# Mereology

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**Abstract**

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

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# 1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.[1] We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].[2]

We will use the following notation throughout.[3]

**typedecl** $i$
**consts** $\mathit{part} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (‹$P$›)
**consts** $\mathit{overlap} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (‹$O$›)
**consts** $\mathit{proper\text{-}part} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (‹$PP$›)
**consts** $\mathit{sum} :: i \Rightarrow i \Rightarrow i$ (**infix** ‹$\oplus$› *52*)
**consts** $\mathit{product} :: i \Rightarrow i \Rightarrow i$ (**infix** ‹$\otimes$› *53*)
**consts** $\mathit{difference} :: i \Rightarrow i \Rightarrow i$ (**infix** ‹$\ominus$› *51*)
**consts** $\mathit{complement} :: i \Rightarrow i$ (‹$-$›)
**consts** $\mathit{universe} :: i$ (‹$u$›)
**consts** $\mathit{general\text{-}sum} :: (i \Rightarrow \mathit{bool}) \Rightarrow i$ (**binder** ‹$\sigma$› *9*)
**consts** $\mathit{general\text{-}product} :: (i \Rightarrow \mathit{bool}) \Rightarrow i$ (**binder** ‹$\pi$› [*8*] *9*)

# 2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.[4] In other words, parthood is assumed to be a partial ordering relation.[5] Overlap is defined as common parthood.[6]

**locale** $PM$ =
  **assumes** *part-reflexivity*: $P\ x\ x$
  **assumes** *part-transitivity* : $P\ x\ y \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$

[1] For similar developments see [Sen, 2017] and [Bittner, 2018].

[2] For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.

[3] See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.

[4] For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].

[5] Hence the name *premereology*, from [Parsons, 2014] p. 6.

[6] See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

**assumes** *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
**begin**

## 2.1 Parthood

**lemma** *identity-implies-part* : $x = y \Longrightarrow P\ x\ y$
⟨*proof*⟩

## 2.2 Overlap

**lemma** *overlap-intro*: $P\ z\ x \Longrightarrow P\ z\ y \Longrightarrow O\ x\ y$
⟨*proof*⟩

**lemma** *part-implies-overlap*: $P\ x\ y \Longrightarrow O\ x\ y$
⟨*proof*⟩

**lemma** *overlap-reflexivity*: $O\ x\ x$
⟨*proof*⟩

**lemma** *overlap-symmetry*: $O\ x\ y \Longrightarrow O\ y\ x$
⟨*proof*⟩

**lemma** *overlap-monotonicity*: $P\ x\ y \Longrightarrow O\ z\ x \Longrightarrow O\ z\ y$
⟨*proof*⟩

The next lemma is from [Hovda, 2009] p. 66.

**lemma** *overlap-lemma*: $\exists x.\ (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$
⟨*proof*⟩

## 2.3 Disjointness

**lemma** *disjoint-implies-distinct*: $\neg\ O\ x\ y \Longrightarrow x \neq y$
⟨*proof*⟩

**lemma** *disjoint-implies-not-part*: $\neg\ O\ x\ y \Longrightarrow \neg\ P\ x\ y$
⟨*proof*⟩

**lemma** *disjoint-symmetry*: $\neg\ O\ x\ y \Longrightarrow \neg\ O\ y\ x$
⟨*proof*⟩

**lemma** *disjoint-demonotonicity*: $P\ x\ y \Longrightarrow \neg\ O\ z\ y \Longrightarrow \neg\ O\ z\ x$
⟨*proof*⟩

**end**

## 3 Ground Mereology

The theory of *ground mereology* adds to premereology the antisymmetry of parthood, and defines proper parthood as nonidentical parthood.[7] In other words, ground mereology assumes that parthood is a partial order.

**locale** $M = PM +$
  **assumes** *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$
  **assumes** *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$
**begin**

### 3.1 Proper Parthood

**lemma** *proper-implies-part*: $PP\ x\ y \Longrightarrow P\ x\ y$
⟨*proof*⟩

**lemma** *proper-implies-distinct*: $PP\ x\ y \Longrightarrow x \neq y$
⟨*proof*⟩

**lemma** *proper-implies-not-part*: $PP\ x\ y \Longrightarrow \neg\ P\ y\ x$
⟨*proof*⟩

**lemma** *proper-part-asymmetry*: $PP\ x\ y \Longrightarrow \neg\ PP\ y\ x$
⟨*proof*⟩

**lemma** *proper-implies-overlap*: $PP\ x\ y \Longrightarrow O\ x\ y$
⟨*proof*⟩

**end**

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.[8] In the presence of antisymmetry, the two definitions of proper parthood are equivalent.[9]

**locale** $M1 = PM +$
  **assumes** *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg\ P\ y\ x$
  **assumes** *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$

[7] For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

[8] See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

[9] See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

**sublocale** $M \subseteq M1$
⟨*proof*⟩

**sublocale** $M1 \subseteq M$
⟨*proof*⟩

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.[10]

**locale** $M2 = PM +$
 **assumes** *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \land x \neq y$
 **assumes** *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \land \neg\ P\ y\ x$

**sublocale** $M \subseteq M2$
⟨*proof*⟩

**sublocale** $M2 \subseteq M$
⟨*proof*⟩

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.[11]

**locale** $M3 = PM +$
 **assumes** *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \land x \neq y$
 **assumes** *part-extensionality*: $x = y \longleftrightarrow (\forall\ z.\ P\ z\ x \longleftrightarrow P\ z\ y)$

**sublocale** $M \subseteq M3$
⟨*proof*⟩

**sublocale** $M3 \subseteq M$
⟨*proof*⟩

The fourth axiomatization adopts proper parthood as primitive.[12] Improper parthood is defined as proper parthood or identity.

**locale** $M4 =$
 **assumes** *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \lor x = y$
 **assumes** *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \land P\ z\ y)$
 **assumes** *proper-part-asymmetry*: $PP\ x\ y \Longrightarrow \neg\ PP\ y\ x$
 **assumes** *proper-part-transitivity*: $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$
**begin**

[10] For this point see especially [Parsons, 2014] pp. 9-10.
[11] For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.
[12] See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

**lemma** *proper-part-irreflexivity*: $\neg$ *PP x x*
⟨*proof*⟩

**end**

**sublocale** *M* $\subseteq$ *M4*
⟨*proof*⟩

**sublocale** *M4* $\subseteq$ *M*
⟨*proof*⟩

# 4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.[13]

**locale** *MM* = *M* +
**assumes** *weak-supplementation*: *PP y x* $\Longrightarrow$ ($\exists$ *z*. *P z x* $\wedge$ $\neg$ *O z y*)

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.[14]

**locale** *MM1* = *M* +
**assumes** *proper-weak-supplementation*:
*PP y x* $\Longrightarrow$ ($\exists$ *z*. *PP z x* $\wedge$ $\neg$ *O z y*)

**sublocale** *MM* $\subseteq$ *MM1*
⟨*proof*⟩

**sublocale** *MM1* $\subseteq$ *MM*
⟨*proof*⟩

The following two corollaries are sometimes found in the literature.[15]

**context** *MM*
**begin**

**corollary** *weak-company*: *PP y x* $\Longrightarrow$ ($\exists$ *z*. *PP z x* $\wedge$ *z* $\neq$ *y*)
⟨*proof*⟩

[13] See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

[14] See [Simons, 1987] p. 28.

[15] See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].

**corollary** *strong-company*: $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y)$
⟨*proof*⟩

**end**

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.[16]

**locale** *MM2* = *PM* +
  **assumes** *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$
  **assumes** *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

**sublocale** *MM2* ⊆ *MM*
⟨*proof*⟩

**sublocale** *MM* ⊆ *MM2*
⟨*proof*⟩

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.[17]

**locale** *MM3* =
  **assumes** *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$
  **assumes** *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
  **assumes** *proper-part-transitivity*: $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$
  **assumes** *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
**begin**

**lemma** *part-reflexivity*: $P\ x\ x$
⟨*proof*⟩

**lemma** *proper-part-irreflexivity*: $\neg\ PP\ x\ x$
⟨*proof*⟩

**end**

**sublocale** *MM3* ⊆ *M4*
⟨*proof*⟩

**sublocale** *MM3* ⊆ *MM*
⟨*proof*⟩

**sublocale** *MM* ⊆ *MM3*

[16] See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.
[17] See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

⟨*proof*⟩

## 5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.[18]

**locale** *EM* = *M* +
 **assumes** *strong-supplementation*:
 $\neg\ P\ x\ y \Longrightarrow (\exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
**begin**

Strong supplementation entails weak supplementation.[19]

**lemma** *weak-supplementation*: $PP\ x\ y \Longrightarrow (\exists\, z.\ P\ z\ y \wedge \neg\ O\ z\ x)$
⟨*proof*⟩

**end**

So minimal mereology is a subtheory of extensional mereology.[20]

**sublocale** $EM \subseteq MM$
⟨*proof*⟩

Strong supplementation also entails the proper parts principle.[21]

**context** *EM*
**begin**

**lemma** *proper-parts-principle*:
$(\exists\, z.\ PP\ z\ x) \Longrightarrow (\forall\, z.\ PP\ z\ x \longrightarrow P\ z\ y) \Longrightarrow P\ x\ y$
⟨*proof*⟩

Which with antisymmetry entails the extensionality of proper parthood.[22]

**theorem** *proper-part-extensionality*:
$(\exists\, z.\ PP\ z\ x \vee PP\ z\ y) \Longrightarrow x = y \longleftrightarrow (\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ y)$
⟨*proof*⟩

It also follows from strong supplementation that parthood is definable in terms of overlap.[23]

**lemma** *part-overlap-eq*: $P\ x\ y \longleftrightarrow (\forall\, z.\ O\ z\ x \longrightarrow O\ z\ y)$
⟨*proof*⟩

Which entails the extensionality of overlap.

[18] See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.
[19] See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.
[20] [Casati and Varzi, 1999] p. 40.
[21] See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.
[22] See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.
[23] See [Parsons, 2014] p. 4.

**theorem** *overlap-extensionality*: $x = y \longleftrightarrow (\forall z.\ O\ z\ x \longleftrightarrow O\ z\ y)$
⟨*proof*⟩

**end**

# 6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.[24]

**locale** $CM = M +$
**assumes** *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \lor O\ v\ y)$
**assumes** *sum-closure*: $\exists z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \lor O\ v\ y$
**assumes** *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \land P\ v\ y)$
**assumes** *product-closure*:
$O\ x\ y \Longrightarrow \exists z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \land P\ v\ y$
**begin**

## 6.1 Products

**lemma** *product-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \land P\ w\ y)) \Longrightarrow x \otimes y = z$
⟨*proof*⟩

**lemma** *product-idempotence*: $x \otimes x = x$
⟨*proof*⟩

**lemma** *product-character*:
$O\ x\ y \Longrightarrow (\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \land P\ w\ y))$
⟨*proof*⟩

**lemma** *product-commutativity*: $O\ x\ y \Longrightarrow x \otimes y = y \otimes x$
⟨*proof*⟩

**lemma** *product-in-factors*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ x \land P\ (x \otimes y)\ y$
⟨*proof*⟩

**lemma** *product-in-first-factor*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ x$
⟨*proof*⟩

**lemma** *product-in-second-factor*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ y$
⟨*proof*⟩

**lemma** *nonpart-implies-proper-product*:

[24] See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

$\neg\ P\ x\ y \wedge O\ x\ y \Longrightarrow PP\ (x \otimes y)\ x$
⟨*proof*⟩

**lemma** *common-part-in-product*: $P\ z\ x \wedge P\ z\ y \Longrightarrow P\ z\ (x \otimes y)$
⟨*proof*⟩

**lemma** *product-part-in-factors*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x \wedge P\ z\ y$
⟨*proof*⟩

**corollary** *product-part-in-first-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x$
⟨*proof*⟩

**corollary** *product-part-in-second-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ y$
⟨*proof*⟩

**lemma** *part-product-identity*: $P\ x\ y \Longrightarrow x \otimes y = x$
⟨*proof*⟩

**lemma** *product-overlap*: $P\ z\ x \Longrightarrow O\ z\ y \Longrightarrow O\ z\ (x \otimes y)$
⟨*proof*⟩

**lemma** *disjoint-from-second-factor*:
$P\ x\ y \wedge \neg\ O\ x\ (y \otimes z) \Longrightarrow \neg\ O\ x\ z$
⟨*proof*⟩

**lemma** *converse-product-overlap*:
$O\ x\ y \Longrightarrow O\ z\ (x \otimes y) \Longrightarrow O\ z\ y$
⟨*proof*⟩

**lemma** *part-product-in-whole-product*:
$O\ x\ y \Longrightarrow P\ x\ v \wedge P\ y\ z \Longrightarrow P\ (x \otimes y)\ (v \otimes z)$
⟨*proof*⟩

**lemma** *right-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
⟨*proof*⟩

**lemma** *left-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$
⟨*proof*⟩

**theorem** *product-associativity*:
$(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow x \otimes (y \otimes z) = (x \otimes y) \otimes z$
⟨*proof*⟩

**end**

### 6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.[25]

**locale** *CMD* = *CM* +
  **assumes** *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
  **assumes** *difference-closure*:
    $(\exists w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
**begin**

**lemma** *difference-intro*:
  $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow x \ominus y = z$
⟨*proof*⟩

**lemma** *difference-idempotence*: $\neg\ O\ x\ y \Longrightarrow (x \ominus y) = x$
⟨*proof*⟩

**lemma** *difference-character*: $(\exists w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow$
  $(\forall w.\ P\ w\ (x \ominus y) \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
⟨*proof*⟩

**lemma** *difference-disjointness*:
  $(\exists z.\ P\ z\ x \land \neg\ O\ z\ y) \Longrightarrow \neg\ O\ y\ (x \ominus y)$
⟨*proof*⟩

**end**

### 6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.[26]

**locale** *CMU* = *CM* +
  **assumes** *universe-eq*: $u = (THE\ z.\ \forall w.\ P\ w\ z)$
  **assumes** *universe-closure*: $\exists y.\ \forall x.\ P\ x\ y$
**begin**

**lemma** *universe-intro*: $(\forall w.\ P\ w\ z) \Longrightarrow u = z$
⟨*proof*⟩

**lemma** *universe-character*: $P\ x\ u$
⟨*proof*⟩

**lemma** $\neg\ PP\ u\ x$
⟨*proof*⟩

[25] See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.
[26] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

**lemma** *product-universe-implies-factor-universe*:
$O\ x\ y \Longrightarrow x \otimes y = u \Longrightarrow x = u$
⟨*proof*⟩

**end**

### 6.4 Complements

As is a condition ensuring the existence of complements.[27]

**locale** *CMC* = *CM* +
**assumes** *complement-eq*: $-x = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
**assumes** *complement-closure*:
$(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
**assumes** *difference-eq*:
$x \ominus y = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
**begin**

**lemma** *complement-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) \Longrightarrow -x = z$
⟨*proof*⟩

**lemma** *complement-character*:
$(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x)$
⟨*proof*⟩

**lemma** *not-complement-part*: $\exists w.\ \neg\ O\ w\ x \Longrightarrow \neg\ P\ x\ (-x)$
⟨*proof*⟩

**lemma** *complement-part*: $\neg\ O\ x\ y \Longrightarrow P\ x\ (-y)$
⟨*proof*⟩

**lemma** *complement-overlap*: $\neg\ O\ x\ y \Longrightarrow O\ x\ (-y)$
⟨*proof*⟩

**lemma** *or-complement-overlap*: $\forall y.\ O\ y\ x \vee O\ y\ (-x)$
⟨*proof*⟩

**lemma** *complement-disjointness*: $\exists v.\ \neg\ O\ v\ x \Longrightarrow \neg\ O\ x\ (-x)$
⟨*proof*⟩

**lemma** *part-disjoint-from-complement*:
$\exists v.\ \neg\ O\ v\ x \Longrightarrow P\ y\ x \Longrightarrow \neg\ O\ y\ (-x)$
⟨*proof*⟩

**lemma** *product-complement-character*: $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y)))$

[27] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

⟨*proof*⟩

**theorem** *difference-closure*: $(\exists\, w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow$
$(\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
⟨*proof*⟩

**end**

**sublocale** *CMC* ⊆ *CMD*
⟨*proof*⟩

**corollary** (**in** *CMC*) *difference-is-product-of-complement*:
$(\exists\, w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow (x \ominus y) = x \otimes (-y)$
⟨*proof*⟩

Universe and difference closure entail complement closure, since the difference of an individual and the universe is the individual's complement.

**locale** *CMUD* = *CMU* + *CMD* +
**assumes** *complement-eq*: $-x = (THE\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
**begin**

**lemma** *universe-difference*:
$(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\forall\, w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x)$
⟨*proof*⟩

**theorem** *complement-closure*:
$(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
⟨*proof*⟩

**end**

**sublocale** *CMUD* ⊆ *CMC*
⟨*proof*⟩

**corollary** (**in** *CMUD*) *complement-universe-difference*:
$(\exists\, y.\ \neg\ O\ y\ x) \Longrightarrow -x = (u \ominus x)$
⟨*proof*⟩

# 7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.[28]

**locale** *CEM* = *CM* + *EM*

Likewise, closed minimal mereology combines closed mereology

[28] See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.

with minimal mereology.[29]

**locale** *CMM* = *CM* + *MM*

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.[30]

**sublocale** *CMM* ⊆ *CEM*
⟨*proof*⟩

**sublocale** *CEM* ⊆ *CMM*⟨*proof*⟩

## 7.1 Sums

**context** *CEM*
**begin**

**lemma** *sum-intro*:
$(\forall\ w.\ O\ w\ z \longleftrightarrow (O\ w\ x \vee O\ w\ y)) \Longrightarrow x \oplus y = z$
⟨*proof*⟩

**lemma** *sum-idempotence*: $x \oplus x = x$
⟨*proof*⟩

**lemma** *part-sum-identity*: $P\ y\ x \Longrightarrow x \oplus y = x$
⟨*proof*⟩

**lemma** *sum-character*: $\forall\ w.\ O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \vee O\ w\ y)$
⟨*proof*⟩

**lemma** *sum-overlap*: $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \vee O\ w\ y)$
⟨*proof*⟩

**lemma** *sum-part-character*:
$P\ w\ (x \oplus y) \longleftrightarrow (\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \vee O\ v\ y)$
⟨*proof*⟩

**lemma** *sum-commutativity*: $x \oplus y = y \oplus x$
⟨*proof*⟩

**lemma** *first-summand-overlap*: $O\ z\ x \Longrightarrow O\ z\ (x \oplus y)$
⟨*proof*⟩

**lemma** *first-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ x$
⟨*proof*⟩

[29] See [Casati and Varzi, 1999] p. 43.
[30] See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

**lemma** *first-summand-in-sum*: $P\ x\ (x \oplus y)$
⟨*proof*⟩

**lemma** *common-first-summand*: $P\ x\ (x \oplus y) \wedge P\ x\ (x \oplus z)$
⟨*proof*⟩

**lemma** *common-first-summand-overlap*: $O\ (x \oplus y)\ (x \oplus z)$
⟨*proof*⟩

**lemma** *second-summand-overlap*: $O\ z\ y \Longrightarrow O\ z\ (x \oplus y)$
⟨*proof*⟩

**lemma** *second-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ y$
⟨*proof*⟩

**lemma** *second-summand-in-sum*: $P\ y\ (x \oplus y)$
⟨*proof*⟩

**lemma** *second-summands-in-sums*: $P\ y\ (x \oplus y) \wedge P\ v\ (z \oplus v)$
⟨*proof*⟩

**lemma** *disjoint-from-sum*: $\neg\ O\ z\ (x \oplus y) \longleftrightarrow \neg\ O\ z\ x \wedge \neg\ O\ z\ y$
⟨*proof*⟩

**lemma** *summands-part-implies-sum-part*:
$P\ x\ z \wedge P\ y\ z \Longrightarrow P\ (x \oplus y)\ z$
⟨*proof*⟩

**lemma** *sum-part-implies-summands-part*:
$P\ (x \oplus y)\ z \Longrightarrow P\ x\ z \wedge P\ y\ z$
⟨*proof*⟩

**lemma** *in-second-summand*: $P\ z\ (x \oplus y) \wedge \neg\ O\ z\ x \Longrightarrow P\ z\ y$
⟨*proof*⟩

**lemma** *disjoint-second-summands*:
$P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z) \Longrightarrow \neg\ O\ y\ z \Longrightarrow P\ v\ x$
⟨*proof*⟩

**lemma** *right-associated-sum*:
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ x \vee (O\ w\ y \vee O\ w\ z)$
⟨*proof*⟩

**lemma** *left-associated-sum*:
$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow (O\ w\ x \vee O\ w\ y) \vee O\ w\ z$
⟨*proof*⟩

**theorem** *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
⟨*proof*⟩

## 7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

**lemma** *common-summand-in-product*: $P\ x\ ((x \oplus y) \otimes (x \oplus z))$
⟨*proof*⟩

**lemma** *product-in-first-summand*:
$\neg\ O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ x$
⟨*proof*⟩

**lemma** *product-is-first-summand*:
$\neg\ O\ y\ z \Longrightarrow (x \oplus y) \otimes (x \oplus z) = x$
⟨*proof*⟩

**lemma** *sum-over-product-left*: $O\ y\ z \Longrightarrow P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
⟨*proof*⟩

**lemma** *sum-over-product-right*:
$O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
⟨*proof*⟩

Sums distribute over products.

**theorem** *sum-over-product*:
$O\ y\ z \Longrightarrow x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
⟨*proof*⟩

**lemma** *product-in-factor-by-sum*:
$O\ x\ y \Longrightarrow P\ (x \otimes y)\ (x \otimes (y \oplus z))$
⟨*proof*⟩

**lemma** *product-of-first-summand*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))\ (x \otimes y)$
⟨*proof*⟩

**theorem** *disjoint-product-over-sum*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow x \otimes (y \oplus z) = x \otimes y$
⟨*proof*⟩

**lemma** *product-over-sum-left*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
⟨*proof*⟩

**lemma** *product-over-sum-right*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
⟨*proof*⟩

**theorem** *product-over-sum*:

$O\ x\ y \wedge O\ x\ z \Longrightarrow x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
⟨*proof*⟩

**lemma** *joint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow O\ x\ v \wedge O\ x\ w \Longrightarrow ((x \otimes v) \oplus (x \otimes w)) = x$
⟨*proof*⟩

**lemma** *disjoint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow \neg\ O\ y\ v \wedge \neg\ O\ w\ x \Longrightarrow x = v \wedge y = w$
⟨*proof*⟩

**end**

## 7.3 Differences

**locale** *CEMD* = *CEM* + *CMD*
**begin**

**lemma** *plus-minus*: $PP\ y\ x \Longrightarrow y \oplus (x \ominus y) = x$
⟨*proof*⟩

**end**

## 7.4 The Universe

**locale** *CEMU* = *CEM* + *CMU*
**begin**

**lemma** *something-disjoint*: $x \neq u \Longrightarrow (\exists\ v.\ \neg\ O\ v\ x)$
⟨*proof*⟩

**lemma** *overlaps-universe*: $O\ x\ u$
⟨*proof*⟩

**lemma** *universe-absorbing*: $x \oplus u = u$
⟨*proof*⟩

**lemma** *second-summand-not-universe*: $x \oplus y \neq u \Longrightarrow y \neq u$
⟨*proof*⟩

**lemma** *first-summand-not-universe*: $x \oplus y \neq u \Longrightarrow x \neq u$
⟨*proof*⟩

**end**

## 7.5 Complements

**locale** *CEMC* = *CEM* + *CMC* +
**assumes** *universe-eq*: $u = (THE\ x.\ \forall\ y.\ P\ y\ x)$
**begin**

**lemma** *complement-sum-character*: $\forall\ y.\ P\ y\ (x \oplus (-x))$
⟨*proof*⟩

**lemma** *universe-closure*: $\exists\ x.\ \forall\ y.\ P\ y\ x$
⟨*proof*⟩

**end**

**sublocale** *CEMC* ⊆ *CEMU*
⟨*proof*⟩

**sublocale** *CEMC* ⊆ *CEMD*
⟨*proof*⟩

**context** *CEMC*
**begin**

**corollary** *universe-is-complement-sum*: $u = x \oplus (-x)$
⟨*proof*⟩

**lemma** *strong-complement-character*:
$x \neq u \Longrightarrow (\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x)$
⟨*proof*⟩

**lemma** *complement-part-not-part*: $x \neq u \Longrightarrow P\ y\ (-x) \Longrightarrow \neg\ P\ y\ x$
⟨*proof*⟩

**lemma** *complement-involution*: $x \neq u \Longrightarrow x = -(-x)$
⟨*proof*⟩

**lemma** *part-complement-reversal*: $y \neq u \Longrightarrow P\ x\ y \Longrightarrow P\ (-y)\ (-x)$
⟨*proof*⟩

**lemma** *complements-overlap*: $x \oplus y \neq u \Longrightarrow O(-x)(-y)$
⟨*proof*⟩

**lemma** *sum-complement-in-complement-product*:
$x \oplus y \neq u \Longrightarrow P(-(x \oplus y))(-x \otimes -y)$
⟨*proof*⟩

**lemma** *complement-product-in-sum-complement*:
$x \oplus y \neq u \Longrightarrow P(-x \otimes -y)(-(x \oplus y))$
⟨*proof*⟩

**theorem** *sum-complement-is-complements-product*:
$x \oplus y \neq u \Longrightarrow -(x \oplus y) = (-x \otimes -y)$
⟨*proof*⟩

**lemma** *complement-sum-in-product-complement*:
$O\ x\ y \Longrightarrow x \neq u \Longrightarrow y \neq u \Longrightarrow P\ ((-x) \oplus (-y))(-(x \otimes y))$
⟨*proof*⟩

**lemma** *product-complement-in-complements-sum*:
$x \neq u \Longrightarrow y \neq u \Longrightarrow P(-(x \otimes y))((-x) \oplus (-y))$
⟨*proof*⟩

**theorem** *complement-of-product-is-sum-of-complements*:
$O\ x\ y \Longrightarrow x \oplus y \neq u \Longrightarrow -(x \otimes y) = (-x) \oplus (-y)$
⟨*proof*⟩

**end**

# 8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.[31]

**locale** *GM* = *M* +
**assumes** *fusion*:
$\exists\ x.\ \varphi\ x \Longrightarrow \exists\ z.\ \forall\ y.\ O\ y\ z \longleftrightarrow (\exists\ x.\ \varphi\ x \wedge O\ y\ x)$
**begin**

Fusion entails sum closure.

**theorem** *sum-closure*: $\exists\ z.\ \forall\ w.\ O\ w\ z \longleftrightarrow (O\ w\ a \vee O\ w\ b)$
⟨*proof*⟩

**end**

# 9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.[32]

**locale** *GMM* = *GM* + *MM*
**begin**

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general minimal mereology and general extensional mereology.[33] But

[31]See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.
[32]See [Casati and Varzi, 1999] p. 46.
[33]For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

**lemma** *product-closure*:
  $O\ x\ y \Longrightarrow (\exists\ z.\ \forall\ v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  **nitpick** [*expect* = *genuine*] ⟨*proof*⟩

**lemma** *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
  **nitpick** [*expect* = *genuine*] ⟨*proof*⟩

**end**

# 10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.[34]

**locale** *GEM* = *GM* + *EM* +
  **assumes** *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
  **assumes** *product-eq*:
    $x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  **assumes** *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
  **assumes** *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
  **assumes** *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
  **assumes** *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
    $(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
  **assumes** *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$

**sublocale** *GEM* ⊆ *GMM*
⟨*proof*⟩

## 10.1 General Sums

**context** *GEM*
**begin**

**lemma** *fusion-intro*:
$(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)) \Longrightarrow (\sigma\ x.\ F\ x) = z$
⟨*proof*⟩

**lemma** *fusion-idempotence*: $(\sigma\ x.\ z = x) = z$

[34] For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

⟨*proof*⟩

The whole is the sum of its parts.

**lemma** *fusion-absorption*: $(\sigma\ x.\ P\ x\ z) = z$
⟨*proof*⟩

**lemma** *part-fusion*: $P\ w\ (\sigma\ v.\ P\ v\ x) \Longrightarrow P\ w\ x$
⟨*proof*⟩

**lemma** *fusion-character*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
⟨*proof*⟩

The next lemma characterises fusions in terms of parthood.[35]

**lemma** *fusion-part-character*: $\exists x.\ F\ x \Longrightarrow$
$(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)))$
⟨*proof*⟩

**lemma** *fusion-part*: $F\ x \Longrightarrow P\ x\ (\sigma\ x.\ F\ x)$
⟨*proof*⟩

**lemma** *common-part-fusion*:
$O\ x\ y \Longrightarrow (\forall w.\ P\ w\ (\sigma\ v.\ (P\ v\ x \wedge P\ v\ y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ y))$
⟨*proof*⟩

**theorem** *product-closure*:
$O\ x\ y \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y))$
⟨*proof*⟩

**end**

**sublocale** *GEM* $\subseteq$ *CEM*
⟨*proof*⟩

**context** *GEM*
**begin**

**corollary** $O\ x\ y \Longrightarrow x \otimes y = (\sigma\ v.\ P\ v\ x \wedge P\ v\ y)$
⟨*proof*⟩

**lemma** *disjoint-fusion*:
$\exists w.\ \neg\ O\ w\ x \Longrightarrow (\forall w.\ P\ w\ (\sigma\ z.\ \neg\ O\ z\ x) \longleftrightarrow \neg\ O\ w\ x)$
⟨*proof*⟩

**theorem** *complement-closure*:
$\exists w.\ \neg\ O\ w\ x \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
⟨*proof*⟩

[35] See [Pontow, 2004] pp. 202-9.

**end**

**sublocale** *GEM* ⊆ *CEMC*
⟨*proof*⟩

**context** *GEM*
**begin**

**corollary** *complement-is-disjoint-fusion*:
$\exists w.\ \neg\ O\ w\ x \Longrightarrow -\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
⟨*proof*⟩

**theorem** *strong-fusion*: $\exists x.\ F\ x \Longrightarrow$
$\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \land (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \land O\ y\ z))$
⟨*proof*⟩

**theorem** *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) =$
$(THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \land (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \land O\ y\ z)))$
⟨*proof*⟩

**lemma** *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y))$
⟨*proof*⟩

## 10.2 General Products

**lemma** *general-product-intro*: $(\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \land O\ y\ z)) \Longrightarrow (\pi\ x.\ F\ x) = x$
⟨*proof*⟩

**lemma** *general-product-idempotence*: $(\pi\ z.\ z = x) = x$
⟨*proof*⟩

**lemma** *general-product-absorption*: $(\pi\ z.\ P\ x\ z) = x$
⟨*proof*⟩

**lemma** *general-product-character*: $\exists z.\ \forall y.\ F\ y \longrightarrow P\ z\ y \Longrightarrow$
$\forall y.\ O\ y\ (\pi\ x.\ F\ x) \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \land O\ y\ z)$
⟨*proof*⟩

**corollary** $\neg\ (\exists x.\ F\ x) \Longrightarrow u = (\pi\ x.\ F\ x)$
⟨*proof*⟩

**end**

## 10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.[36]

**locale** *GEM1* = *MM* +
**assumes** *strong-fusion*: $\exists x.\ F\ x \Longrightarrow \exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
**assumes** *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$
**assumes** *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
**assumes** *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
**assumes** *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
**assumes** *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
**assumes** *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
**assumes** *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$
**begin**

**theorem** *fusion*:
$\exists x.\ \varphi\ x \Longrightarrow (\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$
⟨*proof*⟩

**lemma** *pair*: $\exists v.\ (\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ v) \wedge (\forall w.\ P\ w\ v \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ w\ z))$
⟨*proof*⟩

**lemma** *or-id*: $(v = x \vee v = y) \wedge O\ w\ v \Longrightarrow O\ w\ x \vee O\ w\ y$
⟨*proof*⟩

**lemma** *strong-sum-closure*:
$\exists z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
⟨*proof*⟩

**end**

**sublocale** *GEM1* $\subseteq$ *GMM*
⟨*proof*⟩

**context** *GEM1*
**begin**

**lemma** *least-upper-bound*:
**assumes** *sf*:

[36] See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

$((\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
**shows** *lub*:
$(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$
⟨*proof*⟩

**corollary** *strong-fusion-intro*: $(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) \Longrightarrow (\sigma\ x.\ F\ x) = x$
⟨*proof*⟩

**lemma** *strong-fusion-character*: $\exists x.\ F\ x \Longrightarrow ((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
⟨*proof*⟩

**lemma** *F-in*: $\exists x.\ F\ x \Longrightarrow (\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x))$
⟨*proof*⟩

**lemma** *parts-overlap-Fs*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
⟨*proof*⟩

**lemma** *in-strong-fusion*: $P\ z\ (\sigma\ x.\ z = x)$
⟨*proof*⟩

**lemma** *strong-fusion-in*: $P\ (\sigma\ x.\ z = x)\ z$
⟨*proof*⟩

**lemma** *strong-fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
⟨*proof*⟩

## 10.4 Strong Sums

**lemma** *pair-fusion*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow (\sigma\ z.\ z = x \vee z = y) = z$
⟨*proof*⟩

**corollary** *strong-sum-fusion*: $x \oplus y = (\sigma\ z.\ z = x \vee z = y)$
⟨*proof*⟩

**corollary** *strong-sum-intro*:
$(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow x \oplus y = z$
⟨*proof*⟩

**corollary** *strong-sum-character*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
⟨*proof*⟩

**corollary** *summands-in*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y))$
⟨*proof*⟩

**corollary** *first-summand-in*: $P\ x\ (x \oplus y)$ ⟨*proof*⟩

**corollary** *second-summand-in*: $P\ y\ (x \oplus y)$ ⟨*proof*⟩

**corollary** *sum-part-overlap*: $(\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
⟨*proof*⟩

**lemma** *strong-sum-absorption*: $y = (x \oplus y) \Longrightarrow P\ x\ y$
⟨*proof*⟩

**theorem** *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
⟨*proof*⟩

**lemma** *sum-character*: $\forall v.\ O\ v\ (x \oplus y) \longleftrightarrow (O\ v\ x \vee O\ v\ y)$
⟨*proof*⟩

**lemma** *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z = (O\ v\ x \vee O\ v\ y))$
⟨*proof*⟩

**theorem** *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
$(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
⟨*proof*⟩

**end**

**sublocale** *GEM1* ⊆ *GEM*
⟨*proof*⟩

**sublocale** *GEM* ⊆ *GEM1*
⟨*proof*⟩

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