The Nash-Williams Theorem

Lawrence C. Paulson

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Abstract

In 1965, Nash-Williams [1] discovered a generalisation of the infinite form of Ramsey's theorem. Where the latter concerns infinite sets of n-element sets for some fixed n, the Nash-Williams theorem concerns infinite sets of finite sets (or lists) subject to a "no initial segment" condition. The present formalisation follows Todorčević [2].

Contents

1 The Pointwise Less-Than Relation Between Two Sets

theory *Nash-Extras*
imports *HOL−Library.Ramsey HOL−Library.Countable-Set*

begin

definition *less-sets* :: $['a::order\ set,\ 'a::order\ set] \Rightarrow bool$ **where**
$less\text{-}sets\ A\ B \equiv \forall x{\in}A.\ \forall y{\in}B.\ x < y$

lemma *less-setsD*: $[\![less\text{-}sets\ A\ B;\ a \in A;\ b \in B]\!] \Longrightarrow a < b$
⟨*proof*⟩

lemma *less-sets-irrefl* [*simp*]: $less\text{-}sets\ A\ A \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *less-sets-trans*: $[\![$*less-sets* A B; *less-sets* B C; $B \neq \{\}]\!] \Longrightarrow$ *less-sets* A C
⟨*proof*⟩

lemma *less-sets-weaken1*: $[\![$*less-sets* A' B; $A \subseteq A']\!] \Longrightarrow$ *less-sets* A B
⟨*proof*⟩

lemma *less-sets-weaken2*: $[\![$*less-sets* A B'; $B \subseteq B']\!] \Longrightarrow$ *less-sets* A B
⟨*proof*⟩

lemma *less-sets-imp-disjnt*: *less-sets* A $B \Longrightarrow$ *disjnt* A B
⟨*proof*⟩

lemma *less-sets-UN1*: *less-sets* $(\bigcup \mathcal{A})$ $B \longleftrightarrow (\forall A{\in}\mathcal{A}.$ *less-sets* A $B)$
⟨*proof*⟩

lemma *less-sets-UN2*: *less-sets* A $(\bigcup \mathcal{B}) \longleftrightarrow (\forall B{\in}\mathcal{B}.$ *less-sets* A $B)$
⟨*proof*⟩

lemma *less-sets-Un1*: *less-sets* $(A \cup A')$ $B \longleftrightarrow$ *less-sets* A B $\wedge$ *less-sets* A' B
⟨*proof*⟩

lemma *less-sets-Un2*: *less-sets* A $(B \cup B') \longleftrightarrow$ *less-sets* A B $\wedge$ *less-sets* A B'
⟨*proof*⟩

lemma *strict-sorted-imp-less-sets*:
strict-sorted (*as* @ *bs*) $\Longrightarrow$ *less-sets* (*list.set as*) (*list.set bs*)
⟨*proof*⟩

lemma *Sup-nat-less-sets-singleton*:
fixes n::*nat*
assumes *Sup* $T < n$ *finite* T
shows *less-sets* T $\{n\}$
⟨*proof*⟩

end

2 The Nash-Williams Theorem

Following S. Todorčević, *Introduction to Ramsey Spaces*, Princeton University Press (2010), 11–12.

theory *Nash-Williams*
imports *Nash-Extras*
begin

lemma *finite-nat-Int-greaterThan-iff*:
fixes N :: *nat set*
shows *finite* $(N \cap \{n{<}..\}) \longleftrightarrow$ *finite* N
⟨*proof*⟩

2.1 Initial segments

definition *init-segment* :: *nat set* $\Rightarrow$ *nat set* $\Rightarrow$ *bool*
where *init-segment* S $T \equiv \exists S'.\ T = S \cup S' \wedge$ *less-sets* S S'

lemma *init-segment-subset*: *init-segment* S $T \Longrightarrow S \subseteq T$
⟨*proof*⟩

lemma *init-segment-refl*: *init-segment* S S
⟨*proof*⟩

lemma *init-segment-antisym*: $[\![$*init-segment* S T; *init-segment* T $S]\!] \Longrightarrow S{=}T$
⟨*proof*⟩

lemma *init-segment-trans*: $[\![$*init-segment* S T; *init-segment* T $U]\!] \Longrightarrow$ *init-segment* S U
⟨*proof*⟩

lemma *init-segment-empty2* [*iff*]: *init-segment* S $\{\} \longleftrightarrow S{=}\{\}$
⟨*proof*⟩

lemma *init-segment-Un*: *less-sets* S $S' \Longrightarrow$ *init-segment* S $(S \cup S')$
⟨*proof*⟩

lemma *init-segment-iff*:
shows *init-segment* S $T \longleftrightarrow S{=}T \vee (\exists m \in T.\ S = \{n \in T.\ n < m\})$ (**is** *?lhs=?rhs*)
⟨*proof*⟩

lemma *init-segment-empty* [*iff*]: *init-segment* $\{\}$ S
⟨*proof*⟩

lemma *init-segment-insert-iff*:
assumes *Sn*: *less-sets* S $\{n\}$ **and** *TS*: $\bigwedge x.\ x \in T{-}S \Longrightarrow n{\le}x$
shows *init-segment* (*insert* n S) $T \longleftrightarrow$ *init-segment* S $T \wedge n \in T$
⟨*proof*⟩

lemma *init-segment-insert*:
assumes *init-segment* S T **and** T: *less-sets* T $\{n\}$
shows *init-segment* S (*insert* n T)
⟨*proof*⟩

2.2 Definitions and basic properties

definition *Ramsey* :: [*nat set set*, *nat*] $\Rightarrow$ *bool*
where *Ramsey* $\mathcal{F}$ $r \equiv \forall f \in \mathcal{F} \to \{..{<}r\}.$
$\forall M.$ *infinite* $M \longrightarrow$
$(\exists N\ i.\ N \subseteq M \wedge$ *infinite* $N \wedge i{<}r \wedge (\forall j{<}r.\ j{\neq}i \longrightarrow f -`\ \{j\} \cap \mathcal{F} \cap Pow\ N = \{\}))$

definition *thin-set* :: *nat set set* $\Rightarrow$ *bool*
where *thin-set* $\mathcal{F} \equiv \mathcal{F} \subseteq$ *Collect finite* $\wedge$ $(\forall\, S{\in}\mathcal{F}.\ \forall\, T{\in}\mathcal{F}.$ *init-segment* $S\ T \longrightarrow S{=}T)$

definition *comparables* :: *nat set* $\Rightarrow$ *nat set* $\Rightarrow$ *nat set set*
where *comparables* $S\ M \equiv \{T.$ *finite* $T \wedge ($*init-segment* $T\ S \vee$ *init-segment* $S\ T \wedge T{-}S \subseteq M)\}$

lemma *comparables-iff*: $T \in$ *comparables* $S\ M \longleftrightarrow$ *finite* $T \wedge ($*init-segment* $T\ S \vee$ *init-segment* $S\ T \wedge T \subseteq S \cup M)$
$\langle$*proof*$\rangle$

lemma *comparables-subset*: $\bigcup($*comparables* $S\ M) \subseteq S \cup M$
$\langle$*proof*$\rangle$

lemma *comparables-empty* [*simp*]: *comparables* $\{\}\ M =$ *Fpow* M
$\langle$*proof*$\rangle$

lemma *comparables-mono*: $N \subseteq M \Longrightarrow$ *comparables* $S\ N \subseteq$ *comparables* $S\ M$
$\langle$*proof*$\rangle$

definition *rejects* $\mathcal{F}\ S\ M \equiv$ *comparables* $S\ M \cap \mathcal{F} = \{\}$

abbreviation *accepts*
where *accepts* $\mathcal{F}\ S\ M \equiv \neg$ *rejects* $\mathcal{F}\ S\ M$

definition *strongly-accepts*
where *strongly-accepts* $\mathcal{F}\ S\ M \equiv (\forall\, N{\subseteq}M.$ *rejects* $\mathcal{F}\ S\ N \longrightarrow$ *finite* $N)$

definition *decides*
where *decides* $\mathcal{F}\ S\ M \equiv$ *rejects* $\mathcal{F}\ S\ M \vee$ *strongly-accepts* $\mathcal{F}\ S\ M$

definition *decides-subsets*
where *decides-subsets* $\mathcal{F}\ M \equiv \forall\, T.\ T \subseteq M \longrightarrow$ *finite* $T \longrightarrow$ *decides* $\mathcal{F}\ T\ M$

lemma *strongly-accepts-imp-accepts*:
$[\![$*strongly-accepts* $\mathcal{F}\ S\ M$; *infinite* $M]\!] \Longrightarrow$ *accepts* $\mathcal{F}\ S\ M$
$\langle$*proof*$\rangle$

lemma *rejects-trivial*: $[\![$*rejects* $\mathcal{F}\ S\ M$; *thin-set* $\mathcal{F}$; *init-segment* $F\ S$; $F \in \mathcal{F}]\!] \Longrightarrow$ *False*
$\langle$*proof*$\rangle$

lemma *rejects-subset*: $[\![$*rejects* $\mathcal{F}\ S\ M$; $N \subseteq M]\!] \Longrightarrow$ *rejects* $\mathcal{F}\ S\ N$
$\langle$*proof*$\rangle$

lemma *strongly-accepts-subset*: $[\![$*strongly-accepts* $\mathcal{F}\ S\ M$; $N \subseteq M]\!] \Longrightarrow$ *strongly-accepts* $\mathcal{F}\ S\ N$

$\langle proof \rangle$

lemma *decides-subset*: $[\![decides\ \mathcal{F}\ S\ M;\ N \subseteq M]\!] \Longrightarrow decides\ \mathcal{F}\ S\ N$
$\langle proof \rangle$

lemma *decides-subsets-subset*: $[\![decides\text{-}subsets\ \mathcal{F}\ M;\ N \subseteq M]\!] \Longrightarrow decides\text{-}subsets\ \mathcal{F}\ N$
$\langle proof \rangle$

lemma *rejects-empty* [*simp*]: $rejects\ \mathcal{F}\ \{\}\ M \longleftrightarrow Fpow\ M \cap \mathcal{F} = \{\}$
$\langle proof \rangle$

lemma *strongly-accepts-empty* [*simp*]: $strongly\text{-}accepts\ \mathcal{F}\ \{\}\ M \longleftrightarrow (\forall N \subseteq M.\ Fpow\ N \cap \mathcal{F} = \{\} \longrightarrow finite\ N)$
$\langle proof \rangle$

lemma *ex-infinite-decides-1*:
assumes *infinite M*
obtains N **where** $N \subseteq M$ *infinite N decides* $\mathcal{F}\ S\ N$
$\langle proof \rangle$

proposition *ex-infinite-decides-finite*:
assumes *infinite M finite S*
obtains N **where** $N \subseteq M$ *infinite N* $\bigwedge T.\ T \subseteq S \Longrightarrow decides\ \mathcal{F}\ T\ N$
$\langle proof \rangle$

lemma *sorted-wrt-subset*: $[\![X \in list.set\ l;\ sorted\text{-}wrt\ (\leq)\ l]\!] \Longrightarrow hd\ l \subseteq X$
$\langle proof \rangle$

Todorčević's Lemma 1.18

proposition *ex-infinite-decides-subsets*:
assumes *thin-set* $\mathcal{F}$ *infinite M*
obtains N **where** $N \subseteq M$ *infinite N decides-subsets* $\mathcal{F}\ N$
$\langle proof \rangle$

Todorčević's Lemma 1.19

proposition *strongly-accepts-1-19*:
assumes *acc*: *strongly-accepts* $\mathcal{F}\ S\ M$
and *thin-set* $\mathcal{F}$ *infinite M* $S \subseteq M$ *finite S*
and *dsM*: *decides-subsets* $\mathcal{F}\ M$
shows *finite* $\{n \in M.\ \neg\ strongly\text{-}accepts\ \mathcal{F}\ (insert\ n\ S)\ M\}$
$\langle proof \rangle$

Much work is needed for this slight strengthening of the previous result!

proposition *strongly-accepts-1-19-plus*:
assumes *thin-set* $\mathcal{F}$ *infinite M*
and *dsM*: *decides-subsets* $\mathcal{F}\ M$
obtains N **where** $N \subseteq M$ *infinite N*

$\bigwedge S\ n.\ [\![S \subseteq N;\ \textit{finite}\ S;\ \textit{strongly-accepts}\ \mathcal{F}\ S\ N;\ n \in N;\ \textit{less-sets}\ S\ \{n\}]\!]$
$\Longrightarrow \textit{strongly-accepts}\ \mathcal{F}\ (\textit{insert}\ n\ S)\ N$
⟨*proof*⟩

2.3 Main Theorem

Weirdly, the assumption $f\ `\ \mathcal{F} \subseteq \{..<2::'a\}$ is not used here; it's perhaps unnecessary due to the particular way that *Ramsey* is defined. It's only needed for $(2::'a) < r$

theorem *Nash-Williams-2*:
assumes *thin-set* $\mathcal{F}$ **shows** *Ramsey* $\mathcal{F}$ *2*
⟨*proof*⟩

theorem *Nash-Williams*:
assumes $\mathcal{F}$: *thin-set* $\mathcal{F}$ $r > 0$ **shows** *Ramsey* $\mathcal{F}$ r
⟨*proof*⟩

end

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References

[1] C. S. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 61(1):33–39, 1965.

[2] S. Todorčević. *Introduction to Ramsey Spaces.* Princeton University Press, 2010.