (erule-tac x=[0, mod a] in allE)
apply (simp add: divisor-mod-0-imp-iMOD-in-i-set-mult)
apply (simp add: iMOD-in-i-set-mult-imp-divisor-mod-0[of - 0 b])
done

lemma i-set-mult-subset-conv:
  a ≠ Suc 0 ⟹ (i-set-mult a ⊆ i-set-mult b) = (a mod b = 0)
apply (rule iffI)
 apply (simp add: i-set-mult-subset-imp-mod-0)
apply (simp add: mod-0-imp-i-set-mult-subset)
done


lemma i-set-mult-mod-0-div:
  ⟦ I ∈ i-set-mult k; k mod d = 0 ⟧ ⟹ I ⊘ d ∈ i-set-mult (k div d)
apply (case-tac d = 0)
 apply (simp add: iT-Div-0[OF i-set-mult-imp-not-empty] i-set-mult.intros)
apply (induct I rule: i-set-mult.induct)
apply (simp-all add: iT-div i-set-mult.intros)
apply (simp-all add: iMOD-div iMODb-div mod-0-imp-mod-mult-left-0 mod-0-imp-div-mult-left-eq
i-set-mult.intros)
done
```

Intervals from *i-set-mult k* remain in *i-set* after division by d a divisor of k.

corollary *i-set-mult-mod-0-div-i-set*:
⟦ *I* ∈ *i-set-mult k*; *k mod d* = *0* ⟧ ⟹ *I* ⊘ *d* ∈ *i-set*
by (*rule subsetD*[*OF i-set-mult-subset-i-set*[*of k div d*]], *rule i-set-mult-mod-0-div*)
corollary *i-set-mult-div-self-i-set*:
I ∈ *i-set-mult k* ⟹ *I* ⊘ *k* ∈ *i-set*
by (*simp add*: *i-set-mult-mod-0-div-i-set*)

lemma *i-set-mod-0-mult-in-i-set-mult*:
⟦ *I* ∈ *i-set*; *m mod k* = *0* ⟧ ⟹ *I* ⊗ *m* ∈ *i-set-mult k*
apply (*case-tac m* = *0*)
apply (*simp add*: *iT-Mult-0 i-set-imp-not-empty i-set-mult.intros*)
apply (*clarsimp simp*: *mult.commute*[*of k*] *elim*!: *dvdE*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rule-tac a*=*I* **in** *i-set-ind.cases*)
apply (*simp-all add*: *iT-mult mult.assoc*[*symmetric*] *i-set-mult.intros*)
done

lemma *i-set-self-mult-in-i-set-mult*:
I ∈ *i-set* ⟹ *I* ⊗ *k* ∈ *i-set-mult k*
by (*rule i-set-mod-0-mult-in-i-set-mult*[*OF - mod-self*])

lemma *i-set-mult-mod-gr0-div-not-in-i-set*:
⟦ *0* < *k*; *0* < *d*; *0* < *k mod d* ⟧ ⟹ ∃ *I*∈*i-set-mult k*. *I* ⊘ *d* ∉ *i-set*
apply (*case-tac d* = *Suc 0*, *simp*)
apply (*frule iMOD-div-mod-gr0-not-ex*[*of d Suc d* ∗ *k 0*])
apply (*rule Suc-le-lessD*, *rule gr0-imp-self-le-mult1*, *assumption*)
apply *simp*
apply (*rule-tac x*=[*0*, *mod Suc d* ∗ *k*] **in** *bexI*)
apply (*rule ccontr*, *simp*)
apply (*frule i-set-infinite-as-iMOD*)
apply (*simp add*: *iT-Div-finite-iff iMOD-infinite*)
apply *fastforce*
apply (*simp add*: *i-set-mult.intros del*: *mult-Suc*)
done

lemma *i-set0-mult-mod-0-div*:
⟦ *I* ∈ *i-set0-mult k*; *k mod d* = *0* ⟧ ⟹ *I* ⊘ *d* ∈ *i-set0-mult* (*k div d*)
apply (*simp add*: *i-set0-mult-def*)
apply (*elim disjE*)
apply (*simp add*: *iT-Div-empty*)
apply (*simp add*: *i-set-mult-mod-0-div*)
done

corollary *i-set0-mult-mod-0-div-i-set0*:
⟦ *I* ∈ *i-set0-mult k*; *k mod d* = *0* ⟧ ⟹ *I* ⊘ *d* ∈ *i-set0*
by (*rule subsetD*[*OF i-set0-mult-subset-i-set0*[*of k div d*]], *rule i-set0-mult-mod-0-div*)

corollary *i-set0-mult-div-self-i-set0*:

I ∈ *i-set0-mult k* ⟹ *I* ⊘ *k* ∈ *i-set0*
by (*simp add*: *i-set0-mult-mod-0-div-i-set0*)

lemma *i-set0-mod-0-mult-in-i-set0-mult*:
⟦ *I* ∈ *i-set0*; *m mod k* = *0* ⟧ ⟹ *I* ⊗ *m* ∈ *i-set0-mult k*
apply (*simp add*: *i-set0-def*)
apply (*erule disjE*)
apply (*simp add*: *iT-Mult-empty i-set0-mult-empty*)
apply (*rule subsetD*[*OF i-set-mult-subset-i-set0-mult*])
apply (*simp add*: *i-set-mod-0-mult-in-i-set-mult*)
done

lemma *i-set0-self-mult-in-i-set0-mult*:
I ∈ *i-set0* ⟹ *I* ⊗ *k* ∈ *i-set0-mult k*
by (*simp add*: *i-set0-mod-0-mult-in-i-set0-mult*)

lemma *i-set0-mult-mod-gr0-div-not-in-i-set0*:
⟦ *0* < *k*; *0* < *d*; *0* < *k mod d* ⟧ ⟹ ∃ *I*∈*i-set0-mult k*. *I* ⊘ *d* ∉ *i-set0*
apply (*frule i-set-mult-mod-gr0-div-not-in-i-set*[*of k d*], *assumption*+)
apply (*erule bexE*, *rename-tac I*, *rule-tac x*=*I* **in** *bexI*)
apply (*simp add*: *i-set0-def iT-Div-not-empty i-set-mult-imp-not-empty*)
apply (*simp add*: *subsetD*[*OF i-set-mult-subset-i-set0-mult*])
done

end

3 Temporal logic operators on natural intervals

theory *IL-TemporalOperators*
imports *IL-IntervalOperators*
begin

Bool : some additional properties

instantiation *bool* :: {*ord*, *zero*, *one*, *plus*, *times*, *order*}
begin

definition *Zero-bool-def* [*simp*]: *0* ≡ *False*
definition *One-bool-def* [*simp*]: *1* ≡ *True*
definition *add-bool-def*: *a* + *b* ≡ *a* ∨ *b*
definition *mult-bool-def*: *a* ∗ *b* ≡ *a* ∧ *b*

instance ..

end

value *False* < *True*
value *True* < *True*
value *True* ≤ *True*

lemmas *bool-op-rel-defs* =
add-bool-def
mult-bool-def
less-bool-def
le-bool-def

instance *bool* :: *wellorder*
apply (*rule wf-wellorderI*)
apply (*rule-tac t*={(*x*, *y*). *x* < *y*} **and** *s*={(*False*, *True*)} **in** *subst*)
apply *fastforce*
apply (*unfold wf-def*, *blast*)
apply *intro-classes*
done

instance *bool* :: *comm-semiring-1*
apply *intro-classes*
apply (*unfold bool-op-rel-defs Zero-bool-def One-bool-def*)
apply *blast*+
done

3.1 Basic definitions

lemma *UNIV-nat*: ℕ = (*UNIV*::*nat set*)
by (*simp add*: *Nats-def*)

Universal temporal operator: Always/Globally

definition *iAll* :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Always
where *iAll I P* ≡ ∀ *t*∈*I*. *P t*

Existential temporal operator: Eventually/Finally

definition *iEx* :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Eventually
where *iEx I P* ≡ ∃ *t*∈*I*. *P t*

syntax
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3□ - -./ -)› [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3◇ - -./ -)› [*0*, *0*, *10*] *10*)
syntax-consts
-iAll ⇌ *iAll* **and**
-iEx ⇌ *iEx*
translations
□ *t I*. *P* ⇌ *CONST iAll I* (λ*t*. *P*)
◇ *t I*. *P* ⇌ *CONST iEx I* (λ*t*. *P*)

Future temporal operator: Next

definition *iNext* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Next
where *iNext t0 I P* ≡ *P* (*inext t0 I*)

Past temporal operator: Last/Previous

definition *iLast* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Last
where *iLast t0 I P* ≡ *P* (*iprev t0 I*)

syntax
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3○ - - -./ -)› [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3⊖ - - -./ -)› [*0*, *0*, *10*] *10*)
syntax-consts
-iNext ⇌ *iNext* **and**
-iLast ⇌ *iLast*
translations
○ *t t0 I*. *P* ⇌ *CONST iNext t0 I* (λ*t*. *P*)
⊖ *t t0 I*. *P* ⇌ *CONST iLast t0 I* (λ*t*. *P*)

lemma ○ *t 10* [*0*...]. (*t* + *10* > *10*)
by (*simp add*: *iNext-def iT-inext-if*)

The following versions of Next and Last operator differ in the cases where no next/previous element exists or specified time point is not in interval: the weak versions return *True* and the strong versions return *False*.

definition *iNextWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Next
where *iNextWeak t0 I P* ≡ (□ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iNextStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Next
where *iNextStrong t0 I P* ≡ (◇ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition *iLastWeak* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Last
where *iLastWeak t0 I P* ≡ (□ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

definition *iLastStrong* :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Last
where *iLastStrong t0 I P* ≡ (◇ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

syntax
-iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$○_W$ - - -./ -)› [*0*, *0*, *10*] *10*)
-iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$○_S$ - - -./ -)› [*0*, *0*, *10*] *10*)
-iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$⊖_W$ - - -./ -)› [*0*, *0*, *10*] *10*)
-iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* (‹(3$⊖_S$ - - -./ -)› [*0*, *0*, *10*] *10*)
syntax-consts
-iNextWeak ⇌ *iNextWeak* **and**
-iNextStrong ⇌ *iNextStrong* **and**
-iLastWeak ⇌ *iLastWeak* **and**
-iLastStrong ⇌ *iLastStrong*
translations

$\bigcirc_W\ t\ t0\ I.\ P \rightleftharpoons CONST\ iNextWeak\ t0\ I\ (\lambda t.\ P)$
$\bigcirc_S\ t\ t0\ I.\ P \rightleftharpoons CONST\ iNextStrong\ t0\ I\ (\lambda t.\ P)$
$\ominus_W\ t\ t0\ I.\ P \rightleftharpoons CONST\ iLastWeak\ t0\ I\ (\lambda t.\ P)$
$\ominus_S\ t\ t0\ I.\ P \rightleftharpoons CONST\ iLastStrong\ t0\ I\ (\lambda t.\ P)$

Some examples for Next and Last operator

lemma $\bigcirc\ t\ 5\ [0\ldots,10].\ ([0::int,10,20,30,40,50,60,70,80,90]\ !\ t < 80)$
by (*simp add*: *iNext-def iIN-inext*)

lemma $\ominus\ t\ 5\ [0\ldots,10].\ ([0::int,10,20,30,40,50,60,70,80,90]\ !\ t < 80)$
by (*simp add*: *iLast-def iIN-iprev*)

Temporal Until operator

definition $iUntil :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Until
where $iUntil\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t')$

Temporal Since operator (past operator corresponding to Until)

definition $iSince :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Since
where $iSince\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t')$

syntax
-iUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - ($3\mathcal{U}$ - -)./ -)› $[10,\ 0,\ 0,\ 0,\ 10]\ 10$)
-iSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - ($3\mathcal{S}$ - -)./ -)› $[10,\ 0,\ 0,\ 0,\ 10]\ 10$)
syntax-consts
-iUntil $\rightleftharpoons$ *iUntil* **and**
-iSince $\rightleftharpoons$ *iSince*
translations
$P.\ t\ \mathcal{U}\ t'\ I.\ Q \rightleftharpoons CONST\ iUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{S}\ t'\ I.\ Q \rightleftharpoons CONST\ iSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition $iWeakUntil :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Until/Wating-for/Unless
where $iWeakUntil\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t'))$

definition $iWeakSince :: iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Since/Back-to
where $iWeakSince\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t'))$

syntax
-iWeakUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$

(‹(-./ - ($3\mathcal{W}$ - -)./ -)› $[10,\ 0,\ 0,\ 0,\ 10]\ 10$)
-iWeakSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$

(‹(-./ - ($3\mathcal{B}$ - -)./ -)› $[10,\ 0,\ 0,\ 0,\ 10]\ 10$)
syntax-consts
-iWeakUntil $\rightleftharpoons$ *iWeakUntil* **and**

-iWeakSince ⇌ *iWeakSince*
translations
$P.\ t\ \mathcal{W}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{B}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition *iRelease* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Release
where $iRelease\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \lor (\Diamond\ t\ I.\ P\ t \land (\Box\ t'\ (I \downarrow\leq t).\ Q\ t'))$

definition *iTrigger* :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Trigger
where $iTrigger\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \lor (\Diamond\ t\ I.\ P\ t \land (\Box\ t'\ (I \downarrow\geq t).\ Q\ t'))$

syntax
-iRelease :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - (3$\mathcal{R}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
-iTrigger :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
(‹(-./ - (3$\mathcal{T}$ - -)./ -)› [10, 0, 0, 0, 10] 10)
syntax-consts
-iRelease ⇌ *iRelease* **and**
-iTrigger ⇌ *iTrigger*
translations
$P.\ t\ \mathcal{R}\ t'\ I.\ Q \rightleftharpoons CONST\ iRelease\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{T}\ t'\ I.\ Q \rightleftharpoons CONST\ iTrigger\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

lemmas *iTL-Next-defs* =
iNext-def iLast-def
iNextWeak-def iLastWeak-def
iNextStrong-def iLastStrong-def
lemmas *iTL-defs* =
iAll-def iEx-def
iUntil-def iSince-def
iWeakUntil-def iWeakSince-def
iRelease-def iTrigger-def
iTL-Next-defs

typed-print-translation ‹
[(***const-syntax*** ‹*iAll*›, *Syntax-Trans.preserve-binder-abs2-tr′* ***syntax-const*** ‹*-iAll*›),
(***const-syntax*** ‹*iEx*›, *Syntax-Trans.preserve-binder-abs2-tr′* ***syntax-const*** ‹*-iEx*›)]
›

print-translation ‹
let
fun btr′ syn ctxt [i,Abs abs,Abs abs′] =
let
val (t,P) = Syntax-Trans.atomic-abs-tr′ ctxt abs;

```
    val (t′,Q) = Syntax-Trans.atomic-abs-tr′ ctxt abs′
  in Syntax.const syn $ P $ t $ t′ $ i $ Q end
in
 [(@{const-syntax iUntil}, btr′ -iUntil),
  (@{const-syntax iSince}, btr′ -iSince)]
end
›
```

print-translation ‹

```
let
  fun btr′ syn ctxt [i,Abs abs,Abs abs′] =
    let
      val (t,P) = Syntax-Trans.atomic-abs-tr′ ctxt abs;
      val (t′,Q) = Syntax-Trans.atomic-abs-tr′ ctxt abs′
    in Syntax.const syn $ P $ t $ t′ $ i $ Q end
in
 [(@{const-syntax iWeakUntil}, btr′ -iWeakUntil),
  (@{const-syntax iWeakSince}, btr′ -iWeakSince)]
end
›
```

print-translation ‹

```
let
  fun btr′ syn ctxt [i,Abs abs,Abs abs′] =
    let
      val (t,P) = Syntax-Trans.atomic-abs-tr′ ctxt abs;
      val (t′,Q) = Syntax-Trans.atomic-abs-tr′ ctxt abs′
    in Syntax.const syn $ P $ t $ t′ $ i $ Q end
in
 [(@{const-syntax iRelease}, btr′ -iRelease),
  (@{const-syntax iTrigger}, btr′ -iTrigger)]
end
›
```

3.2 Basic lemmata for temporal operators

3.2.1 Intro/elim rules

lemma
iexI[*intro*]: $\llbracket\ P\ t;\ t \in I\ \rrbracket \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
rev-iexI[*intro?*]: $\llbracket\ t \in I;\ P\ t\ \rrbracket \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
iexE[*elim!*]: $\llbracket\ \Diamond\ t\ I.\ P\ t;\ \bigwedge t.\ \llbracket\ t \in I;\ P\ t\ \rrbracket \Longrightarrow Q\ \rrbracket \Longrightarrow Q$
by (*unfold iEx-def*, *blast+*)

lemma
iallI[*intro!*]: $(\bigwedge t.\ t \in I \Longrightarrow P\ t) \Longrightarrow \Box\ t\ I.\ P\ t$ **and**
ispec[*dest?*]: $\llbracket\ \Box\ t\ I.\ P\ t;\ t \in I\ \rrbracket \Longrightarrow P\ t$ **and**
iallE[*elim*]: $\llbracket\ \Box\ t\ I.\ P\ t;\ P\ t \Longrightarrow Q;\ t \notin I \Longrightarrow Q\ \rrbracket \Longrightarrow Q$
by (*unfold iAll-def*, *blast+*)

lemma
iuntilI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$ **and**
rev-iuntilI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
by (*unfold iUntil-def*, *blast+*)

lemma
iuntilE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{U}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
by (*unfold iUntil-def*, *blast*)

lemma
isinceI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$ **and**
rev-isinceI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
by (*unfold iSince-def*, *blast+*)
lemma
isinceE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{S}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
by (*unfold iSince-def*, *blast*)

3.2.2 Rewrite rules for trivial simplification

lemma *iall-triv*[*simp*]: $(\Box\ t\ I.\ P) = ((\exists\, t.\ t \in I) \longrightarrow P)$
by (*simp add*: *iAll-def*)

lemma *iex-triv*[*simp*]: $(\Diamond\ t\ I.\ P) = ((\exists\, t.\ t \in I) \wedge P)$
by (*simp add*: *iEx-def*)

lemma *iex-conjL1*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*

lemma *iex-conjR1*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*

lemma *iex-conjL2*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*

lemma *iex-conjR2*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$

by *blast*

lemma *iex-commute*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P\ t1\ t2) =$
$(\Diamond\ t2\ I2.\ \Diamond\ t1\ I1.\ P\ t1\ t2)$
by *blast*

lemma *iall-conjL1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*

lemma *iall-conjR1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*

lemma *iall-conjL2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*

lemma *iall-conjR2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*

lemma *iall-commute*:
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P\ t1\ t2) =$
$(\Box\ t2\ I2.\ \Box\ t1\ I1.\ P\ t1\ t2)$
by *blast*

lemma *iall-conj-distrib*:
$(\Box\ t\ I.\ P\ t \wedge Q\ t) = ((\Box\ t\ I.\ P\ t) \wedge (\Box\ t\ I.\ Q\ t))$
by *blast*

lemma *iex-disj-distrib*:
$(\Diamond\ t\ I.\ P\ t \vee Q\ t) = ((\Diamond\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t))$
by *blast*

lemma *iall-conj-distrib2*:
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2 \wedge Q\ t1\ t2) =$
$((\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2) \wedge (\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ Q\ t1\ t2))$
by *blast*

lemma *iex-disj-distrib2*:
(◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P t1 t2* ∨ *Q t1 t2*) =
((◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P t1 t2*) ∨ (◇ *t1 I1*. ◇ *t2* (*I2 t1*). *Q t1 t2*))
by *blast*

lemma *iUntil-disj-distrib*:
(*P t1*. *t1* $\mathcal{U}$ *t2 I*. (*Q1 t2* ∨ *Q2 t2*)) = ((*P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q1 t2*) ∨ (*P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q2 t2*))
unfolding *iUntil-def* **by** *blast*

lemma *iSince-disj-distrib*:
(*P t1*. *t1* $\mathcal{S}$ *t2 I*. (*Q1 t2* ∨ *Q2 t2*)) = ((*P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q1 t2*) ∨ (*P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q2 t2*))
unfolding *iSince-def* **by** *blast*

lemma
iNext-iff: (○ *t t0 I*. *P t*) = (□ *t* [...*0*] ⊕ (*inext t0 I*). *P t*) **and**
iLast-iff: (⊖ *t t0 I*. *P t*) = (□ *t* [...*0*] ⊕ (*iprev t0 I*). *P t*)
by (*fastforce simp*: *iTL-Next-defs iT-add iIN-0*)+

lemma
iNext-iEx-iff: (○ *t t0 I*. *P t*) = (◇ *t* [...*0*] ⊕ (*inext t0 I*). *P t*) **and**
iLast-iEx-iff: (⊖ *t t0 I*. *P t*) = (◇ *t* [...*0*] ⊕ (*iprev t0 I*). *P t*)
by (*fastforce simp*: *iTL-Next-defs iT-add iIN-0*)+

lemma *inext-singleton-cut-greater-not-empty-iff*:
({*inext t0 I*} ↓> *t0* ≠ {}) = (*I* ↓> *t0* ≠ {} ∧ *t0* ∈ *I*)
apply (*simp add*: *cut-greater-singleton*)
apply (*case-tac t0* ∈ *I*)
 prefer *2*
 apply (*simp add*: *not-in-inext-fix*)
apply *simp*
apply (*case-tac I* ↓> *t0* = {})
 apply (*simp add*: *cut-greater-empty-iff inext-all-le-fix*)
apply (*simp add*: *cut-greater-not-empty-iff inext-mono2*)
done

lemma *iprev-singleton-cut-less-not-empty-iff*:
({*iprev t0 I*} ↓< *t0* ≠ {}) = (*I* ↓< *t0* ≠ {} ∧ *t0* ∈ *I*)
apply (*simp add*: *cut-less-singleton*)
apply (*case-tac t0* ∈ *I*)
 prefer *2*
 apply (*simp add*: *not-in-iprev-fix*)
apply *simp*
apply (*case-tac I* ↓< *t0* = {})
 apply (*simp add*: *cut-less-empty-iff iprev-all-ge-fix*)
apply (*simp add*: *cut-less-not-empty-iff iprev-mono2*)
done

lemma *inext-singleton-cut-greater-empty-iff*:
 $(\{inext\ t0\ I\} \downarrow> t0 = \{\}) = (I \downarrow> t0 = \{\} \vee t0 \notin I)$
apply (*subst Not-eq-iff*[*symmetric*])
apply (*simp add*: *inext-singleton-cut-greater-not-empty-iff*)
done

lemma *iprev-singleton-cut-less-empty-iff*:
 $(\{iprev\ t0\ I\} \downarrow< t0 = \{\}) = (I \downarrow< t0 = \{\} \vee t0 \notin I)$
apply (*subst Not-eq-iff*[*symmetric*])
apply (*simp add*: *iprev-singleton-cut-less-not-empty-iff*)
done

lemma *iNextWeak-iff* : $(\bigcirc_W\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \vee (I \downarrow> t0 = \{\}) \vee t0 \notin I)$
apply (*unfold iTL-defs*)
apply (*simp add*: *inext-singleton-cut-greater-empty-iff*[*symmetric*] *cut-greater-singleton*)
done

lemma *iNextStrong-iff* : $(\bigcirc_S\ t\ t0\ I.\ P\ t) = ((\bigcirc\ t\ t0\ I.\ P\ t) \wedge (I \downarrow> t0 \neq \{\}) \wedge t0 \in I)$
apply (*unfold iTL-defs*)
apply (*simp add*: *inext-singleton-cut-greater-not-empty-iff*[*symmetric*] *cut-greater-singleton*)
done

lemma *iLastWeak-iff* : $(\ominus_W\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \vee (I \downarrow< t0 = \{\}) \vee t0 \notin I)$
apply (*unfold iTL-defs*)
apply (*simp add*: *iprev-singleton-cut-less-empty-iff*[*symmetric*] *cut-less-singleton*)
done

lemma *iLastStrong-iff* : $(\ominus_S\ t\ t0\ I.\ P\ t) = ((\ominus\ t\ t0\ I.\ P\ t) \wedge (I \downarrow< t0 \neq \{\}) \wedge t0 \in I)$
apply (*unfold iTL-defs*)
apply (*simp add*: *iprev-singleton-cut-less-not-empty-iff*[*symmetric*] *cut-less-singleton*)
done

lemmas *iTL-Next-iff* =
 iNext-iff iLast-iff
 iNextWeak-iff iNextStrong-iff
 iLastWeak-iff iLastStrong-iff

lemma
 iNext-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
 iLast-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{iprev\ t0\ I\}.\ P\ t)$
by (*fastforce simp*: *iTL-Next-defs iT-add iIN-0*)+
lemmas *iNextLast-iff-singleton* = *iNext-iff-singleton iLast-iff-singleton*

lemma
iNext-iEx-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iEx-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{iprev\ t0\ I\}.\ P\ t)$
by (*fastforce simp*: *iTL-Next-defs iT-add iIN-0*)+

lemma
iNextWeak-iAll-conv: $(\bigcirc_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iNextStrong-iEx-conv: $(\bigcirc_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iLastWeak-iAll-conv: $(\ominus_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$ **and**
iLastStrong-iEx-conv: $(\ominus_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$
by (*simp-all add*: *iTL-Next-defs*)

lemma
iAll-True[*simp*]: $\Box\ t\ I.\ True$ **and**
iAll-False[*simp*]: $(\Box\ t\ I.\ False) = (I = \{\})$ **and**
iEx-True[*simp*]: $(\Diamond\ t\ I.\ True) = (I \neq \{\})$ **and**
iEx-False[*simp*]: $\neg\ (\Diamond\ t\ I.\ False)$
by *blast*+

lemma *empty-iff-iAll-False*: $(I = \{\}) = (\Box\ t\ I.\ False)$ **by** *blast*
lemma *not-empty-iff-iEx-True*: $(I \neq \{\}) = (\Diamond\ t\ I.\ True)$ **by** *blast*

lemma
iNext-True: $\bigcirc\ t\ t0\ I.\ True$ **and**
iNextWeak-True: $(\bigcirc_W\ t\ t0\ I.\ True)$ **and**
iNext-False: $\neg\ (\bigcirc\ t\ t0\ I.\ False)$ **and**
iNextStrong-False: $\neg\ (\bigcirc_S\ t\ t0\ I.\ False)$
by (*simp-all add*: *iTL-defs*)

lemma
iNextStrong-True: $(\bigcirc_S\ t\ t0\ I.\ True) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$ **and**
iNextWeak-False: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ False)) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
by (*simp-all add*: *iTL-defs ex-in-conv inext-singleton-cut-greater-not-empty-iff*)

lemma
iLast-True: $\ominus\ t\ t0\ I.\ True$ **and**
iLastWeak-True: $(\ominus_W\ t\ t0\ I.\ True)$ **and**
iLast-False: $\neg\ (\ominus\ t\ t0\ I.\ False)$ **and**
iLastStrong-False: $\neg\ (\ominus_S\ t\ t0\ I.\ False)$
by (*simp-all add*: *iTL-defs*)

lemma
iLastStrong-True: $(\ominus_S\ t\ t0\ I.\ True) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$ **and**
iLastWeak-False: $(\neg\ (\ominus_W\ t\ t0\ I.\ False)) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
by (*simp-all add*: *iTL-defs ex-in-conv iprev-singleton-cut-less-not-empty-iff*)

lemma *iUntil-True-left*[*simp*]: (*True*. t' $\mathcal{U}$ t I. Q t) = ($\diamond$ t I. Q t)
by *blast*

lemma *iUntil-True*[*simp*]: (P t'. t' $\mathcal{U}$ t I. *True*) = ($I \neq \{\}$)
apply (*unfold iTL-defs*)
apply (*rule iffI*)
 apply *blast*
apply (*rule-tac x=iMin I* **in** *bexI*)
apply (*simp add*: *cut-less-Min-empty iMinI-ex2*)+
done

lemma *iUntil-False-left*[*simp*]: (*False*. t' $\mathcal{U}$ t I. Q t) = ($I \neq \{\} \wedge Q$ (*iMin I*))
apply (*case-tac* $I = \{\}$, *blast*)
apply (*simp add*: *iTL-defs*)
apply (*rule iffI*)
 apply *clarsimp*
 apply (*drule iMin-equality*)
  apply (*simp add*: *cut-less-empty-iff*)
 apply *simp*
apply (*rule-tac x=iMin I* **in** *bexI*)
 apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done

lemma *iUntil-False*[*simp*]: $\neg$ (P t'. t' $\mathcal{U}$ t I. *False*)
by *blast*

lemma *iSince-True-left*[*simp*]: (*True*. t' $\mathcal{S}$ t I. Q t) = ($\diamond$ t I. Q t)
by *blast*

lemma *iSince-True-if*:
 (P t'. t' $\mathcal{S}$ t I. *True*) =
 (*if finite I then* $I \neq \{\}$ *else* $\diamond$ *t1 I*. $\square$ *t2* (I ↓> *t1*). P *t2*)
apply (*clarsimp simp*: *iTL-defs*)
apply (*rule iffI*)
 apply *clarsimp*
apply (*rule-tac x=Max I* **in** *bexI*)
 apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done

corollary *iSince-True-finite*[*simp*]: *finite I* $\Longrightarrow$ (P t'. t' $\mathcal{S}$ t I. *True*) = ($I \neq \{\}$)
by (*simp add*: *iSince-True-if*)

lemma *iSince-False-left*[*simp*]: (*False*. t' $\mathcal{S}$ t I. Q t) = (*finite I* $\wedge$ $I \neq \{\}$ $\wedge$ Q (*Max I*))
apply (*simp add*: *iTL-defs*)
apply (*case-tac* $I = \{\}$, *simp*)
apply (*case-tac infinite I*)

 apply (*simp add*: *nat-cut-greater-infinite-not-empty*)
apply (*rule iffI*)
 apply *clarsimp*
 apply (*drule Max-equality*)
   apply *simp*
  apply (*simp add*: *cut-greater-empty-iff*)
 apply *simp*
apply (*rule-tac x=Max I* **in** *bexI*)
 apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done

lemma *iSince-False*[*simp*]: ¬ (*P t'*. *t'* $\mathcal{S}$ *t I*. *False*)
by *blast*

lemma *iWeakUntil-True-left*[*simp*]: *True*. *t'* $\mathcal{W}$ *t I*. *Q t*
by (*simp add*: *iWeakUntil-def*)

lemma *iWeakUntil-True*[*simp*]: *P t'*. *t'* $\mathcal{W}$ *t I*. *True*
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule disjI2*)
apply (*rule-tac x=iMin I* **in** *bexI*)
 apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done

lemma *iWeakUntil-False-left*[*simp*]: (*False*. *t'* $\mathcal{W}$ *t I*. *Q t*) = (*I* = {} ∨ *Q* (*iMin I*))
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule iffI*)
 apply (*clarsimp simp*: *cut-less-empty-iff*)
 apply (*frule iMin-equality*)
 apply *simp+*
apply (*rule-tac x=iMin I* **in** *bexI*)
 apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done

lemma *iWeakUntil-False*[*simp*]: (*P t'*. *t'* $\mathcal{W}$ *t I*. *False*) = (□ *t I*. *P t*)
by (*simp add*: *iWeakUntil-def*)

lemma *iWeakSince-True-left*[*simp*]: *True*. *t'* $\mathcal{B}$ *t I*. *Q t*
by (*simp add*: *iTL-defs*)

lemma *iWeakSince-True-disj*:
  (*P t'*. *t'* $\mathcal{B}$ *t I*. *True*) =
  (*I* = {} ∨ (◇ *t1 I*. □ *t2* (*I* ↓> *t1*). *P t2*))

unfolding *iTL-defs* **by** *blast*

lemma *iWeakSince-True-finite*[*simp*]: *finite I* $\Longrightarrow$ ($P\ t'.\ t'\ \mathcal{B}\ t\ I.\ True$)
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule disjI2*)
apply (*rule-tac x=Max I* **in** *bexI*)
 apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done

lemma *iWeakSince-False-left*[*simp*]: ($False.\ t'\ \mathcal{B}\ t\ I.\ Q\ t$) = ($I$ = {} $\vee$ (*finite I* $\wedge$ Q (*Max I*)))
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*case-tac infinite I*)
 apply (*simp add*: *nat-cut-greater-infinite-not-empty*)
apply (*rule iffI*)
 apply *clarsimp*
 apply (*drule Max-equality*)
   apply *simp*
  apply (*simp add*: *cut-greater-empty-iff*)
 apply *simp*
apply *simp*
apply (*rule-tac x=Max I* **in** *bexI*)
 apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done

lemma *iWeakSince-False*[*simp*]: ($P\ t'.\ t'\ \mathcal{B}\ t\ I.\ False$) = ($\Box\ t\ I.\ P\ t$)
by (*simp add*: *iWeakSince-def*)

lemma *iRelease-True-left*[*simp*]: ($True.\ t'\ \mathcal{R}\ t\ I.\ Q\ t$) = ($I$ = {} $\vee$ Q (*iMin I*))
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule iffI*)
 apply (*erule disjE*)
  apply (*blast intro*: *iMinI2-ex2*)
 apply *clarsimp*
 apply (*drule-tac x=iMin I* **in** *bspec*)
  apply (*blast intro*: *iMinI-ex2*)
 apply *simp*
apply (*rule disjI2*)
apply (*rule-tac x=iMin I* **in** *bexI*)
 apply *fastforce*
apply (*simp add*: *iMinI-ex2*)
done

lemma *iRelease-True*[*simp*]: $P\ t'.\ t'\ \mathcal{R}\ t\ I.\ True$

by (*simp add*: *iTL-defs*)

lemma *iRelease-False-left*[*simp*]: $(False.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
by (*simp add*: *iTL-defs*)

lemma *iRelease-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ False) = (I = \{\})$
unfolding *iTL-defs* **by** *blast*

lemma *iTrigger-True-left*[*simp*]: $(True.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (I = \{\} \lor (\Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow\geq t1).\ Q\ t2))$
unfolding *iTL-defs* **by** *blast*

lemma *iTrigger-True*[*simp*]: $P\ t'.\ t'\ \mathcal{T}\ t\ I.\ True$
by (*simp add*: *iTL-defs*)

lemma *iTrigger-False-left*[*simp*]: $(False.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
by (*simp add*: *iTL-defs*)

lemma *iTrigger-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ False) = (I = \{\})$
unfolding *iTL-defs* **by** *blast*

lemma
iUntil-TrueTrue[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$ **and**
iSince-TrueTrue[*simp*]: $(True.\ t'\ \mathcal{S}\ t\ I.\ True) = (I \neq \{\})$ **and**
iWeakUntil-TrueTrue[*simp*]: $True.\ t'\ \mathcal{W}\ t\ I.\ True$ **and**
iWeakSince-TrueTrue[*simp*]: $True.\ t'\ \mathcal{B}\ t\ I.\ True$ **and**
iRelease-TrueTrue[*simp*]: $True.\ t'\ \mathcal{R}\ t\ I.\ True$ **and**
iTrigger-TrueTrue[*simp*]: $True.\ t'\ \mathcal{T}\ t\ I.\ True$
by (*simp-all add*: *iTL-defs ex-in-conv*)

3.2.3 Empty sets and singletons

lemma *iAll-empty*[*simp*]: $\Box\ t\ \{\}.\ P\ t$ **by** *blast*
lemma *iEx-empty*[*simp*]: $\neg\ (\Diamond\ t\ \{\}.\ P\ t)$ **by** *blast*

lemma *iUntil-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{U}\ t1\ \{\}.\ Q\ t1)$ **by** *blast*
lemma *iSince-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{S}\ t1\ \{\}.\ Q\ t1)$ **by** *blast*
lemma *iWeakUntil-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{W}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iWeakUntil-def*)
lemma *iWeakSince-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{B}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iWeakSince-def*)

lemma *iRelease-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{R}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iRelease-def*)
lemma *iTrigger-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{T}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iTrigger-def*)

lemmas *iTL-empty* =
iAll-empty iEx-empty
iUntil-empty iSince-empty
iWeakUntil-empty iWeakSince-empty
iRelease-empty iTrigger-empty

lemma *iAll-singleton*[*simp*]: $(\Box\ t'\ \{t\}.\ P\ t') = P\ t$ **by** *blast*
lemma *iEx-singleton*[*simp*]: $(\Diamond\ t'\ \{t\}.\ P\ t') = P\ t$ **by** *blast*

lemma *iUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{U}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
by (*simp add*: *iUntil-def cut-less-singleton*)

lemma *iSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{S}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
by (*simp add*: *iSince-def cut-greater-singleton*)

lemma *iWeakUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{W}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
by (*simp add*: *iWeakUntil-def cut-less-singleton*)

lemma *iWeakSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{B}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
by (*simp add*: *iWeakSince-def cut-greater-singleton*)

lemma *iRelease-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{R}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
unfolding *iRelease-def* **by** *blast*

lemma *iTrigger-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{T}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
unfolding *iTrigger-def* **by** *blast*

lemmas *iTL-singleton* =
iAll-singleton iEx-singleton
iUntil-singleton iSince-singleton
iWeakUntil-singleton iWeakSince-singleton
iRelease-singleton iTrigger-singleton

3.2.4 Conversions between temporal operators

lemma *iAll-iEx-conv*: $(\Box\ t\ I.\ P\ t) = (\neg\ (\Diamond\ t\ I.\ \neg\ P\ t))$ **by** *blast*
lemma *iEx-iAll-conv*: $(\Diamond\ t\ I.\ P\ t) = (\neg\ (\Box\ t\ I.\ \neg\ P\ t))$ **by** *blast*

lemma *not-iAll*[*simp*]: $(\neg\ (\Box\ t\ I.\ P\ t)) = (\Diamond\ t\ I.\ \neg\ P\ t)$ **by** *blast*
lemma *not-iEx*[*simp*]: $(\neg\ (\Diamond\ t\ I.\ P\ t)) = (\Box\ t\ I.\ \neg\ P\ t)$ **by** *blast*

lemma *iUntil-iEx-conv*: $(\mathit{True}.\ t'\ \mathcal{U}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ **by** *blast*
lemma *iSince-iEx-conv*: $(\mathit{True}.\ t'\ \mathcal{S}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ **by** *blast*

lemma *iRelease-iAll-conv*: $(\mathit{False}.\ t'\ \mathcal{R}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
by (*simp add*: *iRelease-def*)

lemma *iTrigger-iAll-conv*: $(\mathit{False}.\ t'\ \mathcal{T}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
by (*simp add*: *iTrigger-def*)

lemma *iWeakUntil-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
unfolding *iWeakUntil-def iUntil-def* **by** *blast*

lemma *iWeakSince-iSince-conv*: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
unfolding *iWeakSince-def iSince-def* **by** *blast*

lemma *iUntil-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
by (*subst iWeakUntil-iUntil-conv*, *blast*)

lemma *iSince-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
by (*subst iWeakSince-iSince-conv*, *blast*)

lemma *iRelease-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{W}\ t\ I.\ (Q\ t \wedge P\ t))$
apply (*unfold iRelease-def iWeakUntil-def*)
apply (*simp add*: *cut-le-less-conv-if*)
apply *blast*
done

lemma *iRelease-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{U}\ t\ I.\ (Q\ t \wedge P\ t)))$
by (*fastforce simp*: *iRelease-iWeakUntil-conv iWeakUntil-iUntil-conv*)

lemma *iTrigger-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{B}\ t\ I.\ (Q\ t \wedge P\ t))$
apply (*unfold iTrigger-def iWeakSince-def*)
apply (*simp add*: *cut-ge-greater-conv-if*)
apply *blast*
done

lemma *iTrigger-iSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{S}\ t\ I.\ (Q\ t \wedge P\ t)))$
by (*fastforce simp*: *iTrigger-iWeakSince-conv iWeakSince-iSince-conv*)

lemma *iRelease-not-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t))$
apply (*simp only*: *iUntil-def iRelease-def not-iAll not-iEx de-Morgan-conj not-not*)
apply (*case-tac* $\Box\ t\ I.\ Q\ t$, *blast*)
apply (*simp* (*no-asm-simp*))
apply *clarsimp*
apply (*rule iffI*)
 apply (*elim iexE*, *intro iallI*, *rename-tac t1 t2*)
 apply (*case-tac* $t2 \le t1$, *blast*)
 apply (*simp add*: *linorder-not-le*, *blast*)
apply (*frule-tac t=t* **in** *ispec*, *assumption*)
apply *clarsimp*
apply (*rule-tac t=iMin* $\{t \in I.\ P\ t\}$ **in** *iexI*)
 prefer *2*

apply (*blast intro*: *subsetD*[*OF* - *iMinI-ex*])
apply (*rule conjI*)
apply (*blast intro*: *iMinI2*)
apply (*clarsimp simp*: *cut-le-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*, *assumption*)
apply (*clarsimp simp*: *cut-less-mem-iff*)
apply (*frule-tac x=t′* **in** *order-less-le-trans*, *assumption*)
apply (*drule not-less-iMin*)
apply *simp*
done
lemma *iUntil-not-iRelease-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{R}\ t\ I.\ \neg\ Q\ t))$
by (*simp add*: *iRelease-not-iUntil-conv*)

The Trigger operator $\mathcal{T}$ is a past operator, so that it is used for time intervals, that are bounded by a current time point, and thus are finite. For an infinite interval the stated relation to the Since operator $\mathcal{S}$ would not be fulfilled.

lemma *iTrigger-not-iSince-conv*: *finite I* $\Longrightarrow (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t))$
apply (*unfold iTrigger-def iSince-def*)
apply (*case-tac* $\Box\ t\ I.\ Q\ t$, *blast*)
apply (*simp* (*no-asm-simp*))
apply *clarsimp*
apply (*rule iffI*)
apply (*elim iexE conjE*, *rule iallI*, *rename-tac t1 t2*)
apply (*case-tac t2* $\geq$ *t1*, *blast*)
apply (*simp add*: *linorder-not-le*, *blast*)
apply (*frule-tac t=t* **in** *ispec*, *assumption*)
apply (*erule disjE*, *blast*)
apply (*erule iexE*)
apply (*subgoal-tac finite* $\{t \in I.\ P\ t\}$)
prefer *2*
apply (*blast intro*: *subset-finite-imp-finite*)
apply (*rule-tac t=Max* $\{t \in I.\ P\ t\}$ **in** *iexI*)
prefer *2*
apply (*blast intro*: *subsetD*[*OF* - *MaxI*])
apply (*rule conjI*)
apply (*blast intro*: *MaxI2*)
apply (*clarsimp simp*: *cut-ge-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*, *assumption*)
apply (*clarsimp simp*: *cut-greater-mem-iff*, *rename-tac t′*)
apply (*frule-tac z=t′* **in** *order-le-less-trans*, *assumption*)
apply (*drule-tac A=*$\{t \in I.\ P\ t\}$ **in** *not-greater-Max*[*rotated 1*])
apply *simp+*
done

lemma *iSince-not-iTrigger-conv*: *finite I* $\Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \neg\ Q\ t))$
by (*simp add*: *iTrigger-not-iSince-conv*)

```
lemma not-iUntil:
  (¬ (P t1. t1 𝒰 t2 I. Q t2)) =
  (□ t I. (Q t ⟶ (◇ t' (I ↓< t). ¬ P t')))
unfolding iTL-defs by blast

lemma not-iSince:
  (¬ (P t1. t1 𝒮 t2 I. Q t2)) =
  (□ t I. (Q t ⟶ (◇ t' (I ↓> t). ¬ P t')))
unfolding iTL-defs by blast


lemma iWeakUntil-conj-iUntil-conv:
  (P t1. t1 𝒲 t2 I. (P t2 ∧ Q t2)) = (¬ (¬ Q t1. t1 𝒰 t2 I. ¬ P t2))
by (simp add: iRelease-not-iUntil-conv[symmetric] iRelease-iWeakUntil-conv)

lemma iUntil-disj-iUntil-conv:
  (P t1 ∨ Q t1. t1 𝒰 t2 I. Q t2) =
  (P t1. t1 𝒰 t2 I. Q t2)
apply (unfold iUntil-def)
apply (rule iffI)
 prefer 2
 apply blast
apply (clarsimp, rename-tac t1)
apply (rule-tac t=iMin {t ∈ I. Q t} in iexI)
 apply (subgoal-tac Q (iMin {t ∈ I. Q t}))
  prefer 2
  apply (blast intro: iMinI2)
 apply (clarsimp, rename-tac t2)
 apply (frule Collect-not-less-iMin, simp)
 apply (subgoal-tac t2 < t1)
  prefer 2
  apply (rule order-less-le-trans, assumption)
  apply (simp add: Collect-iMin-le)
 apply blast
apply (rule subsetD[OF - iMinI])
apply blast+
done

lemma iWeakUntil-disj-iWeakUntil-conv:
  (P t1 ∨ Q t1. t1 𝒲 t2 I. Q t2) =
  (P t1. t1 𝒲 t2 I. Q t2)
apply (simp only: iWeakUntil-iUntil-conv iUntil-disj-iUntil-conv)
apply (case-tac P t1. t1 𝒰 t2 I. Q t2, simp+)
apply (case-tac □ t I. P t, blast)
apply (simp add: not-iUntil)
apply (clarsimp, rename-tac t1)
apply (case-tac ¬ Q t1, blast)
```

apply (*subgoal-tac iMin* $\{t \in I.\ Q\ t\} \in I$)
 prefer *2*
 apply (*blast intro*: *subsetD*[*OF* - *iMinI*])
apply (*frule-tac t=iMin* $\{t \in I.\ Q\ t\}$ **in** *ispec*, *assumption*)
apply (*drule mp*)
 apply (*blast intro*: *iMinI2*)
apply (*clarsimp*, *rename-tac t2*)
apply (*subgoal-tac* ¬ *Q t2*)
 prefer *2*
 apply (*drule Collect-not-less-iMin*)
 apply (*simp add*: *cut-less-mem-iff*)
apply *blast*
done

lemma *iWeakUntil-iUntil-conj-conv*:
 $(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
 $(\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (\neg\ P\ t2 \wedge \neg\ Q\ t2)))$
apply (*subst iWeakUntil-disj-iWeakUntil-conv*[*symmetric*])
apply (*subst de-Morgan-disj*[*symmetric*])
apply (*subst iWeakUntil-conj-iUntil-conv*[*symmetric*])
apply (*simp add*: *conj-disj-distribR conj-disj-absorb*)
done

Negation and temporal operators

lemma
 not-iNext[*simp*]: $(\neg\ (\bigcirc\ t\ t0\ I.\ P\ t)) = (\bigcirc\ t\ t0\ I.\ \neg\ P\ t)$ **and**
 not-iNextWeak[*simp*]: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ P\ t)) = (\bigcirc_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
 not-iNextStrong[*simp*]: $(\neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)) = (\bigcirc_W\ t\ t0\ I.\ \neg\ P\ t)$ **and**
 not-iLast[*simp*]: $(\neg\ (\ominus\ t\ t0\ I.\ P\ t)) = (\ominus\ t\ t0\ I.\ \neg\ P\ t)$ **and**
 not-iLastWeak[*simp*]: $(\neg\ (\ominus_W\ t\ t0\ I.\ P\ t)) = (\ominus_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
 not-iLastStrong[*simp*]: $(\neg\ (\ominus_S\ t\ t0\ I.\ P\ t)) = (\ominus_W\ t\ t0\ I.\ \neg\ P\ t)$
by (*simp-all add*: *iTL-Next-defs*)

lemma *not-iWeakUntil*:
 $(\neg\ (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)) =$
 $((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I \downarrow< t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
by (*simp add*: *iWeakUntil-iUntil-conv not-iUntil*)
lemma *not-iWeakSince*:
 $(\neg\ (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)) =$
 $((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I \downarrow> t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
by (*simp add*: *iWeakSince-iSince-conv not-iSince*)

lemma *not-iRelease*:
 $(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) =$
 $((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ P\ t \longrightarrow (\Diamond\ t\ I \downarrow\le t.\ \neg\ Q\ t)))$
by (*simp add*: *iRelease-def*)

lemma *not-iRelease-iUntil-conv*:
 $(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t)$

by (*simp add*: *iUntil-not-iRelease-conv*)

lemma *not-iTrigger*:
$(\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ \neg\ P\ t \vee (\Diamond\ t\ I\ {\downarrow}{\geq}\ t.\ \neg\ Q\ t)))$
by (*simp add*: *iTrigger-def*)

lemma *not-iTrigger-iSince-conv*:
finite $I \Longrightarrow (\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t)$
by (*simp add*: *iSince-not-iTrigger-conv*)

3.2.5 Some implication results

lemma *all-imp-iall*: $\forall x.\ P\ x \Longrightarrow \Box\ t\ I.\ P\ t$ **by** *blast*
lemma *bex-imp-lex*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \exists x.\ P\ x$ **by** *blast*

lemma *iAll-imp-iEx*: $I \neq \{\} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$ **by** *blast*
lemma *i-set-iAll-imp-iEx*: $I \in$ *i-set* $\Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$
by (*rule iAll-imp-iEx*, *rule i-set-imp-not-empty*)

lemmas *iT-iAll-imp-iEx* = *iT-not-empty*[*THEN iAll-imp-iEx*]

lemma *iUntil-imp-iEx*: $P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
unfolding *iTL-defs* **by** *blast*

lemma *iSince-imp-iEx*: $P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
unfolding *iTL-defs* **by** *blast*

lemma *iall-subset-imp-iall*: $[\![\ \Box\ t\ B.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
by *blast*

lemma *iex-subset-imp-iex*: $[\![\ \Diamond\ t\ A.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
by *blast*

lemma *iall-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ I.\ Q\ t$ **by** *blast*
lemma *iex-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Diamond\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ Q\ t$ **by** *blast*

lemma *iUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iSince-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iWeakUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iWeakSince-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iRelease-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iTrigger-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iMin-imp-iUntil*:
$[\![\ I \neq \{\};\ Q\ (iMin\ I)\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
apply (*unfold iUntil-def*)
apply (*rule-tac t=iMin I* **in** *iexI*)
 apply (*simp add*: *cut-less-Min-empty*)
apply (*blast intro*: *iMinI-ex2*)
done

lemma *Max-imp-iSince*:
$[\![\ finite\ I;\ I \neq \{\};\ Q\ (Max\ I)\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
apply (*unfold iSince-def*)
apply (*rule-tac t=Max I* **in** *iexI*)
 apply (*simp add*: *cut-greater-Max-empty*)
apply (*blast intro*: *Max-in*)
done

3.2.6 Congruence rules for temporal operators' predicates

lemma *iAll-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Box\ t\ I.\ P\ (f\ t)\ t) = (\Box\ t\ I.\ P\ (g\ t)\ t)$
unfolding *iTL-defs* **by** *simp*

lemma *iEx-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Diamond\ t\ I.\ P\ (f\ t)\ t) = (\Diamond\ t\ I.\ P\ (g\ t)\ t)$
unfolding *iTL-defs* **by** *simp*

lemma *iUntil-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*)
apply (*rule-tac f=λx.* $(Q\ t \wedge x)$ **in** *arg-cong*, *rename-tac t*)
apply (*rule iAll-cong*[*OF iall-subset-imp-iall*[*OF - cut-less-subset*]])
apply (*rule iallI*, *rename-tac t′*)
apply (*drule-tac t=t′* **in** *ispec*)
apply *simp+*

done

lemma *iUntil-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (g\ t2)\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*drule-tac t=t* **in** *ispec*)
apply *simp+*
done

lemma *iSince-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
apply (*unfold iSince-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*rule-tac f=λx.* (*Q t* ∧ *x*) **in** *arg-cong*)
apply (*rule iAll-cong*[*OF iall-subset-imp-iall*[*OF - cut-greater-subset*]])
apply (*rule iallI*, *rename-tac t′*)
apply (*drule-tac t=t′* **in** *ispec*)
apply *simp+*
done

lemma *iSince-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (g\ t2)\ t2)$
apply (*unfold iSince-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*drule-tac t=t* **in** *ispec*)
apply *simp+*
done

lemma *bex-subst*:
$\forall x{\in}A.\ P\ x \longrightarrow (Q\ x = Q'\ x) \Longrightarrow$
$(\exists x{\in}A.\ P\ x \wedge Q\ x) = (\exists x{\in}A.\ P\ x \wedge Q'\ x)$
by *blast*

lemma *iEx-subst*:
$\Box\ t\ I.\ P\ t \longrightarrow (Q\ t = Q'\ t) \Longrightarrow$
$(\Diamond\ t\ I.\ P\ t \wedge Q\ t) = (\Diamond\ t\ I.\ P\ t \wedge Q'\ t)$
by *blast*

3.2.7 Temporal operators with set unions/intersections and subsets

lemma *iAll-subset*: $\llbracket\ A \subseteq B;\ \Box\ t\ B.\ P\ t\ \rrbracket \Longrightarrow \Box\ t\ A.\ P\ t$
by (*rule iall-subset-imp-iall*)

lemma *iEx-subset*: $\llbracket\ A \subseteq B;\ \Diamond\ t\ A.\ P\ t\ \rrbracket \Longrightarrow \Diamond\ t\ B.\ P\ t$
by (*rule iex-subset-imp-iex*)

lemma *iUntil-append*:
$\llbracket$ *finite* A; *Max* $A \leq$ *iMin* B $\rrbracket \Longrightarrow$
$P\ t1.\ t1\ \mathcal{U}\ t2\ A.\ Q\ t2 \Longrightarrow P\ t1.\ t1\ \mathcal{U}\ t2\ (A \cup B).\ Q\ t2$
apply (*case-tac* $A = \{\}$, *simp*)
apply (*unfold iUntil-def*)
apply (*rule iEx-subset*[*OF Un-upper1*])
apply (*rule-tac* $f{=}\lambda t.\ A \downarrow< t$ **and** $g{=}\lambda t.\ (A \cup B) \downarrow< t$ **in** *subst*[*OF iEx-cong*, *rule-format*])
apply (*clarsimp simp*: *cut-less-Un*, *rename-tac t t'*)
apply (*cut-tac* $t{=}t$ **and** $I{=}B$ **in** *cut-less-Min-empty*)
apply *simp+*
done

lemma *iSince-prepend*:
$\llbracket$ *finite* A; *Max* $A \leq$ *iMin* B $\rrbracket \Longrightarrow$
$P\ t1.\ t1\ \mathcal{S}\ t2\ B.\ Q\ t2 \Longrightarrow P\ t1.\ t1\ \mathcal{S}\ t2\ (A \cup B).\ Q\ t2$
apply (*case-tac* $B = \{\}$, *simp*)
apply (*unfold iSince-def*)
apply (*rule iEx-subset*[*OF Un-upper2*])
apply (*rule-tac* $f{=}\lambda t.\ B \downarrow> t$ **and** $g{=}\lambda t.\ (A \cup B) \downarrow> t$ **in** *subst*[*OF iEx-cong*, *rule-format*])
apply (*clarsimp simp*: *cut-greater-Un*, *rename-tac t t'*)
apply (*cut-tac* $t{=}t$ **and** $I{=}A$ **in** *cut-greater-Max-empty*)
apply (*simp add*: *iMin-ge-iff*)+
done

lemma
iAll-union: $\llbracket\ \Box\ t\ A.\ P\ t;\ \Box\ t\ B.\ P\ t\ \rrbracket \Longrightarrow \Box\ t\ (A \cup B).\ P\ t$ **and**
iAll-union-conv: $(\Box\ t\ A \cup B.\ P\ t) = ((\Box\ t\ A.\ P\ t) \wedge (\Box\ t\ B.\ P\ t))$
by *blast+*

lemma
iEx-union: $(\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t) \Longrightarrow \Diamond\ t\ (A \cup B).\ P\ t$ **and**
iEx-union-conv: $(\Diamond\ t\ (A \cup B).\ P\ t) = ((\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t))$
by *blast+*

lemma *iAll-inter*: $(\Box\ t\ A.\ P\ t) \vee (\Box\ t\ B.\ P\ t) \Longrightarrow \Box\ t\ (A \cap B).\ P\ t$ **by** *blast*

lemma *not-iEx-inter*:
$\exists\, A\ B\ P.\ (\Diamond\ t\ A.\ P\ t) \wedge (\Diamond\ t\ B.\ P\ t) \wedge \neg\ (\Diamond\ t\ (A \cap B).\ P\ t)$
by (*rule-tac* $x{=}\{0\}$ **in** *exI*, *rule-tac* $x{=}\{Suc\ 0\}$ **in** *exI*, *blast*)

lemma
iAll-insert: $\llbracket\ P\ t;\ \Box\ t\ I.\ P\ t\ \rrbracket \Longrightarrow \Box\ t'\ (insert\ t\ I).\ P\ t'$ **and**
iAll-insert-conv: $(\Box\ t'\ (insert\ t\ I).\ P\ t') = (P\ t \wedge (\Box\ t'\ I.\ P\ t'))$
by *blast+*

lemma
iEx-insert: $\llbracket\ P\ t \vee (\Diamond\ t\ I.\ P\ t)\ \rrbracket \Longrightarrow \Diamond\ t'\ (insert\ t\ I).\ P\ t'$ **and**
iEx-insert-conv: $(\Diamond\ t'\ (insert\ t\ I).\ P\ t') = (P\ t \vee (\Diamond\ t'\ I.\ P\ t'))$
by *blast+*

3.3 Further results for temporal operators

lemma *Collect-minI-iEx*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow< t).\ \neg\ P\ t')$
by (*unfold iAll-def iEx-def*, *rule Collect-minI-ex-cut*)

lemma *iUntil-disj-conv1*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (Q\ (iMin\ I) \vee (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t \wedge iMin\ I < t))$
apply (*case-tac Q* (*iMin I*))
apply (*simp add*: *iMin-imp-iUntil*)
apply (*unfold iUntil-def*, *blast*)
done

lemma *iSince-disj-conv1*:
$\llbracket\ finite\ I;\ I \neq \{\}\ \rrbracket \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (Q\ (Max\ I) \vee (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t \wedge t < Max\ I))$
apply (*case-tac Q* (*Max I*))
apply (*simp add*: *Max-imp-iSince*)
apply (*unfold iSince-def*, *blast*)
done

lemma *iUntil-next*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) =$
$(Q\ (iMin\ I) \vee (P\ (iMin\ I) \wedge (P\ t'.\ t'\ \mathcal{U}\ t\ (I \downarrow> (iMin\ I)).\ Q\ t)))$
apply (*case-tac Q* (*iMin I*))
apply (*simp add*: *iMin-imp-iUntil*)
apply (*simp add*: *iUntil-def*)
apply (*frule iMinI-ex2*)
apply *blast*
done

lemma *iSince-prev*: $\llbracket\ finite\ I;\ I \neq \{\}\ \rrbracket \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) =$
$(Q\ (Max\ I) \vee (P\ (Max\ I) \wedge (P\ t'.\ t'\ \mathcal{S}\ t\ (I \downarrow< Max\ I).\ Q\ t)))$
apply (*case-tac Q* (*Max I*))
apply (*simp add*: *Max-imp-iSince*)
apply (*simp add*: *iSince-def*)

apply (*frule Max-in*, *assumption*)
apply *blast*
done

lemma *iNext-induct-rule*:
⟦ P (*iMin* I); □ t I. (P t ⟶ (○ t' t I. P t')); $t \in I$ ⟧ ⟹ P t
apply (*rule inext-induct*[*of* - *I*])
apply *simp*
apply (*drule-tac t*=*n* **in** *ispec*, *assumption*)
apply (*simp add*: *iNext-def*)
apply *assumption*
done

lemma *iNext-induct*:
⟦ P (*iMin* I); □ t I. (P t ⟶ (○ t' t I. P t')) ⟧ ⟹ □ t I. P t
by (*rule iallI*, *rule iNext-induct-rule*)

lemma *iLast-induct-rule*:
⟦ P (*Max* I); □ t I. (P t ⟶ (⊖ t' t I. P t')); *finite* I; $t \in I$ ⟧ ⟹ P t
apply (*rule iprev-induct*[*of* - *I*])
apply *assumption*
apply (*drule-tac t*=*n* **in** *ispec*, *assumption*)
apply (*simp add*: *iLast-def*)
apply *assumption*+
done

lemma *iLast-induct*:
⟦ P (*Max* I); □ t I. (P t ⟶ (⊖ t' t I. P t')); *finite* I ⟧ ⟹ □ t I. P t
by (*rule iallI*, *rule iLast-induct-rule*)

lemma *iUntil-conj-not*: ((P *t1* ∧ ¬ Q *t1*). *t1* $\mathcal{U}$ *t2* I. Q *t2*) = (P *t1*. *t1* $\mathcal{U}$ *t2* I. Q *t2*)
apply (*unfold iUntil-def*)
apply (*rule iffI*)
apply *blast*
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t*=*iMin* {$x \in I$. Q x} **in** *iexI*)
apply (*rule conjI*)
apply (*blast intro*: *iMinI2*)
apply (*clarsimp simp*: *cut-less-mem-iff*, *rename-tac t1*)
apply (*subgoal-tac iMin* {$x \in I$. Q x} $\le t$)
prefer *2*
apply (*simp add*: *iMin-le*)
apply (*frule order-less-le-trans*, *assumption*)
apply (*drule-tac t*=*t1* **in** *ispec*, *simp add*: *cut-less-mem-iff*)
apply (*rule ccontr*, *simp*)
apply (*subgoal-tac t1* ∈ {$x \in I$. Q x})
prefer *2*

apply *blast*
apply (*drule-tac k=t1* **and** *I*={*x* ∈ *I*. *Q x*} **in** *iMin-le*)
apply *simp*
apply (*blast intro*: *subsetD*[*OF* - *iMinI*])
done

lemma *iWeakUntil-conj-not*: ((*P t1* ∧ ¬ *Q t1*). *t1* 𝒲 *t2 I*. *Q t2*) = (*P t1*. *t1* 𝒲 *t2 I*. *Q t2*)
by (*simp only*: *iWeakUntil-iUntil-conv iUntil-conj-not*, *blast*)

lemma *iSince-conj-not*: *finite I* ⟹
((*P t1* ∧ ¬ *Q t1*). *t1* 𝒮 *t2 I*. *Q t2*) = (*P t1*. *t1* 𝒮 *t2 I*. *Q t2*)
apply (*simp only*: *iSince-def*)
apply (*case-tac I* = {}, *simp*)
apply (*rule iffI*)
apply *blast*
apply (*clarsimp*, *rename-tac t*)
apply (*subgoal-tac finite* {*x* ∈ *I*. *Q x*})
prefer *2*
apply *fastforce*
apply (*rule-tac t=Max* {*x* ∈ *I*. *Q x*} **in** *iexI*)
apply (*rule conjI*)
apply (*blast intro*: *MaxI2*)
apply (*clarsimp simp*: *cut-greater-mem-iff*, *rename-tac t1*)
apply (*subgoal-tac t* ≤ *Max* {*x* ∈ *I*. *Q x*})
prefer *2*
apply *simp*
apply (*frule order-le-less-trans*, *assumption*)
apply (*drule-tac t=t1* **in** *ispec*, *simp add*: *cut-greater-mem-iff*)
apply (*rule ccontr*, *simp*)
apply (*subgoal-tac t1* ∈ {*x* ∈ *I*. *Q x*})
prefer *2*
apply *blast*
apply (*drule not-greater-Max*[*rotated 1*], *simp+*)
apply (*rule subsetD*[*OF* - *MaxI*], *fastforce+*)
done

lemma *iWeakSince-conj-not*: *finite I* ⟹
((*P t1* ∧ ¬ *Q t1*). *t1* ℬ *t2 I*. *Q t2*) = (*P t1*. *t1* ℬ *t2 I*. *Q t2*)
by (*simp only*: *iWeakSince-iSince-conv iSince-conj-not*, *blast*)

lemma *iNextStrong-imp-iNextWeak*: ($\bigcirc_S$ *t t0 I*. *P t*) ⟶ ($\bigcirc_W$ *t t0 I*. *P t*)
unfolding *iTL-Next-defs* **by** *blast*
lemma *iLastStrong-imp-iLastWeak*: ($\ominus_S$ *t t0 I*. *P t*) ⟶ ($\ominus_W$ *t t0 I*. *P t*)
unfolding *iTL-Next-defs* **by** *blast*

lemma *infin-imp-iNextWeak-iNextStrong-eq-iNext*:
⟦ *infinite I*; *t0* ∈ *I* ⟧ ⟹

$((\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)) \wedge ((\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t))$
by (*simp add*: *iTL-Next-iff nat-cut-greater-infinite-not-empty*)

lemma *infin-imp-iNextWeak-eq-iNext*: $[\![$ *infinite I*; $t0 \in I$ $]\!] \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-iNextStrong-eq-iNext*)
lemma *infin-imp-iNextStrong-eq-iNext*: $[\![$ *infinite I*; $t0 \in I$ $]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-iNextStrong-eq-iNext*)
lemma *infin-imp-iNextStrong-eq-iNextWeak*: $[\![$ *infinite I*; $t0 \in I$ $]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc_W\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-eq-iNext infin-imp-iNextStrong-eq-iNext*)

lemma
not-in-iNext-eq: $t0 \notin I \Longrightarrow (\bigcirc\ t\ t0\ I.\ P\ t) = (P\ t0)$ **and**
not-in-iLast-eq: $t0 \notin I \Longrightarrow (\ominus\ t\ t0\ I.\ P\ t) = (P\ t0)$
by (*simp-all add*: *iTL-defs not-in-inext-fix not-in-iprev-fix*)

lemma
not-in-iNextWeak-eq: $t0 \notin I \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastWeak-eq: $t0 \notin I \Longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
by (*simp-all add*: *iNextWeak-iff iLastWeak-iff*)

lemma
not-in-iNextStrong-eq: $t0 \notin I \Longrightarrow \neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastStrong-eq: $t0 \notin I \Longrightarrow \neg\ (\ominus_S\ t\ t0\ I.\ P\ t)$
by (*simp-all add*: *iNextStrong-iff iLastStrong-iff*)

lemma
iNext-UNIV: $(\bigcirc\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextWeak-UNIV: $(\bigcirc_W\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextStrong-UNIV: $(\bigcirc_S\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$
by (*simp-all add*: *iTL-Next-defs inext-UNIV cut-greater-singleton*)

lemma
iLast-UNIV: $(\ominus\ t\ t0\ UNIV.\ P\ t) = P\ (t0 - Suc\ 0)$ **and**
iLastWeak-UNIV: $(\ominus_W\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else\ True)$ **and**
iLastStrong-UNIV: $(\ominus_S\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else\ False)$
by (*simp-all add*: *iTL-Next-defs iprev-UNIV cut-less-singleton*)

lemmas *iTL-Next-UNIV* =
iNext-UNIV iNextWeak-UNIV iNextStrong-UNIV
iLast-UNIV iLastWeak-UNIV iLastStrong-UNIV

lemma *inext-nth-iNext-Suc*: $(\bigcirc\ t\ (I \to n)\ I.\ P\ t) = P\ (I \to Suc\ n)$
by (*simp add*: *iNext-def*)

lemma *iprev-nth-iLast-Suc*: $(\ominus\ t\ (I \leftarrow n)\ I.\ P\ t) = P\ (I \leftarrow Suc\ n)$
by (*simp add*: *iLast-def*)

3.4 Temporal operators and arithmetic interval operators

Shifting intervals through addition and subtraction of constants. Mirroring intervals through subtraction of intervals from constants. Expanding and compressing intervals through multiplication and division by constants.

Always operator

lemma *iT-Plus-iAll-conv*: $(\Box\ t\ I \oplus k.\ P\ t) = (\Box\ t\ I.\ P\ (t + k))$
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
 apply (*clarify*, *rename-tac x*)
 apply (*drule-tac x=x + k* **in** *spec*)
 apply (*simp add*: *iT-Plus-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x − k* **in** *spec*)
apply (*simp add*: *iT-Plus-mem-iff*)
done

lemma *iT-Mult-iAll-conv*: $(\Box\ t\ I \otimes k.\ P\ t) = (\Box\ t\ I.\ P\ (t * k))$
apply (*unfold iAll-def Ball-def*)
apply (*case-tac I = {}*)
 apply (*simp add*: *iT-Mult-empty*)
apply (*case-tac k = 0*)
 apply (*force simp*: *iT-Mult-0 iTILL-0*)
apply (*rule iffI*)
 apply (*clarify*, *rename-tac x*)
 apply (*drule-tac x=x ∗ k* **in** *spec*)
 apply (*simp add*: *iT-Mult-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x div k* **in** *spec*)
apply (*simp add*: *iT-Mult-mem-iff mod-0-div-mult-cancel*)
done

lemma *iT-Plus-neg-iAll-conv*: $(\Box\ t\ I \oplus- k.\ P\ t) = (\Box\ t\ (I \downarrow\geq k).\ P\ (t - k))$
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
 apply (*clarify*, *rename-tac x*)
 apply (*drule-tac x=x − k* **in** *spec*)
 apply (*simp add*: *iT-Plus-neg-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x + k* **in** *spec*)
apply (*simp add*: *iT-Plus-neg-mem-iff cut-ge-mem-iff*)
done

lemma *iT-Minus-iAll-conv*: $(\Box\ t\ k \ominus I.\ P\ t) = (\Box\ t\ (I \downarrow\leq k).\ P\ (k - t))$
apply (*unfold iAll-def Ball-def*)

apply (*rule iffI*)
 apply (*clarify*, *rename-tac x*)
 apply (*drule-tac x=k − x* **in** *spec*)
 apply (*simp add*: *iT-Minus-mem-iff*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=k − x* **in** *spec*)
apply (*simp add*: *iT-Minus-mem-iff cut-le-mem-iff*)
done

lemma *iT-Div-iAll-conv*: (□ *t I* ⊘ *k*. *P t*) = (□ *t I*. *P* (*t div k*))
apply (*case-tac I* = {})
 apply (*simp add*: *iT-Div-empty*)
apply (*case-tac k* = *0*)
 apply (*force simp*: *iT-Div-0 iTILL-0*)
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
 apply (*clarify*, *rename-tac x*)
 apply (*drule-tac x=x div k* **in** *spec*)
 apply (*simp add*: *iT-Div-imp-mem*)
apply (*blast dest*: *iT-Div-mem-iff*[*THEN iffD1*])
done

lemmas *iT-arith-iAll-conv* =
  iT-Plus-iAll-conv
  iT-Mult-iAll-conv
  iT-Plus-neg-iAll-conv
  iT-Minus-iAll-conv
  iT-Div-iAll-conv

Eventually operator

lemma
  iT-Plus-iEx-conv: (◇ *t I* ⊕ *k*. *P t*) = (◇ *t I*. *P* (*t* + *k*)) **and**
  iT-Mult-iEx-conv: (◇ *t I* ⊗ *k*. *P t*) = (◇ *t I*. *P* (*t* ∗ *k*)) **and**
  iT-Plus-neg-iEx-conv: (◇ *t I* ⊕− *k*. *P t*) = (◇ *t* (*I* ↓≥ *k*). *P* (*t* − *k*)) **and**
  iT-Minus-iEx-conv: (◇ *t k* ⊖ *I*. *P t*) = (◇ *t* (*I* ↓≤ *k*). *P* (*k* − *t*)) **and**
  iT-Div-iEx-conv: (◇ *t I* ⊘ *k*. *P t*) = (◇ *t I*. *P* (*t div k*))
by (*simp-all only*: *iEx-iAll-conv iT-arith-iAll-conv*)

Until and Since operators

lemma *iT-Plus-iUntil-conv*: (*P t1*. *t1* 𝒰 *t2* (*I* ⊕ *k*). *Q t2*) = (*P* (*t1* + *k*). *t1* 𝒰 *t2 I*. *Q* (*t2* + *k*))
by (*simp add*: *iUntil-def iT-Plus-iAll-conv iT-Plus-iEx-conv iT-Plus-cut-less2*)

lemma *iT-Mult-iUntil-conv*: (*P t1*. *t1* 𝒰 *t2* (*I* ⊗ *k*). *Q t2*) = (*P* (*t1* ∗ *k*). *t1* 𝒰 *t2 I*. *Q* (*t2* ∗ *k*))
apply (*case-tac I* = {})
 apply (*simp add*: *iT-Mult-empty*)
apply (*case-tac k* = *0*)
 apply (*force simp add*: *iT-Mult-0 iTILL-0*)

apply (*simp add*: *iUntil-def iT-Mult-iAll-conv iT-Mult-iEx-conv iT-Mult-cut-less2*)
done

lemma *iT-Plus-neg-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{U}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
apply (*simp add*: *iUntil-def iT-Plus-neg-iAll-conv iT-Plus-neg-iEx-conv iT-Plus-neg-cut-less2*)
apply (*simp add*: *i-cut-commute-disj*)
done

lemma *iT-Minus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
apply (*simp add*: *iUntil-def iSince-def iT-Minus-iAll-conv iT-Minus-iEx-conv iT-Minus-cut-less2*)
apply (*simp add*: *i-cut-commute-disj*)
done

lemma *iT-Div-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2\ div\ k))$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Div-empty*)
apply (*case-tac* $k = 0$)
 apply (*force simp add*: *iT-Div-0 iTILL-0*)
apply (*simp add*: *iUntil-def iT-Div-iAll-conv iT-Div-iEx-conv iT-Div-cut-less2*)
apply (*rule iffI*)
 apply (*clarsimp*, *rename-tac t*)
 apply (*subgoal-tac* $I \downarrow\geq (t - t\ mod\ k) \neq \{\}$)
  prefer *2*
  apply (*simp add*: *cut-ge-not-empty-iff*)
  apply (*rule-tac x=t* **in** *bexI*)
  apply *simp+*
 apply (*case-tac* $t\ mod\ k = 0$)
  apply (*rule-tac t=t* **in** *iexI*)
  apply *simp+*
 apply (*rule-tac t=iMin* $(I \downarrow\geq (t - t\ mod\ k))$ **in** *iexI*)
  apply (*subgoal-tac*
    $t - t\ mod\ k \leq iMin\ (I \downarrow\geq (t - t\ mod\ k)) \wedge$
    $iMin\ (I \downarrow\geq (t - t\ mod\ k)) \leq t$)
   prefer *2*
   apply (*rule conjI*)
    apply (*blast intro*: *cut-ge-Min-greater*)
   apply (*simp add*: *iMin-le cut-ge-mem-iff*)
  apply *clarify*
  apply (*rule-tac t=iMin* $(I \downarrow\geq (t - t\ mod\ k))$ *div k* **and** *s=t div k* **in** *subst*)
   apply (*rule order-antisym*)
    apply (*drule-tac m*=$t - t\ mod\ k$ **and** *k=k* **in** *div-le-mono*)
    apply (*simp add*: *sub-mod-div-eq-div*)
   apply (*rule div-le-mono*, *assumption*)
  apply (*clarsimp*, *rename-tac t1*)
  apply (*subgoal-tac* $t1 \in I \downarrow< (t - t\ mod\ k) \cup I \downarrow\geq (t - t\ mod\ k)$)
   prefer *2*

apply (*simp add*: *cut-less-cut-ge-ident*)
apply (*subgoal-tac t1* $\notin$ *I* $\downarrow\geq$ (*t* $-$ *t mod k*))
prefer *2*
apply (*blast dest*: *not-less-iMin*)
apply *blast*
apply (*blast intro*: *subsetD*[*OF* - *iMinI-ex2*])
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t*=*t* **in** *iexI*)
apply *simp*
apply (*rule-tac B*=*I* $\downarrow<$ *t* **in** *iAll-subset*)
apply (*simp add*: *cut-less-mono*)
apply *simp*+
done

Until and Since operators can be converted into each other through substraction of intervals from constants

lemma *iUntil-iSince-conv*:
⟦ *finite I*; *Max I* $\leq$ *k* ⟧ $\Longrightarrow$
(*P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2*) = (*P* (*k* $-$ *t1*). *t1* $\mathcal{S}$ *t2* (*k* $\ominus$ *I*). *Q* (*k* $-$ *t2*))
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty*)
apply (*frule le-trans*[*OF iMin-le-Max*], *assumption*+)
apply (*subgoal-tac Max* (*k* $\ominus$ *I*) $\leq$ *k*)
prefer *2*
apply (*simp add*: *iT-Minus-Max*)
apply (*subgoal-tac iMin* (*k* $\ominus$ *I*) $\leq$ *k*)
prefer *2*
apply (*rule order-trans*[*OF iMin-le-Max*])
apply (*simp add*: *iT-Minus-finite iT-Minus-empty-iff del*: *Max-le-iff*)+
apply (*rule-tac t*=*P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2* **and** *s*=*P t1*. *t1* $\mathcal{U}$ *t2* (*k* $\ominus$ (*k* $\ominus$ *I*)). *Q t2* **in** *subst*)
apply (*simp add*: *iT-Minus-Minus-eq*)
apply (*simp add*: *iT-Minus-iUntil-conv cut-le-Max-all iT-Minus-finite*)
done

lemma *iSince-iUntil-conv*:
⟦ *finite I*; *Max I* $\leq$ *k* ⟧ $\Longrightarrow$
(*P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2*) = (*P* (*k* $-$ *t1*). *t1* $\mathcal{U}$ *t2* (*k* $\ominus$ *I*). *Q* (*k* $-$ *t2*))
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty*)
apply (*simp* (*no-asm-simp*) *add*: *iT-Minus-iUntil-conv*)
apply (*simp* (*no-asm-simp*) *add*: *cut-le-Max-all*)
apply (*unfold iSince-def*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t*=*t* **in** *iexI*)
apply (*frule-tac x*=*t* **in** *bspec*, *assumption*)
apply (*clarsimp*, *rename-tac t1*)
apply (*drule-tac t*=*t1* **in** *ispec*)

```
   apply (simp add: cut-greater-mem-iff)
  apply simp+
apply (clarsimp, rename-tac t)
apply (rule-tac t=t in iexI)
 apply (clarsimp, rename-tac t')
 apply (drule-tac t=t' in ispec)
  apply (simp add: cut-greater-mem-iff)
 apply simp+
done
```

lemma *iT-Plus-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iSince-def iT-Plus-iAll-conv iT-Plus-iEx-conv iT-Plus-cut-greater2*)

lemma *iT-Mult-iSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iSince-def iT-Mult-iAll-conv iT-Mult-iEx-conv iT-Mult-cut-greater2*)

lemma *iT-Plus-neg-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{S}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$

```
apply (simp add: iSince-def iT-Plus-neg-iAll-conv iT-Plus-neg-iEx-conv)
apply (rule iffI)
 apply (clarsimp, rename-tac t)
 apply (simp add: iT-Plus-neg-cut-greater2)
 apply (rule-tac t=t in iexI)
  apply (clarsimp, rename-tac t')
  apply (drule-tac t=t' − k in ispec)
   apply (simp add: iT-Plus-neg-mem-iff2 cut-greater-mem-iff)
  apply simp
 apply blast
apply (clarsimp, rename-tac t)
apply (rule-tac t=t in iexI)
 apply (clarsimp, rename-tac t')
 apply (drule-tac t=t' + k in ispec)
  apply (simp add: iT-Plus-neg-mem-iff i-cut-mem-iff)
 apply simp
apply blast
done
```

lemma *iT-Minus-iSince-conv*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$

```
apply (case-tac I = {})
 apply (simp add: iT-Minus-empty cut-le-empty)
apply (case-tac I ↓≤ k = {})
 apply (simp add: iT-Minus-image-conv)
apply (subst iT-Minus-cut-eq[OF order-refl, symmetric])
apply (subst iSince-iUntil-conv[where k=k])
  apply (rule iT-Minus-finite)
 apply (subst iT-Minus-Max)
```

apply *simp*
apply (*rule cut-le-bound*, *rule iMinI-ex2*, *simp*)
apply *simp*
apply (*simp add*: *iT-Minus-Minus-cut-eq*)
done

lemma *iT-Div-iSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2\ div\ k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Div-empty*)
apply (*simp add*: *iSince-def iT-Div-iAll-conv iT-Div-iEx-conv*)
apply (*simp add*: *iT-Div-cut-greater*)
apply (*subgoal-tac* $\forall t.\ t \le t\ div\ k * k + (k - Suc\ 0)$)
prefer *2*
apply *clarsimp*
apply (*simp add*: *div-mult-cancel add.commute*[*of - k*])
apply (*simp add*: *le-add-diff Suc-mod-le-divisor*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac t*)
apply (*drule-tac x=t* **in** *spec*)
apply (*subgoal-tac* $I \downarrow\le (t\ div\ k * k + (k - Suc\ 0)) \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-le-not-empty-iff*)
apply (*rule-tac x=t* **in** *bexI*, *assumption+*)
apply (*subgoal-tac* $t \le Max\ (I \downarrow\le (t\ div\ k * k + (k - Suc\ 0)))$)
prefer *2*
apply (*simp add*: *nat-cut-le-finite cut-le-mem-iff*)
apply (*subgoal-tac* $Max\ (I \downarrow\le (t\ div\ k * k + (k - Suc\ 0))) \le t\ div\ k * k + (k - Suc\ 0)$)
prefer *2*
apply (*simp add*: *nat-cut-le-finite cut-le-mem-iff*)
apply (*subgoal-tac* $Max\ (I \downarrow\le (t\ div\ k * k + (k - Suc\ 0)))\ div\ k = t\ div\ k$)
prefer *2*
apply (*rule order-antisym*)
apply (*rule-tac t=t div k* **and** *s=*$(t\ div\ k * k + (k - Suc\ 0))\ div\ k$ **in** *subst*)
apply (*simp only*: *div-add1-eq1-mod-0-left*[*OF mod-mult-self2-is-0*])
apply *simp*
apply (*rule div-le-mono*)
apply (*simp only*: *div-add1-eq1-mod-0-left*[*OF mod-mult-self2-is-0*])
apply *simp*
apply (*rule div-le-mono*, *assumption*)
apply (*rule-tac t=*$Max\ (I \downarrow\le (t\ div\ k * k + (k - Suc\ 0)))$ **in** *iexI*)
apply (*clarsimp*, *rename-tac t1*)
apply (*subgoal-tac* $t1 \in I$)
prefer *2*
apply *assumption*
apply (*subgoal-tac* $t\ div\ k * k + (k - Suc\ 0) < t1$)
prefer *2*

apply (*rule ccontr*)
apply (*drule not-greater-Max*[*OF nat-cut-le-finite*])
apply (*simp add*: *i-cut-mem-iff*)
apply (*drule-tac t=t1 div k* **in** *ispec*)
apply (*simp add*: *iT-Div-imp-mem cut-greater-mem-iff*)
apply *assumption*
apply (*blast intro*: *subsetD*[*OF - Max-in*[*OF nat-cut-le-finite*]])
apply (*clarsimp*, *rename-tac t*)
apply (*drule-tac x=t* **in** *spec*)
apply (*rule-tac t=t* **in** *iexI*)
apply (*clarsimp simp*: *iT-Div-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*)
apply (*simp add*: *cut-greater-mem-iff*)
apply *simp+*
done

Weak Until and Weak Since operators

lemma *iT-Plus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iWeakUntil-iUntil-conv iT-Plus-iUntil-conv iT-Plus-iAll-conv*)

lemma *iT-Mult-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iWeakUntil-iUntil-conv iT-Mult-iUntil-conv iT-Mult-iAll-conv*)

lemma *iT-Plus-neg-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{W}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iWeakUntil-iUntil-conv iT-Plus-neg-iUntil-conv iT-Plus-neg-iAll-conv*)

lemma *iT-Minus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{B}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
by (*simp add*: *iWeakUntil-iUntil-conv iWeakSince-iSince-conv iT-Minus-iUntil-conv iT-Minus-iAll-conv*)

lemma *iT-Div-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iWeakUntil-iUntil-conv iT-Div-iUntil-conv iT-Div-iAll-conv*)

lemma *iT-Plus-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Plus-iSince-conv iT-Plus-iAll-conv*)

lemma *iT-Mult-iWeakSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Mult-iSince-conv iT-Mult-iAll-conv*)

lemma *iT-Plus-neg-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{B}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$

by (*simp add*: *iWeakSince-iSince-conv iT-Plus-neg-iSince-conv iT-Plus-neg-iAll-conv*)

lemma *iT-Minus-iWeakSince-conv*:
$(P\ t1.\ t1\ \mathcal{B}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{W}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
by (*simp add*: *iWeakSince-iSince-conv iT-Minus-iSince-conv iT-Minus-iAll-conv iWeakUntil-iUntil-conv*)

lemma *iT-Div-iWeakSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Div-iSince-conv iT-Div-iAll-conv*)

Release and Trigger operators

lemma *iT-Plus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Plus-iWeakUntil-conv*)

lemma *iT-Mult-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Mult-iWeakUntil-conv*)

lemma *iT-Plus-neg-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{R}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Plus-neg-iWeakUntil-conv*)

lemma *iT-Minus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{T}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Minus-iWeakUntil-conv iTrigger-iSince-conv iWeakSince-iSince-conv disj-commute*)

lemma *iT-Div-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Div-iWeakUntil-conv*)

lemma *iT-Plus-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Plus-iWeakSince-conv*)

lemma *iT-Mult-iTrigger-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Mult-iWeakSince-conv*)

lemma *iT-Plus-neg-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{T}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Plus-neg-iWeakSince-conv*)

lemma *iT-Minus-iTrigger-conv*:
$(P\ t1.\ t1\ \mathcal{T}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{R}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$

by (*fastforce simp add*: *iTrigger-iWeakSince-conv iT-Minus-iWeakSince-conv iRelease-iUntil-conv iWeakUntil-iUntil-conv*)

lemma *iT-Div-iTrigger-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Div-iWeakSince-conv*)

end