Maximum Cardinality Matching

Christine Rizkallah

August 16, 2018

Abstract

A *matching* in a graph G is a subset M of the edges of G such that no two share an endpoint. A matching has maximum cardinality if its cardinality is at least as large as that of any other matching. An *odd-set cover* OSC of a graph G is a labeling of the nodes of G with integers such that every edge of G is either incident to a node labeled 1 or connects two nodes labeled with the same number $i \geq 2$.

Theorem 1 (Edmonds [2])**.** Let M be a matching in a graph G and let OSC be an odd-set cover of G. For any $i \geq 0$, let n_i be the number of nodes labeled i. If

$$|M| = n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor$$

then M is a maximum cardinality matching.

We provide an Isabelle proof of Edmonds theorem. For an explanation of the proof see [1].

Contents

theory *Matching*
imports *Main*
begin

type-synonym *label* = *nat*

1 Definitions

definition $\mathit{finite\text{-}graph}$ :: $'v\ \mathit{set} => ('v * 'v)\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{finite\text{-}graph}\ V\ E = (\mathit{finite}\ V \wedge \mathit{finite}\ E \wedge$
$(\forall\ e \in E.\ \mathit{fst}\ e \in V \wedge \mathit{snd}\ e \in V \wedge \mathit{fst}\ e\ {}^{\sim}{=}\ \mathit{snd}\ e))$

definition degree :: $('v * 'v)\ \mathit{set} \Rightarrow 'v \Rightarrow \mathit{nat}$ **where**
$\mathit{degree}\ E\ v = \mathit{card}\ \{e \in E.\ \mathit{fst}\ e = v \vee \mathit{snd}\ e = v\}$

definition $\mathit{edge\text{-}as\text{-}set}$ :: $('v * 'v) \Rightarrow 'v\ \mathit{set}$ **where**
$\mathit{edge\text{-}as\text{-}set}\ e = \{\mathit{fst}\ e,\ \mathit{snd}\ e\}$

definition N :: $'v\ \mathit{set} \Rightarrow ('v \Rightarrow \mathit{label}) \Rightarrow \mathit{nat} \Rightarrow \mathit{nat}$ **where**
$N\ V\ L\ i = \mathit{card}\ \{v \in V.\ L\ v = i\}$

definition weight:: $\mathit{label}\ \mathit{set} \Rightarrow (\mathit{label} \Rightarrow \mathit{nat}) \Rightarrow \mathit{nat}$ **where**
$\mathit{weight}\ LV\ f = f\ 1 + (\sum i{\in}LV.\ (f\ i)\ \mathit{div}\ 2)$

definition OSC :: $('v \Rightarrow \mathit{label}) \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{OSC}\ L\ E = (\forall\, e \in E.\ L\ (\mathit{fst}\ e) = 1 \vee L\ (\mathit{snd}\ e) = 1 \vee$
$L\ (\mathit{fst}\ e) = L\ (\mathit{snd}\ e) \wedge L\ (\mathit{fst}\ e) > 1)$

definition $\mathit{disjoint\text{-}edges}$ :: $('v * 'v) \Rightarrow ('v * 'v) \Rightarrow \mathit{bool}$ **where**
$\mathit{disjoint\text{-}edges}\ e1\ e2 = (\mathit{fst}\ e1 \neq \mathit{fst}\ e2 \wedge \mathit{fst}\ e1 \neq \mathit{snd}\ e2 \wedge$
$\mathit{snd}\ e1 \neq \mathit{fst}\ e2 \wedge \mathit{snd}\ e1 \neq \mathit{snd}\ e2)$

definition $\mathit{matching}$ :: $'v\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow \mathit{bool}$ **where**
$\mathit{matching}\ V\ E\ M = (M \subseteq E \wedge \mathit{finite\text{-}graph}\ V\ E \wedge$
$(\forall\, e1 \in M.\ \forall\ e2 \in M.\ e1 \neq e2 \longrightarrow \mathit{disjoint\text{-}edges}\ e1\ e2))$

definition $\mathit{matching\text{-}i}$ :: $\mathit{nat} \Rightarrow 'v\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow$
$('v \Rightarrow \mathit{label}) \Rightarrow ('v * 'v)\ \mathit{set}$ **where**
$\mathit{matching\text{-}i}\ i\ V\ E\ M\ L = \{e \in M.\ i{=}1 \wedge (L\ (\mathit{fst}\ e) = i \vee L\ (\mathit{snd}\ e) = i)$
$\vee\ i{>}1 \wedge L\ (\mathit{fst}\ e) = i \wedge L\ (\mathit{snd}\ e) = i\}$

definition $\mathit{V\text{-}i}$:: $\mathit{nat} \Rightarrow 'v\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow ('v * 'v)\ \mathit{set} \Rightarrow$
$('v \Rightarrow \mathit{label}) \Rightarrow 'v\ \mathit{set}$ **where**
$\mathit{V\text{-}i}\ i\ V\ E\ M\ L = \bigcup\ (\mathit{edge\text{-}as\text{-}set}\ `\ \mathit{matching\text{-}i}\ i\ V\ E\ M\ L)$

definition $\mathit{endpoint\text{-}inV}$ :: $'v\ \mathit{set} \Rightarrow ('v * 'v) \Rightarrow 'v$ **where**
$\mathit{endpoint\text{-}inV}\ V\ e = (\mathit{if}\ \mathit{fst}\ e \in V\ \mathit{then}\ \mathit{fst}\ e\ \mathit{else}\ \mathit{snd}\ e)$

definition $\mathit{relevant\text{-}endpoint}$ :: $('v \Rightarrow \mathit{label}) \Rightarrow 'v\ \mathit{set} \Rightarrow$
$('v * 'v) \Rightarrow 'v$ **where**
$\mathit{relevant\text{-}endpoint}\ L\ V\ e = (\mathit{if}\ L\ (\mathit{fst}\ e) = 1\ \mathit{then}\ \mathit{fst}\ e\ \mathit{else}\ \mathit{snd}\ e)$

2 Lemmas

lemma *definition-of-range*:

endpoint-inV V1 ‘ matching-i 1 V E M L =
{ *v*. ∃ *e* ∈ *matching-i 1 V E M L*. *endpoint-inV V1 e* = *v* } ⟨*proof*⟩

lemma *matching-i-edges-as-sets*:
edge-as-set ‘ matching-i i V E M L =
{ *e1*. ∃ (*u*, *v*) ∈ *matching-i i V E M L*. *edge-as-set* (*u*, *v*) = *e1*} ⟨*proof*⟩

lemma *matching-disjointness*:
assumes *matching V E M*
assumes *e1* ∈ *M*
assumes *e2* ∈ *M*
assumes *e1* ≠ *e2*
shows *edge-as-set e1* ∩ *edge-as-set e2* = {}
⟨*proof*⟩

lemma *expand-set-containment*:
assumes *matching V E M*
assumes *e* ∈ *M*
shows *e* ∈ *E*
⟨*proof*⟩

theorem *injectivity*:
assumes *is-osc*: *OSC L E*
assumes *is-m*: *matching V E M*
assumes *e1-in-M1*: *e1* ∈ *matching-i 1 V E M L*
and *e2-in-M1*: *e2* ∈ *matching-i 1 V E M L*
assumes *diff*: (*e1* ≠ *e2*)
shows *endpoint-inV* {*v* ∈ *V*. *L v* = *1*} *e1* ≠ *endpoint-inV* {*v* ∈ *V*. *L v* = *1*} *e2*
⟨*proof*⟩

2.1 |*M1*| ≤ *n1*

lemma *card-M1-le-NVL1*:
assumes *matching V E M*
assumes *OSC L E*
shows *card* (*matching-i 1 V E M L*) ≤ (*N V L 1*)
⟨*proof*⟩

lemma *edge-as-set-inj-on-Mi*:
assumes *matching V E M*
shows *inj-on edge-as-set* (*matching-i i V E M L*)
⟨*proof*⟩

lemma *card-Mi-eq-card-edge-as-set-Mi*:
assumes *matching V E M*
shows *card* (*matching-i i V E M L*) = *card* (*edge-as-set‘ matching-i i V E M L*)
(**is** *card ?Mi* = *card* (*?f* ‘ -))
⟨*proof*⟩

lemma *card-edge-as-set-Mi-twice-card-partitions*:
  assumes $OSC\ L\ E \wedge matching\ V\ E\ M \wedge i > 1$
  shows $2 * card\ (edge\text{-}as\text{-}set`matching\text{-}i\ i\ V\ E\ M\ L)$
  $= card\ (V\text{-}i\ i\ V\ E\ M\ L)$ (**is** $2 * card\ ?C = card\ ?Vi$)
⟨*proof*⟩

lemma *card-Mi-twice-card-Vi*:
  assumes $OSC\ L\ E \wedge matching\ V\ E\ M \wedge i > 1$
  shows $2 * card\ (matching\text{-}i\ i\ V\ E\ M\ L) = card\ (V\text{-}i\ i\ V\ E\ M\ L)$
⟨*proof*⟩

lemma *card-Mi-le-floor-div-2-Vi*:
  assumes $OSC\ L\ E \wedge matching\ V\ E\ M \wedge i > 1$
  shows $card\ (matching\text{-}i\ i\ V\ E\ M\ L) \leq (card\ (V\text{-}i\ i\ V\ E\ M\ L))\ div\ 2$
  ⟨*proof*⟩

lemma *card-Vi-le-NVLi*:
  assumes $i{>}1 \wedge matching\ V\ E\ M$
  shows $card\ (V\text{-}i\ i\ V\ E\ M\ L) \leq N\ V\ L\ i$
  ⟨*proof*⟩

2.2 $|Mi| \leq \lfloor ni/2 \rfloor$

lemma *card-Mi-le-floor-div-2-NVLi*:
  assumes $OSC\ L\ E \wedge matching\ V\ E\ M \wedge i > 1$
  shows $card\ (matching\text{-}i\ i\ V\ E\ M\ L) \leq (N\ V\ L\ i)\ div\ 2$
⟨*proof*⟩

2.3 $|M| \leq \sum |Mi|$

lemma *card-M-le-sum-card-Mi*:
assumes *matching V E M* **and** *OSC L E*
shows $card\ M \leq (\sum\ i \in L`V.\ card\ (matching\text{-}i\ i\ V\ E\ M\ L))$
  (**is** $card\ \text{-} \leq\ ?CardMi$)
⟨*proof*⟩

theorem *card-M-le-weight-NVLi*:
  assumes *matching V E M* **and** *OSC L E*
  shows $card\ M \leq weight\ \{i \in L\ `\ V.\ i > 1\}\ (N\ V\ L)$ (**is** $\text{-} \leq\ ?W$)
⟨*proof*⟩

3 Final Theorem

The following theorem is due to Edmond [2]:

theorem *maximum-cardinality-matching*:
  assumes *matching V E M* **and** *OSC L E*
  and $card\ M = weight\ \{i \in L\ `\ V.\ i > 1\}\ (N\ V\ L)$

and *matching* $V\ E\ M'$
shows *card* $M' \leq$ *card* M
$\langle proof \rangle$

The widely used algorithmic library LEDA has a certifying algorithm for maximum cardinality matching. This Isabelle proof is part of the work done to verify the checker of this certifying algorithm. For more information see [1].

end

References

[1] E. Alkassar, S. Böhme, K. Mehlhorn, and C. Rizkallah. Verification of certifying computations. In *Proceedings of the 23rd International Conference on Computer Aided Verfication (CAV2011)*, Cliff Lodge, Snowbird, Utah, USA, 2011. To Appear.

[2] J. Edmonds. Maximum matching and a polyhedron with 0,1 - vertices. *Journal of Research of the National Bureau of Standards*, 69B:125–130, 1965.