# Hall's Marriage Theorem

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**Abstract**

A proof of Hall's Marriage Theorem due to Halmos and Vaughan [1].

**theory** *Marriage*
**imports** *Main*
**begin**

**theorem** *marriage-necessary*:
  **fixes** $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  **assumes** $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  **and** $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$ (**is** $\exists R.\ ?R\ R\ A\ \&\ ?inj\ R\ A$)
  **shows** $\forall J{\subseteq}I.\ card\ J \leq card\ (\bigcup (A\ `\ J))$
⟨*proof*⟩

The proof by Halmos and Vaughan:

**theorem** *marriage-HV*:
  **fixes** $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  **assumes** $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  **and** $\forall J{\subseteq}I.\ card\ J \leq card\ (\bigcup (A\ `\ J))$ (**is** $?M\ A\ I$)
  **shows** $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$
    (**is** $?SDR\ A\ I$ **is** $\exists R.\ ?R\ R\ A\ I\ \&\ ?inj\ R\ A\ I$)
⟨*proof*⟩

The proof by Rado:

**theorem** *marriage-Rado*:
  **fixes** $A :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
  **assumes** $finite\ I$ **and** $\forall\ i{\in}I.\ finite\ (A\ i)$
  **and** $\forall J{\subseteq}I.\ card\ J \leq card\ (\bigcup (A\ `\ J))$ (**is** $?M\ A$)
  **shows** $\exists R.\ (\forall i{\in}I.\ R\ i \in A\ i) \wedge inj\text{-}on\ R\ I$
    (**is** $?SDR\ A$ **is** $\exists R.\ ?R\ R\ A\ \&\ ?inj\ R\ A$)
⟨*proof*⟩

**end**

## References

[1] P. R. Halmos and H. E. Vaughan. The marriage problem. *American Journal of Mathematics*, 72:214–215, 1950.