Matroids

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Abstract

This article defines combinatorial structures known as *Independence Systems* and *Matroids* and provides basic concepts and theorems related to them. These structures play an important role in combinatorial optimisation, e. g. greedy algorithms such as Kruskal's algorithm. The development is based on Oxley's 'What is a Matroid?' [1].

Contents

1 Independence systems

theory *Indep-System*
  imports *Main*
begin

lemma *finite-psubset-inc-induct*:
  assumes *finite* A $X \subseteq A$
  assumes $\bigwedge X.\ (\bigwedge Y.\ X \subset Y \Longrightarrow Y \subseteq A \Longrightarrow P\ Y) \Longrightarrow P\ X$
  shows $P\ X$
proof −
  have *wf*: *wf* $\{(X,Y).\ Y \subset X \wedge X \subseteq A\}$
    by (*rule wf-bounded-set*[**where** *ub* = λ-. *A* **and** *f* = *id*]) (*auto simp add*: ‹*finite A*›)
  show *?thesis*
  proof (*induction X rule*: *wf-induct*[*OF wf*, *case-names step*])
    case (*step X*)
    then show *?case* **using** *assms*(*3*)[*of X*] **by** *blast*
  qed
qed

An *independence system* consists of a finite ground set together with an independence predicate over the sets of this ground set. At least one set of the carrier is independent and subsets of independent sets are also independent.

locale *indep-system* =
  fixes *carrier* :: *'a set*
  fixes *indep* :: *'a set* ⇒ *bool*
  assumes *carrier-finite*: *finite carrier*
  assumes *indep-subset-carrier*: *indep* $X \Longrightarrow X \subseteq$ *carrier*
  assumes *indep-ex*: $\exists X.$ *indep* X
  assumes *indep-subset*: *indep* $X \Longrightarrow Y \subseteq X \Longrightarrow$ *indep* Y
begin

lemmas *psubset-inc-induct* [*case-names carrier step*] = *finite-psubset-inc-induct*[*OF carrier-finite*]
lemmas *indep-finite* [*simp*] = *finite-subset*[*OF indep-subset-carrier carrier-finite*]

The empty set is independent.

lemma *indep-empty* [*simp*]: *indep* {}
  using *indep-ex indep-subset* **by** *auto*

1.1 Sub-independence systems

A subset of the ground set induces an independence system.

definition *indep-in* **where** *indep-in* $\mathcal{E}$ $X \longleftrightarrow X \subseteq \mathcal{E} \wedge$ *indep* X

lemma *indep-inI*:
  assumes $X \subseteq \mathcal{E}$

assumes *indep* X
shows *indep-in* $\mathcal{E}$ X
using *assms* **unfolding** *indep-in-def* **by** *auto*

lemma *indep-in-subI*: *indep-in* $\mathcal{E}$ $X \Longrightarrow$ *indep-in* $\mathcal{E}'$ $(X \cap \mathcal{E}')$
using *indep-subset* **unfolding** *indep-in-def* **by** *auto*

lemma *dep-in-subI*:
assumes $X \subseteq \mathcal{E}'$
shows $\neg$ *indep-in* $\mathcal{E}'$ $X \Longrightarrow \neg$ *indep-in* $\mathcal{E}$ X
using *assms* **unfolding** *indep-in-def* **by** *auto*

lemma *indep-in-subset-carrier*: *indep-in* $\mathcal{E}$ $X \Longrightarrow X \subseteq \mathcal{E}$
unfolding *indep-in-def* **by** *auto*

lemma *indep-in-subI-subset*:
assumes $\mathcal{E}' \subseteq \mathcal{E}$
assumes *indep-in* $\mathcal{E}'$ X
shows *indep-in* $\mathcal{E}$ X
proof $-$
have *indep-in* $\mathcal{E}$ $(X \cap \mathcal{E})$ **using** *assms indep-in-subI* **by** *auto*
moreover have $X \cap \mathcal{E} = X$ **using** *assms indep-in-subset-carrier* **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemma *indep-in-supI*:
assumes $X \subseteq \mathcal{E}'$ $\mathcal{E}' \subseteq \mathcal{E}$
assumes *indep-in* $\mathcal{E}$ X
shows *indep-in* $\mathcal{E}'$ X
proof $-$
have $X \cap \mathcal{E}' = X$ **using** *assms* **by** *auto*
then show *?thesis* **using** *assms indep-in-subI*[**where** $\mathcal{E} = \mathcal{E}$ **and** $\mathcal{E}' = \mathcal{E}'$ **and** $X = X$] **by** *auto*
qed

lemma *indep-in-indep*: *indep-in* $\mathcal{E}$ $X \Longrightarrow$ *indep* X
unfolding *indep-in-def* **by** *auto*

lemmas *indep-inD* = *indep-in-subset-carrier indep-in-indep*

lemma *indep-system-subset* [*simp*, *intro*]:
assumes $\mathcal{E} \subseteq$ *carrier*
shows *indep-system* $\mathcal{E}$ (*indep-in* $\mathcal{E}$)
unfolding *indep-system-def indep-in-def*
using *finite-subset*[*OF assms carrier-finite*] *indep-subset* **by** *auto*

We will work a lot with different sub structures. Therefore, every definition 'foo' will have a counterpart 'foo_in' which has the ground set as an additional parameter. Furthermore, every result about 'foo' will have another result about 'foo_in'. With this, we usually don't have to work with **interpretation** in proofs.

context
  fixes $\mathcal{E}$
  assumes $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *indep-in-sub-cong*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$
  shows $\mathcal{E}$.*indep-in* $\mathcal{E}'$ $X \longleftrightarrow$ *indep-in* $\mathcal{E}'$ X
  unfolding $\mathcal{E}$.*indep-in-def* *indep-in-def* **using** *assms* **by** *auto*

lemmas *indep-in-ex* = $\mathcal{E}$.*indep-ex*
lemmas *indep-in-subset* = $\mathcal{E}$.*indep-subset*
lemmas *indep-in-empty* = $\mathcal{E}$.*indep-empty*

end

1.2 Bases

A *basis* is a maximal independent set, i. e. an independent set which becomes dependent on inserting any element of the ground set.

definition *basis* **where** *basis* $X \longleftrightarrow$ *indep* $X \wedge (\forall x \in$ *carrier* $- X.\ \neg$ *indep* (*insert* x X))

lemma *basisI*:
  assumes *indep* X
  assumes $\bigwedge x.\ x \in$ *carrier* $- X \Longrightarrow \neg$ *indep* (*insert* x X)
  shows *basis* X
  using *assms* **unfolding** *basis-def* **by** *auto*

lemma *basis-indep*: *basis* $X \Longrightarrow$ *indep* X
  unfolding *basis-def* **by** *auto*

lemma *basis-max-indep*: *basis* $X \Longrightarrow x \in$ *carrier* $- X \Longrightarrow \neg$ *indep* (*insert* x X)
  unfolding *basis-def* **by** *auto*

lemmas *basisD* = *basis-indep basis-max-indep*
lemmas *basis-subset-carrier* = *indep-subset-carrier*[*OF basis-indep*]
lemmas *basis-finite* [*simp*] = *indep-finite*[*OF basis-indep*]

lemma *indep-not-basis*:
  assumes *indep* X
  assumes $\neg$ *basis* X
  shows $\exists x \in$ *carrier* $- X.$ *indep* (*insert* x X)

using *assms basisI* **by** *auto*

lemma *basis-subset-eq*:
assumes *basis* B_1
assumes *basis* B_2
assumes $B_1 \subseteq B_2$
shows $B_1 = B_2$
proof (*rule ccontr*)
assume $B_1 \neq B_2$
then obtain x **where** x: $x \in B_2 - B_1$ **using** *assms* **by** *auto*
then have *insert* x $B_1 \subseteq B_2$ **using** *assms* **by** *auto*
then have *indep* (*insert* x B_1) **using** *assms basis-indep*[*of* B_2] *indep-subset* **by** *auto*
moreover have $x \in$ *carrier* $- B_1$ **using** *assms x basis-subset-carrier* **by** *auto*
ultimately show *False* **using** *assms basisD* **by** *auto*
qed

definition *basis-in* **where**
basis-in $\mathcal{E}$ $X \longleftrightarrow$ *indep-system.basis* $\mathcal{E}$ (*indep-in* $\mathcal{E}$) X

lemma *basis-iff-basis-in*: *basis* $B \longleftrightarrow$ *basis-in carrier* B
proof −
interpret $\mathcal{E}$: *indep-system carrier indep-in carrier*
by *auto*

show *basis* $B \longleftrightarrow$ *basis-in carrier* B
unfolding *basis-in-def*
proof (*standard*, *goal-cases LTR RTL*)
case *LTR*
show *?case*
proof (*rule* $\mathcal{E}$*.basisI*)
show *indep-in carrier* B **using** *LTR basisD indep-subset-carrier indep-inI* **by** *auto*
next
fix x
assume $x \in$ *carrier* $- B$
then have $\neg$ *indep* (*insert* x B) **using** *LTR basisD* **by** *auto*
then show $\neg$ *indep-in carrier* (*insert* x B) **using** *indep-inD* **by** *auto*
qed
next
case *RTL*
show *?case*
proof (*rule basisI*)
show *indep* B **using** *RTL* $\mathcal{E}$*.basis-indep indep-inD* **by** *blast*
next
fix x
assume $x \in$ *carrier* $- B$
then have $\neg$ *indep-in carrier* (*insert* x B) **using** *RTL* $\mathcal{E}$*.basisD* **by** *auto*
then show $\neg$ *indep* (*insert* x B) **using** *indep-subset-carrier indep-inI* **by** *blast*

    qed
  qed
qed

context
  fixes $\mathcal{E}$
  assumes $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *basis-inI-aux*: $\mathcal{E}$.*basis* $X \Longrightarrow$ *basis-in* $\mathcal{E}$ X
  unfolding *basis-in-def* **by** *auto*

lemma *basis-inD-aux*: *basis-in* $\mathcal{E}$ $X \Longrightarrow$ $\mathcal{E}$.*basis* X
  unfolding *basis-in-def* **by** *auto*

lemma *not-basis-inD-aux*: $\neg$ *basis-in* $\mathcal{E}$ $X \Longrightarrow \neg$ $\mathcal{E}$.*basis* X
  using *basis-inI-aux* **by** *auto*

lemmas *basis-inI* = *basis-inI-aux*[*OF* $\mathcal{E}$.*basisI*]
lemmas *basis-in-indep-in* = $\mathcal{E}$.*basis-indep*[*OF basis-inD-aux*]
lemmas *basis-in-max-indep-in* = $\mathcal{E}$.*basis-max-indep*[*OF basis-inD-aux*]
lemmas *basis-inD* = $\mathcal{E}$.*basisD*[*OF basis-inD-aux*]
lemmas *basis-in-subset-carrier* = $\mathcal{E}$.*basis-subset-carrier*[*OF basis-inD-aux*]
lemmas *basis-in-finite* = $\mathcal{E}$.*basis-finite*[*OF basis-inD-aux*]
lemmas *indep-in-not-basis-in* = $\mathcal{E}$.*indep-not-basis*[*OF - not-basis-inD-aux*]
lemmas *basis-in-subset-eq* = $\mathcal{E}$.*basis-subset-eq*[*OF basis-inD-aux basis-inD-aux*]

end

context
  fixes $\mathcal{E}$
  assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using $*$ **by** *auto*

lemma *basis-in-sub-cong*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$
  shows $\mathcal{E}$.*basis-in* $\mathcal{E}'$ $B \longleftrightarrow$ *basis-in* $\mathcal{E}'$ B
proof (*safe*, *goal-cases LTR RTL*)
  case *LTR*
  show *?case*
  proof (*rule basis-inI*)
    show $\mathcal{E}' \subseteq$ *carrier* **using** *assms* $*$ **by** *auto*
  next

show *indep-in* $\mathcal{E}'$ *B*
using $*$ *assms LTR* $\mathcal{E}$*.basis-in-subset-carrier* $\mathcal{E}$*.basis-in-indep-in indep-in-sub-cong*
by *auto*
next
fix *x*
assume $x \in \mathcal{E}' - B$
then show $\neg$ *indep-in* $\mathcal{E}'$ (*insert x B*)
using $*$ *assms LTR* $\mathcal{E}$*.basis-in-max-indep-in* $\mathcal{E}$*.basis-in-subset-carrier indep-in-sub-cong* **by** *auto*
qed
next
case *RTL*
show *?case*
proof (*rule* $\mathcal{E}$*.basis-inI*)
show $\mathcal{E}' \subseteq \mathcal{E}$ **using** *assms* **by** *auto*
next
show $\mathcal{E}$*.indep-in* $\mathcal{E}'$ *B*
using $*$ *assms RTL basis-in-subset-carrier basis-in-indep-in indep-in-sub-cong*
by *auto*
next
fix *x*
assume $x \in \mathcal{E}' - B$
then show $\neg$ $\mathcal{E}$*.indep-in* $\mathcal{E}'$ (*insert x B*)
using $*$ *assms RTL basis-in-max-indep-in basis-in-subset-carrier indep-in-sub-cong*
by *auto*
qed
qed

end

1.3 Circuits

A *circuit* is a minimal dependent set, i. e. a set which becomes independent on removing any element of the ground set.

definition *circuit* **where** *circuit X* $\longleftrightarrow$ $X \subseteq$ *carrier* $\wedge$ $\neg$ *indep X* $\wedge$ ($\forall x \in X$. *indep* ($X - \{x\}$))

lemma *circuitI*:
assumes $X \subseteq$ *carrier*
assumes $\neg$ *indep X*
assumes $\bigwedge x.\ x \in X \Longrightarrow$ *indep* ($X - \{x\}$)
shows *circuit X*
using *assms* **unfolding** *circuit-def* **by** *auto*

lemma *circuit-subset-carrier*: *circuit X* $\Longrightarrow X \subseteq$ *carrier*
unfolding *circuit-def* **by** *auto*
lemmas *circuit-finite* [*simp*] = *finite-subset*[*OF circuit-subset-carrier carrier-finite*]

lemma *circuit-dep*: *circuit X* $\Longrightarrow \neg$ *indep X*

unfolding *circuit-def* **by** *auto*

lemma *circuit-min-dep*: *circuit* $X \Longrightarrow x \in X \Longrightarrow$ *indep* $(X - \{x\})$
unfolding *circuit-def* **by** *auto*

lemmas *circuitD* = *circuit-subset-carrier circuit-dep circuit-min-dep*

lemma *circuit-nonempty*: *circuit* $X \Longrightarrow X \neq \{\}$
using *circuit-dep indep-empty* **by** *blast*

lemma *dep-not-circuit*:
assumes $X \subseteq$ *carrier*
assumes $\neg$ *indep* X
assumes $\neg$ *circuit* X
shows $\exists x \in X. \neg$ *indep* $(X - \{x\})$
using *assms circuitI* **by** *auto*

lemma *circuit-subset-eq*:
assumes *circuit* C_1
assumes *circuit* C_2
assumes $C_1 \subseteq C_2$
shows $C_1 = C_2$
proof (*rule ccontr*)
assume $C_1 \neq C_2$
then obtain x **where** $x \notin C_1$ $x \in C_2$ **using** *assms* **by** *auto*
then have *indep* C_1 **using** *indep-subset* ‹$C_1 \subseteq C_2$› *circuit-min-dep*[*OF* ‹*circuit* C_2›, *of* x] **by** *auto*
then show *False* **using** *assms circuitD* **by** *auto*
qed

definition *circuit-in* **where**
circuit-in $\mathcal{E}$ $X \longleftrightarrow$ *indep-system.circuit* $\mathcal{E}$ (*indep-in* $\mathcal{E}$) X

context
fixes $\mathcal{E}$
assumes $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *circuit-inI-aux*: $\mathcal{E}$.*circuit* $X \Longrightarrow$ *circuit-in* $\mathcal{E}$ X
unfolding *circuit-in-def* **by** *auto*

lemma *circuit-inD-aux*: *circuit-in* $\mathcal{E}$ $X \Longrightarrow \mathcal{E}$.*circuit* X
unfolding *circuit-in-def* **by** *auto*

lemma *not-circuit-inD-aux*: $\neg$ *circuit-in* $\mathcal{E}$ $X \Longrightarrow \neg$ $\mathcal{E}$.*circuit* X
using *circuit-inI-aux* **by** *auto*

lemmas *circuit-inI* = *circuit-inI-aux*[*OF* $\mathcal{E}$.*circuitI*]

lemmas *circuit-in-subset-carrier* = $\mathcal{E}$.*circuit-subset-carrier*[*OF circuit-inD-aux*]
lemmas *circuit-in-finite* = $\mathcal{E}$.*circuit-finite*[*OF circuit-inD-aux*]
lemmas *circuit-in-dep-in* = $\mathcal{E}$.*circuit-dep*[*OF circuit-inD-aux*]
lemmas *circuit-in-min-dep-in* = $\mathcal{E}$.*circuit-min-dep*[*OF circuit-inD-aux*]
lemmas *circuit-inD* = $\mathcal{E}$.*circuitD*[*OF circuit-inD-aux*]
lemmas *circuit-in-nonempty* = $\mathcal{E}$.*circuit-nonempty*[*OF circuit-inD-aux*]
lemmas *dep-in-not-circuit-in* = $\mathcal{E}$.*dep-not-circuit*[*OF - - not-circuit-inD-aux*]
lemmas *circuit-in-subset-eq* = $\mathcal{E}$.*circuit-subset-eq*[*OF circuit-inD-aux circuit-inD-aux*]

end

lemma *circuit-in-subI*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$ $\mathcal{E} \subseteq$ *carrier*
  assumes *circuit-in* $\mathcal{E}'$ C
  shows *circuit-in* $\mathcal{E}$ C
proof (*rule circuit-inI*)
  show $\mathcal{E} \subseteq$ *carrier* **using** *assms* **by** *auto*
next
  show $C \subseteq \mathcal{E}$ **using** *assms circuit-in-subset-carrier*[*of* $\mathcal{E}'$ C] **by** *auto*
next
  show $\neg$ *indep-in* $\mathcal{E}$ C
    using *assms*
      circuit-in-dep-in[**where** $\mathcal{E} = \mathcal{E}'$ **and** $X = C$]
      circuit-in-subset-carrier dep-in-subI[**where** $\mathcal{E}' = \mathcal{E}'$ **and** $\mathcal{E} = \mathcal{E}$]
    by *auto*
next
  fix x
  assume $x \in C$
  then show *indep-in* $\mathcal{E}$ $(C - \{x\})$
    using *assms circuit-in-min-dep-in indep-in-subI-subset* **by** *auto*
qed

lemma *circuit-in-supI*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$ $\mathcal{E} \subseteq$ *carrier* $C \subseteq \mathcal{E}'$
  assumes *circuit-in* $\mathcal{E}$ C
  shows *circuit-in* $\mathcal{E}'$ C
proof (*rule circuit-inI*)
  show $\mathcal{E}' \subseteq$ *carrier* **using** *assms* **by** *auto*
next
  show $C \subseteq \mathcal{E}'$ **using** *assms* **by** *auto*
next
  have $\neg$ *indep-in* $\mathcal{E}$ C **using** *assms circuit-in-dep-in* **by** *auto*
  then show $\neg$ *indep-in* $\mathcal{E}'$ C **using** *assms dep-in-subI*[*of* C $\mathcal{E}$] **by** *auto*
next
  fix x
  assume $x \in C$

then have *indep-in* $\mathcal{E}$ $(C - \{x\})$ **using** *assms circuit-in-min-dep-in* **by** *auto*
then have *indep-in* $\mathcal{E}'$ $((C - \{x\}) \cap \mathcal{E}')$ **using** *indep-in-subI* **by** *auto*
moreover have $(C - \{x\}) \cap \mathcal{E}' = C - \{x\}$ **using** *assms circuit-in-subset-carrier* **by** *auto*
ultimately show *indep-in* $\mathcal{E}'$ $(C - \{x\})$ **by** *auto*
qed

context
fixes $\mathcal{E}$
assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
using $*$ **by** *auto*

lemma *circuit-in-sub-cong*:
assumes $\mathcal{E}' \subseteq \mathcal{E}$
shows $\mathcal{E}$*.circuit-in* $\mathcal{E}'$ $C \longleftrightarrow$ *circuit-in* $\mathcal{E}'$ C
proof (*safe*, *goal-cases LTR RTL*)
case *LTR*
show *?case*
proof (*rule circuit-inI*)
show $\mathcal{E}' \subseteq$ *carrier* **using** *assms* $*$ **by** *auto*
next
show $C \subseteq \mathcal{E}'$
using *assms LTR* $\mathcal{E}$*.circuit-in-subset-carrier* **by** *auto*
next
show $\neg$ *indep-in* $\mathcal{E}'$ C
using *assms LTR* $\mathcal{E}$*.circuit-in-dep-in indep-in-sub-cong*[*OF* $*$] **by** *auto*
next
fix x
assume $x \in C$
then show *indep-in* $\mathcal{E}'$ $(C - \{x\})$
using *assms LTR* $\mathcal{E}$*.circuit-in-min-dep-in indep-in-sub-cong*[*OF* $*$] **by** *auto*
qed
next
case *RTL*
show *?case*
proof (*rule* $\mathcal{E}$*.circuit-inI*)
show $\mathcal{E}' \subseteq \mathcal{E}$ **using** *assms* $*$ **by** *auto*
next
show $C \subseteq \mathcal{E}'$
using *assms* $*$ *RTL circuit-in-subset-carrier* **by** *auto*
next
show $\neg$ $\mathcal{E}$*.indep-in* $\mathcal{E}'$ C
using *assms* $*$ *RTL circuit-in-dep-in indep-in-sub-cong*[*OF* $*$] **by** *auto*
next
fix x
assume $x \in C$

then show $\mathcal{E}$.*indep-in* $\mathcal{E}'$ $(C - \{x\})$
using *assms* $*$ *RTL circuit-in-min-dep-in indep-in-sub-cong*[*OF* $*$] by *auto*
qed
qed

end

lemma *circuit-imp-circuit-in*:
assumes *circuit C*
shows *circuit-in carrier C*
proof (*rule circuit-inI*)
show $C \subseteq$ *carrier* using *circuit-subset-carrier*[*OF assms*] .
next
show $\neg$ *indep-in carrier C* using *circuit-dep*[*OF assms*] *indep-in-indep* by *auto*
next
fix *x*
assume $x \in C$
then have *indep* $(C - \{x\})$ using *circuit-min-dep*[*OF assms*] by *auto*
then show *indep-in carrier* $(C - \{x\})$ using *circuit-subset-carrier*[*OF assms*]
by (*auto intro*: *indep-inI*)
qed *auto*

1.4 Relation between independence and bases

A set is independent iff it is a subset of a basis.

lemma *indep-imp-subset-basis*:
assumes *indep X*
shows $\exists B.$ *basis* $B \wedge X \subseteq B$
using *assms*
proof (*induction X rule*: *psubset-inc-induct*)
case *carrier*
show *?case* using *indep-subset-carrier*[*OF assms*] .
next
case (*step X*)
{
assume $\neg$ *basis X*
then obtain *x* where $x \in$ *carrier* $x \notin X$ *indep* (*insert x X*)
using *step.prems indep-not-basis* by *auto*
then have *?case* using *step.IH*[*of insert x X*] *indep-subset-carrier* by *auto*
}
then show *?case* by *auto*
qed

lemmas *subset-basis-imp-indep* = *indep-subset*[*OF basis-indep*]

lemma *indep-iff-subset-basis*: *indep* $X \longleftrightarrow (\exists B.$ *basis* $B \wedge X \subseteq B)$
using *indep-imp-subset-basis subset-basis-imp-indep* by *auto*

lemma *basis-ex*: $\exists B.$ *basis B* using *indep-imp-subset-basis*[*OF indep-empty*] **by** *auto*

context
  fixes $\mathcal{E}$
  assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using $*$ **by** *auto*

lemma *indep-in-imp-subset-basis-in*:
  assumes *indep-in* $\mathcal{E}$ X
  shows $\exists B.$ *basis-in* $\mathcal{E}$ $B \wedge X \subseteq B$
  unfolding *basis-in-def* **using** $\mathcal{E}$.*indep-imp-subset-basis*[*OF assms*] **.**

lemmas *subset-basis-in-imp-indep-in* = *indep-in-subset*[*OF* $*$ *basis-in-indep-in*[*OF* $*$]]

lemma *indep-in-iff-subset-basis-in*: *indep-in* $\mathcal{E}$ $X \longleftrightarrow (\exists B.$ *basis-in* $\mathcal{E}$ $B \wedge X \subseteq B)$
  using *indep-in-imp-subset-basis-in subset-basis-in-imp-indep-in* **by** *auto*

lemma *basis-in-ex*: $\exists B.$ *basis-in* $\mathcal{E}$ B
  unfolding *basis-in-def* **using** $\mathcal{E}$.*basis-ex* **.**

lemma *basis-in-subI*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$ $\mathcal{E} \subseteq$ *carrier*
  assumes *basis-in* $\mathcal{E}'$ B
  shows $\exists B' \subseteq \mathcal{E} - \mathcal{E}'.$ *basis-in* $\mathcal{E}$ $(B \cup B')$
proof $-$
  have *indep-in* $\mathcal{E}$ B **using** *assms basis-in-indep-in indep-in-subI-subset* **by** *auto*
  then obtain B' **where** B': *basis-in* $\mathcal{E}$ B' $B \subseteq B'$
    using *assms indep-in-imp-subset-basis-in*[*of* B] **by** *auto*
  show *?thesis*
  proof (*rule exI*)
    have $B' - B \subseteq \mathcal{E} - \mathcal{E}'$
    proof
      fix x
      assume $*$: $x \in B' - B$
      then have $x \in \mathcal{E}$ $x \notin B$
        using *assms* ‹*basis-in* $\mathcal{E}$ B'› *basis-in-subset-carrier*[*of* $\mathcal{E}$] **by** *auto*
      moreover {
        assume $x \in \mathcal{E}'$
        moreover have *indep-in* $\mathcal{E}$ (*insert* x B)
          using $*$ *assms indep-in-subset*[*OF - basis-in-indep-in*] B' **by** *auto*
        ultimately have *indep-in* $\mathcal{E}'$ (*insert* x B)
          using *assms basis-in-subset-carrier* **unfolding** *indep-in-def* **by** *auto*
        then have *False* **using** *assms* $*$ ‹$x \in \mathcal{E}'$› *basis-in-max-indep-in* **by** *auto*
      }

    ultimately show $x \in \mathcal{E} - \mathcal{E}'$ **by** *auto*
   qed
   moreover have $B \cup (B' - B) = B'$ **using** ‹$B \subseteq B'$› **by** *auto*
   ultimately show $B' - B \subseteq \mathcal{E} - \mathcal{E}' \wedge$ *basis-in* $\mathcal{E}$ $(B \cup (B' - B))$
    using ‹*basis-in* $\mathcal{E}$ B'› **by** *auto*
  qed
qed

lemma *basis-in-supI*:
  assumes $B \subseteq \mathcal{E}'$ $\mathcal{E}' \subseteq \mathcal{E}$ $\mathcal{E} \subseteq$ *carrier*
  assumes *basis-in* $\mathcal{E}$ B
  shows *basis-in* $\mathcal{E}'$ B
proof (*rule basis-inI*)
  show $\mathcal{E}' \subseteq$ *carrier* **using** *assms* **by** *auto*
next
  show *indep-in* $\mathcal{E}'$ B
  proof $-$
   have *indep-in* $\mathcal{E}'$ $(B \cap \mathcal{E}')$
    using *assms basis-in-indep-in*[*of* $\mathcal{E}$ B] *indep-in-subI* **by** *auto*
   moreover have $B \cap \mathcal{E}' = B$ **using** *assms* **by** *auto*
   ultimately show *?thesis* **by** *auto*
  qed
next
  show $\bigwedge x.\ x \in \mathcal{E}' - B \Longrightarrow \neg$ *indep-in* $\mathcal{E}'$ (*insert* x B)
   using *assms basis-in-subset-carrier basis-in-max-indep-in dep-in-subI*[*of* - $\mathcal{E}$ $\mathcal{E}'$]
by *auto*
qed

end

1.5 Relation between dependence and circuits

A set is dependent iff it contains a circuit.

lemma *dep-imp-supset-circuit*:
  assumes $X \subseteq$ *carrier*
  assumes $\neg$ *indep* X
  shows $\exists C.$ *circuit* $C \wedge C \subseteq X$
  using *assms*
proof (*induction* X *rule*: *remove-induct*)
  case (*remove* X)
  {
   assume $\neg$ *circuit* X
   then obtain x **where** $x \in X$ $\neg$ *indep* $(X - \{x\})$
    using *remove.prems dep-not-circuit* **by** *auto*
   then obtain C **where** *circuit* C $C \subseteq X - \{x\}$
    using *remove.prems remove.IH*[*of* x] **by** *auto*
   then have *?case* **by** *auto*
  }
  then show *?case* **using** *remove.prems* **by** *auto*

qed (*auto simp add*: *carrier-finite finite-subset*)

lemma *supset-circuit-imp-dep*:
  assumes *circuit* $C \wedge C \subseteq X$
  shows $\neg$ *indep* X
  using *assms indep-subset circuit-dep* **by** *auto*

lemma *dep-iff-supset-circuit*:
  assumes $X \subseteq$ *carrier*
  shows $\neg$ *indep* $X \longleftrightarrow (\exists C.$ *circuit* $C \wedge C \subseteq X)$
  using *assms dep-imp-supset-circuit supset-circuit-imp-dep* **by** *auto*

context
  fixes $\mathcal{E}$
  assumes $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *dep-in-imp-supset-circuit-in*:
  assumes $X \subseteq \mathcal{E}$
  assumes $\neg$ *indep-in* $\mathcal{E}$ X
  shows $\exists C.$ *circuit-in* $\mathcal{E}$ $C \wedge C \subseteq X$
  unfolding *circuit-in-def* **using** $\mathcal{E}$.*dep-imp-supset-circuit*[*OF assms*] **.**

lemma *supset-circuit-in-imp-dep-in*:
  assumes *circuit-in* $\mathcal{E}$ $C \wedge C \subseteq X$
  shows $\neg$ *indep-in* $\mathcal{E}$ X
  using *assms* $\mathcal{E}$.*supset-circuit-imp-dep* **unfolding** *circuit-in-def* **by** *auto*

lemma *dep-in-iff-supset-circuit-in*:
  assumes $X \subseteq \mathcal{E}$
  shows $\neg$ *indep-in* $\mathcal{E}$ $X \longleftrightarrow (\exists C.$ *circuit-in* $\mathcal{E}$ $C \wedge C \subseteq X)$
  using *assms dep-in-imp-supset-circuit-in supset-circuit-in-imp-dep-in* **by** *auto*

end

1.6 Ranks

definition *lower-rank-of* :: *'a set* $\Rightarrow$ *nat* **where**
  lower-rank-of carrier' $\equiv$ *Min* {*card* B | B. *basis-in carrier'* B}

definition *upper-rank-of* :: *'a set* $\Rightarrow$ *nat* **where**
  upper-rank-of carrier' $\equiv$ *Max* {*card* B | B. *basis-in carrier'* B}

lemma *collect-basis-finite*: *finite* (*Collect basis*)
proof −
  have *Collect basis* $\subseteq \{X.\ X \subseteq$ *carrier*$\}$

using *basis-subset-carrier* **by** *auto*
moreover have *finite* ...
using *carrier-finite* **by** *auto*
ultimately show *?thesis* **using** *finite-subset* **by** *auto*
qed

context
fixes $\mathcal{E}$
assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *indep-system* $\mathcal{E}$ *indep-in* $\mathcal{E}$
using $*$ **by** *auto*

lemma *collect-basis-in-finite*: *finite* (*Collect* (*basis-in* $\mathcal{E}$))
unfolding *basis-in-def* **using** $\mathcal{E}$.*collect-basis-finite* **.**

lemma *lower-rank-of-le*: *lower-rank-of* $\mathcal{E} \leq$ *card* $\mathcal{E}$
proof –
have $\exists\, n \in \{card\ B \mid B.\ basis\text{-}in\ \mathcal{E}\ B\}.\ n \leq card\ \mathcal{E}$
using *card-mono*[*OF* $\mathcal{E}$.*carrier-finite* *basis-in-subset-carrier*[*OF* $*$]] *basis-in-ex*[*OF* $*$] **by** *auto*
moreover have *finite* $\{card\ B \mid B.\ basis\text{-}in\ \mathcal{E}\ B\}$
using *collect-basis-in-finite* **by** *auto*
ultimately show *?thesis*
unfolding *lower-rank-of-def* **using** *basis-ex* *Min-le-iff* **by** *auto*
qed

lemma *upper-rank-of-le*: *upper-rank-of* $\mathcal{E} \leq$ *card* $\mathcal{E}$
proof –
have $\forall\, n \in \{card\ B \mid B.\ basis\text{-}in\ \mathcal{E}\ B\}.\ n \leq card\ \mathcal{E}$
using *card-mono*[*OF* $\mathcal{E}$.*carrier-finite* *basis-in-subset-carrier*[*OF* $*$]] **by** *auto*
then show *?thesis*
unfolding *upper-rank-of-def* **using** *basis-in-ex*[*OF* $*$] *collect-basis-in-finite* **by** *auto*
qed

context
fixes $\mathcal{E}'$
assumes $**$: $\mathcal{E}' \subseteq \mathcal{E}$
begin

interpretation $\mathcal{E}'_1$: *indep-system* $\mathcal{E}'$ *indep-in* $\mathcal{E}'$
using $*$ $**$ **by** *auto*
interpretation $\mathcal{E}'_2$: *indep-system* $\mathcal{E}'$ $\mathcal{E}$.*indep-in* $\mathcal{E}'$
using $*$ $**$ **by** *auto*

lemma *lower-rank-of-sub-cong*:
shows $\mathcal{E}$.*lower-rank-of* $\mathcal{E}'$ = *lower-rank-of* $\mathcal{E}'$

proof −
  have $\bigwedge B.\ \mathcal{E}'_1.basis\ B \longleftrightarrow \mathcal{E}'_2.basis\ B$
    using $**$ *basis-in-sub-cong*$[OF\ *,\ of\ \mathcal{E}']$
    unfolding *basis-in-def* $\mathcal{E}$.*basis-in-def* **by** *auto*
  then show *?thesis*
    unfolding *lower-rank-of-def* $\mathcal{E}$.*lower-rank-of-def*
    using *basis-in-sub-cong*$[OF\ *\ **]$
    by *auto*
qed

lemma *upper-rank-of-sub-cong*:
  shows $\mathcal{E}$.*upper-rank-of* $\mathcal{E}'$ = *upper-rank-of* $\mathcal{E}'$
proof −
  have $\bigwedge B.\ \mathcal{E}'_1.basis\ B \longleftrightarrow \mathcal{E}'_2.basis\ B$
    using $**$ *basis-in-sub-cong*$[OF\ *,\ of\ \mathcal{E}']$
    unfolding *basis-in-def* $\mathcal{E}$.*basis-in-def* **by** *auto*
  then show *?thesis*
    unfolding *upper-rank-of-def* $\mathcal{E}$.*upper-rank-of-def*
    using *basis-in-sub-cong*$[OF\ *\ **]$
    by *auto*
qed

end

end

end

end

2 Matroids

theory *Matroid*
  imports *Indep-System*
begin

lemma *card-subset-ex*:
  assumes *finite A* $n \leq$ *card A*
  shows $\exists B \subseteq A.$ *card B* = *n*
using *assms*
proof (*induction A arbitrary*: *n rule*: *finite-induct*)
  case (*insert x A*)
  show *?case*
  proof (*cases n*)
    case *0*
    then show *?thesis* **using** *card.empty* **by** *blast*
  next
    case (*Suc k*)
    then have $\exists B \subseteq A.$ *card B* = *k* **using** *insert* **by** *auto*

then obtain B **where** $B \subseteq A$ *card* $B = k$ **by** *auto*
moreover from *this* **have** *finite* B **using** *insert.hyps finite-subset* **by** *auto*
ultimately have *card* (*insert* x B) $= n$
using *Suc insert.hyps card-insert-disjoint* **by** *fastforce*
then show *?thesis* **using** ‹$B \subseteq A$› **by** *blast*
qed
qed *auto*

locale *matroid* = *indep-system* +
assumes *augment-aux*:
indep $X \Longrightarrow$ *indep* $Y \Longrightarrow$ *card* X = *Suc* (*card* Y) $\Longrightarrow \exists x \in X - Y$. *indep* (*insert* x Y)
begin

lemma *augment*:
assumes *indep* X *indep* Y *card* $Y <$ *card* X
shows $\exists x \in X - Y$. *indep* (*insert* x Y)
proof −
obtain X' **where** $X' \subseteq X$ *card* X' = *Suc* (*card* Y)
using *assms card-subset-ex*[*of* X *Suc* (*card* Y)] *indep-finite* **by** *auto*
then obtain x **where** $x \in X' - Y$ *indep* (*insert* x Y)
using *assms augment-aux*[*of* X' Y] *indep-subset* **by** *auto*
then show *?thesis* **using** ‹$X' \subseteq X$› **by** *auto*
qed

lemma *augment-psubset*:
assumes *indep* X *indep* Y $Y \subset X$
shows $\exists x \in X - Y$. *indep* (*insert* x Y)
using *assms augment psubset-card-mono indep-finite* **by** *blast*

2.1 Minors

A subset of the ground set induces a matroid.

lemma *matroid-subset* [*simp*, *intro*]:
assumes $\mathcal{E} \subseteq$ *carrier*
shows *matroid* $\mathcal{E}$ (*indep-in* $\mathcal{E}$)
unfolding *matroid-def matroid-axioms-def*
proof (*safe*, *goal-cases indep-system augment*)
case *indep-system*
then show *?case* **using** *indep-system-subset*[*OF assms*] **.**
next
case (*augment* X Y)
then show *?case* **using** *augment-aux*[*of* X Y] **unfolding** *indep-in-def* **by** *auto*
qed

context
fixes $\mathcal{E}$
assumes $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemmas *augment-aux-indep-in* = $\mathcal{E}$.*augment-aux*
lemmas *augment-indep-in* = $\mathcal{E}$.*augment*
lemmas *augment-psubset-indep-in* = $\mathcal{E}$.*augment-psubset*

end

2.2 Bases

lemma *basis-card*:
  assumes *basis* B_1
  assumes *basis* B_2
  shows *card* B_1 = *card* B_2
proof (*rule ccontr*, *goal-cases False*)
  case *False*
  then have *card* B_1 < *card* B_2 ∨ *card* B_2 < *card* B_1 **by** *auto*
  moreover {
    fix B_1 B_2
    assume *basis* B_1 *basis* B_2 *card* B_1 < *card* B_2
    then obtain x **where** $x \in B_2 - B_1$ *indep* (*insert* x B_1)
      using *augment basisD* **by** *blast*
    then have $x \in$ *carrier* $- B_1$
      using ‹*basis* B_1› *basisD indep-subset-carrier* **by** *blast*
    then have ¬ *indep* (*insert* x B_1) **using** ‹*basis* B_1› *basisD* **by** *auto*
    then have *False* **using** ‹*indep* (*insert* x B_1)› **by** *auto*
  }
  ultimately show *?case* **using** *assms* **by** *auto*
qed

lemma *basis-indep-card*:
  assumes *indep* X
  assumes *basis* B
  shows *card* $X \leq$ *card* B
proof −
  obtain B' **where** *basis* B' $X \subseteq B'$ **using** *assms indep-imp-subset-basis* **by** *auto*
  then show *?thesis* **using** *assms basis-finite basis-card*[*of* B B'] **by** (*auto intro*: *card-mono*)
qed

lemma *basis-augment*:
  assumes *basis* B_1 *basis* B_2 $x \in B_1 - B_2$
  shows $\exists\, y \in B_2 - B_1$. *basis* (*insert* y ($B_1 - \{x\}$))
proof −
  let *?*B_1 = $B_1 - \{x\}$
  have *card ?*B_1 < *card* B_2
    using *assms basis-card*[*of* B_1 B_2] *card-Diff1-less*[*OF basis-finite*, *of* B_1] **by** *auto*

moreover have *indep* $?B_1$ **using** *assms basis-indep*[*of* B_1] *indep-subset*[*of* B_1 $?B_1$] **by** *auto*
ultimately obtain *y* **where** *y*: $y \in B_2 - ?B_1$ *indep* (*insert y* $?B_1$)
using *assms augment*[*of* B_2 $?B_1$] *basis-indep* **by** *auto*
let $?B_1{}' =$ *insert y* $?B_1$
have *basis* $?B_1{}'$ **using** ‹*indep* $?B_1{}'$›
proof (*rule basisI*, *goal-cases insert*)
case (*insert x*)
have *card* (*insert x* $?B_1{}'$) $>$ *card* B_1
proof $-$
have *card* (*insert x* $?B_1{}'$) $=$ *Suc* (*card* $?B_1{}'$)
using *insert card.insert-remove*[*OF indep-finite*, *of* $?B_1{}'$] *y* **by** *auto*
also have $\ldots =$ *Suc* (*Suc* (*card* $?B_1$))
using *card.insert-remove*[*OF indep-finite*, *of* $?B_1$] ‹*indep* $?B_1$› *y* **by** *auto*
also have $\ldots =$ *Suc* (*card* B_1)
using *assms basis-finite*[*of* B_1] *card.remove*[*of* B_1] **by** *auto*
finally show *?thesis* **by** *auto*
qed
then have $\neg$*indep* (*insert x* (*insert y* $?B_1$))
using *assms basis-indep-card*[*of insert x* (*insert y* $?B_1$) B_1] **by** *auto*
moreover have *insert x* (*insert y* $?B_1$) $\subseteq$ *carrier*
using *assms insert y basis-finite indep-subset-carrier* **by** *auto*
ultimately show *?case* **by** *auto*
qed
then show *?thesis* **using** *assms y* **by** *auto*
qed

context
fixes $\mathcal{E}$
assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemmas *basis-in-card* $=$ $\mathcal{E}$.*basis-card*[*OF basis-inD-aux*[*OF* $*$] *basis-inD-aux*[*OF* $*$]]
lemmas *basis-in-indep-in-card* $=$ $\mathcal{E}$.*basis-indep-card*[*OF - basis-inD-aux*[*OF* $*$]]

lemma *basis-in-augment*:
assumes *basis-in* $\mathcal{E}$ B_1 *basis-in* $\mathcal{E}$ B_2 $x \in B_1 - B_2$
shows $\exists\, y \in B_2 - B_1$. *basis-in* $\mathcal{E}$ (*insert y* ($B_1 - \{x\}$))
using *assms* $\mathcal{E}$.*basis-augment* **unfolding** *basis-in-def* **by** *auto*

end

2.3 Circuits

lemma *circuit-elim*:

assumes *circuit* C_1 *circuit* C_2 $C_1 \neq C_2$ $x \in C_1 \cap C_2$
shows $\exists\, C_3 \subseteq (C_1 \cup C_2) - \{x\}$. *circuit* C_3
proof $-$
let *?C* $= (C_1 \cup C_2) - \{x\}$
let *?carrier* $= C_1 \cup C_2$

have *assms'*: *circuit-in carrier* C_1 *circuit-in carrier* C_2
using *assms circuit-imp-circuit-in* **by** *auto*

have *?C* $\subseteq$ *carrier* **using** *assms circuit-subset-carrier* **by** *auto*
show *?thesis*
proof (*cases indep ?C*)
case *False*
then show *?thesis* **using** *dep-iff-supset-circuit* ‹*?C* $\subseteq$ *carrier*› **by** *auto*
next
case *True*
then have *indep-in ?carrier ?C* **using** ‹*?C* $\subseteq$ *carrier*› **by** (*auto intro*: *indep-inI*)

have $*$: *?carrier* $\subseteq$ *carrier* **using** *assms circuit-subset-carrier* **by** *auto*
obtain y **where** y: $y \in C_2$ $y \notin C_1$ **using** *assms circuit-subset-eq* **by** *blast*
then have *indep-in ?carrier* $(C_2 - \{y\})$
using *assms'* *circuit-in-min-dep-in*[*OF* $*$ *circuit-in-supI*[*OF* $*$, *of* C_2]] **by** *auto*
then obtain B **where** B: *basis-in ?carrier B* $C_2 - \{y\} \subseteq B$
using $*$ *assms indep-in-imp-subset-basis-in*[*of ?carrier* $C_2 - \{y\}$] **by** *auto*

have $y \notin B$
proof (*rule ccontr*, *goal-cases False*)
case *False*
then have $C_2 \subseteq B$ **using** B **by** *auto*
moreover have *circuit-in ?carrier* C_2 **using** $*$ *assms'* *circuit-in-supI* **by** *auto*
ultimately have $\neg$ *indep-in ?carrier B*
using B *basis-in-subset-carrier*[*OF* $*$] *supset-circuit-in-imp-dep-in*[*OF* $*$] **by** *auto*
then show *False* **using** *assms B basis-in-indep-in*[*OF* $*$] **by** *auto*
qed

have $C_1 - B \neq \{\}$
proof (*rule ccontr*, *goal-cases False*)
case *False*
then have $C_1 - (C_1 \cap B) = \{\}$ **by** *auto*
then have $C_1 = C_1 \cap B$ **using** *assms circuit-subset-eq* **by** *auto*
moreover have *indep* $(C_1 \cap B)$
using *assms B basis-in-indep-in*[*OF* $*$] *indep-in-subset*[*OF* $*$, *of B* $C_1 \cap B$] *indep-in-indep*
by *auto*
ultimately show *?case* **using** *assms circuitD* **by** *auto*
qed
then obtain z **where** z: $z \in C_1$ $z \notin B$ **by** *auto*

have $y \neq z$ **using** y z **by** *auto*
have $x \in C_1$ $x \in C_2$ **using** *assms* **by** *auto*

have *finite ?carrier* **using** *assms carrier-finite finite-subset* **by** *auto*
have *card* $B \leq$ *card* (*?carrier* $-$ $\{y, z\}$)
proof (*rule card-mono*)
show *finite* ($C_1 \cup C_2 - \{y, z\}$) **using** ‹*finite ?carrier*› **by** *auto*
next
show $B \subseteq C_1 \cup C_2 - \{y, z\}$
using B *basis-in-subset-carrier*[*OF* $*$, *of* B] ‹$y \notin B$› ‹$z \notin B$› **by** *auto*
qed
also have ... = *card ?carrier* $-$ *2*
using ‹*finite ?carrier*› ‹$y \in C_2$› ‹$z \in C_1$› ‹$y \neq z$› *card-Diff-subset-Int* **by** *auto*
also have ... < *card ?carrier* $-$ *1*
proof $-$
have *card ?carrier* = *card* C_1 + *card* C_2 $-$ *card* ($C_1 \cap C_2$)
using *assms* ‹*finite ?carrier*› *card-Un-Int*[*of* C_1 C_2] **by** *auto*
also have ... = *card* C_1 + (*card* C_2 $-$ *card* ($C_1 \cap C_2$))
using *assms* ‹*finite ?carrier*› *card-mono*[*of* C_2] **by** *auto*
also have ... = *card* C_1 + *card* ($C_2 - C_1$)
proof $-$
have *card* ($C_2 - C_1$) = *card* C_2 $-$ *card* ($C_2 \cap C_1$)
using *assms* ‹*finite ?carrier*› *card-Diff-subset-Int*[*of* C_2 C_1] **by** *auto*
also have ... = *card* C_2 $-$ *card* ($C_1 \cap C_2$) **by** (*simp add*: *inf-commute*)
finally show *?thesis* **by** *auto*
qed
finally have *card* ($C_1 \cup C_2$) = *card* C_1 + *card* ($C_2 - C_1$) **.**
moreover have *card* C_1 > *0* **using** *assms circuit-nonempty* ‹*finite ?carrier*› **by** *auto*
moreover have *card* ($C_2 - C_1$) > *0* **using** *assms* ‹*finite ?carrier*› ‹$y \in C_2$› ‹$y \notin C_1$› **by** *auto*
ultimately show *?thesis* **by** *auto*
qed
also have ... = *card ?C*
using ‹*finite ?carrier*› *card-Diff-singleton* ‹$x \in C_1$› ‹$x \in C_2$› **by** *auto*
finally have *card* B < *card ?C* **.**
then have *False*
using *basis-in-indep-in-card*[*OF* $*$, *of ?C B*] B ‹*indep-in ?carrier ?C*› **by** *auto*
then show *?thesis* **by** *auto*
qed
qed

lemma *min-dep-imp-supset-circuit*:
assumes *indep* X
assumes *circuit* C
assumes $C \subseteq$ *insert* x X
shows $x \in C$

proof (*rule ccontr*)
  assume $x \notin C$
  then have $C \subseteq X$ **using** *assms* **by** *auto*
  then have *indep C* **using** *assms indep-subset* **by** *auto*
  then show *False* **using** *assms circuitD* **by** *auto*
qed

lemma *min-dep-imp-ex1-supset-circuit*:
  assumes $x \in$ *carrier*
  assumes *indep X*
  assumes $\neg$ *indep* (*insert x X*)
  shows $\exists!C.$ *circuit C* $\wedge$ $C \subseteq$ *insert x X*
proof −
  obtain *C* **where** *C*: *circuit C* $C \subseteq$ *insert x X*
    using *assms indep-subset-carrier dep-iff-supset-circuit* **by** *auto*

  show *?thesis*
  proof (*rule ex1I*, *goal-cases ex unique*)
    show *circuit C* $\wedge$ $C \subseteq$ *insert x X* **using** *C* **by** *auto*
  next
    {
      fix C'
      assume C': *circuit* C' $C' \subseteq$ *insert x X*
      have $C' = C$
      proof (*rule ccontr*)
        assume $C' \neq C$
        moreover have $x \in C' \cap C$ **using** *C* C' *assms min-dep-imp-supset-circuit*
by *auto*
        ultimately have $\neg$ *indep* $(C' \cup C - \{x\})$
          using *circuit-elim*[*OF C*(*1*) C'(*1*), *of x*] *supset-circuit-imp-dep*[*of* - $C' \cup$
$C - \{x\}$] **by** *auto*
        moreover have $C' \cup C - \{x\} \subseteq X$ **using** *C* C' **by** *auto*
        ultimately show *False* **using** *assms indep-subset* **by** *auto*
      qed
    }
    then show $\bigwedge C'.$ *circuit* C' $\wedge$ $C' \subseteq$ *insert x X* $\Longrightarrow C' = C$
      by *auto*
  qed
qed

lemma *basis-ex1-supset-circuit*:
  assumes *basis B*
  assumes $x \in$ *carrier* $- B$
  shows $\exists!C.$ *circuit C* $\wedge$ $C \subseteq$ *insert x B*
  using *assms min-dep-imp-ex1-supset-circuit basisD* **by** *auto*

definition *fund-circuit* :: $'a \Rightarrow 'a$ *set* $\Rightarrow 'a$ *set* **where**
  fund-circuit x B $\equiv$ (*THE C. circuit C* $\wedge$ $C \subseteq$ *insert x B*)

lemma *circuit-iff-fund-circuit*:
  circuit $C \longleftrightarrow (\exists\, x\ B.\ x \in$ *carrier* $- B \wedge$ *basis* $B \wedge C =$ *fund-circuit* $x\ B)$
proof (*safe*, *goal-cases LTR RTL*)
  case *LTR*
  then obtain x **where** $x \in C$ **using** *circuit-nonempty* **by** *auto*
  then have *indep* $(C - \{x\})$ **using** *LTR* **unfolding** *circuit-def* **by** *auto*
  then obtain B **where** B: *basis* B $C - \{x\} \subseteq B$ **using** *indep-imp-subset-basis* **by** *auto*
  then have $x \in$ *carrier* **using** *LTR circuit-subset-carrier* ‹$x \in C$› **by** *auto*
  moreover have $x \notin B$
  proof (*rule ccontr*, *goal-cases False*)
    case *False*
    then have $C \subseteq B$ **using** ‹$C - \{x\} \subseteq B$› **by** *auto*
    then have $\neg$ *indep* B **using** *LTR B basis-subset-carrier supset-circuit-imp-dep* **by** *auto*
    then show *?case* **using** *B basis-indep* **by** *auto*
  qed
  ultimately show *?case*
    unfolding *fund-circuit-def*
    using *LTR B theI-unique*[*OF basis-ex1-supset-circuit*[*of B x*], *of C*] **by** *auto*
next
  case (*RTL x B*)
  then have $\exists!C.$ *circuit* $C \wedge C \subseteq$ *insert* $x\ B$
    using *min-dep-imp-ex1-supset-circuit basisD*[*of B*] **by** *auto*
  then show *?case*
    unfolding *fund-circuit-def*
    using *theI*[*of* $\lambda C.$ *circuit* $C \wedge C \subseteq$ *insert* $x\ B$] **by** *fastforce*
qed

lemma *fund-circuitI*:
  assumes *basis* B
  assumes $x \in$ *carrier* $- B$
  assumes *circuit* C
  assumes $C \subseteq$ *insert* $x\ B$
  shows *fund-circuit* $x\ B = C$
  unfolding *fund-circuit-def*
  using *assms theI-unique*[*OF basis-ex1-supset-circuit*, *of B x C*] **by** *auto*

definition *fund-circuit-in* **where** *fund-circuit-in* $\mathcal{E}$ x $B \equiv$ *matroid.fund-circuit* $\mathcal{E}$ (*indep-in* $\mathcal{E}$) x B

context
  fixes $\mathcal{E}$
  assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *fund-circuit-inI-aux*: $\mathcal{E}$*.fund-circuit* x B = *fund-circuit-in* $\mathcal{E}$ x B
  unfolding *fund-circuit-in-def* **by** *auto*

lemma *circuit-in-elim*:
  assumes *circuit-in* $\mathcal{E}$ C_1 *circuit-in* $\mathcal{E}$ C_2 $C_1 \neq C_2$ $x \in C_1 \cap C_2$
  shows $\exists\, C_3 \subseteq (C_1 \cup C_2) - \{x\}$. *circuit-in* $\mathcal{E}$ C_3
  using *assms* $\mathcal{E}$*.circuit-elim* **unfolding** *circuit-in-def* **by** *auto*

lemmas *min-dep-in-imp-supset-circuit-in* = $\mathcal{E}$*.min-dep-imp-supset-circuit*[*OF* - *circuit-inD-aux*[*OF* $*$]]

lemma *min-dep-in-imp-ex1-supset-circuit-in*:
  assumes $x \in \mathcal{E}$
  assumes *indep-in* $\mathcal{E}$ X
  assumes $\neg$ *indep-in* $\mathcal{E}$ (*insert* x X)
  shows $\exists!C$. *circuit-in* $\mathcal{E}$ C $\wedge$ $C \subseteq$ *insert* x X
  using *assms* $\mathcal{E}$*.min-dep-imp-ex1-supset-circuit* **unfolding** *circuit-in-def* **by** *auto*

lemma *basis-in-ex1-supset-circuit-in*:
  assumes *basis-in* $\mathcal{E}$ B
  assumes $x \in \mathcal{E} - B$
  shows $\exists!C$. *circuit-in* $\mathcal{E}$ C $\wedge$ $C \subseteq$ *insert* x B
  using *assms* $\mathcal{E}$*.basis-ex1-supset-circuit* **unfolding** *circuit-in-def basis-in-def* **by** *auto*

lemma *fund-circuit-inI*:
  assumes *basis-in* $\mathcal{E}$ B
  assumes $x \in \mathcal{E} - B$
  assumes *circuit-in* $\mathcal{E}$ C
  assumes $C \subseteq$ *insert* x B
  shows *fund-circuit-in* $\mathcal{E}$ x B = C
  using *assms* $\mathcal{E}$*.fund-circuitI*
  unfolding *basis-in-def circuit-in-def fund-circuit-in-def* **by** *auto*

end

context
  fixes $\mathcal{E}$
  assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ‹$\mathcal{E} \subseteq$ *carrier*› **by** *auto*

lemma *fund-circuit-in-sub-cong*:
  assumes $\mathcal{E}' \subseteq \mathcal{E}$
  assumes $x \in \mathcal{E}' - B$
  assumes *basis-in* $\mathcal{E}'$ B
  shows $\mathcal{E}$*.fund-circuit-in* $\mathcal{E}'$ x B = *fund-circuit-in* $\mathcal{E}'$ x B

proof −
  obtain *C* **where** *C*: *circuit-in* $\mathcal{E}'$ *C* $C \subseteq$ *insert x B*
    using ∗ *assms basis-in-ex1-supset-circuit-in*[*of* $\mathcal{E}'$ *B x*] **by** *auto*
  then have *fund-circuit-in* $\mathcal{E}'$ *x B* = *C*
    using ∗ *assms fund-circuit-inI* **by** *auto*
  also have ... = $\mathcal{E}$.*fund-circuit-in* $\mathcal{E}'$ *x B*
   using ∗ *assms C* $\mathcal{E}$.*fund-circuit-inI basis-in-sub-cong*[*of* $\mathcal{E}$] *circuit-in-sub-cong*[*of* $\mathcal{E}$] **by** *auto*
  finally show *?thesis* **by** *auto*
qed

end

2.4 Ranks

abbreviation *rank-of* **where** *rank-of* ≡ *lower-rank-of*

lemmas *rank-of-def* = *lower-rank-of-def*
lemmas *rank-of-sub-cong* = *lower-rank-of-sub-cong*
lemmas *rank-of-le* = *lower-rank-of-le*

context
  fixes $\mathcal{E}$
  assumes ∗: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using ∗ **by** *auto*

lemma *lower-rank-of-eq-upper-rank-of*: *lower-rank-of* $\mathcal{E}$ = *upper-rank-of* $\mathcal{E}$
proof −
  obtain *B* **where** *basis-in* $\mathcal{E}$ *B* **using** *basis-in-ex*[*OF* ∗] **by** *auto*
  then have {*card B* | *B*. *basis-in* $\mathcal{E}$ *B*} = {*card B*}
    by *safe* (*auto dest*: *basis-in-card*[*OF* ∗])
  then show *?thesis* **unfolding** *lower-rank-of-def upper-rank-of-def* **by** *auto*
qed

lemma *rank-of-eq-card-basis-in*:
  assumes *basis-in* $\mathcal{E}$ *B*
  shows *rank-of* $\mathcal{E}$ = *card B*
proof −
  have {*card B* | *B*. *basis-in* $\mathcal{E}$ *B*} = {*card B*} **using** *assms* **by** *safe* (*auto dest*: *basis-in-card*[*OF* ∗])
  then show *?thesis* **unfolding** *rank-of-def* **by** *auto*
qed

lemma *rank-of-indep-in-le*:
  assumes *indep-in* $\mathcal{E}$ *X*
  shows *card X* ≤ *rank-of* $\mathcal{E}$

proof −
  {
    fix B
    assume *basis-in* $\mathcal{E}$ B
    moreover obtain B' **where** *basis-in* $\mathcal{E}$ B' $X \subseteq B'$
      using *assms indep-in-imp-subset-basis-in*[*OF* ∗] **by** *auto*
    ultimately have *card* $X \leq$ *card* B
      using *card-mono*[*OF basis-in-finite*[*OF* ∗]] *basis-in-card*[*OF* ∗, *of* B B'] **by** *auto*
  }
  moreover have *finite* $\{$*card* B | B. *basis-in* $\mathcal{E}$ $B\}$
    using *collect-basis-in-finite*[*OF* ∗] **by** *auto*
  ultimately show *?thesis*
    unfolding *rank-of-def* **using** *basis-in-ex*[*OF* ∗] **by** *auto*
qed

end

lemma *rank-of-mono*:
  assumes $X \subseteq Y$
  assumes $Y \subseteq$ *carrier*
  shows *rank-of* $X \leq$ *rank-of* Y
proof −
  obtain B_X **where** B_X: *basis-in* X B_X **using** *assms basis-in-ex*[*of* X] **by** *auto*
  moreover obtain B_Y **where** B_Y: *basis-in* Y B_Y **using** *assms basis-in-ex*[*of* Y] **by** *auto*
  moreover have *card* $B_X \leq$ *card* B_Y
    using *assms basis-in-indep-in-card*[*OF* - - B_Y] *basis-in-indep-in*[*OF* - B_X] *indep-in-subI-subset*
    by *auto*
  ultimately show *?thesis* **using** *assms rank-of-eq-card-basis-in* **by** *auto*
qed

lemma *rank-of-insert-le*:
  assumes $X \subseteq$ *carrier*
  assumes $x \in$ *carrier*
  shows *rank-of* (*insert* x X) $\leq$ *Suc* (*rank-of* X)
proof −
  obtain B **where** B: *basis-in* X B **using** *assms basis-in-ex*[*of* X] **by** *auto*
  have *basis-in* (*insert* x X) $B \vee$ *basis-in* (*insert* x X) (*insert* x B)
  proof −
    obtain B' **where** B': $B' \subseteq$ *insert* x $X - X$ *basis-in* (*insert* x X) $(B \cup B')$
      using *assms B basis-in-subI*[*of insert* x X X B] **by** *auto*
    then have $B' = \{\} \vee B' = \{x\}$ **by** *auto*
    then show *?thesis*
    proof
      assume $B' = \{\}$
      then have *basis-in* (*insert* x X) B **using** B' **by** *auto*
      then show *?thesis* **by** *auto*

next
assume $B' = \{x\}$
then have *basis-in* (*insert x X*) (*insert x B*) **using** B' **by** *auto*
then show *?thesis* **by** *auto*
qed
qed
then show *?thesis*
proof
assume *basis-in* (*insert x X*) *B*
then show *?thesis*
using *assms B rank-of-eq-card-basis-in* **by** *auto*
next
assume *basis-in* (*insert x X*) (*insert x B*)
then have *rank-of* (*insert x X*) = *card* (*insert x B*)
using *assms rank-of-eq-card-basis-in* **by** *auto*
also have ... = *Suc* (*card* ($B - \{x\}$))
using *assms card.insert-remove*[*of B x*] **using** *B basis-in-finite* **by** *auto*
also have ... $\leq$ *Suc* (*card B*)
using *assms B basis-in-finite card-Diff1-le*[*of B*] **by** *auto*
also have ... = *Suc* (*rank-of X*)
using *assms B rank-of-eq-card-basis-in* **by** *auto*
finally show *?thesis* **.**
qed
qed

lemma *rank-of-Un-Int-le*:
assumes $X \subseteq$ *carrier*
assumes $Y \subseteq$ *carrier*
shows *rank-of* ($X \cup Y$) + *rank-of* ($X \cap Y$) $\leq$ *rank-of X* + *rank-of Y*
proof −
obtain *B-Int* **where** *B-Int*: *basis-in* ($X \cap Y$) *B-Int* **using** *assms basis-in-ex*[*of* $X \cap Y$] **by** *auto*
then have *indep-in* ($X \cup Y$) *B-Int*
using *assms indep-in-subI-subset*[*OF* - *basis-in-indep-in*[*of* $X \cap Y$ *B-Int*], *of* $X \cup Y$] **by** *auto*
then obtain *B-Un* **where** *B-Un*: *basis-in* ($X \cup Y$) *B-Un B-Int* $\subseteq$ *B-Un*
using *assms indep-in-imp-subset-basis-in*[*of* $X \cup Y$ *B-Int*] **by** *auto*

have *card* (*B-Un* $\cap$ ($X \cup Y$)) + *card* (*B-Un* $\cap$ ($X \cap Y$)) = *card* ((*B-Un* $\cap X$) $\cup$ (*B-Un* $\cap Y$)) + *card* ((*B-Un* $\cap X$) $\cap$ (*B-Un* $\cap Y$))
by (*simp add*: *inf-assoc inf-left-commute inf-sup-distrib1*)
also have ... = *card* (*B-Un* $\cap X$) + *card* (*B-Un* $\cap Y$)
proof −
have *finite* (*B-Un* $\cap X$) *finite* (*B-Un* $\cap Y$)
using *assms finite-subset*[*OF* - *carrier-finite*] **by** *auto*
then show *?thesis* **using** *card-Un-Int*[*of B-Un* $\cap X$ *B-Un* $\cap Y$] **by** *auto*
qed
also have ... $\leq$ *rank-of X* + *rank-of Y*
proof −

have *card* $(B\text{-}Un \cap X) \leq$ *rank-of* X
proof −
have *indep-in* X $(B\text{-}Un \cap X)$ **using** *assms basis-in-indep-in*[*OF* - *B-Un*(*1*)] *indep-in-subI* **by** *auto*
then show *?thesis* **using** *assms rank-of-indep-in-le* **by** *auto*
qed
moreover have *card* $(B\text{-}Un \cap Y) \leq$ *rank-of* Y
proof −
have *indep-in* Y $(B\text{-}Un \cap Y)$ **using** *assms basis-in-indep-in*[*OF* - *B-Un*(*1*)] *indep-in-subI* **by** *auto*
then show *?thesis* **using** *assms rank-of-indep-in-le* **by** *auto*
qed
ultimately show *?thesis* **by** *auto*
qed
finally have *rank-of* X + *rank-of* $Y \geq$ *card* $(B\text{-}Un \cap (X \cup Y))$ + *card* $(B\text{-}Un \cap (X \cap Y))$ **.**
moreover have $B\text{-}Un \cap (X \cup Y) = B\text{-}Un$ **using** *assms basis-in-subset-carrier*[*OF* - *B-Un*(*1*)] **by** *auto*
moreover have $B\text{-}Un \cap (X \cap Y) = B\text{-}Int$
proof −
have *card* $(B\text{-}Un \cap (X \cap Y)) \leq$ *card* $B\text{-}Int$
proof −
have *indep-in* $(X \cap Y)$ $(B\text{-}Un \cap (X \cap Y))$
using *assms basis-in-indep-in*[*OF* - *B-Un*(*1*)] *indep-in-subI* **by** *auto*
then show *?thesis* **using** *assms basis-in-indep-in-card*[*of* $X \cap Y$ - *B-Int*] *B-Int* **by** *auto*
qed
moreover have *finite* $(B\text{-}Un \cap (X \cap Y))$
using *assms carrier-finite finite-subset*[*of* $B\text{-}Un \cap (X \cap Y)$] **by** *auto*
moreover have $B\text{-}Int \subseteq B\text{-}Un \cap (X \cap Y)$
using *assms B-Un B-Int basis-in-subset-carrier*[*of* $X \cap Y$ *B-Int*] **by** *auto*
ultimately show *?thesis* **using** *card-seteq* **by** *blast*
qed
ultimately have *rank-of* X + *rank-of* $Y \geq$ *card* $B\text{-}Un$ + *card* $B\text{-}Int$ **by** *auto*
moreover have *card* $B\text{-}Un$ = *rank-of* $(X \cup Y)$
using *assms rank-of-eq-card-basis-in*[*OF* - *B-Un*(*1*)] **by** *auto*
moreover have *card* $B\text{-}Int$ = *rank-of* $(X \cap Y)$
using *assms rank-of-eq-card-basis-in*[*OF* - *B-Int*] **by** *fastforce*
ultimately show *rank-of* X + *rank-of* $Y \geq$ *rank-of* $(X \cup Y)$ + *rank-of* $(X \cap Y)$ **by** *auto*
qed

lemma *rank-of-Un-absorbI*:
assumes $X \subseteq$ *carrier* $Y \subseteq$ *carrier*
assumes $\bigwedge y.\ y \in Y - X \Longrightarrow$ *rank-of* (*insert* y X) = *rank-of* X
shows *rank-of* $(X \cup Y)$ = *rank-of* X
proof −
have *finite* $(Y - X)$ **using** *finite-subset*[*OF* ‹$Y \subseteq$ *carrier*›] *carrier-finite* **by** *auto*

```
  then show ?thesis using assms
  proof (induction Y − X arbitrary: Y rule: finite-induct )
    case empty
    then have X ∪ Y = X by auto
    then show ?case by auto
  next
    case (insert y F)
    have rank-of (X ∪ Y) + rank-of X ≤ rank-of X + rank-of X
    proof −
      have rank-of (X ∪ Y) + rank-of X = rank-of ((X ∪ (Y − {y})) ∪ (insert
y X)) + rank-of ((X ∪ (Y − {y})) ∩ (insert y X))
      proof −
        have X ∪ Y = (X ∪ (Y − {y})) ∪ (insert y X) X = (X ∪ (Y − {y})) ∩
(insert y X) using insert by auto
        then show ?thesis by auto
      qed
      also have . . . ≤ rank-of (X ∪ (Y − {y})) + rank-of (insert y X)
      proof (rule rank-of-Un-Int-le)
        show X ∪ (Y − {y}) ⊆ carrier using insert by auto
      next
        show insert y X ⊆ carrier using insert by auto
      qed
      also have . . . = rank-of (X ∪ (Y − {y})) + rank-of X
      proof −
        have y ∈ Y − X using insert by auto
        then show ?thesis using insert by auto
      qed
      also have . . . = rank-of X + rank-of X
      proof −
        have F = (Y − {y}) − X Y − {y} ⊆ carrier using insert by auto
        then show ?thesis using insert insert(3)[of Y − {y}] by auto
      qed
      finally show ?thesis .
    qed
    moreover have rank-of (X ∪ Y) + rank-of X ≥ rank-of X + rank-of X
      using insert rank-of-mono by auto
    ultimately show ?case by auto
  qed
qed

lemma indep-iff-rank-of:
  assumes X ⊆ carrier
  shows indep X ⟷ rank-of X = card X
proof (standard, goal-cases LTR RTL)
  case LTR
  then have indep-in X X by (auto intro: indep-inI)
  then have basis-in X X by (auto intro: basis-inI[OF assms])
  then show ?case using rank-of-eq-card-basis-in[OF assms] by auto
next
```

case *RTL*
obtain *B* **where** *B*: *basis-in X B* **using** *basis-in-ex*[*OF assms*] **by** *auto*
then have *card B* = *card X* **using** *RTL rank-of-eq-card-basis-in*[*OF assms*] **by** *auto*
then have *B* = *X*
using *basis-in-subset-carrier*[*OF assms B*] *card-seteq*[*OF finite-subset*[*OF assms carrier-finite*]]
by *auto*
then show *?case* **using** *basis-in-indep-in*[*OF assms B*] *indep-in-indep* **by** *auto*
qed

lemma *basis-iff-rank-of*:
assumes $X \subseteq$ *carrier*
shows *basis X* $\longleftrightarrow$ *rank-of X* = *card X* $\wedge$ *rank-of X* = *rank-of carrier*
proof (*standard*, *goal-cases LTR RTL*)
case *LTR*
then have *rank-of X* = *card X* **using** *assms indep-iff-rank-of basis-indep* **by** *auto*
moreover have ... = *rank-of carrier*
using *LTR rank-of-eq-card-basis-in*[*of carrier X*] *basis-iff-basis-in* **by** *auto*
ultimately show *?case* **by** *auto*
next
case *RTL*
show *?case*
proof (*rule basisI*)
show *indep X* **using** *assms RTL indep-iff-rank-of* **by** *blast*
next
fix *x*
assume *x*: $x \in$ *carrier* $-$ *X*
show $\neg$ *indep* (*insert x X*)
proof (*rule ccontr*, *goal-cases False*)
case *False*
then have *card* (*insert x X*) $\leq$ *rank-of carrier*
using *assms x indep-inI rank-of-indep-in-le* **by** *auto*
also have ... = *card X* **using** *RTL* **by** *auto*
finally show *?case* **using** *finite-subset*[*OF assms carrier-finite*] *x* **by** *auto*
qed
qed
qed

lemma *circuit-iff-rank-of*:
assumes $X \subseteq$ *carrier*
shows *circuit X* $\longleftrightarrow$ $X \neq \{\}$ $\wedge$ ($\forall x \in X$. *rank-of* ($X - \{x\}$) = *card* ($X - \{x\}$) $\wedge$ *card* ($X - \{x\}$) = *rank-of X*)
proof (*standard*, *goal-cases LTR RTL*)
case *LTR*
then have $X \neq \{\}$ **using** *circuit-nonempty* **by** *auto*
moreover have *indep-remove*: $\bigwedge x.\ x \in X \Longrightarrow$ *rank-of* ($X - \{x\}$) = *card* ($X - \{x\}$)

proof −
fix x
assume $x \in X$
then have *indep* $(X - \{x\})$ **using** *circuit-min-dep*[*OF LTR*] **by** *auto*
moreover have $X - \{x\} \subseteq$ *carrier* **using** *assms* **by** *auto*
ultimately show *rank-of* $(X - \{x\})$ = *card* $(X - \{x\})$ **using** *indep-iff-rank-of* **by** *auto*
qed
moreover have $\bigwedge x.\ x \in X \Longrightarrow$ *rank-of* $(X - \{x\})$ = *rank-of* X
proof −
fix x
assume $*$: $x \in X$
have *rank-of* $X \leq$ *card* X **using** *assms rank-of-le* **by** *auto*
moreover have *rank-of* $X \neq$ *card* X **using** *assms LTR circuitD indep-iff-rank-of*[*of* X] **by** *auto*
ultimately have *rank-of* $X <$ *card* X **by** *auto*
then have *rank-of* $X \leq$ *card* $(X - \{x\})$ **using** $*$ *finite-subset*[*OF assms*] *carrier-finite* **by** *auto*
also have ... = *rank-of* $(X - \{x\})$ **using** *indep-remove* ‹$x \in X$› **by** *auto*
finally show *rank-of* $(X - \{x\})$ = *rank-of* X **using** *assms rank-of-mono*[*of* $X - \{x\}$ X] **by** *auto*
qed
ultimately show *?case* **by** *auto*
next
case *RTL*
then have $X \neq \{\}$
and *indep-remove*: $\bigwedge x.\ x \in X \Longrightarrow$ *rank-of* $(X - \{x\})$ = *card* $(X - \{x\})$
and *dep*: $\bigwedge x.\ x \in X \Longrightarrow$ *rank-of* $(X - \{x\})$ = *rank-of* X
by *auto*
show *?case* **using** *assms*
proof (*rule circuitI*)
obtain x **where** x: $x \in X$ **using** ‹$X \neq \{\}$› **by** *auto*
then have *rank-of* X = *card* $(X - \{x\})$ **using** *dep indep-remove* **by** *auto*
also have ... $<$ *card* X **using** *card-Diff1-less*[*OF finite-subset*[*OF assms carrier-finite*] x] **.**
finally show $\neg$ *indep* X **using** *indep-iff-rank-of*[*OF assms*] **by** *auto*
next
fix x
assume $x \in X$
then show *indep* $(X - \{x\})$ **using** *assms indep-remove*[*of* x] *indep-iff-rank-of*[*of* $X - \{x\}$]
by *auto*
qed
qed

context
fixes $\mathcal{E}$
assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using $*$ **by** *auto*

lemma *indep-in-iff-rank-of*:
  assumes $X \subseteq \mathcal{E}$
  shows *indep-in* $\mathcal{E}$ $X \longleftrightarrow$ *rank-of* X = *card* X
  using *assms* $\mathcal{E}$.*indep-iff-rank-of* *rank-of-sub-cong*[*OF* $*$ *assms*] **by** *auto*

lemma *basis-in-iff-rank-of*:
  assumes $X \subseteq \mathcal{E}$
  shows *basis-in* $\mathcal{E}$ $X \longleftrightarrow$ *rank-of* X = *card* X $\wedge$ *rank-of* X = *rank-of* $\mathcal{E}$
  using $\mathcal{E}$.*basis-iff-rank-of*[*OF assms*] *rank-of-sub-cong*[*OF* $*$] *assms*
  unfolding *basis-in-def* **by** *auto*

lemma *circuit-in-iff-rank-of*:
  assumes $X \subseteq \mathcal{E}$
  shows *circuit-in* $\mathcal{E}$ $X \longleftrightarrow X \neq \{\} \wedge (\forall x \in X.$ *rank-of* $(X - \{x\})$ = *card* $(X - \{x\}) \wedge$ *card* $(X - \{x\})$ = *rank-of* $X)$
proof −
  have *circuit-in* $\mathcal{E}$ $X \longleftrightarrow \mathcal{E}$.*circuit* X **unfolding** *circuit-in-def* **..**
  also have ... $\longleftrightarrow X \neq \{\} \wedge (\forall x \in X.\ \mathcal{E}$.*rank-of* $(X - \{x\})$ = *card* $(X - \{x\}) \wedge$ *card* $(X - \{x\}) = \mathcal{E}$.*rank-of* $X)$
    using $\mathcal{E}$.*circuit-iff-rank-of*[*OF assms*] **.**
  also have ... $\longleftrightarrow X \neq \{\} \wedge (\forall x \in X.$ *rank-of* $(X - \{x\})$ = *card* $(X - \{x\}) \wedge$ *card* $(X - \{x\})$ = *rank-of* $X)$
  proof −
    {
      fix x
      have $\mathcal{E}$.*rank-of* $(X - \{x\})$ = *rank-of* $(X - \{x\})$ $\mathcal{E}$.*rank-of* X = *rank-of* X
        using *assms* *rank-of-sub-cong*[*OF* $*$, *of* $X - \{x\}$] *rank-of-sub-cong*[*OF* $*$, *of* X] **by** *auto*
      then have $\mathcal{E}$.*rank-of* $(X - \{x\})$ = *card* $(X - \{x\}) \wedge$ *card* $(X - \{x\}) = \mathcal{E}$.*rank-of* $X \longleftrightarrow$ *rank-of* $(X - \{x\})$ = *card* $(X - \{x\}) \wedge$ *card* $(X - \{x\})$ = *rank-of* X
        by *auto*
    }
    then show *?thesis*
      by (*auto simp*: *simp del*: *card-Diff-insert*)
  qed
  finally show *?thesis* **.**
qed

end

2.5 Closure

definition *cl* :: $'a$ *set* $\Rightarrow$ $'a$ *set* **where**
  cl $X \equiv \{x \in$ *carrier*. *rank-of* (*insert* x X) = *rank-of* $X\}$

lemma *clI*:
  assumes $x \in$ *carrier*
  assumes *rank-of* (*insert x X*) = *rank-of X*
  shows $x \in$ *cl X*
  unfolding *cl-def* **using** *assms* **by** *auto*

lemma *cl-altdef*:
  assumes $X \subseteq$ *carrier*
  shows *cl X* = $\bigcup${*Y* ∈ *Pow carrier*. *X* ⊆ *Y* ∧ *rank-of Y* = *rank-of X*}
proof −
  {
    fix *x*
    assume ∗: $x \in$ *cl X*
    have $x \in \bigcup${*Y* ∈ *Pow carrier*. *X* ⊆ *Y* ∧ *rank-of Y* = *rank-of X*}
    proof
      show *insert x X* ∈ {*Y* ∈ *Pow carrier*. *X* ⊆ *Y* ∧ *rank-of Y* = *rank-of X*}
        using *assms* ∗ **unfolding** *cl-def* **by** *auto*
    qed *auto*
  }
  moreover {
    fix *x*
    assume ∗: $x \in \bigcup${*Y* ∈ *Pow carrier*. *X* ⊆ *Y* ∧ *rank-of Y* = *rank-of X*}
    then obtain *Y* **where** *Y*: *x* ∈ *Y* *Y* ⊆ *carrier* *X* ⊆ *Y* *rank-of Y* = *rank-of X*
by *auto*
    have *rank-of* (*insert x X*) = *rank-of X*
    proof −
      have *rank-of* (*insert x X*) ≤ *rank-of X*
      proof −
        have *insert x X* ⊆ *Y* **using** *Y* **by** *auto*
        then show *?thesis* **using** *rank-of-mono*[*of insert x X Y*] *Y* **by** *auto*
      qed
      moreover have *rank-of X* ≤ *rank-of* (*insert x X*) **using** *Y* **by** (*auto intro*:
rank-of-mono)
      ultimately show *?thesis* **by** *auto*
    qed
    then have *x* ∈ *cl X* **using** ∗ **unfolding** *cl-def* **by** *auto*
  }
  ultimately show *?thesis* **by** *blast*
qed

lemma *cl-rank-of*: *x* ∈ *cl X* ⟹ *rank-of* (*insert x X*) = *rank-of X*
  unfolding *cl-def* **by** *auto*

lemma *cl-subset-carrier*: *cl X* ⊆ *carrier*
  unfolding *cl-def* **by** *auto*

lemmas *clD* = *cl-rank-of cl-subset-carrier*

lemma *cl-subset*:
  assumes $X \subseteq$ *carrier*
  shows $X \subseteq$ *cl* X
  using *assms* **using** *insert-absorb*[*of* - X] **by** (*auto intro*!: *clI*)

lemma *cl-mono*:
  assumes $X \subseteq Y$
  assumes $Y \subseteq$ *carrier*
  shows *cl* $X \subseteq$ *cl* Y
proof
  fix x
  assume $x \in$ *cl* X
  then have $x \in$ *carrier* **using** *cl-subset-carrier* **by** *auto*

  have *insert* x $X \subseteq$ *carrier*
    using *assms* ‹$x \in$ *cl* X› *cl-subset-carrier*[*of* X] **by** *auto*
  then interpret *X-insert*: *matroid insert* x X *indep-in* (*insert* x X) **by** *auto*

  have *insert* x $Y \subseteq$ *carrier*
    using *assms* ‹$x \in$ *cl* X› *cl-subset-carrier*[*of* X] **by** *auto*
  then interpret *Y-insert*: *matroid insert* x Y *indep-in* (*insert* x Y) **by** *auto*

  show $x \in$ *cl* Y **using** ‹$x \in$ *carrier*›
  proof (*rule clI*, *cases* $x \in X$)
    case *True*
    then show *rank-of* (*insert* x Y) = *rank-of* Y **using** *assms insert-absorb*[*of* x
Y] **by** *auto*
  next
    case *False*
    obtain B_X **where** B_X: *basis-in* X B_X **using** *assms basis-in-ex*[*of* X] **by** *auto*

    have *basis-in* (*insert* x X) B_X
    proof −
      have *rank-of* B_X = *card* B_X ∧ *rank-of* B_X = *rank-of* (*insert* x X)
      proof −
        have *rank-of* B_X = *card* B_X ∧ *rank-of* B_X = *rank-of* X
          using *assms* B_X
            basis-in-subset-carrier[**where** $\mathcal{E} = X$ **and** $X = B_X$]
            basis-in-iff-rank-of[**where** $\mathcal{E} = X$ **and** $X = B_X$]
          by *blast*
        then show *?thesis* **using** *cl-rank-of*[*OF* ‹$x \in$ *cl* X›] **by** *auto*
      qed
      then show *?thesis*
      using *assms basis-in-subset-carrier*[*OF* - B_X] ‹$x \in$ *carrier*› *basis-in-iff-rank-of*[*of*
insert x X B_X]
        by *auto*
    qed

have *indep-in* (*insert* x Y) B_X
using *assms basis-in-indep-in*[*OF* - B_X] *indep-in-subI-subset*[*of* X *insert* x Y] **by** *auto*
then obtain B_Y **where** B_Y: *basis-in* (*insert* x Y) B_Y $B_X \subseteq B_Y$
using *assms* ‹$x \in$ *carrier*› *indep-in-iff-subset-basis-in*[*of insert* x Y B_X] **by** *auto*

have *basis-in* Y B_Y
proof −
have $x \notin B_Y$
proof (*rule ccontr*, *goal-cases False*)
case *False*
then have *insert* x $B_X \subseteq B_Y$ **using** ‹$B_X \subseteq B_Y$› **by** *auto*
then have *indep-in* (*insert* x Y) (*insert* x B_X)
using *assms* ‹$x \in$ *carrier*›
B_Y *basis-in-indep-in*[**where** $\mathcal{E}$ = *insert* x Y **and** $X = B_Y$]
indep-in-subset[**where** $\mathcal{E}$ = *insert* x Y **and** $X = B_Y$ **and** Y = *insert* x B_X]
by *auto*
then have *indep-in* (*insert* x X) (*insert* x B_X)
using *assms* B_X
basis-in-subset-carrier[**where** $\mathcal{E} = X$ **and** $X = B_X$]
indep-in-supI[**where** $\mathcal{E}$ = *insert* x Y **and** $\mathcal{E}'$ = *insert* x X **and** X = *insert* x B_X]
by *auto*
moreover have $x \in$ *insert* x $X - B_X$
using *assms* ‹$x \notin X$› B_X *basis-in-subset-carrier*[**where** $\mathcal{E} = X$ **and** $X = B_X$] **by** *auto*
ultimately show *?case*
using *assms* ‹$x \in$ *carrier*› ‹*basis-in* (*insert* x X) B_X›
basis-in-max-indep-in[**where** $\mathcal{E}$ = *insert* x X **and** $X = B_X$ **and** $x = x$]
by *auto*
qed
then show *?thesis*
using *assms* ‹$x \in$ *carrier*› B_Y *basis-in-subset-carrier*[*of insert* x Y B_Y]
basis-in-supI[**where** $\mathcal{E}$ = *insert* x Y **and** $\mathcal{E}' = Y$ **and** $B = B_Y$] **by** *auto*
qed

show *rank-of* (*insert* x Y) = *rank-of* Y
proof −
have *rank-of* (*insert* x Y) = *card* B_Y
using *assms* ‹$x \in$ *carrier*› ‹*basis-in* (*insert* x Y) B_Y› *basis-in-subset-carrier*
using *basis-in-iff-rank-of*[**where** $\mathcal{E}$ = *insert* x Y **and** $X = B_Y$]
by *auto*
also have . . . = *rank-of* Y
using *assms* ‹$x \in$ *carrier*› ‹*basis-in* Y B_Y› *basis-in-subset-carrier*
using *basis-in-iff-rank-of*[**where** $\mathcal{E} = Y$ **and** $X = B_Y$]
by *auto*
finally show *?thesis* **.**

```
    qed
  qed
qed

lemma cl-insert-absorb:
  assumes X ⊆ carrier
  assumes x ∈ cl X
  shows cl (insert x X) = cl X
proof
  show cl (insert x X) ⊆ cl X
  proof (standard, goal-cases elem)
    case (elem y)
    then have *: x ∈ carrier y ∈ carrier using assms cl-subset-carrier by auto

    have rank-of (insert y X) = rank-of (insert y (insert x X))
    proof –
      have rank-of (insert y X) ≤ rank-of (insert y (insert x X))
        using assms * by (auto intro: rank-of-mono)
      moreover have rank-of (insert y (insert x X)) = rank-of (insert y X)
      proof –
        have insert y (insert x X) = insert x (insert y X) by auto
       then have rank-of (insert y (insert x X)) = rank-of (insert x (insert y X))
by auto
        also have ... = rank-of (insert y X)
        proof –
          have cl X ⊆ cl (insert y X) by (rule cl-mono) (auto simp add: assms ‹y
∈ carrier›)
          then have x ∈ cl (insert y X) using assms by auto
          then show ?thesis unfolding cl-def by auto
        qed
        finally show ?thesis .
      qed
      ultimately show ?thesis by auto
    qed
    also have ... = rank-of (insert x X) using elem using cl-rank-of by auto
    also have ... = rank-of X using assms cl-rank-of by auto
    finally show y ∈ cl X using * by (auto intro: clI)
  qed
next
  have insert x X ⊆ carrier using assms cl-subset-carrier by auto
  moreover have X ⊆ insert x X using assms by auto
  ultimately show cl X ⊆ cl (insert x X) using assms cl-subset-carrier cl-mono
by auto
qed

lemma cl-cl-absorb:
  assumes X ⊆ carrier
  shows cl (cl X) = cl X
proof
```

show *cl* (*cl X*) ⊆ *cl X*
proof (*standard*, *goal-cases elem*)
case (*elem x*)
then have $x \in$ *carrier* **using** *cl-subset-carrier* **by** *auto*
then show *?case*
proof (*rule clI*)
have *rank-of* (*insert x X*) ≥ *rank-of X*
using *assms* ‹$x \in$ *carrier*› *rank-of-mono*[*of X insert x X*] **by** *auto*
moreover have *rank-of* (*insert x X*) ≤ *rank-of X*
proof −
have *rank-of* (*insert x X*) ≤ *rank-of* (*insert x* (*cl X*))
using *assms* ‹$x \in$ *carrier*› *cl-subset-carrier cl-subset*[*of X*]
rank-of-mono[*of insert x X insert x* (*cl X*)] **by** *auto*
also have ... = *rank-of* (*cl X*) **using** *elem cl-rank-of* **by** *auto*
also have ... = *rank-of* (*X* ∪ (*cl X* − *X*))
using *Diff-partition*[*OF cl-subset*[*OF assms*]] **by** *auto*
also have ... = *rank-of X* **using** ‹*X* ⊆ *carrier*›
proof (*rule rank-of-Un-absorbI*)
show *cl X* − *X* ⊆ *carrier* **using** *assms cl-subset-carrier* **by** *auto*
next
fix *y*
assume $y \in$ *cl X* − *X* − *X*
then show *rank-of* (*insert y X*) = *rank-of X* **unfolding** *cl-def* **by** *auto*
qed
finally show *?thesis* **.**
qed
ultimately show *rank-of* (*insert x X*) = *rank-of X* **by** *auto*
qed
qed
next
show *cl X* ⊆ *cl* (*cl X*) **using** *cl-subset*[*OF cl-subset-carrier*] **by** *auto*
qed

lemma *cl-augment*:
assumes *X* ⊆ *carrier*
assumes $x \in$ *carrier*
assumes $y \in$ *cl* (*insert x X*) − *cl X*
shows $x \in$ *cl* (*insert y X*)
using ‹$x \in$ *carrier*›
proof (*rule clI*)
have *rank-of* (*insert y X*) ≤ *rank-of* (*insert x* (*insert y X*))
using *assms cl-subset-carrier* **by** (*auto intro*: *rank-of-mono*)
moreover have *rank-of* (*insert x* (*insert y X*)) ≤ *rank-of* (*insert y X*)
proof −
have *rank-of* (*insert x* (*insert y X*)) = *rank-of* (*insert y* (*insert x X*))
proof −
have *insert x* (*insert y X*) = *insert y* (*insert x X*) **by** *auto*
then show *?thesis* **by** *auto*
qed

```
    also have rank-of (insert y (insert x X)) = rank-of (insert x X)
      using assms cl-def by auto
    also have ... ≤ Suc (rank-of X)
      using assms cl-subset-carrier by (auto intro: rank-of-insert-le)
    also have ... = rank-of (insert y X)
    proof −
      have rank-of (insert y X) ≤ Suc (rank-of X)
        using assms cl-subset-carrier by (auto intro: rank-of-insert-le)
      moreover have rank-of (insert y X) ≠ rank-of X
        using assms cl-def by auto
      moreover have rank-of X ≤ rank-of (insert y X)
        using assms cl-subset-carrier by (auto intro: rank-of-mono)
      ultimately show ?thesis by auto
    qed
    finally show ?thesis .
  qed
  ultimately show rank-of (insert x (insert y X)) = rank-of (insert y X) by auto
qed

lemma clI-insert:
  assumes x ∈ carrier
  assumes indep X
  assumes ¬ indep (insert x X)
  shows x ∈ cl X
  using ‹x ∈ carrier›
proof (rule clI)
  have ∗: X ⊆ carrier using assms indep-subset-carrier by auto
  then have ∗∗: insert x X ⊆ carrier using assms by auto

  have indep-in (insert x X) X using assms by (auto intro: indep-inI)
  then obtain B where B: basis-in (insert x X) B X ⊆ B
    using assms indep-in-iff-subset-basis-in[OF ∗∗] by auto
  have x ∉ B
  proof (rule ccontr, goal-cases False)
    case False
    then have indep-in (insert x X) (insert x X)
      using B indep-in-subset[OF ∗∗ basis-in-indep-in[OF ∗∗]] by auto
    then show ?case using assms indep-in-indep by auto
  qed

  have basis-in X B using ∗
  proof (rule basis-inI, goal-cases indep max-indep)
    case indep
    show ?case
    proof (rule indep-in-supI[where ℰ = insert x X])
      show B ⊆ X using B basis-in-subset-carrier[OF ∗∗] ‹x ∉ B› by auto
    next
      show indep-in (insert x X) B using basis-in-indep-in[OF ∗∗ B(1)] .
    qed auto
```

next
  case (*max-indep y*)
  then have ¬ *indep-in* (*insert x X*) (*insert y B*)
    using *B basis-in-max-indep-in*[*OF* ∗∗] **by** *auto*
  then show *?case* **by** (*auto intro*: *indep-in-subI-subset*)
qed
then show *rank-of* (*insert x X*) = *rank-of X*
  using *B rank-of-eq-card-basis-in*[*OF* ∗] *rank-of-eq-card-basis-in*[*OF* ∗∗] **by** *auto*
qed

lemma *indep-in-carrier* [*simp*]: *indep-in carrier* = *indep*
  using *indep-subset-carrier* **by** (*auto simp*: *indep-in-def fun-eq-iff*)

context
  fixes *I*
  defines $I \equiv (\lambda X.\ X \subseteq carrier \wedge (\forall x{\in}X.\ x \notin cl\ (X - \{x\})))$
begin

lemma *I-mono*: *I Y* **if** $Y \subseteq X$ *I X* **for** *X Y* :: *'a set*
proof −
  have $\forall x{\in}Y.\ x \notin cl\ (Y - \{x\})$
  proof (*intro ballI*)
    fix *x* **assume** *x*: $x \in Y$
    with *that* **have** $cl\ (Y - \{x\}) \subseteq cl\ (X - \{x\})$
      by (*intro cl-mono*) (*auto simp*: *I-def*)
    with *that* **and** *x* **show** $x \notin cl\ (Y - \{x\})$ **by** (*auto simp*: *I-def*)
  qed
  with *that* **show** *?thesis* **by** (*auto simp*: *I-def*)
qed

lemma *clI'*:
  assumes *I X* $x \in carrier$ ¬*I* (*insert x X*)
  shows $x \in cl\ X$
proof −
  from *assms* **have** *x*: $x \notin X$ **by** (*auto simp*: *insert-absorb*)
  from *assms* **obtain** *y* **where** *y*: $y \in insert\ x\ X$ $y \in cl\ (insert\ x\ X - \{y\})$
    by (*force simp*: *I-def*)
  show $x \in cl\ X$
  proof (*cases* $x = y$)
    case *True*
    thus *?thesis* **using** *assms x y* **by** (*auto simp*: *I-def*)
  next
    case *False*
    have $y \in cl\ (insert\ x\ X - \{y\})$ **by** *fact*
    also from *False* **have** $insert\ x\ X - \{y\} = insert\ x\ (X - \{y\})$ **by** *auto*
    finally have $y \in cl\ (insert\ x\ (X - \{y\})) - cl\ (X - \{y\})$
      using *assms False y* **unfolding** *I-def* **by** *blast*
    hence $x \in cl\ (insert\ y\ (X - \{y\}))$
      using *cl-augment*[*of* $X - \{y\}$ *x y*] *assms False y* **by** (*auto simp*: *I-def*)

also from *y* **and** *False* **have** *insert y* $(X - \{y\}) = X$ **by** *auto*
finally show *?thesis* **.**
qed
qed

lemma *matroid-I*: *matroid carrier I*
proof (*unfold-locales*, *goal-cases*)
show *finite carrier* **by** (*rule carrier-finite*)
next
case (*4 X Y*)
have $\forall x \in Y.\ x \notin$ *cl* $(Y - \{x\})$
proof (*intro ballI*)
fix *x* **assume** *x*: $x \in Y$
with *4* **have** *cl* $(Y - \{x\}) \subseteq$ *cl* $(X - \{x\})$
by (*intro cl-mono*) (*auto simp*: *I-def*)
with *4* **and** *x* **show** $x \notin$ *cl* $(Y - \{x\})$ **by** (*auto simp*: *I-def*)
qed
with *4* **show** *?case* **by** (*auto simp*: *I-def*)
next
case (*5 X Y*)
have ~$(\exists X\ Y.\ I\ X \wedge I\ Y \wedge$ *card* $X <$ *card* $Y \wedge (\forall x \in Y - X.\ \neg I$ (*insert x X*)))
proof
assume $*$: $\exists X\ Y.\ I\ X \wedge I\ Y \wedge$ *card* $X <$ *card* $Y \wedge (\forall x \in Y - X.\ \neg I$ (*insert x* X)) (**is** $\exists X\ Y.$ *?P X Y*)
define *n* **where** $n =$ *Max* $((\lambda(X, Y).$ *card* $(X \cap Y))$ ‘ $\{(X, Y).$ *?P X Y*$\})$
have $\{(X, Y).$ *?P X Y*$\} \subseteq$ *Pow carrier* $\times$ *Pow carrier*
by (*auto simp*: *I-def*)
hence *finite*: *finite* $\{(X, Y).$ *?P X Y*$\}$
by (*rule finite-subset*) (*insert carrier-finite*, *auto*)
hence $n \in ((\lambda(X, Y).$ *card* $(X \cap Y))$ ‘ $\{(X, Y).$ *?P X Y*$\})$
unfolding *n-def* **using** $*$ **by** (*intro Max-in finite-imageI*) *auto*
then obtain *X Y* **where** *XY*: *?P X Y* $n =$ *card* $(X \cap Y)$
by *auto*
hence *finite'*: *finite X finite Y*
using *finite-subset*[*OF - carrier-finite*] *XY* **by** (*auto simp*: *I-def*)
from *XY finite'* **have** ~$(Y \subseteq X)$
using *card-mono*[*of X Y*] **by** *auto*
then obtain *y* **where** *y*: $y \in Y - X$ **by** *blast*

have *False*
proof (*cases* $X \subseteq$ *cl* $(Y - \{y\})$)
case *True*
from *y XY* **have** [*simp*]: $y \in$ *carrier* **by** (*auto simp*: *I-def*)
assume $X \subseteq$ *cl* $(Y - \{y\})$
hence *cl* $X \subseteq$ *cl* (*cl* $(Y - \{y\})$)
by (*intro cl-mono cl-subset-carrier*)
also have $\ldots =$ *cl* $(Y - \{y\})$
using *XY* **by** (*intro cl-cl-absorb*) (*auto simp*: *I-def*)

finally have *cl* $X \subseteq$ *cl* $(Y - \{y\})$ **.**
moreover have $y \notin$ *cl* $(Y - \{y\})$
using *y I-def XY*(*1*) **by** *blast*
ultimately have $y \notin$ *cl X* **by** *blast*
thus *False* **unfolding** *I-def*
using *XY y clI′* ‹$y \in$ *carrier*› **by** *blast*
next
case *False*
with *y XY* **have** [*simp*]: $y \in$ *carrier* **by** (*auto simp*: *I-def*)
assume $\neg(X \subseteq$ *cl* $(Y - \{y\}))$
then obtain *t* **where** *t*: $t \in X$ $t \notin$ *cl* $(Y - \{y\})$
by *auto*
with *XY* **have** [*simp*]: $t \in$ *carrier* **by** (*auto simp*: *I-def*)

have $t \in X - Y$
using *t y clI*[*of t Y* $-$ $\{y\}$] **by** (*cases* $t = y$) (*auto simp*: *insert-absorb*)
moreover have *I* $(Y - \{y\})$ **using** *XY*(*1*) *I-mono*[*of Y* $-$ $\{y\}$ *Y*] **by** *blast*
ultimately have ∗: *I* (*insert t* $(Y - \{y\})$)
using *clI′*[*of Y* $-$ $\{y\}$ *t*] *t* **by** *auto*

from *XY* **have** *finite Y*
by (*intro finite-subset*[*OF - carrier-finite*]) (*auto simp*: *I-def*)
moreover from *y* **have** $Y \neq \{\}$ **by** *auto*
ultimately have [*simp*]: *card* (*insert t* $(Y - \{y\})$) = *card Y* **using** ‹$t \in X$
$-$ *Y*› *y*
by (*simp add*: *Suc-diff-Suc card-gt-0-iff*)

have $\exists x \in Y - X$. *I* (*insert x X*)
proof (*rule ccontr*)
assume ¬ *?thesis*
hence *?P X* (*insert t* $(Y - \{y\})$) **using** *XY* ∗ ‹$t \in X - Y$›
by *auto*
hence *card* ($X \cap$ *insert t* $(Y - \{y\})$) $\leq n$
unfolding *n-def* **using** *finite* **by** (*intro Max-ge*) *auto*
also have $X \cap$ *insert t* $(Y - \{y\})$ = *insert t* $((X \cap Y) - \{y\})$
using *y* ‹$t \in X - Y$› **by** *blast*
also have *card* ... = *Suc* (*card* $(X \cap Y)$)
using *y* ‹$t \in X - Y$› ‹*finite Y*› **by** (*simp add*: *card.insert-remove*)
finally show *False* **using** *XY* **by** *simp*
qed
with *XY* **show** *False* **by** *blast*
qed
thus *False* **.**
qed
with *5* **show** *?case* **by** *auto*
qed (*auto simp*: *I-def*)

end

definition *cl-in* **where** $cl\text{-}in\ \mathcal{E}\ X = matroid.cl\ \mathcal{E}\ (indep\text{-}in\ \mathcal{E})\ X$

lemma *cl-eq-cl-in*:
  assumes $X \subseteq carrier$
  shows $cl\ X = cl\text{-}in\ carrier\ X$
proof −
  interpret $\mathcal{E}$: *matroid carrier indep-in carrier*
    by (*intro matroid-subset*) *auto*
  have $cl\ X = \{x \in carrier.\ rank\text{-}of\ (insert\ x\ X) = rank\text{-}of\ X\}$
    unfolding *cl-def* **by** *auto*
  also have $\ldots = \{x \in carrier.\ \mathcal{E}.rank\text{-}of\ (insert\ x\ X) = \mathcal{E}.rank\text{-}of\ X\}$
    using *rank-of-sub-cong*[*of carrier*] *assms* **by** *auto*
  also have $\ldots = cl\text{-}in\ carrier\ X$
    unfolding *cl-in-def* $\mathcal{E}$*.cl-def* **by** *auto*
  finally show *?thesis* **.**
qed

context
  fixes $\mathcal{E}$
  assumes $*$: $\mathcal{E} \subseteq carrier$
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
  using $*$ **by** *auto*

lemma *cl-inI-aux*: $x \in \mathcal{E}.cl\ X \Longrightarrow x \in cl\text{-}in\ \mathcal{E}\ X$
  unfolding *cl-in-def* **by** *auto*

lemma *cl-inD-aux*: $x \in cl\text{-}in\ \mathcal{E}\ X \Longrightarrow x \in \mathcal{E}.cl\ X$
  unfolding *cl-in-def* **by** *auto*

lemma *cl-inI*:
  assumes $X \subseteq \mathcal{E}$
  assumes $x \in \mathcal{E}$
  assumes $rank\text{-}of\ (insert\ x\ X) = rank\text{-}of\ X$
  shows $x \in cl\text{-}in\ \mathcal{E}\ X$
proof −
  have $\mathcal{E}.rank\text{-}of\ (insert\ x\ X) = rank\text{-}of\ (insert\ x\ X)\ \mathcal{E}.rank\text{-}of\ X = rank\text{-}of\ X$
    using *assms rank-of-sub-cong*[*OF* $*$] **by** *auto*
  then show *?thesis* **unfolding** *cl-in-def* **using** *assms* **by** (*auto intro*: $\mathcal{E}$*.clI*)
qed

lemma *cl-in-altdef*:
  assumes $X \subseteq \mathcal{E}$
  shows $cl\text{-}in\ \mathcal{E}\ X = \bigcup\{Y \in Pow\ \mathcal{E}.\ X \subseteq Y \wedge rank\text{-}of\ Y = rank\text{-}of\ X\}$
  unfolding *cl-in-def*
proof (*safe*, *goal-cases LTR RTL*)
  case (*LTR x*)
  then have $x \in \bigcup\{Y \in Pow\ \mathcal{E}.\ X \subseteq Y \wedge \mathcal{E}.rank\text{-}of\ Y = \mathcal{E}.rank\text{-}of\ X\}$

using $\mathcal{E}$.*cl-altdef*[*OF assms*] **by** *auto*
then obtain Y **where** Y: $x \in Y$ $Y \in$ *Pow* $\mathcal{E}$ $X \subseteq Y$ $\mathcal{E}$.*rank-of* Y = $\mathcal{E}$.*rank-of* X **by** *auto*
then show *?case* **using** *rank-of-sub-cong*[*OF* $*$] **by** *auto*
next
case (*RTL* x Y)
then have $x \in \bigcup\{Y \in$ *Pow* $\mathcal{E}$. $X \subseteq Y \wedge \mathcal{E}$.*rank-of* Y = $\mathcal{E}$.*rank-of* $X\}$
using *rank-of-sub-cong*[*OF* $*$, *of* X] *rank-of-sub-cong*[*OF* $*$, *of* Y] **by** *auto*
then show *?case* **using** $\mathcal{E}$.*cl-altdef*[*OF assms*] **by** *auto*
qed

lemma *cl-in-subset-carrier*: *cl-in* $\mathcal{E}$ $X \subseteq \mathcal{E}$
using $\mathcal{E}$.*cl-subset-carrier* **unfolding** *cl-in-def* **.**

lemma *cl-in-rank-of*:
assumes $X \subseteq \mathcal{E}$
assumes $x \in$ *cl-in* $\mathcal{E}$ X
shows *rank-of* (*insert* x X) = *rank-of* X
proof −
have $\mathcal{E}$.*rank-of* (*insert* x X) = $\mathcal{E}$.*rank-of* X
using *assms* $\mathcal{E}$.*cl-rank-of* **unfolding** *cl-in-def* **by** *auto*
moreover have $\mathcal{E}$.*rank-of* (*insert* x X) = *rank-of* (*insert* x X)
using *assms rank-of-sub-cong*[*OF* $*$, *of insert* x X] *cl-in-subset-carrier* **by** *auto*
moreover have $\mathcal{E}$.*rank-of* X = *rank-of* X
using *assms rank-of-sub-cong*[*OF* $*$] **by** *auto*
ultimately show *?thesis* **by** *auto*
qed

lemmas *cl-inD* = *cl-in-rank-of cl-in-subset-carrier*

lemma *cl-in-subset*:
assumes $X \subseteq \mathcal{E}$
shows $X \subseteq$ *cl-in* $\mathcal{E}$ X
using $\mathcal{E}$.*cl-subset*[*OF assms*] **unfolding** *cl-in-def* **.**

lemma *cl-in-mono*:
assumes $X \subseteq Y$
assumes $Y \subseteq \mathcal{E}$
shows *cl-in* $\mathcal{E}$ $X \subseteq$ *cl-in* $\mathcal{E}$ Y
using $\mathcal{E}$.*cl-mono*[*OF assms*] **unfolding** *cl-in-def* **.**

lemma *cl-in-insert-absorb*:
assumes $X \subseteq \mathcal{E}$
assumes $x \in$ *cl-in* $\mathcal{E}$ X
shows *cl-in* $\mathcal{E}$ (*insert* x X) = *cl-in* $\mathcal{E}$ X
using *assms* $\mathcal{E}$.*cl-insert-absorb* **unfolding** *cl-in-def* **by** *auto*

lemma *cl-in-augment*:
assumes $X \subseteq \mathcal{E}$

assumes $x \in \mathcal{E}$
assumes $y \in$ *cl-in* $\mathcal{E}$ (*insert* x X) $-$ *cl-in* $\mathcal{E}$ X
shows $x \in$ *cl-in* $\mathcal{E}$ (*insert* y X)
using *assms* $\mathcal{E}$.*cl-augment* **unfolding** *cl-in-def* **by** *auto*

lemmas *cl-inI-insert* = *cl-inI-aux*[*OF* $\mathcal{E}$.*clI-insert*]

end

lemma *cl-in-subI*:
assumes $X \subseteq \mathcal{E}'$ $\mathcal{E}' \subseteq \mathcal{E}$ $\mathcal{E} \subseteq$ *carrier*
shows *cl-in* $\mathcal{E}'$ $X \subseteq$ *cl-in* $\mathcal{E}$ X
proof (*safe*, *goal-cases elem*)
case (*elem* x)
then have $x \in \mathcal{E}'$ *rank-of* (*insert* x X) = *rank-of* X
using *assms* *cl-inD*[**where** $\mathcal{E} = \mathcal{E}'$ **and** $X = X$] **by** *auto*
then show $x \in$ *cl-in* $\mathcal{E}$ X **using** *assms* **by** (*auto intro*: *cl-inI*)
qed

context
fixes $\mathcal{E}$
assumes $*$: $\mathcal{E} \subseteq$ *carrier*
begin

interpretation $\mathcal{E}$: *matroid* $\mathcal{E}$ *indep-in* $\mathcal{E}$
using $*$ **by** *auto*

lemma *cl-in-sub-cong*:
assumes $X \subseteq \mathcal{E}'$ $\mathcal{E}' \subseteq \mathcal{E}$
shows $\mathcal{E}$.*cl-in* $\mathcal{E}'$ X = *cl-in* $\mathcal{E}'$ X
proof (*safe*, *goal-cases LTR RTL*)
case (*LTR* x)
then have $x \in \mathcal{E}'$ $\mathcal{E}$.*rank-of* (*insert* x X) = $\mathcal{E}$.*rank-of* X
using *assms*
$\mathcal{E}$.*cl-in-rank-of*[**where** $\mathcal{E} = \mathcal{E}'$ **and** $X = X$ **and** $x = x$]
$\mathcal{E}$.*cl-in-subset-carrier*[**where** $\mathcal{E} = \mathcal{E}'$]
by *auto*
moreover have $\mathcal{E}$.*rank-of* X = *rank-of* X
using *assms* *rank-of-sub-cong*[*OF* $*$] **by** *auto*
moreover have $\mathcal{E}$.*rank-of* (*insert* x X) = *rank-of* (*insert* x X)
using *assms* *rank-of-sub-cong*[*OF* $*$, *of insert* x X] ‹$x \in \mathcal{E}'$› **by** *auto*
ultimately show *?case* **using** *assms* $*$ **by** (*auto intro*: *cl-inI*)
next
case (*RTL* x)
then have $x \in \mathcal{E}'$ *rank-of* (*insert* x X) = *rank-of* X
using $*$ *assms* *cl-inD*[**where** $\mathcal{E} = \mathcal{E}'$ **and** $X = X$] **by** *auto*
moreover have $\mathcal{E}$.*rank-of* X = *rank-of* X
using *assms* *rank-of-sub-cong*[*OF* $*$] **by** *auto*
moreover have $\mathcal{E}$.*rank-of* (*insert* x X) = *rank-of* (*insert* x X)

using *assms rank-of-sub-cong*[*OF* $*$, *of insert x X*] ‹$x \in \mathcal{E}'$› **by** *auto*
ultimately show *?case* **using** *assms* **by** (*auto intro*: $\mathcal{E}$.*cl-inI*)
qed

end
end
end

References

[1] J. Oxley. What is a matroid?, 2003.