Markov Models

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Abstract

This is a formalization of various Markov models in Isabelle/HOL. It builds on Isabelle's probability theory. The available models are currently discrete-time and continuous-time Markov chains as well as Markov decision processes. As application of these models we formalize probabilistic model checking of pCTL formulas, analysis of IPv4 address allocation in ZeroConf and an analysis of the anonymity of the Crowds protocol.

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1 Introduction

This is a formalization of probabilistic models in Isabelle/HOL. It builds on Isabelle's probability theory (HOL-Probability). It provides formalizations for the following models:

- Discrete-time Markov processes with measurable state spaces [2]
- Markov decision processes on discrete spaces [5]
- Continuous-time Markov chains on discrete spaces [2]

As application of these models we formalize

- a probabilistic model checking of pCTL formulas [4],
- an analysis of IPv4 address allocation in ZeroConf [3],
- an analysis of the anonymity of the Crowds protocol [3],
- the reachability analysis on finite-state MDPs [5], and
- expected running-time semantics for pGCL [1].

The formalization of rewarded DTMCs and pCTL model checking is discussed in detail in our paper.

2 Auxiliary Theory

Parts of it should be moved to the Isabelle repository

theory *Markov-Models-Auxiliary*
imports
HOL−Probability.Probability
HOL−Library.Rewrite
HOL−Library.Linear-Temporal-Logic-on-Streams
Coinductive.Coinductive-Stream
Coinductive.Coinductive-Nat
begin

lemma *lfp-upperbound*: $(\bigwedge y.\ x \leq f\ y) \Longrightarrow x \leq lfp\ f$
⟨*proof*⟩

lemma *lfp-arg*: $(\lambda t.\ lfp\ (F\ t)) = lfp\ (\lambda x\ t.\ F\ t\ (x\ t))$
⟨*proof*⟩

lemma *lfp-pair*: $lfp\ (\lambda f\ (a,\ b).\ F\ (\lambda a\ b.\ f\ (a,\ b))\ a\ b)\ (a,\ b) = lfp\ F\ a\ b$
⟨*proof*⟩

lemma *all-Suc-split*: $(\forall i.\ P\ i) \longleftrightarrow (P\ 0 \wedge (\forall i.\ P\ (Suc\ i)))$
⟨*proof*⟩

definition *with* $P\ f\ d = (if\ \exists x.\ P\ x\ then\ f\ (SOME\ x.\ P\ x)\ else\ d)$

lemma *withI*[*case-names default exists*]:
$((\bigwedge x.\ \neg\ P\ x) \Longrightarrow Q\ d) \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ (f\ x)) \Longrightarrow Q\ (with\ P\ f\ d)$
⟨*proof*⟩

context *order*
begin

definition
maximal f S = {*x*∈*S*. ∀ *y*∈*S*. *f y* ≤ *f x*}

lemma *maximalI*: *x* ∈ *S* ⟹ (⋀*y*. *y* ∈ *S* ⟹ *f y* ≤ *f x*) ⟹ *x* ∈ *maximal f S*
⟨*proof*⟩

lemma *maximalI-trans*: *x* ∈ *maximal f S* ⟹ *f x* ≤ *f y* ⟹ *y* ∈ *S* ⟹ *y* ∈ *maximal f S*
⟨*proof*⟩

lemma *maximalD1*: *x* ∈ *maximal f S* ⟹ *x* ∈ *S*
⟨*proof*⟩

lemma *maximalD2*: *x* ∈ *maximal f S* ⟹ *y* ∈ *S* ⟹ *f y* ≤ *f x*
⟨*proof*⟩

lemma *maximal-inject*: *x* ∈ *maximal f S* ⟹ *y* ∈ *maximal f S* ⟹ *f x* = *f y*
⟨*proof*⟩

lemma *maximal-empty*[*simp*]: *maximal f* {} = {}
⟨*proof*⟩

lemma *maximal-singleton*[*simp*]: *maximal f* {*x*} = {*x*}
⟨*proof*⟩

lemma *maximal-in-S*: *maximal f S* ⊆ *S*
⟨*proof*⟩

end

context *linorder*
begin

lemma *maximal-ne*:
assumes *finite S S* ≠ {}
shows *maximal f S* ≠ {}
⟨*proof*⟩

end

lemma *mono-les*:
fixes *s S N* **and** *l1 l2* :: *'a* ⇒ *real* **and** *K* :: *'a* ⇒ *'a pmf*
defines Δ *x* ≡ *l2 x* − *l1 x*
assumes *s*: *s* ∈ *S* **and** *S*: (⋃*s*∈*S*. *set-pmf* (*K s*)) ⊆ *S* ∪ *N*
assumes *int-l1*[*simp*]: ⋀*s*. *s* ∈ *S* ⟹ *integrable* (*K s*) *l1*
assumes *int-l2*[*simp*]: ⋀*s*. *s* ∈ *S* ⟹ *integrable* (*K s*) *l2*
assumes *to-N*: ⋀*s*. *s* ∈ *S* ⟹ ∃ *t*∈*N*. (*s*, *t*) ∈ (*SIGMA s*:*UNIV*. *K s*)*
assumes *l1*: ⋀*s*. *s* ∈ *S* ⟹ (∫ *t*. *l1 t* ∂*K s*) + *c s* ≤ *l1 s*
assumes *l2*: ⋀*s*. *s* ∈ *S* ⟹ *l2 s* ≤ (∫ *t*. *l2 t* ∂*K s*) + *c s*

assumes *eq*: $\bigwedge s.\ s \in N \Longrightarrow l2\ s \leq l1\ s$
assumes *finitary*: *finite* $(\Delta\ `\ (S \cup N))$
shows $l2\ s \leq l1\ s$
⟨*proof*⟩

lemma *unique-les*:
fixes $s\ S\ N$ **and** $l1\ l2 :: 'a \Rightarrow real$ **and** $K :: 'a \Rightarrow 'a\ pmf$
defines $\Delta\ x \equiv l2\ x - l1\ x$
assumes *s*: $s \in S$ **and** *S*: $(\bigcup s{\in}S.\ set\text{-}pmf\ (K\ s)) \subseteq S \cup N$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l1$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l2$
assumes $\bigwedge s.\ s \in S \Longrightarrow \exists\, t{\in}N.\ (s,\ t) \in (SIGMA\ s{:}UNIV.\ K\ s)^*$
assumes $\bigwedge s.\ s \in S \Longrightarrow l1\ s = (\int t.\ l1\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in S \Longrightarrow l2\ s = (\int t.\ l2\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in N \Longrightarrow l2\ s = l1\ s$
assumes *1*: *finite* $(\Delta\ `\ (S \cup N))$
shows $l2\ s = l1\ s$
⟨*proof*⟩

lemma *inf-continuous-suntil-disj*[*order-continuous-intros*]:
assumes *Q*: *inf-continuous Q*
assumes *disj*: $\bigwedge x\ \omega.\ \neg\ (P\ \omega \wedge Q\ x\ \omega)$
shows *inf-continuous* $(\lambda x.\ P\ suntil\ Q\ x)$
⟨*proof*⟩

lemma *inf-continuous-nxt*[*order-continuous-intros*]: *inf-continuous P* $\Longrightarrow$ *inf-continuous* $(\lambda x.\ nxt\ (P\ x)\ \omega)$
⟨*proof*⟩

lemma *sup-continuous-nxt*[*order-continuous-intros*]: *sup-continuous P* $\Longrightarrow$ *sup-continuous* $(\lambda x.\ nxt\ (P\ x)\ \omega)$
⟨*proof*⟩

lemma *mcont-ennreal-of-enat*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ *ennreal-of-enat*
⟨*proof*⟩

lemma *mcont2mcont-ennreal-of-enat*[*cont-intro*]:
mcont lub ord Sup $(\leq)$ *f* $\Longrightarrow$ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ ennreal\text{-}of\text{-}enat\ (f\ x))$
⟨*proof*⟩

declare *stream.exhaust*[*cases type*: *stream*]

lemma *scount-eq-emeasure*: *scount P* ω = *emeasure* (*count-space UNIV*) $\{i.\ P\ (sdrop\ i\ \omega)\}$
⟨*proof*⟩

lemma *measurable-scount*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *scount P* $\in$ *measurable* (*stream-space M*) (*count-space UNIV*)

⟨*proof*⟩

lemma *measurable-sfirst2*:
assumes [*measurable*]: *Measurable.pred* $(N \bigotimes_M$ *stream-space* $M)$ $(\lambda(x, \omega).\ P\ x\ \omega)$
shows $(\lambda(x, \omega).$ *sfirst* $(P\ x)\ \omega) \in$ *measurable* $(N \bigotimes_M$ *stream-space* $M)$ (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-sfirst2'*[*measurable* (*raw*)]:
assumes [*measurable* (*raw*)]: $f \in N \to_M$ *stream-space* M *Measurable.pred* $(N \bigotimes_M$ *stream-space* $M)$ $(\lambda x.\ P$ (*fst* x) (*snd* x))
shows $(\lambda x.$ *sfirst* $(P\ x)\ (f\ x)) \in$ *measurable* N (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-sfirst*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space* M) P
shows *sfirst* $P \in$ *measurable* (*stream-space* M) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-epred*[*measurable*]: *epred* $\in$ *count-space UNIV* $\to_M$ *count-space UNIV*
⟨*proof*⟩

lemma *nn-integral-stretch*:
$f \in$ *borel* $\to_M$ *borel* $\Longrightarrow c \neq 0 \Longrightarrow (\int^+ x.\ f\ (c * x)\ \partial$*lborel*$) = (1\ /\ |c|$::*real*$) * (\int^+ x.\ f\ x\ \partial$*lborel*$)$
⟨*proof*⟩

lemma *prod-sum-distrib*:
fixes $f\ g ::\ 'a \Rightarrow\ 'b \Rightarrow\ 'c$::*comm-semiring-1*
assumes *finite* I **shows** $(\bigwedge i.\ i \in I \Longrightarrow$ *finite* $(J\ i)) \Longrightarrow (\prod i{\in}I.\ \sum j{\in}J\ i.\ f\ i\ j) = (\sum m{\in}Pi_E\ I\ J.\ \prod i{\in}I.\ f\ i\ (m\ i))$
⟨*proof*⟩

lemma *prod-add-distrib*:
fixes $f\ g ::\ 'a \Rightarrow\ 'b$::*comm-semiring-1*
assumes *finite* I **shows** $(\prod i{\in}I.\ f\ i + g\ i) = (\sum J{\in}Pow\ I.\ (\prod i{\in}J.\ f\ i) * (\prod i{\in}I - J.\ g\ i))$
⟨*proof*⟩

subclass (**in** *linordered-nonzero-semiring*) *ordered-semiring-0*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-nonneg*: $(\forall\, a{\in}A.\ 0 \le f\ a) \Longrightarrow 0 \le$ *prod* $f\ A$
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono*:

$\forall i{\in}A.\ 0 \le f\ i \wedge f\ i \le g\ i \Longrightarrow prod\ f\ A \le prod\ g\ A$
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono2*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *g i* ∧ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *1* ≤ *f i*)
shows *prod g I* ≤ *prod f J*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *prod-mono3*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *J* ⟹ *0* ≤ *g i* ⋀*i*. *i* ∈ *I* ⟹ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *g i* ≤ *1*)
shows *prod g J* ≤ *prod f I*
⟨*proof*⟩

lemma (**in** *linordered-nonzero-semiring*) *one-le-prod*: (⋀*i*. *i* ∈ *I* ⟹ *1* ≤ *f i*) ⟹ *1* ≤ *prod f I*
⟨*proof*⟩

lemma *sum-plus-one-le-prod-plus-one*:
fixes *p* :: *'a* ⇒ *'b*::*linordered-nonzero-semiring*
assumes ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *p i*
shows (∑ *i*∈*I*. *p i*) + *1* ≤ (∏ *i*∈*I*. *p i* + *1*)
⟨*proof*⟩

lemma *summable-iff-convergent-prod*:
fixes *p* :: *nat* ⇒ *real* **assumes** *p*: ⋀*i*. *0* ≤ *p i*
shows *summable p* ⟷ *convergent* (λ*n*. ∏ *i*<*n*. *p i* + *1*)
⟨*proof*⟩

primrec *eexp* :: *ereal* ⇒ *ennreal*
where
eexp MInfty = *0*
| *eexp* (*ereal r*) = *ennreal* (*exp r*)
| *eexp PInfty* = *top*

lemma
shows *eexp-minus-infty*[*simp*]: *eexp* (−∞) = *0*
and *eexp-infty*[*simp*]: *eexp* ∞ = *top*
⟨*proof*⟩

lemma *eexp-0*[*simp*]: *eexp 0* = *1*
⟨*proof*⟩

lemma *eexp-inj*[*simp*]: *eexp x* = *eexp y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *eexp-mono*[*simp*]: *eexp x* ≤ *eexp y* ⟷ *x* ≤ *y*
⟨*proof*⟩

lemma *eexp-strict-mono*[*simp*]: *eexp x* < *eexp y* ⟷ *x* < *y*
⟨*proof*⟩

lemma *exp-eq-0-iff*[*simp*]: *eexp x* = *0* ⟷ *x* = −∞
⟨*proof*⟩

lemma *eexp-surj*: *range eexp* = *UNIV*
⟨*proof*⟩

lemma *continuous-on-eexp'*: *continuous-on UNIV eexp*
⟨*proof*⟩

lemma *continuous-on-eexp*[*continuous-intros*]: *continuous-on A f* ⟹ *continuous-on A* (λ*x*. *eexp* (*f x*))
⟨*proof*⟩

lemma *tendsto-eexp*[*tendsto-intros*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *eexp* (*f x*)) ⟶ *eexp x*) *F*
⟨*proof*⟩

lemma *measurable-eexp*[*measurable*]: *eexp* ∈ *borel* →$_M$ *borel*
⟨*proof*⟩

lemma *eexp-add*: ¬ ((*x* = ∞ ∧ *y* = −∞) ∨ (*x* = −∞ ∧ *y* = ∞)) ⟹ *eexp* (*x* + *y*) = *eexp x* ∗ *eexp y*
⟨*proof*⟩

lemma *sum-Pinfty*:
fixes *f* :: *'a* ⇒ *ereal*
shows *sum f I* = ∞ ⟷ (*finite I* ∧ (∃ *i*∈*I*. *f i* = ∞))
⟨*proof*⟩

lemma *sum-Minfty*:
fixes *f* :: *'a* ⇒ *ereal*
shows *sum f I* = −∞ ⟷ (*finite I* ∧ ¬ (∃ *i*∈*I*. *f i* = ∞) ∧ (∃ *i*∈*I*. *f i* = −∞))
⟨*proof*⟩

lemma *eexp-sum*: ¬ (∃ *i*∈*I*. ∃ *j*∈*I*. *f i* = −∞ ∧ *f j* = ∞) ⟹ *eexp* (∑ *i*∈*I*. *f i*) = (∏ *i*∈*I*. *eexp* (*f i*))
⟨*proof*⟩

lemma *eexp-suminf*:
assumes *wf-f*: ¬ {−∞, ∞} ⊆ *range f* **and** *f*: *summable f*
shows (λ*n*. ∏ *i*<*n*. *eexp* (*f i*)) ⟶ *eexp* (∑ *i*. *f i*)
⟨*proof*⟩

lemma *continuous-onI-antimono*:
fixes *f* :: *'a*::*linorder-topology* ⇒ *'b*::{*dense-order*,*linorder-topology*}

assumes *open* (*f*‘*A*)
and *mono*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \leq y \Longrightarrow f\ y \leq f\ x$
shows *continuous-on A f*
⟨*proof*⟩

lemma *minus-add-eq-ereal*: $\neg\ ((a = \infty \wedge b = -\infty) \vee (a = -\infty \wedge b = \infty)) \Longrightarrow -\ (a + b::ereal) = -a - b$
⟨*proof*⟩

lemma *setsum-negf-ereal*: $\neg\ \{-\infty, \infty\} \subseteq f`I \Longrightarrow (\sum i{\in}I.\ -\ f\ i) = -\ (\sum i{\in}I.\ f\ i::ereal)$
⟨*proof*⟩

lemma *convergent-minus-iff-ereal*: *convergent* $(\lambda x.\ -\ f\ x::ereal) \longleftrightarrow$ *convergent f*
⟨*proof*⟩

lemma *summable-minus-ereal*: $\neg\ \{-\infty, \infty\} \subseteq$ *range f* $\Longrightarrow$ *summable* $(\lambda n.\ f\ n) \Longrightarrow$ *summable* $(\lambda n.\ -\ f\ n::ereal)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *product-nn-integral-component*:
assumes $f \in$ *borel-measurable* $(M\ i) i \in I$
shows $integral^N\ (Pi_M\ I\ M)\ (\lambda x.\ f\ (x\ i)) = integral^N\ (M\ i)\ f$
⟨*proof*⟩

lemma *ennreal-inverse-le*[*simp*]: *inverse x* $\leq$ *inverse y* $\longleftrightarrow$ $y \leq (x::ennreal)$
⟨*proof*⟩

lemma *inverse-inverse-ennreal*[*simp*]: *inverse* (*inverse x*::*ennreal*) = *x*
⟨*proof*⟩

lemma *range-inverse-ennreal*: *range inverse* = (*UNIV*::*ennreal set*)
⟨*proof*⟩

lemma *continuous-on-inverse-ennreal*′: *continuous-on* (*UNIV* :: *ennreal set*) *inverse*
⟨*proof*⟩

lemma *sums-minus-ereal*: $\neg\ \{-\ \infty, \infty\} \subseteq f\ `\ UNIV \Longrightarrow (\lambda n.\ -\ f\ n::ereal)$ *sums* $x \Longrightarrow f$ *sums* $-\ x$
⟨*proof*⟩

lemma *suminf-minus-ereal*: $\neg\ \{-\ \infty, \infty\} \subseteq f\ `\ UNIV \Longrightarrow$ *summable f* $\Longrightarrow (\sum n.\ -\ f\ n :: ereal) = -\ suminf\ f$
⟨*proof*⟩

end

3 Discrete-Time Markov Chain

theory *Discrete-Time-Markov-Chain*
imports *Markov-Models-Auxiliary*
begin

Markov chain with discrete time steps and discrete state space.

lemma *sstart-eq'*: *sstart* Ω $(x \# xs) = \{\omega.\ shd\ \omega = x \wedge stl\ \omega \in sstart\ \Omega\ xs\}$
⟨*proof*⟩

lemma *measure-eq-stream-space-coinduct*[*consumes 1*, *case-names left right cont*]:
assumes *R N M*
assumes *R-1*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow N \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))$
and *R-2*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow M \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))$
and *cont*: $\bigwedge N\ M.\ R\ N\ M \Longrightarrow \exists\, N'\ M'\ p.\ (\forall\, y{\in}set\text{-}pmf\ p.\ R\ (N'\ y)\ (M'\ y)) \wedge$
$(\forall\, x.\ N'\ x \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))) \wedge (\forall\, x.\ M'\ x \in space\ (prob\text{-}algebra\ (stream\text{-}space\ (count\text{-}space\ UNIV)))) \wedge$
$N = (measure\text{-}pmf\ p \gg\!= (\lambda y.\ distr\ (N'\ y)\ (stream\text{-}space\ (count\text{-}space\ UNIV))\ ((\#\#)\ y))) \wedge$
$M = (measure\text{-}pmf\ p \gg\!= (\lambda y.\ distr\ (M'\ y)\ (stream\text{-}space\ (count\text{-}space\ UNIV))\ ((\#\#)\ y)))$
shows $N = M$
⟨*proof*⟩

3.1 Discrete Markov Kernel

locale *MC-syntax* =
fixes $K :: 's \Rightarrow 's\ pmf$
begin

abbreviation *acc* :: $('s \times 's)\ set$ **where**
$acc \equiv (SIGMA\ s{:}UNIV.\ K\ s)^*$

abbreviation *acc-on* :: $'s\ set \Rightarrow ('s \times 's)\ set$ **where**
$acc\text{-}on\ S \equiv (SIGMA\ s{:}UNIV.\ K\ s \cap S)^*$

lemma *countable-reachable*: *countable* $(acc\ ``\ \{s\})$
⟨*proof*⟩

lemma *countable-acc*: *countable* $X \Longrightarrow$ *countable* $(acc\ ``\ X)$
⟨*proof*⟩

context
notes [[*inductive-internals*]]
begin

coinductive *enabled* **where**
$enabled\ (shd\ \omega)\ (stl\ \omega) \Longrightarrow shd\ \omega \in K\ s \Longrightarrow enabled\ s\ \omega$

end

lemma *alw-enabled*: *enabled* (*shd* ω) (*stl* ω) $\Longrightarrow$ *alw* ($\lambda\omega$. *enabled* (*shd* ω) (*stl* ω)) ω
⟨*proof*⟩

abbreviation $S \equiv$ *stream-space* (*count-space UNIV*)

lemma *in-S* [*measurable* (*raw*)]: $x \in$ *space* S
⟨*proof*⟩

inductive-simps *enabled-iff*: *enabled* s ω

lemma *enabled-Stream*: *enabled* x (y ## ω) $\longleftrightarrow$ $y \in K\ x \wedge$ *enabled* y ω
⟨*proof*⟩

lemma *measurable-enabled*[*measurable*]:
Measurable.pred (*stream-space* (*count-space UNIV*)) (*enabled* s) (**is** *Measurable.pred ?S* -)
⟨*proof*⟩

lemma *enabled-iff-snth*: *enabled* s ω $\longleftrightarrow$ ($\forall\, i.\ \omega$!! $i \in K$ ((s ## ω) !! i))
⟨*proof*⟩

primcorec *force-enabled* **where**
force-enabled x ω =
(*let* y = *if shd* $\omega \in K\ x$ *then shd* ω *else* (*SOME* y. $y \in K\ x$) *in* y ## *force-enabled* y (*stl* ω))

lemma *force-enabled-in-set-pmf*[*simp*, *intro*]: *shd* (*force-enabled* x ω) $\in K\ x$
⟨*proof*⟩

lemma *enabled-force-enabled*: *enabled* x (*force-enabled* x ω)
⟨*proof*⟩

lemma *force-enabled*: *enabled* x $\omega \Longrightarrow$ *force-enabled* x $\omega = \omega$
⟨*proof*⟩

lemma *Ex-enabled*: $\exists\omega$. *enabled* x ω
⟨*proof*⟩

lemma *measurable-force-enabled*: *force-enabled* $x \in$ *measurable* S S
⟨*proof*⟩

abbreviation $D \equiv$ *stream-space* (Π_M $s \in$ *UNIV*. $K\ s$)

lemma *sets-D*: *sets* D = *sets* (*stream-space* (Π_M $s \in$ *UNIV*. *count-space UNIV*))
⟨*proof*⟩

lemma *space-D*: *space D* = *space* (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*))
⟨*proof*⟩

lemma *measurable-D-D*: *measurable D D* =
measurable (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*)) (*stream-space* (Π_M *s*∈*UNIV*. *count-space UNIV*))
⟨*proof*⟩

primcorec *walk* :: *'s* ⇒ (*'s* ⇒ *'s*) *stream* ⇒ *'s stream* **where**
shd (*walk s ω*) = (*if shd ω s* ∈ *K s then shd ω s else* (*SOME t*. *t* ∈ *K s*))
| *stl* (*walk s ω*) = *walk* (*if shd ω s* ∈ *K s then shd ω s else* (*SOME t*. *t* ∈ *K s*)) (*stl ω*)

lemma *enabled-walk*: *enabled s* (*walk s ω*)
⟨*proof*⟩

lemma *measurable-walk*[*measurable*]: *walk s* ∈ *measurable D S*
⟨*proof*⟩

3.2 Trace Space for Discrete-Time Markov Chains

definition *T* :: *'s* ⇒ *'s stream measure* **where**
T s = *distr* (*stream-space* (Π_M *s*∈*UNIV*. *K s*)) *S* (*walk s*)

lemma *space-T*[*simp*]: *space* (*T s*) = *space S*
⟨*proof*⟩

lemma *sets-T*[*simp*, *measurable-cong*]: *sets* (*T s*) = *sets S*
⟨*proof*⟩

lemma *measurable-T1*[*simp*]: *measurable* (*T s*) *M* = *measurable S M*
⟨*proof*⟩

lemma *measurable-T2*[*simp*]: *measurable M* (*T s*) = *measurable M S*
⟨*proof*⟩

lemma *in-measurable-T1*[*measurable* (*raw*)]: *f* ∈ *measurable S M* ⟹ *f* ∈ *measurable* (*T s*) *M*
⟨*proof*⟩

lemma *in-measurable-T2*[*measurable* (*raw*)]: *f* ∈ *measurable M S* ⟹ *f* ∈ *measurable M* (*T s*)
⟨*proof*⟩

lemma *AE-T-enabled*: *AE ω in T s*. *enabled s ω*
⟨*proof*⟩

sublocale *T*: *prob-space T s* **for** *s*

⟨*proof*⟩

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space S*) = *1*
⟨*proof*⟩

lemma *nn-integral-T*:
assumes *f*[*measurable*]: *f* ∈ *borel-measurable S*
shows (∫⁺*X*. *f X* ∂*T s*) = (∫⁺*t*. (∫⁺ω. *f* (*t* ## ω) ∂*T t*) ∂*K s*)
⟨*proof*⟩

lemma *nn-integral-T-gfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *borel*)
assumes *cont-g*[*THEN inf-continuous-compose*, *order-continuous-intros*]: ⋀*s*. *inf-continuous* (*g s*)
assumes *int-g*: ⋀*f s*. *f* ∈ *borel-measurable S* ⟹ (∫⁺ω. *g s* (*f* ω) ∂*T s*) = *g s* (∫⁺ω. *f* ω ∂*T s*)
assumes *bnd-g*: ⋀*f s*. *g s f* ≤ *b* 0 ≤ *b b* < ∞
shows (∫⁺ω. *gfp l* ω ∂*T s*) = *gfp* (λ*f s*. ∫⁺*t*. *g t* (*f t*) ∂*K s*) *s*
⟨*proof*⟩

lemma *nn-integral-T-lfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *borel*)
assumes *cont-g*[*THEN sup-continuous-compose*, *order-continuous-intros*]: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f s*. *f* ∈ *borel-measurable S* ⟹ (∫⁺ω. *g s* (*f* ω) ∂*T s*) = *g s* (∫⁺ω. *f* ω ∂*T s*)
shows (∫⁺ω. *lfp l* ω ∂*T s*) = *lfp* (λ*f s*. ∫⁺*t*. *g t* (*f t*) ∂*K s*) *s*
⟨*proof*⟩

lemma *emeasure-Collect-T*:
assumes *f*[*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*x*∈*space* (*T s*). *P x*} = (∫⁺*t*. *emeasure* (*T t*) {*x*∈*space* (*T t*). *P* (*t* ## *x*)} ∂*K s*)
⟨*proof*⟩

lemma *AE-T-iff*:
assumes [*measurable*]: *Measurable.pred S P*
shows (*AE* ω *in T x*. *P* ω) ⟷ (∀ *y*∈*K x*. *AE* ω *in T y*. *P* (*y* ## ω))
⟨*proof*⟩

lemma *AE-T-alw*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *P*: ⋀*s*. (*x*, *s*) ∈ *acc* ⟹ *AE* ω *in T s*. *P* ω

shows *AE ω in T x. alw P ω*
⟨*proof*⟩

lemma *emeasure-suntil-disj*:
assumes [*measurable*]: *Measurable.pred S P*
assumes ∗: ⋀*t. AE ω in T t. ¬ (P ⊓ (HLD X ⊓ nxt (HLD X suntil P))) ω*
shows *emeasure (T s) {ω∈space (T s). (HLD X suntil P) ω} =*
lfp (λF s. emeasure (T s) {ω∈space (T s). P ω} + (∫⁺t. F t ∗ indicator X t ∂K s)) s
⟨*proof*⟩

lemma *emeasure-HLD-nxt*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure (T s) {ω∈space (T s). (X · P) ω} =*
(∫⁺x. emeasure (T x) {ω∈space (T x). P ω} ∗ indicator X x ∂K s)
⟨*proof*⟩

lemma *emeasure-HLD*:
emeasure (T s) {ω∈space (T s). HLD X ω} = emeasure (K s) X
⟨*proof*⟩

lemma *emeasure-suntil-HLD*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure (T s) {x∈space (T s). (not (HLD {t}) suntil (HLD {t} aand nxt P)) x} =*
emeasure (T s) {x∈space (T s). ev (HLD {t}) x} ∗ emeasure (T t) {x∈space (T t). P x}
⟨*proof*⟩

lemma *AE-suntil*:
assumes [*measurable*]: *Measurable.pred S P*
shows *(AE x in T s. (not (HLD {t}) suntil (HLD {t} aand nxt P)) x) ⟷*
(AE x in T s. ev (HLD {t}) x) ∧ (AE x in T t. P x)
⟨*proof*⟩

3.3 Fairness

definition *fair* :: *'s ⇒ 's ⇒ 's stream ⇒ bool* **where**
fair s t = alw (ev (HLD {s})) impl alw (ev (HLD {s} aand nxt (HLD {t})))

lemma *AE-T-fair*:
assumes *t' ∈ K t*
shows *AE ω in T s. fair t t' ω*
⟨*proof*⟩

lemma *enabled-imp-trancl*:
assumes *alw (HLD B) ω enabled s ω*
shows *alw (HLD (acc-on B `` {s})) ω*
⟨*proof*⟩

lemma *AE-T-reachable*: $AE\ \omega\ in\ T\ s.\ alw\ (HLD\ (acc\ ``\ \{s\}))\ \omega$
⟨*proof*⟩

lemma *AE-T-all-fair*: $AE\ \omega\ in\ T\ s.\ \forall (t,t')\in SIGMA\ t{:}UNIV.\ K\ t.\ fair\ t\ t'\ \omega$
⟨*proof*⟩

lemma *fair-imp*: **assumes** $fair\ t\ t'\ \omega\ alw\ (ev\ (HLD\ \{t\}))\ \omega$ **shows** $alw\ (ev\ (HLD\ \{t'\}))\ \omega$
⟨*proof*⟩

lemma *AE-T-ev-HLD*:
assumes *exiting*: $\bigwedge t.\ (s,\ t) \in acc\text{-}on\ (-B) \Longrightarrow \exists\, t'\in B.\ (t,\ t') \in acc$
assumes *fin*: *finite* $(acc\text{-}on\ (-B)\ ``\ \{s\})$
shows $AE\ \omega\ in\ T\ s.\ ev\ (HLD\ B)\ \omega$
⟨*proof*⟩

lemma *AE-T-ev-HLD′*:
assumes *exiting*: $\bigwedge s.\ s \notin X \Longrightarrow \exists\, t\in X.\ (s,\ t) \in acc$
assumes *fin*: *finite* $(-X)$
shows $AE\ \omega\ in\ T\ s.\ ev\ (HLD\ X)\ \omega$
⟨*proof*⟩

lemma *AE-T-max-sfirst*:
assumes [*measurable*]: *Measurable.pred S X*
assumes *AE*: $AE\ \omega\ in\ T\ c.\ sfirst\ X\ (c\ \#\#\ \omega) < \infty$ **and** $0 < e$
shows $\exists\, N{::}nat.\ \mathcal{P}(\omega\ in\ T\ c.\ N < sfirst\ X\ (c\ \#\#\ \omega)) < e$ (**is** $\exists N.\ ?P\ N < e$)
⟨*proof*⟩

3.4 First Hitting Time

lemma *nn-integral-sfirst-finite′*:
assumes $s \notin H$
assumes [*simp*]: *finite* $(acc\text{-}on\ (-H)\ ``\ \{s\})$
assumes *until*: $AE\ \omega\ in\ T\ s.\ ev\ (HLD\ H)\ \omega$
shows $(\int^+ \omega.\ sfirst\ (HLD\ H)\ \omega\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *nn-integral-sfirst-finite*:
assumes [*simp*]: *finite* $(acc\text{-}on\ (-H)\ ``\ \{s\})$
assumes *until*: $AE\ \omega\ in\ T\ s.\ ev\ (HLD\ H)\ \omega$
shows $(\int^+ \omega.\ sfirst\ (HLD\ H)\ (s\ \#\#\ \omega)\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *prob-T*:
assumes *P*: *Measurable.pred S P*
shows $\mathcal{P}(\omega\ in\ T\ s.\ P\ \omega) = (\int t.\ \mathcal{P}(\omega\ in\ T\ t.\ P\ (t\ \#\#\ \omega))\ \partial K\ s)$
⟨*proof*⟩

lemma *T-subprob*[*measurable*]: $T \in$ *measurable* (*measure-pmf I*) (*subprob-algebra S*)
⟨*proof*⟩

3.5 Markov chain with Initial Distribution

definition $T' :: 's\ pmf \Rightarrow 's\ stream\ measure$ **where**
$T'\ I = bind\ I\ (\lambda s.\ distr\ (T\ s)\ S\ ((\#\#)\ s))$

lemma *distr-Stream-subprob*:
$(\lambda s.\ distr\ (T\ s)\ S\ ((\#\#)\ s)) \in measurable\ (measure\text{-}pmf\ I)\ (subprob\text{-}algebra\ S)$
⟨*proof*⟩

lemma *sets-T'*: $sets\ (T'\ I) = sets\ S$
⟨*proof*⟩

lemma *prob-space-T'*: *prob-space* $(T'\ I)$
⟨*proof*⟩

lemma *AE-T'*:
assumes [*measurable*]: *Measurable.pred S P*
shows $(AE\ x\ in\ T'\ I.\ P\ x) \longleftrightarrow (\forall s \in I.\ AE\ x\ in\ T\ s.\ P\ (s\ \#\#\ x))$
⟨*proof*⟩

lemma *emeasure-T'*:
assumes [*measurable*]: $X \in sets\ S$
shows $emeasure\ (T'\ I)\ X = (\int^+ s.\ emeasure\ (T\ s)\ \{\omega \in space\ S.\ s\ \#\#\ \omega \in X\}\ \partial I)$
⟨*proof*⟩

lemma *prob-T'*:
assumes [*measurable*]: *Measurable.pred S P*
shows $\mathcal{P}(x\ in\ T'\ I.\ P\ x) = (\int s.\ \mathcal{P}(x\ in\ T\ s.\ P\ (s\ \#\#\ x))\ \partial I)$
⟨*proof*⟩

lemma *T-eq-T'*: $T\ s = T'\ (K\ s)$
⟨*proof*⟩

lemma *T-eq-bind*: $T\ s = (measure\text{-}pmf\ (K\ s) \gg\!= (\lambda t.\ distr\ (T\ t)\ S\ ((\#\#)\ t)))$
⟨*proof*⟩

lemma *T-split*:
$T\ s = (T\ s \gg\!= (\lambda \omega.\ distr\ (T\ ((s\ \#\#\ \omega)\ !!\ n))\ S\ (\lambda \omega'.\ stake\ n\ \omega\ @-\ \omega')))$
⟨*proof*⟩

lemma *nn-integral-T-split*:
assumes *f*[*measurable*]: $f \in borel\text{-}measurable\ S$
shows $(\int^+ \omega.\ f\ \omega\ \partial T\ s) = (\int^+ \omega.\ (\int^+ \omega'.\ f\ (stake\ n\ \omega\ @-\ \omega')\ \partial T\ ((s\ \#\#\ \omega)\ !!\ n))\ \partial T\ s)$

⟨*proof*⟩

lemma *emeasure-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*ω*∈*space* (*T s*). *P ω*} =
(∫⁺*ω*. *emeasure* (*T* ((*s* ## *ω*) !! *n*)) {*ω*′∈*space* (*T* ((*s* ## *ω*) !! *n*)). *P* (*stake n ω* @− *ω*′)} *∂T s*)
⟨*proof*⟩

lemma *prob-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows *P*(*ω in T s*. *P ω*) = (∫ *ω*. *P*(*ω*′ *in T* ((*s* ## *ω*) !! *n*). *P* (*stake n ω* @− *ω*′)) *∂T s*)
⟨*proof*⟩

lemma *enabled-imp-alw*:
(⋃*s*∈*X*. *set-pmf* (*K s*)) ⊆ *X* ⟹ *x* ∈ *X* ⟹ *enabled x ω* ⟹ *alw* (*HLD X*) *ω*
⟨*proof*⟩

lemma *alw-HLD-iff-sconst*:
alw (*HLD* {*x*}) *ω* ⟷ *ω* = *sconst x*
⟨*proof*⟩

lemma *enabled-iff-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*} **shows** *enabled x ω* ⟷ *ω* = *sconst x*
⟨*proof*⟩

lemma *AE-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*}
shows (*AE ω in T x*. *P ω*) ⟷ *P* (*sconst x*)
⟨*proof*⟩

lemma *ev-eq-lfp*: *ev P* = *lfp* (*λF ω*. *P ω* ∨ (¬ *P ω* ∧ *F* (*stl ω*)))
⟨*proof*⟩

lemma *INF-eq-zero-iff-ennreal*: ((∏ *i*∈*A*. *f i*) = (*0*::*ennreal*)) = (∀ *x*>*0*. ∃ *i*∈*A*. *f i* < *x*)
⟨*proof*⟩

lemma *inf-continuous-cmul*:
fixes *c* :: *ennreal*
assumes *f*: *inf-continuous f* **and** *c*: *c* < ⊤
shows *inf-continuous* (*λx*. *c* ∗ *f x*)
⟨*proof*⟩

lemma *AE-T-ev-HLD-infinite*:
fixes *X* :: *'s set* **and** *r* :: *real*
assumes *r* < *1*
assumes *r*: ⋀*x*. *x* ∈ *X* ⟹ *measure* (*K x*) *X* ≤ *r*

shows *AE* ω *in* *T x*. *ev* (*HLD* (− *X*)) ω
⟨*proof*⟩

3.6 Trace space with Restriction

definition *rT x* = *restrict-space* (*T x*) {ω. *enabled x* ω}

lemma *space-rT*: $\omega \in$ *space* (*rT x*) ⟷ *enabled x* ω
⟨*proof*⟩

lemma *Collect-enabled-S*[*measurable*]: *Collect* (*enabled x*) ∈ *sets S*
⟨*proof*⟩

lemma *space-rT-in-S*: *space* (*rT x*) ∈ *sets S*
⟨*proof*⟩

lemma *sets-rT*: *A* ∈ *sets* (*rT x*) ⟷ *A* ∈ *sets S* ∧ *A* ⊆ {ω. *enabled x* ω}
⟨*proof*⟩

lemma *prob-space-rT*: *prob-space* (*rT x*)
⟨*proof*⟩

lemma *measurable-force-enabled2*[*measurable*]: *force-enabled x* ∈ *measurable S* (*rT x*)
⟨*proof*⟩

lemma *space-rT-not-empty*[*simp*]: *space* (*rT x*) ≠ {}
⟨*proof*⟩

lemma *T-eq-bind'*: *T x* = *do* { *y* ← *measure-pmf* (*K x*) ; ω ← *T y* ; *return S* (*y* ## ω) }
⟨*proof*⟩

lemma *rT-eq-bind*: *rT x* = *do* { *y* ← *measure-pmf* (*K x*) ; ω ← *rT y* ; *return* (*rT x*) (*y* ## ω) }
⟨*proof*⟩

lemma *snth-rT*: (λ*x*. *x* !! *n*) ∈ *measurable* (*rT x*) (*count-space* (*acc* `` {*x*}))
⟨*proof*⟩

3.7 Bisimulation

lemma *T-coinduct*[*consumes 1*, *case-names prob sets cont*]:
assumes *R x M*
assumes *prob*: ⋀*x M*. *R x M* ⟹ *prob-space M*
and *sets*: ⋀*x M*. *R x M* ⟹ *sets M* = *sets S*
and *cont'*: ⋀*x M*. *R x M* ⟹ ∃ *M'*. (∀ *y*∈*K x*. *R y* (*M' y*)) ∧ (∀ *y*. *sets* (*M' y*) = *S* ∧ *prob-space* (*M' y*)) ∧
M = (*measure-pmf* (*K x*) ⤜ (λ*y*. *distr* (*M' y*) *S* ((##) *y*)))
shows *T x* = *M*

⟨*proof*⟩

lemma *T-bisim*:
assumes *M*: ⋀*x*. *prob-space* (*M x*) ⋀*x*. *sets* (*M x*) = *sets S*
and *M-eq*: ⋀*x*. *M x* = (*measure-pmf* (*K x*) ⤜ (λ*s*. *distr* (*M s*) *S* ((##) *s*)))
shows *T* = *M*
⟨*proof*⟩

lemma *T-subprob'*[*measurable*]: *T* ∈ *measurable* (*count-space UNIV*) (*subprob-algebra S*)
⟨*proof*⟩

lemma *T-subprob''*[*simp*]: *T a* ∈ *space* (*subprob-algebra S*)
⟨*proof*⟩

lemma *AE-not-suntil-coinduct* [*consumes 1*, *case-names ψ φ*]:
assumes *P s*
assumes *ψ*: ⋀*s*. *P s* ⟹ *s* ∉ *ψ*
assumes *φ*: ⋀*s t*. *P s* ⟹ *s* ∈ *φ* ⟹ *t* ∈ *K s* ⟹ *P t*
shows *AE ω in T s*. *not* (*HLD φ suntil HLD ψ*) (*s ## ω*)
⟨*proof*⟩

lemma *AE-not-suntil-coinduct-strong* [*consumes 1*, *case-names ψ φ*]:
assumes *P s*
assumes *P-ψ*: ⋀*s*. *P s* ⟹ *s* ∉ *ψ*
assumes *P-φ*: ⋀*s t*. *P s* ⟹ *s* ∈ *φ* ⟹ *t* ∈ *K s* ⟹ *P t* ∨
(*AE ω in T t*. *not* (*HLD φ suntil HLD ψ*) (*t ## ω*))
shows *AE ω in T s*. *not* (*HLD φ suntil HLD ψ*) (*s ## ω*) (**is** *?nuntil s*)
⟨*proof*⟩

end

3.8 Reward Structure on Markov Chains

locale *MC-with-rewards* = *MC-syntax K* **for** *K* :: *'s* ⇒ *'s pmf* +
fixes *ι* :: *'s* ⇒ *'s* ⇒ *ennreal* **and** *ϱ* :: *'s* ⇒ *ennreal*
assumes *ι-nonneg*: ⋀*s t*. *0* ≤ *ι s t* **and** *ϱ-nonneg*: ⋀*s*. *0* ≤ *ϱ s*
assumes *measurable-ι*[*measurable*]: (λ(*a*, *b*). *ι a b*) ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *count-space UNIV*)
begin

definition *reward-until* :: *'s set* ⇒ *'s* ⇒ *'s stream* ⇒ *ennreal* **where**
reward-until X = *lfp* (λ*F s ω*. *if s* ∈ *X then 0 else ϱ s* + *ι s* (*shd ω*) + (*F* (*shd ω*) (*stl ω*)))

lemma *measurable-ϱ*[*measurable*]: *ϱ* ∈ *borel-measurable* (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-reward-until*[*measurable* (*raw*)]:

assumes [*measurable*]: $f \in$ *measurable* M (*count-space UNIV*)
assumes [*measurable*]: $g \in$ *measurable* M S
shows $(\lambda x.$ *reward-until* X $(f\ x)$ $(g\ x)) \in$ *borel-measurable* M
⟨*proof*⟩

lemma *continuous-reward-until*:
sup-continuous $(\lambda F\ s\ \omega.$ *if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ $(shd\ \omega) + (F$ $(shd\ \omega)$ $(stl\ \omega)))$
⟨*proof*⟩

lemma
shows *reward-until-unfold*: *reward-until* $X\ s\ \omega =$
(if $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ $(shd\ \omega) +$ *reward-until* X $(shd\ \omega)$ $(stl\ \omega))$
(**is** *?unfold*)
⟨*proof*⟩

lemma *reward-until-simps*[*simp*]:
shows $s \in X \Longrightarrow$ *reward-until* $X\ s\ \omega = 0$
and $s \notin X \Longrightarrow$ *reward-until* $X\ s\ \omega = \varrho\ s + \iota\ s$ $(shd\ \omega) +$ *reward-until* X $(shd\ \omega)$ $(stl\ \omega)$
⟨*proof*⟩

lemma *reward-until-SCons*[*simp*]:
reward-until $X\ s$ $(t\ \#\#\ \omega) =$ *(if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s\ t +$ *reward-until* X $t\ \omega)$
⟨*proof*⟩

lemma *nn-integral-reward-until-finite*:
assumes [*simp*]: *finite* $(acc\ ``\ \{s\})$ (**is** *finite* $(?R\ ``\ \{s\}))$
assumes ϱ: $\bigwedge t.\ (s,\ t) \in$ *acc-on* $(-H) \Longrightarrow \varrho\ t < \infty$
assumes ι: $\bigwedge t\ t'.\ (s,\ t) \in$ *acc-on* $(-H) \Longrightarrow t' \in K\ t \Longrightarrow \iota\ t\ t' < \infty$
assumes *ev*: *AE* ω *in* T $s.$ *ev* $(HLD\ H)\ \omega$
shows $(\int^+ \omega.$ *reward-until* $H\ s\ \omega\ \partial T\ s) \neq \infty$
⟨*proof*⟩

end

3.9 Bisimulation on a relation

definition *rel-set-strong* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where *rel-set-strong* $R\ A\ B \longleftrightarrow (\forall x\ y.\ R\ x\ y \longrightarrow (x \in A \longleftrightarrow y \in B))$

lemma *T-eq-rel-half*[*consumes 4*, *case-names prob sets cont*]:
fixes $R :: 's \Rightarrow 't \Rightarrow bool$ **and** $f :: 's \Rightarrow 't$ **and** $S :: 's\ set$
assumes *R-def*: $\bigwedge s\ t.\ R\ s\ t \longleftrightarrow (s \in S \wedge f\ s = t)$
assumes *A*[*measurable*]: $A \in sets$ $(stream\text{-}space$ $(count\text{-}space\ UNIV))$
and *B*[*measurable*]: $B \in sets$ $(stream\text{-}space$ $(count\text{-}space\ UNIV))$
and *AB*: *rel-set-strong* $(stream\text{-}all2\ R)$ A B **and** *KL*: *rel-fun* R $(rel\text{-}pmf\ R)$ K L **and** *xy*: $R\ x\ y$

shows *MC-syntax.T K x A = MC-syntax.T L y B*
⟨*proof*⟩

3.10 Product Construction

locale *MC-pair* =
K1: *MC-syntax K1* + *K2*: *MC-syntax K2* **for** *K1 K2*
begin

definition *Kp* ≡ λ(*a*, *b*). *pair-pmf* (*K1 a*) (*K2 b*)

sublocale *MC-syntax Kp* ⟨*proof*⟩

definition
$szip_E$ *a b* ≡ λ(*ω1*, *ω2*). *szip* (*K1.force-enabled a ω1*) (*K2.force-enabled b ω2*)

lemma *szip-rT*[*measurable*]: (λ(*ω1*, *ω2*). *szip ω1 ω2*) ∈ *measurable* (*K1.rT x1* ⨂$_M$ *K2.rT x2*) *S*
⟨*proof*⟩

lemma *measurable-szipE*[*measurable*]: $szip_E$ *a b* ∈ *measurable* (*K1.S* ⨂$_M$ *K2.S*) *S*
⟨*proof*⟩

lemma *T-eq-prod*: *T* = (λ(*x1*, *x2*). *do* { *ω1* ← *K1.T x1* ; *ω2* ← *K2.T x2* ; *return S* ($szip_E$ *x1 x2* (*ω1*, *ω2*)) })
(**is** - = *?B*)
⟨*proof*⟩

lemma *nn-integral-pT*:
fixes *f* **assumes** [*measurable*]: *f* ∈ *borel-measurable S*
shows (∫$^+$*ω*. *f ω* ∂*T* (*x*, *y*)) = (∫$^+$*ω1*. ∫$^+$*ω2*. *f* ($szip_E$ *x y* (*ω1*, *ω2*)) ∂*K2.T y* ∂*K1.T x*)
⟨*proof*⟩

lemma *prod-eq-prob-T*:
assumes [*measurable*]: *Measurable.pred K1.S P1 Measurable.pred K2.S P2*
shows $\mathcal{P}$(*ω in K1.T x1*. *P1 ω*) ∗ $\mathcal{P}$(*ω in K2.T x2*. *P2 ω*) =
$\mathcal{P}$(*ω in T* (*x1*, *x2*). *P1* (*smap fst ω*) ∧ *P2* (*smap snd ω*))
⟨*proof*⟩

end

end

3.11 Trace Space equal to Markov Chains

theory *Trace-Space-Equals-Markov-Processes*
imports *Discrete-Time-Markov-Chain*
begin

We can construct for each time-homogeneous discrete-time Markov chain a corresponding probability space using *Markov-Models.Discrete-Time-Markov-Chain*. The constructed probability space has the same probabilities.

locale *Time-Homogeneous-Discrete-Markov-Process = M?: prob-space +*
fixes $S :: 's\ set$ **and** $X :: nat \Rightarrow 'a \Rightarrow 's$
assumes X [*measurable*]: $\bigwedge t.\ X\ t \in measurable\ M\ (count\text{-}space\ UNIV)$
assumes S: $countable\ S\ \bigwedge n.\ AE\ x\ in\ M.\ X\ n\ x \in S$
assumes MC: $\bigwedge n\ s\ s'.$
$\mathcal{P}(\omega\ in\ M.\ \forall\, t{\leq}n.\ X\ t\ \omega = s\ t\ ) \neq 0 \Longrightarrow$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid \forall\, t{\leq}n.\ X\ t\ \omega = s\ t\ ) =$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid X\ n\ \omega = s\ n\ )$
assumes TH: $\bigwedge n\ m\ s\ t.$
$\mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = t) \neq 0 \Longrightarrow \mathcal{P}(\omega\ in\ M.\ X\ m\ \omega = t) \neq 0 \Longrightarrow$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s \mid X\ n\ \omega = t) = \mathcal{P}(\omega\ in\ M.\ X\ (Suc\ m)\ \omega = s \mid X\ m\ \omega = t)$
begin

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lift-definition $I :: 's\ pmf$ **is** $distr\ M\ (count\text{-}space\ UNIV)\ (X\ 0)$
⟨*proof*⟩

lemma *I-in-S*:
assumes $pmf\ I\ s \neq 0$ **shows** $s \in S$
⟨*proof*⟩

lift-definition $K :: 's \Rightarrow 's\ pmf$ **is**
$\lambda s.\ with\ (\lambda n.\ \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s) \neq 0)$
$(\lambda n.\ distr\ (uniform\text{-}measure\ M\ \{\omega{\in}space\ M.\ X\ n\ \omega = s\})\ (count\text{-}space\ UNIV)\ (X\ (Suc\ n)))$
$(uniform\text{-}measure\ (count\text{-}space\ UNIV)\ \{s\})$
⟨*proof*⟩

lemma *pmf-K*:
assumes n: $0 < \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s)$
shows $pmf\ (K\ s)\ t = \mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = t \mid X\ n\ \omega = s)$
⟨*proof*⟩

lemma *pmf-K2*:
$(\bigwedge n.\ \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s) = 0) \Longrightarrow pmf\ (K\ s)\ t = indicator\ \{t\}\ s$
⟨*proof*⟩

end

sublocale K: *MC-syntax K* ⟨*proof*⟩

lemma *bind-I-K-eq-M*: $K.T'\ I = distr\ M\ K.S\ (\lambda\omega.\ to\text{-}stream\ (\lambda n.\ X\ n\ \omega))$ (**is** *- = ?D*)
⟨*proof*⟩

end

lemma (**in** *MC-syntax*) *is-THDTMC*:
fixes $I :: 's\ pmf$
defines $U \equiv (SIGMA\ s{:}UNIV.\ K\ s)^*\ ``\ I$
shows *Time-Homogeneous-Discrete-Markov-Process* $(T'\ I)\ U\ (\lambda n\ \omega.\ \omega\ !!\ n)$
⟨*proof*⟩

end

4 Classifying Markov Chain States

theory *Classifying-Markov-Chain-States*
imports
HOL−Computational-Algebra.Group-Closure
Discrete-Time-Markov-Chain
begin

lemma *eventually-mult-Gcd*:
fixes $S :: nat\ set$
assumes S: $\bigwedge s\ t.\ s \in S \Longrightarrow t \in S \Longrightarrow s + t \in S$
assumes s: $s \in S\ s > 0$
shows *eventually* $(\lambda m.\ m * Gcd\ S \in S)$ *sequentially*
⟨*proof*⟩

context *MC-syntax*
begin

4.1 Expected number of visits

definition $G\ s\ t = (\int^+\omega.\ scount\ (HLD\ \{t\})\ (s\ \#\#\ \omega)\ \partial T\ s)$

lemma *G-eq*: $G\ s\ t = (\int^+\omega.\ emeasure\ (count\text{-}space\ UNIV)\ \{i.\ (s\ \#\#\ \omega)\ !!\ i = t\}\ \partial T\ s)$
⟨*proof*⟩

definition $p\ s\ t\ n = \mathcal{P}(\omega\ in\ T\ s.\ (s\ \#\#\ \omega)\ !!\ n = t)$

definition *gf-G* $s\ t\ z = (\sum n.\ p\ s\ t\ n *_R z \hat{}\ n)$

definition *convergence-G* $s\ t\ z \longleftrightarrow summable\ (\lambda n.\ p\ s\ t\ n * norm\ z \hat{}\ n)$

lemma *p-nonneg*[*simp*]: $0 \le p\ x\ y\ n$
⟨*proof*⟩

lemma *p-le-1*: $p\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma *p-x-x-0*[*simp*]: $p\ x\ x\ 0 = 1$
⟨*proof*⟩

lemma *p-0*: $p\ x\ y\ 0 = (\mathit{if}\ x = y\ \mathit{then}\ 1\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *p-in-reachable*: **assumes** $(x, y) \notin (\mathit{SIGMA}\ x{:}\mathit{UNIV}.\ K\ x)^*$ **shows** $p\ x\ y\ n = 0$
⟨*proof*⟩

lemma *p-Suc*: $\mathit{ennreal}\ (p\ x\ y\ (\mathit{Suc}\ n)) = (\int^+ w.\ p\ w\ y\ n\ \partial K\ x)$
⟨*proof*⟩

lemma *p-Suc'*:
$p\ x\ y\ (\mathit{Suc}\ n) = (\int x'.\ p\ x'\ y\ n\ \partial K\ x)$
⟨*proof*⟩

lemma *p-add*: $p\ x\ y\ (n + m) = (\int^+ w.\ p\ x\ w\ n * p\ w\ y\ m\ \partial \mathit{count\text{-}space}\ \mathit{UNIV})$
⟨*proof*⟩

lemma *prob-reachable-le*:
assumes [*simp*]: $m \leq n$
shows $p\ x\ y\ m * p\ y\ w\ (n - m) \leq p\ x\ w\ n$
⟨*proof*⟩

lemma *G-eq-suminf*: $G\ x\ y = (\sum i.\ \mathit{ennreal}\ (p\ x\ y\ i))$
⟨*proof*⟩

lemma *G-eq-real-suminf*:
$\mathit{convergence\text{-}G}\ x\ y\ (1{::}\mathit{real}) \Longrightarrow G\ x\ y = \mathit{ennreal}\ (\sum i.\ p\ x\ y\ i)$
⟨*proof*⟩

lemma *convergence-norm-G*:
$\mathit{convergence\text{-}G}\ x\ y\ z \Longrightarrow \mathit{summable}\ (\lambda n.\ p\ x\ y\ n * \mathit{norm}\ z\ \widehat{}\ n)$
⟨*proof*⟩

lemma *convergence-G*:
$\mathit{convergence\text{-}G}\ x\ y\ (z{::}'a{::}\{\mathit{banach}, \mathit{real\text{-}normed\text{-}div\text{-}algebra}\}) \Longrightarrow \mathit{summable}\ (\lambda n.\ p\ x\ y\ n *_R z\ \widehat{}\ n)$
⟨*proof*⟩

lemma *convergence-G-less-1*:
fixes $z :: - :: \{\mathit{banach}, \mathit{real\text{-}normed\text{-}field}\}$
assumes z: $\mathit{norm}\ z < 1$ **shows** $\mathit{convergence\text{-}G}\ x\ y\ z$
⟨*proof*⟩

lemma *lim-gf-G*: $((\lambda z.\ ennreal\ (gf\text{-}G\ x\ y\ z)) \longrightarrow G\ x\ y)\ (at\text{-}left\ (1::real))$
⟨*proof*⟩

4.2 Reachability probability

definition $u\ x\ y\ n = \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ n\ \omega)$

definition $U\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{t\})\ \omega)$

definition $gf\text{-}U\ x\ y\ z = (\sum n.\ u\ x\ y\ n *_R z \mathbin{\hat{}} Suc\ n)$

definition $f\ x\ y\ n = \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ n\ (x\ \#\#\ \omega))$

definition $F\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{t\})\ (s\ \#\#\ \omega))$

definition $gf\text{-}F\ x\ y\ z = (\sum n.\ f\ x\ y\ n * z \mathbin{\hat{}} n)$

lemma *f-Suc*: $x \neq y \Longrightarrow f\ x\ y\ (Suc\ n) = u\ x\ y\ n$
⟨*proof*⟩

lemma *f-Suc-eq*: $f\ x\ x\ (Suc\ n) = 0$
⟨*proof*⟩

lemma *f-0*: $f\ x\ y\ 0 = (if\ x = y\ then\ 1\ else\ 0)$
⟨*proof*⟩

lemma shows *u-nonneg*: $0 \leq u\ x\ y\ n$ **and** *u-le-1*: $u\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma shows *f-nonneg*: $0 \leq f\ x\ y\ n$ **and** *f-le-1*: $f\ x\ y\ n \leq 1$
⟨*proof*⟩

lemma *U-nonneg*[*simp*]: $0 \leq U\ x\ y$
⟨*proof*⟩

lemma *U-le-1*: $U\ s\ t \leq 1$
⟨*proof*⟩

lemma *U-cases*: $U\ s\ s = 1 \vee U\ s\ s < 1$
⟨*proof*⟩

lemma *u-sums-U*: $u\ x\ y\ sums\ U\ x\ y$
⟨*proof*⟩

lemma *gf-U-eq-U*: $gf\text{-}U\ x\ y\ 1 = U\ x\ y$
⟨*proof*⟩

lemma *f-sums-F*: $f\ x\ y\ sums\ F\ x\ y$
⟨*proof*⟩

lemma *F-nonneg*[*simp*]: $0 \leq F\ x\ y$
⟨*proof*⟩

lemma *F-le-1*: $F\ x\ y \leq 1$
⟨*proof*⟩

lemma *gf-F-eq-F*: $gf\text{-}F\ x\ y\ 1 = F\ x\ y$
⟨*proof*⟩

lemma *gf-F-le-1*:
fixes $z :: real$
assumes z: $0 \leq z\ z \leq 1$
shows $gf\text{-}F\ x\ y\ z \leq 1$
⟨*proof*⟩

lemma *u-le-p*: $u\ x\ y\ n \leq p\ x\ y\ (Suc\ n)$
⟨*proof*⟩

lemma *f-le-p*: $f\ x\ y\ n \leq p\ x\ y\ n$
⟨*proof*⟩

lemma *convergence-norm-U*:
fixes $z :: - :: real\text{-}normed\text{-}div\text{-}algebra$
assumes z: $convergence\text{-}G\ x\ y\ z$
shows $summable\ (\lambda n.\ u\ x\ y\ n * norm\ z \mathbin{\hat{}} Suc\ n)$
⟨*proof*⟩

lemma *convergence-norm-F*:
fixes $z :: - :: real\text{-}normed\text{-}div\text{-}algebra$
assumes z: $convergence\text{-}G\ x\ y\ z$
shows $summable\ (\lambda n.\ f\ x\ y\ n * norm\ z \mathbin{\hat{}} n)$
⟨*proof*⟩

lemma *gf-G-nonneg*:
fixes $z :: real$
shows $0 \leq z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}G\ x\ y\ z$
⟨*proof*⟩

lemma *gf-F-nonneg*:
fixes $z :: real$
shows $0 \leq z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}F\ x\ y\ z$
⟨*proof*⟩

lemma *convergence-U*:
fixes $z :: - :: banach$
shows $convergence\text{-}G\ x\ y\ z \Longrightarrow summable\ (\lambda n.\ u\ x\ y\ n * z \mathbin{\hat{}} Suc\ n)$
⟨*proof*⟩

lemma *p-eq-sum-p-u*: $p\ x\ y\ (Suc\ n) = (\sum i{\leq}n.\ p\ y\ y\ (n - i) * u\ x\ y\ i)$
⟨*proof*⟩

lemma *p-eq-sum-p-f*: $p\ x\ y\ n = (\sum i{\leq}n.\ p\ y\ y\ (n - i) * f\ x\ y\ i)$
⟨*proof*⟩

lemma *gf-G-eq-gf-F*:
assumes *z*: $norm\ z < 1$
shows $gf\text{-}G\ x\ y\ z = gf\text{-}F\ x\ y\ z * gf\text{-}G\ y\ y\ z$
⟨*proof*⟩

lemma *gf-G-eq-gf-U*:
fixes $z :: 'z :: \{banach, real\text{-}normed\text{-}field\}$
assumes *z*: *convergence-G x x z*
shows $gf\text{-}G\ x\ x\ z = 1\ /\ (1 - gf\text{-}U\ x\ x\ z)\ gf\text{-}U\ x\ x\ z \neq 1$
⟨*proof*⟩

lemma *gf-U*: $(gf\text{-}U\ x\ y \longrightarrow U\ x\ y)\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *gf-U-le-1*: **assumes** *z*: $0 < z\ z < 1$ **shows** $gf\text{-}U\ x\ y\ z \leq (1{::}real)$
⟨*proof*⟩

lemma *gf-F*: $(gf\text{-}F\ x\ y \longrightarrow F\ x\ y)\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *U-bounded*: $0 \leq U\ x\ y\ U\ x\ y \leq 1$
⟨*proof*⟩

4.3 Recurrent states

definition *recurrent* :: $'s \Rightarrow bool$ **where**
$recurrent\ s \longleftrightarrow (AE\ \omega\ in\ T\ s.\ ev\ (HLD\ \{s\})\ \omega)$

lemma *recurrent-iff-U-eq-1*: $recurrent\ s \longleftrightarrow U\ s\ s = 1$
⟨*proof*⟩

definition $H\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ alw\ (ev\ (HLD\ \{t\}))\ \omega)$

lemma *H-eq*:
$recurrent\ s \longleftrightarrow H\ s\ s = 1$
$\neg\ recurrent\ s \longleftrightarrow H\ s\ s = 0$
$H\ s\ t = U\ s\ t * H\ t\ t$
⟨*proof*⟩

lemma *recurrent-iff-G-infinite*: $recurrent\ x \longleftrightarrow G\ x\ x = \infty$
⟨*proof*⟩

definition *communicating* :: $('s \times 's)\ set$ **where**

communicating = *acc* ∩ *acc*$^{-1}$

definition *essential-class* :: *'s set* ⇒ *bool* **where**
essential-class C ⟷ *C* ∈ *UNIV* // *communicating* ∧ *acc* `` *C* ⊆ *C*

lemma *accI-U*:
assumes *0* < *U x y* **shows** (*x*, *y*) ∈ *acc*
⟨*proof*⟩

lemma *accD-pos*:
assumes (*x*, *y*) ∈ *acc*
shows ∃ *n*. *0* < *p x y n*
⟨*proof*⟩

lemma *accI-pos*: *0* < *p x y n* ⟹ (*x*, *y*) ∈ *acc*
⟨*proof*⟩

lemma *recurrent-iffI-communicating*:
assumes (*x*, *y*) ∈ *communicating*
shows *recurrent x* ⟷ *recurrent y*
⟨*proof*⟩

lemma *recurrent-acc*:
assumes *recurrent x* (*x*, *y*) ∈ *acc*
shows *U y x* = *1 H y x* = *1 recurrent y* (*x*, *y*) ∈ *communicating*
⟨*proof*⟩

lemma *equiv-communicating*: *equiv UNIV communicating*
⟨*proof*⟩

lemma *recurrent-class*:
assumes *recurrent x*
shows *acc* `` {*x*} = *communicating* `` {*x*}
⟨*proof*⟩

lemma *irreduccible-recurrent-class*:
assumes *recurrent x* **shows** *acc* `` {*x*} ∈ *UNIV* // *communicating*
⟨*proof*⟩

lemma *essential-classI*:
assumes *C*: *C* ∈ *UNIV* // *communicating*
assumes *eq*: ⋀*x y*. *x* ∈ *C* ⟹ (*x*, *y*) ∈ *acc* ⟹ *y* ∈ *C*
shows *essential-class C*
⟨*proof*⟩

lemma *essential-recurrent-class*:
assumes *recurrent x* **shows** *essential-class* (*communicating* `` {*x*})
⟨*proof*⟩

lemma *essential-classD2*:
essential-class $C \Longrightarrow x \in C \Longrightarrow (x, y) \in acc \Longrightarrow y \in C$
⟨*proof*⟩

lemma *essential-classD3*:
essential-class $C \Longrightarrow x \in C \Longrightarrow y \in C \Longrightarrow (x, y) \in communicating$
⟨*proof*⟩

lemma *AE-acc*:
shows $AE\ \omega\ in\ T\ x.\ \forall m.\ (x, (x\ \#\#\ \omega)\ !!\ m) \in acc$
⟨*proof*⟩

lemma *finite-essential-class-imp-recurrent*:
assumes *C*: *essential-class C finite C* **and** *x*: $x \in C$
shows *recurrent x*
⟨*proof*⟩

lemma *irreducibleD*:
$C \in UNIV\ //\ communicating \Longrightarrow a \in C \Longrightarrow b \in C \Longrightarrow (a, b) \in communicating$
⟨*proof*⟩

lemma *irreducibleD2*:
$C \in UNIV\ //\ communicating \Longrightarrow a \in C \Longrightarrow (a, b) \in communicating \Longrightarrow b \in C$
⟨*proof*⟩

lemma *essential-class-iff-recurrent*:
$finite\ C \Longrightarrow C \in UNIV\ //\ communicating \Longrightarrow essential\text{-}class\ C \longleftrightarrow (\forall x{\in}C.\ recurrent\ x)$
⟨*proof*⟩

definition $U'\ x\ y = (\int{}^{+}\omega.\ eSuc\ (sfirst\ (HLD\ \{y\})\ \omega)\ \partial T\ x)$

lemma *U′-neq-zero*[*simp*]: $U'\ x\ y \neq 0$
⟨*proof*⟩

definition *gf-U′* $x\ y\ z = (\sum n.\ u\ x\ y\ n * Suc\ n * z\ \widehat{}\ n)$

definition *pos-recurrent* $x \longleftrightarrow recurrent\ x \wedge U'\ x\ x \neq \infty$

lemma *summable-gf-U′*:
assumes *z*: $norm\ z < 1$
shows $summable\ (\lambda n.\ u\ x\ y\ n * Suc\ n * z\ \widehat{}\ n)$
⟨*proof*⟩

lemma *gf-U′-nonneg*[*simp*]: $0 < z \Longrightarrow z < 1 \Longrightarrow 0 \leq gf\text{-}U'\ x\ y\ z$
⟨*proof*⟩

lemma *DERIV-gf-U*:

fixes z :: *real* **assumes** z: $0 < z\ z < 1$
shows *DERIV* (*gf-U x y*) z :> *gf-U′ x y z*
⟨*proof*⟩

lemma *sfirst-finiteI-recurrent*:
recurrent x ⟹ $(x, y) \in$ *acc* ⟹ *AE ω in T x. sfirst* (*HLD* {*y*}) $\omega < \infty$
⟨*proof*⟩

lemma *U′-eq-suminf*:
assumes x: *recurrent x* $(x, y) \in$ *acc*
shows *U′ x y* = $(\sum i.$ *ennreal* (*u x y i* ∗ *Suc i*))
⟨*proof*⟩

lemma *gf-U′-tendsto-U′*:
assumes x: *recurrent x* $(x, y) \in$ *acc*
shows $((\lambda z.$ *ennreal* (*gf-U′ x y z*)) ⟶ *U′ x y*) (*at-left 1*)
⟨*proof*⟩

lemma *one-le-integral-t*:
assumes x: *recurrent x* **shows** $1 \leq$ *U′ x x*
⟨*proof*⟩

lemma *gf-U′-pos*:
fixes z :: *real*
assumes z: $0 < z\ z < 1$ **and** *U x y* ≠ *0*
shows *0* < *gf-U′ x y z*
⟨*proof*⟩

lemma *inverse-gf-U′-tendsto*:
assumes *recurrent y*
shows $((\lambda x. -1 / -$ *gf-U′ y y x*) ⟶ *enn2real* (*1 / U′ y y*)) (*at-left* (*1*::*real*))
⟨*proof*⟩

lemma *gf-G-pos*:
fixes z :: *real*
assumes z: $0 < z\ z < 1$ **and** ∗: $(x, y) \in$ *acc*
shows *0* < *gf-G x y z*
⟨*proof*⟩

lemma *pos-recurrentI-communicating*:
assumes y: *pos-recurrent y* **and** x: $(y, x) \in$ *communicating*
shows *pos-recurrent x*
⟨*proof*⟩

lemma *pos-recurrent-iffI-communicating*:
$(y, x) \in$ *communicating* ⟹ *pos-recurrent y* ⟷ *pos-recurrent x*
⟨*proof*⟩

lemma *U-le-F*: *U x y* ≤ *F x y*

⟨*proof*⟩

lemma *not-empty-irreducible*: $C \in UNIV\ //\ communicating \Longrightarrow C \neq \{\}$
⟨*proof*⟩

4.4 Stationary distribution

definition *stat* :: *'s set* ⇒ *'s measure* **where**
stat C = *point-measure UNIV* (λ*x*. *indicator C x* / *U' x x*)

lemma *sets-stat*[*simp*]: *sets* (*stat C*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *space-stat*[*simp*]: *space* (*stat C*) = *UNIV*
⟨*proof*⟩

lemma *stat-subprob*:
assumes *C*: *essential-class C* **and** *countable C* **and** *pos*: ∀ *c*∈*C*. *pos-recurrent c*
shows *emeasure* (*stat C*) *C* ≤ *1*
⟨*proof*⟩

lemma *emeasure-stat-not-C*:
assumes *y* ∉ *C*
shows *emeasure* (*stat C*) {*y*} = *0*
⟨*proof*⟩

definition *stationary-distribution* :: *'s pmf* ⇒ *bool* **where**
stationary-distribution N ⟷ *N* = *bind-pmf N K*

lemma *stationary-distributionI*:
assumes *le*: ⋀*y*. (∫ *x*. *pmf* (*K x*) *y* ∂*measure-pmf N*) ≤ *pmf N y*
shows *stationary-distribution N*
⟨*proof*⟩

lemma *stationary-distribution-iterate*:
assumes *N*: *stationary-distribution N*
shows *ennreal* (*pmf N y*) = (∫ $^+$*x*. *p x y n* ∂*N*)
⟨*proof*⟩

lemma *stationary-distribution-iterate'*:
assumes *stationary-distribution N*
shows *measure N* {*y*} = (∫ *x*. *p x y n* ∂*N*)
⟨*proof*⟩

lemma *stationary-distributionD*:
assumes *C*: *essential-class C countable C*
assumes *N*: *stationary-distribution N N* ⊆ *C*
shows ∀ *x*∈*C*. *pos-recurrent x measure-pmf N* = *stat C*
⟨*proof*⟩

lemma *measure-point-measure-singleton*:
$x \in A \Longrightarrow$ *measure* (*point-measure A X*) $\{x\}$ = *enn2real* (*X x*)
⟨*proof*⟩

lemma *stationary-distribution-imp-int-t*:
assumes *C*: *essential-class C countable C stationary-distribution N N* $\subseteq$ *C*
assumes *x*: $x \in C$ **shows** *U' x x* = *1 / ennreal* (*pmf N x*)
⟨*proof*⟩

definition *period-set x* = $\{i.\ 0 < i \wedge 0 < p\ x\ x\ i\ \}$
definition *period C* = (*SOME d*. $\forall x{\in}C.\ d$ = *Gcd* (*period-set x*))

lemma *Gcd-period-set-invariant*:
assumes *c*: $(x, y) \in$ *communicating*
shows *Gcd* (*period-set x*) = *Gcd* (*period-set y*)
⟨*proof*⟩

lemma *period-eq*:
assumes *C* $\in$ *UNIV // communicating x* $\in$ *C*
shows *period C* = *Gcd* (*period-set x*)
⟨*proof*⟩

definition *aperiodic C* $\longleftrightarrow$ *C* $\in$ *UNIV // communicating* $\wedge$ *period C* = *1*

definition *not-ephemeral C* $\longleftrightarrow$ *C* $\in$ *UNIV // communicating* $\wedge$ $\neg$ ($\exists x.$ *C* = $\{x\}$ $\wedge$ *p x x 1* = *0*)

lemma *not-ephemeralD*:
assumes *C*: *not-ephemeral C x* $\in$ *C*
shows $\exists\, n{>}0.\ 0 < p\ x\ x\ n$
⟨*proof*⟩

lemma *not-ephemeralD-pos-period*:
assumes *C*: *not-ephemeral C*
shows *0* < *period C*
⟨*proof*⟩

lemma *period-posD*:
assumes *C*: *C* $\in$ *UNIV // communicating* **and** *0* < *period C x* $\in$ *C*
shows $\exists\, n{>}0.\ 0 < p\ x\ x\ n$
⟨*proof*⟩

lemma *not-ephemeralD-pos-period'*:
assumes *C*: *C* $\in$ *UNIV // communicating*
shows *not-ephemeral C* $\longleftrightarrow$ *0* < *period C*
⟨*proof*⟩

lemma *eventually-periodic*:
assumes *C*: $C \in UNIV$ // *communicating* $0 < period\ C\ x \in C$
shows *eventually* $(\lambda m.\ 0 < p\ x\ x\ (m * period\ C))$ *sequentially*
⟨*proof*⟩

lemma *aperiodic-eventually-recurrent*:
$aperiodic\ C \longleftrightarrow C \in UNIV$ // *communicating* $\wedge$ $(\forall x{\in}C.\ eventually\ (\lambda m.\ 0 < p\ x\ x\ m)\ sequentially)$
⟨*proof*⟩

lemma *stationary-distributionD-emeasure*:
assumes *N*: *stationary-distribution N*
shows $emeasure\ N\ A = (\int^+ s.\ emeasure\ (K\ s)\ A\ \partial N)$
⟨*proof*⟩

lemma *communicatingD1*:
$C \in UNIV$ // $communicating \Longrightarrow (a,\ b) \in communicating \Longrightarrow a \in C \Longrightarrow b \in C$
⟨*proof*⟩

lemma *communicatingD2*:
$C \in UNIV$ // $communicating \Longrightarrow (a,\ b) \in communicating \Longrightarrow b \in C \Longrightarrow a \in C$
⟨*proof*⟩

lemma *acc-iff*: $(x,\ y) \in acc \longleftrightarrow (\exists n.\ 0 < p\ x\ y\ n)$
⟨*proof*⟩

lemma *communicating-iff*: $(x,\ y) \in communicating \longleftrightarrow (\exists n.\ 0 < p\ x\ y\ n) \wedge (\exists n.\ 0 < p\ y\ x\ n)$
⟨*proof*⟩

end

context *MC-pair*
begin

lemma *p-eq-p1-p2*:
$p\ (x1,\ x2)\ (y1,\ y2)\ n = K1.p\ x1\ y1\ n * K2.p\ x2\ y2\ n$
⟨*proof*⟩

lemma *P-accD*:
assumes $((x1,\ x2),\ (y1,\ y2)) \in acc$**shows** $(x1,\ y1) \in K1.acc\ (x2,\ y2) \in K2.acc$
⟨*proof*⟩

lemma *aperiodicI-pair*:
assumes *C1*: *K1.aperiodic C1* **and** *C2*: *K2.aperiodic C2*

shows *aperiodic* (*C1* × *C2*)
⟨*proof*⟩

lemma *stationary-distributionI-pair*:
assumes *N1*: *K1.stationary-distribution N1*
assumes *N2*: *K2.stationary-distribution N2*
shows *stationary-distribution* (*pair-pmf N1 N2*)
⟨*proof*⟩

end

context *MC-syntax*
begin

lemma *stationary-distribution-imp-limit*:
assumes *C*: *aperiodic C essential-class C countable C* **and** *N*: *stationary-distribution N N* ⊆ *C*
assumes [*simp*]: *y* ∈ *C*
shows (λ*n*. ∫ *x*. |*p y x n* − *pmf N x*| ∂*count-space C*) ⟶ *0*
(**is** *?L* ⟶ *0*)
⟨*proof*⟩

lemma *stationary-distribution-imp-p-limit*:
assumes *aperiodic C essential-class C* **and** [*simp*]: *countable C*
assumes *N*: *stationary-distribution N N* ⊆ *C*
assumes [*simp*]: *x* ∈ *C y* ∈ *C*
shows *p x y* ⟶ *pmf N y*
⟨*proof*⟩

end

lemma (**in** *MC-syntax*) *essential-classI2*:
assumes *X* ≠ {}
assumes *accI*: ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *X* ⟹ (*x*, *y*) ∈ *acc*
assumes *ED*: ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *set-pmf* (*K x*) ⟹ *y* ∈ *X*
shows *essential-class X*
⟨*proof*⟩

end

5 Markov Decision Processes

theory *Markov-Decision-Process*
imports *Discrete-Time-Markov-Chain*
begin

lemma *some-elem-ne*: *s* ≠ {} ⟹ *some-elem s* ∈ *s*
⟨*proof*⟩

5.1 Configurations

We want to construct a *non-free* codatatype *'s cfg = Cfg (state: 's) (action: 's pmf) (cont: 's ⇒ 's cfg).* with the restriction *state (cont cfg s) = s*

hide-const *cont*

codatatype *'s scheduler = Scheduler (action-sch: 's pmf) (cont-sch: 's ⇒ 's scheduler)*

lemma *equivp-rel-prod: equivp R ⟹ equivp Q ⟹ equivp (rel-prod R Q)*
⟨*proof*⟩

coinductive *eq-scheduler :: 's scheduler ⇒ 's scheduler ⇒ bool*
where
⋀D. action-sch sc1 = D ⟹ action-sch sc2 = D ⟹
(∀ s∈D. eq-scheduler (cont-sch sc1 s) (cont-sch sc2 s)) ⟹ eq-scheduler sc1 sc2

lemma *eq-scheduler-refl[intro]: eq-scheduler sc sc*
⟨*proof*⟩

quotient-type *'s cfg = 's × 's scheduler / rel-prod (=) eq-scheduler*
⟨*proof*⟩

lift-definition *state :: 's cfg ⇒ 's* **is** *fst*
⟨*proof*⟩

lift-definition *action :: 's cfg ⇒ 's pmf* **is** *λ(s, sc). action-sch sc*
⟨*proof*⟩

lift-definition *cont :: 's cfg ⇒ 's ⇒ 's cfg* **is**
λ(s, sc) t. if t ∈ action-sch sc then (t, cont-sch sc t) else
(t, cont-sch sc (some-elem (action-sch sc)))
⟨*proof*⟩

lift-definition *Cfg :: 's ⇒ 's pmf ⇒ ('s ⇒ 's cfg) ⇒ 's cfg* **is**
λs D c. (s, Scheduler D (λt. snd (c t)))
⟨*proof*⟩

lift-definition *cfg-corec :: 's ⇒ ('a ⇒ 's pmf) ⇒ ('a ⇒ 's ⇒ 'a) ⇒ 'a ⇒ 's cfg*
is
λs D C x. (s, corec-scheduler D (λx s. Inr (C x s)) x) ⟨*proof*⟩

lemma *state-cont[simp]: state (cont cfg s) = s*
⟨*proof*⟩

lemma *state-Cfg[simp]: state (Cfg s d' c') = s*
⟨*proof*⟩

lemma *action-Cfg[simp]: action (Cfg s d' c') = d'*

⟨*proof*⟩

lemma *cont-Cfg*[*simp*]: $t \in$ *set-pmf* $d' \Longrightarrow$ *state* $(c'\ t) = t \Longrightarrow$ *cont* $(Cfg\ s\ d'\ c')\ t = c'\ t$
⟨*proof*⟩

lemma *state-cfg-corec*[*simp*]: *state* $(cfg\text{-}corec\ s\ d\ c\ x) = s$
⟨*proof*⟩

lemma *action-cfg-corec*[*simp*]: *action* $(cfg\text{-}corec\ s\ d\ c\ x) = d\ x$
⟨*proof*⟩

lemma *cont-cfg-corec*[*simp*]: $t \in$ *set-pmf* $(d\ x) \Longrightarrow$ *cont* $(cfg\text{-}corec\ s\ d\ c\ x)\ t = cfg\text{-}corec\ t\ d\ c\ (c\ x\ t)$
⟨*proof*⟩

lemma *cfg-coinduct*[*consumes 1*, *case-names state action cont*, *coinduct pred*]:
$X\ c\ d \Longrightarrow (\bigwedge c\ d.\ X\ c\ d \Longrightarrow state\ c = state\ d) \Longrightarrow (\bigwedge c\ d.\ X\ c\ d \Longrightarrow action\ c = action\ d) \Longrightarrow$
$(\bigwedge c\ d\ t.\ X\ c\ d \Longrightarrow t \in set\text{-}pmf\ (action\ c) \Longrightarrow X\ (cont\ c\ t)\ (cont\ d\ t)) \Longrightarrow c = d$
⟨*proof*⟩

coinductive *rel-cfg* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ cfg \Rightarrow 'b\ cfg \Rightarrow bool$ **for** $P :: 'a \Rightarrow 'b \Rightarrow bool$
where
$P\ (state\ cfg1)\ (state\ cfg2) \Longrightarrow$
rel-pmf $(\lambda s\ t.\ rel\text{-}cfg\ P\ (cont\ cfg1\ s)\ (cont\ cfg2\ t))\ (action\ cfg1)\ (action\ cfg2) \Longrightarrow$
rel-cfg P cfg1 cfg2

lemma *rel-cfg-state*: *rel-cfg P cfg1 cfg2* $\Longrightarrow$ *P* $(state\ cfg1)\ (state\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-cont*:
rel-cfg P cfg1 cfg2 $\Longrightarrow$
rel-pmf $(\lambda s\ t.\ rel\text{-}cfg\ P\ (cont\ cfg1\ s)\ (cont\ cfg2\ t))\ (action\ cfg1)\ (action\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-action*:
assumes *P*: *rel-cfg P cfg1 cfg2* **shows** *rel-pmf P* $(action\ cfg1)\ (action\ cfg2)$
⟨*proof*⟩

lemma *rel-cfg-eq*: *rel-cfg* $(=)$ *cfg1 cfg2* $\longleftrightarrow$ *cfg1* $=$ *cfg2*
⟨*proof*⟩

5.2 Configuration with Memoryless Scheduler

definition *memoryless-on f s* $=$ *cfg-corec s f* $(\lambda\text{-}\ t.\ t)\ s$

lemma
shows *state-memoryless-on*[*simp*]: *state* (*memoryless-on f s*) = *s*
and *action-memoryless-on*[*simp*]: *action* (*memoryless-on f s*) = *f s*
and *cont-memoryless-on*[*simp*]: *t* ∈ (*f s*) ⟹ *cont* (*memoryless-on f s*) *t* = *memoryless-on f t*
⟨*proof*⟩

definition *K-cfg* :: *'s cfg* ⇒ *'s cfg pmf* **where**
K-cfg cfg = *map-pmf* (*cont cfg*) (*action cfg*)

lemma *set-K-cfg*: *set-pmf* (*K-cfg cfg*) = *cont cfg* ‘ *set-pmf* (*action cfg*)
⟨*proof*⟩

lemma *nn-integral-K-cfg*: (∫⁺*cfg*. *f cfg* ∂*K-cfg cfg*) = (∫⁺*s*. *f* (*cont cfg s*) ∂*action cfg*)
⟨*proof*⟩

5.3 MDP Kernel and Induced Configurations

locale *Markov-Decision-Process* =
fixes *K* :: *'s* ⇒ *'s pmf set*
assumes *K-wf*: ⋀*s*. *K s* ≠ {}
begin

definition *E* = (*SIGMA s*:*UNIV*. ⋃*D*∈*K s*. *set-pmf D*)

coinductive *cfg-onp* :: *'s* ⇒ *'s cfg* ⇒ *bool* **where**
⋀*s*. *state cfg* = *s* ⟹ *action cfg* ∈ *K s* ⟹ (⋀*t*. *t* ∈ *action cfg* ⟹ *cfg-onp t* (*cont cfg t*)) ⟹
cfg-onp s cfg

definition *cfg-on s* = {*cfg*. *cfg-onp s cfg*}

lemma
shows *cfg-onD-action*[*intro*, *simp*]: *cfg* ∈ *cfg-on s* ⟹ *action cfg* ∈ *K s*
and *cfg-onD-cont*[*intro*, *simp*]: *cfg* ∈ *cfg-on s* ⟹ *t* ∈ *action cfg* ⟹ *cont cfg t* ∈ *cfg-on t*
and *cfg-onD-state*[*simp*]: *cfg* ∈ *cfg-on s* ⟹ *state cfg* = *s*
and *cfg-onI*: *state cfg* = *s* ⟹ *action cfg* ∈ *K s* ⟹ (⋀*t*. *t* ∈ *action cfg* ⟹ *cont cfg t* ∈ *cfg-on t*) ⟹ *cfg* ∈ *cfg-on s*
⟨*proof*⟩

lemma *cfg-on-coinduct*[*coinduct set*: *cfg-on*]:
assumes *P s cfg*
assumes ⋀*cfg s*. *P s cfg* ⟹ *state cfg* = *s*
assumes ⋀*cfg s*. *P s cfg* ⟹ *action cfg* ∈ *K s*
assumes ⋀*cfg s t*. *P s cfg* ⟹ *t* ∈ *action cfg* ⟹ *P t* (*cont cfg t*)
shows *cfg* ∈ *cfg-on s*

⟨*proof*⟩

lemma *memoryless-on-cfg-onI*:
assumes $\bigwedge s.\ f\ s \in K\ s$
shows *memoryless-on f s* ∈ *cfg-on s*
⟨*proof*⟩

lemma *cfg-of-cfg-onI*:
$D \in K\ s \Longrightarrow (\bigwedge t.\ t \in D \Longrightarrow c\ t \in$ *cfg-on t*$) \Longrightarrow$ *Cfg s D c* ∈ *cfg-on s*
⟨*proof*⟩

definition *arb-act s* = (*SOME D*. $D \in K\ s$)

lemma *arb-actI*[*simp*]: *arb-act s* ∈ *K s*
⟨*proof*⟩

lemma *cfg-on-not-empty*[*intro*, *simp*]: *cfg-on s* ≠ {}
⟨*proof*⟩

sublocale *MC*: *MC-syntax K-cfg* ⟨*proof*⟩

abbreviation *St* :: *'s stream measure* **where**
St ≡ *stream-space* (*count-space UNIV*)

5.4 Trace Space

definition *T cfg* = *distr* (*MC.T cfg*) *St* (*smap state*)

sublocale *T*: *prob-space T cfg* **for** *cfg*
⟨*proof*⟩

lemma *space-T*[*simp*]: *space* (*T cfg*) = *space St*
⟨*proof*⟩

lemma *sets-T*[*simp*]: *sets* (*T cfg*) = *sets St*
⟨*proof*⟩

lemma *measurable-T1*[*simp*]: *measurable* (*T cfg*) *N* = *measurable St N*
⟨*proof*⟩

lemma *measurable-T2*[*simp*]: *measurable N* (*T cfg*) = *measurable N St*
⟨*proof*⟩

lemma *nn-integral-T*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows $(\int^+ X.\ f\ X\ \partial T\ cfg) = (\int^+ cfg'.\ (\int^+ x.\ f\ (state\ cfg' \#\#\ x)\ \partial T\ cfg')\ \partial K\text{-}cfg\ cfg)$
⟨*proof*⟩

lemma *T-eq*:
T cfg = (*measure-pmf* (*K-cfg cfg*) ⤜ (λ*cfg'*. *distr* (*T cfg'*) *St* (λω. *state cfg'* ## ω)))
⟨*proof*⟩

lemma *T-memoryless-on*: *T* (*memoryless-on ct s*) = *MC-syntax.T ct s*
⟨*proof*⟩

lemma *nn-integral-T-lfp*:
assumes [*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *borel*)
assumes *cont-g*: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f cfg*. *f* ∈ *borel-measurable* (*stream-space* (*count-space UNIV*))
⟹
(∫$^+$ω. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (∫$^+$ω. *f* ω ∂*T cfg*)
shows (∫$^+$ω. *lfp* (λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))) ω ∂*T cfg*) =
lfp (λ*f cfg*. ∫$^+$*t*. *g* (*state t*) (*f t*) ∂*K-cfg cfg*) *cfg*
⟨*proof*⟩

lemma *emeasure-Collect-T*:
assumes [*measurable*]: *Measurable.pred St P*
shows *emeasure* (*T cfg*) {*x*∈*space St*. *P x*} =
(∫$^+$*cfg'*. *emeasure* (*T cfg'*) {*x*∈*space St*. *P* (*state cfg'* ## *x*)} ∂*K-cfg cfg*)
⟨*proof*⟩

definition *E-sup* :: *'s* ⇒ (*'s stream* ⇒ *ennreal*) ⇒ *ennreal*
where
E-sup s f = (⨆ *cfg*∈*cfg-on s*. ∫$^+$*x*. *f x* ∂*T cfg*)

lemma *E-sup-const*: *0* ≤ *c* ⟹ *E-sup s* (λ-. *c*) = *c*
⟨*proof*⟩

lemma *E-sup-mult-right*:
assumes [*measurable*]: *f* ∈ *borel-measurable St* **and** [*simp*]: *0* ≤ *c*
shows *E-sup s* (λ*x*. *c* ∗ *f x*) = *c* ∗ *E-sup s f*
⟨*proof*⟩

lemma *E-sup-mono*:
(⋀ω. *f* ω ≤ *g* ω) ⟹ *E-sup s f* ≤ *E-sup s g*
⟨*proof*⟩

lemma *E-sup-add*:
assumes [*measurable*]: *f* ∈ *borel-measurable St g* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *g x*) ≤ *E-sup s f* + *E-sup s g*
⟨*proof*⟩

lemma *E-sup-add-left*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *c*) = *E-sup s f* + *c*

⟨*proof*⟩

lemma *E-sup-add-right*:
f ∈ *borel-measurable St* ⟹ *E-sup s* (λ*x*. *c* + *f x*) = *c* + *E-sup s f*
⟨*proof*⟩

lemma *E-sup-SUP*:
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel-measurable St* **and** [*simp*]: *incseq f*
shows *E-sup s* (λ*x*. ⨆ *i*. *f i x*) = (⨆ *i*. *E-sup s* (*f i*))
⟨*proof*⟩

lemma *E-sup-iterate*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows *E-sup s f* = (⨆ *D*∈*K s*. ∫⁺ *t*. *E-sup t* (λω. *f* (*t* ## ω)) ∂*measure-pmf D*)
⟨*proof*⟩

lemma *E-sup-bot*: *E-sup s* ⊥ = *0*
⟨*proof*⟩

lemma *E-sup-lfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes *measurable-g*[*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *borel*)
assumes *cont-g*: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f cfg*. *f* ∈ *borel-measurable St* ⟹
(∫⁺ ω. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (*integral*^{N} (*T cfg*) *f*)
shows (λ*s*. *E-sup s* (*lfp l*)) = *lfp* (λ*f s*. ⨆ *D*∈*K s*. ∫⁺*t*. *g t* (*f t*) ∂*measure-pmf D*)
⟨*proof*⟩

definition *P-sup s P* = (⨆ *cfg*∈*cfg-on s*. *emeasure* (*T cfg*) {*x*∈*space St*. *P x*})

lemma *P-sup-eq-E-sup*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-sup s P* = *E-sup s* (*indicator* {*x*∈*space St*. *P x*})
⟨*proof*⟩

lemma *P-sup-True*[*simp*]: *P-sup t* (λω. *True*) = *1*
⟨*proof*⟩

lemma *P-sup-False*[*simp*]: *P-sup t* (λω. *False*) = *0*
⟨*proof*⟩

lemma *P-sup-SUP*:
fixes *P* :: *nat* ⇒ *'s stream* ⇒ *bool*
assumes *mono P* **and** *P*[*measurable*]: ⋀*i*. *Measurable.pred St* (*P i*)
shows *P-sup s* (λ*x*. ∃ *i*. *P i x*) = (⨆ *i*. *P-sup s* (*P i*))

⟨*proof*⟩

lemma *P-sup-lfp*:
assumes *Q*: *sup-continuous Q*
assumes *f*: $f \in$ *measurable St M*
assumes *Q-m*: $\bigwedge P$. *Measurable.pred M P* $\Longrightarrow$ *Measurable.pred M* (*Q P*)
shows *P-sup s* (λx. *lfp Q* (*f x*)) = ($\bigsqcup i$. *P-sup s* (λx. (*Q* $\widehat{\ }\widehat{\ }$ *i*) $\bot$ (*f x*)))
⟨*proof*⟩

lemma *P-sup-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-sup s P* = ($\bigsqcup D{\in}K$ *s*. $\int^+$ *t*. *P-sup t* ($\lambda\omega$. *P* (*t* ## ω)) ∂*measure-pmf D*)
⟨*proof*⟩

definition *E-inf s f* = ($\bigsqcap$ *cfg*∈*cfg-on s*. $\int^+ x$. *f x* ∂*T cfg*)

lemma *E-inf-const*: $0 \le c \Longrightarrow$ *E-inf s* (λ-. *c*) = *c*
⟨*proof*⟩

lemma *E-inf-mono*:
($\bigwedge\omega$. *f* $\omega \le$ *g* ω) $\Longrightarrow$ *E-inf s f* $\le$ *E-inf s g*
⟨*proof*⟩

lemma *E-inf-iterate*:
assumes [*measurable*]: $f \in$ *borel-measurable St*
shows *E-inf s f* = ($\bigsqcap D{\in}K$ *s*. $\int^+$ *t*. *E-inf t* ($\lambda\omega$. *f* (*t* ## ω)) ∂*measure-pmf D*)
⟨*proof*⟩

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space St*) = *1*
⟨*proof*⟩

lemma *E-inf-greatest*:
($\bigwedge$*cfg*. *cfg* $\in$ *cfg-on s* $\Longrightarrow x \le$ ($\int^+ x$. *f x* ∂*T cfg*)) $\Longrightarrow x \le$ *E-inf s f*
⟨*proof*⟩

lemma *E-inf-lower2*:
cfg $\in$ *cfg-on s* $\Longrightarrow$ ($\int^+ x$. *f x* ∂*T cfg*) $\le x \Longrightarrow$ *E-inf s f* $\le x$
⟨*proof*⟩

Maybe the following statement can be generalized to infinite *K s*.

lemma *E-inf-lfp*:
fixes *g*
defines *l* $\equiv \lambda f\ \omega$. *g* (*shd* ω) (*f* (*stl* ω))
assumes *measurable-g*[*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*: $\bigwedge s$. *sup-continuous* (*g s*)
assumes *int-g*: $\bigwedge f$ *cfg*. $f \in$ *borel-measurable St* $\Longrightarrow$
($\int^+ \omega$. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (*integral*^{N} (*T cfg*) *f*)

assumes *K-finite*: $\bigwedge s.$ *finite* $(K\ s)$
shows $(\lambda s.$ *E-inf* s $(lfp\ l)) = lfp\ (\lambda f\ s.\ \prod D{\in}K\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial$*measure-pmf* $D)$
⟨*proof*⟩

definition *P-inf* $s\ P = (\prod cfg{\in}$*cfg-on* $s.$ *emeasure* $(T\ cfg)\ \{x{\in}$*space St*. $P\ x\})$

lemma *P-inf-eq-E-inf*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf* $s\ P =$ *E-inf* s (*indicator* $\{x{\in}$*space St*. $P\ x\})$
⟨*proof*⟩

lemma *P-inf-True*[*simp*]: *P-inf* $t\ (\lambda\omega.$ *True*$) = 1$
⟨*proof*⟩

lemma *P-inf-False*[*simp*]: *P-inf* $t\ (\lambda\omega.$ *False*$) = 0$
⟨*proof*⟩

lemma *P-inf-INF*:
fixes $P ::$ *nat* $\Rightarrow$ *'s stream* $\Rightarrow$ *bool*
assumes *decseq* P **and** P[*measurable*]: $\bigwedge i.$ *Measurable.pred St* $(P\ i)$
shows *P-inf* $s\ (\lambda x.\ \forall\, i.\ P\ i\ x) = (\prod i.$ *P-inf* $s\ (P\ i))$
⟨*proof*⟩

lemma *P-inf-gfp*:
assumes *Q*: *inf-continuous Q*
assumes *f*: $f \in$ *measurable St M*
assumes *Q-m*: $\bigwedge P.$ *Measurable.pred M P* $\Longrightarrow$ *Measurable.pred M* $(Q\ P)$
shows *P-inf* $s\ (\lambda x.$ *gfp* $Q\ (f\ x)) = (\prod i.$ *P-inf* $s\ (\lambda x.\ (Q\ \frown\ i)\ \top\ (f\ x)))$
⟨*proof*⟩

lemma *P-inf-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf* $s\ P = (\prod D{\in}K\ s.\ \int^+ t.$ *P-inf* $t\ (\lambda\omega.\ P\ (t\ \#\#\ \omega))\ \partial$*measure-pmf* $D)$
⟨*proof*⟩

end

5.5 Finite MDPs

locale *Finite-Markov-Decision-Process* = *Markov-Decision-Process* K **for** $K ::$ *'s* $\Rightarrow$ *'s pmf set* +
fixes $S ::$ *'s set*
assumes *S-not-empty*: $S \neq \{\}$
assumes *S-finite*: *finite S*
assumes *K-closed*: $\bigwedge s.\ s \in S \Longrightarrow (\bigcup D{\in}K\ s.$ *set-pmf* $D) \subseteq S$
assumes *K-finite*: $\bigwedge s.\ s \in S \Longrightarrow$ *finite* $(K\ s)$
begin

lemma *action-closed*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow t \in action\ cfg \Longrightarrow t \in S$
⟨*proof*⟩

lemma *set-pmf-closed*: $s \in S \Longrightarrow D \in K\ s \Longrightarrow t \in D \Longrightarrow t \in S$
⟨*proof*⟩

lemma *Pi-closed*: $ct \in Pi\ S\ K \Longrightarrow s \in S \Longrightarrow t \in ct\ s \Longrightarrow t \in S$
⟨*proof*⟩

lemma *E-closed*: $s \in S \Longrightarrow (s, t) \in E \Longrightarrow t \in S$
⟨*proof*⟩

lemma *set-pmf-finite*: $s \in S \Longrightarrow D \in K\ s \Longrightarrow finite\ D$
⟨*proof*⟩

definition *valid-cfg* = $(\bigcup s{\in}S.\ cfg\text{-}on\ s)$

lemma *valid-cfgI*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow cfg \in valid\text{-}cfg$
⟨*proof*⟩

lemma *valid-cfgD*: $cfg \in valid\text{-}cfg \Longrightarrow cfg \in cfg\text{-}on\ (state\ cfg)$
⟨*proof*⟩

lemma
shows *valid-cfg-state-in-S*: $cfg \in valid\text{-}cfg \Longrightarrow state\ cfg \in S$
and *valid-cfg-action*: $cfg \in valid\text{-}cfg \Longrightarrow s \in action\ cfg \Longrightarrow s \in S$
and *valid-cfg-cont*: $cfg \in valid\text{-}cfg \Longrightarrow s \in action\ cfg \Longrightarrow cont\ cfg\ s \in valid\text{-}cfg$
⟨*proof*⟩

lemma *valid-K-cfg*[*intro*]: $cfg \in valid\text{-}cfg \Longrightarrow cfg' \in K\text{-}cfg\ cfg \Longrightarrow cfg' \in valid\text{-}cfg$
⟨*proof*⟩

definition *simple ct* = *memoryless-on* $(\lambda s.\ if\ s \in S\ then\ ct\ s\ else\ arb\text{-}act\ s)$

lemma *simple-cfg-on*[*simp*]: $ct \in Pi\ S\ K \Longrightarrow simple\ ct\ s \in cfg\text{-}on\ s$
⟨*proof*⟩

lemma *simple-valid-cfg*[*simp*]: $ct \in Pi\ S\ K \Longrightarrow s \in S \Longrightarrow simple\ ct\ s \in valid\text{-}cfg$
⟨*proof*⟩

lemma *cont-simple*[*simp*]: $s \in S \Longrightarrow t \in set\text{-}pmf\ (ct\ s) \Longrightarrow cont\ (simple\ ct\ s)\ t = simple\ ct\ t$
⟨*proof*⟩

lemma *state-simple*[*simp*]: $state\ (simple\ ct\ s) = s$
⟨*proof*⟩

lemma *action-simple*[*simp*]: $s \in S \Longrightarrow action\ (simple\ ct\ s) = ct\ s$
⟨*proof*⟩

lemma *simple-valid-cfg-iff*: $ct \in Pi\ S\ K \Longrightarrow simple\ ct\ s \in valid\text{-}cfg \longleftrightarrow s \in S$
⟨*proof*⟩

end

end
theory *MDP-Reachability-Problem*
imports *Markov-Decision-Process*
begin

inductive-set *directed-towards* :: $'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow 'a\ set$ **for** $A\ r$ **where**
start: $\bigwedge x.\ x \in A \Longrightarrow x \in directed\text{-}towards\ A\ r$
| *step*: $\bigwedge x\ y.\ y \in directed\text{-}towards\ A\ r \Longrightarrow (x, y) \in r \Longrightarrow x \in directed\text{-}towards\ A\ r$

hide-fact (**open**) *start step*

lemma *directed-towards-mono*:
assumes $s \in directed\text{-}towards\ A\ F\ F \subseteq G$ **shows** $s \in directed\text{-}towards\ A\ G$
⟨*proof*⟩

lemma *directed-eq-rtrancl*: $x \in directed\text{-}towards\ A\ r \longleftrightarrow (\exists\, a{\in}A.\ (x, a) \in r^*)$
⟨*proof*⟩

lemma *directed-eq-rtrancl-Image*: $directed\text{-}towards\ A\ r = (r^*)^{-1}\ ``\ A$
⟨*proof*⟩

locale *Reachability-Problem* = *Finite-Markov-Decision-Process K S* **for** $K :: 's \Rightarrow$ *'s pmf set* **and** $S +$
fixes *S1 S2* :: *'s set*
assumes *S1*: $S1 \subseteq S$ **and** *S2*: $S2 \subseteq S$ **and** *S1-S2*: $S1 \cap S2 = \{\}$
begin

lemma [*measurable*]:
$S \in sets\ (count\text{-}space\ UNIV)\ S1 \in sets\ (count\text{-}space\ UNIV)\ S2 \in sets\ (count\text{-}space\ UNIV)$
⟨*proof*⟩

definition
$v = (\lambda cfg{\in}valid\text{-}cfg.\ emeasure\ (T\ cfg)\ \{x{\in}space\ St.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ x)\})$

lemma *v-eq*: $cfg \in valid\text{-}cfg \Longrightarrow$
$v\ cfg = emeasure\ (T\ cfg)\ \{x{\in}space\ St.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ x)\}$
⟨*proof*⟩

lemma *real-v*: $cfg \in valid\text{-}cfg \Longrightarrow enn2real\ (v\ cfg) = \mathcal{P}(\omega\ in\ T\ cfg.\ (HLD\ S1\ suntil\ HLD\ S2)\ (state\ cfg\ \#\#\ \omega))$

⟨*proof*⟩

lemma *v-le-1*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≤ *1*
⟨*proof*⟩

lemma *v-neq-Pinf*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *top*
⟨*proof*⟩

lemma *v-1-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *1* ⟷ (*AE* ω *in T cfg*. (*HLD S1 suntil HLD S2*) (*state cfg* ## ω))
⟨*proof*⟩

lemma *v-0-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *0* ⟷ (*AE x in T cfg*. *not* (*HLD S1 suntil HLD S2*) (*state cfg* ## *x*))
⟨*proof*⟩

lemma *v-S2*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ *S2* ⟹ *v cfg* = *1*
⟨*proof*⟩

lemma *v-nS12*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∉ *S1* ⟹ *state cfg* ∉ *S2* ⟹ *v cfg* = *0*
⟨*proof*⟩

lemma *v-nS*[*simp*]: *cfg* ∉ *valid-cfg* ⟹ *v cfg* = *undefined*
⟨*proof*⟩

lemma *v-S1*:
assumes *cfg*[*simp*, *intro*]: *cfg* ∈ *valid-cfg* **and** *cfg-S1*[*simp*]: *state cfg* ∈ *S1*
shows *v cfg* = (∫⁺*s*. *v* (*cont cfg s*) ∂*action cfg*)
⟨*proof*⟩

lemma *real-v-integrable*:
integrable (*action cfg*) (λ*s*. *enn2real* (*v* (*cont cfg s*)))
⟨*proof*⟩

lemma *real-v-integral-eq*:
assumes *cfg*[*simp*]: *cfg* ∈ *valid-cfg*
shows *enn2real* (∫⁺ *s*. *v* (*cont cfg s*) ∂*action cfg*) = ∫ *s*. *enn2real* (*v* (*cont cfg s*)) ∂*action cfg*
⟨*proof*⟩

lemma *v-eq-0-coinduct*[*consumes 3*, *case-names valid nS2 cont*]:
assumes ∗: *P cfg*
assumes *valid*: ⋀*cfg*. *P cfg* ⟹ *cfg* ∈ *valid-cfg*
assumes *nS2*: ⋀*cfg*. *P cfg* ⟹ *state cfg* ∉ *S2*
assumes *cont*: ⋀*cfg cfg'*. *P cfg* ⟹ *state cfg* ∈ *S1* ⟹ *cfg'* ∈ *K-cfg cfg* ⟹ *P cfg'* ∨ *v cfg'* = *0*
shows *v cfg* = *0*
⟨*proof*⟩

definition $p = (\lambda s{\in}S.\ P\text{-}sup\ s\ (\lambda\omega.\ (HLD\ S1\ suntil\ HLD\ S2)\ (s\ \#\#\ \omega)))$

lemma *p-eq-SUP-v*: $s \in S \Longrightarrow p\ s = \bigsqcup\ (v\ `\ cfg\text{-}on\ s)$
⟨*proof*⟩

lemma *v-le-p*: $cfg \in valid\text{-}cfg \Longrightarrow v\ cfg \leq p\ (state\ cfg)$
⟨*proof*⟩

lemma *p-eq-0-imp*: $cfg \in valid\text{-}cfg \Longrightarrow p\ (state\ cfg) = 0 \Longrightarrow v\ cfg = 0$
⟨*proof*⟩

lemma *p-eq-0-iff*: $s \in S \Longrightarrow p\ s = 0 \longleftrightarrow (\forall\ cfg{\in}cfg\text{-}on\ s.\ v\ cfg = 0)$
⟨*proof*⟩

lemma *p-le-1*: $s \in S \Longrightarrow p\ s \leq 1$
⟨*proof*⟩

lemma *p-undefined*[*simp*]: $s \notin S \Longrightarrow p\ s = undefined$
⟨*proof*⟩

lemma *p-not-inf*[*simp*]: $s \in S \Longrightarrow p\ s \neq top$
⟨*proof*⟩

lemma *p-S1*: $s \in S1 \Longrightarrow p\ s = (\bigsqcup D{\in}K\ s.\ \int^+ t.\ p\ t\ \partial measure\text{-}pmf\ D)$
⟨*proof*⟩

lemma *p-S2*[*simp*]: $s \in S2 \Longrightarrow p\ s = 1$
⟨*proof*⟩

lemma *p-nS12*: $s \in S \Longrightarrow s \notin S1 \Longrightarrow s \notin S2 \Longrightarrow p\ s = 0$
⟨*proof*⟩

lemma *p-pos*:
assumes $(s,\ t) \in (SIGMA\ s{:}S1.\ \bigcup D{\in}K\ s.\ set\text{-}pmf\ D)^*\ t \in S2$ **shows** $0 < p\ s$
⟨*proof*⟩

definition *F-sup* :: $('s \Rightarrow ennreal) \Rightarrow 's \Rightarrow ennreal$ **where**
$F\text{-}sup\ f = (\lambda s{\in}S.\ if\ s \in S2\ then\ 1\ else\ if\ s \in S1\ then\ SUP\ D{\in}K\ s.\ \int^+ t.\ f\ t\ \partial measure\text{-}pmf\ D\ else\ 0)$

lemma *F-sup-cong*: $(\bigwedge s.\ s \in S \Longrightarrow f\ s = g\ s) \Longrightarrow F\text{-}sup\ f\ s = F\text{-}sup\ g\ s$
⟨*proof*⟩

lemma *continuous-F-sup*: *sup-continuous F-sup*
⟨*proof*⟩

lemma *mono-F-sup*: *mono F-sup*

⟨*proof*⟩

lemma *lfp-F-sup-iterate*: *lfp F-sup* = (*SUP i*. (*F-sup* ^^ *i*) (λ*x*∈*S*. *0*))
⟨*proof*⟩

lemma *p-eq-lfp-F-sup*: *p* = *lfp F-sup*
⟨*proof*⟩

definition S_e = {*s*∈*S*. *p s* = *0*}

lemma S_e: $S_e \subseteq S$
⟨*proof*⟩

lemma *v-*S_e: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ S_e ⟹ *v cfg* = *0*
⟨*proof*⟩

lemma S_e*-nS2*: $S_e \cap S2 = \{\}$
⟨*proof*⟩

lemma S_e*-E1*: *s* ∈ S_e ∩ *S1* ⟹ (*s*, *t*) ∈ *E* ⟹ *t* ∈ S_e
⟨*proof*⟩

lemma S_e*-E2*: *s* ∈ *S1* ⟹ (⋀*t*. (*s*, *t*) ∈ *E* ⟹ *t* ∈ S_e) ⟹ *s* ∈ S_e
⟨*proof*⟩

lemma S_e*-E-iff*: *s* ∈ *S1* ⟹ *s* ∈ S_e ⟷ (∀ *t*. (*s*, *t*) ∈ *E* ⟶ *t* ∈ S_e)
⟨*proof*⟩

definition S_r = *S* − (S_e ∪ *S2*)

lemma S_r: $S_r \subseteq S$
⟨*proof*⟩

lemma S_r*-S1*: $S_r \subseteq S1$
⟨*proof*⟩

lemma S_r*-eq*: S_r = *S1* − S_e
⟨*proof*⟩

lemma *v-neq-0-imp*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *0* ⟹ *state cfg* ∈ S_r ∪ *S2*
⟨*proof*⟩

lemma *valid-cfg-action-in-K*: *cfg* ∈ *valid-cfg* ⟹ *action cfg* ∈ *K* (*state cfg*)
⟨*proof*⟩

lemma *K-cfg-E*: *cfg* ∈ *valid-cfg* ⟹ *cfg*′ ∈ *K-cfg cfg* ⟹ (*state cfg*, *state cfg*′) ∈ *E*
⟨*proof*⟩

lemma *S_r-directed-towards-S2*:
assumes *s*: $s \in S_r$
shows $s \in$ *directed-towards S2* $\{(s,\ t) \mid s\ t.\ s \in S_r \wedge (s,\ t) \in E\}$ (**is** $s \in ?D$)
⟨*proof*⟩

definition *proper ct* $\longleftrightarrow$ $ct \in Pi_E\ S\ K \wedge (\forall\, s{\in}S_r.\ v\ (simple\ ct\ s) > 0)$

lemma *S_r-nS2*: $s \in S_r \Longrightarrow s \notin S2$
⟨*proof*⟩

lemma *properD1*: *proper ct* $\Longrightarrow$ $ct \in Pi_E\ S\ K$
⟨*proof*⟩

lemma *proper-eq*:
assumes *ct*[*simp*, *intro*]: $ct \in Pi_E\ S\ K$
shows *proper ct* $\longleftrightarrow$ $S_r \subseteq$ *directed-towards S2* (*SIGMA* $s{:}S_r.$ *ct s*)
(**is** - $\longleftrightarrow$ - $\subseteq$ *?D*)
⟨*proof*⟩

lemma *exists-proper*:
obtains *ct* **where** *proper ct*
⟨*proof*⟩

definition *l-desc X ct l s* $\longleftrightarrow$
$s \in$ *directed-towards S2* (*SIGMA* $s : X.\ \{l\ s\}$) $\wedge$
v (*simple ct s*) $\leq$ *v* (*simple ct* (*l s*)) $\wedge$
l s $\in$ *maximal* ($\lambda s.$ *v* (*simple ct s*)) (*ct s*)

lemma *exists-l-desc*:
assumes *ct*: *proper ct*
shows $\exists\, l{\in}S_r \to S_r \cup S2.\ \forall\, s{\in}S_r.$ *l-desc* S_r *ct l s*
⟨*proof*⟩

lemma *F-v-memoryless*:
obtains *ct* **where** $ct \in Pi_E\ S\ K$ *v*∘*simple ct* = *F-sup* (*v*∘*simple ct*)
⟨*proof*⟩

lemma *p-v-memoryless*:
obtains *ct* **where** $ct \in Pi_E\ S\ K\ p$ = *v*∘*simple ct*
⟨*proof*⟩

definition $n = (\lambda s{\in}S.$ *P-inf s* $(\lambda\omega.$ (*HLD S1 suntil HLD S2*) (*s ## ω*)))

lemma *n-eq-INF-v*: $s \in S \Longrightarrow n\ s = (\bigsqcap cfg{\in}$*cfg-on s*. *v cfg*)
⟨*proof*⟩

lemma *n-le-v*: $s \in S \Longrightarrow$ *cfg* $\in$ *cfg-on s* $\Longrightarrow$ $n\ s \leq$ *v cfg*
⟨*proof*⟩

lemma *n-eq-1-imp*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow n\ s = 1 \Longrightarrow v\ cfg = 1$
⟨*proof*⟩

lemma *n-eq-1-iff*: $s \in S \Longrightarrow n\ s = 1 \longleftrightarrow (\forall\, cfg{\in}cfg\text{-}on\ s.\ v\ cfg = 1)$
⟨*proof*⟩

lemma *n-le-1*: $s \in S \Longrightarrow n\ s \leq 1$
⟨*proof*⟩

lemma *n-undefined*[*simp*]: $s \notin S \Longrightarrow n\ s = undefined$
⟨*proof*⟩

lemma *n-eq-0*: $s \in S \Longrightarrow cfg \in cfg\text{-}on\ s \Longrightarrow v\ cfg = 0 \Longrightarrow n\ s = 0$
⟨*proof*⟩

lemma *n-not-inf*[*simp*]: $s \in S \Longrightarrow n\ s \neq top$
⟨*proof*⟩

lemma *n-S1*: $s \in S1 \Longrightarrow n\ s = (\prod D{\in}K\ s.\ \int^+ t.\ n\ t\ \partial measure\text{-}pmf\ D)$
⟨*proof*⟩

lemma *n-S2*[*simp*]: $s \in S2 \Longrightarrow n\ s = 1$
⟨*proof*⟩

lemma *n-nS12*: $s \in S \Longrightarrow s \notin S1 \Longrightarrow s \notin S2 \Longrightarrow n\ s = 0$
⟨*proof*⟩

lemma *n-pos*:
assumes $P\ s\ s \in S1\ wf\ R$
assumes *cont*: $\bigwedge s\ D.\ P\ s \Longrightarrow s \in S1 \Longrightarrow D \in K\ s \Longrightarrow \exists\, w{\in}D.\ ((w,\ s) \in R \wedge w \in S1 \wedge P\ w) \vee 0 < n\ w$
shows $0 < n\ s$
⟨*proof*⟩

definition *F-inf* :: $('s \Rightarrow ennreal) \Rightarrow ('s \Rightarrow ennreal)$ **where**
$F\text{-}inf\ f = (\lambda s{\in}S.\ if\ s \in S2\ then\ 1\ else\ if\ s \in S1\ then\ (\prod D{\in}K\ s.\ \int^+ t.\ f\ t\ \partial measure\text{-}pmf\ D)\ else\ 0)$

lemma *F-inf-n*: $F\text{-}inf\ n = n$
⟨*proof*⟩

lemma *F-inf-nS*[*simp*]: $s \notin S \Longrightarrow F\text{-}inf\ f\ s = undefined$
⟨*proof*⟩

lemma *mono-F-inf*: *mono F-inf*
⟨*proof*⟩

lemma *S1-nS2*: $s \in S1 \Longrightarrow s \notin S2$
⟨*proof*⟩

lemma *n-eq-lfp-F-inf*: $n = lfp\ F\text{-}inf$
⟨*proof*⟩

lemma *real-n*: $s \in S \Longrightarrow ennreal\ (enn2real\ (n\ s)) = n\ s$
⟨*proof*⟩

lemma *real-p*: $s \in S \Longrightarrow ennreal\ (enn2real\ (p\ s)) = p\ s$
⟨*proof*⟩

lemma *p-ub*:
fixes x
assumes $s \in S$
assumes *solution*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow (\sum t{\in}S.\ pmf\ D\ t * x\ t) \leq x\ s$
assumes *solution-0*: $\bigwedge s.\ s \in S \Longrightarrow p\ s = 0 \Longrightarrow x\ s = 0$
assumes *solution-S2*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $enn2real\ (p\ s) \leq x\ s$ (**is** *?y s* ≤ -)
⟨*proof*⟩

lemma *n-lb*:
fixes x
assumes $s \in S$
assumes *solution*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \leq (\sum t{\in}S.\ pmf\ D\ t * x\ t)$
assumes *solution-n0*: $\bigwedge s.\ s \in S \Longrightarrow n\ s = 0 \Longrightarrow x\ s = 0$
assumes *solution-S2*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $x\ s \leq enn2real\ (n\ s)$ (**is** - ≤ *?y s*)
⟨*proof*⟩

end

end

6 Discrete-time Markov Processes

In this file we construct discrete-time Markov processes, e.g. with arbitrary state spaces.

theory *Discrete-Time-Markov-Process*
imports *Markov-Models-Auxiliary*
begin

lemma *measure-eqI-PiM-sequence*:
fixes $M :: nat \Rightarrow 'a\ measure$
assumes $*[simp]$: $sets\ P = PiM\ UNIV\ M\ sets\ Q = PiM\ UNIV\ M$
assumes *eq*: $\bigwedge A\ n.\ (\bigwedge i.\ A\ i \in sets\ (M\ i)) \Longrightarrow$
$P\ (prod\text{-}emb\ UNIV\ M\ \{..n\}\ (Pi_E\ \{..n\}\ A)) = Q\ (prod\text{-}emb\ UNIV\ M\ \{..n\}\ (Pi_E\ \{..n\}\ A))$
assumes *A*: *finite-measure P*

shows $P = Q$
⟨*proof*⟩

lemma *distr-cong-simp*:
$M = K \Longrightarrow sets\ N = sets\ L \Longrightarrow (\bigwedge x.\ x \in space\ M \Rightarrow f\ x = g\ x) \Longrightarrow distr\ M\ N\ f = distr\ K\ L\ g$
⟨*proof*⟩

6.1 Constructing Discrete-Time Markov Processes

locale *discrete-Markov-process* =
fixes $M :: 'a\ measure$ **and** $K :: 'a \Rightarrow 'a\ measure$
assumes $K[measurable]$: $K \in M \to_M prob\text{-}algebra\ M$
begin

lemma *space-K*: $x \in space\ M \Longrightarrow space\ (K\ x) = space\ M$
⟨*proof*⟩

lemma *sets-K*[*measurable-cong*]: $x \in space\ M \Longrightarrow sets\ (K\ x) = sets\ M$
⟨*proof*⟩

lemma *prob-space-K*: $x \in space\ M \Longrightarrow prob\text{-}space\ (K\ x)$
⟨*proof*⟩

definition $K' :: 'a \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a\ measure$
where
$K'\ x\ n'\ \omega' = K\ (case\text{-}nat\ x\ \omega'\ n')$

lemma *IT-K'*:
assumes *x*: $x \in space\ M$ **shows** *Ionescu-Tulcea* $(K'\ x)\ (\lambda\text{-}.\ M)$
⟨*proof*⟩

definition *lim-sequence* $:: 'a \Rightarrow (nat \Rightarrow 'a)\ measure$
where
lim-sequence $x = projective\text{-}family.lim\ UNIV\ (Ionescu\text{-}Tulcea.CI\ (K'\ x)\ (\lambda\text{-}.\ M))\ (\lambda\text{-}.\ M)$

lemma
assumes *x*: $x \in space\ M$
shows *space-lim-sequence*: $space\ (lim\text{-}sequence\ x) = space\ (\Pi_M\ i \in UNIV.\ M)$
and *sets-lim-sequence*[*measurable-cong*]: $sets\ (lim\text{-}sequence\ x) = sets\ (\Pi_M\ i \in UNIV.\ M)$
and *emeasure-lim-sequence-emb*: $\bigwedge J\ X.\ finite\ J \Longrightarrow X \in sets\ (\Pi_M\ j \in J.\ M) \Longrightarrow$
$emeasure\ (lim\text{-}sequence\ x)\ (prod\text{-}emb\ UNIV\ (\lambda\text{-}.\ M)\ J\ X) =$
$emeasure\ (Ionescu\text{-}Tulcea.CI\ (K'\ x)\ (\lambda\text{-}.\ M)\ J)\ X$
and *emeasure-lim-sequence-emb-I0o*: $\bigwedge n\ X.\ X \in sets\ (\Pi_M\ i \in \{0..<n\}.\ M) \Longrightarrow$
$emeasure\ (lim\text{-}sequence\ x)\ (prod\text{-}emb\ UNIV\ (\lambda\text{-}.\ M)\ \{0..<n\}\ X) =$

emeasure (*Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *0* *n* (λ*x*. *undefined*)) *X*
⟨*proof*⟩

lemma *lim-sequence*[*measurable*]: *lim-sequence* ∈ *M* →$_M$ *prob-algebra* (Π$_M$ *i*∈*UNIV*. *M*)
⟨*proof*⟩

lemma *step-C*:
assumes *x*: *x* ∈ *space M*
shows *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *0* *1* (λ-. *undefined*) ⤜ *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *1* *n* =
K *x* ⤜ (λ*y*. *Ionescu-Tulcea.C* (*K'* *x*) (λ-. *M*) *1* *n* (*case-nat* *y* (λ-. *undefined*)))
⟨*proof*⟩

lemma *lim-sequence-eq*:
assumes *x*: *x* ∈ *space M*
shows *lim-sequence* *x* = *bind* (*K* *x*) (λ*y*. *distr* (*lim-sequence* *y*) (Π$_M$ *j*∈*UNIV*. *M*) (*case-nat* *y*))
(**is** - = *?B* *x*)
⟨*proof*⟩

lemma *AE-lim-sequence*:
assumes *x*[*simp*]: *x* ∈ *space M* **and** *P*[*measurable*]: *Measurable.pred* (Π$_M$ *i*∈*UNIV*. *M*) *P*
shows (*AE* ω *in lim-sequence* *x*. *P* ω) ⟷ (*AE* *y* *in* *K* *x*. *AE* ω *in lim-sequence* *y*. *P* (*case-nat* *y* ω))
⟨*proof*⟩

definition *lim-stream* :: *'a* ⇒ *'a stream measure*
where
lim-stream *x* = *distr* (*lim-sequence* *x*) (*stream-space* *M*) *to-stream*

lemma *space-lim-stream*: *space* (*lim-stream* *x*) = *streams* (*space* *M*)
⟨*proof*⟩

lemma *sets-lim-stream*[*measurable-cong*]: *sets* (*lim-stream* *x*) = *sets* (*stream-space* *M*)
⟨*proof*⟩

lemma *lim-stream*[*measurable*]: *lim-stream* ∈ *M* →$_M$ *prob-algebra* (*stream-space* *M*)
⟨*proof*⟩

lemma *space-stream-space-M-ne*: *x* ∈ *space M* ⟹ *space* (*stream-space* *M*) ≠ {}
⟨*proof*⟩

lemma *prob-space-lim-stream*: *x* ∈ *space M* ⟹ *prob-space* (*lim-stream* *x*)
⟨*proof*⟩

lemma *lim-stream-eq*:
assumes *x*: $x \in$ *space M*
shows *lim-stream x* = *do* { $y \leftarrow K\ x$; $\omega \leftarrow$ *lim-stream y*; *return* (*stream-space M*) (y ## ω) }
⟨*proof*⟩

lemma *AE-lim-stream*:
assumes *x*[*simp*]: $x \in$ *space M* **and** *P*[*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows (*AE* ω *in lim-stream x*. *P* ω) $\longleftrightarrow$ (*AE y in K x*. *AE* ω *in lim-stream y*. *P* (y ## ω))
⟨*proof*⟩

lemma *emeasure-lim-stream*:
assumes *x*[*measurable*, *simp*]: $x \in$ *space M* **and** *A*[*measurable*, *simp*]: *A* ∈ *sets* (*stream-space M*)
shows *lim-stream x A* = ($\int^+ y$. *emeasure* (*lim-stream y*) (((##) y) $-$ ‘ *A* ∩ *space* (*stream-space M*)) *∂K x*)
⟨*proof*⟩

lemma *lim-stream-eq-coinduct*[*case-names in-space step*]:
fixes *R* :: *'a* ⇒ *'a stream measure* ⇒ *bool*
assumes *x*: *R x B x* ∈ *space M*
assumes *R*: $\bigwedge x\ B.\ R\ x\ B \Longrightarrow \exists B' \in M \to_M$ *prob-algebra* (*stream-space M*).
(*AE y in K x*. *R y* (*B' y*) ∨ *lim-stream y* = *B' y*) ∧
B = *do* { $y \leftarrow K\ x$; $\omega \leftarrow B'\ y$; *return* (*stream-space M*) (y ## ω) }
shows *lim-stream x* = *B*
⟨*proof*⟩

lemma *prob-space-lim-sequence*: $x \in$ *space M* ⟹ *prob-space* (*lim-sequence x*)
⟨*proof*⟩

end

6.2 Strong Markov Property for Discrete-Time Markov Processes

The filtration adopted to streams, i.e. to the n-th projection.

definition *stream-filtration* :: *'a measure* ⇒ *enat* ⇒ *'a stream measure*
where *stream-filtration M n* = (*SUP i*∈{*i*::*nat*. *i* ≤ *n*}. *vimage-algebra* (*streams* (*space M*)) ($\lambda\omega$. ω !! *i*) *M*)

lemma *measurable-stream-filtration1*: *enat i* ≤ *n* ⟹ ($\lambda\omega$. ω !! *i*) ∈ *stream-filtration M n* →$_M$ *M*
⟨*proof*⟩

lemma *measurable-stream-filtration2*:
f ∈ *space N* → *streams* (*space M*) ⟹ ($\bigwedge i$. *enat i* ≤ *n* ⟹ (λx. *f x* !! *i*) ∈ *N* →$_M$ *M*) ⟹ *f* ∈ *N* →$_M$ *stream-filtration M n*

⟨*proof*⟩

lemma *space-stream-filtration*: *space* (*stream-filtration M n*) = *space* (*stream-space M*)
⟨*proof*⟩

lemma *sets-stream-filteration-le-stream-space*: *sets* (*stream-filtration M n*) ⊆ *sets* (*stream-space M*)
⟨*proof*⟩

interpretation *stream-filtration*: *filtration space* (*stream-space M*) *stream-filtration M*
⟨*proof*⟩

lemma *measurable-stopping-time-stream*:
stopping-time (*stream-filtration M*) *T* ⟹ *T* ∈ *stream-space M* →$_M$ *count-space UNIV*
⟨*proof*⟩

lemma *measurable-stopping-time-All-eq-0*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
shows {*x*∈*space M*. ∀ω∈*streams* (*space M*). *T* (*x* ## ω) = *0*} ∈ *sets M*
⟨*proof*⟩

lemma *stopping-time-0*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** ω: ω ∈ *streams* (*space M*) *T* (*x* ## ω) > *0*
and ω′: ω′ ∈ *streams* (*space M*)
shows *T* (*x* ## ω′) > *0*
⟨*proof*⟩

lemma *stopping-time-epred-SCons*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** ω: ω ∈ *streams* (*space M*) *T* (*x* ## ω) > *0*
shows *stopping-time* (*stream-filtration M*) (λω. *epred* (*T* (*x* ## ω)))
⟨*proof*⟩

context *discrete-Markov-process*
begin

lemma *lim-stream-strong-Markov*:
assumes *x*: *x* ∈ *space M* **and** *T*: *stopping-time* (*stream-filtration M*) *T*
shows *lim-stream x* =
lim-stream x ⤜ (λω. *case T* ω *of*
enat i ⇒ *distr* (*lim-stream* (ω !! *i*)) (*stream-space M*) (λω′. *stake* (*Suc i*) ω @− ω′)
| ∞ ⇒ *return* (*stream-space M*) ω)
(**is** - = *?L T x*)
⟨*proof*⟩

end

end

7 Continuous-time Markov chains

theory *Continuous-Time-Markov-Chain*
imports *Discrete-Time-Markov-Process Discrete-Time-Markov-Chain*
begin

7.1 Trace Operations: relate $('a \times real)\ stream$ and $real \Rightarrow 'a$

partial-function (*tailrec*) *trace-at* :: $'a \Rightarrow (real \times 'a)\ stream \Rightarrow real \Rightarrow 'a$
where
trace-at s ω j = (*case ω of* (t', s')##*ω* ⇒ *if* $t' \leq j$ *then trace-at* s' *ω j else s*)

lemma *trace-at-simp*[*simp*]: *trace-at s* ((t', s')##*ω*) *j* = (*if* $t' \leq j$ *then trace-at* s' *ω j else s*)
⟨*proof*⟩

lemma *trace-at-eq*:
trace-at s ω j = (*case sfirst* (λ*x*. *j* < *fst* (*shd x*)) *ω of* ∞ ⇒ *undefined* | *enat i* ⇒ (*s* ## *smap snd ω*) !! *i*)
⟨*proof*⟩

lemma *trace-at-shift*: *trace-at s* (*smap* (λ(t, s'). (t + t', s')) *ω*) *t* = *trace-at s ω* (*t* − t')
⟨*proof*⟩

primcorec *merge-at* :: $(real \times 'a)\ stream \Rightarrow real \Rightarrow (real \times 'a)\ stream \Rightarrow (real \times 'a)\ stream$
where
merge-at ω j ω' = (*case ω of* (t, s) ## *ω* ⇒ *if t* ≤ *j then* (t, s)##*merge-at ω j ω' else ω'*)

lemma *merge-at-simp*[*simp*]: *merge-at* (*x*##*ω*) *j ω'* = (*if fst x* ≤ *j then x*##*merge-at ω j ω' else ω'*)
⟨*proof*⟩

7.2 Exponential Distribution

definition *exponential* :: *real* ⇒ *real measure*
where
exponential l = *density lborel* (*exponential-density l*)

lemma *space-exponential*: *space* (*exponential l*) = *UNIV*
⟨*proof*⟩

lemma *sets-exponential*[*measurable-cong*]: *sets* (*exponential l*) = *sets borel*
⟨*proof*⟩

lemma *prob-space-exponential*: *0* < *l* ⟹ *prob-space* (*exponential l*)
⟨*proof*⟩

lemma *AE-exponential*: *0* < *l* ⟹ *AE x in exponential l*. *0* < *x*
⟨*proof*⟩

lemma *emeasure-exponential-Ioi-cutoff*:
assumes *0* < *l*
shows *emeasure* (*exponential l*) {*x* <..} = *exp* (− (*max 0 x*) ∗ *l*)
⟨*proof*⟩

lemma *emeasure-exponential-Ioi*:
0 < *l* ⟹ *0* ≤ *x* ⟹ *emeasure* (*exponential l*) {*x* <..} = *exp* (− *x* ∗ *l*)
⟨*proof*⟩

lemma *exponential-eq-stretch*:
assumes *0* < *l*
shows *exponential l* = *distr* (*exponential 1*) *borel* (λ*x*. (*1*/*l*) ∗ *x*)
⟨*proof*⟩

lemma *uniform-measure-exponential*:
assumes *0* < *l 0* ≤ *t*
shows *uniform-measure* (*exponential l*) {*t* <..} = *distr* (*exponential l*) *borel* ((+) *t*) (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-finite*:
assumes *J* ⊆ *I finite J* ⋀*i*. *i* ∈ *I* ⟹ *0* < *R i 0* ≤ *x*
shows *emeasure* (Π$_M$ *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π$_E$ *j*∈*J*. {*x*<..})) = *exp* (− *x* ∗ (∑ *i*∈*J*. *R i*))
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-sequence*:
assumes *R*: *summable R* ⋀*i*. *0* < *R i 0* ≤ *x*
shows *emeasure* (Π$_M$ *i*∈*UNIV*. *exponential* (*R i*)) (Π *i*∈*UNIV*. {*x*<..}) = *exp* (− *x* ∗ *suminf R*)
⟨*proof*⟩

lemma *emeasure-PiM-exponential-Ioi-countable*:
assumes *R*: *J* ⊆ *I countable J* ⋀*i*. *i* ∈ *I* ⟹ *0* < *R i 0* ≤ *x* **and** *finite*: *integrable* (*count-space J*) *R*
shows *emeasure* (Π$_M$ *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π$_E$ *j*∈*J*. {*x*<..})) =
exp (− *x* ∗ (*LINT i*|*count-space J*. *R i*))
⟨*proof*⟩

lemma *AE-PiM-exponential-suminf-infty*:
fixes $R :: nat \Rightarrow real$
assumes *R*: $\bigwedge n.\ 0 < R\ n$ **and** *finite*: $(\sum n.\ ennreal\ (1\ /\ R\ n)) = top$
shows $AE\ \omega\ in\ \Pi_M\ n{\in}UNIV.\ exponential\ (R\ n).\ (\sum n.\ ereal\ (\omega\ n)) = \infty$
⟨*proof*⟩

7.3 Transition Rates

locale *transition-rates* =
fixes $R :: 'a \Rightarrow 'a \Rightarrow real$
assumes *R-nonneg*[*simp*]: $\bigwedge x\ y.\ 0 \leq R\ x\ y$
assumes *R-diagonal-0*[*simp*]: $\bigwedge x.\ R\ x\ x = 0$
assumes *finite-weight*: $\bigwedge x.\ (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV) < \infty$
assumes *positive-weight*: $\bigwedge x.\ 0 < (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV)$
begin

abbreviation $S :: (real \times 'a)\ measure$
where $S \equiv (borel \bigotimes_M count\text{-}space\ UNIV)$

abbreviation $T :: (real \times 'a)\ stream\ measure$
where $T \equiv stream\text{-}space\ S$

abbreviation $I :: 'a \Rightarrow 'a\ set$
where $I\ x \equiv \{y.\ 0 < R\ x\ y\}$

lemma *I-countable*: *countable* $(I\ x)$
⟨*proof*⟩

definition *escape-rate* $:: 'a \Rightarrow real$ **where**
$escape\text{-}rate\ x = \int y.\ R\ x\ y\ \partial count\text{-}space\ UNIV$

lemma *ennreal-escape-rate*: $ennreal\ (escape\text{-}rate\ x) = (\int^+ y.\ R\ x\ y\ \partial count\text{-}space\ UNIV)$
⟨*proof*⟩

lemma *escape-rate-pos*: $0 < escape\text{-}rate\ x$
⟨*proof*⟩

lemma *nonneg-escape-rate*[*simp*]: $0 \leq escape\text{-}rate\ x$
⟨*proof*⟩

lemma *prob-space-exponential-escape-rate*: *prob-space* $(exponential\ (escape\text{-}rate\ x))$
⟨*proof*⟩

lemma *measurable-escape-rate*[*measurable*]: $escape\text{-}rate \in count\text{-}space\ UNIV \rightarrow_M borel$
⟨*proof*⟩

lemma *measurable-exponential-escape-rate*[*measurable*]: $(\lambda x.\ exponential\ (escape\text{-}rate$

$x)) \in$ *count-space UNIV* $\rightarrow_M$ *prob-algebra borel*
⟨*proof*⟩

interpretation *pmf-as-function* ⟨*proof*⟩

lift-definition $J :: 'a \Rightarrow 'a$ *pmf* **is** $\lambda x\ y.\ \ R\ x\ y\ /$ *escape-rate* x
⟨*proof*⟩

lemma *set-pmf-J*: *set-pmf* $(J\ x) = I\ x$
⟨*proof*⟩

interpretation *exp-esc*: *pair-prob-space distr* (*exponential* (*escape-rate* x)) *borel*
$((+)\ t)\ J\ x$ **for** x
⟨*proof*⟩

7.4 Continuous-time Kernel

definition $K :: (real \times 'a) \Rightarrow (real \times 'a)$ *measure* **where**
$K = (\lambda(t, x).$ (*distr* (*exponential* (*escape-rate* x)) *borel* $((+)\ t)) \bigotimes_M\ J\ x)$

interpretation K: *discrete-Markov-process borel* $\bigotimes_M$ *count-space UNIV K*
⟨*proof*⟩

interpretation *DTMC*: *MC-syntax J* ⟨*proof*⟩

lemma *in-space-S*[*simp*]: $x \in$ *space S*
⟨*proof*⟩

lemma *in-space-T*[*simp*]: $x \in$ *space T*
⟨*proof*⟩

lemma *in-space-lim-stream*: $\omega \in$ *space* (*K.lim-stream* x)
⟨*proof*⟩

lemma *prob-space-K-lim*: *prob-space* (*K.lim-stream* x)
⟨*proof*⟩

definition *select-first* $:: 'a \Rightarrow ('a \Rightarrow real) \Rightarrow 'a \Rightarrow bool$
where *select-first* $x\ p\ y = (y \in I\ x \wedge (\forall\, y' \in I\ x - \{y\}.\ p\ y < p\ y'))$

lemma *select-firstD1*: *select-first* $x\ p\ y \Longrightarrow y \in I\ x$
⟨*proof*⟩

lemma *select-first-unique*:
assumes *y*: *select-first* $x\ p\ y1$ *select-first* $x\ p\ y2$ **shows** $y1 = y2$
⟨*proof*⟩

lemma *The-select-first*[*simp*]: *select-first* $x\ p\ y \Longrightarrow$ *The* (*select-first* $x\ p$) $= y$
⟨*proof*⟩

lemma *select-first-INF*:
select-first x p y $\Longrightarrow$ $(INF\ x{\in}I\ x.\ p\ x) = p\ y$
⟨*proof*⟩

lemma *measurable-select-first*[*measurable*]:
$(\lambda p.\ select\text{-}first\ x\ p\ y) \in (\Pi_M\ y{\in}I\ x.\ borel) \to_M\ count\text{-}space\ UNIV$
⟨*proof*⟩

lemma *measurable-THE-select-first*[*measurable*]:
$(\lambda p.\ The\ (select\text{-}first\ x\ p)) \in (\Pi_M\ y{\in}I\ x.\ borel) \to_M\ count\text{-}space\ UNIV$
⟨*proof*⟩

lemma *sets-S-eq*: *sets S* = *sigma-sets UNIV* { $\{t\ ..\} \times A \mid t\ A.\ A \subseteq -\ I\ x \vee (\exists\, s{\in}I\ x.\ A = \{s\})$ }
⟨*proof*⟩

7.5 Kernel equals Parallel Choice

abbreviation *PAR* :: *'a* ⇒ (*'a* ⇒ *real*) *measure*
where
PAR x ≡ $(\Pi_M\ y{\in}I\ x.\ exponential\ (R\ x\ y))$

lemma *PAR-least*:
assumes *y*: *y* ∈ *I x*
shows *PAR x* {*p*∈*space* (*PAR x*). *t* ≤ *p y* ∧ *select-first x p y*} =
emeasure (*exponential* (*escape-rate x*)) {*t* ..} ∗ *ennreal* (*pmf* (*J x*) *y*)
⟨*proof*⟩

lemma *AE-PAR-least*: *AE p in PAR x*. ∃ *y*∈*I x*. *select-first x p y*
⟨*proof*⟩

lemma *K-alt*: *K* (*t*, *x*) = *distr* ($\Pi_M\ y{\in}I\ x.\ exponential\ (R\ x\ y)$) *S* ($\lambda p.\ (t + (INF\ y{\in}I\ x.\ p\ y),\ The\ (select\text{-}first\ x\ p))$) (**is** - = *?R*)
⟨*proof*⟩

lemma *AE-K*: *AE y in K x*. *fst x* < *fst y* ∧ *snd y* ∈ *J* (*snd x*)
⟨*proof*⟩

lemma *AE-lim-stream*:
AE ω in K.lim-stream x. ∀ *i*. *snd* ((*x* ## *ω*) !! *i*) ∈ *DTMC.acc*‘‘{*snd x*} ∧ *snd* (*ω* !! *i*) ∈ *J* (*snd* ((*x* ## *ω*) !! *i*)) ∧ *fst* ((*x* ## *ω*) !! *i*) < *fst* (*ω* !! *i*)
(**is** *AE ω in K.lim-stream x*. ∀ *i*. *?P ω i*)
⟨*proof*⟩

lemma *measurable-merge-at*[*measurable*]: $(\lambda(\omega, \omega').\ merge\text{-}at\ \omega\ j\ \omega') \in (T \bigotimes_M T) \to_M\ T$
⟨*proof*⟩

lemma *measurable-trace-at*[*measurable*]: $(\lambda(s,\ \omega).\ \textit{trace-at}\ s\ \omega\ j) \in (\textit{count-space}\ \textit{UNIV} \bigotimes_M T) \to_M \textit{count-space}\ \textit{UNIV}$
⟨*proof*⟩

lemma *measurable-trace-at'*: $(\lambda((s,\ j),\ \omega).\ \textit{trace-at}\ s\ \omega\ j) \in ((\textit{count-space}\ \textit{UNIV} \bigotimes_M \textit{borel}) \bigotimes_M T) \to_M \textit{count-space}\ \textit{UNIV}$
⟨*proof*⟩

lemma *K-time-split*:
assumes $t \leq j$ **and** [*measurable*]: $f \in S \to_M \textit{borel}$
shows $(\int^+ x.\ f\ x * \textit{indicator}\ \{j <..\}\ (\textit{fst}\ x)\ \partial K\ (t,\ s)) = (\int^+ x.\ f\ x\ \partial K\ (j,\ s)) * \textit{exponential}\ (\textit{escape-rate}\ s)\ \{j - t <..\}$
⟨*proof*⟩

lemma *K-in-space*[*simp*]: $K\ x \in \textit{space}\ (\textit{prob-algebra}\ S)$
⟨*proof*⟩

lemma *L-in-space*[*simp*]: $K.\textit{lim-stream}\ x \in \textit{space}\ (\textit{prob-algebra}\ T)$
⟨*proof*⟩

7.6 Markov Chain Property

lemma *lim-time-split*:
$t \leq j \Longrightarrow K.\textit{lim-stream}\ (t,\ s) = \textit{do}\ \{\ \omega \leftarrow K.\textit{lim-stream}\ (t,\ s)\ ;\ \omega' \leftarrow K.\textit{lim-stream}\ (j,\ \textit{trace-at}\ s\ \omega\ j)\ ;\ \textit{return}\ T\ (\textit{merge-at}\ \omega\ j\ \omega')\}$
(**is** - $\Longrightarrow$ - = *?DO t s*)
⟨*proof*⟩

lemma *K-eq*: $K\ (t,\ s) = \textit{distr}\ (\textit{exponential}\ (\textit{escape-rate}\ s) \bigotimes_M J\ s)\ S\ (\lambda(t',\ s).\ (t + t',\ s))$
⟨*proof*⟩

lemma *K-shift*: $K\ (t + t',\ s) = \textit{distr}\ (K\ (t,\ s))\ S\ (\lambda(t,\ s).\ (t + t',\ s))$
⟨*proof*⟩

lemma *K-not-empty*: $\textit{space}\ (K\ x) \neq \{\}$
⟨*proof*⟩

lemma *lim-stream-not-empty*: $\textit{space}\ (K.\textit{lim-stream}\ x) \neq \{\}$
⟨*proof*⟩

lemma *lim-shift*: — Generalize to bijective function on *K.lim-stream* invariant on *K*
$K.\textit{lim-stream}\ (t + t',\ s) = \textit{distr}\ (K.\textit{lim-stream}\ (t,\ s))\ T\ (\textit{smap}\ (\lambda(t,\ s).\ (t + t',\ s)))$
(**is** - = *?D t s*)
⟨*proof*⟩

lemma *lim-0*: $K.\textit{lim-stream}\ (t,\ s) = \textit{distr}\ (K.\textit{lim-stream}\ (0,\ s))\ T\ (\textit{smap}\ (\lambda(t',$

s). $(t' + t, s)$)))
⟨*proof*⟩

7.7 Explosion time

definition *explosion* :: $(real \times 'a)$ *stream* $\Rightarrow$ *ereal*
where *explosion* $\omega = (SUP\ i.\ ereal\ (fst\ (\omega\ !!\ i)))$

lemma *ball-less-Suc-eq*: $(\forall\, i{<}Suc\ n.\ P\ i) \longleftrightarrow (P\ 0 \wedge (\forall\, i{<}n.\ P\ (Suc\ i)))$
⟨*proof*⟩

lemma *lim-stream-timediff-eq-exponential-1*:
distr (*K.lim-stream ts*) (*PiM UNIV* (λ-. *borel*))
($\lambda\omega$ *i. escape-rate* (*snd* ((*ts*##ω) !! *i*)) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*ts*##ω) !! *i*))) =
PiM UNIV (λ-. *exponential 1*)
(**is** *?D* = *?P*)
⟨*proof*⟩

lemma *AE-explosion-infty*:
assumes *bdd*: *bdd-above* (*range escape-rate*)
shows *AE* ω *in K.lim-stream x. explosion* $\omega = \infty$
⟨*proof*⟩

7.8 Transition probability p_t

context
begin

declare [[*inductive-internals* = *true*]]

inductive *trace-in* :: $'a$ *set* $\Rightarrow$ *real* $\Rightarrow$ $'a$ $\Rightarrow$ $(real \times 'a)$ *stream* $\Rightarrow$ *bool* **for** *S t*
where
$t < t' \Longrightarrow s \in S \Longrightarrow$ *trace-in S t s* $((t', s')\#\#\omega)$
| $t \geq t' \Longrightarrow$ *trace-in S t s'* $\omega \Longrightarrow$ *trace-in S t s* $((t', s')\#\#\omega)$

end

lemma *trace-in-simps*[*simp*]:
trace-in ss t s (x##ω) = (*if t* $<$ *fst x then s* $\in$ *ss else trace-in ss t* (*snd x*) ω)
⟨*proof*⟩

lemma *trace-in-eq-lfp*:
trace-in ss t = *lfp* ($\lambda F\ s.\ \lambda(t', s')\#\#\omega \Rightarrow$ *if t* $<$ *t' then s* $\in$ *ss else F s'* ω)
⟨*proof*⟩

lemma *trace-in-shiftD*: *trace-in ss t s* $\omega \Longrightarrow$ *trace-in ss* $(t + t')$ *s* (*smap* ($\lambda(t, s')$.
$(t + t', s')$) ω)
⟨*proof*⟩

lemma *trace-in-shift*[*simp*]: *trace-in ss t s* (*smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) *ω*) ⟷ *trace-in ss* (*t* − *t'*) *s ω*
⟨*proof*⟩

lemma *measurable-trace-in'*:
Measurable.pred (*borel* ⨂$_M$ *count-space UNIV* ⨂$_M$ *T*) (λ(*t*, *s*, *ω*). *trace-in ss t s ω*)
(**is** *?M* (λ(*t*, *s*, *ω*). *trace-in ss t s ω*))
⟨*proof*⟩

lemma *measurable-trace-in*[*measurable* (*raw*)]:
assumes [*measurable*]: *f* ∈ *M* →$_M$ *borel g* ∈ *M* →$_M$ *count-space UNIV h* ∈ *M* →$_M$ *T*
shows *Measurable.pred M* (λ*x*. *trace-in ss* (*f x*) (*g x*) (*h x*))
⟨*proof*⟩

definition *p* :: *'a* ⇒ *'a* ⇒ *real* ⇒ *real*
where *p s s' t* = *P*(*ω in K.lim-stream* (*0*, *s*). *trace-in* {*s'*} *t s ω*)

lemma *p*[*measurable*]: (λ(*s*, *t*). *p s s' t*) ∈ (*count-space UNIV* ⨂$_M$ *borel*) →$_M$ *borel*
⟨*proof*⟩

lemma *p-nonpos*: **assumes** *t* ≤ *0* **shows** *p s s' t* = *of-bool* (*s* = *s'*)
⟨*proof*⟩

lemma *p-0*: *p s s' 0* = *of-bool* (*s* = *s'*)
⟨*proof*⟩

lemma *in-sets-T*[*measurable* (*raw*)]: *Measurable.pred T P* ⟹ {*ω*. *P ω*} ∈ *sets T*
⟨*proof*⟩

lemma *distr-id'*: *sets M* = *sets N* ⟹ *distr M N* (λ*x*. *x*) = *M*
⟨*proof*⟩

lemma *p-nonneg*[*simp*]: *0* ≤ *p s s' t*
⟨*proof*⟩

lemma *p-le-1*[*simp*]: *p s s' t* ≤ *1*
⟨*proof*⟩

lemma *p-eq*:
assumes *0* ≤ *t*
shows *p s s'' t* = (*of-bool* (*s* = *s''*) + (*LINT u*:{*0..t*}|*lborel*. *escape-rate s* ∗ *exp* (*escape-rate s* ∗ *u*) ∗ (*LINT s'*|*J s*. *p s' s'' u*))) / *exp* (*t* ∗ *escape-rate s*)
⟨*proof*⟩

lemma *continuous-on-p*: *continuous-on A* (*p s s'*)
⟨*proof*⟩

lemma *p-vector-derivative*: — Backward equation
assumes $0 \leq t$
shows ($p\ s\ s'$ *has-vector-derivative* (*LINT* $s''|$*count-space UNIV*. $R\ s\ s'' * p\ s''\ s'\ t) - $ *escape-rate* $s * p\ s\ s'\ t$)
(*at* t *within* $\{0..\}$)
(**is** (- *has-vector-derivative ?A*) -)
⟨*proof*⟩

coinductive *wf-times* :: *real* ⇒ (*real* × $'a$) *stream* ⇒ *bool*
where
$t < t' \Longrightarrow$ *wf-times* $t'\ \omega \Longrightarrow$ *wf-times* t ((t', s') *##* ω)

lemma *wf-times-simp*[*simp*]: *wf-times* t (x *##* ω) $\longleftrightarrow$ $t <$ *fst* $x \wedge$ *wf-times* (*fst* x) ω
⟨*proof*⟩

lemma *trace-in-merge-at*:
assumes ω': *wf-times* $t'\ \omega'$
shows *trace-in ss t x* (*merge-at* $\omega\ t'\ \omega'$) $\longleftrightarrow$
(*if* $t < t'$ *then trace-in ss t x* ω *else* $\exists\, y$. *trace-in* $\{y\}\ t'\ x\ \omega \wedge$ *trace-in ss t y* ω')
(**is** *?merge* $\longleftrightarrow$ *?cases*)
⟨*proof*⟩

lemma *AE-lim-wf-times*: *AE* ω *in K.lim-stream* (t, s). *wf-times* $t\ \omega$
⟨*proof*⟩

lemma *wf-times-shiftD*: *wf-times* t' (*smap* ($\lambda(t', y)$. ($t' + t$, y)) ω) $\Longrightarrow$ *wf-times* ($t' - t$) ω
⟨*proof*⟩

lemma *wf-times-shift*[*simp*]: *wf-times* t' (*smap* ($\lambda(t', y)$. ($t' + t$, y)) ω) = *wf-times* ($t' - t$) ω
⟨*proof*⟩

lemma *trace-in-unique*: *trace-in* $\{y1\}\ t\ x\ \omega \Longrightarrow$ *trace-in* $\{y2\}\ t\ x\ \omega \Longrightarrow y1 = y2$
⟨*proof*⟩

lemma *trace-at-eq*: *trace-in* $\{z\}\ t\ x\ \omega \Longrightarrow$ *trace-at* $x\ \omega\ t = z$
⟨*proof*⟩

lemma *AE-lim-acc*: *AE* ω *in K.lim-stream* (t, x). $\forall\, t\ z$. *trace-in* $\{z\}\ t\ x\ \omega \longrightarrow (x, z) \in$ *DTMC.acc*
⟨*proof*⟩

lemma *p-add*:
assumes $0 \leq t\ 0 \leq t'$
shows $p\ x\ y\ (t + t') =$ (*LINT* $z|$*count-space* (*DTMC.acc*‘‘$\{x\}$). $p\ x\ z\ t * p\ z\ y\ t'$)

⟨*proof*⟩

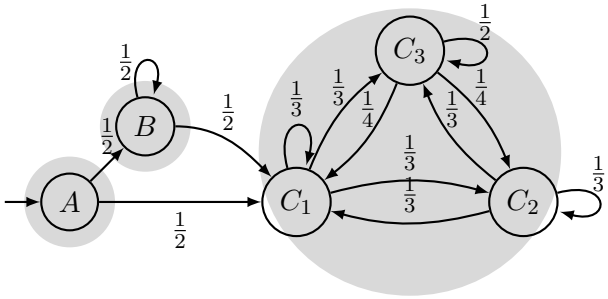
end

end
theory *Markov-Models*
imports
Markov-Models-Auxiliary
Discrete-Time-Markov-Chain
Trace-Space-Equals-Markov-Processes
Classifying-Markov-Chain-States
Markov-Decision-Process
MDP-Reachability-Problem
Discrete-Time-Markov-Process
Continuous-Time-Markov-Chain
begin

end
theory *Example-A*
imports *../Classifying-Markov-Chain-States*
begin

8 Example A

We formalize the following Markov chain:



First we define the state space as its own type:

datatype *state* = *A* | *B* | *C1* | *C2* | *C3*

Now the state space is *UNIV* :: *state set*

lemma *UNIV-state*: *UNIV* = {*A*, *B*, *C1*, *C2*, *C3*}
⟨*proof*⟩

instance *state* :: *finite*
⟨*proof*⟩

The transition function *tau* is easily defined using the case statement, this allows us to give a sparse specification as all *0* cases are collected at the end.

definition *tau* :: *state* ⇒ *state* ⇒ *real* **where**
tau s t = (*case* (*s*, *t*) *of*
(*A*, *B*) ⇒ *1 / 2* | (*A*, *C1*) ⇒ *1 / 2*
| (*B*, *B*) ⇒ *1 / 2* | (*B*, *C1*) ⇒ *1 / 2*
| (*C1*, *C1*) ⇒ *1 / 3* | (*C1*, *C2*) ⇒ *1 / 3* | (*C1*, *C3*) ⇒ *1 / 3*
| (*C2*, *C1*) ⇒ *1 / 3* | (*C2*, *C2*) ⇒ *1 / 3* | (*C2*, *C3*) ⇒ *1 / 3*
| (*C3*, *C1*) ⇒ *1 / 4* | (*C3*, *C2*) ⇒ *1 / 4* | (*C3*, *C3*) ⇒ *1 / 2*
| - ⇒ *0*)

lift-definition *K* :: *state* ⇒ *state pmf* **is** *tau*
⟨*proof*⟩

We use the *finite-pmf*-locale which introduces the point measure *tau.M*, and provides us with the necessary simplifier setup.

interpretation *A*: *MC-syntax K* ⟨*proof*⟩

8.1 The essential classs {*C1*, *C2*, *C3*}

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *A-E-eq*:
set-pmf (*K x*) = (*case x of A* ⇒ {*B*, *C1*} | *B* ⇒ {*B*, *C1*} | - ⇒ {*C1*, *C2*, *C3*})
⟨*proof*⟩

lemma *A-essential*: *A.essential-class* {*C1*, *C2*, *C3*}
⟨*proof*⟩

lemma *A-aperiodic*: *A.aperiodic* {*C1*, *C2*, *C3*}
⟨*proof*⟩

8.2 The stationary distribution *n*

Similar to *tau* we introduce *n* using the *finite-pmf*-locale.

lift-definition *n* :: *state pmf* **is** λ*C1* ⇒ *0.3* | *C2* ⇒ *0.3* | *C3* ⇒ *0.4* | - ⇒ *0*
⟨*proof*⟩

lemma *stationary-distribution-N*: *A.stationary-distribution n*
⟨*proof*⟩

lemma *exclusive-N*[*simp*]: *set-pmf n* = {*C1*, *C2*, *C3*}
⟨*proof*⟩

end

lemma *n-is-limit*:
assumes *x*: *x* ∈ {*C1*, *C2*, *C3*} **and** *y*: *y* ∈ {*C1*, *C2*, *C3*}

shows $(A.p\ x\ y) \longrightarrow pmf\ n\ y$
⟨*proof*⟩

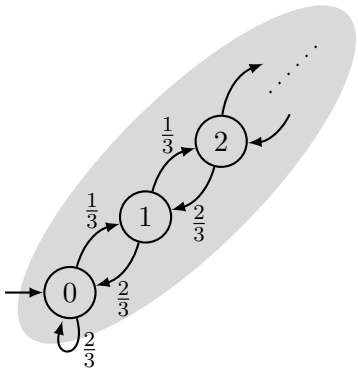
lemma *C-is-pos-recurrent*: $x \in \{C1,\ C2,\ C3\} \Longrightarrow A.pos\text{-}recurrent\ x$
⟨*proof*⟩

lemma *C-recurrence-time*:
assumes *x*: $x \in \{C1,\ C2,\ C3\}$
shows $A.U'\ x\ x = 1\ /\ pmf\ n\ x$
⟨*proof*⟩

end
theory *Example-B*
imports *../Classifying-Markov-Chain-States*
begin

9 Example B

We now formalize the following Markov chain:



As state space we have the set of natural numbers, the transition function *tau* has three cases:

definition $K :: nat \Rightarrow nat\ pmf$ **where**
$K\ x = map\text{-}pmf\ (\lambda True \Rightarrow x + 1 \mid False \Rightarrow x - 1)\ (bernoulli\text{-}pmf\ (1/3))$

For the special case when $x = 0$ we have $x - 1 = 0$ and hence $tau\ 0\ 0 = (2::'a)\ /\ (3::'a)$.

We pack this transition function into a discrete Markov kernel.

We call the locale of the Markov chain *B*, hence all constants and theorems from this Markov chain get a *B* prefix.

interpretation *B*: *MC-syntax K* ⟨*proof*⟩

9.1 Enabled, accessible and communicating states

For each step the predecessor and the successor are enabled (in the *0* case, the predecessor is again *0*. Hence every state is accessible from everywhere and every states is communicating with each other state. Finally we know that the state space is an essential class.

lemma *B-E-eq*: *set-pmf* $(K\ x) = \{x - 1, x + 1\}$
⟨*proof*⟩

lemma *B-E-Suc*: *Suc* $x \in$ *set-pmf* $(K\ x)$ $x \in$ *set-pmf* $(K\ (Suc\ x))$
⟨*proof*⟩

lemma *B-accessible*[*intro*]: $(i, j) \in B.acc$
⟨*proof*⟩

lemma *B-communicating*[*intro*]: $(i, j) \in B.communicating$
⟨*proof*⟩

lemma *B-essential*: *B.essential-class UNIV*
⟨*proof*⟩

9.2 B is aperiodic

lemma *B-aperiodic*: *B.aperiodic UNIV*
⟨*proof*⟩

9.3 The stationary distribution N

abbreviation N :: *nat pmf* **where**
$N \equiv$ *geometric-pmf* $(1\ /\ 2)$

lemma *stationary-distribution-N*: *B.stationary-distribution N*
⟨*proof*⟩

9.4 Limit behavior and recurrence times

lemma *limit*: $(B.p\ i\ j) \longrightarrow (1/2)\hat{}Suc\ j$
⟨*proof*⟩

lemma *pos-recurrent*: *B.pos-recurrent i*
⟨*proof*⟩

lemma *recurrence-time*: $B.U'\ i\ i = 2\hat{}Suc\ i$
⟨*proof*⟩

end

theory *PCTL*
imports

../Discrete-Time-Markov-Chain
Gauss−Jordan−Elim−Fun.Gauss-Jordan-Elim-Fun
HOL−Library.While-Combinator
HOL−Library.Monad-Syntax
begin

10 Adapt Gauss-Jordan elimination to DTMCs

locale *Finite-DTMC* =
fixes $K :: 's \Rightarrow 's\ pmf$ **and** $S :: 's\ set$ **and** $\varrho :: 's \Rightarrow real$ **and** $\iota :: 's \Rightarrow 's \Rightarrow real$
assumes *ι-nonneg*[*simp*]: $\bigwedge s\ t.\ 0 \leq \iota\ s\ t$ **and** *ϱ-nonneg*[*simp*]: $\bigwedge s.\ 0 \leq \varrho\ s$
assumes *measurable-ι*: $(\lambda(a,\ b).\ \iota\ a\ b) \in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *count-space UNIV*)
assumes *finite-S*[*simp*]: *finite S* **and** *S-not-empty*: $S \neq \{\}$
assumes *E-closed*: $(\bigcup s{\in}S.\ set\text{-}pmf\ (K\ s)) \subseteq S$
begin

lemma *measurable-ι'*[*measurable* (*raw*)]:
$f \in$ *measurable M* (*count-space UNIV*) $\Longrightarrow$ $g \in$ *measurable M* (*count-space UNIV*) $\Longrightarrow$
$(\lambda x.\ \iota\ (f\ x)\ (g\ x)) \in$ *borel-measurable M*
⟨*proof*⟩

lemma *measurable-ϱ*[*measurable*]: $\varrho \in$ *borel-measurable* (*count-space UNIV*)
⟨*proof*⟩

sublocale *R?*: *MC-with-rewards K ι ϱ*
⟨*proof*⟩

lemma *single-l*:
fixes *s* **and** *x* :: *real* **assumes** $s \in S$
shows $(\sum s'{\in}S.\ (if\ s' = s\ then\ 1\ else\ 0) * l\ s') = x \longleftrightarrow l\ s = x$
⟨*proof*⟩

definition *order* = (*SOME f*. *bij-betw f* {..< *card S*} *S*)

lemma
shows *bij-order*[*simp*]: *bij-betw order* {..< *card S*} *S*
and *inj-order*[*simp*]: *inj-on order* {..<*card S*}
and *image-order*[*simp*]: *order* ' {..<*card S*} = *S*
and *order-S*[*simp*, *intro*]: $\bigwedge i.\ i < card\ S \Longrightarrow order\ i \in S$
⟨*proof*⟩

lemma *order-Ex*:
assumes $s \in S$ **obtains** *i* **where** $i < card\ S\ s = order\ i$
⟨*proof*⟩

definition *iorder* = *the-inv-into* {..<*card S*} *order*

lemma *bij-iorder*: *bij-betw iorder S* {..<*card S*}
⟨*proof*⟩

lemma *iorder-image-eq*: *iorder ' S* = {..<*card S*}
and *inj-iorder*: *inj-on iorder S*
⟨*proof*⟩

lemma *order-iorder*: ⋀*s*. *s* ∈ *S* ⟹ *order* (*iorder s*) = *s*
⟨*proof*⟩

definition *gauss-jordan'* :: ('*s* ⇒ '*s* ⇒ *real*) ⇒ ('*s* ⇒ *real*) ⇒ ('*s* ⇒ *real*) *option*
where
gauss-jordan' M a = *do* {
let M' = (λ*i j*. *if j* = *card S then a* (*order i*) *else M* (*order i*) (*order j*)) ;
sol ← *gauss-jordan M'* (*card S*) ;
Some (λ*i*. *sol* (*iorder i*) (*card S*))
}

lemma *gauss-jordan'-correct*:
assumes *gauss-jordan' M a* = *Some f*
shows ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *f s'*) = *a s*
⟨*proof*⟩

lemma *gauss-jordan'-complete*:
assumes *exists*: ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *x s'*) = *a s*
assumes *unique*: ⋀*y*. ∀ *s*∈*S*. (∑ *s'*∈*S*. *M s s'* ∗ *y s'*) = *a s* ⟹ ∀ *s*∈*S*. *y s* = *x s*
shows ∃ *y*. *gauss-jordan' M a* = *Some y*
⟨*proof*⟩

end

11 pCTL model checking

11.1 Syntax

datatype *realrel* = *LessEqual* | *Less* | *Greater* | *GreaterEqual* | *Equal*

datatype '*s sform* = *true*
| *Label* '*s set*
| *Neg* '*s sform*
| *And* '*s sform* '*s sform*
| *Prob realrel real* '*s pform*
| *Exp realrel real* '*s eform*
and '*s pform* = *X* '*s sform*
| *U nat* '*s sform* '*s sform*
| *UInfinity* '*s sform* '*s sform* (‹U^{∞}›)
and '*s eform* = *Cumm nat* (‹$C^{\leq}$›)
| *State nat* (‹$I^{=}$›)
| *Future* '*s sform*

primrec *bound-until* **where**
bound-until 0 φ ψ = ψ
| *bound-until* (*Suc n*) φ ψ = ψ *or* (φ *aand nxt* (*bound-until n* φ ψ))

lemma *measurable-bound-until*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (*bound-until n* φ ψ)
⟨*proof*⟩

11.2 Semantics

primrec *inrealrel* :: *realrel* ⇒ 'a ⇒ ('a::*linorder*) ⇒ *bool* **where**
inrealrel LessEqual r q ⟷ $q \leq r$ |
inrealrel Less r q ⟷ $q < r$ |
inrealrel Greater r q ⟷ $q > r$ |
inrealrel GreaterEqual r q ⟷ $q \geq r$ |
inrealrel Equal r q ⟷ $q = r$

context *Finite-DTMC*
begin

abbreviation *prob s P* ≡ *measure* (*T s*) {x∈*space* (*T s*). *P x*}
abbreviation *E s* ≡ *set-pmf* (*K s*)

primrec *svalid* :: 's *sform* ⇒ 's *set*
and *pvalid* :: 's *pform* ⇒ 's *stream* ⇒ *bool*
and *reward* :: 's *eform* ⇒ 's *stream* ⇒ *ennreal* **where**
svalid true = *S* |
svalid (*Label L*) = {$s \in S.\ s \in L$} |
svalid (*Neg F*) = *S* − *svalid F* |
svalid (*And F1 F2*) = *svalid F1* ∩ *svalid F2* |
svalid (*Prob rel r F*) = {$s \in S.$ *inrealrel rel r* $\mathcal{P}(\omega$ *in T s. pvalid F* (*s* ## $\omega)$) } |
svalid (*Exp rel r F*) = {$s \in S.$ *inrealrel rel* (*ennreal r*) ($\int^+ \omega.$ *reward F* (*s* ## ω) ∂*T s*) } |

pvalid (*X F*) = *nxt* (*HLD* (*svalid F*)) |
pvalid (*U k F1 F2*) = *bound-until k* (*HLD* (*svalid F1*)) (*HLD* (*svalid F2*)) |
pvalid (U^∞ *F1 F2*) = *HLD* (*svalid F1*) *suntil HLD* (*svalid F2*) |

reward ($C^{\leq}$ *k*) = ($\lambda\omega.$ ($\sum i{<}k.$ ϱ (ω !! *i*) + ι (ω !! *i*) (ω !! (*Suc i*)))) |
reward ($I^{=}$ *k*) = ($\lambda\omega.$ ϱ (ω !! *k*)) |
reward (*Future F*) = ($\lambda\omega.$ *if ev* (*HLD* (*svalid F*)) ω *then reward-until* (*svalid F*) (*shd* ω) (*stl* ω) *else* ∞)

lemma *svalid-subset-S*: *svalid F* ⊆ *S*
⟨*proof*⟩

lemma *finite-svalid*[*simp*, *intro*]: *finite* (*svalid F*)
⟨*proof*⟩

lemma *svalid-sets*[*measurable*]: *svalid F* ∈ *sets* (*count-space S*)
⟨*proof*⟩

lemma *pvalid-sets*[*measurable*]: *Measurable.pred R.S* (*pvalid F*)
⟨*proof*⟩

lemma *reward-measurable*[*measurable*]: *reward F* ∈ *borel-measurable R.S*
⟨*proof*⟩

11.3 Implementation of *Sat*

11.3.1 *Prob0*

definition *Prob0* **where**
Prob0 Φ Ψ = *S* − *while* (λ*R*. ∃ *s*∈Φ. *R* ∩ *E s* ≠ {} ∧ *s* ∉ *R*) (λ*R*. *R* ∪ {*s*∈Φ. *R* ∩ *E s* ≠ {}}) Ψ

lemma *Prob0-subset-S*: *Prob0* Φ Ψ ⊆ *S*
⟨*proof*⟩

lemma *Prob0-iff-reachable*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob0* Φ Ψ = {*s* ∈ *S*. ((*SIGMA x*:Φ. *E x*)* `` {*s*}) ∩ Ψ = {}} (**is** - = *?U*)
⟨*proof*⟩

lemma *Prob0-iff*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob0* Φ Ψ = {*s*∈*S*. *AE* ω *in T s*. ¬ (*HLD* Φ *suntil HLD* Ψ) (*s* ## ω)} (**is** - = *?U*)
⟨*proof*⟩

lemma *E-rtrancl-closed*:
assumes *s* ∈ *S* (*s*, *t*) ∈ (*SIGMA x*:*A*. *B x*)* ⋀*x*. *x* ∈ *A* ⟹ *B x* ⊆ *E x* **shows** *t* ∈ *S*
⟨*proof*⟩

11.3.2 *Prob1*

definition *Prob1* **where**
Prob1 Y Φ Ψ = *Prob0* (Φ − Ψ) *Y*

lemma *Prob1-iff*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob1* (*Prob0* Φ Ψ) Φ Ψ = {*s*∈*S*. *AE* ω *in T s*. (*HLD* Φ *suntil HLD* Ψ) (*s* ## ω)}
(**is** *Prob1 ?P0* - - = {*s*∈*S*. *?pU s*})
⟨*proof*⟩

11.3.3 *ProbU*, *ExpCumm*, and *ExpState*

abbreviation $\tau\ s\ t \equiv pmf\ (K\ s)\ t$

fun $ProbU :: 's \Rightarrow nat \Rightarrow 's\ set \Rightarrow 's\ set \Rightarrow real$ **where**
$ProbU\ q\ 0\ S1\ S2 = (if\ q \in S2\ then\ 1\ else\ 0)$ |
$ProbU\ q\ (Suc\ k)\ S1\ S2 =$
$(if\ q \in S1 - S2\ then\ (\sum q' \in S.\ \tau\ q\ q' * ProbU\ q'\ k\ S1\ S2)$
$else\ if\ q \in S2\ then\ 1\ else\ 0)$

fun $ExpCumm :: 's \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpCumm\ s\ 0 = 0$ |
$ExpCumm\ s\ (Suc\ k) = \varrho\ s + (\sum s' \in S.\ \tau\ s\ s' * (\iota\ s\ s' + ExpCumm\ s'\ k))$

fun $ExpState :: 's \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpState\ s\ 0 = \varrho\ s$ |
$ExpState\ s\ (Suc\ k) = (\sum s' \in S.\ \tau\ s\ s' * ExpState\ s'\ k)$

11.3.4 *LES*

definition $LES :: 's\ set \Rightarrow 's \Rightarrow 's \Rightarrow real$ **where**
$LES\ F\ r\ c =$
$(if\ r \in F\ then\ (if\ c = r\ then\ 1\ else\ 0)$
$else\ (if\ c = r\ then\ \tau\ r\ c - 1\ else\ \tau\ r\ c\ ))$

11.3.5 *ProbUinfty*, compute unbounded until

definition $ProbUinfty :: 's\ set \Rightarrow 's\ set \Rightarrow ('s \Rightarrow real)\ option$ **where**
$ProbUinfty\ S1\ S2 = gauss\text{-}jordan'\ (LES\ (Prob0\ S1\ S2 \cup S2))$
$(\lambda i.\ if\ i \in S2\ then\ 1\ else\ 0)$

11.3.6 *ExpFuture*, compute unbounded reward

definition $ExpFuture :: 's\ set \Rightarrow ('s \Rightarrow ennreal)\ option$ **where**
$ExpFuture\ F = do\ \{$
$let\ N = Prob0\ S\ F\ ;$
$let\ Y = Prob1\ N\ S\ F\ ;$
$sol \leftarrow gauss\text{-}jordan'\ (LES\ (S - Y \cup F))$
$(\lambda i.\ if\ i \in Y \wedge i \notin F\ then - \varrho\ i - (\sum s' \in S.\ \tau\ i\ s' * \iota\ i\ s')\ else\ 0)\ ;$
$Some\ (\lambda s.\ if\ s \in Y\ then\ ennreal\ (sol\ s)\ else\ \infty)$
$\}$

11.3.7 *Sat*

fun $Sat :: 's\ sform \Rightarrow 's\ set\ option$ **where**
$Sat\ true = Some\ S$ |
$Sat\ (Label\ L) = Some\ \{s \in S.\ s \in L\}$ |
$Sat\ (Neg\ F) = do\ \{\ F \leftarrow Sat\ F\ ;\ Some\ (S - F)\ \}$ |
$Sat\ (And\ F1\ F2) = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ Some\ (F1 \cap F2)$
$\}$ |

Sat (Prob rel r (X F)) = *do* { *F* ← *Sat F* ; *Some* {*q* ∈ *S*. *inrealrel rel r* (∑ *q*′∈*F*. *τ q q*′)} } |
Sat (Prob rel r (U k F1 F2)) = *do* { *F1* ← *Sat F1* ; *F2* ← *Sat F2* ; *Some* {*q* ∈ *S*. *inrealrel rel r* (*ProbU q k F1 F2*) } } |
Sat (Prob rel r (U$^{\infty}$ *F1 F2))* = *do* { *F1* ← *Sat F1* ; *F2* ← *Sat F2* ; *P* ← *ProbUinfty F1 F2* ; *Some* {*q* ∈ *S*. *inrealrel rel r* (*P q*) } } |

Sat (Exp rel r (Cumm k)) = *Some* {*s* ∈ *S*. *inrealrel rel r* (*ExpCumm s k*) } |
Sat (Exp rel r (State k)) = *Some* {*s* ∈ *S*. *inrealrel rel r* (*ExpState s k*) } |
Sat (Exp rel r (Future F)) = *do* { *F* ← *Sat F* ; *E* ← *ExpFuture F* ; *Some* {*q* ∈ *S*. *inrealrel rel* (*ennreal r*) (*E q*) } }

lemma *prob-sum*:
s ∈ *S* ⟹ *Measurable.pred R.S P* ⟹ $\mathcal{P}$(*ω in T s*. *P ω*) = (∑ *t*∈*S*. *τ s t* ∗ $\mathcal{P}$(*ω in T t*. *P* (*t ## ω*)))
⟨*proof*⟩

lemma *nn-integral-eq-sum*:
s ∈ *S* ⟹ *f* ∈ *borel-measurable R.S* ⟹ (∫$^{+}$*x*. *f x* ∂*T s*) = (∑ *t*∈*S*. *τ s t* ∗ (∫$^{+}$*x*. *f* (*t ## x*) ∂*T t*))
⟨*proof*⟩

lemma *T-space*[*simp*]: *measure* (*T s*) (*space R.S*) = *1*
⟨*proof*⟩

lemma *emeasure-T-space*[*simp*]: *emeasure* (*T s*) (*space R.S*) = *1*
⟨*proof*⟩

lemma *τ-distr*[*simp*]: *s* ∈ *S* ⟹ (∑ *t*∈*S*. *τ s t*) = *1*
⟨*proof*⟩

lemma *ProbU*:
q ∈ *S* ⟹ *ProbU q k* (*svalid F1*) (*svalid F2*) = $\mathcal{P}$(*ω in T q*. *pvalid* (*U k F1 F2*) (*q ## ω*))
⟨*proof*⟩

lemma *Prob0-imp-not-Psi*:
assumes Φ ⊆ *S* Ψ ⊆ *S s* ∈ *Prob0* Φ Ψ **shows** *s* ∉ Ψ
⟨*proof*⟩

lemma *Psi-imp-not-Prob0*:
assumes Φ ⊆ *S* Ψ ⊆ *S* **shows** *s* ∈ Ψ ⟹ *s* ∉ *Prob0* Φ Ψ
⟨*proof*⟩

11.3.8 Finite expected reward

abbreviation *s0* ≡ *SOME s*. *s* ∈ *S*

lemma *s0-in-S*: $s0 \in S$
⟨*proof*⟩

lemma *nn-integral-reward-finite*:
assumes $s \in S$
assumes *until*: $AE\ \omega\ in\ T\ s.\ (HLD\ S\ suntil\ HLD\ (svalid\ F))\ (s\ \#\#\ \omega)$
shows $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) \neq \infty$
⟨*proof*⟩

lemma *unique*:
assumes *in-S*: $\Phi \subseteq S\ \Psi \subseteq S\ N \subseteq S\ Prob0\ \Phi\ \Psi \subseteq N\ \Psi \subseteq N$
assumes *l1*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l1\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l1\ s')$
assumes *l2*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l2\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l2\ s')$
assumes *eq*: $\bigwedge s.\ s \in N \Longrightarrow l1\ s = l2\ s$
shows $\forall s \in S.\ l1\ s = l2\ s$
⟨*proof*⟩

lemma *uniqueness-of-ProbU*:
assumes *sol*:
$\forall s \in S.\ (\sum s' \in S.\ LES\ (Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2)\ s\ s' * l\ s') =$
$(if\ s \in svalid\ F2\ then\ 1\ else\ 0)$
shows $\forall s \in S.\ l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$
⟨*proof*⟩

lemma *infinite-reward*:
fixes *s F*
defines $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - ≡ *Prob0 S ?F*)
defines $Y \equiv Prob1\ N\ S\ (svalid\ F)$
assumes *s*: $s \in S\ s \notin Y$
shows $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) = \infty$
⟨*proof*⟩

11.3.9 The expected reward implies a unique LES

lemma *existence-of-ExpFuture*:
fixes *s F*
assumes *N-def*: $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - ≡ *Prob0 S ?F*)
assumes *Y-def*: $Y \equiv Prob1\ N\ S\ (svalid\ F)$
assumes *s*: $s \in S\ s \notin S - (Y - ?F)$
shows $enn2real\ (\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) - (\varrho\ s + (\sum s' \in S.\ \tau\ s\ s' * \iota\ s\ s')) =$
$(\sum s' \in S.\ \tau\ s\ s' * enn2real\ (\int^+ \omega.\ reward\ (Future\ F)\ (s'\ \#\#\ \omega)\ \partial T\ s'))$
⟨*proof*⟩

lemma *uniqueness-of-ExpFuture*:
fixes *F*
assumes *N-def*: $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - ≡ *Prob0 S ?F*)
assumes *Y-def*: $Y \equiv Prob1\ N\ S\ (svalid\ F)$

assumes *const-def*: $const \equiv \lambda s.\ if\ s \in Y \wedge s \notin svalid\ F\ then - \varrho\ s - (\sum s' \in S.\ \tau\ s\ s' * \iota\ s\ s')\ else\ 0$
assumes *sol*: $\bigwedge s.\ s \in S \Longrightarrow (\sum s' \in S.\ LES\ (S - Y \cup ?F)\ s\ s' * l\ s') = const\ s$
shows $\forall s \in S.\ l\ s = enn2real\ (\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s)$
(**is** $\forall s \in S.\ l\ s = enn2real\ (\int^+ \omega.\ ?R\ (s\ \#\#\ \omega)\ \partial T\ s)$)
⟨*proof*⟩

11.4 Soundness of *Sat*

theorem *Sat-sound*:
$Sat\ F \neq None \Longrightarrow Sat\ F = Some\ (svalid\ F)$
⟨*proof*⟩

11.5 Completeness of *Sat*

theorem *Sat-complete*:
$Sat\ F \neq None$
⟨*proof*⟩

11.6 Completeness and Soundness *Sat*

corollary *Sat*: $Sat\ \Phi = Some\ (svalid\ \Phi)$
⟨*proof*⟩

end

end

12 Probabilistic Guarded Command Language (pGCL)

theory *PGCL*
imports *../Markov-Decision-Process*
begin

12.1 Syntax

datatype *'s pgcl* =
Skip
| *Abort*
| *Assign 's ⇒ 's*
| *Seq 's pgcl 's pgcl*
| *Par 's pgcl 's pgcl*
| *If 's ⇒ bool 's pgcl 's pgcl*
| *Prob bool pmf 's pgcl 's pgcl*
| *While 's ⇒ bool 's pgcl*

12.2 Denotational Semantics

primrec *wp* :: *'s pgcl ⇒ ('s ⇒ ennreal) ⇒ ('s ⇒ ennreal)* **where**

wp Skip f = *f*
| *wp Abort f* = (λ-. *0*)
| *wp* (*Assign u*) *f* = *f* ∘ *u*
| *wp* (*Seq* c_1 c_2) *f* = *wp* c_1 (*wp* c_2 *f*)
| *wp* (*If b* c_1 c_2) *f* = (λ*s*. *if b s then wp* c_1 *f s else wp* c_2 *f s*)
| *wp* (*Par* c_1 c_2) *f* = *wp* c_1 *f* ⊓ *wp* c_2 *f*
| *wp* (*Prob p* c_1 c_2) *f* = (λ*s*. *pmf p True* ∗ *wp* c_1 *f s* + *pmf p False* ∗ *wp* c_2 *f s*)
| *wp* (*While b c*) *f* = *lfp* (λ*X s*. *if b s then wp c X s else f s*)

lemma *wp-mono*: *mono* (*wp c*)
⟨*proof*⟩

abbreviation *det* :: *'s pgcl* ⇒ *'s* ⇒ (*'s pgcl* × *'s*) *pmf set* (‹≪ -, - ≫›) **where**
det c s ≡ {*return-pmf* (*c*, *s*)}

12.3 Operational Semantics

fun *step* :: (*'s pgcl* × *'s*) ⇒ (*'s pgcl* × *'s*) *pmf set* **where**
step (*Skip*, *s*) = ≪*Skip*, *s*≫
| *step* (*Abort*, *s*) = ≪*Abort*, *s*≫
| *step* (*Assign u*, *s*) = ≪*Skip*, *u s*≫
| *step* (*Seq* c_1 c_2, *s*) = (*map-pmf* (λ(*p1'*, *s'*). (*if p1' = Skip then* c_2 *else Seq p1'* c_2, *s'*))) ‘ *step* (c_1, *s*)
| *step* (*If b* c_1 c_2, *s*) = (*if b s then* ≪c_1, *s*≫ *else* ≪c_2, *s*≫)
| *step* (*Par* c_1 c_2, *s*) = ≪c_1, *s*≫ ∪ ≪c_2, *s*≫
| *step* (*Prob p* c_1 c_2, *s*) = {*map-pmf* (λ*b*. *if b then* (c_1, *s*) *else* (c_2, *s*)) *p*}
| *step* (*While b c*, *s*) = (*if b s then* ≪*Seq c* (*While b c*), *s*≫ *else* ≪*Skip*, *s*≫)

lemma *step-finite*: *finite* (*step x*)
⟨*proof*⟩

lemma *step-non-empty*: *step x* ≠ {}
⟨*proof*⟩

interpretation *step*: *Markov-Decision-Process step*
⟨*proof*⟩

definition *rF* :: (*'s* ⇒ *ennreal*) ⇒ ((*'s pgcl* × *'s*) *stream* ⇒ *ennreal*) ⇒ (*'s pgcl* × *'s*) *stream* ⇒ *ennreal* **where**
rF f F ω = (*if fst* (*shd ω*) = *Skip then f* (*snd* (*shd ω*)) *else F* (*stl ω*))

abbreviation *r* :: (*'s* ⇒ *ennreal*) ⇒ (*'s pgcl* × *'s*) *stream* ⇒ *ennreal* **where**
r f ≡ *lfp* (*rF f*)

lemma *continuous-rF*: *sup-continuous* (*rF f*)
⟨*proof*⟩

lemma *mono-rF*: *mono* (*rF f*)
⟨*proof*⟩

lemma *r-unfold*: $r\ f\ \omega = (\mathit{if}\ \mathit{fst}\ (\mathit{shd}\ \omega) = \mathit{Skip}\ \mathit{then}\ f\ (\mathit{snd}\ (\mathit{shd}\ \omega))\ \mathit{else}\ r\ f\ (\mathit{stl}\ \omega))$
⟨*proof*⟩

lemma *mono-r*: $F \leq G \Longrightarrow r\ F\ \omega \leq r\ G\ \omega$
⟨*proof*⟩

lemma *measurable-rF*:
assumes *F*[*measurable*]: $F \in \mathit{borel\text{-}measurable}\ \mathit{step.St}$
shows $\mathit{rF}\ f\ F \in \mathit{borel\text{-}measurable}\ \mathit{step.St}$
⟨*proof*⟩

lemma *measurable-r*[*measurable*]: $r\ f \in \mathit{borel\text{-}measurable}\ \mathit{step.St}$
⟨*proof*⟩

lemma *mono-r'*: $\mathit{mono}\ (\lambda F\ s.\ \prod D{\in}\mathit{step}\ s.\ \int^+ t.\ (\mathit{if}\ \mathit{fst}\ t = \mathit{Skip}\ \mathit{then}\ f\ (\mathit{snd}\ t)\ \mathit{else}\ F\ t)\ \partial\mathit{measure\text{-}pmf}\ D)$
⟨*proof*⟩

lemma *E-inf-r*:
$\mathit{step.E\text{-}inf}\ s\ (r\ f) =$
$\mathit{lfp}\ (\lambda F\ s.\ \prod D{\in}\mathit{step}\ s.\ \int^+ t.\ (\mathit{if}\ \mathit{fst}\ t = \mathit{Skip}\ \mathit{then}\ f\ (\mathit{snd}\ t)\ \mathit{else}\ F\ t)\ \partial\mathit{measure\text{-}pmf}\ D)\ s$
⟨*proof*⟩

lemma *E-inf-r-unfold*:
$\mathit{step.E\text{-}inf}\ s\ (r\ f) = (\prod D{\in}\mathit{step}\ s.\ \int^+ t.\ (\mathit{if}\ \mathit{fst}\ t = \mathit{Skip}\ \mathit{then}\ f\ (\mathit{snd}\ t)\ \mathit{else}\ \mathit{step.E\text{-}inf}\ t\ (r\ f))\ \partial\mathit{measure\text{-}pmf}\ D)$
⟨*proof*⟩

lemma *E-inf-r-induct*[*consumes 1*, *case-names step*]:
assumes $P\ s\ y$
assumes $*$: $\bigwedge F\ s\ y.\ P\ s\ y \Longrightarrow$
$(\bigwedge s\ y.\ P\ s\ y \Longrightarrow F\ s \leq y) \Longrightarrow (\bigwedge s.\ F\ s \leq \mathit{step.E\text{-}inf}\ s\ (r\ f)) \Longrightarrow$
$(\prod D{\in}\mathit{step}\ s.\ \int^+ t.\ (\mathit{if}\ \mathit{fst}\ t = \mathit{Skip}\ \mathit{then}\ f\ (\mathit{snd}\ t)\ \mathit{else}\ F\ t)\ \partial\mathit{measure\text{-}pmf}\ D) \leq y$
shows $\mathit{step.E\text{-}inf}\ s\ (r\ f) \leq y$
⟨*proof*⟩

lemma *E-inf-Skip*: $\mathit{step.E\text{-}inf}\ (\mathit{Skip}, s)\ (r\ f) = f\ s$
⟨*proof*⟩

lemma *E-inf-Seq*:
assumes [*simp*]: $\bigwedge x.\ 0 \leq f\ x$
shows $\mathit{step.E\text{-}inf}\ (\mathit{Seq}\ a\ b, s)\ (r\ f) = \mathit{step.E\text{-}inf}\ (a, s)\ (r\ (\lambda s.\ \mathit{step.E\text{-}inf}\ (b, s)\ (r\ f)))$
⟨*proof*⟩

lemma *E-inf-While*:
step.E-inf (*While g c*, *s*) (*r f*) =
lfp (λ*F s*. *if g s then step.E-inf* (*c*, *s*) (*r F*) *else f s*) *s*
⟨*proof*⟩

12.4 Equate Both Semantics

lemma *E-inf-r-eq-wp*: *step.E-inf* (*c*, *s*) (*r f*) = *wp c f s*
⟨*proof*⟩

end

13 Formalization of the Crowds-Protocol

theory *Crowds-Protocol*
imports *../Discrete-Time-Markov-Chain*
begin

lemma *cond-prob-nonneg*[*simp*]: *0* ≤ *cond-prob M A B*
⟨*proof*⟩

lemma (**in** *MC-syntax*) *emeasure-suntil-geometric*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *s* ∈ *X* **and** ∗[*simp*]: *0* ≤ *p 0* ≤ *r*
assumes *r*: ⋀*s*. *s* ∈ *X* ⟹ *emeasure* (*T s*) {ω∈*space* (*T s*). *P* ω} = *ennreal r*
assumes *p*: ⋀*s*. *s* ∈ *X* ⟹ *emeasure* (*K s*) *X* = *ennreal p p* < *1*
assumes ⋀*t*. *AE* ω *in T t*. ¬ (*P* ⊓ (*HLD X* ⊓ *nxt* (*HLD X suntil P*))) ω
shows *emeasure* (*T s*) {ω∈*space* (*T s*). (*HLD X suntil P*) ω} = *r* / (*1* − *p*)
⟨*proof*⟩

13.1 Definition of the Crowds-Protocol

datatype *'a state* = *Start* | *Init 'a* | *Mix 'a* | *End*

lemma *inj-Mix*[*simp*]: *inj-on Mix A*
⟨*proof*⟩

lemma *inj-Init*[*simp*]: *inj-on Init A*
⟨*proof*⟩

lemma *distinct-state-image*[*simp*]:
Start ∉ *Mix* ‘ *A Init j* ∉ *Mix* ‘ *A End* ∉ *Mix* ‘ *A Mix j* ∈ *Mix* ‘ *A* ⟷ *j* ∈ *A*
Start ∉ *Init* ‘ *A Mix j* ∉ *Init* ‘ *A End* ∉ *Init* ‘ *A Init j* ∈ *Init* ‘ *A* ⟷ *j* ∈ *A*
⟨*proof*⟩

lemma *Init-cut-Mix*[*simp*]:
Init ‘ *H* ∩ *Mix* ‘ *J* = {}
⟨*proof*⟩

abbreviation *Jondo B* ≡ *Init‘B* ∪ *Mix‘B*

locale *Crowds-Protocol* =
  fixes *J* :: *'a set* **and** *C* :: *'a set* **and** *p-f* :: *real* **and** *p-i* :: *'a* ⇒ *real*
  assumes *J-not-empty*: *J* ≠ {} **and** *finite-J*[*simp*]: *finite J*
  assumes *C-smaller*: *C* ⊂ *J* **and** *C-non-empty*: *C* ≠ {}
  assumes *p-f*: *0* < *p-f p-f* < *1*
  assumes *p-i-nonneg*[*simp*]: ⋀*j*. *j* ∈ *J* ⟹ *0* ≤ *p-i j*
  assumes *p-i-distr*: (∑ *j*∈*J*. *p-i j*) = *1*
  assumes *p-i-C*: ⋀*j*. *j* ∈ *C* ⟹ *p-i j* = *0*
begin

abbreviation *H* :: *'a set* **where**
  H ≡ *J* − *C*

definition *p-j* = *1* / *card J*

lemma *p-f-nonneg*[*simp*]: *0* ≤ *p-f p-f* ≤ *1*
  ⟨*proof*⟩

lemma *p-j-nonneg*[*simp*]: *0* ≤ *p-j*
  ⟨*proof*⟩

definition *p-H* = *card H* / *card J*

lemma *p-H-nonneg*[*simp*]: *0* ≤ *p-H p-H* ≤ *1*
  ⟨*proof*⟩

definition *next-prob* :: *'a state* ⇒ *'a state* ⇒ *real* **where**
  next-prob s t = (*case* (*s*, *t*) *of* (*Start*, *Init j*) ⇒ *if j* ∈ *H then p-i j else 0*
  | (*Init j*, *Mix j'*) ⇒ *if j'* ∈ *J then p-j else 0*
  | (*Mix j*, *Mix j'*) ⇒ *if j'* ∈ *J then p-f* ∗ *p-j else 0*
  | (*Mix j*, *End*) ⇒ *1* − *p-f*
  | (*End*, *End*) ⇒ *1*
  | - ⇒ *0*)

definition *N s* = *embed-pmf* (*next-prob s*)

interpretation *MC-syntax N* ⟨*proof*⟩

abbreviation 𝔓 ≡ *T Start*

abbreviation *E s* ≡ *set-pmf* (*N s*)

lemma *finite-C*[*simp*]: *finite C*
  ⟨*proof*⟩

lemma *sum-p-i-C*[*simp*]: *sum p-i C* = *0*
  ⟨*proof*⟩

lemma *sum-p-i-H*[*simp*]: *sum p-i H = 1*
⟨*proof*⟩

lemma *possible-jondo*:
obtains *j* **where** *j ∈ J j ∉ C p-i j ≠ 0*
⟨*proof*⟩

lemma *C-le-J*[*simp*]: *card C < card J*
⟨*proof*⟩

lemma *p-H*: *0 < p-H p-H < 1*
⟨*proof*⟩

lemma *p-H-p-f-pos*: *0 < p-H * p-f*
⟨*proof*⟩

lemma *p-H-p-f-less-1*: *p-H * p-f < 1*
⟨*proof*⟩

lemma *p-j-pos*: *0 < p-j*
⟨*proof*⟩

lemma *H-compl*: *1 − p-H = real (card C) / real (card J)*
⟨*proof*⟩

lemma *H-compl2*: *1 − p-H = card C * p-j*
⟨*proof*⟩

lemma *H-eq2*: *card H * p-j = p-H*
⟨*proof*⟩

lemma *pmf-next-pmf*[*simp*]: *pmf (N s) t = next-prob s t*
⟨*proof*⟩

lemma *next-prob-Start*[*simp*]: *next-prob Start (Init j) = (if j ∈ H then p-i j else 0)*
⟨*proof*⟩

lemma *next-prob-to-Init*[*simp*]: *j ∈ H ⟹ next-prob s (Init j) =*
(case s of Start ⇒ p-i j | - ⇒ 0)
⟨*proof*⟩

lemma *next-prob-to-Mix*[*simp*]: *j ∈ J ⟹ next-prob s (Mix j) =*
*(case s of Init j ⇒ p-j | Mix j ⇒ p-f * p-j | - ⇒ 0)*
⟨*proof*⟩

lemma *next-prob-to-End*[*simp*]: *next-prob s End =*
(case s of Mix j ⇒ 1 − p-f | End ⇒ 1 | - ⇒ 0)

⟨*proof*⟩

lemma *next-prob-from-End*[*simp*]: *next-prob End s* = *0* ⟷ *s* ≠ *End*
⟨*proof*⟩

lemma *next-prob-Mix-MixI*: ∃ *j*. *s* = *Mix j* ⟹ ∃*j*∈*J*. *s'* = *Mix j* ⟹ *next-prob s s'* = *p-f* ∗ *p-j*
⟨*proof*⟩

lemma *E-Start*: *E Start* = {*Init j* | *j*. *j* ∈ *H* ∧ *p-i j* ≠ *0* }
⟨*proof*⟩

lemma *E-Init*: *E* (*Init j*) = {*Mix j* | *j*. *j* ∈ *J* }
⟨*proof*⟩

lemma *E-Mix*: *E* (*Mix j*) = {*Mix j* | *j*. *j* ∈ *J* } ∪ {*End*}
⟨*proof*⟩

lemma *E-End*: *E End* = {*End*}
⟨*proof*⟩

lemma *enabled-End*:
enabled End ω ⟷ *ω* = *sconst End*
⟨*proof*⟩

lemma *AE-End*: (*AE ω in T End*. *P ω*) ⟷ *P* (*sconst End*)
⟨*proof*⟩

lemma *emeasure-Init-eq-Mix*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *AE-End*: *AE x in T End*. ¬ *P* (*End* ## *x*)
shows *emeasure* (*T* (*Init j*)) {*x*∈*space* (*T* (*Init j*)). *P x*} =
emeasure (*T* (*Mix j*)) {*x*∈*space* (*T* (*Mix j*)). *P x*} / *p-f*
⟨*proof*⟩

What is the probability that the server sees a specific jondo (including the initiator) as sender.

definition *visit* :: *'a set* ⇒ *'a set* ⇒ *'a state stream* ⇒ *bool* **where**
visit I L = *Init*‘(*I* ∩ *H*) · (*HLD* (*Mix*‘*J*) *suntil* (*Mix*‘(*L* ∩ *J*) · *HLD* {*End*}))

lemma *visit-unique1*:
visit I1 L1 ω ⟹ *visit I2 L2 ω* ⟹ *I1* ∩ *I2* ≠ {}
⟨*proof*⟩

lemma *visit-unique2*:
assumes *visit I1 L1 ω visit I2 L2 ω*
shows *L1* ∩ *L2* ≠ {}
⟨*proof*⟩

lemma *visit-imp-in-H*: *visit* $\{i\}$ J $\omega \Longrightarrow i \in H$
⟨*proof*⟩

lemma *emeasure-visit*:
assumes *I*: $I \subseteq H$ **and** *L*: $L \subseteq J$
shows *emeasure* $\mathfrak{P}$ $\{\omega \in space\ \mathfrak{P}.\ visit\ I\ L\ \omega\} = (\sum i \in I.\ p\text{-}i\ i) * (card\ L * p\text{-}j)$
⟨*proof*⟩

lemma *measurable-visit*[*measurable*]: *Measurable.pred S* (*visit I L*)
⟨*proof*⟩

lemma *AE-visit*: *AE* ω *in* $\mathfrak{P}$. *visit H J* ω
⟨*proof*⟩

13.2 Server gets no information

lemma *server-view1*: $j \in J \Longrightarrow \mathcal{P}(\omega\ in\ \mathfrak{P}.\ visit\ H\ \{j\}\ \omega) = p\text{-}j$
⟨*proof*⟩

lemma *server-view-indep*:
$L \subseteq J \Longrightarrow I \subseteq H \Longrightarrow \mathcal{P}(\omega\ in\ \mathfrak{P}.\ visit\ I\ L\ \omega) = \mathcal{P}(\omega\ in\ \mathfrak{P}.\ visit\ H\ L\ \omega) * \mathcal{P}(\omega\ in\ \mathfrak{P}.\ visit\ I\ J\ \omega)$
⟨*proof*⟩

lemma *server-view*: $\mathcal{P}(\omega\ in\ \mathfrak{P}.\ \exists j \in H.\ visit\ \{j\}\ \{j\}\ \omega) = p\text{-}j$
⟨*proof*⟩

13.3 Probability that collaborators gain information

definition *hit-C* = *Init*‘*H* · *ev* (*HLD* (*Mix*‘*C*))
definition *before-C B* = (*HLD* (*Jondo H*)) *suntil* ((*Jondo* (*B* ∩ *H*)) · *HLD* (*Mix* ‘ *C*))

lemma *measurable-hit-C*[*measurable*]: *Measurable.pred S hit-C*
⟨*proof*⟩

lemma *measurable-before-C*[*measurable*]: *Measurable.pred S* (*before-C B*)
⟨*proof*⟩

lemma *before-C*:
assumes ω: *enabled Start* ω
shows *before-C B* $\omega \longleftrightarrow$
((*Init*‘*H* · (*HLD* (*Mix*‘*H*) *suntil* (*Mix*‘(*B* ∩ *H*) · *HLD* (*Mix*‘*C*)))) *or* (*Init*‘(*B* ∩ *H*) · *HLD* (*Mix*‘*C*))) ω
⟨*proof*⟩

lemma *before-C-unique*:
assumes ω: *before-C I1* ω *before-C I2* ω **shows** *I1* ∩ *I2* ≠ {}
⟨*proof*⟩

lemma *hit-C-imp-before-C*:
assumes *enabled Start* ω *hit-C* ω **shows** *before-C H* ω
⟨*proof*⟩

lemma *before-C-single*:
assumes *before-C I* ω **shows** $\exists\, i{\in}I \cap H.$ *before-C* $\{i\}$ ω
⟨*proof*⟩

lemma *before-C-imp-in-H*: *before-C* $\{i\}$ $\omega \Longrightarrow i \in H$
⟨*proof*⟩

13.4 The probability that the sender hits a collaborator

lemma *Pr-hit-C*: $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *hit-C* $\omega) = (1 - p\text{-}H) \,/\, (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *before-C-imp-hit-C*:
assumes *enabled Start* ω *before-C B* ω
shows *hit-C* ω
⟨*proof*⟩

lemma *negE*: $\neg\ P \Longrightarrow P \Longrightarrow$ *False*
⟨*proof*⟩

lemma *Pr-visit-before-C*:
assumes *L*: $L \subseteq H$ **and** *I*: $I \subseteq H$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *visit I J* $\omega \wedge$ *before-C L* $\omega \mid$ *hit-C* $\omega\ ) =$
$(\sum i{\in}I.\ p\text{-}i\ i) * card\ L * p\text{-}j * p\text{-}f + (\sum i{\in}I \cap L.\ p\text{-}i\ i) * (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *Pr-visit-eq-before-C*:
$\mathcal{P}(\omega$ *in* $\mathfrak{P}.\ \exists j{\in}H.$ *visit* $\{j\}$ *J* $\omega \wedge$ *before-C* $\{j\}$ $\omega \mid$ *hit-C* $\omega\ ) = 1 - (p\text{-}H - p\text{-}j)$
$* p\text{-}f$
⟨*proof*⟩

lemma *probably-innocent*:
assumes *approx*: $1 \,/\, (2 * (p\text{-}H - p\text{-}j)) \leq p\text{-}f$ **and** $p\text{-}H \neq p\text{-}j$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.\ \exists j{\in}H.$ *visit* $\{j\}$ *J* $\omega \wedge$ *before-C* $\{j\}$ $\omega \mid$ *hit-C* $\omega\ ) \leq 1 \,/\, 2$
⟨*proof*⟩

lemma *Pr-before-C*:
assumes *L*: $L \subseteq H$
shows $\mathcal{P}(\omega$ *in* $\mathfrak{P}.$ *before-C L* $\omega \mid$ *hit-C* $\omega\ ) =$
$card\ L * p\text{-}j * p\text{-}f + (\sum l{\in}L.\ p\text{-}i\ l) * (1 - p\text{-}H * p\text{-}f)$
⟨*proof*⟩

lemma *P-visit*:
assumes *I*: $I \subseteq H$

shows $\mathcal{P}(\omega \text{ in } \mathfrak{P}.\ \textit{visit } I\ J\ \omega \mid \textit{hit-C } \omega\ ) = (\sum i{\in}I.\ p\text{-}i\ i)$
⟨*proof*⟩

13.5 Probability space of hitting a collaborator

definition $hC = \textit{uniform-measure } \mathfrak{P}\ \{\omega{\in}\textit{space } \mathfrak{P}.\ \textit{hit-C } \omega\}$

lemma *emeasure-hit-C-not-0*: $\textit{emeasure } \mathfrak{P}\ \{\omega \in \textit{space } \mathfrak{P}.\ \textit{hit-C } \omega\} \neq 0$
⟨*proof*⟩

lemma *measurable-hC*[*measurable* (*raw*)]:
$A \in \textit{sets } S \Longrightarrow A \in \textit{sets } hC$
$f \in \textit{measurable } M\ S \Longrightarrow f \in \textit{measurable } M\ hC$
$g \in \textit{measurable } S\ M \Longrightarrow g \in \textit{measurable } hC\ M$
$A \cap \textit{space } S \in \textit{sets } S \Longrightarrow A \cap \textit{space } hC \in \textit{sets } S$
⟨*proof*⟩

lemma *vimage-Int-space-C*[*simp*]:
$f -` \{x\} \cap \textit{space } hC = \{\omega{\in}\textit{space } S.\ f\ \omega = x\}$
⟨*proof*⟩

sublocale *hC*: *information-space hC 2*
⟨*proof*⟩

abbreviation
mutual-information-Pow-CP (‹$\mathcal{I}'(\text{-}\ ;\ \text{-}')$›) **where**
$\mathcal{I}(X\ ;\ Y) \equiv hC.\textit{mutual-information } 2\ (\textit{count-space } (X`\textit{space } hC))\ (\textit{count-space } (Y`\textit{space } hC))\ X\ Y$

lemma *simple-functionI*:
assumes *finite* (*range f*)
assumes [*measurable*]: $\bigwedge x.\ \{\omega{\in}\textit{space } S.\ f\ \omega = x\} \in \textit{sets } S$
shows *simple-function hC f*
⟨*proof*⟩

13.6 Estimate the information to the collaborators

lemma *measure-hC*[*simp*]:
assumes *A*[*measurable*]: $A \in \textit{sets } S$
shows $\textit{measure } hC\ A = \mathcal{P}(\omega \text{ in } \mathfrak{P}.\ \omega \in A \mid \textit{hit-C } \omega\ )$
⟨*proof*⟩

13.6.1 Setup random variables for mutual information

definition $\textit{first-J } \omega = (\textit{THE } i.\ \textit{visit } \{i\}\ J\ \omega)$

lemma *first-J-eq*:
$\textit{visit } \{i\}\ J\ \omega \Longrightarrow \textit{first-J } \omega = i$
⟨*proof*⟩

lemma *AE-first-J*:
AE ω in ℙ. visit {i} J ω ⟷ first-J ω = i
⟨*proof*⟩

lemma *measurbale-first-J*[*measurable*]: *first-J ∈ measurable S* (*count-space UNIV*)
⟨*proof*⟩

definition *last-H ω* = (*THE i. before-C {i} ω*)

lemma *measurbale-last-H*[*measurable*]: *last-H ∈ measurable S* (*count-space UNIV*)
⟨*proof*⟩

lemma *last-H-eq*:
before-C {i} ω ⟹ last-H ω = i
⟨*proof*⟩

lemma *last-H*:
assumes *enabled Start ω hit-C ω*
shows *before-C {last-H ω} ω last-H ω ∈ H*
⟨*proof*⟩

lemma *AE-last-H*:
AE ω in ℙ. hit-C ω ⟶ before-C {i} ω ⟷ last-H ω = i
⟨*proof*⟩

lemma *information-flow*:
defines *h ≡ real* (*card H*)
assumes *init-uniform*: ⋀*i. i ∈ H ⟹ p-i i = 1 / h*
shows *ℐ*(*first-J ; last-H*) ≤ (*1 − (h − 1) ∗ p-j ∗ p-f*) ∗ *log 2 h*
⟨*proof*⟩

end

end

14 Formalizing the IPv4-address allocation in ZeroConf

theory *Zeroconf-Analysis*
imports *../Discrete-Time-Markov-Chain*
begin

declare *UNIV-bool*[*simp*]

14.1 Definition of a ZeroConf allocation run

datatype *zc-state* = *start*
| *probe nat*

| *ok*
| *error*

lemma *inj-probe*: *inj-on probe X*
⟨*proof*⟩

Countability of *zc-state* simplifies measurability of functions on *zc-state*.

instance *zc-state* :: *countable*
⟨*proof*⟩

locale *Zeroconf-Analysis* =
fixes *N* :: *nat* **and** *p q r e* :: *real*
assumes *p*: *0* < *p p* < *1* **and** *q*: *0* < *q q* < *1*
assumes *r*[*simp*]: *0* ≤ *r* **and** *e*[*simp*]: *0* ≤ *e*
begin

lemma *p-bounds*[*simp*]: *0* ≤ *p p* ≤ *1*
⟨*proof*⟩

lemma *q-bounds*[*simp*]: *0* ≤ *q q* ≤ *1*
⟨*proof*⟩

abbreviation *states* **where**
states ≡ *probe* ' {.. *N*} ∪ {*start*, *ok*, *error*}

primrec *τ* :: *zc-state* ⇒ *zc-state pmf* **where**
τ start = *map-pmf* (λ*True* ⇒ *probe 0* | *False* ⇒ *ok*) (*bernoulli-pmf q*)
| *τ* (*probe n*) = *map-pmf* (λ*True* ⇒ (*if n* < *N then probe* (*Suc n*) *else error*) | *False* ⇒ *start*) (*bernoulli-pmf p*)
| *τ ok* = *return-pmf ok*
| *τ error* = *return-pmf error*

primrec *ϱ* :: *zc-state* ⇒ *zc-state* ⇒ *real* **where**
ϱ start = (λ-. *0*) (*probe 0* := *r*, *ok* := *r* ∗ (*N* + *1*))
| *ϱ* (*probe n*) = (*if n* < *N then* (λ-. *0*) (*probe* (*Suc n*) := *r*) *else* (λ-. *0*) (*error* := *e*))
| *ϱ ok* = (λ-. *0*) (*ok* := *0*)
| *ϱ error* = (λ-. *0*) (*error* := *0*)

lemma *ϱ-nonneg'*[*simp*]: *0* ≤ *ϱ s t*
⟨*proof*⟩

sublocale *MC-with-rewards τ ϱ* λ*s*. *0*
⟨*proof*⟩

14.2 The allocation run is a rewarded DTMC

abbreviation *E s* ≡ *set-pmf* (*τ s*)

lemma *enabled-ok*: *enabled ok* $\omega \longleftrightarrow \omega = sconst\ ok$
⟨*proof*⟩

lemma *finite-E*[*intro*, *simp*]: *finite* (*E s*)
⟨*proof*⟩

lemma *E-closed*: $s \in states \Longrightarrow E\ s \subseteq states$
⟨*proof*⟩

lemma *enabled-error*: *enabled error* $\omega \longleftrightarrow \omega = sconst\ error$
⟨*proof*⟩

lemma *pos-neg-q-pn*: $0 < 1 - q * (1 - p \hat{} Suc\ N)$
⟨*proof*⟩

lemma *to-error*: **assumes** $n \leq N$ **shows** $(probe\ n, error) \in acc$
⟨*proof*⟩

14.3 Probability of a erroneous allocation

definition *P-err* $s = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{error\})\ (s\ \#\#\ \omega))$

lemma *P-err*:
defines *p-start* $== (q * p \hat{} Suc\ N) / (1 - q * (1 - p \hat{} Suc\ N))$
defines *p-probe* $== (\lambda n.\ p \hat{} Suc\ (N - n) + (1 - p \hat{} Suc\ (N - n)) * p\text{-}start)$
assumes *s*: $s \in states - \{ok, error\}$
shows *P-err s* = (*case s of ok* ⇒ *0* | *error* ⇒ *1* | *probe n* ⇒ *p-probe n* | *start* ⇒ *p-start*)
(**is** ... = *?E s*)
⟨*proof*⟩

lemma *P-err-start*: *P-err start* $= (q * p \hat{} Suc\ N) / (1 - q * (1 - p \hat{} Suc\ N))$
⟨*proof*⟩

14.4 An allocation run terminates almost surely

lemma *states-closed*:
assumes $s \in states$
assumes $(s, t) \in acc\text{-}on\ (- \{error, ok\})$
shows $t \in states$
⟨*proof*⟩

lemma *finite-reached*:
assumes *s*: $s \in states$ **shows** *finite* (*acc-on* (− {*error*, *ok*}) `` {*s*})
⟨*proof*⟩

lemma *AE-reaches-error-or-ok*:
assumes *s*: $s \in states$
shows *AE* ω *in T s. ev* (*HLD* {*error*, *ok*}) ω
⟨*proof*⟩

14.5 Expected runtime of an allocation run

definition $R\ s = (\int^+ \omega.\ \textit{reward-until}\ \{\textit{error},\ \textit{ok}\}\ s\ \omega\ \partial T\ s)$

definition $R'\ s = \textit{enn2real}\ (R\ s)$

lemma *R-iter*: $s \neq \textit{error} \Longrightarrow s \neq \textit{ok} \Longrightarrow R\ s = (\int^+ t.\ \textit{ennreal}\ (\varrho\ s\ t) + R\ t\ \partial\tau\ s)$
⟨*proof*⟩

lemma *R-finite*:
assumes *s*: $s \in \textit{states}$
shows $R\ s \neq \infty$
⟨*proof*⟩

lemma *R-less-top*: $s \in \textit{states} \Longrightarrow R\ s < \textit{top}$
⟨*proof*⟩

lemma *R′-iter*: **assumes** *s*: $s \in \textit{states}\ s \neq \textit{error}\ s \neq \textit{ok}$ **shows** $R'\ s = (\int t.\ \varrho\ s\ t + R'\ t\ \partial\tau\ s)$
⟨*proof*⟩

lemma *cost-from-start*:
$R'\ \textit{start} =$
$(q * (r + p\hat{}\textit{Suc}\ N * e + r * p * (1 - p\hat{}N) / (1 - p)) + (1 - q) * (r * \textit{Suc}\ N)) /$
$(1 - q + q * p\hat{}\textit{Suc}\ N)$
⟨*proof*⟩

end

interpretation *ZC*: *Zeroconf-Analysis 2 16 / 65024* :: *real 0.01 0.002 3600*
⟨*proof*⟩

lemma $\textit{ZC.P-err start} \leq 1 / 10\hat{}12$
⟨*proof*⟩

lemma $\textit{ZC.R' start} \leq 0.007$
⟨*proof*⟩

end

15 Formalization of the Gossip-Broadcast

theory *Gossip-Broadcast*
imports *../Discrete-Time-Markov-Chain*
begin

lemma *inj-on-upd-PiE*:

assumes $i \notin I$ **shows** *inj-on* $(\lambda(x,f).\ f(i := x))\ (M \times (\Pi_E\ i{\in}I.\ A\ i))$
⟨*proof*⟩

lemma *sum-folded-product*:
fixes $I :: 'i\ set$ **and** $f :: 's \Rightarrow 'i \Rightarrow 'a{::}\{semiring\text{-}0,\ comm\text{-}monoid\text{-}mult\}$
assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow$ *finite* $(S\ i)$
shows $(\sum x{\in}Pi_E\ I\ S.\ \prod i{\in}I.\ f\ (x\ i)\ i) = (\prod i{\in}I.\ \sum s{\in}S\ i.\ f\ s\ i)$
⟨*proof*⟩

15.1 Definition of the Gossip-Broadcast

datatype *state* = *listening* | *sending* | *sleeping*

type-synonym *sys-state* = $(nat \times nat) \Rightarrow state$

lemma *state-UNIV*: $UNIV = \{listening,\ sending,\ sleeping\}$
⟨*proof*⟩

locale *gossip-broadcast* =
fixes *size* :: *nat* **and** *p* :: *real*
assumes *size*: $0 < size$
assumes *p*: $0 < p\ p < 1$
begin

interpretation *pmf-as-function* ⟨*proof*⟩

definition *states* :: *sys-state set* **where**
$states = (\{..< size\} \times \{..< size\}) \rightarrow_E \{listening,\ sending,\ sleeping\}$

definition *start* :: *sys-state* **where**
$start = (\lambda x{\in}\{..< size\}{\times}\{..< size\}.\ listening)((0,\ 0) := sending)$

definition *neighbour-sending* **where**
$neighbour\text{-}sending\ s = (\lambda(x,y).$
$(x > 0 \wedge s\ (x - 1,\ y) = sending) \vee$
$(x < size \wedge s\ (x + 1,\ y) = sending) \vee$
$(y > 0 \wedge s\ (x,\ y - 1) = sending) \vee$
$(y < size \wedge s\ (x,\ y + 1) = sending))$

definition *node-trans* :: *sys-state* ⇒ $(nat \times nat)$ ⇒ *state* ⇒ *state* ⇒ *real* **where**
node-trans g x s = (*case s of*
listening ⇒ (*if neighbour-sending g x*
then (λ-.*0*) (*sending* := *p*, *sleeping* := *1* − *p*)
else (λ-.*0*) (*listening* := *1*))
| *sending* ⇒ (λ-.*0*) (*sleeping* := *1*)
| *sleeping* ⇒ (λ-.*0*) (*sleeping* := *1*))

lemma *node-trans-sum-eq-1*[*simp*]:
node-trans g x s′ listening + (*node-trans g x s′ sending* + *node-trans g x s′*

sleeping) = *1*
⟨*proof*⟩

lemma *node-trans-nonneg*[*simp*]: *0* ≤ *node-trans s x i j*
⟨*proof*⟩

lift-definition *proto-trans* :: *sys-state* ⇒ *sys-state pmf* **is**
λ*s s'*. *if s'* ∈ *states then* (∏ *x*∈{..< *size*}×{..< *size*}. *node-trans s x* (*s x*) (*s' x*))
else 0
⟨*proof*⟩

end

15.2 The Gossip-Broadcast forms a DTMC

sublocale *gossip-broadcast* ⊆ *MC-syntax proto-trans* ⟨*proof*⟩

end

16 Certification of Reachability Problems on MDPs

theory *MDP-RP-Certification*
imports
../MDP-Reachability-Problem
HOL−Library.IArray
HOL−Library.Code-Target-Numeral
begin

context *Reachability-Problem*
begin

lemma *p-ub'*:
fixes *x*
assumes *1*: *s* ∈ *S* ⋀*s D*. *s* ∈ *S1* ⟹ *D* ∈ *K s* ⟹ (∑ *t*∈*S*. *pmf D t* ∗ *x t*) ≤ *x s*
assumes *2*: ⋀*s*. *s* ∈ *S1* ⟹ *x s* ≠ *0* ⟹ (∃ *t*∈*S2*. (*s*, *t*) ∈ (*SIGMA s*:*S1*. ⋃ *D*∈*K s*. *set-pmf D*)*)
assumes *3*: ⋀*s*. *s* ∈ *S* − *S1* − *S2* ⟹ *x s* = *0*
assumes *4*: ⋀*s*. *s* ∈ *S2* ⟹ *x s* = *1*
shows *enn2real* (*p s*) ≤ *x s*
⟨*proof*⟩

lemma *n-lb'*:
fixes *x*
assumes *wf R*
assumes *1*: *s* ∈ *S* ⋀*s D*. *s* ∈ *S1* ⟹ *D* ∈ *K s* ⟹ *x s* ≤ (∑ *t*∈*S*. *pmf D t* ∗ *x t*)
assumes *2*: ⋀*s D*. *s* ∈ *S1* ⟹ *D* ∈ *K s* ⟹ *x s* ≠ *0* ⟹ ∃ *t*∈*D*. ((*t*, *s*) ∈ *R* ∧ *t* ∈ *S1* ∧ *x t* ≠ *0*) ∨ *t* ∈ *S2*
assumes *3*: ⋀*s*. *s* ∈ *S* − *S1* − *S2* ⟹ *x s* = *0*
assumes *4*: ⋀*s*. *s* ∈ *S2* ⟹ *x s* = *1*

shows $x\ s \leq enn2real\ (n\ s)$
⟨*proof*⟩

end

no-notation *Stream.snth* (**infixl** ‹!!› *100*) — we use !! for IArray

16.1 Computable representation

record *mdp-reachability-problem* =
state-count :: *nat*
distrs :: (*nat* × *rat*) *list list iarray*
states1 :: *bool iarray*
states2 :: *bool iarray*

record *'a RP-sub-cert* =
solution :: *rat iarray*
witness :: (*'a* × *nat*) *iarray*

record *RP-cert* =
pos-cert :: (*nat* × *nat*) *RP-sub-cert*
neg-cert :: *nat list RP-sub-cert*

definition *sparse-mult sx y* = *sum-list* (*map* (λ(*n*, *x*). *x* ∗ *y* !! *n*) *sx*)

primrec *lookup* **where**
lookup d [] *x* = *d*
| *lookup d* (*y*#*ys*) *x* = (*if fst y* = *x then snd y else lookup d ys x*)

lemma *lookup-eq-map-of*: *lookup d xs x* = (*case map-of xs x of Some x* ⇒ *x* | *None* ⇒ *d*)
⟨*proof*⟩

lemma *lookup-in-set*:
distinct (*map fst xs*) ⟹ *x* ∈ *set xs* ⟹ *lookup d xs* (*fst x*) = *snd x*
⟨*proof*⟩

lemma *lookup-not-in-set*:
x ∉ *fst* ‘ *set xs* ⟹ *lookup d xs x* = *d*
⟨*proof*⟩

lemma *lookup-nonneg*:
(⋀*x v*. (*x*, *v*) ∈ *set xs* ⟹ *0* ≤ *v*) ⟹ (*0*::*'a*::*ordered-comm-monoid-add*) ≤ *lookup 0 xs x*
⟨*proof*⟩

lemma *sparse-mult-eq-sum-lookup*:
fixes *xs* :: (*nat* × *'a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)

shows *sparse-mult xs y* = ($\sum$ *i*<*M*. *lookup 0 xs i* * *y* !! *i*)
⟨*proof*⟩

lemma *sum-list-eq-sum-lookup*:
fixes *xs* :: (*nat* × *'a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)
shows *sum-list* (*map snd xs*) = ($\sum$ *i*<*M*. *lookup 0 xs i*)
⟨*proof*⟩

definition
valid-mdp-rp mdp ⟷
0 < *state-count mdp* ∧
IArray.length (*distrs mdp*) = *state-count mdp* ∧
IArray.length (*states1 mdp*) = *state-count mdp* ∧
IArray.length (*states2 mdp*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*. ¬ (*states1 mdp* !! *i* ∧ *states2 mdp* !! *i*) ∧
list-all (λ*ds*. *distinct* (*map fst ds*) ∧ *list-all* (λ(*n*, *x*). *0* ≤ *x* ∧ *n* < *state-count mdp*) *ds* ∧
sum-list (*map snd ds*) = *1*) (*distrs mdp* !! *i*) ∧
¬ *List.null* (*distrs mdp* !! *i*))

definition
valid-sub-cert mdp c ord check ⟷
IArray.length (*witness c*) = *state-count mdp* ∧
IArray.length (*solution c*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*.
if states2 mdp !! *i then solution c* !! *i* = *1*
else if states1 mdp !! *i then 0* ≤ *solution c* !! *i* ∧
(*list-all* (λ*ds*. *ord* (*sparse-mult ds* (*solution c*)) (*solution c* !! *i*)) (*distrs mdp* !! *i*)) ∧
(*0* < *solution c* !! *i* ⟶ *check* (*distrs mdp* !! *i*) (*witness c* !! *i*))
else solution c !! *i* = *0*)

definition
valid-pos-cert mdp c ⟷
valid-sub-cert mdp c (≤)
(λ*D* ((*j*, *a*), *n*). *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧ *0* < *solution c* !! *j* ∧
a < *length D* ∧ *lookup 0* (*D* ! *a*) *j* ≠ *0*)

definition
valid-neg-cert mdp c ⟷
valid-sub-cert mdp c (≥)
(λ*D* (*J*, *n*). *list-all2* (λ*j d*. *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧
lookup 0 d j ≠ *0* ∧ *0* < *solution c* !! *j*) *J D*)

definition
valid-cert mdp c ⟷ *valid-pos-cert mdp* (*pos-cert c*) ∧ *valid-neg-cert mdp* (*neg-cert c*)

lemma *valid-mdp-rpD-length*:
assumes *valid-mdp-rp mdp*
shows *0 < state-count mdp IArray.length (distrs mdp) = state-count mdp*
IArray.length (states1 mdp) = state-count mdp IArray.length (states2 mdp) = state-count mdp
⟨*proof*⟩

lemma *valid-mdp-rpD*:
assumes *valid-mdp-rp mdp i < state-count mdp*
shows ¬ (*states1 mdp !! i ∧ states2 mdp !! i*)
and ⋀*ds n x. ds ∈ set (distrs mdp !! i) ⟹ (n, x) ∈ set ds ⟹ n < state-count mdp*
and ⋀*ds n x. ds ∈ set (distrs mdp !! i) ⟹ (n, x) ∈ set ds ⟹ 0 ≤ x*
and ⋀*ds. ds ∈ set (distrs mdp !! i) ⟹ sum-list (map snd ds) = 1*
and ⋀*ds. ds ∈ set (distrs mdp !! i) ⟹ distinct (map fst ds)*
and *distrs mdp !! i ≠ []*
⟨*proof*⟩

lemma *valid-mdp-rp-sparse-mult*:
assumes *valid-mdp-rp mdp i < state-count mdp ds ∈ set (distrs mdp !! i)*
shows *sparse-mult ds y = (∑ i<state-count mdp. lookup 0 ds i * y !! i)*
⟨*proof*⟩

lemma *valid-sub-certD*:
assumes *valid-mdp-rp mdp valid-sub-cert mdp c ord check i < state-count mdp*
shows ¬ *states1 mdp !! i ⟹* ¬ *states2 mdp !! i ⟹ solution c !! i = 0*
and *states2 mdp !! i ⟹ solution c !! i = 1*
and *states1 mdp !! i ⟹ 0 ≤ solution c !! i*
and ⋀*ds. states1 mdp !! i ⟹ ds ∈ set (distrs mdp !! i) ⟹ ord (sparse-mult ds (solution c)) (solution c !! i)*
and ⋀*ds. states1 mdp !! i ⟹ 0 < solution c !! i ⟶ check (distrs mdp !! i) (witness c !! i)*
⟨*proof*⟩

lemma *valid-pos-certD*:
assumes *valid-mdp-rp mdp valid-pos-cert mdp c i < state-count mdp states1 mdp !! i*
0 < solution c !! i witness c !! i = ((j, a), n)
shows *snd (witness c !! j) < n ∧ j < state-count mdp ∧ a < length (distrs mdp !! i) ∧*
lookup 0 ((distrs mdp !! i) ! a) j ≠ 0 ∧ 0 < solution c !! j
⟨*proof*⟩

lemma *valid-neg-certD*:
assumes *valid-mdp-rp mdp valid-neg-cert mdp c i < state-count mdp states1 mdp !! i*
0 < solution c !! i witness c !! i = (js, n)
shows *list-all2 (λj ds. j < state-count mdp ∧ snd (witness c !! j) < n ∧ lookup*

0 ds j ≠ 0 ∧ 0 < solution c !! j) js (distrs mdp !! i)
⟨*proof*⟩

context
fixes *mdp c*
assumes *rp*: *valid-mdp-rp mdp*
assumes *cert*: *valid-cert mdp c*
begin

interpretation *pmf-as-function* ⟨*proof*⟩

abbreviation *S* ≡ {..< *state-count mdp*}
abbreviation *S1* ≡ {*i*. *i* < *state-count mdp* ∧ (*states1 mdp*) !! *i*}
abbreviation *S2* ≡ {*i*. *i* < *state-count mdp* ∧ (*states2 mdp*) !! *i*}

lift-definition *K* :: *nat* ⇒ *nat pmf set* **is**
λi. if i < state-count mdp then
{ (*λj. of-rat* (*lookup 0 D j*) :: *real*) | *D*. *D* ∈ *set* (*distrs mdp* !! *i*) }
else { *indicator* {*0*} }
⟨*proof*⟩

interpretation *MDP*: *Reachability-Problem K S S1 S2*
⟨*proof*⟩

definition *P-max s* = *enn2real* (*MDP.p s*)
definition *P-min s* = *enn2real* (*MDP.n s*)

lemma
assumes *i* < *state-count mdp*
shows *P-max*: *P-max i* ≤ *real-of-rat* (*solution* (*pos-cert c*) !! *i*) (**is** *?max*)
and *P-min*: *P-min i* ≥ *real-of-rat* (*solution* (*neg-cert c*) !! *i*) (**is** *?min*)
⟨*proof*⟩

end

end

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