

# The Nash-Williams Theorem

Lawrence C. Paulson

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## Abstract

In 1965, Nash-Williams [1] discovered a generalisation of the infinite form of Ramsey’s theorem. Where the latter concerns infinite sets of  $n$ -element sets for some fixed  $n$ , the Nash-Williams theorem concerns infinite sets of finite sets (or lists) subject to a “no initial segment” condition. The present formalisation follows Todorčević [2].

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## 1 The Pointwise Less-Than Relation Between Two Sets

**theory** *Nash-Extras*

**imports** *HOL-Library.Ramsey* *HOL-Library.Countable-Set*

**begin**

**definition** *less-sets* :: [*'a::order set, 'a::order set*]  $\Rightarrow$  *bool* (**infixr**  $\ll$  50)  
**where**  $A \ll B \equiv \forall x \in A. \forall y \in B. x < y$

**lemma** *less-setsD*:  $\llbracket A \ll B; a \in A; b \in B \rrbracket \Longrightarrow a < b$   
*<proof>*

**lemma** *less-sets-irrefl* [*simp*]:  $A \ll A \longleftrightarrow A = \{\}$   
*<proof>*

**lemma** *less-sets-trans*:  $\llbracket A \ll B; B \ll C; B \neq \{\} \rrbracket \implies A \ll C$   
*<proof>*

**lemma** *less-sets-weaken1*:  $\llbracket A' \ll B; A \subseteq A' \rrbracket \implies A \ll B$   
*<proof>*

**lemma** *less-sets-weaken2*:  $\llbracket A \ll B'; B \subseteq B' \rrbracket \implies A \ll B$   
*<proof>*

**lemma** *less-sets-imp-disjnt*:  $A \ll B \implies \text{disjnt } A \ B$   
*<proof>*

**lemma** *less-sets-UN1*: *less-sets*  $(\bigcup \mathcal{A}) \ B \longleftrightarrow (\forall A \in \mathcal{A}. A \ll B)$   
*<proof>*

**lemma** *less-sets-UN2*: *less-sets*  $A \ (\bigcup \mathcal{B}) \longleftrightarrow (\forall B \in \mathcal{B}. A \ll B)$   
*<proof>*

**lemma** *less-sets-Un1*: *less-sets*  $(A \cup A') \ B \longleftrightarrow A \ll B \wedge A' \ll B$   
*<proof>*

**lemma** *less-sets-Un2*: *less-sets*  $A \ (B \cup B') \longleftrightarrow A \ll B \wedge A \ll B'$   
*<proof>*

**lemma** *strict-sorted-imp-less-sets*:  
*strict-sorted*  $(as \ @ \ bs) \implies (\text{list.set } as) \ll (\text{list.set } bs)$   
*<proof>*

**lemma** *Sup-nat-less-sets-singleton*:  
**fixes**  $n :: \text{nat}$   
**assumes**  $\text{Sup } T < n$  *finite*  $T$   
**shows** *less-sets*  $T \ \{\ n \}$   
*<proof>*

**end**

## 2 The Nash-Williams Theorem

Following S. Todorćević, *Introduction to Ramsey Spaces*, Princeton University Press (2010), 11–12.

**theory** *Nash-Williams*  
**imports** *Nash-Extras*  
**begin**

**lemma** *finite-nat-Int-greaterThan-iff*:  
**fixes**  $N :: \text{nat set}$   
**shows** *finite*  $(N \cap \{n < ..\}) \longleftrightarrow \text{finite } N$   
*<proof>*

## 2.1 Initial segments

**definition** *init-segment* :: *nat set*  $\Rightarrow$  *nat set*  $\Rightarrow$  *bool*  
**where** *init-segment* *S T*  $\equiv \exists S'. T = S \cup S' \wedge S \ll S'$

**lemma** *init-segment-subset*: *init-segment* *S T*  $\Longrightarrow S \subseteq T$   
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-refl*: *init-segment* *S S*  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-antisym*:  $\llbracket$ *init-segment* *S T*; *init-segment* *T S* $\rrbracket \Longrightarrow S=T$   
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-trans*:  $\llbracket$ *init-segment* *S T*; *init-segment* *T U* $\rrbracket \Longrightarrow$  *init-segment* *S U*  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-empty2* [*iff*]: *init-segment* *S*  $\{\}$   $\longleftrightarrow S=\{\}$   
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-Un*:  $S \ll S' \Longrightarrow$  *init-segment* *S* (*S*  $\cup$  *S'*)  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-iff*:  
**shows** *init-segment* *S T*  $\longleftrightarrow S=T \vee (\exists m \in T. S = \{n \in T. n < m\})$  (**is**  $?lhs=?rhs$ )  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-empty* [*iff*]: *init-segment*  $\{\}$  *S*  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-insert-iff*:  
**assumes** *Sn*:  $S \ll \{n\}$  **and** *TS*:  $\bigwedge x. x \in T-S \Longrightarrow n \leq x$   
**shows** *init-segment* (*insert* *n S*) *T*  $\longleftrightarrow$  *init-segment* *S T*  $\wedge n \in T$  (**is**  $?lhs=?rhs$ )  
 $\langle$ *proof* $\rangle$

**lemma** *init-segment-insert*:  
**assumes** *init-segment* *S T* **and** *T*:  $T \ll \{n\}$   
**shows** *init-segment* *S* (*insert* *n T*)  
 $\langle$ *proof* $\rangle$

## 2.2 Definitions and basic properties

**definition** *Ramsey* :: [*nat set set*, *nat*]  $\Rightarrow$  *bool*  
**where** *Ramsey*  $\mathcal{F} r \equiv \forall f \in \mathcal{F} \rightarrow \{..<r\}.$   
 $\forall M. \text{infinite } M \longrightarrow$   
 $(\exists N i. N \subseteq M \wedge \text{infinite } N \wedge i < r \wedge$   
 $(\forall j < r. j \neq i \longrightarrow f - \{j\} \cap \mathcal{F} \cap \text{Pow } N = \{\}))$

Alternative, simpler definition suggested by a referee.

**lemma** *Ramsey-eq*:

*Ramsey*  $\mathcal{F} r \longleftrightarrow (\forall f \in \mathcal{F} \rightarrow \{..<r\}.$   
 $\forall M. \text{infinite } M \rightarrow$   
 $(\exists N i. N \subseteq M \wedge \text{infinite } N \wedge i < r \wedge \mathcal{F} \cap \text{Pow } N \subseteq f -'$   
 $\{i\}))$   
*<proof>*

**definition** *thin-set* :: *nat set set*  $\Rightarrow$  *bool*

**where** *thin-set*  $\mathcal{F} \equiv \mathcal{F} \subseteq \text{Collect finite} \wedge (\forall S \in \mathcal{F}. \forall T \in \mathcal{F}. \text{init-segment } S T \rightarrow S = T)$

**definition** *comparables* :: *nat set*  $\Rightarrow$  *nat set*  $\Rightarrow$  *nat set set*

**where** *comparables*  $S M \equiv \{T. \text{finite } T \wedge (\text{init-segment } T S \vee \text{init-segment } S T \wedge T - S \subseteq M)\}$

**lemma** *comparables-iff*:  $T \in \text{comparables } S M \longleftrightarrow \text{finite } T \wedge (\text{init-segment } T S \vee \text{init-segment } S T \wedge T \subseteq S \cup M)$   
*<proof>*

**lemma** *comparables-subset*:  $\bigcup (\text{comparables } S M) \subseteq S \cup M$   
*<proof>*

**lemma** *comparables-empty* [*simp*]: *comparables*  $\{\}$   $M = \text{Fpow } M$   
*<proof>*

**lemma** *comparables-mono*:  $N \subseteq M \implies \text{comparables } S N \subseteq \text{comparables } S M$   
*<proof>*

**definition** *rejects*  $\mathcal{F} S M \equiv \text{comparables } S M \cap \mathcal{F} = \{\}$

**abbreviation** *accepts*

**where** *accepts*  $\mathcal{F} S M \equiv \neg \text{rejects } \mathcal{F} S M$

**definition** *strongly-accepts*

**where** *strongly-accepts*  $\mathcal{F} S M \equiv (\forall N \subseteq M. \text{rejects } \mathcal{F} S N \rightarrow \text{finite } N)$

**definition** *decides*

**where** *decides*  $\mathcal{F} S M \equiv \text{rejects } \mathcal{F} S M \vee \text{strongly-accepts } \mathcal{F} S M$

**definition** *decides-subsets*

**where** *decides-subsets*  $\mathcal{F} M \equiv \forall T. T \subseteq M \rightarrow \text{finite } T \rightarrow \text{decides } \mathcal{F} T M$

**lemma** *strongly-accepts-imp-accepts*:

$[\text{strongly-accepts } \mathcal{F} S M; \text{infinite } M] \implies \text{accepts } \mathcal{F} S M$   
*<proof>*

**lemma** *rejects-trivial*:  $[\text{rejects } \mathcal{F} S M; \text{thin-set } \mathcal{F}; \text{init-segment } F S; F \in \mathcal{F}] \implies$

*False*  
*<proof>*

**lemma** *rejects-subset*:  $\llbracket \text{rejects } \mathcal{F} S M; N \subseteq M \rrbracket \implies \text{rejects } \mathcal{F} S N$   
*<proof>*

**lemma** *strongly-accepts-subset*:  $\llbracket \text{strongly-accepts } \mathcal{F} S M; N \subseteq M \rrbracket \implies \text{strongly-accepts } \mathcal{F} S N$   
*<proof>*

**lemma** *decides-subset*:  $\llbracket \text{decides } \mathcal{F} S M; N \subseteq M \rrbracket \implies \text{decides } \mathcal{F} S N$   
*<proof>*

**lemma** *decides-subsets-subset*:  $\llbracket \text{decides-subsets } \mathcal{F} M; N \subseteq M \rrbracket \implies \text{decides-subsets } \mathcal{F} N$   
*<proof>*

**lemma** *rejects-empty [simp]*:  $\text{rejects } \mathcal{F} \{\} M \iff \text{Fpow } M \cap \mathcal{F} = \{\}$   
*<proof>*

**lemma** *strongly-accepts-empty [simp]*:  $\text{strongly-accepts } \mathcal{F} \{\} M \iff (\forall N \subseteq M. \text{Fpow } N \cap \mathcal{F} = \{\} \longrightarrow \text{finite } N)$   
*<proof>*

**lemma** *ex-infinite-decides-1*:  
**assumes** *infinite M*  
**obtains** *N* **where**  $N \subseteq M$  *infinite N* *decides*  $\mathcal{F} S N$   
*<proof>*

**proposition** *ex-infinite-decides-finite*:  
**assumes** *infinite M* *finite S*  
**obtains** *N* **where**  $N \subseteq M$  *infinite N*  $\wedge T. T \subseteq S \implies \text{decides } \mathcal{F} T N$   
*<proof>*

**lemma** *sorted-wrt-subset*:  $\llbracket X \in \text{list.set } l; \text{sorted-wrt } (\leq) l \rrbracket \implies \text{hd } l \subseteq X$   
*<proof>*

Todorčević's Lemma 1.18

**proposition** *ex-infinite-decides-subsets*:  
**assumes** *thin-set*  $\mathcal{F}$  *infinite M*  
**obtains** *N* **where**  $N \subseteq M$  *infinite N* *decides-subsets*  $\mathcal{F} N$   
*<proof>*

Todorčević's Lemma 1.19

**proposition** *strongly-accepts-1-19*:  
**assumes** *acc*: *strongly-accepts*  $\mathcal{F} S M$   
**and** *thin-set*  $\mathcal{F}$  *infinite M*  $S \subseteq M$  *finite S*  
**and** *dsM*: *decides-subsets*  $\mathcal{F} M$

**shows**  $\{n \in M. \neg \text{strongly-accepts } \mathcal{F} (\text{insert } n \ S) \ M\}$   
(proof)

Much work is needed for this slight strengthening of the previous result!

**proposition** *strongly-accepts-1-19-plus:*

**assumes** *thin-set*  $\mathcal{F}$  *infinite*  $M$

**and**  $dsM$ : *decides-subsets*  $\mathcal{F}$   $M$

**obtains**  $N$  **where**  $N \subseteq M$  *infinite*  $N$

$\bigwedge S n. \llbracket S \subseteq N; \text{finite } S; \text{strongly-accepts } \mathcal{F} \ S \ N; n \in N; S \ll \{n\} \rrbracket$

$\implies \text{strongly-accepts } \mathcal{F} (\text{insert } n \ S) \ N$

(proof)

## 2.3 Main Theorem

**lemma** *Nash-Williams-1: Ramsey*  $\mathcal{F}$  1

(proof)

**theorem** *Nash-Williams-2:*

**assumes** *thin-set*  $\mathcal{F}$  **shows** *Ramsey*  $\mathcal{F}$  2

(proof)

**theorem** *Nash-Williams:*

**assumes**  $\mathcal{F}$ : *thin-set*  $\mathcal{F}$   $r > 0$  **shows** *Ramsey*  $\mathcal{F}$   $r$

(proof)

end

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## References

- [1] C. S. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 61(1):33–39, 1965.
- [2] S. Todorčević. *Introduction to Ramsey Spaces*. Princeton University Press, 2010.