Markov Models

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Abstract

This is a formalization of various Markov models in Isabelle/HOL. It builds on Isabelle's probability theory. The available models are currently discrete-time and continuous-time Markov chains as well as Markov decision processes. As application of these models we formalize probabilistic model checking of pCTL formulas, analysis of IPv4 address allocation in ZeroConf and an analysis of the anonymity of the Crowds protocol.

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1 Introduction

This is a formalization of probabilistic models in Isabelle/HOL. It builds on Isabelle's probability theory (HOL-Probability). It provides formalizations for the following models:

- Discrete-time Markov processes with measurable state spaces [2]
- Markov decision processes on discrete spaces [5]
- Continuous-time Markov chains on discrete spaces [2]

As application of these models we formalize

- a probabilistic model checking of pCTL formulas [4],
- an analysis of IPv4 address allocation in ZeroConf [3],
- an analysis of the anonymity of the Crowds protocol [3],
- the reachability analysis on finite-state MDPs [5], and
- expected running-time semantics for pGCL [1].

The formalization of rewarded DTMCs and pCTL model checking is discussed in detail in our paper.

2 Auxiliary Theory

Parts of it should be moved to the Isabelle repository

theory *Markov-Models-Auxiliary*
imports
  HOL−Probability.Probability
  HOL−Library.Rewrite
  HOL−Library.Linear-Temporal-Logic-on-Streams
  Coinductive.Coinductive-Stream
  Coinductive.Coinductive-Nat
begin

lemma *lfp-upperbound*: $(\bigwedge y.\ x \le f\ y) \Longrightarrow x \le \mathit{lfp}\ f$
  unfolding *lfp-def* **by** (*intro Inf-greatest*) (*auto intro*: *order-trans*)

lemma *lfp-arg*: $(\lambda t.\ \mathit{lfp}\ (F\ t)) = \mathit{lfp}\ (\lambda x\ t.\ F\ t\ (x\ t))$
  apply (*auto simp*: *lfp-def le-fun-def fun-eq-iff intro*!: *Inf-eqI Inf-greatest*)
  subgoal for $x\ y$
    by (*rule INF-lower2*[*of top*$(x := y)$]) *auto*
  done

lemma *lfp-pair*: $\mathit{lfp}\ (\lambda f\ (a,\ b).\ F\ (\lambda a\ b.\ f\ (a,\ b))\ a\ b)\ (a,\ b) = \mathit{lfp}\ F\ a\ b$
  unfolding *lfp-def*
  by (*auto intro*!: *INF-eq simp*: *le-fun-def*)
    (*auto intro*!: *exI*[*of* - $\lambda(a,\ b).\ x\ a\ b$ **for** x])

lemma *all-Suc-split*: $(\forall i.\ P\ i) \longleftrightarrow (P\ 0 \wedge (\forall i.\ P\ (\mathit{Suc}\ i)))$
  using *nat-induct* **by** *auto*

definition $\mathit{with}\ P\ f\ d = (\mathit{if}\ \exists x.\ P\ x\ \mathit{then}\ f\ (\mathit{SOME}\ x.\ P\ x)\ \mathit{else}\ d)$

lemma *withI*[*case-names default exists*]:
  $((\bigwedge x.\ \neg\ P\ x) \Longrightarrow Q\ d) \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ (f\ x)) \Longrightarrow Q\ (\mathit{with}\ P\ f\ d)$

unfolding *with-def* **by** (*auto intro*: *someI2*)

context *order*
begin

definition
maximal f S = $\{x{\in}S.\ \forall\, y{\in}S.\ f\ y \le f\ x\}$

lemma *maximalI*: $x \in S \Longrightarrow (\bigwedge y.\ y \in S \Longrightarrow f\ y \le f\ x) \Longrightarrow x \in$ *maximal f S*
by (*simp add*: *maximal-def*)

lemma *maximalI-trans*: $x \in$ *maximal f S* $\Longrightarrow f\ x \le f\ y \Longrightarrow y \in S \Longrightarrow y \in$ *maximal f S*
unfolding *maximal-def* **by** (*blast intro*: *antisym order-trans*)

lemma *maximalD1*: $x \in$ *maximal f S* $\Longrightarrow x \in S$
by (*simp add*: *maximal-def*)

lemma *maximalD2*: $x \in$ *maximal f S* $\Longrightarrow y \in S \Longrightarrow f\ y \le f\ x$
by (*simp add*: *maximal-def*)

lemma *maximal-inject*: $x \in$ *maximal f S* $\Longrightarrow y \in$ *maximal f S* $\Longrightarrow f\ x = f\ y$
by (*rule order.antisym*) (*simp-all add*: *maximal-def*)

lemma *maximal-empty*[*simp*]: *maximal f* $\{\} = \{\}$
by (*simp add*: *maximal-def*)

lemma *maximal-singleton*[*simp*]: *maximal f* $\{x\} = \{x\}$
by (*auto simp add*: *maximal-def*)

lemma *maximal-in-S*: *maximal f S* $\subseteq S$
by (*auto simp*: *maximal-def*)

end

context *linorder*
begin

lemma *maximal-ne*:
assumes *finite S* $S \neq \{\}$
shows *maximal f S* $\neq \{\}$
using *assms*
proof (*induct rule*: *finite-ne-induct*)
case (*insert s S*)
show *?case*
proof *cases*
assume $\forall\, x{\in}S.\ f\ x \le f\ s$
with *insert* **have** $s \in$ *maximal f* (*insert s S*)
by (*auto intro*!: *maximalI*)

```
    then show ?thesis
      by auto
  next
    assume ¬ (∀x∈S. f x ≤ f s)
    then have maximal f (insert s S) = maximal f S
      by (auto simp: maximal-def)
    with insert show ?thesis
      by auto
  qed
qed simp

end

lemma mono-les:
  fixes s S N and l1 l2 :: 'a ⇒ real and K :: 'a ⇒ 'a pmf
  defines Δ x ≡ l2 x − l1 x
  assumes s: s ∈ S and S: (⋃s∈S. set-pmf (K s)) ⊆ S ∪ N
  assumes int-l1[simp]: ⋀s. s ∈ S ⟹ integrable (K s) l1
  assumes int-l2[simp]: ⋀s. s ∈ S ⟹ integrable (K s) l2
  assumes to-N: ⋀s. s ∈ S ⟹ ∃t∈N. (s, t) ∈ (SIGMA s:UNIV. K s)*
  assumes l1: ⋀s. s ∈ S ⟹ (∫t. l1 t ∂K s) + c s ≤ l1 s
  assumes l2: ⋀s. s ∈ S ⟹ l2 s ≤ (∫t. l2 t ∂K s) + c s
  assumes eq: ⋀s. s ∈ N ⟹ l2 s ≤ l1 s
  assumes finitary: finite (Δ ‘ (S∪N))
  shows l2 s ≤ l1 s
proof −
  define M where M = {s∈S∪N. ∀t∈S∪N. Δ t ≤ Δ s}

  have [simp]: ⋀s. s∈S ⟹ integrable (K s) Δ
    by (simp add: Δ-def[abs-def])

  have M-unqiue: ⋀s t. s ∈ M ⟹ t ∈ M ⟹ Δ s = Δ t
    by (auto intro!: antisym simp: M-def)
  have M1: ⋀s. s ∈ M ⟹ s ∈ S ∪ N
    by (auto simp: M-def)
  have M2: ⋀s t. s ∈ M ⟹ t ∈ S ∪ N ⟹ Δ t ≤ Δ s
    by (auto simp: M-def)
  have M3: ⋀s t. s ∈ M ⟹ t ∈ S ∪ N ⟹ t ∉ M ⟹ Δ t < Δ s
    by (auto simp: M-def less-le)

  have N: ∀s∈N. Δ s ≤ 0
    using eq by (simp add: Δ-def)

  { fix s assume s: s ∈ M M ∩ N = {}
    then have s ∈ S − N
      by (auto dest: M1)
    with to-N[of s] obtain t where (s, t) ∈ (SIGMA s:UNIV. K s)* and t ∈ N
      by (auto simp: M-def)
    from this(1) ‹s ∈ M› have Δ s ≤ 0
```

```
  proof (induction rule: converse-rtrancl-induct)
    case (step s s')
    then have s: s ∈ M s ∈ S s ∉ N and s': s' ∈ S ∪ N s' ∈ K s
      using S ‹M ∩ N = {}› by (auto dest: M1)
    have s' ∈ M
    proof (rule ccontr)
      assume s' ∉ M
      with ‹s ∈ S› s' ‹s ∈ M›
      have 0 < pmf (K s) s' Δ s' < Δ s
        by (auto intro: M2 M3 pmf-positive)

      have Δ s ≤ ((∫ t. l2 t ∂K s) + c s) − ((∫ t. l1 t ∂K s) + c s)
        unfolding Δ-def using ‹s ∈ S› ‹s ∉ N› by (intro diff-mono l1 l2) auto
      then have Δ s ≤ (∫ s'. Δ s' ∂K s)
        using ‹s ∈ S› by (simp add: Δ-def)
      also have ... < (∫ s'. Δ s ∂K s)
        using ‹s' ∈ K s› ‹Δ s' < Δ s› ‹s∈S› S ‹s∈M›
        by (intro measure-pmf.integral-less-AE[where A={s'}])
       (auto simp: emeasure-measure-pmf-finite AE-measure-pmf-iff set-pmf-iff [symmetric]
            intro!: M2)
      finally show False
        using measure-pmf.prob-space[of K s] by simp
    qed
    with step.IH ‹t∈N› N have Δ s' ≤ 0 s' ∈ M
      by auto
    with ‹s∈S› show Δ s ≤ 0
      by (force simp: M-def)
  qed (insert N ‹t∈N›, auto) }

show ?thesis
proof cases
  assume M ∩ N = {}
  have Max (Δ ‘(S∪N)) ∈ Δ ‘(S∪N)
    using ‹s ∈ S› by (intro Max-in finitary) auto
  then obtain t where t ∈ S ∪ N Δ t = Max (Δ ‘(S∪N))
    unfolding image-iff by metis
  then have t ∈ M
    by (auto simp: M-def finitary intro!: Max-ge)
  have Δ s ≤ Δ t
    using ‹t∈M› ‹s∈S› by (auto dest: M2)
  also have Δ t ≤ 0
    using ‹t∈M› ‹M ∩ N = {}› by fact
  finally show ?thesis
    by (simp add: Δ-def)
next
  assume M ∩ N ≠ {}
  then obtain t where t ∈ M t ∈ N by auto
  with N ‹s∈S› have Δ s ≤ 0
    by (intro order-trans[of Δ s Δ t 0]) (auto simp: M-def)
```

then show *?thesis*
by (*simp add*: Δ-*def*)
qed
qed

lemma *unique-les*:
fixes *s S N* **and** *l1 l2* :: $'a \Rightarrow real$ **and** *K* :: $'a \Rightarrow 'a\ pmf$
defines $\Delta\ x \equiv l2\ x - l1\ x$
assumes *s*: $s \in S$ **and** *S*: $(\bigcup s{\in}S.\ set\text{-}pmf\ (K\ s)) \subseteq S \cup N$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l1$
assumes $\bigwedge s.\ s \in S \Longrightarrow integrable\ (K\ s)\ l2$
assumes $\bigwedge s.\ s \in S \Longrightarrow \exists t{\in}N.\ (s, t) \in (SIGMA\ s{:}UNIV.\ K\ s)^*$
assumes $\bigwedge s.\ s \in S \Longrightarrow l1\ s = (\int t.\ l1\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in S \Longrightarrow l2\ s = (\int t.\ l2\ t\ \partial K\ s) + c\ s$
assumes $\bigwedge s.\ s \in N \Longrightarrow l2\ s = l1\ s$
assumes *1*: *finite* (Δ ‘ (*S*∪*N*))
shows $l2\ s = l1\ s$
proof −
have *finite* ((λ*x*. *l2 x* − *l1 x*) ‘ (*S*∪*N*))
using *1* **by** (*auto simp*: Δ-*def*[*abs-def*])
moreover then have *finite* (*uminus* ‘ (λ*x*. *l2 x* − *l1 x*) ‘ (*S*∪*N*))
by *auto*
ultimately show *?thesis*
using *assms*
by (*intro antisym mono-les*[*of s S K N l2 l1 c*] *mono-les*[*of s S K N l1 l2 c*])
(*auto simp*: *image-comp comp-def*)
qed

lemma *inf-continuous-suntil-disj*[*order-continuous-intros*]:
assumes *Q*: *inf-continuous Q*
assumes *disj*: $\bigwedge x\ \omega.\ \neg\ (P\ \omega \wedge Q\ x\ \omega)$
shows *inf-continuous* (λ*x*. *P suntil Q x*)
unfolding *inf-continuous-def*
proof (*safe intro*!: *ext*)
fix *M ω i* **assume** $(P\ suntil\ Q\ (\bigsqcap i.\ M\ i))\ \omega$ *decseq M* **then show** (*P suntil Q* (*M i*)) *ω*
unfolding *inf-continuousD*[*OF Q* ‹*decseq M*›] **by** *induction* (*auto intro*: *suntil.intros*)
next
fix *M ω* **assume** ∗: $(\bigsqcap i.\ P\ suntil\ Q\ (M\ i))\ \omega$ *decseq M*
then have (*P suntil Q* (*M 0*)) *ω*
by *auto*
from *this* ∗ **show** $(P\ suntil\ Q\ (\bigsqcap i.\ M\ i))\ \omega$
unfolding *inf-continuousD*[*OF Q* ‹*decseq M*›]
proof *induction*
case (*base ω*) **with** *disj*[*of ω M* -] **show** *?case* **by** (*auto intro*: *suntil.intros elim*: *suntil.cases*)
next
case (*step ω*) **with** *disj*[*of ω M* -] **show** *?case* **by** (*auto intro*: *suntil.intros*

elim: *suntil.cases*)
  qed
qed

lemma *inf-continuous-nxt*[*order-continuous-intros*]: *inf-continuous P* $\Longrightarrow$ *inf-continuous* (λx. *nxt* (*P x*) ω)
  by (*auto simp*: *inf-continuous-def image-comp*)

lemma *sup-continuous-nxt*[*order-continuous-intros*]: *sup-continuous P* $\Longrightarrow$ *sup-continuous* (λx. *nxt* (*P x*) ω)
  by (*auto simp*: *sup-continuous-def image-comp*)

lemma *mcont-ennreal-of-enat*: *mcont Sup* ($\le$) *Sup* ($\le$) *ennreal-of-enat*
  by (*auto intro*!: *mcontI monotoneI contI ennreal-of-enat-Sup*)

lemma *mcont2mcont-ennreal-of-enat*[*cont-intro*]:
  mcont lub ord Sup ($\le$) *f* $\Longrightarrow$ *mcont lub ord Sup* ($\le$) (λx. *ennreal-of-enat* (*f x*))
  by (*auto intro*: *ccpo.mcont2mcont*[*OF complete-lattice-ccpo'*] *mcont-ennreal-of-enat*)

declare *stream.exhaust*[*cases type*: *stream*]

lemma *scount-eq-emeasure*: *scount P* ω = *emeasure* (*count-space UNIV*) {*i*. *P* (*sdrop i* ω)}
proof *cases*
  assume *alw* (*ev P*) ω
  moreover then have *infinite* {*i*. *P* (*sdrop i* ω)}
    using *infinite-iff-alw-ev*[*of P* ω] **by** *simp*
  ultimately show *?thesis*
    by (*simp add*: *scount-infinite-iff*[*symmetric*])
next
  assume ¬ *alw* (*ev P*) ω
  moreover then have *finite* {*i*. *P* (*sdrop i* ω)}
    using *infinite-iff-alw-ev*[*of P* ω] **by** *simp*
  ultimately show *?thesis*
    by (*simp add*: *not-alw-iff not-ev-iff scount-eq-card*)
qed

lemma *measurable-scount*[*measurable*]:
  assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
  shows *scount P* $\in$ *measurable* (*stream-space M*) (*count-space UNIV*)
  unfolding *scount-eq*[*abs-def*] **by** *measurable*

lemma *measurable-sfirst2*:
  assumes [*measurable*]: *Measurable.pred* ($N \bigotimes_M$ *stream-space M*) ($\lambda(x, \omega)$. *P x* ω)
  shows ($\lambda(x, \omega)$. *sfirst* (*P x*) ω) $\in$ *measurable* ($N \bigotimes_M$ *stream-space M*) (*count-space UNIV*)
  apply (*coinduction rule*: *measurable-enat-coinduct*)
  apply *simp*

apply (*rule exI*[*of* - $\lambda x.\ 0$])
apply (*rule exI*[*of* - $\lambda(x, \omega).\ (x,\ stl\ \omega)$])
apply (*rule exI*[*of* - $\lambda(x, \omega).\ P\ x\ \omega$])
apply (*subst sfirst.simps*[*abs-def*])
apply (*simp add*: *fun-eq-iff*)
done

lemma *measurable-sfirst2'*[*measurable* (*raw*)]:
assumes [*measurable* (*raw*)]: $f \in N \to_M$ *stream-space M Measurable.pred* ($N \bigotimes_M$ *stream-space M*) ($\lambda x.\ P\ (fst\ x)\ (snd\ x)$)
shows ($\lambda x.$ *sfirst* $(P\ x)\ (f\ x)) \in$ *measurable N* (*count-space UNIV*)
using *measurable-sfirst2*[*measurable*] **by** *measurable*

lemma *measurable-sfirst*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *sfirst* $P \in$ *measurable* (*stream-space M*) (*count-space UNIV*)
by *measurable*

lemma *measurable-epred*[*measurable*]: *epred* $\in$ *count-space UNIV* $\to_M$ *count-space UNIV*
by (*rule measurable-count-space*)

lemma *nn-integral-stretch*:
$f \in borel \to_M borel \Longrightarrow c \neq 0 \Longrightarrow (\int^+ x.\ f\ (c * x)\ \partial lborel) = (1\ /\ |c|::real) * (\int^+ x.\ f\ x\ \partial lborel)$
using *nn-integral-real-affine*[*of f c 0*] **by** (*simp add*: *mult.assoc*[*symmetric*] *ennreal-mult*[*symmetric*])

lemma *prod-sum-distrib*:
fixes *f g* :: $'a \Rightarrow 'b \Rightarrow 'c$::*comm-semiring-1*
assumes *finite I* **shows** $(\bigwedge i.\ i \in I \Longrightarrow finite\ (J\ i)) \Longrightarrow (\prod i{\in}I.\ \sum j{\in}J\ i.\ f\ i\ j) = (\sum m{\in}Pi_E\ I\ J.\ \prod i{\in}I.\ f\ i\ (m\ i))$
using ‹*finite I*›
proof *induction*
case (*insert i I*) **then show** *?case*
by (*auto simp*: *PiE-insert-eq finite-PiE sum.reindex inj-combinator sum.swap*[*of* - Pi_E *I J*]
sum.cartesian-product' *sum-distrib-left sum-distrib-right*
intro!: *sum.cong prod.cong arg-cong*[**where** *f*=(∗) *x* **for** *x*])
qed *simp*

lemma *prod-add-distrib*:
fixes *f g* :: $'a \Rightarrow 'b$::*comm-semiring-1*
assumes *finite I* **shows** $(\prod i{\in}I.\ f\ i + g\ i) = (\sum J{\in}Pow\ I.\ (\prod i{\in}J.\ f\ i) * (\prod i{\in}I - J.\ g\ i))$
proof −
have $(\prod i{\in}I.\ f\ i + g\ i) = (\prod i{\in}I.\ \sum b{\in}\{True,\ False\}.\ if\ b\ then\ f\ i\ else\ g\ i)$
by *simp*
also have ... $= (\sum m{\in}I \to_E \{True,\ False\}.\ \prod i{\in}I.\ if\ m\ i\ then\ f\ i\ else\ g\ i)$

using ‹*finite I*› **by** (*rule prod-sum-distrib*) *simp*
also have ... = ($\sum$ *J*∈*Pow I*. ($\prod$ *i*∈*J*. *f i*) ∗ ($\prod$ *i*∈*I* − *J*. *g i*))
by (*rule sum.reindex-bij-witness*[**where** *i*=λ*J*. λ*i*∈*I*. *i*∈*J* **and** *j*=λ*m*. {*i*∈*I*. *m i*}])
(*auto simp*: *fun-eq-iff prod.If-cases* ‹*finite I*› *intro*!: *arg-cong2*[**where** *f*=(∗)] *prod.cong*)
finally show *?thesis* **.**
qed

subclass (**in** *linordered-nonzero-semiring*) *ordered-semiring-0*
proof qed

lemma (**in** *linordered-nonzero-semiring*) *prod-nonneg*: (∀ *a*∈*A*. *0* ≤ *f a*) ⟹ *0* ≤ *prod f A*
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma (**in** *linordered-nonzero-semiring*) *prod-mono*:
∀ *i*∈*A*. *0* ≤ *f i* ∧ *f i* ≤ *g i* ⟹ *prod f A* ≤ *prod g A*
by (*induct A rule*: *infinite-finite-induct*) (*auto intro*!: *prod-nonneg mult-mono*)

lemma (**in** *linordered-nonzero-semiring*) *prod-mono2*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *g i* ∧ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *1* ≤ *f i*)
shows *prod g I* ≤ *prod f J*
proof −
have *prod g I* = ($\prod$ *i*∈*J*. *if i* ∈ *I then g i else 1*)
using ‹*finite J*› ‹*I* ⊆ *J*› **by** (*simp add*: *prod.If-cases Int-absorb1*)
also have ... ≤ *prod f J*
using *assms* **by** (*intro prod-mono*) *auto*
finally show *?thesis* **.**
qed

lemma (**in** *linordered-nonzero-semiring*) *prod-mono3*:
assumes *finite J I* ⊆ *J* ⋀*i*. *i* ∈ *J* ⟹ *0* ≤ *g i* ⋀*i*. *i* ∈ *I* ⟹ *g i* ≤ *f i* (⋀*i*. *i* ∈ *J* − *I* ⟹ *g i* ≤ *1*)
shows *prod g J* ≤ *prod f I*
proof −
have *prod g J* ≤ ($\prod$ *i*∈*J*. *if i* ∈ *I then f i else 1*)
using *assms* **by** (*intro prod-mono*) *auto*
also have ... = *prod f I*
using ‹*finite J*› ‹*I* ⊆ *J*› **by** (*simp add*: *prod.If-cases Int-absorb1*)
finally show *?thesis* **.**
qed

lemma (**in** *linordered-nonzero-semiring*) *one-le-prod*: (⋀*i*. *i* ∈ *I* ⟹ *1* ≤ *f i*) ⟹ *1* ≤ *prod f I*
proof (*induction I rule*: *infinite-finite-induct*)
case (*insert i I*) **then show** *?case*
using *mult-mono*[*of 1 f i 1 prod f I*]

by (*auto intro*: *order-trans*[*OF zero-le-one*])
qed *auto*

lemma *sum-plus-one-le-prod-plus-one*:
fixes p :: *'a* ⇒ *'b*::*linordered-nonzero-semiring*
assumes $\bigwedge i.\ i \in I \Longrightarrow 0 \leq p\ i$
shows $(\sum i{\in}I.\ p\ i) + 1 \leq (\prod i{\in}I.\ p\ i + 1)$
proof *cases*
assume [*simp*]: *finite I*
with *assms* **have** [*simp*]: $J \subseteq I \Longrightarrow 0 \leq$ *prod p J* **for** *J*
by (*intro prod-nonneg*) *auto*
have $1 + (\sum i{\in}I.\ p\ i) = (\sum J{\in}insert\ \{\}\ ((\lambda x.\ \{x\})\ `I).\ (\prod i{\in}J.\ p\ i) * (\prod i{\in}I - J.\ 1))$
by (*subst sum.insert*) (*auto simp*: *sum.reindex*)
also have $\ldots \leq (\sum J{\in}Pow\ I.\ (\prod i{\in}J.\ p\ i) * (\prod i{\in}I - J.\ 1))$
using *assms* **by** (*intro sum-mono2*) *auto*
finally show *?thesis*
by (*subst prod-add-distrib*) (*auto simp*: *add.commute*)
qed *simp*

lemma *summable-iff-convergent-prod*:
fixes p :: *nat* ⇒ *real* **assumes** p: $\bigwedge i.\ 0 \leq p\ i$
shows *summable p* ⟷ *convergent* $(\lambda n.\ \prod i{<}n.\ p\ i + 1)$
unfolding *summable-iff-convergent*
proof
assume *convergent* $(\lambda n.\ \prod i{<}n.\ p\ i + 1)$
then obtain x **where** x: $(\lambda n.\ \prod i{<}n.\ p\ i + 1) \longrightarrow x$
by (*auto simp*: *convergent-def*)
then have $1 \leq x$
by (*rule tendsto-lowerbound*) (*auto intro*!: *always-eventually one-le-prod p*)

have *convergent* $(\lambda n.\ 1 + (\sum i{<}n.\ p\ i))$
proof (*intro Bseq-mono-convergent BseqI allI*)
show $0 < x$ **using** ‹$1 \leq x$› **by** *auto*
next
fix n
have *norm* $((\sum i{<}n.\ p\ i) + 1) \leq (\prod i{<}n.\ p\ i + 1)$
using p **by** (*simp add*: *sum-nonneg sum-plus-one-le-prod-plus-one p*)
also have $\ldots \leq x$
using *assms*
by (*intro tendsto-lowerbound*[*OF x*])
(*auto simp*: *eventually-sequentially intro*!: *exI*[*of - n*] *prod-mono2*)
finally show *norm* $(1 + sum\ p\ \{..{<}n\}) \leq x$
by (*simp add*: *add.commute*)
qed (*insert p*, *auto intro*!: *sum-mono2*)
then show *convergent* $(\lambda n.\ \sum i{<}n.\ p\ i)$
unfolding *convergent-add-const-iff* **.**
next
assume *convergent* $(\lambda n.\ \sum i{<}n.\ p\ i)$

```
  then obtain x where x: (λn. exp (∑ i<n. p i)) ⟶ exp x
    by (force simp: convergent-def intro!: tendsto-exp)
  show convergent (λn. ∏ i<n. p i + 1)
  proof (intro Bseq-mono-convergent BseqI allI)
    show 0 < exp x by simp
  next
    fix n
    have norm (∏ i<n. p i + 1) ≤ exp (∑ i<n. p i)
      using p exp-ge-add-one-self[of p -] by (auto simp add: prod-nonneg exp-sum
add.commute intro!: prod-mono)
    also have ... ≤ exp x
      using p
     by (intro tendsto-lowerbound[OF x]) (auto simp: eventually-sequentially intro!:
sum-mono2 )
    finally show norm (∏ i<n. p i + 1) ≤ exp x .
  qed (insert p, auto intro!: prod-mono2)
qed

primrec eexp :: ereal ⇒ ennreal
  where
    eexp MInfty = 0
  | eexp (ereal r) = ennreal (exp r)
  | eexp PInfty = top

lemma
  shows eexp-minus-infty[simp]: eexp (−∞) = 0
    and eexp-infty[simp]: eexp ∞ = top
  using eexp.simps by simp-all

lemma eexp-0[simp]: eexp 0 = 1
  by (simp add: zero-ereal-def)

lemma eexp-inj[simp]: eexp x = eexp y ⟷ x = y
  by (cases x; cases y; simp)

lemma eexp-mono[simp]: eexp x ≤ eexp y ⟷ x ≤ y
  by (cases x; cases y; simp add: top-unique)

lemma eexp-strict-mono[simp]: eexp x < eexp y ⟷ x < y
  by (simp add: less-le)

lemma exp-eq-0-iff[simp]: eexp x = 0 ⟷ x = −∞
  using eexp-inj[of x −∞] unfolding eexp-minus-infty .

lemma eexp-surj: range eexp = UNIV
proof −
  have part: UNIV = {0} ∪ {0 <..< top} ∪ {top::ennreal}
    by (auto simp: less-top)
  show ?thesis
```

unfolding *part*
by (*force simp*: *image-iff less-top less-top-ennreal intro*!: *eexp.simps*[*symmetric*] *eexp.simps dest*: *exp-total*)
qed

lemma *continuous-on-eexp*′: *continuous-on UNIV eexp*
by (*rule continuous-onI-mono*) (*auto simp*: *eexp-surj*)

lemma *continuous-on-eexp*[*continuous-intros*]: *continuous-on A f* $\Longrightarrow$ *continuous-on A* ($\lambda x.$ *eexp* (*f x*))
by (*rule continuous-on-compose2*[*OF continuous-on-eexp*′]) *auto*

lemma *tendsto-eexp*[*tendsto-intros*]: ($f \longrightarrow x$) $F \Longrightarrow$ (($\lambda x.$ *eexp* (*f x*)) $\longrightarrow$ *eexp x*) *F*
by (*rule continuous-on-tendsto-compose*[*OF continuous-on-eexp*′]) *auto*

lemma *measurable-eexp*[*measurable*]: *eexp* $\in$ *borel* $\rightarrow_M$ *borel*
using *continuous-on-eexp*′ **by** (*rule borel-measurable-continuous-onI*)

lemma *eexp-add*: $\neg$ (($x = \infty \wedge y = -\infty$) $\vee$ ($x = -\infty \wedge y = \infty$)) $\Longrightarrow$ *eexp* ($x + y$) = *eexp x* $*$ *eexp y*
by (*cases x*; *cases y*; *simp add*: *exp-add ennreal-mult ennreal-top-mult ennreal-mult-top*)

lemma *sum-Pinfty*:
fixes *f* :: ′*a* $\Rightarrow$ *ereal*
shows *sum f I* = $\infty \longleftrightarrow$ (*finite I* $\wedge$ ($\exists\, i \in I.\ f\ i = \infty$))
by (*induction I rule*: *infinite-finite-induct*) *auto*

lemma *sum-Minfty*:
fixes *f* :: ′*a* $\Rightarrow$ *ereal*
shows *sum f I* = $-\infty \longleftrightarrow$ (*finite I* $\wedge \neg$ ($\exists\, i \in I.\ f\ i = \infty$) $\wedge$ ($\exists\, i \in I.\ f\ i = -\infty$))
by (*induction I rule*: *infinite-finite-induct*)
(*auto simp*: *sum-Pinfty*)

lemma *eexp-sum*: $\neg$ ($\exists\, i \in I.\ \exists\, j \in I.\ f\ i = -\infty \wedge f\ j = \infty$) $\Longrightarrow$ *eexp* ($\sum i \in I.\ f\ i$) = ($\prod i \in I.$ *eexp* (*f i*))
proof (*induction I rule*: *infinite-finite-induct*)
case (*insert i I*)
have *eexp* (*sum f* (*insert i I*)) = *eexp* (*f i*) $*$ *eexp* (*sum f I*)
using *insert.prems insert.hyps* **by** (*auto simp*: *sum-Pinfty sum-Minfty intro*!: *eexp-add*)
then show *?case*
using *insert* **by** *auto*
qed *simp-all*

lemma *eexp-suminf*:
assumes *wf-f*: $\neg$ $\{-\infty, \infty\} \subseteq$ *range f* **and** *f*: *summable f*
shows ($\lambda n.\ \prod i<n.$ *eexp* (*f i*)) $\longrightarrow$ *eexp* ($\sum i.\ f\ i$)
proof –

```
  have (λn. eexp (∑ i<n. f i)) ⟶ eexp (∑ i. f i)
    by (intro tendsto-eexp summable-LIMSEQ f)
  also have (λn. eexp (∑ i<n. f i)) = (λn. ∏ i<n. eexp (f i))
    using wf-f by (auto simp: fun-eq-iff image-iff eq-commute intro!: eexp-sum)
  finally show ?thesis .
qed

lemma continuous-onI-antimono:
  fixes f :: 'a::linorder-topology ⇒ 'b::{dense-order,linorder-topology}
  assumes open (f'A)
    and mono: ⋀x y. x ∈ A ⟹ y ∈ A ⟹ x ≤ y ⟹ f y ≤ f x
  shows continuous-on A f
proof (rule continuous-on-generate-topology[OF open-generated-order], safe)
  have monoD: ⋀x y. x ∈ A ⟹ y ∈ A ⟹ f y < f x ⟹ x < y
    by (auto simp: not-le[symmetric] mono)
  have ∃ x. x ∈ A ∧ f x < b ∧ x < a if a: a ∈ A and fa: f a < b for a b
  proof −
    obtain y where f a < y {f a ..< y} ⊆ f'A
      using open-right[OF ‹open (f'A)›, of f a b] a fa
      by auto
    obtain z where z: f a < z z < min b y
      using dense[of f a min b y] ‹f a < y› ‹f a < b› by auto
    then obtain c where z = f c c ∈ A
      using ‹{f a ..< y} ⊆ f'A›[THEN subsetD, of z] by (auto simp: less-imp-le)
    with a z show ?thesis
      by (auto intro!: exI[of - c] simp: monoD)
  qed
  then show ∃ C. open C ∧ C ∩ A = f −' {..<b} ∩ A for b
    by (intro exI[of - (⋃x∈{x∈A. f x < b}. {x <..})])
      (auto intro: le-less-trans[OF mono] less-imp-le)

  have ∃ x. x ∈ A ∧ b < f x ∧ x > a if a: a ∈ A and fa: b < f a for a b
  proof −
    note a fa
    moreover
    obtain y where y < f a {y <.. f a} ⊆ f'A
      using open-left[OF ‹open (f'A)›, of f a b]  a fa
      by auto
    then obtain z where z: max b y < z z < f a
      using dense[of max b y f a] ‹y < f a› ‹b < f a› by auto
    then obtain c where z = f c c ∈ A
      using ‹{y <.. f a} ⊆ f'A›[THEN subsetD, of z] by (auto simp: less-imp-le)
    with a z show ?thesis
      by (auto intro!: exI[of - c] simp: monoD)
  qed
  then show ∃ C. open C ∧ C ∩ A = f −' {b <..} ∩ A for b
    by (intro exI[of - (⋃x∈{x∈A. b < f x}. {..< x})])
      (auto intro: less-le-trans[OF - mono] less-imp-le)
qed
```

lemma *minus-add-eq-ereal*: $\neg$ (($a = \infty \wedge b = -\infty$) $\vee$ ($a = -\infty \wedge b = \infty$)) $\Longrightarrow$ $-$ (a + b::*ereal*) = $-a$ $-$ b
by (*cases a*; *cases b*; *simp*)

lemma *setsum-negf-ereal*: $\neg$ $\{-\infty, \infty\} \subseteq f`I \Longrightarrow (\sum i{\in}I. - f\ i) = - (\sum i{\in}I. f$ i::*ereal*)
by (*induction I rule*: *infinite-finite-induct*)
(*auto simp*: *minus-add-eq-ereal sum-Minfty sum-Pinfty*,
(*subst minus-add-eq-ereal*; *auto simp*: *sum-Pinfty sum-Minfty image-iff minus-ereal-def*)+)

lemma *convergent-minus-iff-ereal*: *convergent* ($\lambda x. - f\ x$::*ereal*) $\longleftrightarrow$ *convergent f*
unfolding *convergent-def* **by** (*metis ereal-uminus-uminus ereal-Lim-uminus*)

lemma *summable-minus-ereal*: $\neg$ $\{-\infty, \infty\} \subseteq$ *range f* $\Longrightarrow$ *summable* ($\lambda n.\ f\ n$) $\Longrightarrow$ *summable* ($\lambda n. - f\ n$::*ereal*)
unfolding *summable-iff-convergent*
by (*subst setsum-negf-ereal*) (*auto simp*: *convergent-minus-iff-ereal*)

lemma (**in** *product-prob-space*) *product-nn-integral-component*:
assumes $f \in$ *borel-measurable* ($M\ i$)$i \in I$
shows $integral^N$ (Pi_M I M) ($\lambda x.\ f\ (x\ i)$) = $integral^N$ ($M\ i$) f
proof $-$
from *assms* **show** *?thesis*
apply (*subst PiM-component*[*symmetric*, *OF* ‹$i \in I$›])
apply (*subst nn-integral-distr*[*OF measurable-component-singleton*])
apply *simp-all*
done
qed

lemma *ennreal-inverse-le*[*simp*]: *inverse x* $\leq$ *inverse y* $\longleftrightarrow$ $y \leq$ (x::*ennreal*)
by (*cases 0* < *x*; *cases x*; *cases 0* < *y*; *cases y*; *auto simp*: *top-unique inverse-ennreal*)

lemma *inverse-inverse-ennreal*[*simp*]: *inverse* (*inverse x*::*ennreal*) = x
by (*cases 0* < *x*; *cases x*; *auto simp*: *inverse-ennreal*)

lemma *range-inverse-ennreal*: *range inverse* = (*UNIV*::*ennreal set*)
proof $-$
have $\exists x.\ y$ = *inverse x* **for** *y* :: *ennreal*
by (*intro exI*[*of - inverse y*]) *simp*
then show *?thesis*
unfolding *surj-def* **by** *auto*
qed

lemma *continuous-on-inverse-ennreal'*: *continuous-on* (*UNIV* :: *ennreal set*) *inverse*
by (*rule continuous-onI-antimono*) (*auto simp*: *range-inverse-ennreal*)

lemma *sums-minus-ereal*: $\neg$ $\{-\ \infty,\ \infty\} \subseteq f$ ‘ *UNIV* $\Longrightarrow$ ($\lambda n.$ $-$ *f n*::*ereal*) *sums* $x \Longrightarrow f$ *sums* $-$ x
  unfolding *sums-def*
  apply (*subst ereal-Lim-uminus*)
  apply (*subst* (*asm*) *setsum-negf-ereal*)
  apply *auto*
  done

lemma *suminf-minus-ereal*: $\neg$ $\{-\ \infty,\ \infty\} \subseteq f$ ‘ *UNIV* $\Longrightarrow$ *summable f* $\Longrightarrow$ ($\sum n.$ $-$ *f n* :: *ereal*) = $-$ *suminf f*
  apply (*rule sums-unique*[*symmetric*])
  apply (*rule sums-minus-ereal*)
  apply (*auto simp*: *ereal-uminus-eq-reorder*)
  done

end

3 Discrete-Time Markov Chain

theory *Discrete-Time-Markov-Chain*
  imports *Markov-Models-Auxiliary*
begin

Markov chain with discrete time steps and discrete state space.

lemma *sstart-eq'*: *sstart* Ω (x # *xs*) = $\{\omega.$ *shd* $\omega = x \land$ *stl* $\omega \in$ *sstart* Ω *xs*$\}$
  by (*auto simp*: *sstart-eq*)

lemma *measure-eq-stream-space-coinduct*[*consumes 1*, *case-names left right cont*]:
  assumes *R N M*
   assumes *R-1*: $\bigwedge N\ M.$ *R N M* $\Longrightarrow$ $N \in$ *space* (*prob-algebra* (*stream-space* (*count-space UNIV*)))
   and *R-2*: $\bigwedge N\ M.$ *R N M* $\Longrightarrow$ $M \in$ *space* (*prob-algebra* (*stream-space* (*count-space UNIV*)))
    and *cont*: $\bigwedge N\ M.$ *R N M* $\Longrightarrow \exists N'\ M'\ p.$ ($\forall y \in$*set-pmf p*. *R* (N' *y*) (M' *y*)) $\land$
      ($\forall x.$ N' $x \in$ *space* (*prob-algebra* (*stream-space* (*count-space UNIV*)))) $\land$ ($\forall x.$ M' $x \in$ *space* (*prob-algebra* (*stream-space* (*count-space UNIV*)))) $\land$
      N = (*measure-pmf p* $\gg=$ ($\lambda y.$ *distr* (N' *y*) (*stream-space* (*count-space UNIV*)) ((##) *y*))) $\land$
      M = (*measure-pmf p* $\gg=$ ($\lambda y.$ *distr* (M' *y*) (*stream-space* (*count-space UNIV*)) ((##) *y*)))
  shows $N = M$
proof $-$
  let *?S* = *stream-space* (*count-space UNIV*)
  have $\forall N\ M.$ *R N M* $\longrightarrow$ ($\exists N'\ M'\ p.$ ($\forall y \in$*set-pmf p*. *R* (N' *y*) (M' *y*)) $\land$
      ($\forall x.$ N' $x \in$ *space* (*prob-algebra ?S*)) $\land$ ($\forall x.$ M' $x \in$ *space* (*prob-algebra ?S*)) $\land$
      N = (*measure-pmf p* $\gg=$ ($\lambda y.$ *distr* (N' *y*) *?S* ((##) *y*))) $\land$
      M = (*measure-pmf p* $\gg=$ ($\lambda y.$ *distr* (M' *y*) *?S* ((##) *y*))))

using *cont* **by** *auto*
then obtain *n m p* **where**
p: $\bigwedge N\ M\ y.\ R\ N\ M \Longrightarrow y \in \textit{set-pmf}\ (p\ N\ M) \Longrightarrow R\ (n\ N\ M\ y)\ (m\ N\ M\ y)$ **and**
n: $\bigwedge N\ M\ x.\ R\ N\ M \Longrightarrow n\ N\ M\ x \in \textit{space}\ (\textit{prob-algebra}\ ?S)$ **and**
n-eq: $\bigwedge N\ M\ y.\ R\ N\ M \Longrightarrow N = (\textit{measure-pmf}\ (p\ N\ M) \gg= (\lambda y.\ \textit{distr}\ (n\ N\ M\ y)\ ?S\ ((\#\#)\ y)))$ **and**
m: $\bigwedge N\ M\ x.\ R\ N\ M \Longrightarrow m\ N\ M\ x \in \textit{space}\ (\textit{prob-algebra}\ ?S)$ **and**
m-eq: $\bigwedge N\ M\ y.\ R\ N\ M \Longrightarrow M = (\textit{measure-pmf}\ (p\ N\ M) \gg= (\lambda y.\ \textit{distr}\ (m\ N\ M\ y)\ ?S\ ((\#\#)\ y)))$
unfolding *choice-iff′ choice-iff* **by** *blast*

define *A* **where** $A = (\textit{SIGMA}\ nm{:}\textit{UNIV}.\ (\lambda x.\ (n\ (\textit{fst}\ nm)\ (\textit{snd}\ nm)\ x,\ m\ (\textit{fst}\ nm)\ (\textit{snd}\ nm)\ x))\ \text{‘}\ p\ (\textit{fst}\ nm)\ (\textit{snd}\ nm))$
have *A-singleton*: $A\ \text{‘‘}\ \{nm\} = (\lambda x.\ (n\ (\textit{fst}\ nm)\ (\textit{snd}\ nm)\ x,\ m\ (\textit{fst}\ nm)\ (\textit{snd}\ nm)\ x))\ \text{‘}\ p\ (\textit{fst}\ nm)\ (\textit{snd}\ nm)$ **for** *nm*
by (*auto simp*: *A-def*)

have *sets-n*[*measurable-cong*, *simp*]: $\textit{sets}\ (n\ N\ M\ y) = \textit{sets}\ ?S$ **if** $R\ N\ M$ **for** $N\ M\ y$
using *n*[*OF that*, *of y*] **by** (*auto simp*: *space-prob-algebra*)
have *sets-m*[*measurable-cong*, *simp*]: $\textit{sets}\ (m\ N\ M\ y) = \textit{sets}\ ?S$ **if** $R\ N\ M$ **for** $N\ M\ y$
using *m*[*OF that*, *of y*] **by** (*auto simp*: *space-prob-algebra*)
have [*simp*]: $R\ N\ M \Longrightarrow \textit{prob-space}\ (n\ N\ M\ y)$ **for** $N\ M\ y$
using *n*[*of N M y*] **by** (*auto simp*: *space-prob-algebra*)
have [*simp*]: $R\ N\ M \Longrightarrow \textit{prob-space}\ (m\ N\ M\ y)$ **for** $N\ M\ y$
using *m*[*of N M y*] **by** (*auto simp*: *space-prob-algebra*)
have [*measurable*]: $R\ N\ M \Longrightarrow n\ N\ M \in \textit{count-space}\ \textit{UNIV} \rightarrow_M \textit{subprob-algebra}\ ?S$ **for** $N\ M$
by (*rule measurable-prob-algebraD*) (*auto intro*: *n*)
have [*measurable*]: $R\ N\ M \Longrightarrow m\ N\ M \in \textit{count-space}\ \textit{UNIV} \rightarrow_M \textit{subprob-algebra}\ ?S$ **for** $N\ M$
by (*rule measurable-prob-algebraD*) (*auto intro*: *m*)

define n' **where** $n'\ N\ M\ y = \textit{distr}\ (n\ N\ M\ y)\ ?S\ ((\#\#)\ y)$ **for** $N\ M\ y$
define m' **where** $m'\ N\ M\ y = \textit{distr}\ (m\ N\ M\ y)\ ?S\ ((\#\#)\ y)$ **for** $N\ M\ y$
have *n′-eq*: $R\ N\ M \Longrightarrow N = (\textit{measure-pmf}\ (p\ N\ M) \gg= n'\ N\ M)$ **for** $N\ M$
unfolding *n′-def* **by** (*rule n-eq*)
have *m′-eq*: $R\ N\ M \Longrightarrow M = (\textit{measure-pmf}\ (p\ N\ M) \gg= m'\ N\ M)$ **for** $N\ M$
unfolding *m′-def* **by** (*rule m-eq*)
have [*measurable*]: $R\ N\ M \Longrightarrow n'\ N\ M \in \textit{count-space}\ \textit{UNIV} \rightarrow_M \textit{subprob-algebra}\ ?S$ **for** $N\ M$
unfolding *n′-def* **by** (*rule measurable-distr2*[**where** *M*=*?S*]) *measurable*
have [*measurable*]: $R\ N\ M \Longrightarrow m'\ N\ M \in \textit{count-space}\ \textit{UNIV} \rightarrow_M \textit{subprob-algebra}\ ?S$ **for** $N\ M$
unfolding *m′-def* **by** (*rule measurable-distr2*[**where** *M*=*?S*]) *measurable*

have *n′-shd*: $R\ N\ M \Longrightarrow \textit{distr}\ (n'\ N\ M\ y)\ (\textit{count-space}\ \textit{UNIV})\ \textit{shd} = \textit{measure-pmf}$

(*return-pmf y*) **for** *N M y*
unfolding *n'-def* **by** (*subst distr-distr*) (*auto simp*: *comp-def prob-space.distr-const return-pmf.rep-eq*)
have *m'-shd*: *R N M* ⟹ *distr* (*m' N M y*) (*count-space UNIV*) *shd* = *measure-pmf* (*return-pmf y*) **for** *N M y*
unfolding *m'-def* **by** (*subst distr-distr*) (*auto simp*: *comp-def prob-space.distr-const return-pmf.rep-eq*)
have *n'-stl*: *R N M* ⟹ *distr* (*n' N M y*) *?S stl* = *n N M y* **for** *N M y*
unfolding *n'-def* **by** (*subst distr-distr*) (*auto simp*: *comp-def distr-id2*)
have *m'-stl*: *R N M* ⟹ *distr* (*m' N M y*) *?S stl* = *m N M y* **for** *N M y*
unfolding *m'-def* **by** (*subst distr-distr*) (*auto simp*: *comp-def distr-id2*)

define *F* **where** *F* = (A^* `` {(*N*, *M*)})
have *countable F*
unfolding *F-def*
apply (*intro countable-rtrancl countable-insert*[*of* - (*N*, *M*)] *countable-empty*)
apply (*rule countable-Image*)
apply (*auto simp*: *A-singleton*)
done
have *F-NM*[*simp*]: (*N*, *M*) ∈ *F* **unfolding** *F-def* **by** *auto*
have *R-F*[*simp*]: *R N' M'* **if** (*N'*, *M'*) ∈ *F* **for** *N' M'*
proof −
have ((*N*, *M*), (*N'*, *M'*)) ∈ A^* **using** *that* **by** (*auto simp*: *F-def*)
then show *R N' M'*
by (*induction p*==(*N'*, *M'*) *arbitrary*: *N' M' rule*: *rtrancl-induct*) (*auto simp*: ‹*R N M*› *A-def p*)
qed
have *nm-F*: (*n N' M' y*, *m N' M' y*) ∈ *F* **if** *y* ∈ *p N' M'* (*N'*, *M'*) ∈ *F* **for** *N' M' y*
proof −
have ∗: ((*N*, *M*), (*N'*, *M'*)) ∈ A^* **using** *that* **by** (*auto simp*: *F-def*)
with *that* **show** *?thesis*
apply (*simp add*: *F-def*)
apply (*intro rtrancl.rtrancl-into-rtrancl*[*OF* ∗])
apply (*auto simp*: *A-def*)
done
qed

define Ω **where** Ω = (⋃(*n*, *m*)∈*F*. *set-pmf* (*p n m*))
have [*measurable*]: Ω ∈ *sets* (*count-space UNIV*) **by** *auto*
have *in*-Ω: (*N*, *M*) ∈ *F* ⟹ *y* ∈ *p N M* ⟹ *y* ∈ Ω **for** *N M y*
by (*auto simp*: Ω-*def Bex-def*)

show *?thesis*
proof (*intro stream-space-eq-sstart*)
from ‹*countable F*› **show** *countable* Ω
by (*auto simp add*: Ω-*def*)
show *prob-space N prob-space M sets N* = *sets ?S sets M* = *sets ?S*
using *R-1*[*OF* ‹*R N M*›] *R-2*[*OF* ‹*R N M*›] **by** (*auto simp add*: *space-prob-algebra*)

have $\bigwedge N\ M.\ (N,\ M) \in F \Longrightarrow AE\ x\ in\ N.\ x\ !!\ i \in \Omega$ **for** *i*
proof (*induction i*)
case *0* **note** *NM* = *0*[*THEN R-F*, *simp*] **show** *?case*
apply (*subst n′-eq*[*OF NM*])
apply (*subst AE-bind*[**where** *B*=*?S*])
apply *measurable*
apply (*auto intro*!: *AE-distrD*[**where** *f*=*shd* **and** *M′*=*count-space UNIV*]
simp: *AE-measure-pmf-iff n*[*OF NM*] *n′-shd in-*Ω[*OF 0*] *cong*:
AE-cong-simp)
done
next
case (*Suc i*) **note** *NM* = *Suc*(*2*)[*THEN R-F*, *simp*]
show *?case*
apply (*subst n′-eq*[*OF NM*])
apply (*subst AE-bind*[**where** *B*=*?S*])
apply *measurable*
apply (*auto intro*!: *AE-distrD*[**where** *f*=*stl* **and** *M′*=*?S*] *Suc*(*1*)[*OF nm-F*]
Suc(*2*)
simp: *AE-measure-pmf-iff n′-stl cong*: *AE-cong-simp*)
done
qed
then have *AE-N*: $\bigwedge N\ M.\ (N,\ M) \in F \Longrightarrow AE\ x\ in\ N.\ x \in streams\ \Omega$
unfolding *streams-iff-snth AE-all-countable* **by** *auto*
then show $AE\ x\ in\ N.\ x \in streams\ \Omega$ **by** (*blast intro*: *F-NM*)

have $\bigwedge N\ M.\ (N,\ M) \in F \Longrightarrow AE\ x\ in\ M.\ x\ !!\ i \in \Omega$ **for** *i*
proof (*induction i arbitrary*: *N M*)
case *0* **note** *NM* = *0*[*THEN R-F*, *simp*] **show** *?case*
apply (*subst m′-eq*[*OF NM*])
apply (*subst AE-bind*[**where** *B*=*?S*])
apply *measurable*
apply (*auto intro*!: *AE-distrD*[**where** *f*=*shd* **and** *M′*=*count-space UNIV*]
simp: *AE-measure-pmf-iff m*[*OF NM*] *m′-shd in-*Ω[*OF 0*] *cong*:
AE-cong-simp)
done
next
case (*Suc i*) **note** *NM* = *Suc*(*2*)[*THEN R-F*, *simp*]
show *?case*
apply (*subst m′-eq*[*OF NM*])
apply (*subst AE-bind*[**where** *B*=*?S*])
apply *measurable*
apply (*auto intro*!: *AE-distrD*[**where** *f*=*stl* **and** *M′*=*?S*] *Suc*(*1*)[*OF nm-F*]
Suc(*2*)
simp: *AE-measure-pmf-iff m′-stl cong*: *AE-cong-simp*)
done
qed
then have *AE-M*: $\bigwedge N\ M.\ (N,\ M) \in F \Longrightarrow AE\ x\ in\ M.\ x \in streams\ \Omega$
unfolding *streams-iff-snth AE-all-countable* **by** *auto*
then show $AE\ x\ in\ M.\ x \in streams\ \Omega$ **by** (*blast intro*: *F-NM*)

fix *xs* **assume** *xs* ∈ *lists* Ω
with ‹(*N*, *M*) ∈ *F*› **show** *emeasure N* (*sstart* Ω *xs*) = *emeasure M* (*sstart* Ω *xs*)
proof (*induction xs arbitrary*: *N M*)
case *Nil*
have *prob-space N prob-space M sets N* = *sets ?S sets M* = *sets ?S*
using *R-1*[*OF R-F*[*OF Nil*(*1*)]] *R-2*[*OF R-F*[*OF Nil*(*1*)]] **by** (*auto simp add*: *space-prob-algebra*)
have *emeasure N* (*streams* Ω) = *1*
by (*rule prob-space.emeasure-eq-1-AE*[*OF* ‹*prob-space N*› - *AE-N*[*OF Nil*(*1*)]]])
(*auto simp add*: ‹*sets N* = *sets ?S*› *intro*!: *streams-sets*)
moreover have *emeasure M* (*streams* Ω) = *1*
by (*rule prob-space.emeasure-eq-1-AE*[*OF* ‹*prob-space M*› - *AE-M*[*OF Nil*(*1*)]]])
(*auto simp add*: ‹*sets M* = *sets ?S*› *intro*!: *streams-sets*)
ultimately show *?case* **by** *simp*
next
case (*Cons x xs*)
note *NM* = *Cons*(*2*)[*THEN R-F*, *simp*]
have ∗: (##) *y* −‘ *sstart* Ω (*x* # *xs*) = (*if x* = *y then sstart* Ω *xs else* {}) **for** *y*
by *auto*
show *?case*
apply (*subst n′-eq*[*OF NM*])
apply (*subst* (*3*) *m′-eq*[*OF NM*])
apply (*subst emeasure-bind*[*OF* - - *sstart-sets*])
apply *simp* []
apply *measurable* []
apply (*subst emeasure-bind*[*OF* - - *sstart-sets*])
apply *simp* []
apply *measurable* []
apply (*intro nn-integral-cong-AE AE-pmfI*)
apply (*subst n′-def*)
apply (*subst m′-def*)
using *Cons*(*3*)
apply (*auto intro*!: *Cons nm-F*
simp add: *emeasure-distr sets-eq-imp-space-eq*[*OF sets-n*] *sets-eq-imp-space-eq*[*OF sets-m*]
space-stream-space ∗)
done
qed
qed
qed

3.1 Discrete Markov Kernel

locale *MC-syntax* =

fixes *K* :: *'s* ⇒ *'s pmf*
begin

abbreviation *acc* :: (*'s* × *'s*) *set* **where**
acc ≡ (*SIGMA s:UNIV. K s*)*

abbreviation *acc-on* :: *'s set* ⇒ (*'s* × *'s*) *set* **where**
acc-on S ≡ (*SIGMA s:UNIV. K s* ∩ *S*)*

lemma *countable-reachable*: *countable* (*acc* ‘‘ {*s*})
by (*auto intro*!: *countable-rtrancl countable-set-pmf simp*: *Sigma-Image*)

lemma *countable-acc*: *countable X* ⟹ *countable* (*acc* ‘‘ *X*)
apply (*rule countable-Image*)
apply (*rule countable-reachable*)
apply *assumption*
done

context
notes [[*inductive-internals*]]
begin

coinductive *enabled* **where**
enabled (*shd* ω) (*stl* ω) ⟹ *shd* ω ∈ *K s* ⟹ *enabled s* ω

end

lemma *alw-enabled*: *enabled* (*shd* ω) (*stl* ω) ⟹ *alw* (λω. *enabled* (*shd* ω) (*stl* ω)) ω
by (*coinduction arbitrary*: ω *rule*: *alw-coinduct*) (*auto elim*: *enabled.cases*)

abbreviation *S* ≡ *stream-space* (*count-space UNIV*)

lemma *in-S* [*measurable* (*raw*)]: *x* ∈ *space S*
by (*simp add*: *space-stream-space*)

inductive-simps *enabled-iff*: *enabled s* ω

lemma *enabled-Stream*: *enabled x* (*y* ## ω) ⟷ *y* ∈ *K x* ∧ *enabled y* ω
by (*subst enabled-iff*) *auto*

lemma *measurable-enabled*[*measurable*]:
Measurable.pred (*stream-space* (*count-space UNIV*)) (*enabled s*) (**is** *Measurable.pred ?S -*)
unfolding *enabled-def*
proof (*coinduction arbitrary*: *s rule*: *measurable-gfp2-coinduct*)
case (*step A s*)
then have [*measurable*]: ⋀*t*. *Measurable.pred ?S* (*A t*) **by** *auto*
have ∗: ⋀*x*. (∃ω *t*. *s* = *t* ∧ *x* = ω ∧ *A* (*shd* ω) (*stl* ω) ∧ *shd* ω ∈ *set-pmf* (*K*

t)) ⟷
    (∃ *t*∈*K s*. *A t* (*stl x*) ∧ *t* = *shd x*)
    by *auto*
  note *countable-set-pmf*[*simp*]
  show *?case*
    unfolding ∗ **by** *measurable*
qed (*auto simp*: *inf-continuous-def*)

lemma *enabled-iff-snth*: *enabled s ω* ⟷ (∀ *i*. *ω* !! *i* ∈ *K* ((*s* ## *ω*) !! *i*))
proof *safe*
  fix *i* **assume** *enabled s ω* **then show** *ω* !! *i* ∈ *K* ((*s* ## *ω*) !! *i*)
    by (*induct i arbitrary*: *s ω*)
      (*force elim*: *enabled.cases*)+
next
  assume ∀ *i*. *ω* !! *i* ∈ *set-pmf* (*K* ((*s* ## *ω*) !! *i*)) **then show** *enabled s ω*
    by (*coinduction arbitrary*: *s ω*)
      (*auto elim*: *allE*[*of - Suc i* **for** *i*] *allE*[*of - 0*])
qed

primcorec *force-enabled* **where**
  force-enabled x ω =
    (*let y* = *if shd ω* ∈ *K x then shd ω else* (*SOME y*. *y* ∈ *K x*) *in y* ## *force-enabled y* (*stl ω*))

lemma *force-enabled-in-set-pmf*[*simp*, *intro*]: *shd* (*force-enabled x ω*) ∈ *K x*
  by (*auto simp*: *some-in-eq set-pmf-not-empty*)

lemma *enabled-force-enabled*: *enabled x* (*force-enabled x ω*)
  by (*coinduction arbitrary*: *x ω*) (*auto simp*: *some-in-eq set-pmf-not-empty*)

lemma *force-enabled*: *enabled x ω* ⟹ *force-enabled x ω* = *ω*
  by (*coinduction arbitrary*: *x ω*) (*auto elim*: *enabled.cases*)

lemma *Ex-enabled*: ∃ *ω*. *enabled x ω*
  by (*rule exI*[*of - force-enabled x undefined*] *enabled-force-enabled*)+

lemma *measurable-force-enabled*: *force-enabled x* ∈ *measurable S S*
proof (*rule measurable-stream-space2*)
  fix *n* **show** (λ*ω*. *force-enabled x ω* !! *n*) ∈ *measurable S* (*count-space UNIV*)
  proof (*induction n arbitrary*: *x*)
    case (*Suc n*) **show** *?case*
      apply *simp*
      apply (*rule measurable-compose-countable'*[*OF measurable-compose*[*OF measurable-stl Suc*], **where** *I*=*set-pmf* (*K x*)])
      apply (*rule measurable-compose*[*OF measurable-shd*])
      apply (*auto simp*: *countable-set-pmf some-in-eq set-pmf-not-empty*)
      done
  qed (*auto intro*!: *measurable-compose*[*OF measurable-shd*])
qed

abbreviation $D \equiv$ *stream-space* (Π_M $s{\in}UNIV$. *K s*)

lemma *sets-D*: *sets D* = *sets* (*stream-space* (Π_M $s{\in}UNIV$. *count-space UNIV*))
  by (*intro sets-stream-space-cong sets-PiM-cong*) *simp-all*

lemma *space-D*: *space D* = *space* (*stream-space* (Π_M $s{\in}UNIV$. *count-space UNIV*))
  using *sets-eq-imp-space-eq*[*OF sets-D*] **.**

lemma *measurable-D-D*: *measurable D D* =
    measurable (*stream-space* (Π_M $s{\in}UNIV$. *count-space UNIV*)) (*stream-space* (Π_M $s{\in}UNIV$. *count-space UNIV*))
  by (*simp add*: *measurable-def space-D sets-D*)

primcorec *walk* :: $'s \Rightarrow ('s \Rightarrow 's)$ *stream* $\Rightarrow$ $'s$ *stream* **where**
  shd (*walk s* ω) = (*if shd* ω *s* $\in$ *K s then shd* ω *s else* (*SOME t*. *t* $\in$ *K s*))
| *stl* (*walk s* ω) = *walk* (*if shd* ω *s* $\in$ *K s then shd* ω *s else* (*SOME t*. *t* $\in$ *K s*)) (*stl* ω)

lemma *enabled-walk*: *enabled s* (*walk s* ω)
  by (*coinduction arbitrary*: *s* ω) (*auto simp*: *some-in-eq set-pmf-not-empty*)

lemma *measurable-walk*[*measurable*]: *walk s* $\in$ *measurable D S*
proof −
  note *measurable-compose*[*OF measurable-snth*, *intro*!]
  note *measurable-compose*[*OF measurable-component-singleton*, *intro*!]
  note *if-weak-cong*[*cong del*]
  note *measurable-g* = *measurable-compose-countable*′[*OF* - - *countable-reachable*]

  define *n* :: *nat* **where** *n* = *0*
  define *g* **where** *g* = (λ-::($'s \Rightarrow 's$) *stream*. *s*)
  then have *g* $\in$ *measurable D* (*count-space* (*acc* ‘‘ {*s*}))
    by *auto*
  then have (λx. *walk* (*g x*) (*sdrop n x*)) $\in$ *measurable D S*
  proof (*coinduction arbitrary*: *g n rule*: *measurable-stream-coinduct*)
    case (*shd g*) **show** *?case*
      by (*fastforce intro*: *measurable-g*[*OF* - *shd*])
  next
    case (*stl g*) **show** *?case*
      by (*fastforce simp add*: *sdrop.simps*[*symmetric*] *some-in-eq set-pmf-not-empty*
              simp del: *sdrop.simps intro*: *rtrancl-into-rtrancl measurable-g*[*OF* - *stl*])
  qed
  then show *?thesis*
    by (*simp add*: *g-def n-def*)
qed

3.2 Trace Space for Discrete-Time Markov Chains

definition *T* :: *′s* ⇒ *′s stream measure* **where**
  T s = *distr* (*stream-space* (Π_M *s*∈*UNIV*. *K s*)) *S* (*walk s*)

lemma *space-T*[*simp*]: *space* (*T s*) = *space S*
  by (*simp add*: *T-def*)

lemma *sets-T*[*simp*, *measurable-cong*]: *sets* (*T s*) = *sets S*
  by (*simp add*: *T-def*)

lemma *measurable-T1*[*simp*]: *measurable* (*T s*) *M* = *measurable S M*
  by (*intro measurable-cong-sets*) *simp-all*

lemma *measurable-T2*[*simp*]: *measurable M* (*T s*) = *measurable M S*
  by (*intro measurable-cong-sets*) *simp-all*

lemma *in-measurable-T1*[*measurable* (*raw*)]: *f* ∈ *measurable S M* ⟹ *f* ∈ *measurable* (*T s*) *M*
  by *simp*

lemma *in-measurable-T2*[*measurable* (*raw*)]: *f* ∈ *measurable M S* ⟹ *f* ∈ *measurable M* (*T s*)
  by *simp*

lemma *AE-T-enabled*: *AE ω in T s*. *enabled s ω*
  unfolding *T-def* **by** (*simp add*: *AE-distr-iff enabled-walk*)

sublocale *T*: *prob-space T s* **for** *s*
proof −
  interpret *P*: *product-prob-space K UNIV* **..**
  interpret *prob-space stream-space* (Π_M *s*∈*UNIV*. *K s*)
    by (*rule P.prob-space-stream-space*)
  fix *s* **show** *prob-space* (*T s*)
    by (*simp add*: *T-def prob-space-distr*)
qed

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space S*) = *1*
  using *T.emeasure-space-1*[*of s*] **by** *simp*

lemma *nn-integral-T*:
  assumes *f*[*measurable*]: *f* ∈ *borel-measurable S*
  shows ($\int^+ X.\ f\ X\ \partial T\ s$) = ($\int^+ t.\ (\int^+ \omega.\ f\ (t\ \#\#\ \omega)\ \partial T\ t)\ \partial K\ s$)
proof −
  interpret *product-prob-space K UNIV* **..**
  interpret *D*: *prob-space stream-space* (Π_M *s*∈*UNIV*. *K s*)
    by (*rule prob-space-stream-space*)

  have *T*: ⋀*f s*. *f* ∈ *borel-measurable S* ⟹ ($\int^+ X.\ f\ X\ \partial T\ s$) = ($\int^+ \omega.\ f\ (walk\ s\ \omega)\ \partial D$)

by (*simp add*: *T-def nn-integral-distr*)

have $(\int^+ X.\ f\ X\ \partial T\ s) = (\int^+ \omega.\ f\ (walk\ s\ \omega)\ \partial D)$
by (*rule T*) *measurable*
also have $\ldots = (\int^+ d.\ \int^+ \omega.\ f\ (walk\ s\ (d\ \#\#\ \omega))\ \partial D\ \partial \Pi_M\ i{\in}UNIV.\ K\ i)$
by (*simp add*: *P.nn-integral-stream-space*)
also have $\ldots = (\int^+ d.\ (\int^+ \omega.\ f\ (d\ s\ \#\#\ walk\ (d\ s)\ \omega) * indicator\ \{t.\ t \in K\ s\}\ (d\ s)\ \partial D)\ \partial \Pi_M\ i{\in}UNIV.\ K\ i)$
apply (*rule nn-integral-cong-AE*)
apply (*subst walk.ctr*)
apply (*simp add*: *frequently-def cong del*: *if-weak-cong*)
apply (*auto simp*: *AE-measure-pmf-iff intro*: *AE-component*)
done
also have $\ldots = (\int^+ d.\ \int^+ \omega.\ f\ (d\ s\ \#\#\ \omega) * indicator\ (K\ s)\ (d\ s)\ \partial T\ (d\ s)\ \partial \Pi_M\ i{\in}UNIV.\ K\ i)$
by (*subst T*) (*simp-all split*: *split-indicator*)
also have $\ldots = (\int^+ t.\ \int^+ \omega.\ f\ (t\ \#\#\ \omega) * indicator\ (K\ s)\ t\ \partial T\ t\ \partial K\ s)$
by (*subst* (*2*) *PiM-component*[*symmetric*]) (*simp-all add*: *nn-integral-distr*)
also have $\ldots = (\int^+ t.\ \int^+ \omega.\ f\ (t\ \#\#\ \omega)\ \partial T\ t\ \partial K\ s)$
by (*rule nn-integral-cong-AE*) (*simp add*: *AE-measure-pmf-iff*)
finally show *?thesis* **.**
qed

lemma *nn-integral-T-gfp*:
fixes *g*
defines $l \equiv \lambda f\ \omega.\ g\ (shd\ \omega)\ (f\ (stl\ \omega))$
assumes [*measurable*]: $case\text{-}prod\ g \in borel\text{-}measurable\ (count\text{-}space\ UNIV \bigotimes_M borel)$
assumes *cont-g*[*THEN inf-continuous-compose*, *order-continuous-intros*]: $\bigwedge s.\ inf\text{-}continuous\ (g\ s)$
assumes *int-g*: $\bigwedge f\ s.\ f \in borel\text{-}measurable\ S \Longrightarrow (\int^+ \omega.\ g\ s\ (f\ \omega)\ \partial T\ s) = g\ s\ (\int^+ \omega.\ f\ \omega\ \partial T\ s)$
assumes *bnd-g*: $\bigwedge f\ s.\ g\ s\ f \le b\ \ 0 \le b\ \ b < \infty$
shows $(\int^+ \omega.\ gfp\ l\ \omega\ \partial T\ s) = gfp\ (\lambda f\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial K\ s)\ s$
proof (*rule nn-integral-gfp*)
show $\bigwedge s.\ sets\ (T\ s) = sets\ S\ \bigwedge F.\ F \in borel\text{-}measurable\ S \Longrightarrow l\ F \in borel\text{-}measurable\ S$
by (*auto simp*: *l-def*)
show $\bigwedge s.\ emeasure\ (T\ s)\ (space\ (T\ s)) \neq 0$
by (*rewrite T.emeasure-space-1*) *simp*
{ fix *s F*
have $integral^N\ (T\ s)\ (l\ F) \le (\int^+ x.\ b\ \partial T\ s)$
by (*intro nn-integral-mono*) (*simp add*: *l-def bnd-g*)
also have $\ldots < \infty$
using *bnd-g* **by** *simp*
finally show $integral^N\ (T\ s)\ (l\ F) < \infty$ **. }**
show *inf-continuous* $(\lambda f\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial K\ s)$
proof (*intro order-continuous-intros*)
fix *f s*

have ($\int^+$ *t. g t* (*f t*) ∂*K s*) $\leq$ ($\int^+$ *t. b* ∂*K s*)
by (*intro nn-integral-mono bnd-g*)
also have ... $< \infty$
using *bnd-g* **by** *simp*
finally show ($\int^+$ *t. g t* (*f t*) ∂*K s*) $\neq \infty$
by *simp*
qed *simp*
next
fix *s* **and** *F* :: *'s stream* $\Rightarrow$ *ennreal* **assume** *F* $\in$ *borel-measurable S*
then show *integral*^{N} (*T s*) (*l F*) = ($\int^+$ *t. g t* (*integral*^{N} (*T t*) *F*) ∂*K s*)
by (*rewrite nn-integral-T*) (*simp-all add*: *l-def int-g*)
qed (*auto intro*!: *order-continuous-intros simp*: *l-def*)

lemma *nn-integral-T-lfp*:
fixes *g*
defines *l* $\equiv$ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes [*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*[*THEN sup-continuous-compose*, *order-continuous-intros*]: $\bigwedge$*s*. *sup-continuous* (*g s*)
assumes *int-g*: $\bigwedge$*f s. f* $\in$ *borel-measurable S* $\Longrightarrow$ ($\int^+\omega$. *g s* (*f* ω) ∂*T s*) = *g s* ($\int^+\omega$. *f* ω ∂*T s*)
shows ($\int^+\omega$. *lfp l* ω ∂*T s*) = *lfp* (λ*f s*. $\int^+$*t. g t* (*f t*) ∂*K s*) *s*
proof (*rule nn-integral-lfp*)
show $\bigwedge$*s. sets* (*T s*) = *sets S* $\bigwedge$*F. F* $\in$ *borel-measurable S* $\Longrightarrow$ *l F* $\in$ *borel-measurable S*
by (*auto simp*: *l-def*)
next
fix *s* **and** *F* :: *'s stream* $\Rightarrow$ *ennreal* **assume** *F* $\in$ *borel-measurable S*
then show *integral*^{N} (*T s*) (*l F*) = ($\int^+$ *t. g t* (*integral*^{N} (*T t*) *F*) ∂*K s*)
by (*rewrite nn-integral-T*) (*simp-all add*: *l-def int-g*)
qed (*auto simp*: *l-def intro*!: *order-continuous-intros*)

lemma *emeasure-Collect-T*:
assumes *f*[*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*x*$\in$*space* (*T s*). *P x*} = ($\int^+$*t. emeasure* (*T t*) {*x*$\in$*space* (*T t*). *P* (*t* ## *x*)} ∂*K s*)
apply (*subst* (*1 2*) *nn-integral-indicator*[*symmetric*])
apply *simp*
apply *simp*
apply (*subst nn-integral-T*)
apply (*auto intro*!: *nn-integral-cong simp add*: *space-stream-space indicator-def*)
done

lemma *AE-T-iff*:
assumes [*measurable*]: *Measurable.pred S P*
shows (*AE* ω *in T x. P* ω) $\longleftrightarrow$ ($\forall$ *y*$\in$*K x. AE* ω *in T y. P* (*y* ## ω))
by (*simp add*: *AE-iff-nn-integral nn-integral-T*[**where** *s*=*x*])
(*auto simp add*: *nn-integral-0-iff-AE AE-measure-pmf-iff split*: *split-indicator*)

lemma *AE-T-alw*:
  assumes [*measurable*]: *Measurable.pred S P*
  assumes *P*: $\bigwedge s.\ (x, s) \in acc \Longrightarrow AE\ \omega\ in\ T\ s.\ P\ \omega$
  shows $AE\ \omega\ in\ T\ x.\ alw\ P\ \omega$
proof −
  define *F* **where** $F = (\lambda p\ x.\ P\ x \wedge p\ (stl\ x))$
  have [*measurable*]: $\bigwedge p.\ Measurable.pred\ S\ p \Longrightarrow Measurable.pred\ S\ (F\ p)$
    by (*auto simp*: *F-def*)
  have *almost-everywhere* $(T\ s)\ ((F\ \widehat{}\widehat{}\ i)\ top)$
    if $(x, s) \in acc$ **for** *i s*
    using *that*
  proof (*induction i arbitrary*: *s*)
    case (*Suc i*) **then show** *?case*
      apply *simp*
      apply (*subst F-def*)
      apply (*simp add*: *P*)
      apply (*subst AE-T-iff*)
      apply (*measurable*; *simp*)
      apply (*auto dest*: *rtrancl-into-rtrancl*)
      done
  qed *simp*
  then have *almost-everywhere* $(T\ x)\ (gfp\ F)$
    by (*subst inf-continuous-gfp*) (*auto simp*: *inf-continuous-def AE-all-countable F-def*)
  then show *?thesis*
    by (*simp add*: *alw-def F-def*)
qed

lemma *emeasure-suntil-disj*:
  assumes [*measurable*]: *Measurable.pred S P*
  assumes ∗: $\bigwedge t.\ AE\ \omega\ in\ T\ t.\ \neg\ (P \sqcap (HLD\ X \sqcap nxt\ (HLD\ X\ suntil\ P)))\ \omega$
  shows $emeasure\ (T\ s)\ \{\omega{\in}space\ (T\ s).\ (HLD\ X\ suntil\ P)\ \omega\} =$
    $lfp\ (\lambda F\ s.\ emeasure\ (T\ s)\ \{\omega{\in}space\ (T\ s).\ P\ \omega\} + (\int^{+}t.\ F\ t * indicator\ X\ t\ \partial K\ s))\ s$
  unfolding *suntil-lfp*
proof (*rule emeasure-lfp*[**where** *s=s*])
  fix *F t* **assume** [*measurable*]: *Measurable.pred* (*T s*) *F* **and**
    F: $F \le lfp\ (\lambda a\ b.\ P\ b \vee HLD\ X\ b \wedge a\ (stl\ b))$
  have $emeasure\ (T\ t)\ \{\omega \in space\ (T\ s).\ P\ \omega \vee HLD\ X\ \omega \wedge F\ (stl\ \omega)\} =$
    $emeasure\ (T\ t)\ \{\omega \in space\ (T\ t).\ P\ \omega\} + emeasure\ (T\ t)\ \{\omega{\in}space\ (T\ t).\ HLD\ X\ \omega \wedge F\ (stl\ \omega)\}$
  proof (*rule emeasure-add-AE*)
    show $AE\ x\ in\ T\ t.\ \neg\ (x \in \{\omega \in space\ (T\ t).\ P\ \omega\} \wedge x \in \{\omega \in space\ (T\ t).\ HLD\ X\ \omega \wedge F\ (stl\ \omega)\})$
      using ∗ **by** *eventually-elim* (*insert F*, *auto simp*: *suntil-lfp*[*symmetric*])
  qed *auto*
  also have $emeasure\ (T\ t)\ \{\omega{\in}space\ (T\ t).\ HLD\ X\ \omega \wedge F\ (stl\ \omega)\} =$
    $(\int^{+}t.\ emeasure\ (T\ t)\ \{\omega \in space\ (T\ s).\ F\ \omega\} * indicator\ X\ t\ \partial K\ t)$

by (*subst emeasure-Collect-T*) (*auto intro*!: *nn-integral-cong split*: *split-indicator*)
finally show *emeasure* (*T t*) {ω ∈ *space* (*T s*). *P* ω ∨ *HLD X* ω ∧ *F* (*stl* ω)} =
emeasure (*T t*) {ω ∈ *space* (*T t*). *P* ω} + ($\int^+$ *t*. *emeasure* (*T t*) {ω ∈ *space* (*T s*). *F* ω} ∗ *indicator X t* ∂*K t*) **.**
qed (*auto intro*!: *order-continuous-intros split*: *split-indicator*)

lemma *emeasure-HLD-nxt*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {ω∈*space* (*T s*). (*X* · *P*) ω} =
($\int^+$*x*. *emeasure* (*T x*) {ω∈*space* (*T x*). *P* ω} ∗ *indicator X x* ∂*K s*)
by (*subst emeasure-Collect-T*)
(*auto intro*!: *nn-integral-cong-AE simp*: *AE-measure-pmf-iff split*: *split-indicator*)

lemma *emeasure-HLD*:
emeasure (*T s*) {ω∈*space* (*T s*). *HLD X* ω} = *emeasure* (*K s*) *X*
using *emeasure-HLD-nxt*[*of* $\lambda\omega$. *True s X*] *T.emeasure-space-1* **by** *simp*

lemma *emeasure-suntil-HLD*:
assumes [*measurable*]: *Measurable.pred S P*
shows *emeasure* (*T s*) {*x*∈*space* (*T s*). (*not* (*HLD* {*t*}) *suntil* (*HLD* {*t*} *aand nxt P*)) *x*} =
emeasure (*T s*) {*x*∈*space* (*T s*). *ev* (*HLD* {*t*}) *x*} ∗ *emeasure* (*T t*) {*x*∈*space* (*T t*). *P x*}
proof −
let *?P* = *emeasure* (*T t*) {ω∈*space* (*T t*). *P* ω}
let *?F* = λ*Q F s*. *emeasure* (*T s*) {ω∈*space* (*T s*). *Q* ω} + ($\int^+$*t′*. *F t′* ∗ *indicator* (− {*t*}) *t′* ∂*K s*)
have *emeasure* (*T s*) {*x*∈*space* (*T s*). (*HLD* (−{*t*}) *suntil* ({*t*} · *P*)) *x*} = *lfp* (*?F* ({*t*} · *P*)) *s*
by (*rule emeasure-suntil-disj*) (*auto simp*: *HLD-iff*)
also have *lfp* (*?F* ({*t*} · *P*)) = (λ*s*. *lfp* (*?F* (*HLD* {*t*})) *s* ∗ *?P*)
proof (*rule lfp-transfer*[*symmetric*, **where** α=λ*x s*. *x s* ∗ *emeasure* (*T t*) {ω∈*space* (*T t*). *P* ω}])
fix *F* **show** (λ*s*. *?F* (*HLD* {*t*}) *F s* ∗ *?P*) = *?F* ({*t*} · *P*) (λ*s*. *F s* ∗ *?P*)
unfolding *emeasure-HLD emeasure-HLD-nxt*[*OF assms*] *distrib-right*
by (*auto simp*: *fun-eq-iff nn-integral-multc*[*symmetric*]
intro!: *arg-cong2*[**where** *f*=(+)] *nn-integral-cong ac-simps*
split: *split-indicator*)
qed (*auto intro*!: *order-continuous-intros sup-continuous-mono lfp-upperbound*
intro: *le-funI add-nonneg-nonneg*
simp: *bot-ennreal split*: *split-indicator*)
also have *lfp* (*?F* (*HLD* {*t*})) *s* = *emeasure* (*T s*) {*x*∈*space* (*T s*). (*HLD* (−{*t*}) *suntil HLD* {*t*}) *x*}
by (*rule emeasure-suntil-disj*[*symmetric*]) (*auto simp*: *HLD-iff*)
finally show *?thesis*
by (*simp add*: *HLD-iff*[*abs-def*] *ev-eq-suntil*)
qed

lemma *AE-suntil*:

assumes [*measurable*]: *Measurable.pred S P*
shows (*AE x in T s.* (*not* (*HLD* {*t*}) *suntil* (*HLD* {*t*} *aand nxt P*)) *x*) ⟷
(*AE x in T s. ev* (*HLD* {*t*}) *x*) ∧ (*AE x in T t. P x*)
apply (*subst* (*1 2 3*) *T.prob-Collect-eq-1*[*symmetric*])
apply *simp*
apply *simp*
apply *simp*
apply (*simp-all add*: *measure-def emeasure-suntil-HLD del*: *space-T nxt.simps*)
apply (*auto simp*: *T.emeasure-eq-measure mult-eq-1*)
done

3.3 Fairness

definition *fair* :: *′s* ⇒ *′s* ⇒ *′s stream* ⇒ *bool* **where**
fair s t = *alw* (*ev* (*HLD* {*s*})) *impl alw* (*ev* (*HLD* {*s*} *aand nxt* (*HLD* {*t*})))

lemma *AE-T-fair*:
assumes $t' \in K\ t$
shows *AE* ω *in T s. fair t t′* ω
proof −
let *?M* = λ*P s. emeasure* (*T s*) {ω∈*space* (*T s*). *P* ω}
let *?t* = *HLD* {*t*} **and** *?t′* = *HLD* {*t′*}
define *N* **where** *N* = *alw* (*ev ?t*) *aand alw* (*not* (*?t aand nxt ?t′*))
let *?until* = *not ?t suntil* (*?t aand nxt* (*not ?t′ aand nxt N*))
have *N-stl*: ⋀ω. *N* ω ⟹ *N* (*stl* ω)
by (*auto simp*: *N-def*)
have [*measurable*]: *Measurable.pred S N*
unfolding *N-def* **by** *measurable*

let *?c* = *pmf* (*K t*) *t′*
let *?R* = λ*x. 1* ⊓ *x* ∗ (*1* − *ennreal ?c*)
have *mono ?R*
by (*intro monoI mult-right-mono inf-mono*) (*auto simp*: *mono-def field-simps*)
have ⋀*s. ?M N s* ≤ *gfp ?R*
proof (*induction rule*: *gfp-ordinal-induct*[*OF* ‹*mono ?R*›])
fix *x s* **assume** *x*: ⋀*s. ?M N s* ≤ *x*
{ fix ω **assume** *N* ω
then have *ev* (*HLD* {*t*}) ω *N* ω
by (*auto simp*: *N-def*)
then have *?until* ω
by (*induct rule*: *ev-induct-strong*) (*auto simp*: *N-def intro*: *suntil.intros dest*: *N-stl*) **}**
then have *?M N s* ≤ *?M ?until s*
by (*intro emeasure-mono-AE*) *auto*
also have ... = *?M* (*ev ?t*) *s* ∗ *?M* (*not ?t′ aand nxt N*) *t*
by (*simp-all add*: *emeasure-suntil-HLD del*: *nxt.simps space-T*)
also have ... ≤ *?M* (*ev ?t*) *s* ∗ (∫⁺*s′*. *1* ⊓ *x* ∗ *indicator* (*UNIV* − {*t′*}) *s′* ∂*K t*)
by (*auto intro*!: *mult-left-mono nn-integral-mono T.measure-le-1 emeasure-mono*

split: *split-indicator simp add*: *x emeasure-Collect-T*[*of - t*] *simp del*: *space-T*)
also have . . . $\leq$ *1* $*$ ($\int^{+}$*s′*. *1* ⊓ *x* $*$ *indicator* (*UNIV* − {*t′*}) *s′* ∂*K t*)
by (*intro mult-right-mono T.measure-le-1*) *simp*
finally show *?M N s* $\leq$ *1* ⊓ *x* $*$ (*1* − *ennreal ?c*)
by (*subst* (*asm*) *nn-integral-cmult-indicator*) (*auto simp*: *emeasure-Diff emeasure-pmf-single*)
qed (*auto intro*: *Inf-greatest*)
also
from ‹*mono ?R*› **have** *gfp ?R* = *?R* (*gfp ?R*) **by** (*rule gfp-unfold*)
then have *gfp ?R* $\leq$ *?R* (*gfp ?R*) **by** *simp*
with *assms*[*THEN pmf-positive*] **have** *gfp ?R* $\leq$ *0*
by (*cases gfp ?R*)
(*auto simp*: *top-unique inf-ennreal.rep-eq field-simps mult-le-0-iff ennreal-1*[*symmetric*]
pmf-le-1 ennreal-minus ennreal-mult[*symmetric*] *ennreal-le-iff2 inf-min min-def*
simp del: *ennreal-1*
split: *if-split-asm*)
finally have ⋀*s*. *AE* ω *in T s*. ¬ *N* ω
by (*subst AE-iff-measurable*[*OF - refl*]) (*auto intro*: *antisym simp*: *le-fun-def*)
then have *AE* ω *in T s*. *alw* (*not N*) ω
by (*intro AE-T-alw*) *auto*
moreover
{ fix ω **assume** *alw* (*ev* (*HLD* {*t*})) ω
then have *alw* (*alw* (*ev* (*HLD* {*t*}))) ω
unfolding *alw-alw* **.**
moreover assume *alw* (*not N*) ω
then have *alw* (*alw* (*ev* (*HLD* {*t*})) *impl ev* (*HLD* {*t*} *aand nxt* (*HLD* {*t′*}))) ω
unfolding *N-def not-alw-iff not-ev-iff de-Morgan-disj de-Morgan-conj not-not imp-conv-disj* **.**
ultimately have *alw* (*ev* (*HLD* {*t*} *aand nxt* (*HLD* {*t′*}))) ω
by (*rule alw-mp*) **}**
then have ∀ω. *alw* (*not N*) ω ⟶ *fair t t′* ω
by (*auto simp*: *fair-def*)
ultimately show *?thesis*
by (*simp add*: *eventually-mono*)
qed

lemma *enabled-imp-trancl*:
assumes *alw* (*HLD B*) ω *enabled s* ω
shows *alw* (*HLD* (*acc-on B* ‘‘ {*s*})) ω
proof −
define *t* **where** *t* = *s*
then have (*s*, *t*) ∈ *acc-on B*
by *auto*
moreover note ‹*alw* (*HLD B*) ω›
moreover note ‹*enabled s* ω›[*unfolded* ‹*t* == *s*›[*symmetric*]]
ultimately show *?thesis*

proof (*coinduction arbitrary*: *t* ω *rule*: *alw-coinduct*)
case *stl* **from** *this*(*1*,*2*,*3*) **show** *?case*
by (*auto simp*: *enabled.simps*[*of* - ω] *alw.simps*[*of* - ω] *HLD-iff*
intro!: *exI*[*of* - *shd* ω] *rtrancl-trans*[*of s t*])
next
case (*alw t* ω) **then show** *?case*
by (*auto simp*: *HLD-iff enabled.simps*[*of* - ω] *alw.simps*[*of* - ω] *intro*!: *rtrancl-trans*[*of* *s t*])
qed
qed

lemma *AE-T-reachable*: *AE* ω *in T s*. *alw* (*HLD* (*acc* `` {*s*})) ω
using *AE-T-enabled*
proof *eventually-elim*
fix ω **assume** *enabled s* ω
from *enabled-imp-trancl*[*of UNIV*, *OF* - *this*]
show *alw* (*HLD* (*acc* `` {*s*})) ω
by (*auto simp*: *HLD-iff*[*abs-def*] *all-imp-alw*)
qed

lemma *AE-T-all-fair*: *AE* ω *in T s*. $\forall$(*t*,*t'*)$\in$*SIGMA t*:*UNIV*. *K t*. *fair t t'* ω
proof −
let *?Rn* = *SIGMA s*:(*acc* `` {*s*}). *K s*
have *AE* ω *in T s*. $\forall$(*t*,*t'*)$\in$*?Rn*. *fair t t'* ω
proof (*subst AE-ball-countable*)
show *countable ?Rn*
by (*intro countable-SIGMA countable-rtrancl*[*OF countable-Image*]) (*auto simp*: *Image-def*)
qed (*auto intro*!: *AE-T-fair*)
then show *?thesis*
using *AE-T-reachable*
proof (*eventually-elim*, *safe*)
fix ω *t t'* **assume** $\forall$(*t*,*t'*)$\in$*?Rn*. *fair t t'* ω *t'* $\in$ *K t* **and** *alw*: *alw* (*HLD* (*acc* `` {*s*})) ω
moreover
{ assume *t* $\notin$ *acc* `` {*s*}
then have *alw* (*not* (*HLD* {*t*})) ω
by (*intro alw-mono*[*OF alw*]) (*auto simp*: *HLD-iff*)
then have *not* (*alw* (*ev* (*HLD* {*t*}))) ω
unfolding *not-alw-iff not-ev-iff* **by** *auto*
then have *fair t t'* ω
unfolding *fair-def* **by** *auto* **}**
ultimately show *fair t t'* ω
by *auto*
qed
qed

lemma *fair-imp*: **assumes** *fair t t'* ω *alw* (*ev* (*HLD* {*t*})) ω **shows** *alw* (*ev* (*HLD* {*t'*})) ω

proof –
  { **fix** *ω* **assume** *ev* (*HLD* {*t*} *aand nxt* (*HLD* {*t*′})) *ω* **then have** *ev* (*HLD* {*t*′}) *ω*
      by *induction auto* }
  with *assms* **show** *?thesis*
    by (*auto simp*: *fair-def elim*!: *alw-mp intro*: *all-imp-alw*)
qed

lemma *AE-T-ev-HLD*:
  assumes *exiting*: ⋀*t*. (*s*, *t*) ∈ *acc-on* (−*B*) ⟹ ∃ *t*′∈*B*. (*t*, *t*′) ∈ *acc*
  assumes *fin*: *finite* (*acc-on* (−*B*) `` {*s*})
  shows *AE ω in T s*. *ev* (*HLD B*) *ω*
  using *AE-T-all-fair AE-T-enabled*
proof *eventually-elim*
  fix *ω* **assume** *fair*: ∀ (*t*, *t*′)∈(*SIGMA s*:*UNIV*. *K s*). *fair t t*′ *ω* **and** *enabled s ω*

  show *ev* (*HLD B*) *ω*
  proof (*rule ccontr*)
    assume ¬ *ev* (*HLD B*) *ω*
    then have *alw* (*HLD* (− *B*)) *ω*
      by (*simp add*: *not-ev-iff HLD-iff*[*abs-def*])
    from *enabled-imp-trancl*[*OF this* ‹*enabled s ω*›]
    have *alw* (*HLD* (*acc-on* (−*B*) `` {*s*})) *ω*
      by (*simp add*: *Diff-eq*)
    from *pigeonhole-stream*[*OF this fin*]
    obtain *t* **where** (*s*, *t*) ∈ *acc-on* (−*B*) *alw* (*ev* (*HLD* {*t*})) *ω*
      by *auto*
    from *exiting*[*OF this*(*1*)] **obtain** *t*′ **where** (*t*, *t*′) ∈ *acc t*′ ∈ *B*
      by *auto*
    from *this*(*1*) **have** *alw* (*ev* (*HLD* {*t*′})) *ω*
    proof *induction*
      case (*step u w*) **then show** *?case*
        using *fair fair-imp*[*of u w ω*] **by** *auto*
    qed *fact*
    { **assume** *ev* (*HLD* {*t*′}) *ω* **then have** *ev* (*HLD B*) *ω*
      by (*rule ev-mono*) (*auto simp*: *HLD-iff* ‹*t*′ ∈ *B*›) }
    then show *False*
      using ‹*alw* (*ev* (*HLD* {*t*′})) *ω*› ‹¬ *ev* (*HLD B*) *ω*› **by** *auto*
  qed
qed

lemma *AE-T-ev-HLD*′:
  assumes *exiting*: ⋀*s*. *s* ∉ *X* ⟹ ∃ *t*∈*X*. (*s*, *t*) ∈ *acc*
  assumes *fin*: *finite* (−*X*)
  shows *AE ω in T s*. *ev* (*HLD X*) *ω*
proof (*rule AE-T-ev-HLD*)
  show ⋀*t*. (*s*, *t*) ∈ *acc-on* (− *X*) ⟹ ∃ *t*′∈*X*. (*t*, *t*′) ∈ *acc*
    using *exiting* **by** (*auto elim*: *rtrancl.cases*)
  have *acc-on* (− *X*) `` {*s*} ⊆ −*X* ∪ {*s*}

by (*auto elim*: *rtrancl.cases*)
with *fin* **show** *finite* (*acc-on* (− *X*) ‘‘ {*s*})
by (*auto dest*: *finite-subset*)
qed

lemma *AE-T-max-sfirst*:
assumes [*measurable*]: *Measurable.pred S X*
assumes *AE*: *AE* ω *in T c. sfirst X* (*c* ## ω) < ∞ **and** *0* < *e*
shows ∃ *N*::*nat*. 𝒫(ω *in T c. N* < *sfirst X* (*c* ## ω)) < *e* (**is** ∃ *N*. *?P N* < *e*)
proof −
have *?P* ⟶ *measure* (*T c*) (⋂ *N*::*nat*. {*bT* ∈ *space* (*T c*). *N* < *sfirst X* (*c* ## *bT*)})
using *dual-order.strict-trans enat-ord-simps*(*2*)
by (*intro T.finite-Lim-measure-decseq*) (*force simp*: *decseq-Suc-iff simp del*: *enat-ord-simps*)+
also have *measure* (*T c*) (⋂ *N*::*nat*. {*bT* ∈ *space* (*T c*). *N* < *sfirst X* (*c* ## *bT*)}) =
𝒫(*bT in T c. sfirst X* (*c* ## *bT*) = ∞)
by (*auto simp del*: *not-infinity-eq intro*!: *arg-cong*[**where** *f*=*measure* (*T c*)])
(*metis less-irrefl not-infinity-eq*)
also have 𝒫(*bT in T c. sfirst X* (*c* ## *bT*) = ∞) = *0*
using *AE* **by** (*intro T.prob-eq-0-AE*) *auto*
finally have ∃ *N*. ∀ *n*≥*N*. *norm* (*?P n* − *0*) < *e*
using ‹*0* < *e*› **by** (*rule LIMSEQ-D*)
then show *?thesis*
by (*auto simp*: *measure-nonneg*)
qed

3.4 First Hitting Time

lemma *nn-integral-sfirst-finite′*:
assumes *s* ∉ *H*
assumes [*simp*]: *finite* (*acc-on* (−*H*) ‘‘ {*s*})
assumes *until*: *AE* ω *in T s. ev* (*HLD H*) ω
shows (∫⁺ ω. *sfirst* (*HLD H*) ω ∂*T s*) ≠ ∞
proof −
have *R-ne*[*simp*]: *acc-on* (−*H*) ‘‘ {*s*} ≠ {}
by *auto*
have [*measurable*]: *H* ∈ *sets* (*count-space UNIV*)
by *simp*

let *?Pf* = λ*n t*. 𝒫(ω *in T t. enat n* < *sfirst* (*HLD H*) (*t* ## ω))
have *Pf-mono*: ⋀*N n t. N* ≤ *n* ⟹ *?Pf n t* ≤ *?Pf N t*
by (*auto intro*!: *T.finite-measure-mono simp del*: *enat-ord-code*(*1*) *simp*: *enat-ord-code*(*1*)[*symmetric*])

have *not-H*: ⋀*t*. (*s*, *t*) ∈ *acc-on* (−*H*) ⟹ *t* ∉ *H*
using ‹*s* ∉ *H*› **by** (*auto elim*: *rtrancl.cases*)

have ∀$_F$ *n in sequentially*. ∀ *t*∈*acc-on* (−*H*)‘‘{*s*}. *?Pf n t* < *1*

proof (*safe intro*!: *eventually-ball-finite*)
fix *t* **assume** (*s*, *t*) ∈ *acc-on* (−*H*)
then have *AE ω in T t*. *sfirst* (*HLD H*) (*t* ## *ω*) < ∞
unfolding *sfirst-finite*
proof *induction*
case (*step t u*) **with** *step.IH* **show** *?case*
by (*subst* (*asm*) *AE-T-iff*) (*auto simp*: *ev-Stream not-H*)
qed (*simp add*: *ev-Stream eventually-frequently-simps until*)
from *AE-T-max-sfirst*[*OF - this*, *of 1*]
obtain *N* **where** *?Pf N t* < *1* **by** *auto*
with *Pf-mono*[*of N*] **show** $\forall_F$ *n in sequentially*. *?Pf n t* < *1*
by (*auto simp*: *eventually-sequentially intro*: *le-less-trans*)
qed *simp*
then obtain *n* **where** ⋀*t*. (*s*, *t*) ∈ *acc-on* (−*H*) ⟹ *?Pf n t* < *1*
by (*auto simp*: *eventually-sequentially*)
moreover define *d* **where** *d* = *Max* (*?Pf n* ‘ *acc-on* (−*H*) ‘‘ {*s*})
ultimately have *d*: *0* ≤ *d d* < *1* ⋀*t*. (*s*, *t*) ∈ *acc-on* (−*H*) ⟹ *?Pf* (*Suc n*) *t* ≤ *d*
using *Pf-mono*[*of n Suc n*] **by** (*auto simp*: *Max-ge-iff measure-nonneg*)

let *?F* = *λF ω*. *if shd ω* ∈ *H then 0 else F* (*stl ω*) + *1* :: *ennreal*
have *sup-continuous ?F*
by (*intro order-continuous-intros*)
then have *mono ?F*
by (*rule sup-continuous-mono*)
have *lfp-nonneg*[*simp*]: ⋀*ω*. *0* ≤ *lfp ?F ω*
by (*subst lfp-unfold*[*OF* ‹*mono ?F*›]) *auto*

let *?I* = *λF s*. $\int^+$*t*. (*if t* ∈ *H then 0 else F t* + *1*) ∂*K s*
have *sup-continuous ?I*
by (*intro order-continuous-intros*) *auto*
then have *mono ?I*
by (*rule sup-continuous-mono*)

define *p* **where** *p* = *Suc n* / (*1* − *d*)
have *p*: *p* = *Suc n* + *d* ∗ *p*
unfolding *p-def* **using** *d*(*1*,*2*) **by** (*auto simp*: *field-simps*)
have [*simp*]: *0* ≤ *p*
using *d*(*1*,*2*) **by** (*auto simp*: *p-def*)

have ($\int^+$ *ω*. *sfirst* (*HLD H*) *ω* ∂*T s*) = ($\int^+$ *ω*. *lfp ?F ω* ∂*T s*)
proof (*intro nn-integral-cong-AE*)
show *AE x in T s*. *sfirst* (*HLD H*) *x* = *lfp ?F x*
using *until*
proof *eventually-elim*
fix *ω* **assume** *ev* (*HLD H*) *ω* **then show** *sfirst* (*HLD H*) *ω* = *lfp ?F ω*
by (*induction rule*: *ev-induct-strong*;
subst lfp-unfold[*OF* ‹*mono ?F*›], *simp add*: *HLD-iff*[*abs-def*] *ac-simps max-absorb2*)

qed
qed
also have ... = *lfp* (*?I*^^*Suc n*) *s*
unfolding *lfp-funpow*[*OF* ‹*mono ?I*›]
by (*subst nn-integral-T-lfp*)
(*auto simp*: *nn-integral-add max-absorb2 intro*!: *order-continuous-intros*)
also have *lfp* (*?I*^^*Suc n*) *t* ≤ *p* **if** (*s*, *t*) ∈ *acc-on* (−*H*) **for** *t*
using *that*
proof (*induction arbitrary*: *t rule*: *lfp-ordinal-induct*[*of ?I*^^*Suc n*])
case (*step S*)
have (*?I*^^*i*) *S t* ≤ *i* + *?Pf i t* ∗ *ennreal p* **for** *i*
using *step*(*3*)
proof (*induction i arbitrary*: *t*)
case *0* **then show** *?case*
using *T.prob-space step*(*1*)
by (*auto simp add*: *zero-ennreal-def*[*symmetric*] *not-H zero-enat-def*[*symmetric*] *one-ennreal-def*[*symmetric*])
next
case (*Suc i*)
then have *t* ∉ *H*
by (*auto simp*: *not-H*)
from *Suc.prems* **have** ⋀*t'*. *t'* ∈ *K t* ⟹ *t'* ∉ *H* ⟹ (*s*, *t'*) ∈ *acc-on* (−*H*)
by (*rule rtrancl-into-rtrancl*) (*insert Suc.prems*, *auto dest*: *not-H*)
then have (*?I* ^^ *Suc i*) *S t* ≤ *?I* (λ*t*. *i* + *ennreal* (*?Pf i t*) ∗ *p*) *t*
by (*auto simp*: *AE-measure-pmf-iff simp del*: *sfirst-eSuc space-T*
intro!: *nn-integral-mono-AE add-mono max.mono Suc*)
also have ... ≤ (∫^+ *t*. *ennreal* (*Suc i*) + *ennreal* 𝒫(ω *in T t*. *enat i* < *sfirst* (*HLD H*) (*t* ## ω)) ∗ *p* ∂*K t*)
by (*intro nn-integral-mono*) *auto*
also have ... ≤ *Suc i* + *ennreal* (*?Pf* (*Suc i*) *t*) ∗ *p*
unfolding *T.emeasure-eq-measure*[*symmetric*]
by (*subst* (*2*) *emeasure-Collect-T*)
(*auto simp*: ‹*t* ∉ *H*› *eSuc-enat*[*symmetric*] *nn-integral-add nn-integral-multc ennreal-of-nat-eq-real-of-nat*)
finally show *?case*
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)
qed
then have (*?I*^^*Suc n*) *S t* ≤ *Suc n* + *?Pf* (*Suc n*) *t* ∗ *ennreal p* **.**
also have ... ≤ *p*
using *d step* **by** (*subst* (*2*) *p*) (*auto intro*!: *mult-right-mono simp*: *ennreal-of-nat-eq-real-of-nat ennreal-mult*)
finally show *?case* **.**
qed (*auto simp*: *SUP-least intro*!: *mono-pow* ‹*mono ?I*› *simp del*: *funpow.simps*)
finally show *?thesis*
unfolding *p-def* **by** (*auto simp*: *top-unique*)
qed

lemma *nn-integral-sfirst-finite*:
assumes [*simp*]: *finite* (*acc-on* (−*H*) ‘‘ {*s*})

assumes *until*: *AE ω in T s. ev (HLD H) ω*
shows ($\int^+$ *ω. sfirst (HLD H) (s ## ω) ∂T s*) ≠ ∞
proof *cases*
assume *s ∉ H* **then show** *?thesis*
using *nn-integral-sfirst-finite′[of s H] until* **by** (*simp add: nn-integral-add*)
qed (*simp add: sfirst.simps*)

lemma *prob-T*:
assumes *P*: *Measurable.pred S P*
shows $\mathcal{P}$(*ω in T s. P ω*) = ($\int$ *t.* $\mathcal{P}$(*ω in T t. P (t ## ω)*) *∂K s*)
using *emeasure-Collect-T[OF P, of s]* **unfolding** *T.emeasure-eq-measure*
by (*subst (asm) nn-integral-eq-integral*)
(*auto intro!: measure-pmf.integrable-const-bound*[**where** *B=1*])

lemma *T-subprob[measurable]*: *T ∈ measurable (measure-pmf I) (subprob-algebra S)*
by (*auto intro!: space-bind simp: space-subprob-algebra*) *unfold-locales*

3.5 Markov chain with Initial Distribution

definition *T′ :: ′s pmf ⇒ ′s stream measure* **where**
T′ I = bind I (λs. distr (T s) S ((##) s))

lemma *distr-Stream-subprob*:
(*λs. distr (T s) S ((##) s)*) ∈ *measurable (measure-pmf I) (subprob-algebra S)*
apply (*intro measurable-distr2[OF - T-subprob]*)
apply (*subst measurable-cong-sets*[**where** *M′=count-space UNIV* $\bigotimes_M$ *S* **and** *N′=S*])
apply (*rule sets-pair-measure-cong*)
apply *auto*
done

lemma *sets-T′*: *sets (T′ I) = sets S*
by (*simp add: T′-def*)

lemma *prob-space-T′*: *prob-space (T′ I)*
unfolding *T′-def*
proof (*rule measure-pmf.prob-space-bind*)
show *AE s in I. prob-space (distr (T s) S ((##) s))*
by (*intro AE-measure-pmf-iff[THEN iffD2] ballI T.prob-space-distr*) *simp*
qed (*rule distr-Stream-subprob*)

lemma *AE-T′*:
assumes [*measurable*]: *Measurable.pred S P*
shows (*AE x in T′ I. P x*) ⟷ (∀ *s∈I. AE x in T s. P (s ## x)*)
unfolding *T′-def* **by** (*simp add: AE-bind[OF distr-Stream-subprob] AE-measure-pmf-iff AE-distr-iff*)

lemma *emeasure-T′*:

assumes [*measurable*]: $X \in$ *sets S*
shows *emeasure* $(T'\ I)\ X = (\int^{+} s.$ *emeasure* $(T\ s)\ \{\omega \in$ *space S*. $s\ \#\#\ \omega \in X\}\ \partial I)$
unfolding *T′-def*
by (*simp add*: *emeasure-bind*[*OF - distr-Stream-subprob*] *emeasure-distr vimage-def Int-def conj-ac*)

lemma *prob-T′*:
assumes [*measurable*]: *Measurable.pred S P*
shows $\mathcal{P}(x\ in\ T'\ I.\ P\ x) = (\int s.\ \mathcal{P}(x\ in\ T\ s.\ P\ (s\ \#\#\ x))\ \partial I)$
proof −
interpret *T′*: *prob-space T′ I* **by** (*rule prob-space-T′*)
show *?thesis*
using *emeasure-T′*[*of* {*x*∈*space* (*T′ I*). *P x*} *I*]
unfolding *T′.emeasure-eq-measure T.emeasure-eq-measure sets-eq-imp-space-eq*[*OF sets-T′*]
apply *simp*
apply (*subst* (*asm*) *nn-integral-eq-integral*)
apply (*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B=1*] *integral-cong arg-cong2*[**where** *f=measure*]
simp: *AE-measure-pmf measure-nonneg space-stream-space*)
done
qed

lemma *T-eq-T′*: *T s = T′* (*K s*)
proof (*rule measure-eqI*)
fix *X* **assume** *X*: $X \in$ *sets* (*T s*)
then have [*measurable*]: $X \in$ *sets S*
by *simp*
have *X-eq*: $X = \{x \in space\ (T\ s).\ x \in X\}$
using *sets.sets-into-space*[*OF X*] **by** *auto*
show *emeasure* (*T s*) *X* = *emeasure* (*T′* (*K s*)) *X*
apply (*subst X-eq*)
apply (*subst emeasure-Collect-T*, *simp*)
apply (*subst emeasure-T′*, *simp*)
apply *simp*
done
qed (*simp add*: *sets-T′*)

lemma *T-eq-bind*: $T\ s = (measure\text{-}pmf\ (K\ s) \gg\!= (\lambda t.\ distr\ (T\ t)\ S\ ((\#\#)\ t)))$
by (*subst T-eq-T′*) (*simp add*: *T′-def*)

lemma *T-split*:
$T\ s = (T\ s \gg\!= (\lambda\omega.\ distr\ (T\ ((s\ \#\#\ \omega)\ !!\ n))\ S\ (\lambda\omega'.\ stake\ n\ \omega\ @\!-\ \omega')))$
proof (*induction n arbitrary*: *s*)
case *0* **then show** *?case*
apply (*simp add*: *distr-cong*[*OF refl sets-T*[*symmetric*, *of s*] *refl*])
apply (*subst bind-const′*)
apply *unfold-locales*

..
next
case (*Suc n*)
let *?K = measure-pmf (K s)* **and** *?m = λn ω ω'. stake n ω @− ω'*
note *sets-stream-space-cong*[*simp*, *measurable-cong*]

have *T s = (?K ≫= (λt. distr (T t) S ((##) t)))*
by (*rule T-eq-bind*)
also have ... = (*?K ≫= (λt. distr (T t ≫= (λω. distr (T ((t ## ω) !! n)) S (?m n ω))) S ((##) t))*)
unfolding *Suc*[*symmetric*] **..**
also have ... = (*?K ≫= (λt. T t ≫= (λω. distr (distr (T ((t ## ω) !! n)) S (?m n ω)) S ((##) t))))*)
by (*simp add*: *distr-bind*[**where** *K=S*, *OF measurable-distr2*[**where** *M=S*]] *space-stream-space*)
also have ... = (*?K ≫= (λt. T t ≫= (λω. distr (T ((t ## ω) !! n)) S (?m (Suc n) (t ## ω)))))*)
by (*simp add*: *distr-distr space-stream-space comp-def*)
also have ... = (*?K ≫= (λt. distr (T t) S ((##) t) ≫= (λω. distr (T (ω !! n)) S (?m (Suc n) ω))))*)
by (*simp add*: *space-stream-space bind-distr*[*OF - measurable-distr2*[**where** *M=S*]] *del*: *stake.simps*)
also have ... = (*T s ≫= (λω. distr (T (ω !! n)) S (?m (Suc n) ω))*)
unfolding *T-eq-bind*[*of s*]
by (*subst bind-assoc*[*OF measurable-distr2*[**where** *M=S*] *measurable-distr2*[**where** *M=S*], *OF - T-subprob*])
(*simp-all add*: *space-stream-space del*: *stake.simps*)
finally show *?case*
by *simp*
qed

lemma *nn-integral-T-split*:
assumes *f*[*measurable*]: *f ∈ borel-measurable S*
shows (*∫⁺ω. f ω ∂T s*) = (*∫⁺ω. (∫⁺ω'. f (stake n ω @− ω') ∂T ((s ## ω) !! n)) ∂T s*)
apply (*subst T-split*[*of s n*])
apply (*simp add*: *nn-integral-bind*[*OF f measurable-distr2*[**where** *M=S*]])
apply (*subst nn-integral-distr*)
apply (*simp-all add*: *space-stream-space*)
done

lemma *emeasure-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows *emeasure (T s) {ω∈space (T s). P ω} =*
(*∫⁺ω. emeasure (T ((s ## ω) !! n)) {ω'∈space (T ((s ## ω) !! n)). P (stake n ω @− ω')} ∂T s*)
apply (*subst T-split*[*of s n*])
apply (*subst emeasure-bind*[*OF - measurable-distr2*[**where** *M=S*]])
apply (*simp-all add*:)

apply (*simp add*: *space-stream-space*)
apply (*subst emeasure-distr*)
apply *simp-all*
apply (*simp-all add*: *space-stream-space*)
done

lemma *prob-T-split*:
assumes *P*[*measurable*]: *Measurable.pred S P*
shows $\mathcal{P}(\omega$ *in T s. P* $\omega) = (\int \omega.\ \mathcal{P}(\omega'$ *in T* $((s\ \#\#\ \omega)\ !!\ n).\ P\ (stake\ n\ \omega\ @-\ \omega'))\ \partial T\ s)$
using *emeasure-T-split*[*OF P*, *of s n*]
unfolding *T.emeasure-eq-measure*
by (*subst* (*asm*) *nn-integral-eq-integral*)
(*auto intro*!: *T.integrable-const-bound*[**where** *B=1*] *measure-measurable-subprob-algebra2*[**where** *N=S*]
simp: *T.emeasure-eq-measure SIGMA-Collect-eq*)

lemma *enabled-imp-alw*:
$(\bigcup s\in X.\ set\text{-}pmf\ (K\ s)) \subseteq X \Longrightarrow x \in X \Longrightarrow enabled\ x\ \omega \Longrightarrow alw\ (HLD\ X)\ \omega$
proof (*coinduction arbitrary*: ω *x*)
case *alw* **then show** *?case*
unfolding *enabled.simps*[*of - ω*]
by (*auto simp*: *HLD-iff*)
qed

lemma *alw-HLD-iff-sconst*:
$alw\ (HLD\ \{x\})\ \omega \longleftrightarrow \omega = sconst\ x$
proof
assume *alw* (*HLD* {*x*}) ω **then show** ω = *sconst x*
by (*coinduction arbitrary*: ω) (*auto simp*: *HLD-iff*)
qed (*auto simp*: *alw-sconst HLD-iff*)

lemma *enabled-iff-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*} **shows** *enabled x* ω ⟷ ω = *sconst x*
proof
assume *enabled x* ω **then show** ω = *sconst x*
by (*coinduction arbitrary*: ω) (*auto elim*: *enabled.cases*)
next
assume ω = *sconst x* **then show** *enabled x* ω
by (*coinduction arbitrary*: ω) *auto*
qed

lemma *AE-sconst*:
assumes [*simp*]: *set-pmf* (*K x*) = {*x*}
shows (*AE* ω *in T x. P* ω) ⟷ *P* (*sconst x*)
proof −
have (*AE* ω *in T x. P* ω) ⟷ (*AE* ω *in T x. P* ω ∧ ω = *sconst x*)
using *AE-T-enabled*[*of x*] **by** (*simp add*: *enabled-iff-sconst*)
also have ... = (*AE* ω *in T x. P* (*sconst x*) ∧ ω = *sconst x*)

by (*simp del*: *AE-conj-iff cong*: *rev-conj-cong*)
also have ... = (*AE* ω *in T x*. *P* (*sconst x*))
using *AE-T-enabled*[*of x*] **by** (*simp add*: *enabled-iff-sconst*)
finally show *?thesis*
by *simp*
qed

lemma *ev-eq-lfp*: *ev P* = *lfp* (λ*F* ω. *P* ω ∨ (¬ *P* ω ∧ *F* (*stl* ω)))
unfolding *ev-def* **by** (*intro antisym lfp-mono*) *blast*+

lemma *INF-eq-zero-iff-ennreal*: ((⨅ *i*∈*A*. *f i*) = (*0*::*ennreal*)) = (∀ *x*>*0*. ∃ *i*∈*A*. *f i* < *x*)
using *INF-eq-bot-iff*[**where** ′*a*=*ennreal*] **unfolding** *bot-ennreal-def zero-ennreal-def*
by *auto*

lemma *inf-continuous-cmul*:
fixes *c* :: *ennreal*
assumes *f*: *inf-continuous f* **and** *c*: *c* < ⊤
shows *inf-continuous* (λ*x*. *c* ∗ *f x*)
proof (*rule inf-continuous-compose*[*OF - f*], *clarsimp simp add*: *inf-continuous-def*)
fix *M* :: *nat* ⇒ *ennreal* **assume** *M*: *decseq M*
show *c* ∗ (⨅ *i*. *M i*) = (⨅ *i*. *c* ∗ *M i*)
using *M*
by (*intro LIMSEQ-unique*[*OF ennreal-tendsto-cmult*[*OF c*] *LIMSEQ-INF*] *LIMSEQ-INF*)
(*auto simp*: *decseq-def mult-left-mono*)
qed

lemma *AE-T-ev-HLD-infinite*:
fixes *X* :: ′*s set* **and** *r* :: *real*
assumes *r* < *1*
assumes *r*: ⋀*x*. *x* ∈ *X* ⟹ *measure* (*K x*) *X* ≤ *r*
shows *AE* ω *in T x*. *ev* (*HLD* (− *X*)) ω
proof −
{ **fix** *x* **assume** *x* ∈ *X*
have *0* ≤ *r* **using** *r*[*OF* ‹*x* ∈ *X*›] *measure-nonneg*[*of K x X*] **by** (*blast intro*: *order.trans*)
define *P* **where** *P F x* = ∫⁺*y*. *indicator X y* ∗ (*F y* ⊓ *1*) ∂*K x* **for** *F x*
have [*measurable*]: *X* ∈ *sets* (*count-space UNIV*) **by** *auto*
have *bnd*: (∫⁺ *y*. *indicator X y* ∗ (*f y* ⊓ *1*) ∂*K x*) ≤ *1* **for** *x f*
by (*intro measure-pmf.nn-integral-le-const AE-pmfI*) (*auto split*: *split-indicator*)
have *emeasure* (*T x*) {ω∈*space* (*T x*). *alw* (*HLD X*) ω} =
emeasure (*T x*) {ω∈*space* (*T x*). *gfp* (λ*F* ω. *shd* ω ∈ *X* ∧ *F* (*stl* ω)) ω}
by (*simp add*: *alw-def HLD-def*)
also have ... = *gfp P x*
apply (*rule emeasure-gfp*)
apply (*auto intro*!: *order-continuous-intros inf-continuous-cmul split*: *split-indicator simp*: *P-def*)
subgoal for *x f* **using** *bnd*[*of x f*] **by** (*auto simp*: *top-unique*)

```
      subgoal for P x
        apply (subst T-eq-bind)
        apply (subst emeasure-bind[where N=S])
        apply simp
        apply (rule measurable-distr2[where M=S])
      apply (auto intro: T-subprob[THEN measurable-space] intro!: nn-integral-cong-AE
AE-pmfI
            simp: emeasure-distr split: split-indicator)
        apply (simp-all add: space-stream-space T.emeasure-le-1 inf.absorb1)
        done
      apply (intro le-funI)
      apply (subst nn-integral-indicator[symmetric])
      apply simp
      apply (intro nn-integral-mono)
      apply (auto split: split-indicator)
      done
    also have ... ≤ (INF n. ennreal r ^ n)
    proof (intro INF-greatest)
      have mono-P: mono P
          by (force simp: le-fun-def mono-def P-def intro!: nn-integral-mono intro:
le-infI1 split: split-indicator)
      fix n show gfp P x ≤ ennreal r ^ n
        using ‹x ∈ X›
      proof (induction n arbitrary: x)
        case 0 then show ?case
       by (subst gfp-unfold[OF mono-P]) (auto intro!: measure-pmf.nn-integral-le-const
AE-pmfI split: split-indicator simp: P-def)
      next
        case (Suc n x)
        have gfp P x = P (gfp P) x by (subst gfp-unfold[OF mono-P]) rule
        also have ... ≤ P (λx. ennreal r ^ n) x
            unfolding P-def[of - x] by (auto intro!: nn-integral-mono le-infI1 Suc
split: split-indicator)
        also have ... ≤ ennreal r ^ (Suc n)
       using Suc by (auto simp: P-def nn-integral-multc measure-pmf.emeasure-eq-measure
intro!: mult-mono ennreal-leI r)
        finally show ?case .
      qed
    qed
    also have ... = 0
      unfolding ennreal-power[OF ‹0 ≤ r›]
    proof (intro LIMSEQ-unique[OF LIMSEQ-INF])
      show decseq (λi. ennreal (r ^ i))
        using ‹0 ≤ r› ‹r < 1› by (auto intro!: ennreal-leI power-decreasing simp:
decseq-def)
      have (λi. ennreal (r ^ i)) ⟶ ennreal 0
       using ‹0 ≤ r› ‹r < 1› by (intro tendsto-ennrealI LIMSEQ-power-zero) auto
      then show (λi. ennreal (r ^ i)) ⟶ 0 by simp
    qed
```

finally have ∗: *emeasure* (*T x*) {ω∈*space* (*T x*). *alw* (*HLD X*) ω} = *0* **by** *auto*
have *AE* ω *in T x*. *ev* (*HLD* (− *X*)) ω
by (*rule AE-I*[*OF* - ∗]) (*auto simp*: *not-ev-iff not-HLD*[*symmetric*]) **}**
note ∗ = *this*
show *?thesis*
apply (*clarsimp simp add*: *AE-T-iff*[*of* - *x*])
subgoal for *x'*
by (*cases x'* ∈ *X*) (*auto simp add*: *ev-Stream* ∗)
done
qed

3.6 Trace space with Restriction

definition *rT x* = *restrict-space* (*T x*) {ω. *enabled x* ω}

lemma *space-rT*: ω ∈ *space* (*rT x*) ⟷ *enabled x* ω
by (*auto simp*: *rT-def space-restrict-space space-stream-space*)

lemma *Collect-enabled-S*[*measurable*]: *Collect* (*enabled x*) ∈ *sets S*
proof −
have *Collect* (*enabled x*) = {ω∈*space S*. *enabled x* ω}
by (*auto simp*: *space-stream-space*)
then show *?thesis*
by *simp*
qed

lemma *space-rT-in-S*: *space* (*rT x*) ∈ *sets S*
by (*simp add*: *rT-def space-restrict-space*)

lemma *sets-rT*: *A* ∈ *sets* (*rT x*) ⟷ *A* ∈ *sets S* ∧ *A* ⊆ {ω. *enabled x* ω}
by (*auto simp*: *rT-def sets-restrict-space space-stream-space*)

lemma *prob-space-rT*: *prob-space* (*rT x*)
unfolding *rT-def* **by** (*auto intro*!: *prob-space-restrict-space T.emeasure-eq-1-AE AE-T-enabled*)

lemma *measurable-force-enabled2*[*measurable*]: *force-enabled x* ∈ *measurable S* (*rT x*)
unfolding *rT-def*
by (*rule measurable-restrict-space2*)
(*auto intro*: *measurable-force-enabled enabled-force-enabled*)

lemma *space-rT-not-empty*[*simp*]: *space* (*rT x*) ≠ {}
by (*simp add*: *rT-def space-restrict-space Ex-enabled*)

lemma *T-eq-bind'*: *T x* = *do* { *y* ← *measure-pmf* (*K x*) ; ω ← *T y* ; *return S* (*y* ## ω) }
apply (*subst T-eq-bind*)
apply (*subst bind-return-distr*[*symmetric*])

apply (*simp-all add*: *space-stream-space comp-def*)
done

lemma *rT-eq-bind*: *rT x* = *do* { *y* ← *measure-pmf* (*K x*) ; *ω* ← *rT y* ; *return* (*rT x*) (*y* ## *ω*) }
unfolding *rT-def*
apply (*subst T-eq-bind*)
apply (*subst restrict-space-bind*[**where** *K*=*S*])
apply (*rule measurable-distr2*[**where** *M*=*S*])
apply (*auto simp del*: *measurable-pmf-measure1*
simp add: *Ex-enabled return-restrict-space intro*!: *bind-cong*)
apply (*subst restrict-space-bind*[*symmetric*, **where** *K*=*S*])
apply (*auto simp add*: *Ex-enabled space-restrict-space return-cong*[*OF sets-T*]
intro!: *measurable-restrict-space1 measurable-compose*[*OF - return-measurable*]
arg-cong2[**where** *f*=*restrict-space*])
apply (*subst bind-return-distr*[*unfolded comp-def*])
apply (*simp add*: *space-restrict-space Ex-enabled*)
apply (*simp add*: *measurable-restrict-space1*)
apply (*rule measure-eqI*)
apply *simp*
apply (*subst* (*1 2*) *emeasure-distr*)
apply (*auto simp*: *measurable-restrict-space1*)
apply (*subst emeasure-restrict-space*)
apply (*auto simp*: *space-restrict-space intro*!: *emeasure-eq-AE*)
using *AE-T-enabled*
apply *eventually-elim*
apply (*simp add*: *space-stream-space*)
apply (*rule sets-Int-pred*)
apply *auto*
apply (*simp add*: *space-stream-space*)
done

lemma *snth-rT*: (*λx. x* !! *n*) ∈ *measurable* (*rT x*) (*count-space* (*acc* \`\` {*x*}))
proof −
have ⋀*ω*. *enabled x ω* ⟹ (*x*, *ω* !! *n*) ∈ *acc*
proof (*induction n arbitrary*: *x*)
case (*Suc n*) **from** *Suc.prems Suc.IH*[*of shd ω stl ω*] **show** *?case*
by (*auto simp*: *enabled.simps*[*of x ω*] *intro*: *rtrancl-trans*)
qed (*auto elim*: *enabled.cases*)
moreover
{ **fix** *X* :: *'s set*
have [*measurable*]: *X* ∈ *count-space UNIV* **by** *simp*
have ∗: (*λx. x* !! *n*) −\` *X* ∩ *space* (*rT x*) = {*ω*∈*space S*. *ω* !! *n* ∈ *X* ∧ *enabled x ω*}
by (*auto simp*: *space-stream-space space-rT*)
have (*λx. x* !! *n*) −\` *X* ∩ *space* (*rT x*) ∈ *sets S*
unfolding ∗ **by** *measurable* }
ultimately show *?thesis*

by (*auto simp*: *measurable-def space-rT sets-rT*)
qed

3.7 Bisimulation

lemma *T-coinduct*[*consumes 1*, *case-names prob sets cont*]:
assumes *R x M*
assumes *prob*: $\bigwedge x\ M.\ R\ x\ M \Longrightarrow$ *prob-space M*
and *sets*: $\bigwedge x\ M.\ R\ x\ M \Longrightarrow$ *sets M* = *sets S*
and *cont'*: $\bigwedge x\ M.\ R\ x\ M \Longrightarrow \exists M'.\ (\forall y{\in}K\ x.\ R\ y\ (M'\ y)) \wedge (\forall y.$ *sets* $(M'\ y)$ $= S \wedge$ *prob-space* $(M'\ y)) \wedge$
$M = ($*measure-pmf* $(K\ x) \gg= (\lambda y.$ *distr* $(M'\ y)\ S\ ((\#\#)\ y)))$
shows $T\ x = M$
using ‹*R x M*›
proof (*coinduction arbitrary*: *x M rule*: *measure-eq-stream-space-coinduct*)
case *left* **then show** *?case* **using** *T.prob-space-axioms*[*of x*] *sets-T*[*of x*] **by** (*auto simp*: *space-prob-algebra*)
next
case (*right M*) **with** *prob*[*of M*] *sets*[*of M*] **show** *?case* **by** (*auto simp*: *space-prob-algebra*)
next
case (*cont x M*) **with** *cont'*[*OF cont*] **obtain** M' **where** ∗:
$(\forall y{\in}K\ x.\ R\ y\ (M'\ y))$
$(\forall y.$ *sets* $(M'\ y) = S \wedge$ *prob-space* $(M'\ y))$
$M = ($*measure-pmf* $(K\ x) \gg= (\lambda y.$ *distr* $(M'\ y)\ S\ ((\#\#)\ y)))$
by *auto*
show *?case*
apply (*rule exI*[*of - T*])
apply (*rule exI*[*of - M'*])
apply (*rule exI*[*of - K x*])
using ∗ *T.prob-space-axioms sets-T*[*of x*]
apply (*auto simp*: *space-prob-algebra intro*: *T-eq-bind*)
done
qed

lemma *T-bisim*:
assumes *M*: $\bigwedge x.$ *prob-space* $(M\ x)$ $\bigwedge x.$ *sets* $(M\ x)$ = *sets S*
and *M-eq*: $\bigwedge x.\ M\ x = ($*measure-pmf* $(K\ x) \gg= (\lambda s.$ *distr* $(M\ s)\ S\ ((\#\#)\ s)))$
shows $T = M$
proof
fix *x* **show** $T\ x = M\ x$
proof (*coinduction arbitrary*: *x rule*: *T-coinduct*)
case (*cont x*) **then show** *?case*
apply (*intro exI*[*of - M*])
apply (*subst M-eq*[*of x*])
apply (*simp add*: *M*)
done
qed *fact*+
qed

```
lemma T-subprob′[measurable]: T ∈ measurable (count-space UNIV) (subprob-algebra
S)
  by (auto intro!: space-bind simp: space-subprob-algebra) unfold-locales

lemma T-subprob′′[simp]: T a ∈ space (subprob-algebra S)
  using measurable-space[OF T-subprob′, of a] by simp

lemma AE-not-suntil-coinduct [consumes 1, case-names ψ φ]:
  assumes P s
  assumes ψ: ⋀s. P s ⟹ s ∉ ψ
  assumes φ: ⋀s t. P s ⟹ s ∈ φ ⟹ t ∈ K s ⟹ P t
  shows AE ω in T s. not (HLD φ suntil HLD ψ) (s ## ω)
proof −
  { fix ω have ¬ (HLD φ suntil HLD ψ) (s ## ω) ⟷
      (∀ n. ¬ ((λR. HLD ψ or (HLD φ aand nxt R)) ^^ n) ⊥ (s ## ω))
      unfolding suntil-def
      by (subst sup-continuous-lfp)
        (auto simp add: sup-continuous-def) }
  moreover
  { fix n from ‹P s› have AE ω in T s. ¬ ((λR. HLD ψ or (HLD φ aand nxt
R)) ^^ n) ⊥ (s ## ω)
    proof (induction n arbitrary: s)
      case (Suc n) then show ?case
        apply (subst AE-T-iff)
       apply (rule measurable-compose[OF measurable-Stream, where M1=count-space
UNIV])
        apply measurable
        apply simp
        apply (auto simp: bot-fun-def intro!: AE-impI dest: φ ψ)
        done
    qed simp }
  ultimately show ?thesis
    by (simp add: AE-all-countable)
qed

lemma AE-not-suntil-coinduct-strong [consumes 1, case-names ψ φ]:
  assumes P s
  assumes P-ψ: ⋀s. P s ⟹ s ∉ ψ
  assumes P-φ: ⋀s t. P s ⟹ s ∈ φ ⟹ t ∈ K s ⟹ P t ∨
    (AE ω in T t. not (HLD φ suntil HLD ψ) (t ## ω))
  shows AE ω in T s. not (HLD φ suntil HLD ψ) (s ## ω) (is ?nuntil s)
proof −
  have P s ∨ ?nuntil s
    using ‹P s› by auto
  then show ?thesis
  proof (coinduction arbitrary: s rule: AE-not-suntil-coinduct)
    case (φ t s) then show ?case
      by (auto simp: AE-T-iff[of - s] suntil-Stream[of - - s] dest: P-φ)
  qed (auto simp: suntil-Stream dest: P-ψ)
```

qed

end

3.8 Reward Structure on Markov Chains

locale *MC-with-rewards* = *MC-syntax K* **for** *K* :: *'s* ⇒ *'s pmf* +
fixes *ι* :: *'s* ⇒ *'s* ⇒ *ennreal* **and** *ϱ* :: *'s* ⇒ *ennreal*
assumes *ι-nonneg*: ⋀*s t. 0* ≤ *ι s t* **and** *ϱ-nonneg*: ⋀*s. 0* ≤ *ϱ s*
assumes *measurable-ι*[*measurable*]: (λ(*a*, *b*). *ι a b*) ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *count-space UNIV*)
begin

definition *reward-until* :: *'s set* ⇒ *'s* ⇒ *'s stream* ⇒ *ennreal* **where**
reward-until X = *lfp* (λ*F s ω. if s* ∈ *X then 0 else ϱ s* + *ι s* (*shd ω*) + (*F* (*shd ω*) (*stl ω*)))

lemma *measurable-ϱ*[*measurable*]: *ϱ* ∈ *borel-measurable* (*count-space UNIV*)
by *simp*

lemma *measurable-reward-until*[*measurable* (*raw*)]:
assumes [*measurable*]: *f* ∈ *measurable M* (*count-space UNIV*)
assumes [*measurable*]: *g* ∈ *measurable M S*
shows (λ*x. reward-until X* (*f x*) (*g x*)) ∈ *borel-measurable M*
proof −
let *?F* = λ*F* (*s*, *ω*). *if s* ∈ *X then 0 else ϱ s* + *ι s* (*shd ω*) + (*F* (*shd ω*, *stl ω*))
{ fix *s ω*
have *reward-until X s ω* = *lfp ?F* (*s*, *ω*)
unfolding *reward-until-def lfp-pair*[*symmetric*] **.. }**
note ∗ = *this*

have [*measurable*]: *lfp ?F* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *S*)
proof (*rule borel-measurable-lfp*)
fix *f* :: (*'s* × *'s stream*) ⇒ *ennreal*
assume [*measurable*]: *f* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *S*)
show *?F f* ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *S*)
unfolding *split-beta'*
apply (*intro measurable-If*)
apply *measurable* []
apply *measurable* []
apply (*rule predE*)
apply (*rule measurable-compose*[*OF measurable-fst*])
apply *measurable* []
done
qed (*auto intro*!: *ι-nonneg ϱ-nonneg order-continuous-intros*)
show *?thesis*
unfolding ∗ **by** *measurable*
qed

lemma *continuous-reward-until*:
sup-continuous ($\lambda F\ s\ \omega$. *if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ (*shd* ω) + (F (*shd* ω) (*stl* ω)))
by (*intro ι-nonneg ϱ-nonneg order-continuous-intros*) (*auto simp*: *sup-continuous-def image-comp*)

lemma
shows *reward-until-unfold*: *reward-until* $X\ s\ \omega$ =
(*if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ (*shd* ω) + *reward-until* X (*shd* ω) (*stl* ω))
(**is** *?unfold*)
proof −
let *?F* = $\lambda F\ s\ \omega$. *if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s$ (*shd* ω) + (F (*shd* ω) (*stl* ω))
{ fix $s\ \omega$ **have** *reward-until* $X\ s\ \omega$ = *?F* (*reward-until* X) $s\ \omega$
unfolding *reward-until-def*
apply (*subst lfp-unfold*)
apply (*rule continuous-reward-until*[*THEN sup-continuous-mono*, *of X*])
apply *rule*
done }
note *step* = *this*
show *?unfold*
by (*subst step*) (*auto intro*!: *arg-cong2*[**where** *f*=(+)])
qed

lemma *reward-until-simps*[*simp*]:
shows $s \in X \Longrightarrow$ *reward-until* $X\ s\ \omega = 0$
and $s \notin X \Longrightarrow$ *reward-until* $X\ s\ \omega = \varrho\ s + \iota\ s$ (*shd* ω) + *reward-until* X (*shd* ω) (*stl* ω)
unfolding *reward-until-unfold*[*of X s ω*] **by** *simp-all*

lemma *reward-until-SCons*[*simp*]:
reward-until $X\ s$ (t ## ω) = (*if* $s \in X$ *then 0 else* $\varrho\ s + \iota\ s\ t$ + *reward-until* X $t\ \omega$)
by *simp*

lemma *nn-integral-reward-until-finite*:
assumes [*simp*]: *finite* (*acc* `` $\{s\}$) (**is** *finite* (*?R* `` $\{s\}$))
assumes ϱ: $\bigwedge t$. $(s, t) \in$ *acc-on* $(-H) \Longrightarrow \varrho\ t < \infty$
assumes ι: $\bigwedge t\ t'$. $(s, t) \in$ *acc-on* $(-H) \Longrightarrow t' \in K\ t \Longrightarrow \iota\ t\ t' < \infty$
assumes *ev*: *AE* ω *in* $T\ s$. *ev* (*HLD* H) ω
shows ($\int^+ \omega$. *reward-until* $H\ s\ \omega\ \partial T\ s$) $\neq \infty$
proof *cases*
assume $s \in H$ **then show** *?thesis*
by *simp*
next
assume $s \notin H$
let *?L* = *acc-on* $(-H)$
define M **where** M = *Max* (($\lambda(s, t)$. $\varrho\ s + \iota\ s\ t$) ` (*SIGMA* t:*?L*``$\{s\}$. $K\ t$))
have *?L* $\subseteq$ *?R*
by (*intro rtrancl-mono*) *auto*

```
with ‹s ∉ H› have subset: (SIGMA t:?L``{s}. K t) ⊆ (?R``{s} × ?R``{s})
  by (auto intro: rtrancl-into-rtrancl elim: rtrancl.cases)
then have [simp, intro!]: finite ((λ(s, t). ϱ s + ι s t) ` (SIGMA t:?L``{s}. K t))
  by (intro finite-imageI) (auto dest: finite-subset)
{ fix t t' assume (s, t) ∈ ?L t ∉ H t' ∈ K t
  then have (t, t') ∈ (SIGMA t:?L``{s}. K t)
    by (auto intro: rtrancl-into-rtrancl)
  then have ϱ t + ι t t' ≤ M
    unfolding M-def by (intro Max-ge) auto }
note le-M = this

have fin-L: finite (?L `` {s})
  by (intro finite-subset[OF - assms(1)] Image-mono ‹?L ⊆ ?R› order-refl)

have M < ∞
  unfolding M-def
proof (subst Max-less-iff, safe)
  show (SIGMA x:?L `` {s}. set-pmf (K x)) = {} ⟹ False
    using ‹s ∉ H› by (auto simp add: Sigma-empty-iff set-pmf-not-empty)
  fix t t' assume (s, t) ∈ ?L t' ∈ K t then show ϱ t + ι t t' < ∞
    using ϱ[of t] ι[of t t'] by simp
qed

from set-pmf-not-empty[of K s] obtain t where t ∈ K s
  by auto
with le-M[of s t] have 0 ≤ M
  using set-pmf-not-empty[of K s] ‹s ∉ H› le-M[of s] ι-nonneg[of s] ϱ-nonneg[of
s]
  by (intro order-trans[OF - le-M]) auto

have AE ω in T s. reward-until H s ω ≤ M ∗ sfirst (HLD H) (s ## ω)
  using ev AE-T-enabled
proof eventually-elim
  fix ω assume ev (HLD H) ω enabled s ω
  moreover define t where t = s
  ultimately have ev (HLD H) ω enabled t ω t ∈ ?L``{s}
    by auto
  then show reward-until H t ω ≤ M ∗ sfirst (HLD H) (t ## ω)
  proof (induction arbitrary: t rule: ev-induct-strong)
    case (base ω t) then show ?case
      by (auto simp: HLD-iff sfirst-Stream elim: enabled.cases intro: le-M)
  next
    case (step ω t) from step.IH[of shd ω] step.prems step.hyps show ?case
      by (auto simp add: HLD-iff enabled.simps[of t] distrib-left sfirst-Stream
                         reward-until-simps[of t]
               simp del: reward-until-simps
               intro!: add-mono le-M intro: rtrancl-into-rtrancl)
  qed
qed
```

then have $(\int^+\omega.\ reward\text{-}until\ H\ s\ \omega\ \partial T\ s) \leq (\int^+\omega.\ M * sfirst\ (HLD\ H)\ (s\ \#\#\ \omega)\ \partial T\ s)$
by (*rule nn-integral-mono-AE*)
also have $\ldots < \infty$
using ‹*0* ≤ *M*› ‹*M* < ∞› *nn-integral-sfirst-finite*[*OF fin-L ev*]
by (*simp add*: *nn-integral-cmult less-top*[*symmetric*] *ennreal-mult-eq-top-iff*)
finally show *?thesis*
by *simp*
qed

end

3.9 Bisimulation on a relation

definition *rel-set-strong* :: ($'a \Rightarrow 'b \Rightarrow bool$) $\Rightarrow$ $'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where *rel-set-strong R A B* $\longleftrightarrow$ $(\forall x\ y.\ R\ x\ y \longrightarrow (x \in A \longleftrightarrow y \in B))$

lemma *T-eq-rel-half*[*consumes 4*, *case-names prob sets cont*]:
fixes $R :: 's \Rightarrow 't \Rightarrow bool$ **and** $f :: 's \Rightarrow 't$ **and** $S :: 's\ set$
assumes *R-def*: $\bigwedge s\ t.\ R\ s\ t \longleftrightarrow (s \in S \wedge f\ s = t)$
assumes *A*[*measurable*]: $A \in$ *sets* (*stream-space* (*count-space UNIV*))
and *B*[*measurable*]: $B \in$ *sets* (*stream-space* (*count-space UNIV*))
and *AB*: *rel-set-strong* (*stream-all2 R*) *A B* **and** *KL*: *rel-fun R* (*rel-pmf R*) *K L* **and** *xy*: *R x y*
shows *MC-syntax.T K x A* = *MC-syntax.T L y B*
proof −
interpret *K*: *MC-syntax K* **by** *unfold-locales*
interpret *L*: *MC-syntax L* **by** *unfold-locales*

have $x \in S$ **using** ‹*R x y*› **by** (*auto simp*: *R-def*)

define *g* **where** $g\ t = (SOME\ s.\ R\ s\ t)$ **for** *t*
have *measurable-g*: $g \in$ *count-space UNIV* $\rightarrow_M$ *count-space UNIV* **by** *auto*
have *g*: $R\ i\ j \Longrightarrow R\ (g\ j)\ j$ **for** *i j*
unfolding *g-def* **by** (*rule someI*)

have *K-subset*: $x \in S \Longrightarrow K\ x \subseteq S$ **for** *x*
using *KL*[*THEN rel-funD*, *of x f x*, *THEN rel-pmf-imp-rel-set*] **by** (*auto simp*: *rel-set-def R-def*)

have *in-S*: *AE* ω *in K.T x*. ω ∈ *streams S*
using *K.AE-T-enabled*
proof *eventually-elim*
case (*elim* ω) **with** ‹*x* ∈ *S*› **show** *?case*
apply (*coinduction arbitrary*: *x* ω)
subgoal for *x* ω **using** *K-subset* **by** (*cases* ω) (*auto simp*: *K.enabled-Stream*)
done
qed

have *L-eq*: *L y* = *map-pmf f* (*K x*) **if** *xy*: *R x y* **for** *x y*
proof −
have *rel-pmf* (λ*x y*. *x* = *y*) (*map-pmf f* (*K x*)) (*L y*)
using *KL*[*THEN rel-funD*, *OF xy*] **by** (*auto intro*: *pmf.rel-mono-strong simp*: *R-def pmf.rel-map*)
then show *?thesis* **unfolding** *pmf.rel-eq* **by** *simp*
qed

let *?D* = λ*x*. *distr* (*K.T x*) *K.S* (*smap f*)
have *prob-space-D*: *?D x* ∈ *space* (*prob-algebra K.S*) **for** *x*
by (*auto simp*: *space-prob-algebra K.T.prob-space-distr*)

have *D-eq-D*: *?D x* = *?D x′* **if** *R x y R x′ y* **for** *x x′ y*
proof (*rule stream-space-eq-sstart*)
define *A* **where** *A* = *K.acc* `` {*x*, *x′*}
have *x-A*: *x* ∈ *A x′* ∈ *A* **by** (*auto simp*: *A-def*)
let *?*Ω = *f* ` *A*
show *countable ?*Ω
unfolding *A-def* **by** (*intro countable-image K.countable-acc*) *auto*
show *prob-space* (*?D x*) *prob-space* (*?D x′*) **by** (*auto intro*!: *K.T.prob-space-distr*)
show *sets* (*?D x*) = *sets L.S sets* (*?D x′*) = *sets L.S* **by** *auto*
have *AE-streams*: *AE x in ?D x″*. *x* ∈ *streams ?*Ω **if** *x″* ∈ *A* **for** *x″*
apply (*simp add*: *space-stream-space streams-sets AE-distr-iff*)
using *K.AE-T-reachable*[*of x″*] **unfolding** *alw-HLD-iff-streams*
proof *eventually-elim*
fix *s* **assume** *s* ∈ *streams* (*K.acc* `` {*x″*})
moreover have *K.acc* `` {*x″*} ⊆ *A*
using ‹*x″* ∈ *A*› **by** (*auto simp*: *A-def Image-def intro*: *rtrancl-trans*)
ultimately show *smap f s* ∈ *streams* (*f* ` *A*)
by (*auto intro*: *smap-streams*)
qed
with *x-A* **show** *AE x in ?D x′*. *x* ∈ *streams ?*Ω *AE x in ?D x*. *x* ∈ *streams ?*Ω
by *auto*
from ‹*x* ∈ *A*› ‹*x′* ∈ *A*› *that* **show** *?D x* (*sstart* (*f* ` *A*) *xs*) = *?D x′* (*sstart* (*f* ` *A*) *xs*) **for** *xs*
proof (*induction xs arbitrary*: *x x′ y*)
case *Nil*
moreover have *?D x* (*streams* (*f* ` *A*)) = *1* **if** *x* ∈ *A* **for** *x*
using *AE-streams*[*of x*] *that*
by (*intro prob-space.emeasure-eq-1-AE*[*OF K.T.prob-space-distr*]) (*auto simp*: *streams-sets*)
ultimately show *?case* **by** *simp*
next
case (*Cons z zs x x′ y*)
have *rel-pmf* (*R OO* R^{-1-1}) (*K x*) (*K x′*)
using *KL*[*THEN rel-funD*, *OF Cons*(*4*)] *KL*[*THEN rel-funD*, *OF Cons*(*5*)]
unfolding *pmf.rel-compp pmf.rel-flip* **by** *auto*
then obtain *p* :: (*′s* × *′s*) *pmf* **where** *p*: ⋀*a b*. (*a*, *b*) ∈ *p* ⟹ (*R OO* R^{-1-1}) *a b* **and**

eq: *map-pmf fst p* = *K x map-pmf snd p* = *K x′*
by (*auto simp*: *pmf.in-rel*)
let *?S* = *stream-space* (*count-space UNIV*)
have ∗: (##) *y* −‘ *smap f* −‘ *sstart* (*f* ‘ *A*) (*z* # *zs*) = (*if f y* = *z then smap f* −‘ *sstart* (*f* ‘ *A*) *zs else* {}) **for** *y z zs*
by *auto*
have ∗∗: *?D x* (*sstart* (*f* ‘ *A*) (*z* # *zs*)) = ($\int^+$ *y′*. (*if f y′* = *z then ?D y′* (*sstart* (*f* ‘ *A*) *zs*) *else 0*) ∂*K x*) **for** *x*
apply (*simp add*: *emeasure-distr*)
apply (*subst K.T-eq-bind*)
apply (*subst emeasure-bind*[**where** *N*=*?S*])
apply *simp*
apply (*rule measurable-distr2*[**where** *M*=*?S*])
apply *measurable*
apply (*intro nn-integral-cong-AE AE-pmfI*)
apply (*auto simp add*: *emeasure-distr*)
apply (*simp-all add*: ∗ *space-stream-space*)
done
have *fst-A*: *fst ab* ∈ *A* **if** *ab* ∈ *p* **for** *ab*
proof −
have *fst ab* ∈ *K x* **using** ‹*ab* ∈ *p*› *set-map-pmf* [*of fst p*] **by** (*auto simp*: *eq*)
with ‹*x* ∈ *A*› **show** *fst ab* ∈ *A*
by (*auto simp*: *A-def intro*: *rtrancl.rtrancl-into-rtrancl*)
qed
have *snd-A*: *snd ab* ∈ *A* **if** *ab* ∈ *p* **for** *ab*
proof −
have *snd ab* ∈ *K x′* **using** ‹*ab* ∈ *p*› *set-map-pmf* [*of snd p*] **by** (*auto simp*: *eq*)
with ‹*x′* ∈ *A*› **show** *snd ab* ∈ *A*
by (*auto simp*: *A-def intro*: *rtrancl.rtrancl-into-rtrancl*)
qed
show *?case*
unfolding ∗∗ *eq*[*symmetric*] *nn-integral-map-pmf*
apply (*intro nn-integral-cong-AE AE-pmfI*)
subgoal for *ab* **using** *p*[*of fst ab snd ab*] **by** (*auto simp*: *R-def intro*!: *Cons*(*1*) *fst-A snd-A*)
done
qed
qed

have *L-eq-D*: *L.T y* = *?D x*
using ‹*R x y*›
proof (*coinduction arbitrary*: *x y rule*: *L.T-coinduct*)
case (*cont x y*)
then have *Kx-Ly*: *rel-pmf R* (*K x*) (*L y*)
by (*rule KL*[*THEN rel-funD*])
then have ∗: *y′* ∈ *L y* ⟹ ∃ *x′*∈*K x*. *R x′ y′* **for** *y′*
by (*auto dest*!: *rel-pmf-imp-rel-set simp*: *rel-set-def*)
have ∗∗: *y′* ∈ *L y* ⟹ *R* (*g y′*) *y′* **for** *y′*

```
      using ∗[of y′] unfolding g-def by (auto intro: someI)

    have D-SCons-eq-D-D: distr (K.T i) K.S (λx. z ## smap f x) = distr (?D i)
K.S (λx. z ## x) for i z
      by (subst distr-distr) (auto simp: comp-def)
    have D-eq-D-gi: ?D i = ?D (g (f i)) if i: i ∈ K x for i
    proof −
      obtain j where j ∈ L y R i j f i = j
        using Kx-Ly i by (force dest!: rel-pmf-imp-rel-set simp: rel-set-def R-def)
      then show ?thesis
        by (auto intro!: D-eq-D[OF ‹R i j›] g)
    qed

    have ∗∗∗: ?D x = measure-pmf (L y) ≫= (λy. distr (?D (g y)) K.S ((##) y))
      apply (subst K.T-eq-bind)
      apply (subst distr-bind[of - - K.S])
         apply (rule measurable-distr2[of - - K.S])
          apply (simp-all add: Pi-iff)
      apply (simp add: distr-distr comp-def L-eq[OF cont] map-pmf-rep-eq)
      apply (subst bind-distr[where K=K.S])
         apply measurable []
        apply (rule measurable-distr2[of - - K.S])
        apply measurable []
        apply (rule measurable-compose[OF measurable-g])
        apply measurable []
       apply simp
      apply (rule bind-measure-pmf-cong[where N=K.S])
      apply (auto simp: space-subprob-algebra space-stream-space intro!: K.T.subprob-space-distr)
        unfolding D-SCons-eq-D-D D-eq-D-gi ..
    show ?case
       by (intro exI[of - λt. distr (K.T (g t)) (stream-space (count-space UNIV))
(smap f)])
         (auto simp add: K.T.prob-space-distr ∗∗∗ dest: ∗∗)
  qed (auto intro: K.T.prob-space-distr)

  have stream-all2 R s t ⟷ (s ∈ streams S ∧ smap f s = t) for s t
  proof safe
    show stream-all2 R s t ⟹ s ∈ streams S
      apply (coinduction arbitrary: s t)
      subgoal for s t by (cases s; cases t) (auto simp: R-def)
      done
    show stream-all2 R s t ⟹ smap f s = t
      apply (coinduction arbitrary: s t)
      subgoal for s t by (cases s; cases t) (auto simp: R-def)
      done
  qed (auto intro!: stream.rel-refl-strong simp: stream.rel-map R-def streams-iff-sset)
  then have ω ∈ streams S ⟹ ω ∈ A ⟷ smap f ω ∈ B for ω
    using AB by (auto simp: rel-set-strong-def)
  with in-S have K.T x A = K.T x (smap f −‘ B ∩ space (K.T x))
```

by (*auto intro*!: *emeasure-eq-AE streams-sets*)
also have ... = (*distr* (*K.T x*) *K.S* (*smap f*)) *B*
by (*intro emeasure-distr*[*symmetric*]) *auto*
also have ... = (*L.T y*) *B* **unfolding** *L-eq-D* **..**
finally show *?thesis* **.**
qed

3.10 Product Construction

locale *MC-pair* =
K1: *MC-syntax K1* + *K2*: *MC-syntax K2* **for** *K1 K2*
begin

definition *Kp* ≡ λ(*a*, *b*). *pair-pmf* (*K1 a*) (*K2 b*)

sublocale *MC-syntax Kp* **.**

definition
$szip_E$ *a b* ≡ λ(*ω1*, *ω2*). *szip* (*K1.force-enabled a ω1*) (*K2.force-enabled b ω2*)

lemma *szip-rT*[*measurable*]: (λ(*ω1*, *ω2*). *szip ω1 ω2*) ∈ *measurable* (*K1.rT x1* $\bigotimes_M$ *K2.rT x2*) *S*
proof (*rule measurable-stream-space2*)
fix *n*
have (λ*x*. (*case x of* (*ω1*, *ω2*) ⇒ *szip ω1 ω2*) !! *n*) = (λ*ω*. (*fst ω* !! *n*, *snd ω* !! *n*))
by *auto*
also have ... ∈ *measurable* (*K1.rT x1* $\bigotimes_M$ *K2.rT x2*) (*count-space UNIV*)
apply (*rule measurable-compose-countable′*[*OF* - *measurable-compose*[*OF measurable-fst K1.snth-rT*, *of n*]])
apply (*rule measurable-compose-countable′*[*OF* - *measurable-compose*[*OF measurable-snd K2.snth-rT*, *of n*]])
apply (*auto intro*!: *K1.countable-acc K2.countable-acc*)
done
finally show (λ*x*. (*case x of* (*ω1*, *ω2*) ⇒ *szip ω1 ω2*) !! *n*) ∈ *measurable* (*K1.rT x1* $\bigotimes_M$ *K2.rT x2*) (*count-space UNIV*)
.
qed

lemma *measurable-szipE*[*measurable*]: $szip_E$ *a b* ∈ *measurable* (*K1.S* $\bigotimes_M$ *K2.S*) *S*
unfolding $szip_E$*-def* **by** *measurable*

lemma *T-eq-prod*: *T* = (λ(*x1*, *x2*). *do* { *ω1* ← *K1.T x1* ; *ω2* ← *K2.T x2* ; *return S* ($szip_E$ *x1 x2* (*ω1*, *ω2*)) })
(**is** - = *?B*)
proof (*rule T-bisim*)
have *T1x*: ⋀*x*. *subprob-space* (*K1.T x*)
by (*rule prob-space-imp-subprob-space*) *unfold-locales*

interpret *T12*: *pair-prob-space K1.T x K2.T y* **for** *x y*
by *unfold-locales*
interpret *T1K2*: *pair-prob-space K1.T x K2 y* **for** *x y*
by *unfold-locales*

let *?P* = *λx1 x2. K1.T x1* $\bigotimes_M$ *K2.T x2*

fix *x* **show** *prob-space* (*?B x*)
by (*auto simp*: *space-stream-space split*: *prod.splits*
intro!: *prob-space.prob-space-bind prob-space-return*
measurable-bind[**where** *N*=*S*] *measurable-compose*[*OF* -
return-measurable] *AE-I2*)
unfold-locales

show *sets* (*?B x*) = *sets S*
by (*simp split*: *prod.splits add*: *measurable-bind*[**where** *N*=*S*] *sets-bind*[**where** *N*=*S*] *space-stream-space*)

obtain *a b* **where** *x-eq*: *x* = (*a*, *b*)
by (*cases x*) *auto*
show *?B x* = (*measure-pmf* (*Kp x*) ≫= (*λs. distr* (*?B s*) *S* ((##) *s*)))
unfolding *x-eq*
apply (*subst K1.T-eq-bind′*)
apply (*subst K2.T-eq-bind′*)
apply (*auto*
simp add: *space-stream-space bind-assoc*[**where** *R*=*S* **and** *N*=*S*] *bind-return-distr*[*symmetric*]
Kp-def T1K2.bind-rotate[**where** *N*=*S*] *split-beta′ set-pair-pmf space-subprob-algebra*
bind-pair-pmf[*of case-prod M* **for** *M*, *unfolded split*, *symmetric*, **where** *N*=*S*] *szip*$_E$*-def*
stream-eq-Stream-iff bind-return[**where** *N*=*S*] *space-bind*[**where** *N*=*S*]
simp del: *measurable-pmf-measure1*
intro!: *bind-measure-pmf-cong*[**where** *N*=*S*] *subprob-space-bind*[**where** *N*=*S*] *subprob-space-measure-pmf*
T1x bind-cong[**where** *M*=*MC-syntax.T K x* **for** *K x*] *arg-cong2*[**where** *f*=*return*])
done
qed

lemma *nn-integral-pT*:
fixes *f* **assumes** [*measurable*]: *f* ∈ *borel-measurable S*
shows (∫⁺*ω. f ω ∂T* (*x*, *y*)) = (∫⁺*ω1.* ∫⁺*ω2. f* (*szip*$_E$ *x y* (*ω1*, *ω2*)) *∂K2.T y ∂K1.T x*)
by (*simp add*: *nn-integral-bind*[**where** *B*=*S*] *nn-integral-return in-S T-eq-prod*)

lemma *prod-eq-prob-T*:
assumes [*measurable*]: *Measurable.pred K1.S P1 Measurable.pred K2.S P2*

shows $\mathcal{P}(\omega$ *in* $K1.T\ x1.\ P1\ \omega) * \mathcal{P}(\omega$ *in* $K2.T\ x2.\ P2\ \omega) =$
$\mathcal{P}(\omega$ *in* $T\ (x1,\ x2).\ P1\ (smap\ fst\ \omega) \wedge P2\ (smap\ snd\ \omega))$
proof −
have $\mathcal{P}(\omega$ *in* $T\ (x1,\ x2).\ P1\ (smap\ fst\ \omega) \wedge P2\ (smap\ snd\ \omega)) =$
$(\int x.\ \int xa.\ indicator\ \{\omega \in space\ S.\ P1\ (smap\ fst\ \omega) \wedge P2\ (smap\ snd\ \omega)\}$ $(szip_E\ x1\ x2\ (x,\ xa))\ \partial MC\text{-}syntax.T\ K2\ x2\ \partial MC\text{-}syntax.T\ K1\ x1)$
by (*subst T-eq-prod*)
(*simp add*: *K1.T.measure-bind*[**where** *N=S*] *K2.T.measure-bind*[**where** *N=S*] *measure-return*)
also have ... $= (\int \omega 1.\ \int \omega 2.\ indicator\ \{\omega \in space\ K1.S.\ P1\ \omega\}\ \omega 1 * indicator\ \{\omega \in space\ K2.S.\ P2\ \omega\}\ \omega 2\ \partial K2.T\ x2\ \partial K1.T\ x1)$
apply (*intro integral-cong-AE*)
apply *measurable*
using *K1.AE-T-enabled*
apply *eventually-elim*
apply (*intro integral-cong-AE*)
apply *measurable*
using *K2.AE-T-enabled*
apply *eventually-elim*
apply (*auto simp*: *space-stream-space* $szip_E$*-def K1.force-enabled K2.force-enabled smap-szip-snd*[**where** $g=\lambda x.\ x$] *smap-szip-fst*[**where** $f=\lambda x.\ x$]
split: *split-indicator*)
done
also have ... $= \mathcal{P}(\omega$ *in* $K1.T\ x1.\ P1\ \omega) * \mathcal{P}(\omega$ *in* $K2.T\ x2.\ P2\ \omega)$
by *simp*
finally show *?thesis* **..**
qed

end

end

3.11 Trace Space equal to Markov Chains

theory *Trace-Space-Equals-Markov-Processes*
imports *Discrete-Time-Markov-Chain*
begin

We can construct for each time-homogeneous discrete-time Markov chain a corresponding probability space using *Markov-Models.Discrete-Time-Markov-Chain*. The constructed probability space has the same probabilities.

locale *Time-Homogeneous-Discrete-Markov-Process* = *M?*: *prob-space* +
fixes $S :: 's\ set$ **and** $X :: nat \Rightarrow 'a \Rightarrow 's$
assumes *X* [*measurable*]: $\bigwedge t.\ X\ t \in measurable\ M\ (count\text{-}space\ UNIV)$
assumes *S*: *countable* S $\bigwedge n.\ AE\ x\ in\ M.\ X\ n\ x \in S$
assumes *MC*: $\bigwedge n\ s\ s'.$
$\mathcal{P}(\omega\ in\ M.\ \forall t \le n.\ X\ t\ \omega = s\ t\ ) \neq 0 \Longrightarrow$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid \forall t \le n.\ X\ t\ \omega = s\ t\ ) =$
$\mathcal{P}(\omega\ in\ M.\ X\ (Suc\ n)\ \omega = s' \mid X\ n\ \omega = s\ n\ )$

```
  assumes TH: ⋀n m s t.
    𝒫(ω in M. X n ω = t) ≠ 0 ⟹ 𝒫(ω in M. X m ω = t) ≠ 0 ⟹
    𝒫(ω in M. X (Suc n) ω = s | X n ω = t) = 𝒫(ω in M. X (Suc m) ω = s | X
m ω = t)
begin

context
begin

interpretation pmf-as-measure .

lift-definition I :: 's pmf is distr M (count-space UNIV) (X 0)
proof −
  let ?X = distr M (count-space UNIV) (X 0)
  interpret X: prob-space ?X
    by (auto simp: prob-space-distr)
  have AE x in ?X. measure ?X {x} ≠ 0
     using S by (subst X.AE-support-countable) (auto simp: AE-distr-iff intro!:
exI[of - S])
  then show prob-space ?X ∧ sets ?X = UNIV ∧ (AE x in ?X. measure ?X {x}
≠ 0)
    by (simp add: prob-space-distr AE-support-countable)
qed

lemma I-in-S:
  assumes pmf I s ≠ 0 shows s ∈ S
proof −
  from ‹pmf I s ≠ 0› have 0 ≠ 𝒫(x in M. X 0 x = s)
    by transfer (auto simp: measure-distr vimage-def Int-def conj-commute)
  also have 𝒫(x in M. X 0 x = s) = 𝒫(x in M. X 0 x = s ∧ s ∈ S)
    using S(2)[of 0] by (intro M.finite-measure-eq-AE) auto
  finally show ?thesis
    by (cases s ∈ S) auto
qed

lift-definition K :: 's ⇒ 's pmf is
  λs. with (λn. 𝒫(ω in M. X n ω = s) ≠ 0)
    (λn. distr (uniform-measure M {ω∈space M. X n ω = s}) (count-space UNIV)
(X (Suc n)))
     (uniform-measure (count-space UNIV) {s})
proof (rule withI)
  fix s n assume ∗: 𝒫(ω in M. X n ω = s) ≠ 0
  let ?D = distr (uniform-measure M {ω∈space M. X n ω = s}) (count-space
UNIV) (X (Suc n))
  have D: prob-space ?D
    by (intro prob-space.prob-space-distr prob-space-uniform-measure)
       (auto simp: M.emeasure-eq-measure ∗)
  then interpret D: prob-space ?D .
  have sets-D: sets ?D = UNIV
```

by *simp*
moreover have *AE x in ?D. measure ?D* $\{x\} \neq 0$
unfolding *D.AE-support-countable*[*OF sets-D*]
proof (*intro exI*[*of - S*] *conjI*)
show *countable S* **by** (*rule S*)
show *AE x in ?D.* $x \in S$
using ∗ *S*(*2*)[*of Suc n*] **by** (*auto simp add*: *AE-distr-iff AE-uniform-measure M.emeasure-eq-measure*)
qed
ultimately show *prob-space ?D* ∧ *sets ?D* = *UNIV* ∧ (*AE x in ?D. measure ?D* $\{x\} \neq 0$)
using *D* **by** *blast*
qed (*auto intro*!: *prob-space-uniform-measure AE-uniform-measureI*)

lemma *pmf-K*:
assumes *n*: $0 < \mathcal{P}(\omega \text{ in } M.\ X\ n\ \omega = s)$
shows *pmf* (*K s*) *t* = $\mathcal{P}(\omega \text{ in } M.\ X\ (Suc\ n)\ \omega = t \mid X\ n\ \omega = s)$
proof (*transfer fixing*: *n s t*)
let *?P* = $\lambda n.\ \mathcal{P}(\omega \text{ in } M.\ X\ n\ \omega = s) \neq 0$
let *?D* = $\lambda n.$ *distr* (*uniform-measure M* $\{\omega \in space\ M.\ X\ n\ \omega = s\}$) (*count-space UNIV*) (*X* (*Suc n*))
let *?U* = *uniform-measure* (*count-space UNIV*) $\{s\}$
show *measure* (*with ?P ?D ?U*) $\{t\}$ = $\mathcal{P}(\omega \text{ in } M.\ X\ (Suc\ n)\ \omega = t \mid X\ n\ \omega = s)$
proof (*rule withI*)
fix n' **assume** *?P* n'
moreover have $X\ (Suc\ n')$ −‘ $\{t\} \cap space\ M = \{x \in space\ M.\ X\ (Suc\ n')\ x = t\}$
by *auto*
ultimately show *measure* (*?D* n') $\{t\}$ = $\mathcal{P}(\omega \text{ in } M.\ X\ (Suc\ n)\ \omega = t \mid X\ n\ \omega = s)$
using *n M.measure-uniform-measure-eq-cond-prob*[*of* $\lambda x.\ X\ (Suc\ n')\ x = t$ $\lambda x.\ X\ n'\ x = s$]
by (*auto simp*: *measure-distr M.emeasure-eq-measure simp del*: *measure-uniform-measure intro*!: *TH*)
qed (*insert n*, *simp*)
qed

lemma *pmf-K2*:
($\bigwedge n.\ \mathcal{P}(\omega \text{ in } M.\ X\ n\ \omega = s) = 0) \Longrightarrow$ *pmf* (*K s*) *t* = *indicator* $\{t\}$ *s*
apply (*transfer fixing*: *s t*)
apply (*rule withI*)
apply (*auto split*: *split-indicator*)
done

end

sublocale *K*: *MC-syntax K* **.**

lemma *bind-I-K-eq-M*: $K.T'$ *I* = *distr M K.S* ($\lambda\omega.$ *to-stream* ($\lambda n.\ X\ n\ \omega$)) (**is** -

= *?D*)
proof (*rule stream-space-eq-sstart*)
  note *streams-sets*[*measurable*]
  note *measurable-abs-UNIV*[*measurable* (*raw*)]
  note *sstart-sets*[*measurable*]

  { fix *s* **assume** $s \in S$
    from *K.AE-T-enabled*[*of s*] **have** *AE* ω *in K.T s.* $\omega \in$ *streams S*
    proof *eventually-elim*
      fix ω **assume** *K.enabled s* ω **from** *this* ‹$s{\in}S$› **show** $\omega \in$ *streams S*
      proof (*coinduction arbitrary*: *s* ω)
        case *streams*
        then have *1*: *pmf* (*K s*) (*shd* ω) $\neq 0$
          by (*simp add*: *K.enabled.simps*[*of s*] *set-pmf-iff*)
        have *shd* $\omega \in S$
        proof *cases*
          assume $\exists n.\ 0 < \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s)$
          then obtain *n* **where** $0 < \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s)$ **by** *auto*
          with *1* **have** *2*: $\mathcal{P}(\omega'\ in\ M.\ X\ (Suc\ n)\ \omega' = shd\ \omega \wedge X\ n\ \omega' = s) \neq 0$
            by (*simp add*: *pmf-K cond-prob-def*)
          show *shd* $\omega \in S$
          proof (*rule ccontr*)
            assume *shd* $\omega \notin S$
            with *S*(*2*)[*of Suc n*] **have** $\mathcal{P}(\omega'\ in\ M.\ X\ (Suc\ n)\ \omega' = shd\ \omega \wedge X\ n\ \omega'$
$= s) = 0$
              by (*intro M.prob-eq-0-AE*) *auto*
            with *2* **show** *False* **by** *contradiction*
          qed
        next
          assume $\neg\ (\exists n.\ 0 < \mathcal{P}(\omega\ in\ M.\ X\ n\ \omega = s))$
          then have *pmf* (*K s*) (*shd* ω) = *indicator* {*shd* ω} *s*
            by (*intro pmf-K2*) (*auto simp*: *not-less measure-le-0-iff*)
          with *1* ‹$s{\in}S$› **show** *?thesis*
            by (*auto split*: *split-indicator-asm*)
        qed
        with *streams* **show** *?case*
          by (*cases* ω) (*auto simp*: *K.enabled.simps*[*of s*])
      qed
    qed }
  note *AE-streams* = *this*

  show *prob-space* (*K.T′ I*)
    by (*rule K.prob-space-T′*)
  show *prob-space ?D*
    by (*rule M.prob-space-distr*) *simp*

  show *AE x in K.T′ I. x* $\in$ *streams S*
    by (*auto simp add*: *K.AE-T′ set-pmf-iff I-in-S AE-distr-iff streams-Stream*
intro!: *AE-streams*)

show *AE x in ?D. x* ∈ *streams S*
by (*simp add*: *AE-distr-iff to-stream-in-streams AE-all-countable S*)
show *sets* (*K.T′ I*) = *sets* (*stream-space* (*count-space UNIV*))
by (*simp add*: *K.sets-T′*)
show *sets ?D* = *sets* (*stream-space* (*count-space UNIV*))
by *simp*

fix *xs′* **assume** *xs′* ≠ [] *xs′* ∈ *lists S*
then obtain *s xs* **where** *xs′*: *xs′* = *s* # *xs* **and** *s*: *s* ∈ *S* **and** *xs*: *xs* ∈ *lists S*
by (*auto simp*: *neq-Nil-conv del*: *in-listsD*)

have *emeasure* (*K.T′ I*) (*sstart S xs′*) = (∫⁺*s. emeasure* (*K.T s*) {ω∈*space K.S. s* ## ω ∈ *sstart S xs′*} ∂*I*)
by (*rule K.emeasure-T′*) *measurable*
also have ... = (∫⁺*s′. emeasure* (*K.T s*) (*sstart S xs*) ∗ *indicator* {*s*} *s′* ∂*I*)
by (*intro arg-cong2*[**where** *f*=*emeasure*] *nn-integral-cong*)
(*auto split*: *split-indicator simp*: *emeasure-distr vimage-def space-stream-space neq-Nil-conv xs′*)
also have ... = *pmf I s* ∗ *emeasure* (*K.T s*) (*sstart S xs*)
by (*auto simp add*: *max-def emeasure-pmf-single intro*: *mult-ac*)
also have *emeasure* (*K.T s*) (*sstart S xs*) = *ennreal* (∏ *i*<*length xs. pmf* (*K* ((*s*#*xs*)!*i*)) (*xs*!*i*))
using *xs s*
proof (*induction arbitrary*: *s*)
case *Nil* **then show** *?case*
by (*simp add*: *K.T.emeasure-eq-1-AE AE-streams*)
next
case (*Cons t xs*)
have *emeasure* (*K.T s*) (*sstart S* (*t* # *xs*)) =
emeasure (*K.T s*) {*x*∈*space* (*K.T s*). *shd x* = *t* ∧ *stl x* ∈ *sstart S xs*}
by (*intro arg-cong2*[**where** *f*=*emeasure*]) (*auto simp*: *space-stream-space*)
also have ... = (∫⁺*t′. emeasure* (*K.T t′*) {*x*∈*space K.S. t′* = *t* ∧ *x* ∈ *sstart S xs*} ∂*K s*)
by (*subst K.emeasure-Collect-T*) *auto*
also have ... = (∫⁺*t′. emeasure* (*K.T t*) (*sstart S xs*) ∗ *indicator* {*t*} *t′* ∂*K s*)
by (*intro nn-integral-cong*) (*auto split*: *split-indicator simp*: *space-stream-space*)
also have ... = *emeasure* (*K.T t*) (*sstart S xs*) ∗ *pmf* (*K s*) *t*
by (*simp add*: *emeasure-pmf-single max-def*)
finally show *?case*
by (*simp add*: *lessThan-Suc-eq-insert-0 zero-notin-Suc-image prod.reindex Cons*
prod-nonneg ennreal-mult[*symmetric*])
qed
also have *pmf I s* ∗ *ennreal* (∏ *i*<*length xs. pmf* (*K* ((*s*#*xs*)!*i*)) (*xs*!*i*)) =
𝒫(*x in M*. ∀ *i*≤*length xs. X i x* = (*s* # *xs*) ! *i*)
using *xs s*
proof (*induction xs rule*: *rev-induct*)
case *Nil*

have *pmf I s* = *prob* {*x* ∈ *space M*. *X 0 x* = *s*}
by *transfer* (*simp add*: *vimage-def Int-def measure-distr conj-commute*)
then show *?case*
by *simp*
next
case (*snoc t xs*)
let *?l* = *length xs* **and** *?lt* = *length* (*xs* @ [*t*]) **and** *?xs'* = *s* # *xs* @ [*t*]
have *ennreal* (*pmf I s*) ∗ (∏ *i*<*?lt*. *pmf* (*K* ((*?xs'*) ! *i*)) ((*xs* @ [*t*]) ! *i*)) =
(*ennreal* (*pmf I s*) ∗ (∏ *i*<*?l*. *pmf* (*K* ((*s* # *xs*) ! *i*)) (*xs* ! *i*))) ∗ *pmf* (*K* ((*s* # *xs*) ! *?l*)) *t*
by (*simp add*: *lessThan-Suc mult-ac nth-append append-Cons*[*symmetric*] *prod-nonneg ennreal-mult*[*symmetric*]
del: *append-Cons*)
also have ... = 𝒫(*x in M*. ∀ *i*≤*?l*. *X i x* = (*s* # *xs*) ! *i*) ∗ *pmf* (*K* ((*s* # *xs*) ! *?l*)) *t*
using *snoc* **by** (*simp add*: *ennreal-mult*[*symmetric*])
also have ... = 𝒫(*x in M*. ∀ *i*≤*?lt*. *X i x* = (*?xs'*) ! *i*)
proof *cases*
assume 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*) = *0*
moreover have 𝒫(*x in M*. ∀ *i*≤*?lt*. *X i x* = (*?xs'*) ! *i*) ≤ 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*)
by (*intro M.finite-measure-mono*) (*auto simp*: *nth-append nth-Cons split*: *nat.split*)
moreover have 𝒫(*x in M*. ∀ *i*≤*?l*. *X i x* = (*s* # *xs*) ! *i*) ≤ 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*)
by (*intro M.finite-measure-mono*) (*auto simp*: *nth-append nth-Cons split*: *nat.split*)
ultimately show *?thesis*
by (*simp add*: *measure-le-0-iff*)
next
assume 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*) ≠ *0*
then have ∗: *0* < 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*)
unfolding *less-le* **by** *simp*
moreover have 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*) ≤ 𝒫(ω *in M*. *X ?l* ω = (*s* # *xs*) ! *?l*)
by (*intro M.finite-measure-mono*) (*auto simp*: *nth-append nth-Cons split*: *nat.split*)
ultimately have 𝒫(ω *in M*. *X ?l* ω = (*s* # *xs*) ! *?l*) ≠ *0*
by *auto*
then have *pmf* (*K* ((*s* # *xs*) ! *?l*)) *t* = 𝒫(ω *in M*. *X ?lt* ω = *?xs'* ! *?lt* | *X ?l* ω = (*s* # *xs*) ! *?l*)
by (*subst pmf-K*) (*auto simp*: *less-le*)
also have ... = 𝒫(ω *in M*. *X ?lt* ω = *?xs'* ! *?lt* | ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*)
using ∗ *MC*[*of ?l* λ*i*. (*s* # *xs*) ! *i ?xs'* ! *?lt*] **by** *simp*
also have ... = 𝒫(ω *in M*. ∀ *i*≤*?lt*. *X i* ω = *?xs'* ! *i*) / 𝒫(ω *in M*. ∀ *i*≤*?l*. *X i* ω = (*s* # *xs*) ! *i*)
unfolding *cond-prob-def*
by (*intro arg-cong2*[**where** *f*=(/)] *arg-cong2*[**where** *f*=*measure*]) (*auto simp*:

nth-Cons nth-append split: *nat.splits*)
      finally show *?thesis*
        using ∗ **by** *simp*
    qed
    finally show *?case* **.**
  qed
  also have . . . = *emeasure ?D* (*sstart S xs*′)
  proof −
    have *AE x in M*. ∀ *i*. *X i x* ∈ *S*
      using *S*(*2*) **by** (*simp add*: *AE-all-countable*)
    then have *AE x in M*. (∀ *i*≤*length xs*. *X i x* = (*s* # *xs*) ! *i*) = (*to-stream* (λ*n*. *X n x*) ∈ *sstart S xs*′)
    proof *eventually-elim*
      fix *x* **assume** ∀ *i*. *X i x* ∈ *S*
      then have *to-stream* (λ*n*. *X n x*) ∈ *streams S*
        by (*auto simp*: *streams-iff-snth to-stream-def*)
      then show (∀ *i*≤*length xs*. *X i x* = (*s* # *xs*) ! *i*) = (*to-stream* (λ*n*. *X n x*) ∈ *sstart S xs*′)
        by (*simp add*: *sstart-eq xs*′ *to-stream-def less-Suc-eq-le del*: *sstart.simps*(*1*) *in-sstart*)
    qed
    then show *?thesis*
     by (*auto simp*: *emeasure-distr M.emeasure-eq-measure intro*!: *M.finite-measure-eq-AE*)
  qed
  finally show *emeasure* (*K.T*′ *I*) (*sstart S xs*′) = *emeasure ?D* (*sstart S xs*′) **.**
qed (*rule S*)

end

lemma (**in** *MC-syntax*) *is-THDTMC*:
  fixes *I* :: ′*s pmf*
  defines *U* ≡ (*SIGMA s*:*UNIV*. *K s*)∗ ‘‘ *I*
  shows *Time-Homogeneous-Discrete-Markov-Process* (*T*′ *I*) *U* (λ*n ω*. *ω* !! *n*)
proof −
  have [*measurable*]: *U* ∈ *sets* (*count-space UNIV*)
    by *auto*

  interpret *prob-space T*′ *I*
    by (*rule prob-space-T*′)

  { fix *s t I*
    have ⋀*t s*. 𝒫(*ω in T s*. *s* = *t*) = *indicator* {*t*} *s*
      using *T.prob-space* **by** (*auto split*: *split-indicator*)
    moreover then have ⋀*t t*′ *s*. 𝒫(*ω in T s*. *shd ω* = *t*′ ∧ *s* = *t*) = *pmf* (*K t*) *t*′ ∗ *indicator* {*t*} *s*
      by (*subst prob-T*) (*auto split*: *split-indicator simp*: *pmf.rep-eq*)
    ultimately have 𝒫(*ω in T*′ *I*. *shd* (*stl ω*) = *t* ∧ *shd ω* = *s*) = 𝒫(*ω in T*′ *I*. *shd ω* = *s*) ∗ *pmf* (*K s*) *t*
      by (*simp add*: *prob-T*′ *pmf.rep-eq*) **}**

note *start-eq* = *this*

{ fix *n s t* **assume** $\mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ n = s) \neq 0$
moreover have $\mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ (Suc\ n) = t \wedge \omega\ !!\ n = s) = \mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ n = s) * pmf\ (K\ s)\ t$
proof (*induction n arbitrary*: *I*)
case (*Suc n*) **then show** *?case*
by (*subst* (*1 2*) *prob-T'*) (*simp-all del*: *space-T add*: *T-eq-T'*)
qed (*simp add*: *start-eq*)
ultimately have $\mathcal{P}(\omega\ in\ T'\ I.\ stl\ \omega\ !!\ n = t \mid \omega\ !!\ n = s) = pmf\ (K\ s)\ t$
by (*simp add*: *cond-prob-def field-simps*) **}**
note *TH* = *this*

{ fix *n ω' t* **assume** $\mathcal{P}(\omega\ in\ T'\ I.\ \forall i \le n.\ \omega\ !!\ i = \omega'\ i) \neq 0$
moreover have $\mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ (Suc\ n) = t \wedge (\forall i \le n.\ \omega\ !!\ i = \omega'\ i)) =$
$\mathcal{P}(\omega\ in\ T'\ I.\ \forall i \le n.\ \omega\ !!\ i = \omega'\ i) * pmf\ (K\ (\omega'\ n))\ t$
proof (*induction n arbitrary*: *I ω'*)
case (*Suc n*)
have ∗[*simp*]: $\bigwedge s\ P.\ measure\ (T'\ (K\ s))\ \{x.\ s = \omega'\ 0 \wedge P\ x\} =$
$measure\ (T'\ (K\ (\omega'\ 0)))\ \{x.\ P\ x\} * indicator\ \{\omega'\ 0\}\ s$
by (*auto split*: *split-indicator*)
from *Suc*[*of* - $\lambda i.\ \omega'\ (Suc\ i)$] **show** *?case*
by (*subst* (*1 2*) *prob-T'*)
(*simp-all add*: *T-eq-T' all-Suc-split*[**where** $P = \lambda i.\ i \le Suc\ n \longrightarrow Q\ i$ **for** *n Q*] *conj-commute conj-left-commute sets-eq-imp-space-eq*[*OF sets-T'*])
qed (*simp add*: *start-eq*)
ultimately have $\mathcal{P}(\omega\ in\ T'\ I.\ stl\ \omega\ !!\ n = t \mid \forall i \le n.\ \omega\ !!\ i = \omega'\ i) = pmf\ (K\ (\omega'\ n))\ t$
by (*simp add*: *cond-prob-def field-simps*) **}**
note *MC* = *this*

{ fix *n ω'* **assume** $\mathcal{P}(\omega\ in\ T'\ I.\ \forall t \le n.\ \omega\ !!\ t = \omega'\ t) \neq 0$
moreover have $\mathcal{P}(\omega\ in\ T'\ I.\ \forall t \le n.\ \omega\ !!\ t = \omega'\ t) \le \mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ n = \omega'\ n)$
by (*auto intro*!: *finite-measure-mono-AE simp*: *sets-T' sets-eq-imp-space-eq*[*OF sets-T'*])
ultimately have $\mathcal{P}(\omega\ in\ T'\ I.\ \omega\ !!\ n = \omega'\ n) \neq 0$
by (*auto simp*: *neq-iff not-less measure-le-0-iff*) **}**
note *MC'* = *this*

show *?thesis*
proof
show *countable U*
unfolding *U-def* **by** (*rule countable-reachable countable-Image countable-set-pmf*)+
show $\bigwedge t.\ (\lambda\omega.\ \omega\ !!\ t) \in measurable\ (T'\ I)\ (count\text{-}space\ UNIV)$
by (*subst measurable-cong-sets*[*OF sets-T' refl*]) *simp*
next
fix *n*
have $\forall x \in I.\ AE\ y\ in\ T\ x.\ (x\ \#\#\ y)\ !!\ n \in U$

```
    unfolding U-def
  proof (induction n arbitrary: I)
    case 0 then show ?case
      by auto
  next
    case (Suc n)
    { fix x assume x ∈ I
      have AE y in T x. y !! n ∈ (SIGMA x:UNIV. K x)* `` K x
        apply (subst AE-T-iff)
        apply (rule measurable-compose[OF measurable-snth], simp)
        apply (rule Suc)
        done
      moreover have (SIGMA x:UNIV. K x)* `` K x ⊆ (SIGMA x:UNIV. K x)*
`` I
        using ‹x ∈ I› by (auto intro: converse-rtrancl-into-rtrancl)
      ultimately have AE y in T x. y !! n ∈ (SIGMA x:UNIV. K x)* `` I
        by (auto simp: subset-eq) }
    then show ?case
      by simp
  qed
  then show AE x in T' I. x !! n ∈ U
    by (simp add: AE-T')
 qed (simp-all add: TH MC MC')
qed

end
```

4 Classifying Markov Chain States

```
theory Classifying-Markov-Chain-States
  imports
    HOL−Computational-Algebra.Group-Closure
    Discrete-Time-Markov-Chain
begin

lemma eventually-mult-Gcd:
  fixes S :: nat set
  assumes S: ⋀s t. s ∈ S ⟹ t ∈ S ⟹ s + t ∈ S
  assumes s: s ∈ S s > 0
  shows eventually (λm. m ∗ Gcd S ∈ S) sequentially
proof −
  define T where T = insert 0 (int ` S)
  with s S have int s ∈ T 0 ∈ T and T: r ∈ T ⟹ t ∈ T ⟹ r + t ∈ T for r t
    by (auto simp del: of-nat-add simp add: of-nat-add [symmetric])
  have Gcd T ∈ group-closure T
    by (rule Gcd-in-group-closure)
  also have group-closure T = {s − t | s t. s ∈ T ∧ t ∈ T}
  proof (auto intro: group-closure.base group-closure.diff)
    fix x assume x ∈ group-closure T
```

then show ∃ *s t*. *x* = *s* − *t* ∧ *s* ∈ *T* ∧ *t* ∈ *T*
proof *induction*
case (*base x*) **with** ‹*0* ∈ *T*› **show** *?case*
apply (*rule-tac x*=*x* **in** *exI*)
apply (*rule-tac x*=*0* **in** *exI*)
apply *auto*
done
next
case (*diff x y*)
then obtain *a b c d* **where**
a ∈ *T b* ∈ *T x* = *a* − *b*
c ∈ *T d* ∈ *T y* = *c* − *d*
by *auto*
then show *?case*
apply (*rule-tac x*=*a* + *d* **in** *exI*)
apply (*rule-tac x*=*b* + *c* **in** *exI*)
apply (*auto intro*: *T*)
done
qed
qed
finally obtain *s*′ *t*′ :: *int*
where *s*′ ∈ *T t*′ ∈ *T Gcd T* = *s*′ − *t*′
by *blast*
moreover define *s* **and** *t* **where** *s* = *nat s*′ **and** *t* = *nat t*′
moreover have *int* (*Gcd S*) = − *int t* ⟷ *S* ⊆ {*0*} ∧ *t* = *0*
by *auto* (*metis Gcd-dvd-nat dvd-0-right dvd-antisym nat-int nat-zminus-int*)
ultimately have
st: *s* = *0* ∨ *s* ∈ *S t* = *0* ∨ *t* ∈ *S* **and** *Gcd-S*: *Gcd S* = *s* − *t*
using *T-def* **by** *safe simp-all*
with *s*
have *t* < *s*
by (*rule-tac ccontr*) *auto*

{ fix *s n* **have** *0* < *n* ⟹ *s* ∈ *S* ⟹ *n* ∗ *s* ∈ *S*
proof (*induct n*)
case (*Suc n*) **then show** *?case*
by (*cases n*) (*auto intro*: *S*)
qed *simp* **}**
note *cmult-S* = *this*

show *?thesis*
unfolding *eventually-sequentially*
proof *cases*
assume *s* = *0* ∨ *t* = *0*
with *st Gcd-S s* **have** ∗: *Gcd S* ∈ *S*
by (*auto simp*: *int-eq-iff*)
then show ∃ *N*. ∀ *n*≥*N*. *n* ∗ *Gcd S* ∈ *S* **by** (*auto intro*!: *exI*[*of - 1*] *cmult-S*)
next
assume ¬ (*s* = *0* ∨ *t* = *0*)

```
    with st have s ∈ S t ∈ S t ≠ 0 by auto
    then have Gcd S dvd t by auto
    then obtain a where a: t = Gcd S * a ..
    with ‹t ≠ 0› have 0 < a by auto

    show ∃N. ∀n≥N. n * Gcd S ∈ S
    proof (safe intro!: exI[of - a * a])
      fix n
      define m where m = (n − a * a) div a
      define r where r = (n − a * a) mod a
      with ‹0 < a› have r < a by simp
      moreover define am where am = a + m
      ultimately have r < am by simp
      assume a * a ≤ n then have n: n = a * a + (m * a + r)
        unfolding m-def r-def by simp
      have n * Gcd S = am * t + r * Gcd S
        unfolding n a by (simp add: field-simps am-def)
      also have ... = r * s + (am − r) * t
        unfolding ‹Gcd S = s − t›
        using ‹t < s› ‹r < am› by (simp add: field-simps diff-mult-distrib2)
      also have ... ∈ S
        using ‹s ∈ S› ‹t ∈ S› ‹r < am›
        by (cases r = 0) (auto intro!: cmult-S S)
      finally show n * Gcd S ∈ S .
    qed
  qed
qed

context MC-syntax
begin
```

4.1 Expected number of visits

```
definition G s t = (∫+ω. scount (HLD {t}) (s ## ω) ∂T s)

lemma G-eq: G s t = (∫+ω. emeasure (count-space UNIV) {i. (s ## ω) !! i =
t} ∂T s)
  by (simp add: G-def scount-eq-emeasure HLD-iff)

definition p s t n = 𝒫(ω in T s. (s ## ω) !! n = t)

definition gf-G s t z = (∑n. p s t n *R z ^ n)

definition convergence-G s t z ⟷ summable (λn. p s t n * norm z ^ n)

lemma p-nonneg[simp]: 0 ≤ p x y n
  by (simp add: p-def)

lemma p-le-1: p x y n ≤ 1
```

by (*simp add*: *p-def*)

lemma *p-x-x-0*[*simp*]: *p x x 0* = *1*
by (*simp add*: *p-def T.prob-space del*: *space-T*)

lemma *p-0*: *p x y 0* = (*if x* = *y then 1 else 0*)
by (*simp add*: *p-def T.prob-space del*: *space-T*)

lemma *p-in-reachable*: **assumes** $(x, y) \notin$ (*SIGMA x*:*UNIV*. *K x*)* **shows** *p x y n* = *0*
unfolding *p-def*
proof (*rule T.prob-eq-0-AE*)
from *AE-T-reachable* **show** *AE* ω *in T x*. (*x* ## ω) !! *n* ≠ *y*
proof *eventually-elim*
fix ω **assume** *alw* (*HLD* ((*SIGMA* ω:*UNIV*. *K* ω)* `` {*x*})) ω
then have *alw* (*HLD* (− {*y*})) ω
using *assms* **by** (*auto intro*: *alw-mono simp*: *HLD-iff*)
then show (*x* ## ω) !! *n* ≠ *y*
using *assms* **by** (*cases n*) (*auto simp*: *alw-HLD-iff-streams streams-iff-snth*)
qed
qed

lemma *p-Suc*: *ennreal* (*p x y* (*Suc n*)) = ($\int^+$ *w*. *p w y n* ∂*K x*)
unfolding *p-def T.emeasure-eq-measure*[*symmetric*] **by** (*subst emeasure-Collect-T*) *simp-all*

lemma *p-Suc′*:
p x y (*Suc n*) = (∫ *x′*. *p x′ y n* ∂*K x*)
using *p-Suc*[*of x y n*]
by (*subst* (*asm*) *nn-integral-eq-integral*)
(*auto simp*: *p-le-1 intro*!: *measure-pmf.integrable-const-bound*[**where** *B*=*1*])

lemma *p-add*: *p x y* (*n* + *m*) = ($\int^+$ *w*. *p x w n* ∗ *p w y m* ∂*count-space UNIV*)
proof (*induction n arbitrary*: *x*)
case *0*
have [*simp*]: ⋀*w*. (*if x* = *w then 1 else 0*) ∗ *p w y m* = *ennreal* (*p x y m*) ∗ *indicator* {*x*} *w*
by *auto*
show *?case*
by (*simp add*: *p-0 one-ennreal-def*[*symmetric*] *max-def*)
next
case (*Suc n*)
define *X* **where** *X* = (*SIGMA x*:*UNIV*. *K x*)* `` *K x*
then have *X*: *countable X*
by (*blast intro*: *countable-Image countable-reachable countable-set-pmf*)

then interpret *X*: *sigma-finite-measure count-space X*
by (*rule sigma-finite-measure-count-space-countable*)
interpret *XK*: *pair-sigma-finite K x count-space X*

by *unfold-locales*

have *ennreal* (*p x y* (*Suc n* + *m*)) = (∫ +*t*. (∫ +*w*. *p t w n* ∗ *p w y m* ∂*count-space UNIV*) ∂*K x*)
by (*simp add*: *p-Suc Suc*)
also have ... = (∫ +*t*. (∫ +*w*. *ennreal* (*p t w n* ∗ *p w y m*) ∗ *indicator X w* ∂*count-space UNIV*) ∂*K x*)
by (*auto intro*!: *nn-integral-cong-AE simp*: *AE-measure-pmf-iff AE-count-space Image-iff p-in-reachable X-def split*: *split-indicator*)
also have ... = (∫ +*t*. (∫ +*w*. *p t w n* ∗ *p w y m* ∂*count-space X*) ∂*K x*)
by (*subst nn-integral-restrict-space*[*symmetric*]) (*simp-all add*: *restrict-count-space*)
also have ... = (∫ +*w*. (∫ +*t*. *p t w n* ∗ *p w y m* ∂*K x*) ∂*count-space X*)
apply (*rule XK.Fubini'*[*symmetric*])
unfolding *measurable-split-conv*
apply (*rule measurable-compose-countable'*[*OF* - *measurable-snd X*])
apply (*rule measurable-compose*[*OF measurable-fst*])
apply *simp*
done
also have ... = (∫ +*w*. (∫ +*t*. *ennreal* (*p t w n* ∗ *p w y m*) ∗ *indicator X w* ∂*K x*) ∂*count-space UNIV*)
by (*simp add*: *nn-integral-restrict-space*[*symmetric*] *restrict-count-space nn-integral-multc*)
also have ... = (∫ +*w*. (∫ +*t*. *ennreal* (*p t w n* ∗ *p w y m*) ∂*K x*) ∂*count-space UNIV*)
by (*auto intro*!: *nn-integral-cong-AE simp*: *AE-measure-pmf-iff AE-count-space Image-iff p-in-reachable X-def split*: *split-indicator*)
also have ... = (∫ +*w*. (∫ +*t*. *p t w n* ∂*K x*) ∗ *p w y m* ∂*count-space UNIV*)
by (*simp add*: *nn-integral-multc*[*symmetric*] *ennreal-mult*)
finally show *?case*
by (*simp add*: *ennreal-mult p-Suc*)
qed

lemma *prob-reachable-le*:
assumes [*simp*]: $m \leq n$
shows $p\ x\ y\ m * p\ y\ w\ (n - m) \leq p\ x\ w\ n$
proof −
have $p\ x\ y\ m * p\ y\ w\ (n - m)$ = (∫ +*y'*. *ennreal* (*p x y m* ∗ *p y w* (*n* − *m*)) ∗ *indicator* {*y*} *y'* ∂*count-space UNIV*)
by *simp*
also have ... ≤ *p x w* (*m* + (*n* − *m*))
by (*subst p-add*)
(*auto intro*!: *nn-integral-mono split*: *split-indicator simp del*: *nn-integral-indicator-singleton*)
finally show *?thesis*
by *simp*
qed

lemma *G-eq-suminf*: *G x y* = (∑ *i*. *ennreal* (*p x y i*))
proof −
have ∗: ⋀*i ω*. *indicator* {*ω* ∈ *space S*. (*x* ## *ω*) !! *i* = *y*} *ω* = *indicator* {*i*. (*x* ## *ω*) !! *i* = *y*} *i*

by (*auto simp*: *space-stream-space split*: *split-indicator*)

have $G\ x\ y = (\int^+ \omega.\ (\sum i.\ indicator\ \{\omega{\in}space\ (T\ x).\ (x\ \#\#\ \omega)\ !!\ i = y\}\ \omega)\ \partial T\ x)$
unfolding *G-eq* **by** (*simp add*: *nn-integral-count-space-nat*[*symmetric*] $*$)
also have $\ldots = (\sum i.\ ennreal\ (p\ x\ y\ i))$
by (*simp add*: *T.emeasure-eq-measure*[*symmetric*] *p-def nn-integral-suminf*)
finally show *?thesis* **.**
qed

lemma *G-eq-real-suminf*:
$convergence\text{-}G\ x\ y\ (1{::}real) \Longrightarrow G\ x\ y = ennreal\ (\sum i.\ p\ x\ y\ i)$
unfolding *G-eq-suminf*
by (*intro suminf-ennreal ennreal-suminf-neq-top p-nonneg*)
(*auto simp*: *convergence-G-def p-def*)

lemma *convergence-norm-G*:
$convergence\text{-}G\ x\ y\ z \Longrightarrow summable\ (\lambda n.\ p\ x\ y\ n * norm\ z\ \hat{}\ n)$
unfolding *convergence-G-def* **.**

lemma *convergence-G*:
$convergence\text{-}G\ x\ y\ (z{::}'a{::}\{banach,\ real\text{-}normed\text{-}div\text{-}algebra\}) \Longrightarrow summable\ (\lambda n.\ p\ x\ y\ n *_R z\ \hat{}\ n)$
unfolding *convergence-G-def*
by (*rule summable-norm-cancel*) (*simp add*: *abs-mult norm-power*)

lemma *convergence-G-less-1*:
fixes $z :: - :: \{banach,\ real\text{-}normed\text{-}field\}$
assumes z: $norm\ z < 1$ **shows** $convergence\text{-}G\ x\ y\ z$
unfolding *convergence-G-def*
proof (*rule summable-comparison-test*)
have $\bigwedge n.\ p\ x\ y\ n * norm\ (z\ \hat{}\ n) \le 1 * norm\ (z\ \hat{}\ n)$
by (*intro mult-right-mono p-le-1*) *simp-all*
then show $\exists N.\ \forall n{\ge}N.\ norm\ (p\ x\ y\ n * norm\ z\ \hat{}\ n) \le norm\ z\ \hat{}\ n$
by (*simp add*: *norm-power*)
qed (*simp add*: *z summable-geometric*)

lemma *lim-gf-G*: $((\lambda z.\ ennreal\ (gf\text{-}G\ x\ y\ z)) \longrightarrow G\ x\ y)\ (at\text{-}left\ (1{::}real))$
unfolding *gf-G-def G-eq-suminf real-scaleR-def*
by (*intro power-series-tendsto-at-left p-nonneg p-le-1 summable-power-series*)

4.2 Reachability probability

definition $u\ x\ y\ n = \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ n\ \omega)$

definition $U\ s\ t = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{t\})\ \omega)$

definition $gf\text{-}U\ x\ y\ z = (\sum n.\ u\ x\ y\ n *_R z\ \hat{}\ Suc\ n)$

definition *f x y n* = $\mathcal{P}$(ω *in T x*. *ev-at* (*HLD* {*y*}) *n* (*x* ## ω))

definition *F s t* = $\mathcal{P}$(ω *in T s*. *ev* (*HLD* {*t*}) (*s* ## ω))

definition *gf-F x y z* = (∑ *n*. *f x y n* ∗ *z* ^ *n*)

lemma *f-Suc*: *x* ≠ *y* ⟹ *f x y* (*Suc n*) = *u x y n*
  by (*simp add*: *u-def f-def*)

lemma *f-Suc-eq*: *f x x* (*Suc n*) = *0*
  by (*simp add*: *f-def*)

lemma *f-0*: *f x y 0* = (*if x* = *y then 1 else 0*)
  using *T.prob-space* **by** (*simp add*: *f-def*)

lemma shows *u-nonneg*: *0* ≤ *u x y n* **and** *u-le-1*: *u x y n* ≤ *1*
  by (*simp-all add*: *u-def*)

lemma shows *f-nonneg*: *0* ≤ *f x y n* **and** *f-le-1*: *f x y n* ≤ *1*
  by (*simp-all add*: *f-def*)

lemma *U-nonneg*[*simp*]: *0* ≤ *U x y*
  by (*simp add*: *U-def*)

lemma *U-le-1*: *U s t* ≤ *1*
  by (*auto simp add*: *U-def intro*!: *antisym*)

lemma *U-cases*: *U s s* = *1* ∨ *U s s* < *1*
  by (*auto simp add*: *U-def intro*!: *antisym*)

lemma *u-sums-U*: *u x y sums U x y*
  unfolding *u-def*[*abs-def*] *U-def ev-iff-ev-at* **by** (*intro T.prob-sums*) (*auto intro*: *ev-at-unique*)

lemma *gf-U-eq-U*: *gf-U x y 1* = *U x y*
  using *u-sums-U*[*THEN sums-unique*] **by** (*simp add*: *gf-U-def U-def*)

lemma *f-sums-F*: *f x y sums F x y*
  unfolding *f-def*[*abs-def*] *F-def ev-iff-ev-at*
  by (*intro T.prob-sums*) (*auto intro*: *ev-at-unique*)

lemma *F-nonneg*[*simp*]: *0* ≤ *F x y*
  by (*auto simp*: *F-def*)

lemma *F-le-1*: *F x y* ≤ *1*
  by (*simp add*: *F-def*)

lemma *gf-F-eq-F*: *gf-F x y 1* = *F x y*
  using *f-sums-F*[*THEN sums-unique*] **by** (*simp add*: *gf-F-def F-def*)

lemma *gf-F-le-1*:
fixes *z* :: *real*
assumes *z*: *0* ≤ *z z* ≤ *1*
shows *gf-F x y z* ≤ *1*
proof −
have *gf-F x y z* ≤ *gf-F x y 1*
using *z* **unfolding** *gf-F-def*
by (*intro suminf-le*[*OF - summable-comparison-test*[*OF - sums-summable*[*OF f-sums-F*[*of x y*]]]] *mult-left-mono allI f-nonneg*)
(*simp-all add*: *power-le-one f-nonneg mult-right-le-one-le f-le-1 sums-summable*[*OF f-sums-F*[*of x y*]])
also have ... ≤ *1*
by (*simp add*: *gf-F-eq-F F-def*)
finally show *?thesis* **.**
qed

lemma *u-le-p*: *u x y n* ≤ *p x y* (*Suc n*)
unfolding *u-def p-def* **by** (*auto intro*!: *T.finite-measure-mono dest*: *ev-at-HLD-imp-snth*)

lemma *f-le-p*: *f x y n* ≤ *p x y n*
unfolding *f-def p-def* **by** (*auto intro*!: *T.finite-measure-mono dest*: *ev-at-HLD-imp-snth*)

lemma *convergence-norm-U*:
fixes *z* :: *-* :: *real-normed-div-algebra*
assumes *z*: *convergence-G x y z*
shows *summable* (λ*n*. *u x y n* ∗ *norm z* ^ *Suc n*)
using *summable-ignore-initial-segment*[*OF convergence-norm-G*[*OF z*], *of 1*]
by (*rule summable-comparison-test*[*rotated*])
(*auto simp add*: *u-nonneg abs-mult intro*!: *exI*[*of - 0*] *mult-right-mono u-le-p*)

lemma *convergence-norm-F*:
fixes *z* :: *-* :: *real-normed-div-algebra*
assumes *z*: *convergence-G x y z*
shows *summable* (λ*n*. *f x y n* ∗ *norm z* ^ *n*)
using *convergence-norm-G*[*OF z*]
by (*rule summable-comparison-test*[*rotated*])
(*auto simp add*: *f-nonneg abs-mult intro*!: *exI*[*of - 0*] *mult-right-mono f-le-p*)

lemma *gf-G-nonneg*:
fixes *z* :: *real*
shows *0* ≤ *z* ⟹ *z* < *1* ⟹ *0* ≤ *gf-G x y z*
unfolding *gf-G-def*
by (*intro suminf-nonneg convergence-G convergence-G-less-1*) *simp-all*

lemma *gf-F-nonneg*:
fixes *z* :: *real*
shows *0* ≤ *z* ⟹ *z* < *1* ⟹ *0* ≤ *gf-F x y z*
unfolding *gf-F-def*

using *convergence-norm-F*[*OF convergence-G-less-1*, *of z x y*]
by (*intro suminf-nonneg*) (*simp-all add*: *f-nonneg*)

lemma *convergence-U*:
fixes z :: - :: *banach*
shows *convergence-G x y z* $\Longrightarrow$ *summable* ($\lambda n.\ u\ x\ y\ n * z$ ^ *Suc n*)
by (*rule summable-norm-cancel*)
(*auto simp add*: *abs-mult u-nonneg power-abs dest*!: *convergence-norm-U*)

lemma *p-eq-sum-p-u*: $p\ x\ y\ (Suc\ n) = (\sum i{\le}n.\ p\ y\ y\ (n - i) * u\ x\ y\ i)$
proof −
have $\bigwedge\omega.\ \omega\ !!\ n = y \Longrightarrow (\exists\, i.\ i \le n \wedge ev\text{-}at\ (HLD\ \{y\})\ i\ \omega)$
proof (*induction n*)
case (*Suc n*)
then obtain *i* **where** $i \le n$ *ev-at* (*HLD* {*y*}) *i* (*stl* ω)
by *auto*
then show *?case*
by (*auto intro*!: *exI*[*of - if HLD* {*y*} ω *then 0 else Suc i*])
qed (*simp add*: *HLD-iff*)
then have $p\ x\ y\ (Suc\ n) = (\sum i{\le}n.\ \mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ i\ \omega \wedge \omega\ !!\ n = y))$
unfolding *p-def* **by** (*intro T.prob-sum*) (*auto intro*: *ev-at-unique*)
also have ... $= (\sum i{\le}n.\ p\ y\ y\ (n - i) * u\ x\ y\ i)$
proof (*intro sum.cong refl*)
fix *i* **assume** *i*: $i \in \{..\ n\}$
then have $\bigwedge\omega.\ (Suc\ i \le n \longrightarrow \omega\ !!\ (n - Suc\ i) = y) \longleftrightarrow ((y\ \#\#\ \omega)\ !!\ (n - i) = y)$
by (*auto simp*: *Stream-snth diff-Suc split*: *nat.split*)
from *i* **have** $i \le n$ **by** *auto*
then have $\mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ i\ \omega \wedge \omega\ !!\ n = y) =$
$(\int \omega'.\ \mathcal{P}(\omega\ in\ T\ y.\ (y\ \#\#\ \omega)\ !!\ (n - i) = y) *$
$indicator\ \{\omega'{\in}space\ (T\ x).\ ev\text{-}at\ (HLD\ \{y\})\ i\ \omega'\ \}\ \omega'\ \partial T\ x)$
by (*subst prob-T-split*[**where** *n*=*Suc i*])
(*auto simp*: *ev-at-shift ev-at-HLD-single-imp-snth shift-snth diff-Suc*
split: *split-indicator nat.split intro*!: *Bochner-Integration.integral-cong arg-cong2*[**where** *f*=*measure*]
simp del: *stake.simps integral-mult-right-zero*)
then show $\mathcal{P}(\omega\ in\ T\ x.\ ev\text{-}at\ (HLD\ \{y\})\ i\ \omega \wedge \omega\ !!\ n = y) = p\ y\ y\ (n - i) * u\ x\ y\ i$
by (*simp add*: *p-def u-def*)
qed
finally show *?thesis* **.**
qed

lemma *p-eq-sum-p-f*: $p\ x\ y\ n = (\sum i{\le}n.\ p\ y\ y\ (n - i) * f\ x\ y\ i)$
by (*cases n*)
(*simp-all del*: *sum.atMost-Suc*
add: *f-0 p-0 p-eq-sum-p-u atMost-Suc-eq-insert-0 zero-notin-Suc-image sum.reindex*

f-Suc f-Suc-eq)

lemma *gf-G-eq-gf-F*:
  assumes *z*: *norm z* < *1*
  shows *gf-G x y z* = *gf-F x y z* ∗ *gf-G y y z*
proof −
  have *gf-G x y z* = ($\sum$ *n*. $\sum$ *i*≤*n*. *p y y* (*n* − *i*) ∗ *f x y i* ∗ *z*^*n*)
    by (*simp add*: *gf-G-def p-eq-sum-p-f*[*of x y*] *sum-distrib-right*)
  also have ... = ($\sum$ *n*. $\sum$ *i*≤*n*. (*f x y i* ∗ *z*^*i*) ∗ (*p y y* (*n* − *i*) ∗ *z*^(*n* − *i*)))
    by (*intro arg-cong*[**where** *f*=*suminf*] *sum.cong ext atLeast0AtMost*[*symmetric*])
       (*simp-all add*: *power-add*[*symmetric*])
  also have ... = ($\sum$ *n*. *f x y n* ∗ *z*^*n*) ∗ ($\sum$ *n*. *p y y n* ∗ *z*^*n*)
    using *convergence-norm-F*[*OF convergence-G-less-1*[*OF z*]] *convergence-norm-G*[*OF convergence-G-less-1*[*OF z*]]
    by (*intro Cauchy-product*[*symmetric*]) (*auto simp*: *f-nonneg abs-mult power-abs*)
  also have ... = *gf-F x y z* ∗ *gf-G y y z*
    by (*simp add*: *gf-F-def gf-G-def*)
  finally show *?thesis* **.**
qed

lemma *gf-G-eq-gf-U*:
  fixes *z* :: *'z* :: {*banach*, *real-normed-field*}
  assumes *z*: *convergence-G x x z*
  shows *gf-G x x z* = *1* / (*1* − *gf-U x x z*) *gf-U x x z* ≠ *1*
proof −
  { **fix** *n*
    have *p x x* (*Suc n*) $*_R$ *z*^*Suc n* = ($\sum$ *i*≤*n*. (*p x x* (*n* − *i*) ∗ *u x x i*) $*_R$ *z*^*Suc n*)
      unfolding *scaleR-sum-left*[*symmetric*] **by** (*simp add*: *p-eq-sum-p-u*)
    also have ... = ($\sum$ *i*≤*n*. (*u x x i* $*_R$ *z*^*Suc i*) ∗ (*p x x* (*n* − *i*) $*_R$ *z*^(*n* − *i*)))
      by (*intro sum.cong refl*) (*simp add*: *field-simps power-diff cong*: *disj-cong*)
    finally have *p x x* (*Suc n*) $*_R$ *z*^(*Suc n*) = ($\sum$ *i*≤*n*. (*u x x i* $*_R$ *z*^*Suc i*) ∗ (*p x x* (*n* − *i*) $*_R$ *z*^(*n* − *i*)))
      unfolding *atLeast0AtMost* **.** }
  note *gfs-Suc-eq* = *this*

  have *gf-G x x z* = *1* + ($\sum$ *n*. *p x x* (*Suc n*) $*_R$ *z*^(*Suc n*))
    unfolding *gf-G-def*
    by (*subst suminf-split-initial-segment*[*OF convergence-G*[*OF z*], *of 1*]) *simp*
  also have ... = *1* + ($\sum$ *n*. $\sum$ *i*≤*n*. (*u x x i* $*_R$ *z*^*Suc i*) ∗ (*p x x* (*n* − *i*) $*_R$ *z*^(*n* − *i*)))
    unfolding *gfs-Suc-eq* **..**
  also have ... = *1* + *gf-U x x z* ∗ *gf-G x x z*
    unfolding *gf-U-def gf-G-def*
    by (*subst Cauchy-product*)
       (*auto simp*: *u-nonneg norm-power simp del*: *power-Suc*
          intro!: *z convergence-norm-G convergence-norm-U*)
  finally show *gf-G x x z* = *1* / (*1* − *gf-U x x z*) *gf-U x x z* ≠ *1*
    apply −

apply (*cases gf-U x x z = 1*)
apply (*auto simp add*: *field-simps*)
done
qed

lemma *gf-U*: (*gf-U x y* ⟶ *U x y*) (*at-left 1*)
proof −
have ((λ*z*. *ennreal* (∑ *n*. *u x y n* ∗ *z* ^ *n*)) ⟶ (∑ *n*. *ennreal* (*u x y n*))) (*at-left 1*)
using *u-le-1 u-nonneg* **by** (*intro power-series-tendsto-at-left summable-power-series*)
also have (∑ *n*. *ennreal* (*u x y n*)) = *ennreal* (*suminf* (*u x y*))
by (*intro u-nonneg suminf-ennreal ennreal-suminf-neq-top sums-summable*[*OF u-sums-U*])
also have *suminf* (*u x y*) = *U x y*
using *u-sums-U* **by** (*rule sums-unique*[*symmetric*])
finally have ((λ*z*. ∑ *n*. *u x y n* ∗ *z* ^ *n*) ⟶ *U x y*) (*at-left 1*)
by (*rule tendsto-ennrealD*)
(*auto simp*: *u-nonneg u-le-1 intro*!: *suminf-nonneg summable-power-series eventually-at-left-1*)
then have ((λ*z*. *z* ∗ (∑ *n*. *u x y n* ∗ *z* ^ *n*)) ⟶ *1* ∗ *U x y*) (*at-left 1*)
by (*intro tendsto-intros*) *simp*
then have ((λ*z*. ∑ *n*. *u x y n* ∗ *z* ^ *Suc n*) ⟶ *1* ∗ *U x y*) (*at-left 1*)
apply (*rule filterlim-cong*[*OF refl refl*, *THEN iffD1*, *rotated*])
apply (*rule eventually-at-left-1*)
apply (*subst suminf-mult*[*symmetric*])
apply (*auto intro*!: *summable-power-series u-le-1 u-nonneg*)
apply (*simp add*: *field-simps*)
done
then show *?thesis*
by (*simp add*: *gf-U-def*[*abs-def*] *U-def*)
qed

lemma *gf-U-le-1*: **assumes** *z*: *0* < *z z* < *1* **shows** *gf-U x y z* ≤ (*1*::*real*)
proof −
note *u* = *u-sums-U*[*of x y*, *THEN sums-summable*]
have *gf-U x y z* ≤ *gf-U x y 1*
using *z*
unfolding *gf-U-def real-scaleR-def*
by (*intro suminf-le allI mult-mono power-mono summable-comparison-test-ev*[*OF - u*] *always-eventually*)
(*auto simp*: *u-nonneg intro*!: *mult-left-le mult-le-one power-le-one*)
also have . . . ≤ *1*
unfolding *gf-U-eq-U* **by** (*rule U-le-1*)
finally show *?thesis* **.**
qed

lemma *gf-F*: (*gf-F x y* ⟶ *F x y*) (*at-left 1*)
proof −
have ((λ*z*. *ennreal* (∑ *n*. *f x y n* ∗ *z* ^ *n*)) ⟶ (∑ *n*. *ennreal* (*f x y n*))) (*at-left*

1)
  using *f-le-1 f-nonneg* **by** (*intro power-series-tendsto-at-left summable-power-series*)
 also have ($\sum$ *n. ennreal* (*f x y n*)) = *ennreal* (*suminf* (*f x y*))
  by (*intro f-nonneg suminf-ennreal ennreal-suminf-neq-top sums-summable*[*OF f-sums-F*])
 also have *suminf* (*f x y*) = *F x y*
  using *f-sums-F* **by** (*rule sums-unique*[*symmetric*])
 finally have ((λ*z.* $\sum$ *n. f x y n* ∗ *z* ^ *n*) ⟶ *F x y*) (*at-left 1*)
  by (*rule tendsto-ennrealD*)
   (*auto simp*: *f-nonneg f-le-1 intro*!: *suminf-nonneg summable-power-series eventually-at-left-1*)
 then show *?thesis*
  by (*simp add*: *gf-F-def*[*abs-def*] *F-def*)
qed

lemma *U-bounded*: *0* ≤ *U x y U x y* ≤ *1*
 unfolding *U-def* **by** *simp-all*

4.3 Recurrent states

definition *recurrent* :: *'s* ⇒ *bool* **where**
 recurrent s ⟷ (*AE* ω *in T s. ev* (*HLD* {*s*}) ω)

lemma *recurrent-iff-U-eq-1*: *recurrent s* ⟷ *U s s* = *1*
  unfolding *recurrent-def U-def* **by** (*subst T.prob-Collect-eq-1*) *simp-all*

definition *H s t* = $\mathcal{P}$(ω *in T s. alw* (*ev* (*HLD* {*t*})) ω)

lemma *H-eq*:
 recurrent s ⟷ *H s s* = *1*
 ¬ *recurrent s* ⟷ *H s s* = *0*
 H s t = *U s t* ∗ *H t t*
proof −
 define *H'* **where** *H' t n* = {ω∈*space S. enat n* ≤ *scount* (*HLD* {*t*::*'s*}) ω} **for** *t n*
 have [*measurable*]: ⋀*y n. H' y n* ∈ *sets S*
  by (*simp add*: *H'-def*)
 let *?H'* = λ*s t n. measure* (*T s*) (*H' t n*)
 { **fix** *x y* :: *'s* **and** ω
  have *Suc 0* ≤ *scount* (*HLD* {*y*}) ω ⟷ *ev* (*HLD* {*y*}) ω
   using *scount-eq-0-iff*[*of HLD* {*y*} ω]
   by (*cases scount* (*HLD* {*y*}) ω *rule*: *enat-coexhaust*)
    (*auto simp*: *not-ev-iff*[*symmetric*] *eSuc-enat*[*symmetric*] *enat-0 HLD-iff*[*abs-def*])
}
 then have *H'-1*: ⋀*x y. ?H' x y 1* = *U x y*
  unfolding *H'-def U-def* **by** *simp*

 { **fix** *n* **and** *x y* :: *'s*
  let *?U* = (*not* (*HLD* {*y*}) *suntil* (*HLD* {*y*} *aand nxt* (λω. *enat n* ≤ *scount*

(*HLD* {*y*}) *ω*)))
{ **fix** *ω*
have *enat* (*Suc n*) ≤ *scount* (*HLD* {*y*}) *ω* ⟷ *?U ω*
proof
assume *enat* (*Suc n*) ≤ *scount* (*HLD* {*y*}) *ω*
with *scount-eq-0-iff*[*of HLD* {*y*} *ω*] **have** *ev* (*HLD* {*y*}) *ω enat* (*Suc n*) ≤ *scount* (*HLD* {*y*}) *ω*
by (*auto simp add*: *not-ev-iff*[*symmetric*] *eSuc-enat*[*symmetric*])
then show *?U ω*
by (*induction rule*: *ev-induct-strong*)
(*auto simp*: *scount-simps eSuc-enat*[*symmetric*] *intro*: *suntil.intros*)
next
assume *?U ω* **then show** *enat* (*Suc n*) ≤ *scount* (*HLD* {*y*}) *ω*
by *induction* (*auto simp*: *scount-simps eSuc-enat*[*symmetric*])
qed }
then have *emeasure* (*T x*) (*H′ y* (*Suc n*)) = *emeasure* (*T x*) {*ω*∈*space* (*T x*). *?U ω*}
by (*simp add*: *H′-def*)
also have ... = *U x y* ∗ *?H′ y y n*
by (*subst emeasure-suntil-HLD*) (*simp-all add*: *T.emeasure-eq-measure U-def H′-def ennreal-mult*)
finally have *?H′ x y* (*Suc n*) = *U x y* ∗ *?H′ y y n*
by (*simp add*: *T.emeasure-eq-measure*) }
note *H′-Suc* = *this*

{ **fix** *m* **and** *x* :: *′s*
have *?H′ x x* (*Suc m*) = *U x x*^*Suc m*
using *H′-1 H′-Suc* **by** (*induct m*) *auto* }
note *H′-eq* = *this*

{ **fix** *x y*
have *?H′ x y* ⟶ *measure* (*T x*) (⋂*i*. *H′ y i*)
proof (*rule T.finite-Lim-measure-decseq*)
show *range* (*H′ y*) ⊆ *T.events x*
by *auto*
next
show *decseq* (*H′ y*)
by (*rule antimonoI*) (*simp add*: *subset-eq H′-def order-subst2*)
qed
also have (⋂*i*. *H′ y i*) = {*ω*∈*space* (*T x*). *alw* (*ev* (*HLD* {*y*})) *ω*}
by (*auto simp*: *H′-def scount-infinite-iff*[*symmetric*]) (*metis Suc-ile-eq enat.exhaust neq-iff*)
finally have *?H′ x y* ⟶ *H x y*
unfolding *H-def* **.** }
note *H′-lim* = *this*

from *H′-lim*[*of s s*, *THEN LIMSEQ-Suc*]
have (*λn*. *U s s* ^ *Suc n*) ⟶ *H s s*
by (*simp add*: *H′-eq*)

then have *lim-H*: $(\lambda n.\ U\ s\ s\ \widehat{}\ n) \longrightarrow H\ s\ s$
  by (*rule LIMSEQ-imp-Suc*)

have $U\ s\ s < 1 \Longrightarrow (\lambda n.\ U\ s\ s\ \widehat{}\ n) \longrightarrow 0$
  by (*rule LIMSEQ-realpow-zero*) (*simp-all add*: *U-def*)
with *lim-H* **have** $U\ s\ s < 1 \Longrightarrow H\ s\ s = 0$
  by (*blast intro*: *LIMSEQ-unique*)
moreover have $U\ s\ s = 1 \Longrightarrow (\lambda n.\ U\ s\ s\ \widehat{}\ n) \longrightarrow 1$
  by *simp*
with *lim-H* **have** $U\ s\ s = 1 \Longrightarrow H\ s\ s = 1$
  by (*blast intro*: *LIMSEQ-unique*)
moreover note *recurrent-iff-U-eq-1 U-cases*
ultimately show *recurrent s* $\longleftrightarrow H\ s\ s = 1$ $\neg$ *recurrent s* $\longleftrightarrow H\ s\ s = 0$
  by (*metis one-neq-zero*)+

from *H′-lim*[*of s t*, *THEN LIMSEQ-Suc*] *H′-Suc*[*of s*]
have $(\lambda n.\ U\ s\ t * ?H'\ t\ t\ n) \longrightarrow H\ s\ t$
  by *simp*
moreover have $(\lambda n.\ U\ s\ t * ?H'\ t\ t\ n) \longrightarrow U\ s\ t * H\ t\ t$
  by (*intro tendsto-intros H′-lim*)
ultimately show $H\ s\ t = U\ s\ t * H\ t\ t$
  by (*blast intro*: *LIMSEQ-unique*)
qed

lemma *recurrent-iff-G-infinite*: *recurrent x* $\longleftrightarrow G\ x\ x = \infty$
proof −
have $((\lambda z.\ ennreal\ (gf\text{-}G\ x\ x\ z)) \longrightarrow G\ x\ x)$ (*at-left 1*)
  by (*rule lim-gf-G*)
then have *G*: $((\lambda z.\ ennreal\ (1\ /\ (1 - gf\text{-}U\ x\ x\ z))) \longrightarrow G\ x\ x)$ (*at-left* (*1*::*real*))
  apply (*rule filterlim-cong*[*OF refl refl*, *THEN iffD1*, *rotated*])
  apply (*rule eventually-at-left-1*)
  apply (*subst gf-G-eq-gf-U*)
  apply (*rule convergence-G-less-1*)
  apply *simp*
  apply *simp*
  done

{ **fix** *z* :: *real* **assume** *z*: $0 < z\ z < 1$
  have *1*: *summable* (*u x x*)
    using *u-sums-U* **by** (*rule sums-summable*)
  have $gf\text{-}U\ x\ x\ z \neq 1$
    using *gf-G-eq-gf-U*[*OF convergence-G-less-1*[*of z*]] *z* **by** *simp*
  moreover
  have $gf\text{-}U\ x\ x\ z \leq U\ x\ x$
    unfolding *gf-U-def gf-U-eq-U*[*symmetric*]
    using *z*
    by (*intro suminf-le*)
      (*auto simp add*: *1 convergence-U convergence-G-less-1 u-nonneg simp del*:
power-Suc

intro!: *mult-right-le-one-le power-le-one*)
ultimately have *gf-U x x z* < *1*
using *U-bounded*[*of x x*] **by** *simp* **}**
note *strict* = *this*

{ assume *U x x* = *1*
moreover have ((λ*xa*. *1* − *gf-U x x xa* :: *real*) ⟶ *1* − *U x x*) (*at-left 1*)
by (*intro tendsto-intros gf-U*)
moreover have *eventually* (λ*z*. *gf-U x x z* < *1*) (*at-left* (*1*::*real*))
by (*auto intro*!: *eventually-at-left-1 strict simp*: ‹*U x x* = *1*› *gf-U-eq-U*)
ultimately have ((λ*z*. *ennreal* (*1* / (*1* − *gf-U x x z*))) ⟶ *top*) (*at-left 1*)
unfolding *ennreal-tendsto-top-eq-at-top*
by (*intro LIM-at-top-divide*[**where** *a*=*1*] *tendsto-const zero-less-one*)
(*auto simp*: *field-simps*)
with *G* **have** *G x x* = *top*
by (*rule tendsto-unique*[*rotated*]) *simp* **}**
moreover
{ assume *U x x* < *1*
then have ((λ*xa*. *ennreal* (*1* / (*1* − *gf-U x x xa*))) ⟶ *1* / (*1* − *U x x*)) (*at-left 1*)
by (*intro tendsto-intros gf-U tendsto-ennrealI*) *simp*
from *tendsto-unique*[*OF* - *G this*] **have** *G x x* ≠ ∞
by *simp* **}**
ultimately show *?thesis*
using *U-cases recurrent-iff-U-eq-1* **by** *auto*
qed

definition *communicating* :: (′*s* × ′*s*) *set* **where**
communicating = *acc* ∩ acc^{-1}

definition *essential-class* :: ′*s set* ⇒ *bool* **where**
essential-class C ⟷ *C* ∈ *UNIV* // *communicating* ∧ *acc* ‘‘ *C* ⊆ *C*

lemma *accI-U*:
assumes *0* < *U x y* **shows** (*x*, *y*) ∈ *acc*
proof (*rule ccontr*)
assume ∗: (*x*, *y*) ∉ *acc*

{ fix ω **assume** *ev* (*HLD* {*y*}) ω *alw* (*HLD* (*acc* ‘‘ {*x*})) ω **from** *this* ∗ **have** *False*
by *induction* (*auto simp*: *HLD-iff*) **}**
with *AE-T-reachable*[*of x*] **have** *U x y* = *0*
unfolding *U-def* **by** (*intro T.prob-eq-0-AE*) *auto*
with ‹*0* < *U x y*› **show** *False* **by** *auto*
qed

lemma *accD-pos*:
assumes (*x*, *y*) ∈ *acc*
shows ∃ *n*. *0* < *p x y n*

using *assms* **proof** *induction*
  case *base* **with** *T.prob-space*[*of x*] **show** *?case*
    by (*auto intro*!: *exI*[*of - 0*])
next
  have [*simp*]: $\bigwedge x\ y.$ (*if* $x = y$ *then 1 else 0*::*real*) = *indicator* $\{y\}$ x
    by *simp*
  case (*step w y*)
  then obtain *n* **where** $0 < p\ x\ w\ n$ **and** $0 < pmf\ (K\ w)\ y$
    by (*auto simp*: *set-pmf-iff less-le*)
  then have $0 < p\ x\ w\ n * pmf\ (K\ w)\ y$
    by (*intro mult-pos-pos*)
  also have $\ldots \leq p\ x\ w\ n * p\ w\ y\ (Suc\ 0)$
    by (*simp add*: *p-Suc′ p-0 pmf.rep-eq*)
  also have $\ldots \leq p\ x\ y\ (Suc\ n)$
    using *prob-reachable-le*[*of n Suc n x w y*] **by** *simp*
  finally show *?case* **..**
qed

lemma *accI-pos*: $0 < p\ x\ y\ n \Longrightarrow (x,\ y) \in acc$
proof (*induct n arbitrary*: *x*)
  case (*Suc n*)
  then have *less*: $0 < (\int x'.\ p\ x'\ y\ n\ \partial K\ x)$
    by (*simp add*: *p-Suc′*)
  have $\exists x' \in K\ x.\ 0 < p\ x'\ y\ n$
  proof (*rule ccontr*)
    assume ¬ *?thesis*
    then have *AE* x' *in K x*. $p\ x'\ y\ n = 0$
      by (*simp add*: *AE-measure-pmf-iff less-le*)
    then have $(\int x'.\ p\ x'\ y\ n\ \partial K\ x) = (\int x'.\ 0\ \partial K\ x)$
      by (*intro integral-cong-AE*) *simp-all*
    with *less* **show** *False* **by** *simp*
  qed
  with *Suc* **show** *?case*
    by (*auto intro*: *converse-rtrancl-into-rtrancl*)
qed (*simp add*: *p-0 split*: *if-split-asm*)

lemma *recurrent-iffI-communicating*:
  assumes $(x,\ y) \in communicating$
  shows *recurrent x* $\longleftrightarrow$ *recurrent y*
proof −
  from *assms* **obtain** *n m* **where** $0 < p\ x\ y\ n$ $0 < p\ y\ x\ m$
    by (*force simp*: *communicating-def dest*: *accD-pos*)
  moreover
  { fix *x y n m* **assume** $0 < p\ x\ y\ n$ $0 < p\ y\ x\ m$ $G\ y\ y = \infty$
    then have $\infty = ennreal\ (p\ x\ y\ n * p\ y\ x\ m) * G\ y\ y$
      by (*auto intro*: *mult-pos-pos simp*: *ennreal-mult-top*)
    also have $ennreal\ (p\ x\ y\ n * p\ y\ x\ m) * G\ y\ y = (\sum i.\ ennreal\ (p\ x\ y\ n * p\ y\ x\ m) * p\ y\ y\ i)$
      unfolding *G-eq-suminf* **by** (*rule ennreal-suminf-cmult*[*symmetric*])

also have ... $\leq$ ($\sum i$. *ennreal* (*p x x* (*n* + *i* + *m*)))
proof (*intro suminf-le allI*)
fix *i*
have (*p x y n* * *p y y* ((*n* + *i*) − *n*)) * *p y x* ((*n* + *i* + *m*) − (*n* + *i*)) $\leq$ *p x y* (*n* + *i*) * *p y x* ((*n* + *i* + *m*) − (*n* + *i*))
by (*intro mult-right-mono prob-reachable-le*) *simp-all*
also have ... $\leq$ *p x x* (*n* + *i* + *m*)
by (*intro prob-reachable-le*) *simp-all*
finally show *ennreal* (*p x y n* * *p y x m*) * *p y y i* $\leq$ *ennreal* (*p x x* (*n* + *i* + *m*))
by (*simp add*: *ac-simps ennreal-mult'*[*symmetric*])
qed *auto*
also have ... $\leq$ ($\sum i$. *ennreal* (*p x x* (*i* + (*n* + *m*))))
by (*simp add*: *ac-simps*)
also have ... $\leq$ ($\sum i$. *ennreal* (*p x x i*))
by (*subst suminf-offset*[*of* λi. *ennreal* (*p x x i*) *n* + *m*]) *auto*
also have ... $\leq$ *G x x*
unfolding *G-eq-suminf* **by** (*auto intro*!: *suminf-le-pos*)
finally have *G x x* = ∞
by (*simp add*: *top-unique*) **}**
ultimately show *?thesis*
using *recurrent-iff-G-infinite* **by** *blast*
qed

lemma *recurrent-acc*:
assumes *recurrent x* (*x*, *y*) $\in$ *acc*
shows *U y x* = *1* *H y x* = *1* *recurrent y* (*x*, *y*) $\in$ *communicating*
proof −
{ fix *w y* **assume** *step*: (*x*, *w*) $\in$ *acc y* $\in$ *K w U w x* = *1* *H w x* = *1* *recurrent w x* $\neq$ *y*
have *measure* (*K w*) *UNIV* = *U w x*
using *step measure-pmf.prob-space*[*of K w*] **by** *simp*
also have ... = ($\int v$. *indicator* {*x*} *v* + *U v x* * *indicator* (− {*x*}) *v* ∂*K w*)
unfolding *U-def*
by (*subst prob-T*)
(*auto intro*!: *Bochner-Integration.integral-cong arg-cong2*[**where** *f*=*measure*] *AE-I2*
simp: *ev-Stream T.prob-eq-1 split*: *split-indicator*)
also have ... = *measure* (*K w*) {*x*} + ($\int v$. *U v x* * *indicator* (− {*x*}) *v* ∂*K w*)
by (*subst Bochner-Integration.integral-add*)
(*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B*=*1*]
simp: *abs-mult mult-le-one U-bounded*(*2*) *measure-pmf.emeasure-eq-measure*)
finally have *measure* (*K w*) *UNIV* − *measure* (*K w*) {*x*} = ($\int v$. *U v x* * *indicator* (− {*x*}) *v* ∂*K w*)
by *simp*
also have *measure* (*K w*) *UNIV* − *measure* (*K w*) {*x*} = *measure* (*K w*) (*UNIV* − {*x*})
by (*subst measure-pmf.finite-measure-Diff*) *auto*

finally have *0* = (∫ *v*. *indicator* (− {*x*}) *v* ∂*K w*) − (∫ *v*. *U v x* ∗ *indicator* (− {*x*}) *v* ∂*K w*)
by (*simp add*: *measure-pmf.emeasure-eq-measure Compl-eq-Diff-UNIV*)
also have . . . = (∫ *v*. (*1* − *U v x*) ∗ *indicator* (− {*x*}) *v* ∂*K w*)
by (*subst Bochner-Integration.integral-diff*[*symmetric*])
(*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B*=*1*] *Bochner-Integration.integral-cong simp*: *abs-mult mult-le-one U-bounded*(*2*) *split*: *split-indicator*)
also have . . . ≥ (∫ *v*. (*1* − *U y x*) ∗ *indicator* {*y*} *v* ∂*K w*) (**is** - ≥ *?rhs*)
using ‹*recurrent x*›
by (*intro integral-mono measure-pmf.integrable-const-bound*[**where** *B*=*1*])
(*auto simp*: *abs-mult mult-le-one U-bounded*(*2*) *recurrent-iff-U-eq-1 field-simps split*: *split-indicator*)
also (*xtrans*) **have** *?rhs* = (*1* − *U y x*) ∗ *pmf* (*K w*) *y*
by (*simp add*: *measure-pmf.emeasure-eq-measure pmf.rep-eq*)
finally have (*1* − *U y x*) ∗ *pmf* (*K w*) *y* = *0*
by (*auto intro*!: *antisym simp*: *U-bounded*(*2*) *mult-le-0-iff*)
with ‹*y* ∈ *K w*› **have** *U y x* = *1*
by (*simp add*: *set-pmf-iff*)
then have *U y x* = *1 H y x* = *1*
using *H-eq*(*3*)[*of y x*] *H-eq*(*1*)[*of x*] **by** (*simp-all add*: ‹*recurrent x*›)
then have (*y*, *x*) ∈ *acc*
by (*intro accI-U*) *auto*
with *step* **have** (*x*, *y*) ∈ *communicating*
by (*auto simp add*: *communicating-def intro*: *rtrancl-trans*)
with ‹*recurrent x*› **have** *recurrent y*
by (*simp add*: *recurrent-iffI-communicating*)
note *this* ‹*U y x* = *1*› ‹*H y x* = *1*› ‹(*x*, *y*) ∈ *communicating*› **}**
note *enabled* = *this*

from ‹(*x*, *y*) ∈ *acc*›
show *U y x* = *1 H y x* = *1 recurrent y* (*x*, *y*) ∈ *communicating*
proof *induction*
case *base* **then show** *U x x* = *1 H x x* = *1 recurrent x* (*x*, *x*) ∈ *communicating*
using ‹*recurrent x*› *H-eq*(*1*)[*of x*] **by** (*auto simp*: *recurrent-iff-U-eq-1 communicating-def*)
next
case (*step w y*)
with *enabled*[*of w y*] ‹*recurrent x*› *H-eq*(*1*)[*of x*]
have *U y x* = *1* ∧ *H y x* = *1* ∧ *recurrent y* ∧ (*x*, *y*) ∈ *communicating*
by (*cases x* = *y*) (*auto simp*: *recurrent-iff-U-eq-1 communicating-def*)
then show *U y x* = *1 H y x* = *1 recurrent y* (*x*, *y*) ∈ *communicating*
by *auto*
qed
qed

lemma *equiv-communicating*: *equiv UNIV communicating*
by (*auto simp*: *equiv-def sym-def communicating-def refl-on-def trans-def*)

lemma *recurrent-class*:

assumes *recurrent x*
shows *acc* `` {*x*} = *communicating* `` {*x*}
using *recurrent-acc*(*4*)[*OF* ‹*recurrent x*›] **by** (*auto simp*: *communicating-def*)

lemma *irreducible-recurrent-class*:
assumes *recurrent x* **shows** *acc* `` {*x*} ∈ *UNIV* // *communicating*
unfolding *recurrent-class*[*OF* ‹*recurrent x*›] **by** (*rule quotientI*) *simp*

lemma *essential-classI*:
assumes *C*: *C* ∈ *UNIV* // *communicating*
assumes *eq*: ⋀*x y*. *x* ∈ *C* ⟹ (*x*, *y*) ∈ *acc* ⟹ *y* ∈ *C*
shows *essential-class C*
by (*auto simp*: *essential-class-def intro*: *C*) (*metis eq*)

lemma *essential-recurrent-class*:
assumes *recurrent x* **shows** *essential-class* (*communicating* `` {*x*})
unfolding *recurrent-class*[*OF* ‹*recurrent x*›, *symmetric*]
apply (*rule essential-classI*)
apply (*rule irreducible-recurrent-class*[*OF assms*])
apply (*auto simp*: *communicating-def*)
done

lemma *essential-classD2*:
essential-class C ⟹ *x* ∈ *C* ⟹ (*x*, *y*) ∈ *acc* ⟹ *y* ∈ *C*
unfolding *essential-class-def* **by** *auto*

lemma *essential-classD3*:
essential-class C ⟹ *x* ∈ *C* ⟹ *y* ∈ *C* ⟹ (*x*, *y*) ∈ *communicating*
unfolding *essential-class-def*
by (*auto elim*!: *quotientE simp*: *communicating-def*)

lemma *AE-acc*:
shows *AE ω in T x*. ∀ *m*. (*x*, (*x* ## *ω*) !! *m*) ∈ *acc*
using *AE-T-reachable*
by *eventually-elim* (*auto simp*: *alw-HLD-iff-streams streams-iff-snth Stream-snth split*: *nat.splits*)

lemma *finite-essential-class-imp-recurrent*:
assumes *C*: *essential-class C finite C* **and** *x*: *x* ∈ *C*
shows *recurrent x*
proof −
have *AE ω in T x*. ∃ *y*∈*C*. *alw* (*ev* (*HLD* {*y*})) *ω*
using *AE-T-reachable*
proof *eventually-elim*
fix *ω* **assume** *alw* (*HLD* (*acc* `` {*x*})) *ω*
then have *alw* (*HLD C*) *ω*
by (*rule alw-mono*) (*auto simp*: *HLD-iff intro*: *assms essential-classD2*)
then show ∃ *y*∈*C*. *alw* (*ev* (*HLD* {*y*})) *ω*
by (*rule pigeonhole-stream*) *fact*

qed
then have *1* = $\mathcal{P}$(*ω in T x*. ∃ *y*∈*C*. *alw* (*ev* (*HLD* {*y*})) *ω*)
by (*subst* (*asm*) *T.prob-Collect-eq-1*[*symmetric*]) (*auto simp*: ‹*finite C*›)
also have ... = *measure* (*T x*) (⋃ *y*∈*C*. {*ω*∈*space* (*T x*). *alw* (*ev* (*HLD* {*y*})) *ω*})
by (*intro arg-cong2*[**where** *f*=*measure*]) *auto*
also have ... ≤ (∑ *y*∈*C*. *H x y*)
unfolding *H-def* **using** ‹*finite C*› **by** (*rule T.finite-measure-subadditive-finite*) *auto*
also have ... = (∑ *y*∈*C*. *U x y* ∗ *H y y*)
by (*auto intro*!: *sum.cong H-eq*)
finally have ∃ *y*∈*C*. *recurrent y*
by (*rule-tac ccontr*) (*simp add*: *H-eq*(*2*))
then obtain *y* **where** *y* ∈ *C recurrent y* **..**
from *essential-classD3*[*OF C*(*1*) *x this*(*1*)] *recurrent-acc*(*3*)[*OF this*(*2*)]
show *recurrent x*
by (*simp add*: *communicating-def*)
qed

lemma *irreducibleD*:
C ∈ *UNIV* // *communicating* =⇒ *a* ∈ *C* =⇒ *b* ∈ *C* =⇒ (*a*, *b*) ∈ *communicating*
by (*auto elim*!: *quotientE simp*: *communicating-def*)

lemma *irreducibleD2*:
C ∈ *UNIV* // *communicating* =⇒ *a* ∈ *C* =⇒ (*a*, *b*) ∈ *communicating* =⇒ *b* ∈ *C*
by (*auto elim*!: *quotientE simp*: *communicating-def*)

lemma *essential-class-iff-recurrent*:
finite C =⇒ *C* ∈ *UNIV* // *communicating* =⇒ *essential-class C* ←→ (∀ *x*∈*C*. *recurrent x*)
by (*metis finite-essential-class-imp-recurrent irreducibleD2 recurrent-acc*(*4*) *essential-classI*)

definition *U′ x y* = (∫⁺*ω*. *eSuc* (*sfirst* (*HLD* {*y*}) *ω*) ∂*T x*)

lemma *U′-neq-zero*[*simp*]: *U′ x y* ≠ *0*
unfolding *U′-def* **by** (*simp add*: *nn-integral-add*)

definition *gf-U′ x y z* = (∑ *n*. *u x y n* ∗ *Suc n* ∗ *z* ^ *n*)

definition *pos-recurrent x* ←→ *recurrent x* ∧ *U′ x x* ≠ ∞

lemma *summable-gf-U′*:
assumes *z*: *norm z* < *1*
shows *summable* (*λn*. *u x y n* ∗ *Suc n* ∗ *z* ^ *n*)
proof −
have *summable* (*λn*. *n* ∗ |*z*| ^ *n*)
proof (*rule root-test-convergence*)

have (*λn. root n n* ∗ |*z*|) ⟶ *1* ∗ |*z*|
by (*intro tendsto-intros LIMSEQ-root*)
then show (*λn. root n* (*norm* (*n* ∗ |*z*| ^ *n*))) ⟶ |*z*|
by (*rule filterlim-cong*[*THEN iffD1*, *rotated 3*])
(*auto intro*!: *exI*[*of* - *1*]
simp add: *abs-mult u-nonneg real-root-mult power-abs eventually-sequentially real-root-power*)
qed (*insert z*, *simp add*: *abs-less-iff*)
note *summable-mult*[*OF this*, *of 1* / |*z*|]
from *summable-ignore-initial-segment*[*OF this*, *of 1*]
show *summable* (*λn. u x y n* ∗ *Suc n* ∗ *z* ^ *n*)
apply (*rule summable-comparison-test*[*rotated*])
using *z*
apply (*auto intro*!: *exI*[*of* - *1*]
simp: *abs-mult u-nonneg power-abs Suc-le-eq gr0-conv-Suc field-simps le-divide-eq u-le-1*
simp del: *of-nat-Suc*)
done
qed

lemma *gf-U*′*-nonneg*[*simp*]: *0* < *z* ⟹ *z* < *1* ⟹ *0* ≤ *gf-U*′ *x y z*
unfolding *gf-U*′*-def*
by (*intro suminf-nonneg summable-gf-U*′) (*auto simp*: *u-nonneg*)

lemma *DERIV-gf-U*:
fixes *z* :: *real* **assumes** *z*: *0* < *z z* < *1*
shows *DERIV* (*gf-U x y*) *z* :> *gf-U*′ *x y z*
unfolding *gf-U-def*[*abs-def*] *gf-U*′*-def real-scaleR-def u-def*[*symmetric*]
using *z* **by** (*intro DERIV-power-series*′[**where** *R*=*1*] *summable-gf-U*′) *auto*

lemma *sfirst-finiteI-recurrent*:
recurrent x ⟹ (*x*, *y*) ∈ *acc* ⟹ *AE ω in T x. sfirst* (*HLD* {*y*}) *ω* < ∞
using *recurrent-acc*(*1*)[*of y x*] *recurrent-acc*[*of x y*]
T.AE-prob-1[*of x* {*ω*∈*space* (*T x*). *ev* (*HLD* {*y*}) *ω*}]
unfolding *sfirst-finite U-def* **by** (*simp add*: *space-stream-space communicating-def*)

lemma *U*′*-eq-suminf*:
assumes *x*: *recurrent x* (*x*, *y*) ∈ *acc*
shows *U*′ *x y* = (∑ *i. ennreal* (*u x y i* ∗ *Suc i*))
proof −
have (∫⁺*ω. eSuc* (*sfirst* (*HLD* {*y*}) *ω*) ∂*T x*) =
(∫⁺*ω.* (∑ *i. ennreal* (*Suc i*) ∗ *indicator* {*ω*∈*space* (*T y*). *ev-at* (*HLD* {*y*}) *i ω*} *ω*) ∂*T x*)
using *sfirst-finiteI-recurrent*[*OF x*]
proof (*intro nn-integral-cong-AE*, *eventually-elim*)
fix *ω* **assume** *sfirst* (*HLD* {*y*}) *ω* < ∞
then obtain *n* :: *nat* **where** [*simp*]: *sfirst* (*HLD* {*y*}) *ω* = *n*
by *auto*

show *eSuc* (*sfirst* (*HLD* {*y*}) *ω*) = (∑ *i*. *ennreal* (*Suc i*) ∗ *indicator* {*ω*∈*space* (*T y*). *ev-at* (*HLD* {*y*}) *i ω*} *ω*)
by (*subst suminf-cmult-indicator*[**where** *i*=*n*])
(*auto simp*: *disjoint-family-on-def ev-at-unique space-stream-space*
sfirst-eq-enat-iff [*symmetric*] *ennreal-of-nat-eq-real-of-nat*
split: *split-indicator*)
qed
also have . . . = (∑ *i*. *ennreal* (*Suc i*) ∗ *emeasure* (*T x*) {*ω*∈*space* (*T x*). *ev-at* (*HLD* {*y*}) *i ω*})
by (*subst nn-integral-suminf*)
(*auto intro*!: *arg-cong*[**where** *f*=*suminf*] *nn-integral-cmult-indicator simp*: *fun-eq-iff*)
finally show *?thesis*
by (*simp add*: *U′-def u-def T.emeasure-eq-measure mult-ac ennreal-mult*)
qed

lemma *gf-U′-tendsto-U′*:
assumes *x*: *recurrent x* (*x*, *y*) ∈ *acc*
shows ((*λz*. *ennreal* (*gf-U′ x y z*)) ⟶ *U′ x y*) (*at-left 1*)
unfolding *U′-eq-suminf*[*OF x*] *gf-U′-def*
by (*auto intro*!: *power-series-tendsto-at-left summable-gf-U′ mult-nonneg-nonneg u-nonneg simp del*: *of-nat-Suc*)

lemma *one-le-integral-t*:
assumes *x*: *recurrent x* **shows** *1* ≤ *U′ x x*
by (*simp add*: *nn-integral-add T.emeasure-space-1 U′-def del*: *space-T*)

lemma *gf-U′-pos*:
fixes *z* :: *real*
assumes *z*: *0* < *z z* < *1* **and** *U x y* ≠ *0*
shows *0* < *gf-U′ x y z*
unfolding *gf-U′-def*
proof (*subst suminf-pos-iff*)
show *summable* (*λn*. *u x y n* ∗ *real* (*Suc n*) ∗ *z* ^ *n*)
using *z* **by** (*intro summable-gf-U′*) *simp*
show *pos*: ⋀*n*. *0* ≤ *u x y n* ∗ *real* (*Suc n*) ∗ *z* ^ *n*
using *u-nonneg z* **by** *auto*
show ∃ *n*. *0* < *u x y n* ∗ *real* (*Suc n*) ∗ *z* ^ *n*
proof (*rule ccontr*)
assume ¬ (∃ *n*. *0* < *u x y n* ∗ *real* (*Suc n*) ∗ *z* ^ *n*)
with *pos* **have** ∀ *n*. *u x y n* ∗ *real* (*Suc n*) ∗ *z* ^ *n* = *0*
by (*intro antisym allI*) (*simp-all add*: *not-less*)
with *z* **have** *u x y* = (*λn*. *0*)
by (*intro ext*) *simp*
with *u-sums-U*[*of x y*, *THEN sums-unique*] ‹*U x y* ≠ *0*› **show** *False*
by *simp*
qed
qed

```
lemma inverse-gf-U′-tendsto:
  assumes recurrent y
  shows ((λx. − 1 / − gf-U′ y y x) ⟶ enn2real (1 / U′ y y)) (at-left (1::real))
proof cases
  assume inf: U′ y y = ∞
  with gf-U′-tendsto-U′[of y y] ‹recurrent y›
  have LIM z (at-left 1). gf-U′ y y z :> at-top
    by (auto simp: ennreal-tendsto-top-eq-at-top U′-def)
  then have LIM z (at-left 1). gf-U′ y y z :> at-infinity
    by (rule filterlim-mono) (auto simp: at-top-le-at-infinity)
  with inf show ?thesis
    by (auto intro!: tendsto-divide-0)
next
  assume fin: U′ y y ≠ ∞
  then obtain r where r: U′ y y = ennreal r and [simp]: 0 ≤ r
    by (cases U′ y y) (auto simp: U′-def)
  then have eq: enn2real (1 / U′ y y) = − 1 / − r and 1 ≤ r
    using one-le-integral-t[OF ‹recurrent y›]
    by (auto simp add: ennreal-1[symmetric] divide-ennreal simp del: ennreal-1)
  have ((λz. ennreal (gf-U′ y y z)) ⟶ ennreal r) (at-left 1)
    using gf-U′-tendsto-U′[OF ‹recurrent y›, of y] r by simp
  then have gf-U′: (gf-U′ y y ⟶ r) (at-left (1::real))
    by (rule tendsto-ennrealD)
      (insert summable-gf-U′, auto intro!: eventually-at-left-1 suminf-nonneg simp:
gf-U′-def u-nonneg)
  show ?thesis
    using ‹1 ≤ r› unfolding eq by (intro tendsto-intros gf-U′) simp
qed

lemma gf-G-pos:
  fixes z :: real
  assumes z: 0 < z z < 1 and ∗: (x, y) ∈ acc
  shows 0 < gf-G x y z
  unfolding gf-G-def
proof (subst suminf-pos-iff)
  show summable (λn. p x y n ∗R z ^ n)
    using z by (intro convergence-G convergence-G-less-1) simp
  show pos: ⋀n. 0 ≤ p x y n ∗R z ^ n
    using z by (auto intro!: mult-nonneg-nonneg p-nonneg)
  show ∃ n. 0 < p x y n ∗R z ^ n
  proof (rule ccontr)
    assume ¬ (∃ n. 0 < p x y n ∗R z ^ n)
    with pos have ∀ n. p x y n ∗ z ^ n = 0
      by (intro antisym allI) (simp-all add: not-less)
    with z have ⋀n. p x y n = 0
      by simp
    with ∗[THEN accD-pos] show False
      by simp
  qed
```

qed

lemma *pos-recurrentI-communicating*:
  assumes *y*: *pos-recurrent y* **and** *x*: (*y*, *x*) ∈ *communicating*
  shows *pos-recurrent x*
proof −
  from *y x* **have** *recurrent*: *recurrent y recurrent x* **and** *fin*: *U′ y y* ≠ ∞
    by (*auto simp*: *pos-recurrent-def recurrent-iffI-communicating nn-integral-add*)
  have *pos*: *0* < *enn2real* (*1* / *U′ y y*)
    using *one-le-integral-t*[*OF* ‹*recurrent y*›] *fin*
    by (*auto simp*: *U′-def enn2real-positive-iff less-top*[*symmetric*] *ennreal-zero-less-divide ennreal-divide-eq-top-iff*)

  from *fin* **obtain** *r* **where** *r*: *U′ y y* = *ennreal r* **and** [*simp*]: *0* ≤ *r*
    by (*cases U′ y y*) (*auto simp*: *U′-def*)

  from *x* **obtain** *n m* **where** *0* < *p x y n 0* < *p y x m*
    by (*auto dest*!: *accD-pos simp*: *communicating-def*)

  let *?L* = *at-left* (*1*::*real*)
  have *le*: *eventually* (λ*z*. *p x y n* ∗ *p y x m* ∗ *z*^(*n* + *m*) ≤ (*1* − *gf-U y y z*) / (*1* − *gf-U x x z*)) *?L*
  proof (*rule eventually-at-left-1*)
    fix *z* :: *real* **assume** *z*: *0* < *z z* < *1*
    then have *conv*: ⋀*x*. *convergence-G x x z*
      by (*intro convergence-G-less-1*) *simp*
    have *sums*: (λ*i*. (*p x y n* ∗ *p y x m* ∗ *z*^(*n* + *m*)) ∗ (*p y y i* ∗ *z*^*i*)) *sums* ((*p x y n* ∗ *p y x m* ∗ *z*^(*n* + *m*)) ∗ *gf-G y y z*)
      unfolding *gf-G-def*
      by (*intro sums-mult summable-sums*) (*auto intro*!: *conv convergence-G*[**where** ′*a*=*real*, *simplified*])
    have (∑ *i*. (*p x y n* ∗ *p y x m* ∗ *z*^(*n* + *m*)) ∗ (*p y y i* ∗ *z*^*i*)) ≤ (∑ *i*. *p x x* (*i* + (*n* + *m*)) ∗ *z*^(*i* + (*n* + *m*)))
    proof (*intro allI suminf-le sums-summable*[*OF sums*] *summable-ignore-initial-segment convergence-G*[**where** ′*a*=*real*, *simplified*] *convergence-G-less-1*)
      show *norm z* < *1* **using** *z* **by** *simp*
      fix *i*
      have (*p x y n* ∗ *p y y* ((*n* + *i*) − *n*)) ∗ *p y x* ((*n* + *i* + *m*) − (*n* + *i*)) ≤ *p x y* (*n* + *i*) ∗ *p y x* ((*n* + *i* + *m*) − (*n* + *i*))
        by (*intro mult-right-mono prob-reachable-le*) *simp-all*
      also have ... ≤ *p x x* (*n* + *i* + *m*)
        by (*intro prob-reachable-le*) *simp-all*
      finally show *p x y n* ∗ *p y x m* ∗ *z* ^ (*n* + *m*) ∗ (*p y y i* ∗ *z* ^ *i*) ≤ *p x x* (*i* + (*n* + *m*)) ∗ *z* ^ (*i* + (*n* + *m*))
        using *z* **by** (*auto simp add*: *ac-simps power-add intro*!: *mult-left-mono*)
    qed
    also have ... ≤ *gf-G x x z*
      unfolding *gf-G-def*
      using *z*

apply (*subst* (*2*) *suminf-split-initial-segment*[**where** *k*=*n* + *m*])
apply (*intro convergence-G conv*)
apply (*simp add*: *sum-nonneg*)
done
finally have ($p\ x\ y\ n * p\ y\ x\ m * z\hat{}(n + m)) * gf\text{-}G\ y\ y\ z \leq gf\text{-}G\ x\ x\ z$
using *sums-unique*[*OF sums*] **by** *simp*
then have ($p\ x\ y\ n * p\ y\ x\ m * z\hat{}(n + m)) \leq gf\text{-}G\ x\ x\ z\ /\ gf\text{-}G\ y\ y\ z$
using *z gf-G-pos*[*of z y y*] **by** (*simp add*: *field-simps*)
also have ... = ($1 - gf\text{-}U\ y\ y\ z) / (1 - gf\text{-}U\ x\ x\ z$)
unfolding *gf-G-eq-gf-U*[*OF conv*] **using** *gf-G-eq-gf-U*(*2*)[*OF conv*] **by** (*simp add*: *field-simps*)
finally show $p\ x\ y\ n * p\ y\ x\ m * z\hat{}(n + m) \leq (1 - gf\text{-}U\ y\ y\ z) / (1 - gf\text{-}U\ x\ x\ z)$ **.**
qed

have $U'\ x\ x \neq \infty$
proof
assume $U'\ x\ x = \infty$
have (($\lambda z.\ (1 - gf\text{-}U\ y\ y\ z) / (1 - gf\text{-}U\ x\ x\ z)) \longrightarrow 0$) *?L*
proof (*rule lhopital-left*)
show (($\lambda z.\ 1 - gf\text{-}U\ y\ y\ z) \longrightarrow 0$) *?L*
using *gf-U*[*of y*] *recurrent-iff-U-eq-1*[*of y*] ‹*recurrent y*› **by** (*auto intro*!: *tendsto-eq-intros*)
show (($\lambda z.\ 1 - gf\text{-}U\ x\ x\ z) \longrightarrow 0$) *?L*
using *gf-U*[*of x*] *recurrent-iff-U-eq-1*[*of x*] ‹*recurrent x*› **by** (*auto intro*!: *tendsto-eq-intros*)
show *eventually* ($\lambda z.\ 1 - gf\text{-}U\ x\ x\ z \neq 0$) *?L*
by (*auto intro*!: *eventually-at-left-1 simp*: *gf-G-eq-gf-U*(*2*) *convergence-G-less-1*)
show *eventually* ($\lambda z.\ -\ gf\text{-}U'\ x\ x\ z \neq 0$) *?L*
using *gf-U'-pos*[*of - x x*] *recurrent-iff-U-eq-1*[*of x*] ‹*recurrent x*›
by (*auto intro*!: *eventually-at-left-1*) (*metis less-le*)
show *eventually* ($\lambda z.$ *DERIV* ($\lambda xa.\ 1 - gf\text{-}U\ x\ x\ xa$) z :> $-\ gf\text{-}U'\ x\ x\ z$) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros DERIV-gf-U*)
show *eventually* ($\lambda z.$ *DERIV* ($\lambda xa.\ 1 - gf\text{-}U\ y\ y\ xa$) z :> $-\ gf\text{-}U'\ y\ y\ z$) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros DERIV-gf-U*)

have ($gf\text{-}U'\ y\ y \longrightarrow U'\ y\ y$) *?L*
using ‹*recurrent y*› **by** (*rule gf-U'-tendsto-U'*) *simp*
then have ∗: ($gf\text{-}U'\ y\ y \longrightarrow r$) *?L*
by (*auto simp add*: *r eventually-at-left-1 dest*!: *tendsto-ennrealD*)
moreover
have ($gf\text{-}U'\ x\ x \longrightarrow U'\ x\ x$) *?L*
using ‹*recurrent x*› **by** (*rule gf-U'-tendsto-U'*) *simp*
then have *LIM z ?L*. $-\ gf\text{-}U'\ x\ x\ z$:> *at-bot*
by (*simp add*: *ennreal-tendsto-top-eq-at-top* ‹$U'\ x\ x = \infty$› *filterlim-uminus-at-top del*: *ennreal-of-enat-eSuc*)
then have *LIM z ?L*. $-\ gf\text{-}U'\ x\ x\ z$:> *at-infinity*
by (*rule filterlim-mono*) (*auto simp*: *at-bot-le-at-infinity*)
ultimately show (($\lambda z.\ -\ gf\text{-}U'\ y\ y\ z\ /\ -\ gf\text{-}U'\ x\ x\ z) \longrightarrow 0$) *?L*

    by (*intro tendsto-divide-0*[**where** *c*=− *r*] *tendsto-intros*)
  qed
  moreover
  have $((\lambda z.\ p\ x\ y\ n * p\ y\ x\ m * z\hat{}(n + m)) \longrightarrow p\ x\ y\ n * p\ y\ x\ m)$ *?L*
    by (*auto intro*!: *tendsto-eq-intros*)
  ultimately have $p\ x\ y\ n * p\ y\ x\ m \le 0$
    using *le* **by** (*rule tendsto-le*[*OF trivial-limit-at-left-real*])
  with ‹$0 < p\ x\ y\ n$› ‹$0 < p\ y\ x\ m$› **show** *False*
    by (*auto simp add*: *mult-le-0-iff*)
 qed
 with ‹*recurrent x*› **show** *?thesis*
  by (*simp add*: *pos-recurrent-def nn-integral-add*)
qed

lemma *pos-recurrent-iffI-communicating*:
 $(y, x) \in$ *communicating* $\Longrightarrow$ *pos-recurrent y* $\longleftrightarrow$ *pos-recurrent x*
 using *pos-recurrentI-communicating*[*of x y*] *pos-recurrentI-communicating*[*of y x*]
 by (*auto simp add*: *communicating-def*)

lemma *U-le-F*: $U\ x\ y \le F\ x\ y$
 by (*auto simp*: *U-def F-def intro*!: *T.finite-measure-mono*)

lemma *not-empty-irreducible*: $C \in$ *UNIV* // *communicating* $\Longrightarrow C \ne \{\}$
 by (*auto simp*: *quotient-def Image-def communicating-def*)

4.4 Stationary distribution

definition *stat* :: *'s set* ⇒ *'s measure* **where**
 stat C = *point-measure UNIV* (λ*x*. *indicator C x* / *U' x x*)

lemma *sets-stat*[*simp*]: *sets* (*stat C*) = *sets* (*count-space UNIV*)
 by (*simp add*: *stat-def sets-point-measure*)

lemma *space-stat*[*simp*]: *space* (*stat C*) = *UNIV*
 by (*simp add*: *stat-def space-point-measure*)

lemma *stat-subprob*:
 assumes *C*: *essential-class C* **and** *countable C* **and** *pos*: ∀ *c*∈*C*. *pos-recurrent c*
 shows *emeasure* (*stat C*) $C \le 1$
proof −
 let *?L* = *at-left* (*1*::*real*)
 from *finite-sequence-to-countable-set*[*OF* ‹*countable C*›]
 obtain *A* **where** *A*: $\bigwedge i.\ A\ i \subseteq C$ $\bigwedge i.\ A\ i \subseteq A\ (Suc\ i)$ $\bigwedge i.$ *finite* (*A i*) $\bigcup$ (*range A*) = *C*
  by *blast*
 then have (λ*n*. *emeasure* (*stat C*) (*A n*)) ⟶ *emeasure* (*stat C*) ($\bigcup$*i*. *A i*)
  by (*intro Lim-emeasure-incseq*) (*auto simp*: *incseq-Suc-iff*)
 then have *emeasure* (*stat C*) ($\bigcup$*i*. *A i*) ≤ 1
 proof (*rule LIMSEQ-le*[*OF - tendsto-const*], *intro exI allI impI*)

fix *n*
from *A*(*1*,*3*) **have** *A-n*: *finite* (*A n*)
by *auto*

from *C* **have** *C* ≠ {}
by (*simp add*: *essential-class-def not-empty-irreducible*)
then obtain *x* **where** *x* ∈ *C* **by** *auto*

have ((λ*z*. (∑ *y*∈*A n*. *gf-F x y z* ∗ ((*1* − *z*) / (*1* − *gf-U y y z*)))) ⟶ (∑ *y*∈*A n*. *F x y* ∗ *enn2real* (*1* / *U′ y y*))) *?L*
proof (*intro tendsto-intros gf-F*, *rule lhopital-left*)
fix *y* **assume** *y* ∈ *A n*
with ‹*A n* ⊆ *C*› **have** *y* ∈ *C*
by *auto*
show ((−) *1* ⟶ *0*) *?L*
by (*intro tendsto-eq-intros*) *simp-all*
have *recurrent y*
using *pos*[*THEN bspec*, *OF* ‹*y*∈*C*›] **by** (*simp add*: *pos-recurrent-def*)
then have *U y y* = *1*
by (*simp add*: *recurrent-iff-U-eq-1*)

show ((λ*x*. *1* − *gf-U y y x*) ⟶ *0*) *?L*
using *gf-U*[*of y y*] ‹*U y y* = *1*› **by** (*intro tendsto-eq-intros*) *auto*
show *eventually* (λ*x*. *1* − *gf-U y y x* ≠ *0*) *?L*
using *gf-G-eq-gf-U*(*2*)[*OF convergence-G-less-1*, **where** ′*z*=*real*] **by** (*auto intro*!: *eventually-at-left-1*)
have *eventually* (λ*x*. *0* < *gf-U′ y y x*) *?L*
by (*intro eventually-at-left-1 gf-U′-pos*) (*simp-all add*: ‹*U y y* = *1*›)
then show *eventually* (λ*x*. − *gf-U′ y y x* ≠ *0*) *?L*
by *eventually-elim simp*
show *eventually* (λ*x*. *DERIV* (λ*x*. *1* − *gf-U y y x*) *x* :> − *gf-U′ y y x*) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros DERIV-gf-U*)
show *eventually* (λ*x*. *DERIV* ((−) *1*) *x* :> − *1*) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros*)
show ((λ*x*. − *1* / − *gf-U′ y y x*) ⟶ *enn2real* (*1* / *U′ y y*)) *?L*
using ‹*recurrent y*› **by** (*rule inverse-gf-U′-tendsto*)
qed
also have (∑ *y*∈*A n*. *F x y* ∗ *enn2real* (*1* / *U′ y y*)) = (∑ *y*∈*A n*. *enn2real* (*1* / *U′ y y*))
proof (*intro sum.cong refl*)
fix *y* **assume** *y* ∈ *A n*
with ‹*A n* ⊆ *C*› **have** *y* ∈ *C* **by** *auto*
with ‹*x* ∈ *C*› **have** (*x*, *y*) ∈ *communicating*
by (*rule essential-classD3*[*OF C*])
with ‹*y*∈*C*› **have** *recurrent y* (*y*, *x*) ∈ *acc*
using *pos*[*THEN bspec*, *of y*] **by** (*auto simp add*: *pos-recurrent-def communicating-def*)
then have *U x y* = *1*
by (*rule recurrent-acc*)

with *F-le-1*[*of x y*] *U-le-F*[*of x y*] **have** $F\ x\ y = 1$ **by** *simp*
then show $F\ x\ y * enn2real\ (1\ /\ U'\ y\ y) = enn2real\ (1\ /\ U'\ y\ y)$
by *simp*
qed
finally have *le*: $(\sum y \in A\ n.\ enn2real\ (1\ /\ U'\ y\ y)) \leq 1$
proof (*rule tendsto-le*[*OF trivial-limit-at-left-real tendsto-const*], *intro eventually-at-left-1*)
fix z :: *real* **assume** *z*: $0 < z\ z < 1$
with ‹$x \in C$› **have** *norm* $z < 1$
by *auto*
then have *conv*: $\bigwedge x\ y.$ *convergence-G* $x\ y\ z$
by (*simp add*: *convergence-G-less-1*)
have $(\sum y \in A\ n.\ gf\text{-}F\ x\ y\ z\ /\ (1 - gf\text{-}U\ y\ y\ z)) = (\sum y \in A\ n.\ gf\text{-}G\ x\ y\ z)$
using ‹*norm* $z < 1$›
apply (*intro sum.cong refl*)
apply (*subst gf-G-eq-gf-F*)
apply *assumption*
apply (*subst gf-G-eq-gf-U*(*1*)[*OF conv*])
apply *auto*
done
also have $\ldots = (\sum y \in A\ n.\ \sum n.\ p\ x\ y\ n * z\hat{}n)$
by (*simp add*: *gf-G-def*)
also have $\ldots = (\sum i.\ \sum y \in A\ n.\ p\ x\ y\ i *_R z\hat{}i)$
by (*subst suminf-sum*[*OF convergence-G*[*OF conv*]]) *simp*
also have $\ldots \leq (\sum i.\ z\hat{}i)$
proof (*intro suminf-le summable-sum convergence-G conv summable-geometric allI*)
fix l
have $(\sum y \in A\ n.\ p\ x\ y\ l *_R z\ \hat{}\ l) = (\sum y \in A\ n.\ p\ x\ y\ l) * z\ \hat{}\ l$
by (*simp add*: *sum-distrib-right*)
also have $\ldots \leq z\ \hat{}\ l$
proof (*intro mult-left-le-one-le*)
have $(\sum y \in A\ n.\ p\ x\ y\ l) = \mathcal{P}(\omega\ in\ T\ x.\ (x\ \#\#\ \omega)\ !!\ l \in A\ n)$
unfolding *p-def* **using** ‹*finite* (*A n*)›
by (*subst T.finite-measure-finite-Union*[*symmetric*])
(*auto simp*: *disjoint-family-on-def intro*!: *arg-cong2*[**where** *f*=*measure*])
then show $(\sum y \in A\ n.\ p\ x\ y\ l) \leq 1$
by *simp*
qed (*insert z*, *auto simp*: *sum-nonneg*)
finally show $(\sum y \in A\ n.\ p\ x\ y\ l *_R z\ \hat{}\ l) \leq z\ \hat{}\ l$ **.**
qed *fact*
also have $\ldots = 1\ /\ (1 - z)$
using *sums-unique*[*OF geometric-sums*, *OF* ‹*norm* $z < 1$›] **..**
finally have $(\sum y \in A\ n.\ gf\text{-}F\ x\ y\ z\ /\ (1 - gf\text{-}U\ y\ y\ z)) \leq 1\ /\ (1 - z)$ **.**
then have $(\sum y \in A\ n.\ gf\text{-}F\ x\ y\ z\ /\ (1 - gf\text{-}U\ y\ y\ z)) * (1 - z) \leq 1$
using *z* **by** (*simp add*: *field-simps*)
then have $(\sum y \in A\ n.\ gf\text{-}F\ x\ y\ z\ /\ (1 - gf\text{-}U\ y\ y\ z) * (1 - z)) \leq 1$
by (*simp add*: *sum-distrib-right*)
then show $(\sum y \in A\ n.\ gf\text{-}F\ x\ y\ z * ((1 - z)\ /\ (1 - gf\text{-}U\ y\ y\ z))) \leq 1$

by *simp*
qed

from *A-n* **have** *emeasure* (*stat C*) (*A n*) = ($\sum$ *y*∈*A n*. *emeasure* (*stat C*) {*y*})
by (*intro emeasure-eq-sum-singleton*) *simp-all*
also have ... = ($\sum$ *y*∈*A n*. *inverse* (*U′ y y*))
unfolding *stat-def U′-def* **using** *A*(*1*)[*of n*]
apply (*intro sum.cong refl*)
apply (*subst emeasure-point-measure-finite2*)
apply (*auto simp*: *divide-ennreal-def Collect-conv-if*)
done
also have ... = *ennreal* ($\sum$ *y*∈*A n*. *enn2real* (*1* / *U′ y y*))
apply (*subst sum-ennreal*[*symmetric*], *simp*)
proof (*intro sum.cong refl*)
fix *y* **assume** *y* ∈ *A n*
with ‹*A n* ⊆ *C*› *pos* **have** *pos-recurrent y*
by *auto*
with *one-le-integral-t*[*of y*] **obtain** *r* **where** *U′ y y* = *ennreal r 1* ≤ *U′ y y* **and** [*simp*]: *0* ≤ *r*
by (*cases U′ y y*) (*auto simp*: *pos-recurrent-def nn-integral-add*)
then show *inverse* (*U′ y y*) = *ennreal* (*enn2real* (*1* / *U′ y y*))
by (*simp add*: *ennreal-1*[*symmetric*] *divide-ennreal inverse-ennreal inverse-eq-divide del*: *ennreal-1*)
qed
also have ... ≤ *1*
using *le* **by** *simp*
finally show *emeasure* (*stat C*) (*A n*) ≤ *1* **.**
qed
with *A* **show** *?thesis*
by *simp*
qed

lemma *emeasure-stat-not-C*:
assumes *y* ∉ *C*
shows *emeasure* (*stat C*) {*y*} = *0*
unfolding *stat-def* **using** ‹*y* ∉ *C*›
by (*subst emeasure-point-measure-finite2*) *auto*

definition *stationary-distribution* :: *′s pmf* ⇒ *bool* **where**
stationary-distribution N ⟷ *N* = *bind-pmf N K*

lemma *stationary-distributionI*:
assumes *le*: ⋀*y*. (∫ *x*. *pmf* (*K x*) *y* ∂*measure-pmf N*) ≤ *pmf N y*
shows *stationary-distribution N*
unfolding *stationary-distribution-def*
proof (*rule pmf-eqI antisym*)+
fix *i*
show *pmf* (*bind-pmf N K*) *i* ≤ *pmf N i*
by (*simp add*: *pmf-bind le*)

define Ω **where** $\Omega = N \cup (\bigcup i{\in}N.\ \textit{set-pmf}\ (K\ i))$
then have Ω: *countable* Ω
by (*auto intro*: *countable-set-pmf*)
then interpret *N*: *sigma-finite-measure count-space* Ω
by (*rule sigma-finite-measure-count-space-countable*)
interpret *pN*: *pair-sigma-finite N count-space* Ω
by *unfold-locales*

have *measurable-pmf*[*measurable*]: $(\lambda(x, y).\ \textit{pmf}\ (K\ x)\ y) \in \textit{borel-measurable}\ (N \bigotimes_M \textit{count-space}\ \Omega)$
unfolding *measurable-split-conv*
apply (*rule measurable-compose-countable'*[*OF - measurable-snd*])
apply (*rule measurable-compose*[*OF measurable-fst*])
apply (*simp-all add*: Ω)
done

{ assume $*$: $(\int y.\ \textit{pmf}\ (K\ y)\ i\ \partial N) < \textit{pmf}\ N\ i$
have $0 \le (\int y.\ \textit{pmf}\ (K\ y)\ i\ \partial N)$
by (*intro integral-nonneg-AE*) *simp*
with $*$ **have** *i*: $i \in \textit{set-pmf}\ N\ i \in \Omega$
by (*auto simp*: *set-pmf-iff* Ω*-def not-le*[*symmetric*])
from $*$ **have** $0 < \textit{pmf}\ N\ i - (\int y.\ \textit{pmf}\ (K\ y)\ i\ \partial N)$
by (*simp add*: *field-simps*)
also have $\ldots = (\int t.\ (\textit{pmf}\ N\ i - (\int y.\ \textit{pmf}\ (K\ y)\ i\ \partial N)) * \textit{indicator}\ \{i\}\ t\ \partial \textit{count-space}\ \Omega)$
by (*simp add*: *i*)
also have $\ldots \le (\int t.\ \textit{pmf}\ N\ t - \int y.\ \textit{pmf}\ (K\ y)\ t\ \partial N\ \partial \textit{count-space}\ \Omega)$
using *le*
by (*intro integral-mono integrable-diff*)
(*auto simp*: *i pmf-bind*[*symmetric*] *integrable-pmf field-simps split*: *split-indicator*)
also have $\ldots = (\int t.\ \textit{pmf}\ N\ t\ \partial \textit{count-space}\ \Omega) - (\int t.\ \int y.\ \textit{pmf}\ (K\ y)\ t\ \partial N\ \partial \textit{count-space}\ \Omega)$
by (*subst Bochner-Integration.integral-diff*) (*auto intro*!: *integrable-pmf simp*: *pmf-bind*[*symmetric*])
also have $(\int t.\ \int y.\ \textit{pmf}\ (K\ y)\ t\ \partial N\ \partial \textit{count-space}\ \Omega) = (\int y.\ \int t.\ \textit{pmf}\ (K\ y)\ t\ \partial \textit{count-space}\ \Omega\ \partial N)$
apply (*intro pN.Fubini-integral integrable-iff-bounded*[*THEN iffD2*] *conjI*)
apply (*auto simp add*: *N.nn-integral-fst*[*symmetric*] *nn-integral-eq-integral integrable-pmf*)
unfolding *less-top*[*symmetric*] **unfolding** *infinity-ennreal-def*[*symmetric*]
apply (*intro integrableD*)
apply (*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B=1*]
simp: *AE-measure-pmf-iff integral-nonneg-AE integral-pmf*)
done
also have $(\int y.\ \int t.\ \textit{pmf}\ (K\ y)\ t\ \partial \textit{count-space}\ \Omega\ \partial N) = (\int y.\ 1\ \partial N)$
by (*intro integral-cong-AE*)
(*auto simp*: *AE-measure-pmf-iff integral-pmf* Ω*-def intro*!: *measure-pmf.prob-eq-1*[*THEN iffD2*])

finally have *False*
  using *measure-pmf.prob-space*[*of N*] **by** (*simp add*: *integral-pmf field-simps not-le*[*symmetric*]) **}**
then show *pmf N i* $\leq$ *pmf* (*bind-pmf N K*) *i*
  by (*auto simp*: *pmf-bind not-le*[*symmetric*])
qed

lemma *stationary-distribution-iterate*:
  assumes *N*: *stationary-distribution N*
  shows *ennreal* (*pmf N y*) = ($\int^+ x.$ *p x y n* ∂N)
proof (*induct n arbitrary*: *y*)
  have [*simp*]: $\bigwedge x\ y.$ *ennreal* (*if x = y then 1 else 0*) = *indicator* {*y*} *x*
    by *simp*
  case *0* **then show** *?case*
    by (*simp add*: *p-0 pmf.rep-eq measure-pmf.emeasure-eq-measure*)
next
  case (*Suc n*) **with** *N* **show** *?case*
    apply (*simp add*: *nn-integral-eq-integral*[*symmetric*] *p-le-1 p-Suc′*
        measure-pmf.integrable-const-bound[**where** *B=1*])
    apply (*subst nn-integral-bind*[*symmetric*, **where** *B=count-space UNIV*])
    apply (*auto simp*: *stationary-distribution-def measure-pmf-bind*[*symmetric*]
        simp del: *measurable-pmf-measure1*)
    done
qed

lemma *stationary-distribution-iterate′*:
  assumes *stationary-distribution N*
  shows *measure N* {*y*} = ($\int x.$ *p x y n* ∂N)
  using *stationary-distribution-iterate*[*OF assms*]
  by (*subst* (*asm*) *nn-integral-eq-integral*)
    (*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B=1*] *simp*: *p-le-1 pmf.rep-eq*)

lemma *stationary-distributionD*:
  assumes *C*: *essential-class C countable C*
  assumes *N*: *stationary-distribution N N* $\subseteq$ *C*
  shows $\forall x \in C.$ *pos-recurrent x measure-pmf N = stat C*
proof −
  have *integrable-K*: $\bigwedge f\ x.$ *integrable N* ($\lambda s.$ *pmf* (*K s*) (*f x*))
   by (*rule measure-pmf.integrable-const-bound*[**where** *B=1*]) (*simp-all add*: *pmf-le-1*)

  have *measure-C*: *measure N C = 1* **and** *ae-C*: *AE x in N*. $x \in C$
    using *N C measure-pmf.prob-eq-1*[*of C*] **by** (*auto simp*: *AE-measure-pmf-iff*)

  have *integrable-p*: $\bigwedge n\ y.$ *integrable N* ($\lambda x.$ *p x y n*)
   by (*rule measure-pmf.integrable-const-bound*[**where** *B=1*]) (*simp-all add*: *p-le-1*)

  { fix *e* :: *real* **assume** *0 < e*
    then have [*simp*]: *0* $\leq$ *e* **by** *simp*

```
    have ∃ A⊆C. finite A ∧ 1 − e < measure N A
    proof (rule ccontr)
      assume contr: ¬ (∃ A ⊆ C. finite A ∧ 1 − e < measure N A)
      from finite-sequence-to-countable-set[OF ‹countable C›]
       obtain F where F: ⋀i. F i ⊆ C ⋀i. F i ⊆ F (Suc i) ⋀i. finite (F i) ⋃
(range F) = C
        by blast
      then have *: (λn. measure N (F n)) ⟶ measure N (⋃ i. F i)
       by (intro measure-pmf.finite-Lim-measure-incseq) (auto simp: incseq-Suc-iff)
      with F contr have measure N (⋃ i. F i) ≤ 1 − e
        by (intro LIMSEQ-le[OF * tendsto-const]) (auto simp: not-less)
      with F ‹0 < e› show False
        by (simp add: measure-C)
    qed
    then obtain A where A ⊆ C finite A and e: 1 − e < measure N A by auto

    { fix y n assume y ∈ C
      from N(1) have measure N {y} = (∫ x. p x y n ∂N)
        by (rule stationary-distribution-iterate')
      also have ... ≤ (∫ x. p x y n * indicator A x + indicator (C − A) x ∂N)
        using ae-C ‹A ⊆ C›
        by (intro integral-mono-AE)
           (auto elim!: eventually-mono
              intro!: integral-add integral-indicator p-le-1 integrable-real-mult-indicator
                 integrable-add
               split: split-indicator simp: integrable-p less-top[symmetric] top-unique)
      also have ... = (∫ x. p x y n * indicator A x ∂N) + measure N (C − A)
        using ae-C ‹A ⊆ C›
        apply (subst Bochner-Integration.integral-add)
        apply (auto elim!: eventually-mono
             intro!: integral-add integral-indicator p-le-1 integrable-real-mult-indicator
              split: split-indicator simp: integrable-p less-top[symmetric] top-unique)
        done
      also have ... ≤ (∫ x. p x y n * indicator A x ∂N) + e
       using e ‹A ⊆ C› by (simp add: measure-pmf.finite-measure-Diff measure-C)
      finally have measure N {y} ≤ (∫ x. p x y n * indicator A x ∂N) + e .
      then have emeasure N {y} ≤ ennreal (∫ x. p x y n * indicator A x ∂N) + e
         by (simp add: measure-pmf.emeasure-eq-measure ennreal-plus[symmetric]
del: ennreal-plus)
      also have ... = (∫+x. ennreal (p x y n) * indicator A x ∂N) + e
        by (subst nn-integral-eq-integral[symmetric])
           (auto intro!: measure-pmf.integrable-const-bound[where B=1]
                simp: abs-mult p-le-1 mult-le-one ennreal-indicator ennreal-mult)
      finally have emeasure N {y} ≤ (∫+x. ennreal (p x y n) * indicator A x ∂N)
+ e . }
    note v-le = this

    { fix y and z :: real assume y: y ∈ C and z: 0 < z z < 1
      have summable-int-p: summable (λn. (∫ x. p x y n * indicator A x ∂N) * (1
```

− *z*) ∗ *z* ^ *n*)
using ‹*y*∈*C*› *z* ‹*A* ⊆ *C*›
by (*auto intro*!: *summable-comparison-test*[*OF* - *summable-mult*[*OF summable-geometric*[*of z*], *of 1*]] *exI*[*of* - *0*] *mult-le-one*
measure-pmf.integral-le-const integrable-real-mult-indicator integrable-p AE-I2 p-le-1
simp: *abs-mult integral-nonneg-AE*)

from *y z* **have** *sums-y*: (λ*n*. *measure N* {*y*} ∗ (*1* − *z*) ∗ *z* ^ *n*) *sums measure N* {*y*}
using *sums-mult*[*OF geometric-sums*[*of z*], *of measure N* {*y*} ∗ (*1* − *z*)] **by** *simp*
then have *emeasure N* {*y*} = *ennreal* (∑ *n*. (*measure N* {*y*} ∗ (*1* − *z*)) ∗ *z* ^ *n*)
by (*auto simp add*: *sums-unique*[*symmetric*] *measure-pmf.emeasure-eq-measure*)
also have ... = (∑ *n*. *emeasure N* {*y*} ∗ (*1* − *z*) ∗ *z* ^ *n*)
using *z summable-mult*[*OF summable-geometric*[*of z*], *of measure-pmf.prob N* {*y*} ∗ (*1* − *z*)]
by (*subst suminf-ennreal*[*symmetric*])
(*auto simp*: *measure-pmf.emeasure-eq-measure ennreal-mult*[*symmetric*] *ennreal-suminf-neq-top*)
also have ... ≤ (∑ *n*. ((∫ ^+^*x*. *ennreal* (*p x y n*) ∗ *indicator A x* ∂*N*) + *e*) ∗ (*1* − *z*) ∗ *z* ^ *n*)
using ‹*y*∈*C*› *z* ‹*A* ⊆ *C*›
by (*intro suminf-le mult-right-mono v-le allI*)
(*auto simp*: *measure-pmf.emeasure-eq-measure*)
also have ... = (∑ *n*. (∫ ^+^*x*. *ennreal* (*p x y n*) ∗ *indicator A x* ∂*N*) ∗ (*1* − *z*) ∗ *z* ^ *n*) + *e*
using ‹*0* < *e*› *z sums-mult*[*OF geometric-sums*[*of z*], *of e* ∗ (*1* − *z*)] ‹*0*<*z*› ‹*z*<*1*›
by (*simp add*: *distrib-right suminf-add*[*symmetric*] *ennreal-suminf-cmult*[*symmetric*]
ennreal-mult[*symmetric*] *suminf-ennreal-eq sums-unique*[*symmetric*]
del: *ennreal-suminf-cmult*)
also have ... = (∑ *n*. *ennreal* (*1* − *z*) ∗ ((∫ ^+^*x*. *ennreal* (*p x y n*) ∗ *indicator A x* ∂*N*) ∗ *z* ^ *n*)) + *e*
by (*simp add*: *ac-simps*)
also have ... = *ennreal* (*1* − *z*) ∗ (∑ *n*. ((∫ ^+^*x*. *ennreal* (*p x y n*) ∗ *indicator A x* ∂*N*) ∗ *z* ^ *n*)) + *e*
using *z* **by** (*subst ennreal-suminf-cmult*) *simp-all*
also have (∑ *n*. ((∫ ^+^*x*. *ennreal* (*p x y n*) ∗ *indicator A x* ∂*N*) ∗ *z* ^ *n*)) =
(∑ *n*. (∫ ^+^*x*. *ennreal* (*p x y n* ∗ *z* ^ *n*) ∗ *indicator A x* ∂*N*))
using *z* **by** (*simp add*: *ac-simps nn-integral-cmult*[*symmetric*] *ennreal-mult*)
also have ... = (∫ ^+^*x*. *ennreal* (*gf-G x y z*) ∗ *indicator A x* ∂*N*)
using *z*
apply (*subst nn-integral-suminf*[*symmetric*])
apply (*auto simp add*: *gf-G-def simp del*: *suminf-ennreal*
intro!: *ennreal-mult-right-cong suminf-ennreal2 nn-integral-cong*)
apply (*intro summable-comparison-test*[*OF* - *summable-mult*[*OF summable-geometric*[*of z*], *of 1*]] *impI*)

apply (*simp-all add*: *abs-mult p-le-1 mult-le-one power-le-one split*: *split-indicator*)
done
also have ... = ($\int^+ x$. *ennreal* (*gf-F x y z* ∗ *gf-G y y z*) ∗ *indicator A x* ∂*N*)
using *z* **by** (*intro nn-integral-cong*) (*simp add*: *gf-G-eq-gf-F*[*symmetric*])
also have ... = *ennreal* (*gf-G y y z*) ∗ ($\int^+ x$. *ennreal* (*gf-F x y z*) ∗ *indicator A x* ∂*N*)
using *z* **by** (*subst nn-integral-cmult*[*symmetric*]) (*simp-all add*: *gf-G-nonneg gf-F-nonneg ac-simps ennreal-mult*)
also have ... = *ennreal* (*1* / (*1* − *gf-U y y z*)) ∗ ($\int^+ x$. *ennreal* (*gf-F x y z*) ∗ *indicator A x* ∂*N*)
using *z* ‹*y* ∈ *C*› **by** (*subst gf-G-eq-gf-U*) (*auto intro*!: *convergence-G-less-1*)
finally have *emeasure N* {*y*} ≤ *ennreal* ((*1* − *z*) / (*1* − *gf-U y y z*)) ∗ ($\int^+ x$. *gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*
using *z*
by (*subst* (*asm*) *mult.assoc*[*symmetric*])
(*simp add*: *ennreal-indicator*[*symmetric*] *ennreal-mult*′[*symmetric*] *gf-F-nonneg*)
then have *measure N* {*y*} ≤ (*1* − *z*) / (*1* − *gf-U y y z*) ∗ ($\int x$. *gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*
using *z*
by (*subst* (*asm*) *nn-integral-eq-integral*[*OF measure-pmf.integrable-const-bound*[**where** *B*=*1*]])
(*auto simp*: *gf-F-nonneg gf-U-le-1 gf-F-le-1 measure-pmf.emeasure-eq-measure mult-le-one*
ennreal-mult′′[*symmetric*] *ennreal-plus*[*symmetric*]
simp del: *ennreal-plus*) **}**
then have ∃ *A* ⊆ *C*. *finite A* ∧ (∀ *y*∈*C*. ∀ *z*. *0* < *z* ⟶ *z* < *1* ⟶ *measure N* {*y*} ≤ (*1* − *z*) / (*1* − *gf-U y y z*) ∗ ($\int x$. *gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*)
using ‹*A* ⊆ *C*› ‹*finite A*› **by** *auto* **}**
note *eps* = *this*

{ fix *y A* **assume** *y* ∈ *C finite A A* ⊆ *C*
then have ((λ*z*. $\int x$. *gf-F x y z* ∗ *indicator A x* ∂*N*) ⟶ $\int x$. *F x y* ∗ *indicator A x* ∂*N*) (*at-left 1*)
by (*subst* (*1 2*) *integral-measure-pmf*[*of A*]) (*auto intro*!: *tendsto-intros gf-F simp*: *indicator-eq-0-iff*) **}**
note *int-gf-F* = *this*

have *all-recurrent*: ∀ *y*∈*C*. *recurrent y*
proof (*rule ccontr*)
assume ¬ (∀ *y*∈*C*. *recurrent y*)
then obtain *x* **where** *x* ∈ *C* ¬ *recurrent x* **by** *auto*
then have *transient*: ⋀*x*. *x* ∈ *C* ⟹ ¬ *recurrent x*
using *C* **by** (*auto simp*: *essential-class-def recurrent-iffI-communicating*[*symmetric*] *elim*!: *quotientE*)

{ fix *y* **assume** *y* ∈ *C*
with *transient* **have** *U y y* < *1*
by (*metis recurrent-iff-U-eq-1 U-cases*)
have *measure N* {*y*} ≤ *0*

proof (*rule dense-ge*)
fix *e* :: *real* **assume** *0 < e*
from *eps*[*OF this*] ‹*y* ∈ *C*› **obtain** *A* **where**
A: *finite A A* ⊆ *C* **and**
le: ⋀*z. 0 < z* ⟹ *z < 1* ⟹ *measure N* {*y*} ≤ (*1* − *z*) / (*1* − *gf-U y y z*) ∗ (∫ *x. gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*
by *auto*
have ((λ*z.* (*1* − *z*) / (*1* − *gf-U y y z*) ∗ (∫ *x. gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*) ⟶
(*1* − *1*) / (*1* − *U y y*) ∗ (∫ *x. F x y* ∗ *indicator A x* ∂*N*) + *e*) (*at-left* (*1*::*real*))
using *A* ‹*U y y < 1*› ‹*y* ∈ *C*› **by** (*intro tendsto-intros gf-U int-gf-F*) *auto*
then have *1*: ((λ*z.* (*1* − *z*) / (*1* − *gf-U y y z*) ∗ (∫ *x. gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*) ⟶ *e*) (*at-left* (*1*::*real*))
by *simp*
with *le* **show** *measure N* {*y*} ≤ *e*
by (*intro tendsto-le*[*OF trivial-limit-at-left-real - tendsto-const*])
(*auto simp*: *eventually-at-left-1*)
qed
then have *measure N* {*y*} = *0*
by (*intro antisym measure-nonneg*) **}**
then have *emeasure N C* = *0*
by (*subst emeasure-countable-singleton*) (*auto simp*: *measure-pmf.emeasure-eq-measure nn-integral-0-iff-AE ae-C C*)
then show *False*
using ‹*measure N C* = *1*› **by** (*simp add*: *measure-pmf.emeasure-eq-measure*)
qed
then have ⋀*x. x* ∈ *C* ⟹ *U x x* = *1*
by (*metis recurrent-iff-U-eq-1*)

{ fix *y* **assume** *y* ∈ *C*
then have *U y y* = *1 recurrent y*
using ‹*y* ∈ *C* ⟹ *U y y* = *1*› *all-recurrent* **by** *auto*
have *measure N* {*y*} ≤ *enn2real* (*1* / *U′ y y*)
proof (*rule field-le-epsilon*)
fix *e* :: *real* **assume** *0 < e*
from *eps*[*OF* ‹*0 < e*›] ‹*y* ∈ *C*› **obtain** *A* **where**
A: *finite A A* ⊆ *C* **and**
le: ⋀*z. 0 < z* ⟹ *z < 1* ⟹ *measure N* {*y*} ≤ (*1* − *z*) / (*1* − *gf-U y y z*) ∗ (∫ *x. gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*
by *auto*
let *?L* = *at-left* (*1*::*real*)
have ((λ*z.* (*1* − *z*) / (*1* − *gf-U y y z*) ∗ (∫ *x. gf-F x y z* ∗ *indicator A x* ∂*N*) + *e*) ⟶
enn2real (*1* / *U′ y y*) ∗ (∫ *x. F x y* ∗ *indicator A x* ∂*N*) + *e*) *?L*
proof (*intro tendsto-add tendsto-const tendsto-mult int-gf-F*,
rule lhopital-left[**where** *f′*=λ*x.* − *1* **and** *g′*=λ*z.* − *gf-U′ y y z*])
show ((−) *1* ⟶ *0*) *?L* ((λ*x. 1* − *gf-U y y x*) ⟶ *0*) *?L*
using *gf-U*[*of y y*] **by** (*auto intro*!: *tendsto-eq-intros simp*: ‹*U y y* = *1*›)

show *y* ∈ *C finite A A* ⊆ *C* **by** *fact+*
show *eventually* (λ*x*. *1* − *gf-U y y x* ≠ *0*) *?L*
using *gf-G-eq-gf-U*(*2*)[*OF convergence-G-less-1*, **where** *'z=real*] **by** (*auto intro*!: *eventually-at-left-1*)
show ((λ*x*. − *1* / − *gf-U' y y x*) ⟶ *enn2real* (*1* / *U' y y*)) *?L*
using ‹*recurrent y*› **by** (*rule inverse-gf-U'-tendsto*)
have *eventually* (λ*x*. *0* < *gf-U' y y x*) *?L*
by (*intro eventually-at-left-1 gf-U'-pos*) (*simp-all add*: ‹*U y y* = *1*›)
then show *eventually* (λ*x*. − *gf-U' y y x* ≠ *0*) *?L*
by *eventually-elim simp*
show *eventually* (λ*x*. *DERIV* (λ*x*. *1* − *gf-U y y x*) *x* :> − *gf-U' y y x*) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros DERIV-gf-U*)
show *eventually* (λ*x*. *DERIV* ((−) *1*) *x* :> − *1*) *?L*
by (*auto intro*!: *eventually-at-left-1 derivative-eq-intros*)
qed
then have *measure N* {*y*} ≤ *enn2real* (*1* / *U' y y*) ∗ (∫ *x*. *F x y* ∗ *indicator A x* ∂*N*) + *e*
by (*rule tendsto-le*[*OF trivial-limit-at-left-real - tendsto-const*]) (*intro eventually-at-left-1 le*)
then have *measure N* {*y*} − *e* ≤ *enn2real* (*1* / *U' y y*) ∗ (∫ *x*. *F x y* ∗ *indicator A x* ∂*N*)
by *simp*
also have . . . ≤ *enn2real* (*1* / *U' y y*)
using *A*
by (*intro mult-left-le measure-pmf.integral-le-const measure-pmf.integrable-const-bound*[**where** *B=1*])
(*auto simp*: *mult-le-one F-le-1 U'-def*)
finally show *measure N* {*y*} ≤ *enn2real* (*1* / *U' y y*) + *e*
by *simp*
qed }
note *measure-y-le* = *this*

show *pos*: ∀ *y*∈*C*. *pos-recurrent y*
proof (*rule ccontr*)
assume ¬ (∀ *y*∈*C*. *pos-recurrent y*)
then obtain *x* **where** *x*: *x* ∈ *C* ¬ *pos-recurrent x* **by** *auto*
{ **fix** *y* **assume** *y* ∈ *C*
with *x* **have** ¬ *pos-recurrent y*
using *C* **by** (*auto simp*: *essential-class-def pos-recurrent-iffI-communicating*[*symmetric*] *elim*!: *quotientE*)
with *all-recurrent* ‹*y* ∈ *C*› **have** *enn2real* (*1* / *U' y y*) = *0*
by (*simp add*: *pos-recurrent-def nn-integral-add*)
with *measure-y-le*[*OF* ‹*y* ∈ *C*›] **have** *measure N* {*y*} = *0*
by (*auto intro*!: *antisym simp*: *pos-recurrent-def*) }
then have *emeasure N C* = *0*
by (*subst emeasure-countable-singleton*) (*auto simp*: *C ae-C measure-pmf.emeasure-eq-measure nn-integral-0-iff-AE*)
then show *False*
using ‹*measure N C* = *1*› **by** (*simp add*: *measure-pmf.emeasure-eq-measure*)

qed

{ **fix** *A* :: *'s set* **assume** [*simp*]: *countable A*
have *emeasure N A* = ($\int^+$*x*. *emeasure N* {*x*} ∂*count-space A*)
by (*intro emeasure-countable-singleton*) *auto*
also have ... ≤ ($\int^+$*x*. *emeasure* (*stat C*) {*x*} ∂*count-space A*)
proof (*intro nn-integral-mono*)
fix *y* **assume** *y* ∈ *space* (*count-space A*)
show *emeasure N* {*y*} ≤ *emeasure* (*stat C*) {*y*}
proof *cases*
assume *y* ∈ *C*
with *pos* **have** *pos-recurrent y*
by *auto*
with *one-le-integral-t*[*of y*] **obtain** *r* **where** *r*: *U′ y y* = *ennreal r 1* ≤ *U′ y y* **and** [*simp*]: *0* ≤ *r*
by (*cases U′ y y*) (*auto simp*: *pos-recurrent-def nn-integral-add*)

from *measure-y-le*[*OF* ‹*y* ∈ *C*›]
have *emeasure N* {*y*} ≤ *ennreal* (*enn2real* (*1* / *U′ y y*))
by (*simp add*: *measure-pmf.emeasure-eq-measure*)
also have ... = *emeasure* (*stat C*) {*y*}
unfolding *stat-def* **using** ‹*y* ∈ *C*› *r*
by (*subst emeasure-point-measure-finite2*)
(*auto simp add*: *ennreal-1*[*symmetric*] *divide-ennreal inverse-ennreal inverse-eq-divide ennreal-mult*[*symmetric*]
simp del: *ennreal-1*)
finally show *emeasure N* {*y*} ≤ *emeasure* (*stat C*) {*y*}
by *simp*
next
assume *y* ∉ *C*
with *ae-C* **have** *emeasure N* {*y*} = *0*
by (*subst AE-iff-measurable*[*symmetric*, **where** *P*=λ*x*. *x* ≠ *y*]) (*auto elim*!: *eventually-mono*)
moreover have *emeasure* (*stat C*) {*y*} = *0*
using *emeasure-stat-not-C*[*OF* ‹*y* ∉ *C*›] **.**
ultimately show *?thesis* **by** *simp*
qed
qed
also have ... = *emeasure* (*stat C*) *A*
by (*intro emeasure-countable-singleton*[*symmetric*]) *auto*
finally have *emeasure N A* ≤ *emeasure* (*stat C*) *A* **.** }
note *N-le-C* = *this*

from *stat-subprob*[*OF C*(*1*) ‹*countable C*› *pos*] *N-le-C*[*OF* ‹*countable C*›] ‹*measure N C* = *1*›
have *stat-C-eq-1*: *emeasure* (*stat C*) *C* = *1*
by (*auto simp add*: *measure-pmf.emeasure-eq-measure one-ennreal-def*)
moreover have *emeasure* (*stat C*) (*UNIV* − *C*) = *0*
by (*subst AE-iff-measurable*[*symmetric*, **where** *P*=λ*x*. *x* ∈ *C*])

(*auto simp*: *stat-def AE-point-measure sets-point-measure space-point-measure*
    split: *split-indicator cong del*: *AE-cong*)
ultimately have *emeasure* (*stat C*) (*space* (*stat C*)) = *1*
  using *plus-emeasure*[*of C stat C UNIV* − *C*] **by** (*simp add*: *Un-absorb1*)
interpret *stat*: *prob-space stat C*
  by *standard fact*

show *measure-pmf N* = *stat C*
proof (*rule measure-eqI-countable-AE*)
  show *sets N* = *UNIV sets* (*stat C*) = *UNIV*
    by *auto*
  show *countable C AE x in N*. *x* ∈ *C* **and** *ae-stat*: *AE x in stat C*. *x* ∈ *C*
  using *C ae-C stat-C-eq-1* **by** (*auto intro*!: *stat.AE-prob-1 simp*: *stat.emeasure-eq-measure*)

  { assume ∃ *x*. *emeasure N* {*x*} ≠ *emeasure* (*stat C*) {*x*}
    then obtain *x* **where** [*simp*]: *emeasure N* {*x*} ≠ *emeasure* (*stat C*) {*x*} **by**
auto
    with *N-le-C*[*of* {*x*}] **have** *x*: *emeasure N* {*x*} < *emeasure* (*stat C*) {*x*}
      by (*auto simp*: *less-le*)
    have *1* = *emeasure N* {*x*} + *emeasure N* (*C* − {*x*})
      using *ae-C*
      by (*subst plus-emeasure*) (*auto intro*!: *measure-pmf.emeasure-eq-1-AE*)
    also have ... < *emeasure* (*stat C*) {*x*} + *emeasure* (*stat C*) (*C* − {*x*})
      using *x N-le-C*[*of C* − {*x*}] *C ae-C*
      by (*simp add*: *stat.emeasure-eq-measure measure-pmf.emeasure-eq-measure*
          ennreal-plus[*symmetric*] *ennreal-less-iff*
        del: *ennreal-plus*)
    also have ... = *1*
      using *ae-stat* **by** (*subst plus-emeasure*) (*auto intro*!: *stat.emeasure-eq-1-AE*)
    finally have *False* **by** *simp* **}**
  then show ⋀*x*. *emeasure N* {*x*} = *emeasure* (*stat C*) {*x*} **by** *auto*
qed
qed

lemma *measure-point-measure-singleton*:
  x ∈ *A* ⟹ *measure* (*point-measure A X*) {*x*} = *enn2real* (*X x*)
  unfolding *measure-def* **by** (*subst emeasure-point-measure-finite2*) *auto*

lemma *stationary-distribution-imp-int-t*:
  assumes *C*: *essential-class C countable C stationary-distribution N N* ⊆ *C*
  assumes *x*: *x* ∈ *C* **shows** *U′ x x* = *1* / *ennreal* (*pmf N x*)
proof −
  from *stationary-distributionD*[*OF C*]
  have *measure-pmf N* = *stat C* **and** ∗: ∀ *x*∈*C*. *pos-recurrent x* **by** *auto*
  show *?thesis*
    unfolding ‹*measure-pmf N* = *stat C*› *pmf.rep-eq stat-def*
    using ∗[*THEN bspec*, *OF x*] *x*
    apply (*simp add*: *measure-point-measure-singleton*)
    apply (*cases U′ x x*)

subgoal for *r*
by (*cases r* = *0*)
(*simp-all add*: *divide-ennreal-def inverse-ennreal*)
apply *simp*
done
qed

definition *period-set x* = {*i*. *0* < *i* ∧ *0* < *p x x i* }
definition *period C* = (*SOME d*. ∀ *x*∈*C*. *d* = *Gcd* (*period-set x*))

lemma *Gcd-period-set-invariant*:
assumes *c*: (*x*, *y*) ∈ *communicating*
shows *Gcd* (*period-set x*) = *Gcd* (*period-set y*)
proof −
{ fix *x y n* **assume** *c*: (*x*, *y*) ∈ *communicating x* ≠ *y* **and** *n*: *n* ∈ *period-set x*
from *c* **obtain** *l k* **where** *0* < *p x y l 0* < *p y x k*
by (*auto simp*: *communicating-def dest*!: *accD-pos*)
moreover with ‹*x* ≠ *y*› **have** *l* ≠ *0* ∧ *k* ≠ *0*
by (*intro notI conjI*) (*auto simp*: *p-0*)
ultimately have *pos*: *0* < *l 0* < *k* **and** *l*: *0* < *p x y l* **and** *k*: *0* < *p y x k*
by *auto*

from *mult-pos-pos*[*OF k l*] *prob-reachable-le*[*of k k* + *l y x y*] *c*
have *k-l*: *0* < *p y y* (*k* + *l*)
by *simp*
then have *Gcd* (*period-set y*) *dvd k* + *l*
using *pos* **by** (*auto intro*!: *Gcd-dvd-nat simp*: *period-set-def*)
moreover
from *n* **have** *0* < *p x x n 0* < *n* **by** (*auto simp*: *period-set-def*)
from *mult-pos-pos*[*OF k this*(*1*)] *prob-reachable-le*[*of k k* + *n y x x*] *c*
have *0* < *p y x* (*k* + *n*)
by *simp*
from *mult-pos-pos*[*OF this*(*1*) *l*] *prob-reachable-le*[*of k* + *n* (*k* + *n*) + *l y x y*] *c*
have *0* < *p y y* (*k* + *n* + *l*)
by *simp*
then have *Gcd* (*period-set y*) *dvd* (*k* + *l*) + *n*
using *pos* **by** (*auto intro*!: *Gcd-dvd-nat simp*: *period-set-def ac-simps*)
ultimately have *Gcd* (*period-set y*) *dvd n*
by (*metis dvd-add-left-iff add.commute*) **}**
note *this*[*of x y*] *this*[*of y x*] *c*
moreover have (*y*, *x*) ∈ *communicating*
using *c* **by** (*simp add*: *communicating-def*)
ultimately show *?thesis*
by (*auto intro*: *dvd-antisym Gcd-greatest Gcd-dvd*)
qed

lemma *period-eq*:
assumes *C* ∈ *UNIV* // *communicating x* ∈ *C*
shows *period C* = *Gcd* (*period-set x*)

unfolding *period-def*
using *assms*
by (*rule-tac someI2*[**where** *a*=*Gcd* (*period-set x*)])
(*auto intro*!: *Gcd-period-set-invariant irreducibleD*)

definition *aperiodic* $C \longleftrightarrow C \in UNIV$ // *communicating* $\wedge$ *period* $C = 1$

definition *not-ephemeral* $C \longleftrightarrow C \in UNIV$ // *communicating* $\wedge \neg (\exists x.\ C = \{x\} \wedge p\ x\ x\ 1 = 0)$

lemma *not-ephemeralD*:
assumes *C*: *not-ephemeral* $C\ x \in C$
shows $\exists n>0.\ 0 < p\ x\ x\ n$
proof *cases*
assume $\exists x.\ C = \{x\}$
with ‹$x \in C$› **have** $C = \{x\}$ **by** *auto*
with *C p-nonneg*[*of x x 1*] **have** $0 < p\ x\ x\ 1$
by (*auto simp*: *not-ephemeral-def less-le*)
with ‹$C = \{x\}$› **show** *?thesis* **by** *auto*
next
from *C* **have** *irr*: $C \in UNIV$ // *communicating*
by (*auto simp*: *not-ephemeral-def*)
assume $\neg(\exists x.\ C = \{x\})$
then have $\forall x.\ C \neq \{x\}$ **by** *auto*
with ‹$x \in C$› **obtain** *y* **where** $y \in C\ x \neq y$
by *blast*
with *irreducibleD*[*OF irr*, *of x y*] *C* ‹$x \in C$› **have** *c*: $(x, y) \in$ *communicating* **by** *auto*
with *accD-pos*[*of x y*] *accD-pos*[*of y x*]
obtain *k l* **where** *pos*: $0 < p\ x\ y\ k\ 0 < p\ y\ x\ l$
by (*auto simp*: *communicating-def*)
with ‹$x \neq y$› **have** $l \neq 0$
by (*intro notI*) (*auto simp*: *p-0*)
have $0 < p\ x\ y\ k * p\ y\ x\ (k + l - k)$
using *pos* **by** *auto*
also have $p\ x\ y\ k * p\ y\ x\ (k + l - k) \leq p\ x\ x\ (k + l)$
using *prob-reachable-le*[*of k k + l x y x*] *c* **by** *auto*
finally show *?thesis*
using ‹$l \neq 0$› ‹$x \in C$› **by** (*auto intro*!: *exI*[*of - k + l*])
qed

lemma *not-ephemeralD-pos-period*:
assumes *C*: *not-ephemeral C*
shows $0 <$ *period C*
proof −
from *C not-empty-irreducible*[*of C*] **obtain** *x* **where** $x \in C$
by (*auto simp*: *not-ephemeral-def*)
from *not-ephemeralD*[*OF C this*]
obtain *n* **where** *n*: $0 < p\ x\ x\ n\ 0 < n$ **by** *auto*

have *C′*: *C* ∈ *UNIV* // *communicating*
  using *C* **by** (*auto simp*: *not-ephemeral-def*)

have *period C* ≠ *0*
  unfolding *period-eq* [*OF C′* ‹*x* ∈ *C*›]
  using *n* **by** (*auto simp*: *period-set-def*)
then show *?thesis* **by** *auto*
qed

lemma *period-posD*:
  assumes *C*: *C* ∈ *UNIV* // *communicating* **and** *0* < *period C x* ∈ *C*
  shows ∃ *n*>*0*. *0* < *p x x n*
proof −
  from ‹*0* < *period C*› **have** *period C* ≠ *0*
    by *auto*
  then show *?thesis*
    unfolding *period-eq* [*OF C* ‹*x* ∈ *C*›]
    unfolding *period-set-def* **by** *auto*
qed

lemma *not-ephemeralD-pos-period′*:
  assumes *C*: *C* ∈ *UNIV* // *communicating*
  shows *not-ephemeral C* ⟷ *0* < *period C*
proof (*auto dest*!: *not-ephemeralD-pos-period intro*: *C*)
  from *C not-empty-irreducible*[*of C*] **obtain** *x* **where** *x* ∈ *C*
    by (*auto simp*: *not-ephemeral-def*)

  assume *0* < *period C*
  then show *not-ephemeral C*
    apply (*auto simp*: *not-ephemeral-def C*)
oops — should be easy to finish

lemma *eventually-periodic*:
  assumes *C*: *C* ∈ *UNIV* // *communicating 0* < *period C x* ∈ *C*
  shows *eventually* (λ*m*. *0* < *p x x* (*m* ∗ *period C*)) *sequentially*
proof −
  from *period-posD*[*OF assms*] **obtain** *n* **where** *n*: *0* < *p x x n 0* < *n* **by** *auto*
  have *C′*: *C* ∈ *UNIV* // *communicating*
    using *C* **by** *auto*

  have *period C* ≠ *0*
    unfolding *period-eq* [*OF C′* ‹*x* ∈ *C*›]
    using *n* **by** (*auto simp*: *period-set-def*)
  have *eventually* (λ*m*. *m* ∗ *Gcd* (*period-set x*) ∈ (*period-set x*)) *sequentially*
  proof (*rule eventually-mult-Gcd*)
    show *n* > *0 n* ∈ *period-set x*
      using *n* **by** (*auto simp add*: *period-set-def*)

fix *k l* **assume** *k* ∈ *period-set x l* ∈ *period-set x*
then have *0* < *p x x k* ∗ *p x x l 0* < *l 0* < *k*
by (*auto simp*: *period-set-def*)
moreover have *p x x k* ∗ *p x x l* ≤ *p x x* (*k* + *l*)
using *prob-reachable-le*[*of k k* + *l x x x*] ‹*x* ∈ *C*›
by *auto*
ultimately show *k* + *l* ∈ *period-set x*
using ‹*0* < *l*› **by** (*auto simp*: *period-set-def*)
qed
with *eventually-ge-at-top*[*of 1*] **show** *eventually* (*λm. 0* < *p x x* (*m* ∗ *period C*)) *sequentially*
by *eventually-elim*
(*insert* ‹*period C* ≠ *0*› *period-eq*[*OF C′* ‹*x* ∈ *C*›, *symmetric*], *auto simp*: *period-set-def*)
qed

lemma *aperiodic-eventually-recurrent*:
aperiodic C ⟷ *C* ∈ *UNIV* // *communicating* ∧ (∀ *x*∈*C*. *eventually* (*λm. 0* < *p x x m*) *sequentially*)
proof *safe*
fix *x* **assume** *x* ∈ *C aperiodic C*
with *eventually-periodic*[*of C x*]
show *eventually* (*λm. 0* < *p x x m*) *sequentially*
by (*auto simp add*: *aperiodic-def*)
next
assume ∀ *x*∈*C*. *eventually* (*λm. 0* < *p x x m*) *sequentially* **and** *C*: *C* ∈ *UNIV* // *communicating*
moreover from *not-empty-irreducible*[*OF C*] **obtain** *x* **where** *x* ∈ *C* **by** *auto*
ultimately obtain *N* **where** ⋀*M*. *M*≥*N* ⟹ *0* < *p x x M*
by (*auto simp*: *eventually-sequentially*)
then have {*N* <..} ⊆ *period-set x*
by (*auto simp*: *period-set-def*)
from *C* **show** *aperiodic C*
unfolding *period-eq* [*OF C* ‹*x* ∈ *C*›] *aperiodic-def*
proof
show *Gcd* (*period-set x*) = *1*
proof (*rule Gcd-eqI*)
from *one-dvd* **show** *1 dvd q* **for** *q* :: *nat* **.**
fix *m*
assume ⋀*q*. *q* ∈ *period-set x* ⟹ *m dvd q*
moreover from ‹{*N* <..} ⊆ *period-set x*›
have {*Suc N*, *Suc* (*Suc N*)} ⊆ *period-set x*
by *auto*
ultimately have *m dvd Suc* (*Suc N*) **and** *m dvd Suc N*
by *auto*
then have *m dvd Suc* (*Suc N*) − *Suc N*
by (*rule dvd-diff-nat*)
then show *is-unit m*

by *simp*
qed *simp*
qed
qed (*simp add*: *aperiodic-def*)

lemma *stationary-distributionD-emeasure*:
assumes *N*: *stationary-distribution N*
shows *emeasure N A* = ($\int^+$*s*. *emeasure* (*K s*) *A* ∂*N*)
proof −
have *prob-space* (*measure-pmf N*)
by *intro-locales*
then interpret *subprob-space measure-pmf N*
by (*rule prob-space-imp-subprob-space*)
show *?thesis*
unfolding *measure-pmf.emeasure-eq-measure*
apply (*subst N*[*unfolded stationary-distribution-def*])
apply (*simp add*: *measure-pmf-bind*)
apply (*subst measure-pmf.measure-bind*[**where** *N*=*count-space UNIV*])
apply (*rule measurable-compose*[*OF - measurable-measure-pmf*])
apply (*auto intro*!: *nn-integral-eq-integral*[*symmetric*] *measure-pmf.integrable-const-bound*[**where** *B*=*1*])
done
qed

lemma *communicatingD1*:
C ∈ *UNIV* // *communicating* ⟹ (*a*, *b*) ∈ *communicating* ⟹ *a* ∈ *C* ⟹ *b* ∈ *C*
by (*auto elim*!: *quotientE*) (*auto simp add*: *communicating-def*)

lemma *communicatingD2*:
C ∈ *UNIV* // *communicating* ⟹ (*a*, *b*) ∈ *communicating* ⟹ *b* ∈ *C* ⟹ *a* ∈ *C*
by (*auto elim*!: *quotientE*) (*auto simp add*: *communicating-def*)

lemma *acc-iff*: (*x*, *y*) ∈ *acc* ⟷ (∃ *n*. *0* < *p x y n*)
by (*blast intro*: *accD-pos accI-pos*)

lemma *communicating-iff*: (*x*, *y*) ∈ *communicating* ⟷ (∃ *n*. *0* < *p x y n*) ∧ (∃ *n*. *0* < *p y x n*)
by (*auto simp add*: *acc-iff communicating-def*)

end

context *MC-pair*
begin

lemma *p-eq-p1-p2*:
p (*x1*, *x2*) (*y1*, *y2*) *n* = *K1.p x1 y1 n* ∗ *K2.p x2 y2 n*
unfolding *p-def K1.p-def K2.p-def*

by (*subst prod-eq-prob-T*)
(*auto intro*!: *arg-cong2*[**where** *f*=*measure*] *split*: *nat.splits simp*: *Stream-snth*)

lemma *P-accD*:
assumes ((*x1*, *x2*), (*y1*, *y2*)) ∈ *acc***shows** (*x1*, *y1*) ∈ *K1.acc* (*x2*, *y2*) ∈ *K2.acc*
using *assms* **by** (*auto simp*: *acc-iff K1.acc-iff K2.acc-iff p-eq-p1-p2 zero-less-mult-iff not-le*[*of 0*, *symmetric*]
cong: *conj-cong*)

lemma *aperiodicI-pair*:
assumes *C1*: *K1.aperiodic C1* **and** *C2*: *K2.aperiodic C2*
shows *aperiodic* (*C1* × *C2*)
unfolding *aperiodic-eventually-recurrent*
proof *safe*
from *C1*[*unfolded K1.aperiodic-eventually-recurrent*] *C2*[*unfolded K2.aperiodic-eventually-recurrent*]
have *C1*: *C1* ∈ *UNIV* // *K1.communicating* **and** *C2*: *C2* ∈ *UNIV* // *K2.communicating* **and**
ev: ⋀*x*. *x* ∈ *C1* ⟹ *eventually* (λ*m*. *0* < *K1.p x x m*) *sequentially* ⋀*x*. *x* ∈ *C2* ⟹ *eventually* (λ*m*. *0* < *K2.p x x m*) *sequentially*
by *auto*
{ **fix** *x1 x2* **assume** *x*: *x1* ∈ *C1 x2* ∈ *C2*
from *ev*(*1*)[*OF x*(*1*)] *ev*(*2*)[*OF x*(*2*)]
show *eventually* (λ*m*. *0* < *p* (*x1*, *x2*) (*x1*, *x2*) *m*) *sequentially*
by *eventually-elim* (*simp add*: *p-eq-p1-p2 x*) }

{ **fix** *x1 x2 y1 y2*
assume *acc*: (*x1*, *y1*) ∈ *K1.acc* (*x2*, *y2*) ∈ *K2.acc x1* ∈ *C1 y1* ∈ *C1 x2* ∈ *C2 y2* ∈ *C2*
then obtain *k l* **where** *0* < *K1.p x1 y1 l 0* < *K2.p x2 y2 k*
by (*auto dest*!: *K1.accD-pos K2.accD-pos*)
with *acc ev*(*1*)[*of y1*] *ev*(*2*)[*of y2*]
have *eventually* (λ*m*. *0* < *K1.p x1 y1 l* ∗ *K1.p y1 y1 m* ∧ *0* < *K2.p x2 y2 k* ∗ *K2.p y2 y2 m*) *sequentially*
by (*auto elim*: *eventually-elim2*)
then have *eventually* (λ*m*. *0* < *K1.p x1 y1* (*m* + *l*) ∧ *0* < *K2.p x2 y2* (*m* + *k*)) *sequentially*
proof *eventually-elim*
fix *m* **assume** *0* < *K1.p x1 y1 l* ∗ *K1.p y1 y1 m* ∧ *0* < *K2.p x2 y2 k* ∗ *K2.p y2 y2 m*
with *acc*
K1.prob-reachable-le[*of l l* + *m x1 y1 y1*]
K2.prob-reachable-le[*of k k* + *m x2 y2 y2*]
show *0* < *K1.p x1 y1* (*m* + *l*) ∧ *0* < *K2.p x2 y2* (*m* + *k*)
by (*auto simp add*: *ac-simps*)
qed
then have *eventually* (λ*m*. *0* < *K1.p x1 y1 m* ∧ *0* < *K2.p x2 y2 m*) *sequentially*
unfolding *eventually-conj-iff* **by** (*subst* (*asm*) (*1 2*) *eventually-sequentially-seg*) (*auto elim*: *eventually-elim2*)
then obtain *N* **where** *0* < *K1.p x1 y1 N 0* < *K2.p x2 y2 N*

```
      by (auto simp: eventually-sequentially)
    with acc have 0 < p (x1, x2) (y1, y2) N
      by (auto simp add: p-eq-p1-p2)
    with acc have ((x1, x2), (y1, y2)) ∈ acc
      by (auto intro!: accI-pos) }
  note 1 = this

  { fix x1 x2 y1 y2 assume acc:((x1, x2), (y1, y2)) ∈ acc
     moreover from acc obtain k where 0 < p (x1, x2) (y1, y2) k by (auto
dest!: accD-pos)
    ultimately have (x1, y1) ∈ K1.acc ∧ (x2, y2) ∈ K2.acc
      by (subst (asm) p-eq-p1-p2)
          (auto intro!: K1.accI-pos K2.accI-pos simp: zero-less-mult-iff not-le[of 0,
symmetric]) }
  note 2 = this

  from K1.not-empty-irreducible[OF C1] K2.not-empty-irreducible[OF C2]
  obtain x1 x2 where xC: x1 ∈ C1 x2 ∈ C2 by auto
  show C1 × C2 ∈ UNIV // communicating
    apply (simp add: quotient-def Image-def)
    apply (safe intro!: exI[of - x1] exI[of - x2])
  proof −
    fix y1 y2 assume yC: y1 ∈ C1 y2 ∈ C2
     from K1.irreducibleD[OF C1 ‹x1 ∈ C1› ‹y1 ∈ C1›] K2.irreducibleD[OF C2
‹x2 ∈ C2› ‹y2 ∈ C2›]
    show ((x1, x2), (y1, y2)) ∈ communicating
      using 1[of x1 y1 x2 y2] 1[of y1 x1 y2 x2] xC yC
     by (auto simp: communicating-def K1.communicating-def K2.communicating-def)
  next
    fix y1 y2 assume ((x1, x2), (y1, y2)) ∈ communicating
    with 2[of x1 x2 y1 y2] 2[of y1 y2 x1 x2]
    have (x1, y1) ∈ K1.communicating (x2, y2) ∈ K2.communicating
     by (auto simp: communicating-def K1.communicating-def K2.communicating-def)
    with xC show y1 ∈ C1 y2 ∈ C2
      using K1.communicatingD1[OF C1] K2.communicatingD1[OF C2] by auto
  qed
qed

lemma stationary-distributionI-pair:
  assumes N1: K1.stationary-distribution N1
  assumes N2: K2.stationary-distribution N2
  shows stationary-distribution (pair-pmf N1 N2)
  unfolding stationary-distribution-def
  unfolding Kp-def pair-pmf-def
  apply (subst N1[unfolded K1.stationary-distribution-def])
  apply (subst N2[unfolded K2.stationary-distribution-def])
  apply (simp add: bind-assoc-pmf bind-return-pmf)
  apply (subst bind-commute-pmf[of N2])
  apply simp
```

done

end

context *MC-syntax*
begin

lemma *stationary-distribution-imp-limit*:
assumes *C*: *aperiodic C essential-class C countable C* **and** *N*: *stationary-distribution N N ⊆ C*
assumes [*simp*]: *y ∈ C*
shows (*λn. ∫ x. |p y x n − pmf N x| ∂count-space C*) ⟶ *0*
(**is** *?L* ⟶ *0*)
proof −
from ‹*essential-class C*› **have** *C-comm*: *C ∈ UNIV // communicating*
by (*simp add*: *essential-class-def*)

define *K′* **where** *K′* = (*λSome x ⇒ map-pmf Some* (*K x*) | *None ⇒ map-pmf Some N*)

interpret *K2*: *MC-syntax K′* **.**
interpret *KN*: *MC-pair K K′* **.**

from *stationary-distributionD*[*OF C*(*2*,*3*) *N*]
have *pos*: ⋀*x. x ∈ C* ⟹ *pos-recurrent x* **and** *measure-pmf N = stat C* **by** *auto*

have *pos*: ⋀*x. x ∈ C* ⟹ *0 < emeasure N* {*x*}
using *pos* **unfolding** *stat-def* ‹*measure-pmf N = stat C*›
by (*subst emeasure-point-measure-finite2*)
(*auto simp*: *U′-def pos-recurrent-def nn-integral-add ennreal-zero-less-divide less-top*)
then have *rpos*: ⋀*x. x ∈ C* ⟹ *0 < pmf N x*
by (*simp add*: *measure-pmf.emeasure-eq-measure pmf.rep-eq*)

have *eq*: ⋀*x y.* (*if x = y then 1 else 0*) = *indicator* {*y*} *x* **by** *auto*

have *intK*: ⋀*f x.* (∫ *x.* (*f x* :: *real*) ∂*K′* (*Some x*)) = (∫ *x. f* (*Some x*) ∂*K x*)
by (*simp add*: *K′-def integral-distr map-pmf-rep-eq*)

{ **fix** *m* **and** *x y* :: *′s*
have *K2.p* (*Some x*) (*Some y*) *m = p x y m*
by (*induct m arbitrary*: *x*)
(*auto intro*!: *integral-cong simp add*: *K2.p-Suc′ p-Suc′ intK K2.p-0 p-0*) }
note *K-p-eq = this*

{ **fix** *n* **and** *x* :: *′s* **have** *K2.p* (*Some x*) *None n = 0*
by (*induct n arbitrary*: *x*) (*auto simp*: *K2.p-Suc′ K2.p-0 intK cong*: *integral-cong*) }
note *K-S-None = this*

```
  from not-empty-irreducible[OF C-comm] obtain c0 where c0: c0 ∈ C by auto

  have K2-acc: ⋀x y. (Some x, y) ∈ K2.acc ⟷ (∃ z. y = Some z ∧ (x, z) ∈ acc)
    apply (auto simp: K2.acc-iff acc-iff K-p-eq)
    apply (case-tac y)
    apply (auto simp: K-p-eq K-S-None)
    done

  have K2-communicating: ⋀c x. c ∈ C ⟹ (Some c, x) ∈ K2.communicating
⟷ (∃ c'∈C. x = Some c')
  proof safe
    fix x c assume c ∈ C (Some c, x) ∈ K2.communicating
    then show ∃ c'∈C. x = Some c'
      by (cases x)
        (auto simp: communicating-iff K2.communicating-iff K-p-eq K-S-None intro!:
irreducibleD2[OF C-comm ‹c∈C›])
  next
    fix c c' x assume c ∈ C c' ∈ C
    with irreducibleD[OF C-comm this] show (Some c, Some c') ∈ K2.communicating
      by (auto simp: K2.communicating-iff communicating-iff K-p-eq)
  qed

  have Some ` C ∈ UNIV // K2.communicating
    by (auto simp add: quotient-def Image-def c0 K2-communicating
             intro!: exI[of - Some c0])
  then have K2.essential-class (Some ` C)
    by (rule K2.essential-classI)
       (auto simp: K2-acc essential-classD2[OF ‹essential-class C›])

  have K2.aperiodic (Some ` C)
    unfolding K2.aperiodic-eventually-recurrent
  proof safe
    fix x assume x ∈ C then show eventually (λm. 0 < K2.p (Some x) (Some
x) m) sequentially
      using ‹aperiodic C› unfolding aperiodic-eventually-recurrent
      by (auto elim!: eventually-mono simp: K-p-eq)
  qed fact
  then have aperiodic: KN.aperiodic (C × Some ` C)
    by (rule KN.aperiodicI-pair[OF ‹aperiodic C›])

  have KN-essential: KN.essential-class (C × Some ` C)
  proof (rule KN.essential-classI)
    show C × Some ` C ∈ UNIV // KN.communicating
      using aperiodic by (simp add: KN.aperiodic-def)
  next
    fix x y assume x ∈ C × Some ` C (x, y) ∈ KN.acc
    with KN.P-accD[of fst x snd x fst y snd y]
    show y ∈ C × Some ` C
```

by (*cases x y rule*: *prod.exhaust*[*case-product prod.exhaust*])
(*auto simp*: *K2-acc essential-classD2*[*OF* ‹*essential-class C*›])
qed

{ fix *n* **and** *x y* :: *'s*
have *measure N* {*y*} = $\mathcal{P}$(*ω in K2.T None*. (*None* ## *ω*) !! (*Suc n*) = *Some y*)
unfolding *stationary-distribution-iterate'*[*OF N*(*1*), *of y n*]
apply (*subst K2.p-def*[*symmetric*])
apply (*subst K2.p-Suc'*)
apply (*subst K'-def*)
apply (*simp add*: *map-pmf-rep-eq integral-distr K-p-eq*)
done
then have *measure N* {*y*} = $\mathcal{P}$(*ω in K2.T None*. *ω* !! *n* = *Some y*)
by *simp* **}**
note *measure-y-eq* = *this*

define *D* **where** *D* = {*x*::*'s* × *'s option*. *Some* (*fst x*) = *snd x*}

have [*measurable*]:
⋀*P*::(*'s* × *'s option* ⇒ *bool*). *P* ∈ *measurable* (*count-space UNIV*) (*count-space UNIV*)
by *simp*

{ fix *n* **and** *x* :: *'s*
have $\mathcal{P}$(*ω in KN.T* (*y*, *None*). ∃ *i*<*n*. *snd* (*ω* !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i ω*) =
(∑ *i*<*n*. $\mathcal{P}$(*ω in KN.T* (*y*, *None*). *snd* (*ω* !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i ω*))
by (*subst KN.T.finite-measure-finite-Union*[*symmetric*])
(*auto simp*: *disjoint-family-on-def intro*!: *arg-cong2*[**where** *f*=*measure*] *dest*: *ev-at-unique*)
also have ... = (∑ *i*<*n*. $\mathcal{P}$(*ω in KN.T* (*y*, *None*). *fst* (*ω* !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i ω*))
proof (*intro sum.cong refl*)
fix *i* **assume** *i*: *i* ∈ {..< *n*}
show $\mathcal{P}$(*ω in KN.T* (*y*, *None*). *snd* (*ω* !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i ω*) =
$\mathcal{P}$(*ω in KN.T* (*y*, *None*). *fst* (*ω* !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i ω*)
apply (*subst* (*1 2*) *KN.prob-T-split*[**where** *n*=*Suc i*])
apply (*simp-all add*: *ev-at-shift snth-Stream del*: *stake.simps KN.space-T*)
unfolding *ev-at-shift snth-Stream*
proof (*intro Bochner-Integration.integral-cong refl*)
fix *ω* :: (*'s* × *'s option*) *stream* **let** *?s* = *λω'*. *stake* (*Suc i*) *ω* @− *ω'*
show $\mathcal{P}$(*ω' in KN.T* (*ω* !! *i*). *snd* (*?s ω'* !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i ω*) =
$\mathcal{P}$(*ω' in KN.T* (*ω* !! *i*). *fst* (*?s ω'* !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i ω*)
proof *cases*
assume *ev-at* (*HLD D*) *i ω*

from *ev-at-imp-snth*[*OF this*]
have *eq*: *snd* (ω !! *i*) = *Some* (*fst* (ω !! *i*))
by (*simp add*: *D-def HLD-iff*)

have $\mathcal{P}$(ω′ *in KN.T* (ω !! *i*). *fst* (ω′ !! (*n* − *Suc i*)) = *x*) =
$\mathcal{P}$(ω′ *in T* (*fst* (ω !! *i*)). ω′ !! (*n* − *Suc i*) = *x*) ∗ $\mathcal{P}$(ω′ *in K2.T* (*snd* (ω !! *i*)). *True*)
by (*subst KN.prod-eq-prob-T*) *simp-all*
also have ... = *p* (*fst* (ω !! *i*)) *x* (*Suc* (*n* − *Suc i*))
using *K2.T.prob-space* **by** (*simp add*: *p-def*)
also have ... = *K2.p* (*snd* (ω !! *i*)) (*Some x*) (*Suc* (*n* − *Suc i*))
by (*simp add*: *K-p-eq eq*)
also have ... = $\mathcal{P}$(ω′ *in T* (*fst* (ω !! *i*)). *True*) ∗ $\mathcal{P}$(ω′ *in K2.T* (*snd* (ω !! *i*)). ω′ !! (*n* − *Suc i*) = *Some x*)
using *T.prob-space* **by** (*simp add*: *K2.p-def*)
also have ... = $\mathcal{P}$(ω′ *in KN.T* (ω !! *i*). *snd* (ω′ !! (*n* − *Suc i*)) = *Some x*)
by (*subst KN.prod-eq-prob-T*) *simp-all*
finally show *?thesis* **using** ‹*ev-at* (*HLD D*) *i* ω› *i*
by (*simp del*: *stake.simps*)
qed *simp*
qed
qed
also have ... = $\mathcal{P}$(ω *in KN.T* (*y*, *None*). (∃ *i*<*n*. *fst* (ω !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i* ω))
by (*subst KN.T.finite-measure-finite-Union*[*symmetric*])
(*auto simp add*: *disjoint-family-on-def dest*: *ev-at-unique*
intro!: *arg-cong2*[**where** *f*=*measure*])
finally have *eq*: $\mathcal{P}$(ω *in KN.T* (*y*, *None*). (∃ *i*<*n*. *snd* (ω !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i* ω)) =
$\mathcal{P}$(ω *in KN.T* (*y*, *None*). (∃ *i*<*n*. *fst* (ω !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i* ω)) **.**

have *p y x* (*Suc n*) − *measure N* {*x*} = $\mathcal{P}$(ω *in T y*. ω !! *n* = *x*) − $\mathcal{P}$(ω *in K2.T None*. ω !! *n* = *Some x*)
unfolding *p-def* **by** (*subst measure-y-eq*) *simp-all*
also have $\mathcal{P}$(ω *in T y*. ω !! *n* = *x*) = $\mathcal{P}$(ω *in T y*. ω !! *n* = *x*) ∗ $\mathcal{P}$(ω *in K2.T None*. *True*)
using *K2.T.prob-space* **by** *simp*
also have ... = $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x*)
by (*subst KN.prod-eq-prob-T*) *auto*
also have ... = $\mathcal{P}$(ω *in KN.T* (*y*, *None*). (∃ *i*<*n*. *fst* (ω !! *n*) = *x* ∧ *ev-at* (*HLD D*) *i* ω)) +
$\mathcal{P}$(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))
by (*subst KN.T.finite-measure-Union*[*symmetric*])
(*auto intro*!: *arg-cong2*[**where** *f*=*measure*])
also have $\mathcal{P}$(ω *in K2.T None*. ω !! *n* = *Some x*) = $\mathcal{P}$(ω *in T y*. *True*) ∗ $\mathcal{P}$(ω *in K2.T None*. ω !! *n* = *Some x*)
using *T.prob-space* **by** *simp*
also have ... = $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x*)

by (*subst KN.prod-eq-prob-T*) *auto*
also have … = 𝒫(ω *in KN.T* (*y*, *None*). (∃ *i*<*n*. *snd* (ω !! *n*) = *Some x* ∧ *ev-at* (*HLD D*) *i* ω)) +
𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))
by (*subst KN.T.finite-measure-Union*[*symmetric*])
(*auto intro*!: *arg-cong2*[**where** *f*=*measure*])
finally have | *p y x* (*Suc n*) − *measure N* {*x*} | =
| 𝒫(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) −
𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) |
unfolding *eq* **by** (*simp add*: *field-simps*)
also have … ≤ | 𝒫(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) | +
| 𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) |
by (*rule abs-triangle-ineq4*)
also have … ≤ 𝒫(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) +
𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))
by *simp*
finally have | *p y x* (*Suc n*) − *measure N* {*x*} | ≤ … . **}**
note *mono* = *this*

{ fix *n* :: *nat*
have (∫⁺*x*. | *p y x* (*Suc n*) − *measure N* {*x*} | ∂*count-space C*) ≤
(∫⁺*x*. *ennreal* (𝒫(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))) +
ennreal (𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))) ∂*count-space C*)
using *mono* **by** (*intro nn-integral-mono*) (*simp add*: *ennreal-plus*[*symmetric*] *del*: *ennreal-plus*)
also have … = (∫⁺*x*. 𝒫(ω *in KN.T* (*y*, *None*). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) ∂*count-space C*) +
(∫⁺*x*. 𝒫(ω *in KN.T* (*y*, *None*). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)) ∂*count-space C*)
by (*subst nn-integral-add*) *auto*
also have … = *emeasure* (*KN.T* (*y*, *None*)) (⋃*x*∈*C*. {ω∈*space* (*KN.T* (*y*, *None*)). *fst* (ω !! *n*) = *x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)}) +
emeasure (*KN.T* (*y*, *None*)) (⋃*x*∈*C*. {ω∈*space* (*KN.T* (*y*, *None*)). *snd* (ω !! *n*) = *Some x* ∧ ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)})
by (*subst* (*1 2*) *emeasure-UN-countable*)
(*auto simp add*: *disjoint-family-on-def KN.T.emeasure-eq-measure C*)
also have … ≤ *ennreal* (𝒫(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))) + *ennreal* (𝒫(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)))
unfolding *KN.T.emeasure-eq-measure*
by (*intro add-mono*) (*auto intro*!: *KN.T.finite-measure-mono*)

also have ... $\leq$ *2* $*$ $\mathcal{P}$(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))
by (*simp add*: *ennreal-plus*[*symmetric*] *del*: *ennreal-plus*)
finally have *?L* (*Suc n*) $\leq$ *2* $*$ $\mathcal{P}$(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))
by (*auto intro*!: *integral-real-bounded simp add*: *pmf.rep-eq*) **}**
note *le-2* = *this*

have *c0-D*: (*c0*, *Some c0*) ∈ *D*
by (*simp add*: *D-def c0*)

let *?N'* = *map-pmf Some N*
interpret *NP*: *pair-prob-space N ?N'* **..**

have *pos-recurrent*: ∀ *x*∈*C* × *Some* ‘ *C*. *KN.pos-recurrent x*
proof (*rule KN.stationary-distributionD*(*1*)[*OF KN-essential* - *KN.stationary-distributionI-pair*[*OF N*(*1*)]])
show *K2.stationary-distribution ?N'*
unfolding *K2.stationary-distribution-def*
by (*subst N*(*1*)[*unfolded stationary-distribution-def*])
(*auto intro*!: *bind-pmf-cong simp*: *K'-def map-pmf-def bind-assoc-pmf bind-return-pmf*)
show *countable* (*C* × *Some*‘*C*)
using *C* **by** *auto*
show *set-pmf* (*pair-pmf N* (*map-pmf Some N*)) ⊆ *C* × *Some* ‘ *C*
using ‹*N* ⊆ *C*› **by** *auto*
qed

from *c0-D* **have** $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *alw* (*not* (*HLD D*)) ω) $\leq$ $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *alw* (*not* (*HLD* {(*c0*, *Some c0*)})) ω)
apply (*auto intro*!: *KN.T.finite-measure-mono*)
apply (*rule alw-mono*, *assumption*)
apply (*auto simp*: *HLD-iff*)
done
also have ... = *0*
apply (*rule KN.T.prob-eq-0-AE*)
apply (*simp add*: *not-ev-iff*[*symmetric*])
apply (*subst KN.AE-T-iff*)
apply *simp*
proof
fix *t* **assume** *t*: *t* ∈ *KN.Kp* (*y*, *None*)
then obtain *a b* **where** *t-eq*: *t* = (*a*, *Some b*) *a* ∈ *K y b* ∈ *N*
unfolding *KN.Kp-def* **by** (*auto simp*: *K'-def*)
with ‹*y* ∈ *C*› **have** *a* ∈ *C*
using *essential-classD2*[*OF* ‹*essential-class C*› ‹*y* ∈ *C*›] **by** *auto*
have *b* ∈ *C*
using ‹*N* ⊆ *C*› ‹*b* ∈ *N*› **by** *auto*

from *pos-recurrent*[*THEN bspec*, *of* (*c0*, *Some c0*)]
have *recurrent-c0*: *KN.recurrent* (*c0*, *Some c0*)

by (*simp add*: *KN.pos-recurrent-def c0*)
have *C* × *Some* ‘ *C* ∈ *UNIV* // *KN.communicating*
using *aperiodic* **by** (*simp add*: *KN.aperiodic-def*)
then have ((*c0*, *Some c0*), *t*) ∈ *KN.communicating*
by (*rule KN.irreducibleD*) (*simp-all add*: *t-eq c0* ‹*b* ∈ *C*› ‹*a* ∈ *C*›)
then have ((*c0*, *Some c0*), *t*) ∈ *KN.acc*
by (*simp add*: *KN.communicating-def*)
then have *KN.U t* (*c0*, *Some c0*) = *1*
by (*rule KN.recurrent-acc*(*1*)[*OF recurrent-c0*])
then show *AE* ω *in KN.T t. ev* (*HLD* {(*c0*, *Some c0*)}) (*t* ## ω)
unfolding *KN.U-def* **by** (*subst* (*asm*) *KN.T.prob-Collect-eq-1*) (*auto simp add*: *ev-Stream*)
qed
finally have $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *alw* (*not* (*HLD D*)) ω) = *0*
by (*intro antisym measure-nonneg*)

have (λ*n*. $\mathcal{P}$(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))) ⟶
measure (*KN.T* (*y*, *None*)) (⋂ *n*. {ω∈*space* (*KN.T* (*y*, *None*)). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)})
by (*rule KN.T.finite-Lim-measure-decseq*) (*auto simp*: *decseq-def*)
also have (⋂ *n*. {ω∈*space* (*KN.T* (*y*, *None*)). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω)}) =
{ω∈*space* (*KN.T* (*y*, *None*)). *alw* (*not* (*HLD D*)) ω}
by (*auto simp*: *not-ev-iff*[*symmetric*] *ev-iff-ev-at*)
also have $\mathcal{P}$(ω *in KN.T* (*y*, *None*). *alw* (*not* (*HLD D*)) ω) = *0* **by** *fact*
finally have ∗: (λ*n*. *2* ∗ $\mathcal{P}$(ω *in KN.T* (*y*, *None*). ¬ (∃ *i*<*n*. *ev-at* (*HLD D*) *i* ω))) ⟶ *0*
by (*intro tendsto-eq-intros*) *auto*

show *?thesis*
apply (*rule LIMSEQ-imp-Suc*)
apply (*rule tendsto-sandwich*[*OF - - tendsto-const* ∗])
using *le-2*
apply (*simp-all add*: *integral-nonneg-AE*)
done
qed

lemma *stationary-distribution-imp-p-limit*:
assumes *aperiodic C essential-class C* **and** [*simp*]: *countable C*
assumes *N*: *stationary-distribution N N* ⊆ *C*
assumes [*simp*]: *x* ∈ *C y* ∈ *C*
shows *p x y* ⟶ *pmf N y*
proof −
define *D* **where** *D y n* = |*p x y n* − *pmf N y*| **for** *y n*

from *stationary-distribution-imp-limit*[*OF assms*(*1*,*2*,*3*,*4*,*5*,*6*)]
have *INT*: (λ*n*. ∫ *y*. *D y n* ∂*count-space C*) ⟶ *0*
unfolding *D-def* **.**

```
  { fix n
    have D y n ≤ (∫ z. D y n * indicator {y} z ∂count-space C)
      by simp
    also have ... ≤ (∫ y. D y n ∂count-space C)
      by (intro integral-mono)
         (auto split: split-indicator simp: D-def p-def disjoint-family-on-def
           intro!: Bochner-Integration.integrable-diff integrable-pmf T.integrable-measure)
    finally have D y n ≤ (∫ y. D y n ∂count-space C) . }
  note * = this

  have D-nonneg: ⋀n. 0 ≤ D y n by (simp add: D-def)

  have D y ⟶ 0
    by (rule tendsto-sandwich[OF - - tendsto-const INT])
       (auto simp: eventually-sequentially * D-nonneg)
  then show ?thesis
    using Lim-null[where l=pmf N y and net=sequentially and f=p x y]
    by (simp add: D-def [abs-def] tendsto-rabs-zero-iff)
qed

end

lemma (in MC-syntax) essential-classI2:
  assumes X ≠ {}
  assumes accI: ⋀x y. x ∈ X ⟹ y ∈ X ⟹ (x, y) ∈ acc
  assumes ED: ⋀x y. x ∈ X ⟹ y ∈ set-pmf (K x) ⟹ y ∈ X
  shows essential-class X
proof (rule essential-classI)
  { fix x y assume (x, y) ∈ acc x ∈ X
    then show y ∈ X
      by induct (auto dest: ED)}
  note accD = this
  from ‹X ≠ {}› obtain x where x ∈ X by auto
  from ‹x ∈ X› show X ∈ UNIV // communicating
    by (auto simp add: quotient-def Image-def communicating-def accI dest: accD
intro!: exI[of - x])
qed

end
```

5 Markov Decision Processes

```
theory Markov-Decision-Process
  imports Discrete-Time-Markov-Chain
begin

lemma some-elem-ne: s ≠ {} ⟹ some-elem s ∈ s
  unfolding some-elem-def by (auto intro: someI)
```

5.1 Configurations

We want to construct a *non-free* codatatype $'s\ cfg = Cfg\ (state{:}\ 's)\ (action{:}\ 's\ pmf)\ (cont{:}\ 's \Rightarrow 's\ cfg)$. with the restriction $state\ (cont\ cfg\ s) = s$

hide-const *cont*

codatatype *'s scheduler* = *Scheduler* (*action-sch*: *'s pmf*) (*cont-sch*: *'s* ⇒ *'s scheduler*)

lemma *equivp-rel-prod*: *equivp R* ⟹ *equivp Q* ⟹ *equivp* (*rel-prod R Q*)
  by (*auto intro*!: *equivpI prod.rel-symp prod.rel-transp prod.rel-reflp elim*: *equivpE*)

coinductive *eq-scheduler* :: *'s scheduler* ⇒ *'s scheduler* ⇒ *bool*
where
  ⋀*D*. *action-sch sc1* = *D* ⟹ *action-sch sc2* = *D* ⟹
    (∀ *s*∈*D*. *eq-scheduler* (*cont-sch sc1 s*) (*cont-sch sc2 s*)) ⟹ *eq-scheduler sc1 sc2*

lemma *eq-scheduler-refl*[*intro*]: *eq-scheduler sc sc*
  by (*coinduction arbitrary*: *sc*) *auto*

quotient-type *'s cfg* = *'s* × *'s scheduler* / *rel-prod* (=) *eq-scheduler*
proof (*intro equivp-rel-prod equivpI reflpI sympI transpI*)
  show *eq-scheduler sc1 sc2* ⟹ *eq-scheduler sc2 sc1* **for** *sc1 sc2* :: *'s scheduler*
    by (*coinduction arbitrary*: *sc1 sc2*) (*auto elim*: *eq-scheduler.cases*)
  show *eq-scheduler sc1 sc2* ⟹ *eq-scheduler sc2 sc3* ⟹ *eq-scheduler sc1 sc3*
    for *sc1 sc2 sc3* :: *'s scheduler*
    by (*coinduction arbitrary*: *sc1 sc2 sc3*)
      (*subst* (*asm*) (*1 2*) *eq-scheduler.simps*, *auto*)
qed *auto*

lift-definition *state* :: *'s cfg* ⇒ *'s* **is** *fst*
  by *auto*

lift-definition *action* :: *'s cfg* ⇒ *'s pmf* **is** λ(*s*, *sc*). *action-sch sc*
  by (*force elim*: *eq-scheduler.cases*)

lift-definition *cont* :: *'s cfg* ⇒ *'s* ⇒ *'s cfg* **is**
  λ(*s*, *sc*) *t*. *if t* ∈ *action-sch sc then* (*t*, *cont-sch sc t*) *else*
    (*t*, *cont-sch sc* (*some-elem* (*action-sch sc*)))
  apply (*simp add*: *rel-prod-conv split*: *prod.splits*)
  apply (*subst* (*asm*) *eq-scheduler.simps*)
  apply (*auto simp*: *Let-def set-pmf-not-empty*[*THEN some-elem-ne*])
  done

lift-definition *Cfg* :: *'s* ⇒ *'s pmf* ⇒ (*'s* ⇒ *'s cfg*) ⇒ *'s cfg* **is**
  λ*s D c*. (*s*, *Scheduler D* (λ*t*. *snd* (*c t*)))
  by (*auto simp*: *rel-prod-conv split-beta' eq-scheduler.simps*[*of Scheduler - -*])

lift-definition *cfg-corec* :: *'s* ⇒ (*'a* ⇒ *'s pmf*) ⇒ (*'a* ⇒ *'s* ⇒ *'a*) ⇒ *'a* ⇒ *'s cfg*

is
  λs D C x. (*s*, *corec-scheduler D* (*λx s. Inr* (*C x s*)) *x*) **.**

lemma *state-cont*[*simp*]: *state* (*cont cfg s*) = *s*
  by *transfer* (*simp split*: *prod.split*)

lemma *state-Cfg*[*simp*]: *state* (*Cfg s d′ c′*) = *s*
  by *transfer simp*

lemma *action-Cfg*[*simp*]: *action* (*Cfg s d′ c′*) = *d′*
  by *transfer simp*

lemma *cont-Cfg*[*simp*]: *t* ∈ *set-pmf d′* ⟹ *state* (*c′ t*) = *t* ⟹ *cont* (*Cfg s d′ c′*) *t* = *c′ t*
  by *transfer* (*auto simp add*: *rel-prod-conv split*: *prod.split*)

lemma *state-cfg-corec*[*simp*]: *state* (*cfg-corec s d c x*) = *s*
  by *transfer auto*

lemma *action-cfg-corec*[*simp*]: *action* (*cfg-corec s d c x*) = *d x*
  by *transfer auto*

lemma *cont-cfg-corec*[*simp*]: *t* ∈ *set-pmf* (*d x*) ⟹ *cont* (*cfg-corec s d c x*) *t* = *cfg-corec t d c* (*c x t*)
  by *transfer auto*

lemma *cfg-coinduct*[*consumes 1*, *case-names state action cont*, *coinduct pred*]:
  X c d ⟹ (⋀*c d. X c d* ⟹ *state c* = *state d*) ⟹ (⋀*c d. X c d* ⟹ *action c* = *action d*) ⟹
    (⋀*c d t. X c d* ⟹ *t* ∈ *set-pmf* (*action c*) ⟹ *X* (*cont c t*) (*cont d t*)) ⟹ *c* = *d*
proof (*transfer*, *clarsimp*)
  fix *X* :: (*′a* × *′a scheduler*) ⇒ (*′a* × *′a scheduler*) ⇒ *bool* **and** *B s1 s2 sc1 sc2*
  assume *X*: *X* (*s1*, *sc1*) (*s2*, *sc2*) **and** *rel-fun cr-cfg* (*rel-fun cr-cfg* (=)) *X B*
    and *1*: ⋀*s1 sc1 s2 sc2*. *X* (*s1*, *sc1*) (*s2*, *sc2*) ⟹ *s1* = *s2*
    and *2*: ⋀*s1 sc1 s2 sc2*. *X* (*s1*, *sc1*) (*s2*, *sc2*) ⟹ *action-sch sc1* = *action-sch sc2*
    and *3*: ⋀*s1 sc1 s2 sc2 t*. *X* (*s1*, *sc1*) (*s2*, *sc2*) ⟹ *t* ∈ *set-pmf* (*action-sch sc2*) ⟹
      X (*t*, *cont-sch sc1 t*) (*t*, *cont-sch sc2 t*)
  from *X* **show** *eq-scheduler sc1 sc2*
    by (*coinduction arbitrary*: *s1 s2 sc1 sc2*)
      (*blast dest*: *2 3*)
qed

coinductive *rel-cfg* :: (*′a* ⇒ *′b* ⇒ *bool*) ⇒ *′a cfg* ⇒ *′b cfg* ⇒ *bool* **for** *P* :: *′a* ⇒ *′b* ⇒ *bool*
where
  P (*state cfg1*) (*state cfg2*) ⟹

rel-pmf (*λs t. rel-cfg P* (*cont cfg1 s*) (*cont cfg2 t*)) (*action cfg1*) (*action cfg2*) ⟹
rel-cfg P cfg1 cfg2

lemma *rel-cfg-state*: *rel-cfg P cfg1 cfg2* ⟹ *P* (*state cfg1*) (*state cfg2*)
by (*auto elim*: *rel-cfg.cases*)

lemma *rel-cfg-cont*:
rel-cfg P cfg1 cfg2 ⟹
rel-pmf (*λs t. rel-cfg P* (*cont cfg1 s*) (*cont cfg2 t*)) (*action cfg1*) (*action cfg2*)
by (*auto elim*: *rel-cfg.cases*)

lemma *rel-cfg-action*:
assumes *P*: *rel-cfg P cfg1 cfg2* **shows** *rel-pmf P* (*action cfg1*) (*action cfg2*)
proof (*rule pmf.rel-mono-strong*)
show *rel-pmf* (*λs t. rel-cfg P* (*cont cfg1 s*) (*cont cfg2 t*)) (*action cfg1*) (*action cfg2*)
using *P* **by** (*rule rel-cfg-cont*)
qed (*auto dest*: *rel-cfg-state*)

lemma *rel-cfg-eq*: *rel-cfg* (=) *cfg1 cfg2* ⟷ *cfg1* = *cfg2*
proof *safe*
show *rel-cfg* (=) *cfg1 cfg2* ⟹ *cfg1* = *cfg2*
proof (*coinduction arbitrary*: *cfg1 cfg2*)
case *cont*
have *action cfg1* = *action cfg2*
using ‹*rel-cfg* (=) *cfg1 cfg2*› **by** (*auto dest*: *rel-cfg-action simp*: *pmf.rel-eq*)
then have *rel-pmf* (*λs t. rel-cfg* (=) (*cont cfg1 s*) (*cont cfg2 t*)) (*action cfg1*) (*action cfg1*)
using *cont* **by** (*auto dest*: *rel-cfg-cont*)
then have *rel-pmf* (*λs t. rel-cfg* (=) (*cont cfg1 s*) (*cont cfg2 t*) ∧ *s* = *t*) (*action cfg1*) (*action cfg1*)
by (*rule pmf.rel-mono-strong*) (*auto dest*: *rel-cfg-state*)
then have *pred-pmf* (*λs. rel-cfg* (=) (*cont cfg1 s*) (*cont cfg2 s*)) (*action cfg1*)
unfolding *pmf.pred-rel* **by** (*rule pmf.rel-mono-strong*) (*auto simp*: *eq-onp-def*)
with ‹*t* ∈ *action cfg1*› **show** *?case*
by (*auto simp*: *pmf.pred-set*)
qed (*auto dest*: *rel-cfg-state rel-cfg-action simp*: *pmf.rel-eq*)
show *rel-cfg* (=) *cfg2 cfg2*
by (*coinduction arbitrary*: *cfg2*) (*auto intro*!: *rel-pmf-reflI*)
qed

5.2 Configuration with Memoryless Scheduler

definition *memoryless-on f s* = *cfg-corec s f* (*λ- t. t*) *s*

lemma
shows *state-memoryless-on*[*simp*]: *state* (*memoryless-on f s*) = *s*
and *action-memoryless-on*[*simp*]: *action* (*memoryless-on f s*) = *f s*

and *cont-memoryless-on*[*simp*]: $t \in (f\ s) \Longrightarrow$ *cont* (*memoryless-on f s*) $t =$ *memoryless-on f t*
by (*simp-all add*: *memoryless-on-def*)

definition *K-cfg* :: *'s cfg* $\Rightarrow$ *'s cfg pmf* **where**
K-cfg cfg = *map-pmf* (*cont cfg*) (*action cfg*)

lemma *set-K-cfg*: *set-pmf* (*K-cfg cfg*) = *cont cfg* ‘ *set-pmf* (*action cfg*)
by (*simp add*: *K-cfg-def*)

lemma *nn-integral-K-cfg*: ($\int^+$*cfg*. *f cfg* ∂*K-cfg cfg*) = ($\int^+$*s*. *f* (*cont cfg s*) ∂*action cfg*)
by (*simp add*: *K-cfg-def map-pmf-rep-eq nn-integral-distr*)

5.3 MDP Kernel and Induced Configurations

locale *Markov-Decision-Process* =
fixes *K* :: *'s* $\Rightarrow$ *'s pmf set*
assumes *K-wf*: $\bigwedge s.\ K\ s \neq \{\}$
begin

definition *E* = (*SIGMA s*:*UNIV*. $\bigcup$*D*$\in$*K s*. *set-pmf D*)

coinductive *cfg-onp* :: *'s* $\Rightarrow$ *'s cfg* $\Rightarrow$ *bool* **where**
$\bigwedge$*s*. *state cfg* = *s* $\Longrightarrow$ *action cfg* $\in$ *K s* $\Longrightarrow$ ($\bigwedge$*t*. *t* $\in$ *action cfg* $\Longrightarrow$ *cfg-onp t* (*cont cfg t*)) $\Longrightarrow$
cfg-onp s cfg

definition *cfg-on s* = {*cfg*. *cfg-onp s cfg*}

lemma
shows *cfg-onD-action*[*intro*, *simp*]: *cfg* $\in$ *cfg-on s* $\Longrightarrow$ *action cfg* $\in$ *K s*
and *cfg-onD-cont*[*intro*, *simp*]: *cfg* $\in$ *cfg-on s* $\Longrightarrow$ *t* $\in$ *action cfg* $\Longrightarrow$ *cont cfg t* $\in$ *cfg-on t*
and *cfg-onD-state*[*simp*]: *cfg* $\in$ *cfg-on s* $\Longrightarrow$ *state cfg* = *s*
and *cfg-onI*: *state cfg* = *s* $\Longrightarrow$ *action cfg* $\in$ *K s* $\Longrightarrow$ ($\bigwedge$*t*. *t* $\in$ *action cfg* $\Longrightarrow$ *cont cfg t* $\in$ *cfg-on t*) $\Longrightarrow$ *cfg* $\in$ *cfg-on s*
by (*auto simp*: *cfg-on-def intro*: *cfg-onp.intros elim*: *cfg-onp.cases*)

lemma *cfg-on-coinduct*[*coinduct set*: *cfg-on*]:
assumes *P s cfg*
assumes $\bigwedge$*cfg s*. *P s cfg* $\Longrightarrow$ *state cfg* = *s*
assumes $\bigwedge$*cfg s*. *P s cfg* $\Longrightarrow$ *action cfg* $\in$ *K s*
assumes $\bigwedge$*cfg s t*. *P s cfg* $\Longrightarrow$ *t* $\in$ *action cfg* $\Longrightarrow$ *P t* (*cont cfg t*)
shows *cfg* $\in$ *cfg-on s*
using *assms cfg-onp.coinduct*[*of P s cfg*] **by** (*simp add*: *cfg-on-def*)

lemma *memoryless-on-cfg-onI*:
assumes $\bigwedge$*s*. *f s* $\in$ *K s*

shows *memoryless-on f s* $\in$ *cfg-on s*
by (*coinduction arbitrary*: *s*) (*auto intro*: *assms*)

lemma *cfg-of-cfg-onI*:
$D \in K\ s \Longrightarrow (\bigwedge t.\ t \in D \Longrightarrow c\ t \in \textit{cfg-on}\ t) \Longrightarrow \textit{Cfg}\ s\ D\ c \in \textit{cfg-on}\ s$
by (*rule cfg-onI*) *auto*

definition *arb-act s* = (*SOME D. D* $\in$ *K s*)

lemma *arb-actI*[*simp*]: *arb-act s* $\in$ *K s*
by (*simp add*: *arb-act-def some-in-eq K-wf*)

lemma *cfg-on-not-empty*[*intro*, *simp*]: *cfg-on s* $\neq$ {}
by (*auto intro*: *memoryless-on-cfg-onI arb-actI*)

sublocale *MC*: *MC-syntax K-cfg* **.**

abbreviation *St* :: *'s stream measure* **where**
St $\equiv$ *stream-space* (*count-space UNIV*)

5.4 Trace Space

definition *T cfg* = *distr* (*MC.T cfg*) *St* (*smap state*)

sublocale *T*: *prob-space T cfg* **for** *cfg*
by (*simp add*: *T-def MC.T.prob-space-distr*)

lemma *space-T*[*simp*]: *space* (*T cfg*) = *space St*
by (*simp add*: *T-def*)

lemma *sets-T*[*simp*]: *sets* (*T cfg*) = *sets St*
by (*simp add*: *T-def*)

lemma *measurable-T1*[*simp*]: *measurable* (*T cfg*) *N* = *measurable St N*
by (*simp add*: *T-def*)

lemma *measurable-T2*[*simp*]: *measurable N* (*T cfg*) = *measurable N St*
by (*simp add*: *T-def*)

lemma *nn-integral-T*:
assumes [*measurable*]: *f* $\in$ *borel-measurable St*
shows $(\int^+ X.\ f\ X\ \partial T\ cfg) = (\int^+ cfg'.\ (\int^+ x.\ f\ (\textit{state}\ cfg'\ \#\#\ x)\ \partial T\ cfg')\ \partial \textit{K-cfg}\ cfg)$
by (*simp add*: *T-def MC.nn-integral-T*[*of - cfg*] *nn-integral-distr*)

lemma *T-eq*:
$T\ cfg = (\textit{measure-pmf}\ (\textit{K-cfg}\ cfg) \gg\!\!= (\lambda cfg'.\ \textit{distr}\ (T\ cfg')\ St\ (\lambda\omega.\ \textit{state}\ cfg'\ \#\#\ \omega)))$
proof (*rule measure-eqI*)

fix *A* **assume** $A \in$ *sets* (*T cfg*)
then show *emeasure* (*T cfg*) *A* =
emeasure (*measure-pmf* (*K-cfg cfg*) $\gg=$ (λ*cfg'*. *distr* (*T cfg'*) *St* ($\lambda\omega$. *state cfg'* ## ω))) *A*
by (*subst emeasure-bind*[**where** *N*=*St*])
(*auto simp*: *space-subprob-algebra nn-integral-distr nn-integral-indicator*[*symmetric*] *nn-integral-T*[*of - cfg*]
simp del: *nn-integral-indicator intro*!: *prob-space-imp-subprob-space T.prob-space-distr*)
qed *simp*

lemma *T-memoryless-on*: *T* (*memoryless-on ct s*) = *MC-syntax.T ct s*
proof −
interpret *ct*: *MC-syntax ct* **.**
have $T \circ$ (*memoryless-on ct*) = *MC-syntax.T ct*
proof (*rule ct.T-bisim*[*symmetric*])
fix *s* **show** ($T \circ$ *memoryless-on ct*) *s* =
measure-pmf (*ct s*) $\gg=$ (λ*s*. *distr* (($T \circ$ *memoryless-on ct*) *s*) *St* ((##) *s*))
by (*auto simp add*: *T-eq*[*of memoryless-on ct s*] *K-cfg-def map-pmf-rep-eq bind-distr*[**where** *K*=*St*]
space-subprob-algebra T.prob-space-distr prob-space-imp-subprob-space
intro!: *bind-measure-pmf-cong*)
qed (*simp-all*, *intro-locales*)
then show *?thesis* **by** (*simp add*: *fun-eq-iff*)
qed

lemma *nn-integral-T-lfp*:
assumes [*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*: $\bigwedge$*s*. *sup-continuous* (*g s*)
assumes *int-g*: $\bigwedge$*f cfg*. $f \in$ *borel-measurable* (*stream-space* (*count-space UNIV*)) $\Longrightarrow$
($\int^+\omega$. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) ($\int^+\omega$. *f* ω ∂*T cfg*)
shows ($\int^+\omega$. *lfp* (λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))) ω ∂*T cfg*) =
lfp (λ*f cfg*. $\int^+$*t*. *g* (*state t*) (*f t*) ∂*K-cfg cfg*) *cfg*
proof (*rule nn-integral-lfp*)
show $\bigwedge$*s*. *sets* (*T s*) = *sets St*
$\bigwedge$*F*. $F \in$ *borel-measurable St* $\Longrightarrow$ (λ*a*. *g* (*shd a*) (*F* (*stl a*))) $\in$ *borel-measurable St*
by *auto*
next
fix *s* **and** *F* :: *'s stream* $\Rightarrow$ *ennreal* **assume** $F \in$ *borel-measurable St*
then show ($\int^+$ *a*. *g* (*shd a*) (*F* (*stl a*)) ∂*T s*) =
($\int^+$ *cfg*. *g* (*state cfg*) (*integral*^{N} (*T cfg*) *F*) ∂*K-cfg s*)
by (*rewrite nn-integral-T*) (*simp-all add*: *int-g*)
qed (*auto intro*!: *order-continuous-intros cont-g*[*THEN sup-continuous-compose*])

lemma *emeasure-Collect-T*:
assumes [*measurable*]: *Measurable.pred St P*

shows *emeasure* (*T cfg*) {*x*∈*space St*. *P x*} =
(∫⁺*cfg'*. *emeasure* (*T cfg'*) {*x*∈*space St*. *P* (*state cfg'* ## *x*)} ∂*K-cfg cfg*)
using *MC.emeasure-Collect-T*[*of* λ*x*. *P* (*smap state x*) *cfg*]
by (*simp add*: *nn-integral-distr emeasure-Collect-distr T-def*)

definition *E-sup* :: '*s* ⇒ ('*s stream* ⇒ *ennreal*) ⇒ *ennreal*
where
E-sup s f = (⨆*cfg*∈*cfg-on s*. ∫⁺*x*. *f x* ∂*T cfg*)

lemma *E-sup-const*: *0* ≤ *c* ⟹ *E-sup s* (λ-. *c*) = *c*
using *T.emeasure-space-1* **by** (*simp add*: *E-sup-def*)

lemma *E-sup-mult-right*:
assumes [*measurable*]: *f* ∈ *borel-measurable St* **and** [*simp*]: *0* ≤ *c*
shows *E-sup s* (λ*x*. *c* * *f x*) = *c* * *E-sup s f*
by (*simp add*: *nn-integral-cmult E-sup-def SUP-mult-left-ennreal*)

lemma *E-sup-mono*:
(⋀ω. *f* ω ≤ *g* ω) ⟹ *E-sup s f* ≤ *E-sup s g*
unfolding *E-sup-def* **by** (*intro SUP-subset-mono order-refl nn-integral-mono*)

lemma *E-sup-add*:
assumes [*measurable*]: *f* ∈ *borel-measurable St g* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *g x*) ≤ *E-sup s f* + *E-sup s g*
proof −
have *E-sup s* (λ*x*. *f x* + *g x*) = (⨆*cfg*∈*cfg-on s*. (∫⁺*x*. *f x* ∂*T cfg*) + (∫⁺*x*. *g x* ∂*T cfg*))
by (*simp add*: *E-sup-def nn-integral-add*)
also have ... ≤ (⨆*cfg*∈*cfg-on s*. ∫⁺*x*. *f x* ∂*T cfg*) + (⨆*cfg*∈*cfg-on s*. (∫⁺*x*. *g x* ∂*T cfg*))
by (*auto simp*: *SUP-le-iff intro*!: *add-mono SUP-upper*)
finally show *?thesis*
by (*simp add*: *E-sup-def*)
qed

lemma *E-sup-add-left*:
assumes [*measurable*]: *f* ∈ *borel-measurable St*
shows *E-sup s* (λ*x*. *f x* + *c*) = *E-sup s f* + *c*
by (*simp add*: *nn-integral-add E-sup-def T.emeasure-space-1*[*simplified*] *ennreal-SUP-add-left*)

lemma *E-sup-add-right*:
f ∈ *borel-measurable St* ⟹ *E-sup s* (λ*x*. *c* + *f x*) = *c* + *E-sup s f*
using *E-sup-add-left*[*of f s c*] **by** (*simp add*: *add.commute*)

lemma *E-sup-SUP*:
assumes [*measurable*]: ⋀*i*. *f i* ∈ *borel-measurable St* **and** [*simp*]: *incseq f*
shows *E-sup s* (λ*x*. ⨆*i*. *f i x*) = (⨆*i*. *E-sup s* (*f i*))
by (*auto simp add*: *E-sup-def nn-integral-monotone-convergence-SUP intro*: *SUP-commute*)

```
lemma E-sup-iterate:
  assumes [measurable]: f ∈ borel-measurable St
  shows E-sup s f = (⨆D∈K s. ∫+ t. E-sup t (λω. f (t ## ω)) ∂measure-pmf D)
proof −
  let ?v = λt. ∫+x. f (state t ## x) ∂T t
  let ?p = λt. E-sup t (λω. f (t ## ω))
  have E-sup s f = (⨆cfg∈cfg-on s. ∫+t. ?v t ∂K-cfg cfg)
    unfolding E-sup-def by (intro SUP-cong refl) (subst nn-integral-T, simp-all add: cfg-on-def)
  also have ... = (⨆D∈K s. ∫+t. ?p t ∂measure-pmf D)
  proof (intro antisym SUP-least)
    fix cfg :: 's cfg assume cfg: cfg ∈ cfg-on s
    then show (∫+ t. ?v t ∂K-cfg cfg) ≤ (SUP D∈K s. ∫+t. ?p t ∂measure-pmf D)
      by (auto simp: E-sup-def nn-integral-K-cfg AE-measure-pmf-iff
               intro!: nn-integral-mono-AE SUP-upper2)
  next
    fix D assume D: D ∈ K s show (∫+t. ?p t ∂D) ≤ (SUP cfg ∈ cfg-on s. ∫+ t. ?v t ∂K-cfg cfg)
    proof cases
      assume p-finite: ∀t∈D. ?p t < ∞
      show ?thesis
      proof (rule ennreal-le-epsilon)
        fix e :: real assume 0 < e
        have ∀t∈D. ∃cfg∈cfg-on t. ?p t ≤ ?v cfg + e
        proof
          fix t assume t ∈ D
          moreover have (SUP cfg ∈ cfg-on t. ?v cfg) = ?p t
            unfolding E-sup-def by (simp add: cfg-on-def)
          ultimately have (SUP cfg ∈ cfg-on t. ?v cfg) ≠ ∞
            using p-finite by auto
          from SUP-approx-ennreal[OF ‹0<e› - refl this]
          show ∃cfg∈cfg-on t. ?p t ≤ ?v cfg + e
            by (auto simp add: E-sup-def intro: less-imp-le)
        qed
        then obtain cfg' where v-cfg': ⋀t. t ∈ D ⟹ ?p t ≤ ?v (cfg' t) + e and
          cfg-on-cfg': ⋀t. t ∈ D ⟹ cfg' t ∈ cfg-on t
          unfolding Bex-def bchoice-iff by blast

        let ?cfg = Cfg s D cfg'
        have cfg: K-cfg ?cfg = map-pmf cfg' D
          by (auto simp add: K-cfg-def fun-eq-iff cfg-on-cfg' intro!: map-pmf-cong)

        have (∫+ t. ?p t ∂D) ≤ (∫+t. ?v (cfg' t) + e ∂D)
          by (intro nn-integral-mono-AE) (simp add: v-cfg' AE-measure-pmf-iff)
        also have ... = (∫+t. ?v (cfg' t) ∂D) + e
          using ‹0 < e› measure-pmf.emeasure-space-1[of D]
          by (subst nn-integral-add) (auto intro: cfg-on-cfg' )
```

```
      also have (∫+t. ?v (cfg' t) ∂D) = (∫+t. ?v t ∂K-cfg ?cfg)
        by (simp add: cfg map-pmf-rep-eq nn-integral-distr)
      also have ... ≤ (SUP cfg∈cfg-on s. (∫+t. ?v t ∂K-cfg cfg))
        by (auto intro!: SUP-upper intro!: cfg-of-cfg-onI D cfg-on-cfg')
       finally show (∫+ t. ?p t ∂D) ≤ (SUP cfg ∈ cfg-on s. ∫+ t. ?v t ∂K-cfg
cfg) + e
        by (blast intro: add-mono)
    qed
  next
    assume ¬ (∀ t∈D. ?p t < ∞)
    then obtain t where t ∈ D ?p t = ∞
      by (auto simp: not-less top-unique)
    then have ∞ = pmf (D) t * ?p t
      by (auto simp: ennreal-mult-top set-pmf-iff)
    also have ... = (SUP cfg ∈ cfg-on t. pmf (D) t * ?v cfg)
      unfolding E-sup-def
      by (auto simp: SUP-mult-left-ennreal[symmetric])
    also have ... ≤ (SUP cfg ∈ cfg-on s. ∫+ t. ?v t ∂K-cfg cfg)
      unfolding E-sup-def
    proof (intro SUP-least SUP-upper2)
      fix cfg :: 's cfg assume cfg: cfg ∈ cfg-on t

      let ?cfg = Cfg s D ((memoryless-on arb-act) (t := cfg))
      have C: K-cfg ?cfg = map-pmf ((memoryless-on arb-act) (t := cfg)) D
        by (auto simp add: K-cfg-def fun-eq-iff intro!: map-pmf-cong simp: cfg)

      show ?cfg ∈ cfg-on s
        by (auto intro!: cfg-of-cfg-onI D cfg memoryless-on-cfg-onI)
      have ennreal (pmf (D) t) * (∫+ x. f (state cfg ## x) ∂T cfg) =
      (∫+t'. (∫+ x. f (state cfg ## x) ∂T cfg) * indicator {t} t' ∂D)
        by (auto simp add:  max-def emeasure-pmf-single intro: mult-ac)
      also have ... = (∫+cfg. ?v cfg * indicator {t} (state cfg) ∂K-cfg ?cfg)
        unfolding C using cfg
        by (auto simp add: nn-integral-distr map-pmf-rep-eq split: split-indicator
                simp del: nn-integral-indicator-singleton
                intro!: nn-integral-cong)
      also have ... ≤ (∫+cfg. ?v cfg ∂K-cfg ?cfg)
        by (auto intro!: nn-integral-mono  split: split-indicator)
      finally show ennreal (pmf (D) t) * (∫+ x. f (state cfg ## x) ∂T cfg)
        ≤ (∫+ t. ∫+ x. f (state t ## x) ∂T t ∂K-cfg ?cfg) .
    qed
    finally show ?thesis
      by (simp add: top-unique del: Sup-eq-top-iff SUP-eq-top-iff)
  qed
 qed
 finally show ?thesis .
qed

lemma E-sup-bot: E-sup s ⊥ = 0
```

by (*auto simp add*: *E-sup-def bot-ennreal*)

lemma *E-sup-lfp*:
fixes *g*
defines $l \equiv \lambda f\ \omega.\ g\ (shd\ \omega)\ (f\ (stl\ \omega))$
assumes *measurable-g*[*measurable*]: *case-prod g* $\in$ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*: $\bigwedge s.$ *sup-continuous* (*g s*)
assumes *int-g*: $\bigwedge f\ cfg.\ f \in$ *borel-measurable St* $\Longrightarrow$
($\int^+ \omega.\ g\ (state\ cfg)\ (f\ \omega)\ \partial T\ cfg) = g\ (state\ cfg)\ (integral^N\ (T\ cfg)\ f)$
shows ($\lambda s.$ *E-sup s* (*lfp l*)) = *lfp* ($\lambda f\ s.\ \bigsqcup D{\in}K\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial$*measure-pmf D*)
proof (*rule lfp-transfer-bounded*[**where** $\alpha{=}\lambda F\ s.$ *E-sup s F* **and** $f{=}l$ **and** $P{=}\lambda f.\ f \in$ *borel-measurable St*])
show *sup-continuous* ($\lambda f\ s.\ \bigsqcup x{\in}K\ s.\ \int^+ t.\ g\ t\ (f\ t)\ \partial$*measure-pmf x*)
using *cont-g*[*THEN sup-continuous-compose*] **by** (*auto intro*!: *order-continuous-intros*)
show *sup-continuous l*
using *cont-g*[*THEN sup-continuous-compose*] **by** (*auto intro*!: *order-continuous-intros simp*: *l-def*)
show $\bigwedge F.\ (\lambda s.$ *E-sup s* $\bot) \le (\lambda s.\ \bigsqcup D{\in}K\ s.\ \int^+ t.\ g\ t\ (F\ t)\ \partial$*measure-pmf D*)
using *K-wf* **by** (*auto simp*: *E-sup-bot le-fun-def intro*: *SUP-upper2*)
next
fix *f* :: *'s stream* $\Rightarrow$ *ennreal* **assume** *f*: $f \in$ *borel-measurable St*
moreover
have *E-sup s* ($\lambda\omega.\ g\ s\ (f\ \omega)$) = *g s* (*E-sup s f*) **for** *s*
unfolding *E-sup-def* **using** *int-g*[*OF f*]
by (*subst SUP-sup-continuous-ennreal*[*OF cont-g*, *symmetric*])
(*auto intro*!: *SUP-cong simp del*: *cfg-onD-state dest*: *cfg-onD-state*[*symmetric*])
ultimately show ($\lambda s.$ *E-sup s* (*l f*)) = ($\lambda s.\ \bigsqcup D{\in}K\ s.\ \int^+ t.\ g\ t$ (*E-sup t f*) ∂*measure-pmf D*)
by (*subst E-sup-iterate*) (*auto simp*: *l-def int-g fun-eq-iff intro*!: *SUP-cong nn-integral-cong*)
qed (*auto simp*: *bot-fun-def l-def SUP-apply*[*abs-def*] *E-sup-SUP*)

definition *P-sup s P* = ($\bigsqcup$ *cfg*$\in$*cfg-on s. emeasure* (*T cfg*) {*x*$\in$*space St. P x*})

lemma *P-sup-eq-E-sup*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-sup s P* = *E-sup s* (*indicator* {*x*$\in$*space St. P x*})
by (*auto simp add*: *P-sup-def E-sup-def intro*!: *SUP-cong nn-integral-cong*)

lemma *P-sup-True*[*simp*]: *P-sup t* ($\lambda\omega$. *True*) = *1*
using *T.emeasure-space-1*
by (*auto simp add*: *P-sup-def SUP-constant*)

lemma *P-sup-False*[*simp*]: *P-sup t* ($\lambda\omega$. *False*) = *0*
by (*auto simp add*: *P-sup-def SUP-constant*)

lemma *P-sup-SUP*:

fixes *P* :: *nat* ⇒ *'s stream* ⇒ *bool*
assumes *mono P* **and** *P*[*measurable*]: $\bigwedge i$. *Measurable.pred St* (*P i*)
shows *P-sup s* (λx. $\exists$ *i*. *P i x*) = ($\bigsqcup i$. *P-sup s* (*P i*))
proof −
have *P-sup s* (λx. $\bigsqcup i$. *P i x*) = ($\bigsqcup$ *cfg*∈*cfg-on s*. *emeasure* (*T cfg*) ($\bigcup i$. {*x*∈*space St*. *P i x*}))
by (*auto simp*: *P-sup-def intro*!: *SUP-cong arg-cong2*[**where** *f*=*emeasure*])
also have ... = ($\bigsqcup$ *cfg*∈*cfg-on s*. $\bigsqcup i$. *emeasure* (*T cfg*) {*x*∈*space St*. *P i x*})
using ‹*mono P*› **by** (*auto intro*!: *SUP-cong SUP-emeasure-incseq*[*symmetric*] *simp*: *mono-def le-fun-def*)
also have ... = ($\bigsqcup i$. *P-sup s* (*P i*))
by (*subst SUP-commute*) (*simp add*: *P-sup-def*)
finally show *?thesis*
by *simp*
qed

lemma *P-sup-lfp*:
assumes *Q*: *sup-continuous Q*
assumes *f*: *f* ∈ *measurable St M*
assumes *Q-m*: $\bigwedge P$. *Measurable.pred M P* ⟹ *Measurable.pred M* (*Q P*)
shows *P-sup s* (λx. *lfp Q* (*f x*)) = ($\bigsqcup i$. *P-sup s* (λx. (*Q* ^^ *i*) ⊥ (*f x*)))
unfolding *sup-continuous-lfp*[*OF Q*]
apply *simp*
proof (*rule P-sup-SUP*)
fix *i* **show** *Measurable.pred St* (λx. (*Q* ^^ *i*) ⊥ (*f x*))
apply (*intro measurable-compose*[*OF f*])
by (*induct i*) (*auto intro*!: *Q-m*)
qed (*intro mono-funpow sup-continuous-mono*[*OF Q*] *mono-compose*[**where** *f*=*f*])

lemma *P-sup-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-sup s P* = ($\bigsqcup$ *D*∈*K s*. $\int^+$ *t*. *P-sup t* ($\lambda\omega$. *P* (*t* ## ω)) ∂*measure-pmf D*)
proof −
have [*simp*]: $\bigwedge x\ s$. *indicator* {*x* ∈ *space St*. *P x*} (*x* ## *s*) = *indicator* {*s* ∈ *space St*. *P* (*x* ## *s*)} *s*
by (*auto simp*: *space-stream-space split*: *split-indicator*)
show *?thesis*
using *E-sup-iterate*[*of indicator* {*x*∈*space St*. *P x*} *s*] **by** (*auto simp*: *P-sup-eq-E-sup*)
qed

definition *E-inf s f* = ($\bigsqcap$ *cfg*∈*cfg-on s*. $\int^+$*x*. *f x* ∂*T cfg*)

lemma *E-inf-const*: *0* ≤ *c* ⟹ *E-inf s* (λ-. *c*) = *c*
using *T.emeasure-space-1* **by** (*simp add*: *E-inf-def*)

lemma *E-inf-mono*:
($\bigwedge\omega$. *f* ω ≤ *g* ω) ⟹ *E-inf s f* ≤ *E-inf s g*
unfolding *E-inf-def* **by** (*intro INF-superset-mono order-refl nn-integral-mono*)

lemma *E-inf-iterate*:
assumes [*measurable*]: $f \in$ *borel-measurable St*
shows *E-inf s f* = ($\prod D{\in}K\ s.\ \int^+ t.$ *E-inf t* ($\lambda\omega.\ f\ (t\ \#\#\ \omega)$) ∂*measure-pmf D*)
proof −
let $?v = \lambda t.\ \int^+ x.\ f\ (state\ t\ \#\#\ x)\ \partial T\ t$
let $?p = \lambda t.$ *E-inf t* ($\lambda\omega.\ f\ (t\ \#\#\ \omega)$)
have *E-inf s f* = ($\prod cfg{\in}cfg\text{-}on\ s.\ \int^+ t.\ ?v\ t\ \partial K\text{-}cfg\ cfg$)
unfolding *E-inf-def* **by** (*intro INF-cong refl*) (*subst nn-integral-T*, *simp-all add*: *cfg-on-def*)
also have ... = ($\prod D{\in}K\ s.\ \int^+ t.\ ?p\ t\ \partial measure\text{-}pmf\ D$)
proof (*intro antisym INF-greatest*)
fix *cfg* :: *'s cfg* **assume** *cfg*: $cfg \in cfg\text{-}on\ s$
then show (*INF* $D \in K\ s.\ \int^+ t.\ ?p\ t\ \partial measure\text{-}pmf\ D$) $\leq$ ($\int^+ t.\ ?v\ t\ \partial K\text{-}cfg\ cfg$)
by (*auto simp add*: *E-inf-def nn-integral-K-cfg AE-measure-pmf-iff intro*!: *nn-integral-mono-AE INF-lower2*)
next
fix *D* **assume** *D*: $D \in K\ s$ **show** (*INF* $cfg \in cfg\text{-}on\ s.\ \int^+ t.\ ?v\ t\ \partial K\text{-}cfg\ cfg$) $\leq$ ($\int^+ t.\ ?p\ t\ \partial D$)
proof (*rule ennreal-le-epsilon*)
fix *e* :: *real* **assume** $0 < e$
have $\forall t{\in}D.\ \exists cfg{\in}cfg\text{-}on\ t.\ ?v\ cfg \leq ?p\ t + e$
proof
fix *t* **assume** $t \in D$
show $\exists cfg{\in}cfg\text{-}on\ t.\ ?v\ cfg \leq ?p\ t + e$
proof *cases*
assume $?p\ t = \infty$ **with** *cfg-on-not-empty*[*of t*] **show** *?thesis*
by (*auto simp*: *top-add simp del*: *cfg-on-not-empty*)
next
assume *p-finite*: $?p\ t \neq \infty$
note ‹$t \in D$›
moreover have (*INF* $cfg \in cfg\text{-}on\ t.\ ?v\ cfg$) = $?p\ t$
unfolding *E-inf-def* **by** (*simp add*: *cfg-on-def*)
ultimately have (*INF* $cfg \in cfg\text{-}on\ t.\ ?v\ cfg$) $\neq \infty$
using *p-finite* **by** *auto*
from *INF-approx-ennreal*[*OF* ‹$0 < e$› *refl this*]
show $\exists cfg{\in}cfg\text{-}on\ t.\ ?v\ cfg \leq ?p\ t + e$
by (*auto simp*: *E-inf-def intro*: *less-imp-le*)
qed
qed
then obtain *cfg'* **where** *v-cfg'*: $\bigwedge t.\ t \in D \Longrightarrow ?v\ (cfg'\ t) \leq ?p\ t + e$ **and** *cfg-on-cfg'*: $\bigwedge t.\ t \in D \Longrightarrow cfg'\ t \in cfg\text{-}on\ t$
unfolding *Bex-def bchoice-iff* **by** *blast*

let $?cfg = Cfg\ s\ D\ cfg'$

have *cfg*: $K\text{-}cfg\ ?cfg = map\text{-}pmf\ cfg'\ D$
by (*auto simp add*: *K-cfg-def cfg-on-cfg' intro*!: *map-pmf-cong*)

have *?cfg* ∈ *cfg-on s*
by (*auto intro*: *D cfg-on-cfg′ cfg-of-cfg-onI*)
then have (*INF cfg* ∈ *cfg-on s*. $\int^+$ *t*. *?v t* ∂*K-cfg cfg*) ≤ ($\int^+$ *t*. *?p t* + *e* ∂*D*)
by (*rule INF-lower2*) (*auto simp*: *cfg map-pmf-rep-eq nn-integral-distr v-cfg′ AE-measure-pmf-iff intro*!: *nn-integral-mono-AE*)
also have ... = ($\int^+$ *t*. *?p t* ∂*D*) + *e*
using ‹*0* < *e*› **by** (*simp add*: *nn-integral-add measure-pmf.emeasure-space-1*[*simplified*])
finally show (*INF cfg* ∈ *cfg-on s*. $\int^+$ *t*. *?v t* ∂*K-cfg cfg*) ≤ ($\int^+$ *t*. *?p t* ∂*D*) + *e* **.**
qed
qed
finally show *?thesis* **.**
qed

lemma *emeasure-T-const*[*simp*]: *emeasure* (*T s*) (*space St*) = *1*
using *T.emeasure-space-1*[*of s*] **by** *simp*

lemma *E-inf-greatest*:
(⋀*cfg*. *cfg* ∈ *cfg-on s* ⟹ *x* ≤ ($\int^+$*x*. *f x* ∂*T cfg*)) ⟹ *x* ≤ *E-inf s f*
unfolding *E-inf-def* **by** (*rule INF-greatest*)

lemma *E-inf-lower2*:
cfg ∈ *cfg-on s* ⟹ ($\int^+$*x*. *f x* ∂*T cfg*) ≤ *x* ⟹ *E-inf s f* ≤ *x*
unfolding *E-inf-def* **by** (*rule INF-lower2*)

Maybe the following statement can be generalized to infinite *K s*.

lemma *E-inf-lfp*:
fixes *g*
defines *l* ≡ λ*f* ω. *g* (*shd* ω) (*f* (*stl* ω))
assumes *measurable-g*[*measurable*]: *case-prod g* ∈ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
assumes *cont-g*: ⋀*s*. *sup-continuous* (*g s*)
assumes *int-g*: ⋀*f cfg*. *f* ∈ *borel-measurable St* ⟹
($\int^+$ ω. *g* (*state cfg*) (*f* ω) ∂*T cfg*) = *g* (*state cfg*) (*integral*^{N} (*T cfg*) *f*)
assumes *K-finite*: ⋀*s*. *finite* (*K s*)
shows (λ*s*. *E-inf s* (*lfp l*)) = *lfp* (λ*f s*. ⨅*D*∈*K s*. $\int^+$*t*. *g t* (*f t*) ∂*measure-pmf D*)
proof (*rule antisym*)
let *?F* = λ*F s*. ⨅*D*∈*K s*. $\int^+$ *t*. *g t* (*F t*) ∂*measure-pmf D*
let *?I* = λ*D*. ($\int^+$*t*. *g t* (*lfp ?F t*) ∂*measure-pmf D*)
have *mono-F*: *mono ?F*
using *sup-continuous-mono*[*OF cont-g*]
by (*force intro*!: *INF-mono nn-integral-mono monoI simp*: *mono-def le-fun-def*)
define *ct* **where** *ct s* = (*SOME D*. *D* ∈ *K s* ∧ (*lfp ?F s* = *?I D*)) **for** *s*
{ fix *s*
have *finite* (*?I* ‘ *K s*)
by (*auto intro*: *K-finite*)
then obtain *D* **where** *D* ∈ *K s* *?I D* = *Min* (*?I* ‘ *K s*)

```
      by (auto simp: K-wf dest!: Min-in)
    note this(2)
    also have ... = (INF D ∈ K s. ?I D)
      using K-wf by (subst Min-Inf) (auto intro: K-finite)
    also have ... = lfp ?F s
      by (rewrite in - = ¤ lfp-unfold[OF mono-F]) auto
    finally have ∃ D. D ∈ K s ∧ (lfp ?F s = ?I D)
      using ‹D ∈ K s› by auto
    then have ct s ∈ K s ∧ (lfp ?F s = ?I (ct s))
      unfolding ct-def by (rule someI-ex)
    then have ct s ∈ K s lfp ?F s = ?I (ct s)
      by auto }
  note ct = this
  then have ct-cfg-on[simp]: ⋀s. memoryless-on ct s ∈ cfg-on s
    by (intro memoryless-on-cfg-onI) simp
  then show (λs. E-inf s (lfp l)) ≤ lfp ?F
  proof (intro le-funI, rule E-inf-lower2)
    fix s
    define P where P f cfg = ∫+ t. g (state t) (f t) ∂K-cfg cfg for f cfg
    have integral^N (T (memoryless-on ct s)) (lfp l) = lfp P (memoryless-on ct s)
     unfolding P-def l-def using measurable-g cont-g int-g by (rule nn-integral-T-lfp)
    also have ... = (SUP i. (P ^^ i) ⊥) (memoryless-on ct s)
      by (rewrite sup-continuous-lfp)
         (auto intro!: order-continuous-intros cont-g[THEN sup-continuous-compose]
simp: P-def)
    also have ... = (SUP i. (P ^^ i) ⊥ (memoryless-on ct s))
      by (simp add: image-comp)
    also have ... ≤ lfp ?F s
    proof (rule SUP-least)
      fix i show (P ^^ i) ⊥ (memoryless-on ct s) ≤ lfp ?F s
      proof (induction i arbitrary: s)
        case 0 then show ?case
          by simp
      next
        case (Suc n)
        have (P ^^ Suc n) ⊥ (memoryless-on ct s) =
          (∫+ t. g t ((P ^^ n) ⊥ (memoryless-on ct t)) ∂ct s)
       by (auto simp add: P-def K-cfg-def AE-measure-pmf-iff intro!: nn-integral-cong-AE)
        also have ... ≤ (∫+ t. g t (lfp ?F t) ∂ct s)
        by (intro nn-integral-mono sup-continuous-mono[OF cont-g, THEN monoD]
Suc)
        also have ... = lfp ?F s
          by (rule ct(2) [symmetric])
        finally show ?case .
      qed
    qed
    finally show integral^N (T (memoryless-on ct s)) (lfp l) ≤ lfp ?F s .
  qed
```

have *cont-l*: *sup-continuous l*
by (*auto simp*: *l-def intro*!: *order-continuous-intros cont-g*[*THEN sup-continuous-compose*])

show *lfp ?F* $\leq$ (λ*s. E-inf s* (*lfp l*))
proof (*intro lfp-lowerbound le-funI*)
fix *s* **show** ($\prod$*x*$\in$*K s.* $\int^+$ *t. g t* (*E-inf t* (*lfp l*)) ∂*measure-pmf x*) $\leq$ *E-inf s* (*lfp l*)
proof (*rewrite* **in** - $\leq$ ⌑ *E-inf-iterate*)
show *l*: *lfp l* $\in$ *borel-measurable St*
using *cont-l* **by** (*rule borel-measurable-lfp*) (*simp add*: *l-def*)
show ($\prod$*D*$\in$*K s.* $\int^+$ *t. g t* (*E-inf t* (*lfp l*)) ∂*measure-pmf D*) $\leq$
($\prod$*D*$\in$*K s.* $\int^+$ *t. E-inf t* ($\lambda\omega$. *lfp l* (*t ## ω*)) ∂*measure-pmf D*)
proof (*rule INF-mono nn-integral-mono bexI*)+
fix *t D* **assume** *D* $\in$ *K s*
{ fix *cfg* **assume** *cfg* $\in$ *cfg-on t*
have ($\int^+$ ω. *g* (*state cfg*) (*lfp l* ω) ∂*T cfg*) = *g* (*state cfg*) ($\int^+$ ω. (*lfp l* ω) ∂*T cfg*)
using *l* **by** (*rule int-g*)
with ‹*cfg* $\in$ *cfg-on t*› **have** $*$: ($\int^+$ ω. *g t* (*lfp l* ω) ∂*T cfg*) = *g t* ($\int^+$ ω. (*lfp l* ω) ∂*T cfg*)
by *simp* **}**
then
have $*$: *g t* ($\prod$*cfg*$\in$*cfg-on t. integral*^{N} (*T cfg*) (*lfp l*)) $\leq$ ($\prod$*cfg*$\in$*cfg-on t.* $\int^+$ ω. *g t* (*lfp l* ω) ∂*T cfg*)
apply *simp*
apply (*rule INF-greatest*)
apply (*rule sup-continuous-mono*[*OF cont-g*, *THEN monoD*])
apply (*rule INF-lower*)
apply *assumption*
done
show *g t* (*E-inf t* (*lfp l*)) $\leq$ *E-inf t* ($\lambda\omega$. *lfp l* (*t ## ω*))
apply (*rewrite* **in** - $\leq$ ⌑ *lfp-unfold*[*OF sup-continuous-mono*[*OF cont-l*]])
apply (*rewrite* **in** - $\leq$ ⌑ *l-def*)
apply (*simp add*: *E-inf-def* $*$)
done
qed
qed
qed
qed

definition *P-inf s P* = ($\prod$*cfg*$\in$*cfg-on s. emeasure* (*T cfg*) {*x*$\in$*space St. P x*})

lemma *P-inf-eq-E-inf*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf s P* = *E-inf s* (*indicator* {*x*$\in$*space St. P x*})
by (*auto simp add*: *P-inf-def E-inf-def intro*!: *SUP-cong nn-integral-cong*)

lemma *P-inf-True*[*simp*]: *P-inf t* ($\lambda\omega$. *True*) = *1*
using *T.emeasure-space-1*

by (*auto simp add*: *P-inf-def SUP-constant*)

lemma *P-inf-False*[*simp*]: *P-inf t* (*λω. False*) = *0*
by (*auto simp add*: *P-inf-def SUP-constant*)

lemma *P-inf-INF*:
fixes *P* :: *nat* ⇒ *'s stream* ⇒ *bool*
assumes *decseq P* **and** *P*[*measurable*]: ⋀*i*. *Measurable.pred St* (*P i*)
shows *P-inf s* (*λx*. ∀ *i*. *P i x*) = (⨅ *i*. *P-inf s* (*P i*))
proof −
have *P-inf s* (*λx*. ⨅ *i*. *P i x*) = (⨅ *cfg*∈*cfg-on s*. *emeasure* (*T cfg*) (⋂ *i*. {*x*∈*space St*. *P i x*}))
by (*auto simp*: *P-inf-def intro*!: *INF-cong arg-cong2*[**where** *f*=*emeasure*])
also have ... = (⨅ *cfg*∈*cfg-on s*. ⨅ *i*. *emeasure* (*T cfg*) {*x*∈*space St*. *P i x*})
using ‹*decseq P*›
by (*auto intro*!: *INF-cong INF-emeasure-decseq*[*symmetric*]
simp: *decseq-def monotone-def le-fun-def*)
also have ... = (⨅ *i*. *P-inf s* (*P i*))
by (*subst INF-commute*) (*simp add*: *P-inf-def*)
finally show *?thesis*
by *simp*
qed

lemma *P-inf-gfp*:
assumes *Q*: *inf-continuous Q*
assumes *f*: *f* ∈ *measurable St M*
assumes *Q-m*: ⋀*P*. *Measurable.pred M P* ⟹ *Measurable.pred M* (*Q P*)
shows *P-inf s* (*λx*. *gfp Q* (*f x*)) = (⨅ *i*. *P-inf s* (*λx*. (*Q* ^^ *i*) ⊤ (*f x*)))
unfolding *inf-continuous-gfp*[*OF Q*]
apply *simp*
proof (*rule P-inf-INF*)
fix *i* **show** *Measurable.pred St* (*λx*. (*Q* ^^ *i*) ⊤ (*f x*))
apply (*intro measurable-compose*[*OF f*])
by (*induct i*) (*auto intro*!: *Q-m*)
next
show *decseq* (*λi x*. (*Q* ^^ *i*) ⊤ (*f x*))
using *inf-continuous-mono*[*OF Q*, *THEN funpow-increasing*[*rotated*]]
unfolding *decseq-def monotone-def le-fun-def* **by** *auto*
qed

lemma *P-inf-iterate*:
assumes [*measurable*]: *Measurable.pred St P*
shows *P-inf s P* = (⨅ *D*∈*K s*. ∫⁺ *t*. *P-inf t* (*λω*. *P* (*t* ## *ω*)) ∂*measure-pmf D*)
proof −
have [*simp*]: ⋀*x s*. *indicator* {*x* ∈ *space St*. *P x*} (*x* ## *s*) = *indicator* {*s* ∈ *space St*. *P* (*x* ## *s*)} *s*
by (*auto simp*: *space-stream-space split*: *split-indicator*)
show *?thesis*

using *E-inf-iterate*[*of indicator* {*x*∈*space St*. *P x*} *s*] **by** (*auto simp*: *P-inf-eq-E-inf*)
qed

end

5.5 Finite MDPs

locale *Finite-Markov-Decision-Process* = *Markov-Decision-Process K* **for** *K* :: *'s* ⇒ *'s pmf set* +
fixes *S* :: *'s set*
assumes *S-not-empty*: *S* ≠ {}
assumes *S-finite*: *finite S*
assumes *K-closed*: ⋀*s*. *s* ∈ *S* ⟹ (⋃*D*∈*K s*. *set-pmf D*) ⊆ *S*
assumes *K-finite*: ⋀*s*. *s* ∈ *S* ⟹ *finite* (*K s*)
begin

lemma *action-closed*: *s* ∈ *S* ⟹ *cfg* ∈ *cfg-on s* ⟹ *t* ∈ *action cfg* ⟹ *t* ∈ *S*
using *cfg-onD-action*[*of cfg s*] *K-closed*[*of s*] **by** *auto*

lemma *set-pmf-closed*: *s* ∈ *S* ⟹ *D* ∈ *K s* ⟹ *t* ∈ *D* ⟹ *t* ∈ *S*
using *K-closed* **by** *auto*

lemma *Pi-closed*: *ct* ∈ *Pi S K* ⟹ *s* ∈ *S* ⟹ *t* ∈ *ct s* ⟹ *t* ∈ *S*
using *set-pmf-closed* **by** *auto*

lemma *E-closed*: *s* ∈ *S* ⟹ (*s*, *t*) ∈ *E* ⟹ *t* ∈ *S*
using *K-closed* **by** (*auto simp*: *E-def*)

lemma *set-pmf-finite*: *s* ∈ *S* ⟹ *D* ∈ *K s* ⟹ *finite D*
using *K-closed* **by** (*intro finite-subset*[*OF - S-finite*]) *auto*

definition *valid-cfg* = (⋃*s*∈*S*. *cfg-on s*)

lemma *valid-cfgI*: *s* ∈ *S* ⟹ *cfg* ∈ *cfg-on s* ⟹ *cfg* ∈ *valid-cfg*
by (*auto simp*: *valid-cfg-def*)

lemma *valid-cfgD*: *cfg* ∈ *valid-cfg* ⟹ *cfg* ∈ *cfg-on* (*state cfg*)
by (*auto simp*: *valid-cfg-def*)

lemma
shows *valid-cfg-state-in-S*: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ *S*
and *valid-cfg-action*: *cfg* ∈ *valid-cfg* ⟹ *s* ∈ *action cfg* ⟹ *s* ∈ *S*
and *valid-cfg-cont*: *cfg* ∈ *valid-cfg* ⟹ *s* ∈ *action cfg* ⟹ *cont cfg s* ∈ *valid-cfg*
by (*auto simp*: *valid-cfg-def intro*!: *bexI*[*of - s*] *intro*: *action-closed*)

lemma *valid-K-cfg*[*intro*]: *cfg* ∈ *valid-cfg* ⟹ *cfg'* ∈ *K-cfg cfg* ⟹ *cfg'* ∈ *valid-cfg*
by (*auto simp add*: *K-cfg-def valid-cfg-cont*)

definition *simple ct* = *memoryless-on* (λ*s*. *if s* ∈ *S then ct s else arb-act s*)

lemma *simple-cfg-on*[*simp*]: *ct* ∈ *Pi S K* ⟹ *simple ct s* ∈ *cfg-on s*
  by (*auto simp*: *simple-def intro*!: *memoryless-on-cfg-onI*)

lemma *simple-valid-cfg*[*simp*]: *ct* ∈ *Pi S K* ⟹ *s* ∈ *S* ⟹ *simple ct s* ∈ *valid-cfg*
  by (*auto intro*: *valid-cfgI*)

lemma *cont-simple*[*simp*]: *s* ∈ *S* ⟹ *t* ∈ *set-pmf* (*ct s*) ⟹ *cont* (*simple ct s*) *t* = *simple ct t*
  by (*simp add*: *simple-def*)

lemma *state-simple*[*simp*]: *state* (*simple ct s*) = *s*
  by (*simp add*: *simple-def*)

lemma *action-simple*[*simp*]: *s* ∈ *S* ⟹ *action* (*simple ct s*) = *ct s*
  by (*simp add*: *simple-def*)

lemma *simple-valid-cfg-iff*: *ct* ∈ *Pi S K* ⟹ *simple ct s* ∈ *valid-cfg* ⟷ *s* ∈ *S*
  using *cfg-onD-state*[*of simple ct s*] **by** (*auto simp add*: *valid-cfg-def intro*!: *bexI*[*of* - *s*])

end

end
theory *MDP-Reachability-Problem*
  imports *Markov-Decision-Process*
begin

inductive-set *directed-towards* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *'a set* **for** *A r* **where**
  start: ⋀*x*. *x* ∈ *A* ⟹ *x* ∈ *directed-towards A r*
| *step*: ⋀*x y*. *y* ∈ *directed-towards A r* ⟹ (*x*, *y*) ∈ *r* ⟹ *x* ∈ *directed-towards A r*

hide-fact (**open**) *start step*

lemma *directed-towards-mono*:
  assumes *s* ∈ *directed-towards A F F* ⊆ *G* **shows** *s* ∈ *directed-towards A G*
  using *assms* **by** *induct* (*auto intro*: *directed-towards.intros*)

lemma *directed-eq-rtrancl*: *x* ∈ *directed-towards A r* ⟷ (∃ *a*∈*A*. (*x*, *a*) ∈ r^*)
proof
  assume *x* ∈ *directed-towards A r* **then show** ∃ *a*∈*A*. (*x*, *a*) ∈ r^*
    by *induction* (*auto intro*: *converse-rtrancl-into-rtrancl*)
next
  assume ∃ *a*∈*A*. (*x*, *a*) ∈ r^*
  then obtain *a* **where** (*x*, *a*) ∈ r^* *a* ∈ *A* **by** *auto*
  then show *x* ∈ *directed-towards A r*
    by (*induction rule*: *converse-rtrancl-induct*)
      (*auto intro*: *directed-towards.start directed-towards.step*)
qed

lemma *directed-eq-rtrancl-Image*: *directed-towards A r* = $(r^*)^{-1}$ ‘‘ *A*
unfolding *set-eq-iff directed-eq-rtrancl Image-iff* **by** *simp*

locale *Reachability-Problem* = *Finite-Markov-Decision-Process K S* **for** *K* :: *'s* ⇒ *'s pmf set* **and** *S* +
fixes *S1 S2* :: *'s set*
assumes *S1*: *S1* ⊆ *S* **and** *S2*: *S2* ⊆ *S* **and** *S1-S2*: *S1* ∩ *S2* = {}
begin

lemma [*measurable*]:
S ∈ *sets* (*count-space UNIV*) *S1* ∈ *sets* (*count-space UNIV*) *S2* ∈ *sets* (*count-space UNIV*)
by *auto*

definition
v = (λ*cfg*∈*valid-cfg*. *emeasure* (*T cfg*) {*x*∈*space St*. (*HLD S1 suntil HLD S2*) (*state cfg* ## *x*)})

lemma *v-eq*: *cfg* ∈ *valid-cfg* ⟹
v cfg = *emeasure* (*T cfg*) {*x*∈*space St*. (*HLD S1 suntil HLD S2*) (*state cfg* ## *x*)}
by (*auto simp add*: *v-def*)

lemma *real-v*: *cfg* ∈ *valid-cfg* ⟹ *enn2real* (*v cfg*) = 𝒫(ω *in T cfg*. (*HLD S1 suntil HLD S2*) (*state cfg* ## ω))
by (*auto simp add*: *v-def T.emeasure-eq-measure*)

lemma *v-le-1*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≤ *1*
by (*auto simp add*: *v-def T.emeasure-eq-measure*)

lemma *v-neq-Pinf*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *top*
by (*auto simp add*: *v-def*)

lemma *v-1-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *1* ⟷ (*AE* ω *in T cfg*. (*HLD S1 suntil HLD S2*) (*state cfg* ## ω))
unfolding *v-eq T.emeasure-eq-measure ennreal-eq-1 space-T*[*symmetric*, *of cfg*]
by (*rule T.prob-Collect-eq-1*) *simp*

lemma *v-0-AE*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* = *0* ⟷ (*AE x in T cfg*. *not* (*HLD S1 suntil HLD S2*) (*state cfg* ## *x*))
unfolding *v-eq T.emeasure-eq-measure space-T*[*symmetric*, *of cfg*] *ennreal-eq-zero-iff*[*OF measure-nonneg*]
by (*rule T.prob-Collect-eq-0*) *simp*

lemma *v-S2*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ *S2* ⟹ *v cfg* = *1*
using *S2* **by** (*subst v-1-AE*) (*auto simp*: *suntil-Stream*)

lemma *v-nS12*[*simp*]: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∉ *S1* ⟹ *state cfg* ∉ *S2* ⟹ *v*

cfg = *0*
  by (*subst v-0-AE*) (*auto simp*: *suntil-Stream*)

lemma *v-nS*[*simp*]: *cfg* ∉ *valid-cfg* ⟹ *v cfg* = *undefined*
  by (*auto simp add*: *v-def*)

lemma *v-S1*:
  assumes *cfg*[*simp*, *intro*]: *cfg* ∈ *valid-cfg* **and** *cfg-S1*[*simp*]: *state cfg* ∈ *S1*
  shows *v cfg* = (∫⁺*s*. *v* (*cont cfg s*) ∂*action cfg*)
proof −
  have [*simp*]: *state cfg* ∉ *S2*
    using *cfg-S1 S1-S2 S1* **by** *blast*
  show *?thesis*
    by (*auto simp*: *v-eq emeasure-Collect-T*[*of - cfg*] *K-cfg-def map-pmf-rep-eq nn-integral-distr*
        AE-measure-pmf-iff suntil-Stream[*of - - state cfg*]
        valid-cfg-cont
      intro!: *nn-integral-cong-AE*)
qed

lemma *real-v-integrable*:
  integrable (*action cfg*) (λ*s*. *enn2real* (*v* (*cont cfg s*)))
  by (*rule measure-pmf.integrable-const-bound*[**where** *B*=*max 1* (*enn2real undefined*)])
    (*auto simp add*: *v-def measure-def*[*symmetric*] *le-max-iff-disj*)

lemma *real-v-integral-eq*:
  assumes *cfg*[*simp*]: *cfg* ∈ *valid-cfg*
  shows *enn2real* (∫⁺ *s*. *v* (*cont cfg s*) ∂*action cfg*) = ∫ *s*. *enn2real* (*v* (*cont cfg s*)) ∂*action cfg*
  by (*subst integral-eq-nn-integral*)
    (*auto simp*: *AE-measure-pmf-iff v-eq T.emeasure-eq-measure valid-cfg-cont*
      intro!: *arg-cong*[**where** *f*=*enn2real*] *nn-integral-cong-AE*)

lemma *v-eq-0-coinduct*[*consumes 3*, *case-names valid nS2 cont*]:
  assumes ∗: *P cfg*
  assumes *valid*: ⋀*cfg*. *P cfg* ⟹ *cfg* ∈ *valid-cfg*
  assumes *nS2*: ⋀*cfg*. *P cfg* ⟹ *state cfg* ∉ *S2*
  assumes *cont*: ⋀*cfg cfg'*. *P cfg* ⟹ *state cfg* ∈ *S1* ⟹ *cfg'* ∈ *K-cfg cfg* ⟹ *P cfg'* ∨ *v cfg'* = *0*
  shows *v cfg* = *0*
proof −
  from ∗ *valid*[*OF* ∗]
  have *AE x in MC-syntax.T K-cfg cfg*. ¬ (*HLD S1 suntil HLD S2*) (*state cfg* ## *smap state x*)
    unfolding *stream.map*[*symmetric*] *suntil-smap hld-smap'*
  proof (*coinduction arbitrary*: *cfg rule*: *MC.AE-not-suntil-coinduct-strong*)
    case (ψ *cfg*) **then show** *?case*
      by (*auto simp del*: *cfg-onD-state dest*: *nS2*)

next
case (φ *cfg' cfg*)
then have $*$: *P cfg state cfg* $\in$ *S1 cfg'* $\in$ *K-cfg cfg* **and** [*simp*, *intro*]: *cfg* $\in$ *valid-cfg*
by *auto*
with *cont*[*OF* $*$] **show** *?case*
by (*subst* (*asm*) *v-0-AE*)
(*auto simp*: *suntil-Stream T-def AE-distr-iff suntil-smap hld-smap' cong del*: *AE-cong*)
qed
then have *AE* ω *in T cfg.* $\neg$ (*HLD S1 suntil HLD S2*) (*state cfg ##* ω)
unfolding *T-def* **by** (*subst AE-distr-iff*) *simp-all*
with *valid*[*OF* $*$] **show** *?thesis*
by (*simp add*: *v-0-AE*)
qed

definition *p* = ($\lambda s{\in}S$. *P-sup s* ($\lambda\omega$. (*HLD S1 suntil HLD S2*) (*s ##* ω)))

lemma *p-eq-SUP-v*: $s \in S \Longrightarrow p\ s = \bigsqcup\ (v\ `\ cfg\text{-}on\ s)$
by (*auto simp add*: *p-def v-def P-sup-def T.emeasure-eq-measure intro*: *valid-cfgI intro*!: *SUP-cong cong*: *SUP-cong-simp*)

lemma *v-le-p*: *cfg* $\in$ *valid-cfg* $\Longrightarrow$ *v cfg* $\leq$ *p* (*state cfg*)
by (*subst p-eq-SUP-v*) (*auto intro*!: *SUP-upper dest*: *valid-cfgD valid-cfg-state-in-S*)

lemma *p-eq-0-imp*: *cfg* $\in$ *valid-cfg* $\Longrightarrow$ *p* (*state cfg*) = *0* $\Longrightarrow$ *v cfg* = *0*
using *v-le-p*[*of cfg*] **by** (*auto intro*: *antisym*)

lemma *p-eq-0-iff*: $s \in S \Longrightarrow p\ s = 0 \longleftrightarrow (\forall\ cfg{\in}cfg\text{-}on\ s.\ v\ cfg = 0)$
unfolding *p-eq-SUP-v* **by** (*subst SUP-eq-iff*) *auto*

lemma *p-le-1*: $s \in S \Longrightarrow p\ s \leq 1$
by (*auto simp*: *p-eq-SUP-v intro*!: *SUP-least v-le-1 intro*: *valid-cfgI*)

lemma *p-undefined*[*simp*]: $s \notin S \Longrightarrow p\ s = undefined$
by (*simp add*: *p-def*)

lemma *p-not-inf*[*simp*]: $s \in S \Longrightarrow p\ s \neq top$
using *p-le-1*[*of s*] **by** (*auto simp*: *top-unique*)

lemma *p-S1*: $s \in S1 \Longrightarrow p\ s = (\bigsqcup D{\in}K\ s.\ \int^+ t.\ p\ t\ \partial measure\text{-}pmf\ D)$
using *S1 S1-S2 K-closed*[*of s*] **unfolding** *p-def*
by (*simp add*: *P-sup-iterate*[*of - s*] *subset-eq set-eq-iff suntil-Stream*[*of - - s*])
(*auto intro*!: *SUP-cong nn-integral-cong-AE simp add*: *AE-measure-pmf-iff*)

lemma *p-S2*[*simp*]: $s \in S2 \Longrightarrow p\ s = 1$
using *S2* **by** (*auto simp*: *v-S2*[*OF valid-cfgI*] *p-eq-SUP-v*)

lemma *p-nS12*: $s \in S \Longrightarrow s \notin S1 \Longrightarrow s \notin S2 \Longrightarrow p\ s = 0$
  by (*auto simp*: *p-eq-SUP-v v-nS12*[*OF valid-cfgI*])

lemma *p-pos*:
  assumes $(s, t) \in (SIGMA\ s{:}S1.\ \bigcup D{\in}K\ s.\ set\text{-}pmf\ D)^*\ t \in S2$ **shows** $0 < p\ s$
using *assms* **proof** (*induction rule*: *converse-rtrancl-induct*)
  case (*step s t′*)
  then obtain *D* **where** $s \in S1\ D \in K\ s\ t' \in D\ 0 < p\ t'$
    by *auto*
  with *S1 set-pmf-closed*[*of s D*] **have** *in-S*: $\bigwedge t.\ t \in D \Longrightarrow t \in S$
    by *auto*
  from ‹$t' \in D$› ‹$0 < p\ t'$› **have** $0 < pmf\ D\ t' * p\ t'$
    by (*auto simp add*: *ennreal-zero-less-mult-iff pmf-positive*)
  also have ... $\le (\int^+ t.\ p\ t' * indicator\ \{t'\}\ t \partial D)$
    using *in-S*[*OF* ‹$t' \in D$›]
   by (*subst nn-integral-cmult-indicator*) (*auto simp*: *ac-simps emeasure-pmf-single*)
  also have ... $\le (\int^+ t.\ p\ t\ \partial D)$
   by (*auto intro*!: *nn-integral-mono-AE split*: *split-indicator simp*: *in-S AE-measure-pmf-iff simp del*: *nn-integral-indicator-singleton*)
  also have ... $\le p\ s$
    using ‹$s \in S1$› ‹$D \in K\ s$› **by** (*auto intro*: *SUP-upper simp add*: *p-S1*)
  finally show *?case* **.**
qed *simp*

definition *F-sup* :: $('s \Rightarrow ennreal) \Rightarrow 's \Rightarrow ennreal$ **where**
  $F\text{-}sup\ f = (\lambda s{\in}S.\ if\ s \in S2\ then\ 1\ else\ if\ s \in S1\ then\ SUP\ D{\in}K\ s.\ \int^+ t.\ f\ t\ \partial measure\text{-}pmf\ D\ else\ 0)$

lemma *F-sup-cong*: $(\bigwedge s.\ s \in S \Longrightarrow f\ s = g\ s) \Longrightarrow F\text{-}sup\ f\ s = F\text{-}sup\ g\ s$
  using *K-closed*[*of s*]
  by (*auto simp*: *F-sup-def AE-measure-pmf-iff subset-eq*
        intro!: *SUP-cong nn-integral-cong-AE*)

lemma *continuous-F-sup*: *sup-continuous F-sup*
  unfolding *sup-continuous-def fun-eq-iff F-sup-def*[*abs-def*]
  by (*auto simp*: *SUP-apply*[*abs-def*] *nn-integral-monotone-convergence-SUP intro*: *SUP-commute*)

lemma *mono-F-sup*: *mono F-sup*
  by (*intro sup-continuous-mono continuous-F-sup*)

lemma *lfp-F-sup-iterate*: $lfp\ F\text{-}sup = (SUP\ i.\ (F\text{-}sup\ \widehat{\ }\widehat{\ }\ i)\ (\lambda x{\in}S.\ 0))$
proof −
  { have $(SUP\ i.\ (F\text{-}sup\ \widehat{\ }\widehat{\ }\ i)\ \bot) = (SUP\ i.\ (F\text{-}sup\ \widehat{\ }\widehat{\ }\ i)\ (\lambda x{\in}S.\ 0))$
    proof (*rule SUP-eq*)
      fix *i* **show** $\exists j{\in}UNIV.\ (F\text{-}sup\ \widehat{\ }\widehat{\ }\ i)\ \bot \le (F\text{-}sup\ \widehat{\ }\widehat{\ }\ j)\ (\lambda x{\in}S.\ 0)$
        by (*intro bexI*[*of - i*] *funpow-mono mono-F-sup*) *auto*
      have $*$: $(\lambda x{\in}S.\ 0) \le F\text{-}sup\ \bot$
        using *K-wf* **by** (*auto simp*: *F-sup-def le-fun-def*)

show ∃*j*∈*UNIV*. (*F-sup* ^^ *i*) (λ*x*∈*S*. *0*) ≤ (*F-sup* ^^ *j*) ⊥
by (*auto intro*!: *exI*[*of* - *Suc i*] *funpow-mono mono-F-sup* *
simp del: *funpow.simps simp add*: *funpow-Suc-right le-funI*)
qed }
then show *?thesis*
by (*auto simp*: *sup-continuous-lfp continuous-F-sup*)
qed

lemma *p-eq-lfp-F-sup*: *p* = *lfp F-sup*
proof −
{ fix *s* **assume** *s* ∈ *S* **let** *?F* = λ*P*. *HLD S2 or* (*HLD S1 aand nxt P*)
have *P-sup s* (λω. (*HLD S1 suntil HLD S2*) (*s* ## ω)) = (⨆*i*. *P-sup s* (λω. (*?F* ^^ *i*) ⊥ (*s* ## ω)))
proof (*simp add*: *suntil-def*, *rule P-sup-lfp*)
show (##) *s* ∈ *measurable St St*
by *simp*

fix *P* **assume** *P*: *Measurable.pred St P*
show *Measurable.pred St* (*HLD S2 or* (*HLD S1 aand* (λω. *P* (*stl* ω))))
by (*intro pred-intros-logic measurable-compose*[*OF* - *P*] *measurable-compose*[*OF measurable-shd*]) *auto*
qed (*auto simp*: *sup-continuous-def*)
also have ... = (*SUP i*. (*F-sup* ^^ *i*) (λ*x*∈*S*. *0*) *s*)
proof (*rule SUP-cong*)
fix *i* **from** ‹*s* ∈ *S*› **show** *P-sup s* (λω. (*?F* ^^ *i*) ⊥ (*s*##ω)) = (*F-sup* ^^ *i*) (λ*x*∈*S*. *0*) *s*
proof (*induct i arbitrary*: *s*)
case (*Suc n*) **show** *?case*
proof (*subst P-sup-iterate*)

show *Measurable.pred St* (λω. (*?F* ^^ *Suc n*) ⊥ (*s* ## ω))
apply (*intro measurable-compose*[*OF measurable-Stream*[*OF measurable-const measurable-ident-sets*[*OF refl*]] *measurable-predpow*])
apply *simp*
apply (*simp add*: *bot-fun-def*[*abs-def*])
apply (*intro pred-intros-logic measurable-compose*[*OF measurable-stl*] *measurable-compose*[*OF measurable-shd*])
apply *auto*
done
next
show (⨆*D*∈*K s*. ∫⁺ *t*. *P-sup t* (λω. (*?F* ^^ *Suc n*) ⊥ (*s* ## *t* ## ω)) ∂*measure-pmf D*) =
(*F-sup* ^^ *Suc n*) (λ*x*∈*S*. *0*) *s*
unfolding *funpow.simps comp-def*
using *S1 S2* ‹*s* ∈ *S*›
by (*subst F-sup-cong*[*OF Suc*(*1*)[*symmetric*]])
(*auto simp add*: *F-sup-def measure-pmf.emeasure-space-1*[*simplified*] *K-wf subset-eq*)
qed

qed *simp*
qed *simp*
finally have *lfp F-sup s = P-sup s* ($\lambda\omega$. (*HLD S1 suntil HLD S2*) (*s ## ω*))
by (*simp add*: *lfp-F-sup-iterate image-comp*) **}**
moreover have $\bigwedge s.\ s \notin S \Longrightarrow$ *lfp F-sup s = undefined*
by (*subst lfp-unfold*[*OF mono-F-sup*]) (*auto simp add*: *F-sup-def*)
ultimately show *?thesis*
by (*auto simp*: *p-def*)
qed

definition $S_e = \{s \in S.\ p\ s = 0\}$

lemma S_e: $S_e \subseteq S$
by (*auto simp add*: S_e*-def*)

lemma *v-*S_e: *cfg* ∈ *valid-cfg* ⟹ *state cfg* ∈ S_e ⟹ *v cfg* = *0*
using *p-eq-0-imp*[*of cfg*] **by** (*auto simp*: S_e*-def*)

lemma S_e*-nS2*: $S_e \cap S2 = \{\}$
by (*auto simp*: S_e*-def*)

lemma S_e*-E1*: $s \in S_e \cap S1 \Longrightarrow (s, t) \in E \Longrightarrow t \in S_e$
unfolding S_e*-def* **using** *S1*
by (*auto simp*: *p-S1 SUP-eq-iff K-wf nn-integral-0-iff-AE AE-measure-pmf-iff E-def*
intro: *set-pmf-closed antisym*
cong: *rev-conj-cong*)

lemma S_e*-E2*: $s \in S1 \Longrightarrow (\bigwedge t.\ (s, t) \in E \Longrightarrow t \in S_e) \Longrightarrow s \in S_e$
unfolding S_e*-def* **using** *S1 S1-S2*
by (*force simp*: *p-S1 SUP-eq-iff K-wf nn-integral-0-iff-AE AE-measure-pmf-iff E-def*
cong: *rev-conj-cong*)

lemma S_e*-E-iff*: $s \in S1 \Longrightarrow s \in S_e \longleftrightarrow (\forall t.\ (s, t) \in E \longrightarrow t \in S_e)$
using S_e*-E1*[*of s*] S_e*-E2*[*of s*] **by** *blast*

definition $S_r = S - (S_e \cup S2)$

lemma S_r: $S_r \subseteq S$
by (*auto simp*: S_r*-def*)

lemma S_r*-S1*: $S_r \subseteq S1$
by (*auto simp*: *p-nS12* S_r*-def* S_e*-def*)

lemma S_r*-eq*: $S_r = S1 - S_e$
using *S1-S2 S1 S2* **by** (*auto simp add*: S_r*-def* S_e*-def p-nS12*)

lemma *v-neq-0-imp*: *cfg* ∈ *valid-cfg* ⟹ *v cfg* ≠ *0* ⟹ *state cfg* ∈ $S_r \cup S2$

using *p-eq-0-imp*[*of cfg*] **by** (*auto simp add*: S_r*-def* S_e*-def valid-cfg-state-in-S*)

lemma *valid-cfg-action-in-K*: $cfg \in valid\text{-}cfg \Longrightarrow action\ cfg \in K\ (state\ cfg)$
by (*auto dest*!: *valid-cfgD*)

lemma *K-cfg-E*: $cfg \in valid\text{-}cfg \Longrightarrow cfg' \in K\text{-}cfg\ cfg \Longrightarrow (state\ cfg,\ state\ cfg') \in E$
by (*auto simp*: *E-def K-cfg-def valid-cfg-action-in-K*)

lemma S_r*-directed-towards-S2*:
assumes *s*: $s \in S_r$
shows $s \in directed\text{-}towards\ S2\ \{(s,\ t) \mid s\ t.\ s \in S_r \wedge (s,\ t) \in E\}$ (**is** $s \in ?D$)
proof −
{ **fix** *cfg* **assume** $s \notin ?D\ cfg \in cfg\text{-}on\ s$
with *s* S_r **have** $state\ cfg \in S_r\ state\ cfg \notin ?D\ cfg \in valid\text{-}cfg$
by (*auto intro*: *valid-cfgI*)
then have $v\ cfg = 0$
proof (*coinduction arbitrary*: *cfg rule*: *v-eq-0-coinduct*)
case (*cont cfg′ cfg*)
with *v-neq-0-imp*[*of cfg′*] **show** *?case*
by (*auto intro*: *directed-towards.intros K-cfg-E*)
qed (*auto intro*: *directed-towards.intros*) }
with *p-eq-0-iff*[*of s*] *s* **show** *?thesis*
unfolding S_r*-def* S_e*-def* **by** *blast*
qed

definition $proper\ ct \longleftrightarrow ct \in Pi_E\ S\ K \wedge (\forall s{\in}S_r.\ v\ (simple\ ct\ s) > 0)$

lemma S_r*-nS2*: $s \in S_r \Longrightarrow s \notin S2$
by (*auto simp*: S_r*-def*)

lemma *properD1*: $proper\ ct \Longrightarrow ct \in Pi_E\ S\ K$
by (*auto simp*: *proper-def*)

lemma *proper-eq*:
assumes *ct*[*simp*, *intro*]: $ct \in Pi_E\ S\ K$
shows $proper\ ct \longleftrightarrow S_r \subseteq directed\text{-}towards\ S2\ (SIGMA\ s{:}S_r.\ ct\ s)$
(**is** - $\longleftrightarrow$ - $\subseteq$ *?D*)
proof −
have $*$[*simp*]: $\bigwedge s.\ s \in S_r \Longrightarrow s \in S$ **and** *ct′*: $ct \in Pi\ S\ K$
using *ct* **by** (*auto simp*: S_r*-def simp del*: *ct*)
{ **fix** *s t* **have** $s \in S \Longrightarrow t \in ct\ s \Longrightarrow t \in S$
using *K-closed*[*of s*] *ct′* **by** (*auto simp add*: *subset-eq*) }
note *ct-closed* = *this*

let $?C = simple\ ct$
from *ct* **have** *valid-C*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow ?C\ s \in valid\text{-}cfg$
by (*auto simp add*: *PiE-def*)
{ **fix** *s* **assume** $s \in ?D$

```
    then have 0 < v (?C s)
    proof induct
      case (step s t)
      then have s: s ∈ S_r and t: t ∈ ct s and [simp]: s ∈ S
        by auto
      with S_r-S1 ct have v (?C s) = (∫⁺t. v (?C t) ∂ct s)
        by (subst v-S1) (auto intro!: nn-integral-cong-AE AE-pmfI)
      also have … ≠ 0
        using ct t step
      by (subst nn-integral-0-iff-AE) (auto simp add: AE-measure-pmf-iff zero-less-iff-neq-zero)
      finally show ?case
        using ct by (auto simp add: less-le)
    qed (subst v-S2, insert S2, auto) }
  moreover
  { fix s assume s: s ∉ ?D s ∈ S_r
    with ct' have C: ?C s ∈ cfg-on s and [simp]: s ∈ S
      by auto
    from s have v (?C s) = 0
    proof (coinduction arbitrary: s rule: v-eq-0-coinduct)
      case (cont cfg s)
      with S1 obtain t where cfg = ?C t t ∈ ct s s ∈ S
        by (auto simp: set-K-cfg subset-eq)
      with cont(1,2) v-neq-0-imp[of ?C t] ct-closed[of s t] show ?case
        by (intro exI[of - t] disjCI) (auto intro: directed-towards.intros)
    qed (auto simp: S_r-nS2) }
  ultimately show ?thesis
    unfolding proper-def using ct by (force simp del: v-nS v-S2 v-nS12 ct)
qed

lemma exists-proper:
  obtains ct where proper ct
proof atomize-elim
  define r where r = rec-nat S2 (λ- S'. {s∈S_r. ∃t∈S'. (s, t) ∈ E})
  then have [simp]: r 0 = S2 ⋀n. r (Suc n) = {s∈S_r. ∃t∈r n. (s, t) ∈ E}
    by simp-all

  { fix s assume s ∈ S_r
    then have s ∈ directed-towards S2 {(s, t) | s t. s ∈ S_r ∧ (s, t) ∈ E}
      by (rule S_r-directed-towards-S2)
    from this ‹s∈S_r› have ∃n. s ∈ r n
    proof induction
      case (step s t)
      show ?case
      proof cases
        assume t ∈ S2 with step.prems step.hyps show ?thesis
          by (intro exI[of - Suc 0]) force
      next
        assume t ∉ S2
        with step obtain n where t ∈ r n t ∈ S_r
```

by (*auto elim*: *directed-towards.cases*)
with ‹$t \in S_r$› *step.hyps* **show** *?thesis*
by (*intro exI*[*of* - *Suc n*]) *force*
qed
qed (*simp add*: S_r*-def*) **}**
note $r = this$

{ fix s **assume** $s \in S$
have $\exists D \in K\ s.\ s \in S_r \longrightarrow (\exists t \in D.\ \exists n.\ t \in r\ n \wedge (\forall m.\ s \in r\ m \longrightarrow n < m))$
proof *cases*
assume s: $s \in S_r$
define n **where** $n = (LEAST\ n.\ s \in r\ n)$
then have $s \in r\ n$ **and** n: $\bigwedge i.\ i < n \Longrightarrow s \notin r\ i$
using $r\ s$ **by** (*auto intro*: *LeastI-ex dest*: *not-less-Least*)
with s **have** $n \neq 0$
by (*intro notI*) (*auto simp*: S_r*-def*)
then obtain n' **where** $n = Suc\ n'$
by (*cases n*) *auto*
with ‹$s \in r\ n$› **obtain** $t\ D$ **where** $D \in K\ s\ t \in D\ t \in r\ n'$
by (*auto simp*: *E-def*)
with n ‹$n = Suc\ n'$› s **show** *?thesis*
by (*auto intro*!: *bexI*[*of* - *D*] *bexI*[*of* - *t*] *exI*[*of* - *n'*] *simp*: *not-less-eq*[*symmetric*])
qed (*insert K-wf* ‹$s \in S$›, *auto*) **}**
then obtain ct **where** ct: $\bigwedge s.\ s \in S \Longrightarrow ct\ s \in K\ s$
$\bigwedge s.\ s \in S \Longrightarrow s \in S_r \Longrightarrow \exists t \in ct\ s.\ \exists n.\ t \in r\ n \wedge (\forall m.\ s \in r\ m \longrightarrow n < m)$
by *metis*
then have $*$: $restrict\ ct\ S \in Pi_E\ S\ K$
by *auto*

moreover
{ fix s **assume** $s \in S_r$
then obtain n **where** $s \in r\ n$
by (*metis r*)
with ‹$s \in S_r$› **have** $s \in$ *directed-towards* $S2$ $(SIGMA\ s : S_r.\ ct\ s)$
proof (*induction n arbitrary*: *s rule*: *less-induct*)
case (*less n s*)
moreover with S_r **have** $s \in S$ **by** *auto*
ultimately obtain $t\ m$ **where** $t \in ct\ s\ t \in r\ m\ m < n$
using *ct*[*of s*] **by** (*auto simp*: *E-def*)
with *less.IH*[*of m t*] ‹$s \in S_r$› **show** *?case*
by (*cases m*) (*auto intro*: *directed-towards.intros*)
qed }

ultimately show $\exists ct.$ *proper ct*
using S_r $S2$
by (*auto simp*: *proper-eq*[*OF* $*$] *subset-eq*
intro!: *exI*[*of* - *restrict ct S*]
cong: *Sigma-cong*)
qed

definition *l-desc X ct l s* $\longleftrightarrow$
s $\in$ *directed-towards S2* (*SIGMA s* : *X*. {*l s*}) $\wedge$
v (*simple ct s*) $\leq$ *v* (*simple ct* (*l s*)) $\wedge$
l s $\in$ *maximal* (λ*s*. *v* (*simple ct s*)) (*ct s*)

lemma *exists-l-desc*:
assumes *ct*: *proper ct*
shows $\exists\, l \in S_r \to S_r \cup S2.\ \forall\, s \in S_r.$ *l-desc* S_r *ct l s*
proof –
have *ct-closed*: $\bigwedge s\ t.\ s \in S \Longrightarrow t \in ct\ s \Longrightarrow t \in S$
using *ct K-closed* **by** (*auto simp*: *proper-def PiE-iff*)
have *ct-Pi*: *ct* $\in$ *Pi S K*
using *ct* **by** (*auto simp*: *proper-def*)

have *finite* S_r
using *S-finite* **by** (*auto simp*: S_r*-def*)
then show *?thesis*
proof (*induct rule*: *finite-induct-select*)
case (*select X*)
then obtain *l* **where** *l*: $l \in X \to X \cup S2$ **and** *desc*: $\bigwedge s.\ s \in X \Longrightarrow$ *l-desc X ct l s*
by *auto*
obtain *x* **where** *x*: $x \in S_r - X$
using ‹$X \subset S_r$› **by** *auto*
then have $x \in S$
by (*auto simp*: S_r*-def*)

let $?C$ = *simple ct*
let $?v = \lambda s.\ v\ (?C\ s)$ **and** $?E = \lambda s.$ *set-pmf* (*ct s*)
let $?M = \lambda s.$ *maximal* $?v\ (?E\ s)$

have *finite-E*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow$ *finite* $(?E\ s)$
using *K-closed ct* **by** (*intro finite-subset*[*OF - S-finite*]) (*auto simp*: *proper-def subset-eq*)

have *valid-C*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow ?C\ s \in$ *valid-cfg*
using *ct* **by** (*auto simp*: *proper-def intro*!: *simple-valid-cfg*)

have *E-ne*[*simp*]: $\bigwedge s.\ ?E\ s \neq \{\}$
by (*rule set-pmf-not-empty*)

have $\exists\, s \in S_r - X.\ \exists\, t \in ?M\ s.\ t \in S2 \cup X$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then have *not-M*: $\bigwedge s.\ s \in S_r - X \Longrightarrow ?M\ s \cap (S2 \cup X) = \{\}$
by *auto*

let $?S_m$ = *maximal* $?v\ (S_r - X)$

```
have finite (Sᵣ − X) Sᵣ − X ≠ {}
  using ‹X ⊂ Sᵣ› by (auto intro!: finite-subset[OF - S-finite] simp: Sᵣ-def)
from maximal-ne[OF this] obtain s_m where s_m: s_m ∈ ?S_m
  by force

have ∃ s0∈?S_m. ∃ t∈?E s0. t ∉ ?S_m
proof (rule ccontr)
  assume ¬ ?thesis
  then have S_m: ⋀s0 t. s0 ∈ ?S_m ⟹ t ∈ ?E s0 ⟹ t ∈ ?S_m by blast
  from ‹s_m ∈ ?S_m› have [simp]: s_m ∈ S and s_m ∈ Sᵣ
    by (auto simp: Sᵣ-def dest: maximalD1)

  from ‹s_m ∈ ?S_m› have v (?C s_m) = 0
  proof (coinduction arbitrary: s_m rule: v-eq-0-coinduct)
    case (cont t s_m) with S1 show ?case
      by (intro exI[of - state t] disjCI conjI S_m[of s_m state t])
         (auto simp: set-K-cfg)
  qed (auto simp: Sᵣ-def ct-Pi dest!: maximalD1)
  with ‹s_m ∈ Sᵣ› ‹proper ct› show False
    by (auto simp: proper-def)
qed
then obtain s0 t where s0 ∈ ?S_m and t: t ∈ ?E s0 t ∉ ?S_m
  by metis
with Sᵣ-S1 have s0: s0 ∈ Sᵣ − X and [simp]: s0 ∈ S and s0 ∈ S1
  by (auto simp: Sᵣ-def dest: maximalD1)

from ‹proper ct› ‹s0 ∈ S› s0 have ?v s0 ≠ 0
  by (auto simp add: proper-def)
then have 0 < ?v s0 by (simp add: zero-less-iff-neq-zero)

{ fix t assume t ∈ S_e ∪ S2 ∪ X t ∈ ?E s0 and ?v s0 ≤ ?v t
  moreover have t ∈ S_e ⟹ ?v t = 0
    by (simp add: p-eq-0-imp S_e-def ct-Pi)
  ultimately have t: t ∈ S2 ∪ X t ∈ ?E s0
    using ‹0 < ?v s0› by (auto simp: S_e-def)

  have maximal ?v (?E s0 ∩ (S2 ∪ X)) ≠ {}
    using finite-E t by (intro maximal-ne) auto
  moreover
  { fix x y assume x: x ∈ S2 ∪ X x ∈ ?E s0
      and ∗: ∀ y∈?E s0 ∩ (S2 ∪ X). ?v y ≤ ?v x and y: y ∈ ?E s0
    with S2 ‹s0 ∈ S›[THEN ct-closed] have [simp]: x ∈ S y ∈ S
      by auto

    have ?v y ≤ ?v x
    proof cases
      assume y ∈ Sᵣ − X
      then have ?v y ≤ ?v s0
```

using ‹$s_0 \in ?S_m$› **by** (*auto intro*: *maximalD2*)
also note ‹$?v\ s_0 \leq ?v\ t$›
also have $?v\ t \leq ?v\ x$
using $*$ t **by** *auto*
finally show *?thesis* **.**
next
assume $y \notin S_r - X$ **with** y $*$ **show** *?thesis*
by (*auto simp*: S_r*-def* v-S_e[*of ?C y*] *ct-Pi*)
qed }
then have *maximal ?v* $(?E\ s_0 \cap (S2 \cup X)) \subseteq$ *maximal ?v* $(?E\ s_0)$
by (*auto simp*: *maximal-def*)
moreover note *not-M*[*OF* s_0]
ultimately have *False*
by (*blast dest*: *maximalD1*) **}**
then have *less-*s_0: $\bigwedge t.\ t \in S_e \cup S2 \cup X \Longrightarrow t \in ?E\ s_0 \Longrightarrow ?v\ t < ?v\ s_0$
by (*auto simp add*: *not-le*[*symmetric*])

let $?K = ct\ s_0$

have $?v\ s_0 = (\int^+ x.\ ?v\ x\ \partial ?K)$
using *v-S1*[*of ?C* s_0] ‹$s_0 \in S1$› ‹$s_0 \in S$›
by (*auto simp add*: *ct-Pi intro*!: *nn-integral-cong-AE AE-pmfI*)
also have $\ldots < (\int^+ x.\ ?v\ s_0\ \partial ?K)$
proof (*intro nn-integral-less*)
have $(\int^+ x.\ ?v\ x\ \partial ?K) \leq (\int^+ x.\ 1\ \partial ?K)$
using *ct ct-closed*[*of* s_0]
by (*intro nn-integral-mono-AE*)
(*auto intro*!: *v-le-1 simp*: *AE-measure-pmf-iff proper-def ct-Pi*)
then show $(\int^+ x.\ ?v\ x\ \partial ?K) \neq \infty$
by (*auto simp*: *top-unique*)
have $?v\ t < ?v\ s_0$
proof *cases*
assume $t \in S_e \cup S2 \cup X$ **then show** *?thesis*
using *less-*s_0[*of t*] t **by** *simp*
next
assume $t \notin S_e \cup S2 \cup X$
with $t(1)$ *ct-closed*[*of* s_0 t] **have** $t \in S_r - X$
unfolding S_r*-def* **by** (*auto simp*: *E-def*)
with $t(2)$ **show** *?thesis*
using ‹$s_0 \in ?S_m$› **by** (*auto simp*: *maximal-def not-le intro*: *less-le-trans*)
qed
then show $\neg$ (*AE x in ?K*. $?v\ s_0 \leq ?v\ x$)
using t **by** (*auto simp*: *not-le AE-measure-pmf-iff E-def cong del*: *AE-cong*
intro!: *exI*[*of - t*])

show *AE x in ?K*. $?v\ x \leq ?v\ s_0$
proof (*subst AE-measure-pmf-iff*, *safe*)
fix t **assume** t: $t \in ?E\ s_0$
show $?v\ t \leq ?v\ s_0$

```
          proof cases
            assume t ∈ S_e ∪ S2 ∪ X then show ?thesis
              using less-s_0[of t] t by simp
          next
            assume t ∉ S_e ∪ S2 ∪ X with t ‹s_0 ∈ ?S_m› ‹s_0 ∈ S› show ?thesis
              by (elim maximalD2) (auto simp: S_r-def intro!: ct-closed[of - t])
          qed
        qed
      qed (insert ct-closed[of s_0], auto simp: AE-measure-pmf-iff)
      also have ... = ?v s_0
        using ‹s_0 ∈ S› measure-pmf.emeasure-space-1[of ct s_0] by simp
      finally show False
        by simp
    qed
    then obtain s t where s: s ∈ S_r − X and t: t ∈ S2 ∪ X t ∈ ?M s
      by auto
    with S2 ‹X ⊂ S_r› have s ∉ S2 and s ∈ S ∧ s ∉ S2 and s ∉ Xand [simp]: t
∈ S
      by (auto simp add: S_r-def)
    define l' where l' = l(s := t)
    then have l'-s[simp, intro]: l' s = t
      by simp

    let ?D = λX l. directed-towards S2 (SIGMA s : X. {l s})
    { fix s' assume s' ∈ ?D X l s' ∈ X
      from this(1) have s' ∈ ?D (insert s X) l'
        by (rule directed-towards-mono) (auto simp: l'-def ‹s ∉ X›) }
    note directed-towards-l' = this

    show ?case
    proof (intro bexI ballI, elim insertE)
      show s ∈ S_r − X by fact
      show l' ∈ insert s X → insert s X ∪ S2
        using s t l by (auto simp: l'-def)
    next
      fix s' assume s': s' ∈ X
      moreover
      from desc[OF s'] have s' ∈ ?D X l and ∗: ?v s' ≤ ?v (l s') l s' ∈ ?M s'
        by (auto simp: l-desc-def)
      moreover have l' s' = l s'
        using ‹s' ∈ X› s by (auto simp add: l'-def)
      ultimately show l-desc (insert s X) ct l' s'
        by (auto simp: l-desc-def intro!: directed-towards-l')
    next
      fix s' assume s' = s
      show l-desc (insert s X) ct l' s'
        unfolding ‹s' = s› l-desc-def l'-s
      proof (intro conjI)
        show s ∈ ?D (insert s X) l'
```

proof *cases*
  assume $t \notin S2$
  with t **have** $t \in X$ **by** *auto*
  with *desc* **have** $t \in ?D\ X\ l$
    by (*simp add*: *l-desc-def*)
  then show *?thesis*
    by (*force intro*: *directed-towards.step*[*OF directed-towards-l'*] ‹$t \in X$›)
qed (*force intro*: *directed-towards.step directed-towards.start*)

from ‹$s \in S_r - X$› S_r-*S1* **have** [*simp*]: $s \in S1\ s \in S$
  by (*auto simp*: S_r-*def*)
show $?v\ s \leq ?v\ t$
  using $t(2)$[*THEN maximalD2*] *ct*
  by (*auto simp add*: *v-S1 AE-measure-pmf-iff proper-def Pi-iff PiE-def*
      intro!: *measure-pmf.nn-integral-le-const*)
qed *fact*
qed
qed *simp*
qed

lemma *F-v-memoryless*:
obtains *ct* **where** $ct \in Pi_E\ S\ K\ v \circ simple\ ct = F\text{-}sup\ (v \circ simple\ ct)$
proof *atomize-elim*
define *R* **where** $R = \{(ct(s := D),\ ct) \mid ct\ s\ D.$
  $ct \in Pi_E\ S\ K \wedge proper\ ct \wedge s \in S_r \wedge D \in K\ s \wedge v\ (simple\ ct\ s) < (\int^+ t.\ v$
$(simple\ ct\ t)\ \partial D)\ \}$

{ fix *ct ct'* **assume** *ct-ct'*: $(ct',\ ct) \in R$
let $?v = \lambda s.\ v\ (simple\ ct\ s)$ **and** $?v' = \lambda s.\ v\ (simple\ ct'\ s)$

from *ct-ct'* **obtain** *s D* **where** $ct \in Pi_E\ S\ K$ *proper ct* **and** *s*: $s \in S_r$ **and** *D*:
$D \in K\ s$
  and *not-maximal*: $?v\ s < (\int^+ t.\ ?v\ t\ \partial D)$ **and** *ct'-eq*: $ct' = ct(s := D)$
  by (*auto simp*: *R-def*)
with S_r-*S1* **have** *ct*: $ct \in Pi\ S\ K$ **and** $s \in S$ **and** $s \in S1$
  by (*auto simp*: S_r-*def*)
then have *valid-ct*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow simple\ ct\ s \in cfg\text{-}on\ s$
  by *simp*

from *ct'-eq* **have** [*simp*]: $ct'\ s = D\ \bigwedge t.\ t \neq s \Longrightarrow ct'\ t = ct\ t$
  by *simp-all*

from *ct-ct'* S_r **have** *ct'-E*: $ct' \in Pi_E\ S\ K$
  by (*auto simp*: *ct'-eq R-def*)
from *ct s D* **have** *ct'*: $ct' \in Pi\ S\ K$
  by (*auto simp*: *ct'-eq*)
then have *valid-ct'*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow simple\ ct'\ s \in cfg\text{-}on\ s$
  by *simp*

from *exists-l-desc*[*OF* ‹*proper ct*›]
obtain *l* **where** *l*: $l \in S_r \to S_r \cup S2$ **and** $\bigwedge s.\ s \in S_r \Longrightarrow$ *l-desc* S_r *ct l s*
by *auto*
then have *directed-l*: $\bigwedge s.\ s \in S_r \Longrightarrow s \in$ *directed-towards S2* (*SIGMA s*:S_r. {*l s*})
and *v-l-mono*: $\bigwedge s.\ s \in S_r \Longrightarrow$ *?v s* $\leq$ *?v* (*l s*)
and *l-in-Ea*: $\bigwedge s.\ s \in S_r \Longrightarrow$ *l s* $\in$ *ct s*
by (*auto simp*: *l-desc-def dest*!: *maximalD1*)

let *?E* = λ*ct*. *SIGMA s*:S_r. *ct s*
let *?D* = λ*ct*. *directed-towards S2* (*?E ct*)

have *finite-E*[*simp*]: $\bigwedge s.\ s \in S \Longrightarrow$ *finite* (*ct′ s*)
using *ct′ K-closed* **by** (*intro rev-finite-subset*[*OF S-finite*]) *auto*

have *maximal ?v* (*ct′ s*) $\neq$ {}
using *ct′ D* ‹*s*∈*S*› *finite-E*[*of s*] **by** (*intro maximal-ne set-pmf-not-empty*) (*auto simp del*: *finite-E*)
then obtain *s′* **where** *s′*: *s′* $\in$ *maximal ?v* (*ct′ s*)
by *blast*
with *K-closed*[*OF* ‹*s* $\in$ *S*›] *D* **have** *s′* $\in$ *S*
by (*auto dest*!: *maximalD1*)

have *s′* $\neq$ *s*
proof
assume [*simp*]: *s′* = *s*
have *?v s* < ($\int^+$*t*. *?v t* ∂*D*)
by *fact*
also have ... $\leq$ ($\int^+$*t*. *?v s* ∂*D*)
using ‹*s* $\in$ *S*› *s′ D* **by** (*intro nn-integral-mono-AE*) (*auto simp*: *AE-measure-pmf-iff intro*: *maximalD2*)
finally show *False*
using *measure-pmf.emeasure-space-1*[*of D*] **by** (*simp add*: ‹*s* $\in$ *S*› *ct*)
qed

have *p s′* $\neq$ *0*
proof
assume *p s′* = *0*
then have *?v s′* = *0*
using *v-le-p*[*of simple ct s′*] *ct* ‹*s′* $\in$ *S*› **by** (*auto intro*!: *antisym ct*)
then have ($\int^+$*t*. *?v t* ∂*D*) = *0*
using *maximalD2*[*OF s′*] **by** (*subst nn-integral-0-iff-AE*) (*auto simp*: ‹*s*∈*S*› *D AE-measure-pmf-iff*)
then have *?v s* < *0*
using *not-maximal* **by** *auto*
then show *False*
using ‹*s*∈*S*› **by** (*simp add*: *ct*)
qed
with ‹*s′* $\in$ *S*› **have** *s′* $\in$ *S2* $\cup$ S_r

```
    by (auto simp: S_r-def S_e-def)

  have l-acyclic: (s', s) ∉ (SIGMA s:S_r. {l s})^+
  proof
    assume (s', s) ∈ (SIGMA s:S_r. {l s})^+
    then have ?v s' ≤ ?v s
      by induct (blast intro: order-trans v-l-mono)+
    also have ... < (∫+t. ?v t ∂D)
      using not-maximal .
    also have ... ≤ (∫+t. ?v s' ∂D)
    using s' by (intro nn-integral-mono-AE) (auto simp: ‹s ∈ S› D AE-measure-pmf-iff
intro: maximalD2)
    finally show False
      using measure-pmf.emeasure-space-1[of D] by (simp add:‹s' ∈ S› ct)
  qed

  from ‹s' ∈ S2 ∪ S_r› have s' ∈ ?D ct'
  proof
    assume s' ∈ S_r
    then have l s' ∈ directed-towards S2 (SIGMA s:S_r. {l s})
      using l directed-l[of l s'] by (auto intro: directed-towards.start)
    moreover from ‹s' ∈ S_r› have (s', l s') ∈ (SIGMA s:S_r. {l s})^+
      by auto
    ultimately have l s' ∈ ?D ct'
    proof induct
      case (step t t')
      then have t: t ≠ s t ∈ S_r t' = l t
        using l-acyclic by auto

      from step have (s', t') ∈ (SIGMA s:S_r. {l s})^+
        by (blast intro: trancl-into-trancl)
      from step(2)[OF this] show ?case
        by (rule directed-towards.step) (simp add: l-in-Ea t)
    qed (rule directed-towards.start)
    then show s' ∈ ?D ct'
      by (rule directed-towards.step)
         (simp add: l-in-Ea ‹s' ∈ S_r› ‹s ∈ S_r› ‹s' ≠ s›)
  qed (rule directed-towards.start)

  have proper: proper ct'
    unfolding proper-eq[OF ct'-E]
  proof
    fix t assume t ∈ S_r
    from directed-l[OF this] show t ∈ ?D ct'
    proof induct
      case (step t t')
      show ?case
      proof cases
        assume t = s
```

with ‹$s \in S_r$› s'[*THEN maximalD1*] **have** $(t,\ s') \in$ *?E ct'*
  by *auto*
with ‹$s' \in$ *?D ct'*› **show** *?thesis*
  by (*rule directed-towards.step*)
next
  assume $t \neq s$
  with *step* **have** $(t,\ t') \in$ *?E ct'*
    by (*auto simp*: *l-in-Ea*)
  with *step.hyps*(*2*) **show** *?thesis*
    by (*rule directed-towards.step*)
qed
qed (*rule directed-towards.start*)
qed

have *?v* $\leq$ *?v'*
proof (*intro le-funI leI notI*)
  fix t' **assume** $*$: *?v' t'* $<$ *?v t'*
  then have $t' \in S$
    by (*metis v-nS simple-valid-cfg-iff ct' ct order.irrefl*)

  define Δ **where** $\Delta\ t =$ *enn2real* (*?v t*) $-$ *enn2real* (*?v' t*) **for** t
  with $*$ ‹$t' \in S$› **have** $0 < \Delta\ t'$
    by (*cases ?v t' ?v' t' rule*: *ennreal2-cases*) (*auto simp add*: *ct' ct ennreal-less-iff*)

  { fix t **assume** t: $t \in$ *maximal* Δ S
    with ‹$t' \in S$› **have** $\Delta\ t' \leq \Delta\ t$
      by (*auto intro*: *maximalD2*)
    with ‹$0 < \Delta\ t'$› **have** $0 < \Delta\ t$ **by** *simp*
    with t **have** $t \in S_r$
      by (*auto simp add*: S_r*-def v-*S_e *ct ct'* Δ*-def dest*!: *maximalD1*) **}**
  note *max-is-*S_r = *this*

  { fix s **assume** $s \in S$
    with *v-le-1*[*of simple ct' s*] *v-le-1*[*of simple ct s*]
    have $|\Delta\ s| \leq 1$
      by (*cases ?v s ?v' s rule*: *ennreal2-cases*) (*auto simp*: Δ*-def ct ct'*) **}**
  note Δ*-le-1*[*simp*] = *this*
  then have *ennreal-*Δ: $\bigwedge s.\ s \in S \Longrightarrow \Delta\ s =$ *?v s* $-$ *?v' s*
   by (*auto simp add*: Δ*-def v-def T.emeasure-eq-measure ct ct' ennreal-minus*)

  from ‹$s \in S$› *S-finite* **have** *maximal* Δ $S \neq \{\}$
    by (*intro maximal-ne*) *auto*
  then obtain t **where** $t \in$ *maximal* Δ S **by** *auto*
  from *max-is-*S_r[*OF this*] *proper* **have** $t \in$ *?D ct'*
    unfolding *proper-eq*[*OF ct'-E*] **by** *auto*
  from *this* ‹$t \in$ *maximal* Δ S› **show** *False*
  proof *induct*
    case (*start t*)

then have $t \in S_r$
by (*intro max-is-*S_r)
with ‹$t \in S2$› **show** *False*
by (*auto simp*: S_r*-def*)
next
case (*step t t'*)
then have t': $t' \in ct'\ t$ **and** $t \in S_r$ **and** t: $t \in maximal\ \Delta\ S$
by (*auto intro*: *max-is-*S_r *simp*: *comp-def*)
then have $t' \in S\ t \in S1\ t \in S$
using S_r*-S1 S1*
by (*auto simp*: *Pi-closed*[*OF ct'*])

have $\Delta\ t \le \Delta\ t'$
proof (*intro leI notI*)
assume *less*: $\Delta\ t' < \Delta\ t$
have $(\int s.\ \Delta\ s\ \partial ct'\ t) < (\int s.\ \Delta\ t\ \partial ct'\ t)$
proof (*intro measure-pmf.integral-less-AE*)
show *emeasure* $(ct'\ t)\ \{t'\} \neq 0\ \{t'\} \in sets\ (ct'\ t)$
AE s in ct' t. $s \in \{t'\} \longrightarrow \Delta\ s \neq \Delta\ t$
using *t' less* **by** (*auto simp add*: *emeasure-pmf-single-eq-zero-iff*)
show *AE s in ct' t.* $\Delta\ s \le \Delta\ t$
using *ct' ct t D*
by (*auto simp add*: *AE-measure-pmf-iff ct* ‹$t \in S$› *Pi-iff E-def Pi-closed*[*OF ct'*]
intro!: *maximalD2*[*of t* Δ] *intro*: *Pi-closed*[*OF ct'*] *maximalD1*)
show *integrable* $(ct'\ t)\ (\lambda\text{-}.\ \Delta\ t)$ *integrable* $(ct'\ t)\ \Delta$
using *ct ct'* ‹$t \in S$› *D*
by (*auto intro*!: *measure-pmf.integrable-const-bound*[**where** *B=1*] Δ*-le-1*
simp: *AE-measure-pmf-iff dest*: *Pi-closed*)
qed
also have ... $= \Delta\ t$
using *measure-pmf.prob-space*[*of ct' t*] **by** *simp*
also have $\Delta\ t \le (\int s.\ enn2real\ (?v\ s)\ \partial ct'\ t) - (\int s.\ enn2real\ (?v'\ s)\ \partial ct'\ t)$
proof −
have $?v\ t \le (\int^+ s.\ ?v\ s\ \partial ct'\ t)$
proof *cases*
assume $t = s$ **with** *not-maximal* **show** *?thesis* **by** *simp*
next
assume $t \neq s$ **with** *S1* ‹$t \in S1$› ‹$t \in S$› *ct ct'* **show** *?thesis*
by (*subst v-S1*) (*auto intro*!: *nn-integral-mono-AE AE-pmfI*)
qed
also have ... = *ennreal* $(\int s.\ enn2real\ (?v\ s)\ \partial ct'\ t)$
using *ct ct'* ‹$t \in S$›
by (*intro measure-pmf.ennreal-integral-real*[*symmetric*, **where** *B=1*])
(*auto simp*: *AE-measure-pmf-iff one-ennreal-def*[*symmetric*]
intro!: *v-le-1 simple-valid-cfg intro*: *Pi-closed*)
finally have $enn2real\ (?v\ t) \le (\int s.\ enn2real\ (?v\ s)\ \partial ct'\ t)$
using *ct* ‹$t \in S$› **by** (*simp add*: *v-def T.emeasure-eq-measure*)

moreover
{ **have** *?v′ t* = (∫⁺*s. ?v′ s ∂ct′ t*)
using *ct ct′* ‹*t* ∈ *S*› ‹*t* ∈ *S1*› *S1* **by** (*subst v-S1*) (*auto intro*!: *nn-integral-cong-AE AE-pmfI*)
also have ... = *ennreal* (∫ *s. enn2real* (*?v′ s*) ∂*ct′ t*)
using *ct′* ‹*t*∈*S*›
by (*intro measure-pmf.ennreal-integral-real*[*symmetric*, **where** *B=1*])
(*auto simp*: *AE-measure-pmf-iff one-ennreal-def*[*symmetric*]
intro!: *v-le-1 simple-valid-cfg intro*: *Pi-closed*)
finally have *enn2real* (*?v′ t*) = (∫ *s. enn2real* (*?v′ s*) ∂*ct′ t*)
using *ct′* ‹*t*∈*S*› **by** (*simp add*: *v-def T.emeasure-eq-measure*) }
ultimately show *?thesis*
using ‹*t* ∈ *S*› **by** (*simp add*: Δ-*def ennreal-minus-mono*)
qed
also have ... = (∫ *s.* Δ *s* ∂*ct′ t*)
unfolding Δ-*def* **using** *Pi-closed*[*OF ct* ‹*t*∈*S*›] *Pi-closed*[*OF ct′* ‹*t*∈*S*›] *ct ct′*
by (*intro Bochner-Integration.integral-diff*[*symmetric*] *measure-pmf.integrable-const-bound*[**where** *B=1*])
(*auto simp*: *AE-measure-pmf-iff real-v*)
finally show *False*
by *simp*
qed
with *t*[*THEN maximalD2*] ‹*t* ∈ *S*› ‹*t′* ∈ *S*› **have** Δ *t* = Δ *t′*
by (*auto intro*: *antisym*)
with *t* ‹*t′* ∈ *S*› **have** *t′* ∈ *maximal* Δ *S*
by (*auto simp*: *maximal-def*)
then show *?case*
by *fact*
qed
qed
moreover have *?v s* < *?v′ s*
proof −
have *?v s* < (∫⁺*t. ?v t ∂D*)
by *fact*
also have ... ≤ (∫⁺*t. ?v′ t ∂D*)
using ‹*?v* ≤ *?v′*› ‹*s*∈*S*› *D ct ct′*
by (*intro nn-integral-mono*) (*auto simp*: *le-fun-def*)
also have ... = *?v′ s*
using ‹*s*∈*S1*› *S1 ct′* ‹*s* ∈ *S*› **by** (*subst* (*2*) *v-S1*) (*auto intro*!: *nn-integral-cong-AE AE-pmfI*)
finally show *?thesis* **.**
qed
ultimately have *?v* < *?v′*
by (*auto simp*: *less-le le-fun-def fun-eq-iff*)
note *this proper ct′* }
note *v-strict* = *this*(*1*) **and** *proper* = *this*(*2*) **and** *sc′-R* = *this*(*3*)

have *finite* (Pi_E *S K* × Pi_E *S K*)

by (*intro finite-PiE S-finite K-finite finite-SigmaI*)
then have *finite R*
by (*rule rev-finite-subset*) (*auto simp add*: *PiE-iff* S_r*-def R-def intro*: *extensional-arb*)
moreover
from *v-strict* **have** *acyclic R*
by (*rule acyclicI-order*)
ultimately have *wf R*
by (*rule finite-acyclic-wf*)

from *exists-proper* **obtain** *ct′* **where** *ct′*: *proper ct′* **.**
define *ct* **where** *ct = restrict ct′ S*
with *ct′* **have** *sc-Pi*: *ct ∈ Pi S K* **and** *ct′ ∈ Pi S K*
by (*auto simp*: *proper-def*)
then have *ct*: *ct ∈ {ct ∈* Pi_E *S K. proper ct}*
using *ct′ directed-towards-mono*[**where** *F=SIGMA s:*S_r*. ct′ s* **and** *G=SIGMA s:*S_r*. ct s*]
apply *simp*
apply (*subst proper-eq*)
by (*auto simp*: *ct-def proper-eq*[*OF properD1*[*OF ct′*]] *subset-eq* S_r*-def*)

show ∃ *ct. ct ∈* Pi_E *S K ∧ v∘simple ct = F-sup* (*v∘simple ct*)
proof (*rule wfE-min*[*OF ‹wf R› ct*])
fix *ct* **assume** *ct*: *ct ∈ {ct ∈* Pi_E *S K. proper ct}*
then have *ct ∈ Pi S K proper ct*
by (*auto simp*: *proper-def*)
assume *min*: ⋀*ct′. (ct′, ct) ∈ R ⟹ ct′ ∉ {ct ∈* Pi_E *S K. proper ct}*
let *?v = λs. v* (*simple ct s*)
{ fix *s* **assume** *s ∈ S s ∈ S1 s ∉ S2*
with *ct* **have** *ct s ∈ K s ?v s ≤* $integral^N$ (*ct s*) *?v*
by (*auto simp*: *v-S1 PiE-def intro*!: *nn-integral-mono-AE AE-pmfI*)
moreover
{ have *0 ≤ ?v s*
using *‹s∈S› ct* **by** (*simp add*: *PiE-def*)
also assume *v-less*: *?v s <* (⨆*D∈K s.* ∫⁺ *s. v* (*simple ct s*) *∂measure-pmf D*)
also have *... ≤ p s*
unfolding *p-S1*[*OF ‹s∈S1›*] **using** *‹s∈S› ct v-le-p*[*OF simple-valid-cfg*, *OF ‹ct ∈ Pi S K›*]
by (*auto intro*!: *SUP-mono nn-integral-mono-AE bexI*
simp: *PiE-def AE-measure-pmf-iff set-pmf-closed*)
finally have *s ∈* S_r
using *‹s∈S› ‹s∉S2›* **by** (*simp add*: S_r*-def* S_e*-def*)

from *v-less* **obtain** *D* **where** *D ∈ K s ?v s <* $integral^N$ *D ?v*
by (*auto simp*: *less-SUP-iff*)
with *ct ‹s∈S› ‹s∈*S_r*›* **have** (*ct*(*s:=D*), *ct*) *∈ R ct*(*s:=D*) *∈* Pi_E *S K*
unfolding *R-def* **by** (*auto simp*: *PiE-def extensional-def*)
from *proper*[*OF this*(*1*)] *min*[*OF this*(*1*)] *ct ‹D ∈ K s› ‹s∈S› this*(*2*)

have *False*
by *simp* **}**
ultimately have *?v s* = ($\bigsqcup$*D*∈*K s*. $\int^+$ *s*. *?v s* ∂*measure-pmf D*)
by (*auto intro*: *antisym SUP-upper2*[**where** *i*=*ct s*] *leI*)
also have ... = ($\bigsqcup$*D*∈*K s*. *integral*^{N} (*measure-pmf D*) (λ*s*∈*S*. *?v s*))
using ‹*s*∈*S*› **by** (*auto intro*!: *SUP-cong nn-integral-cong v-nS simp*: *ct simple-valid-cfg-iff* ‹*ct* ∈ *Pi S K*›)
finally have *?v s* = ($\bigsqcup$*D*∈*K s*. *integral*^{N} (*measure-pmf D*) (λ*s*∈*S*. *?v s*)) **. }**
then have *?v* = *F-sup ?v*
unfolding *F-sup-def* **using** *ct*
by (*auto intro*!: *ext v-S2 simple-cfg-on v-nS v-nS12 SUP-cong nn-integral-cong simp*: *PiE-def simple-valid-cfg-iff*)
with *ct* **show** *?thesis*
by (*auto simp*: *comp-def*)
qed
qed

lemma *p-v-memoryless*:
obtains *ct* **where** *ct* ∈ Pi_E *S K p* = *v*∘*simple ct*
proof −
obtain *ct* **where** *ct-PiE*: *ct* ∈ Pi_E *S K* **and** *eq*: *v*∘*simple ct* = *F-sup* (*v*∘*simple ct*)
by (*rule F-v-memoryless*)
then have *ct*: *ct* ∈ *Pi S K*
by (*simp add*: *PiE-def*)
have *p* = *v*∘*simple ct*
proof (*rule antisym*)
show *p* ≤ *v*∘*simple ct*
unfolding *p-eq-lfp-F-sup* **by** (*rule lfp-lowerbound*) (*metis order-refl eq*)
show *v*∘*simple ct* ≤ *p*
proof (*rule le-funI*)
fix *s* **show** (*v*∘*simple ct*) *s* ≤ *p s*
using *v-le-p*[*of simple ct s*]
by (*cases s* ∈ *S*) (*auto simp del*: *simp add*: *v-def ct*)
qed
qed
with *ct-PiE that* **show** *thesis* **by** *auto*
qed

definition *n* = (λ*s*∈*S*. *P-inf s* (λω. (*HLD S1 suntil HLD S2*) (*s* ## ω)))

lemma *n-eq-INF-v*: *s* ∈ *S* ⟹ *n s* = ($\bigsqcap$*cfg*∈*cfg-on s*. *v cfg*)
by (*auto simp add*: *n-def v-def P-inf-def T.emeasure-eq-measure valid-cfgI intro*!: *INF-cong*)

lemma *n-le-v*: *s* ∈ *S* ⟹ *cfg* ∈ *cfg-on s* ⟹ *n s* ≤ *v cfg*
by (*subst n-eq-INF-v*) (*blast intro*!: *INF-lower*)+

lemma *n-eq-1-imp*: *s* ∈ *S* ⟹ *cfg* ∈ *cfg-on s* ⟹ *n s* = *1* ⟹ *v cfg* = *1*

using *n-le-v*[*of s cfg*] *v-le-1*[*of cfg*] **by** (*auto intro*: *antisym valid-cfgI*)

lemma *n-eq-1-iff*: *s* ∈ *S* ⟹ *n s* = *1* ⟷ (∀ *cfg*∈*cfg-on s*. *v cfg* = *1*)
apply *rule*
apply (*metis n-eq-1-imp*)
apply (*auto simp*: *n-eq-INF-v intro*!: *INF-eqI*)
done

lemma *n-le-1*: *s* ∈ *S* ⟹ *n s* ≤ *1*
by (*auto simp*: *n-eq-INF-v intro*!: *INF-lower2*[*OF simple-cfg-on*[*of arb-act*]] *v-le-1*)

lemma *n-undefined*[*simp*]: *s* ∉ *S* ⟹ *n s* = *undefined*
by (*simp add*: *n-def*)

lemma *n-eq-0*: *s* ∈ *S* ⟹ *cfg* ∈ *cfg-on s* ⟹ *v cfg* = *0* ⟹ *n s* = *0*
using *n-le-v*[*of s cfg*] **by** *auto*

lemma *n-not-inf*[*simp*]: *s* ∈ *S* ⟹ *n s* ≠ *top*
using *n-le-1*[*of s*] **by** (*auto simp*: *top-unique*)

lemma *n-S1*: *s* ∈ *S1* ⟹ *n s* = (⨅*D*∈*K s*. ∫⁺ *t*. *n t* ∂*measure-pmf D*)
using *S1 S1-S2* **unfolding** *n-def*
apply *auto*
apply (*subst P-inf-iterate*)
apply (*auto intro*!: *nn-integral-cong-AE INF-cong intro*: *set-pmf-closed*
simp: *AE-measure-pmf-iff suntil-Stream set-eq-iff*)
done

lemma *n-S2*[*simp*]: *s* ∈ *S2* ⟹ *n s* = *1*
using *S2* **by** (*auto simp add*: *n-eq-INF-v valid-cfgI*)

lemma *n-nS12*: *s* ∈ *S* ⟹ *s* ∉ *S1* ⟹ *s* ∉ *S2* ⟹ *n s* = *0*
by (*auto simp add*: *n-eq-INF-v valid-cfgI*)

lemma *n-pos*:
assumes *P s s* ∈ *S1 wf R*
assumes *cont*: ⋀*s D*. *P s* ⟹ *s* ∈ *S1* ⟹ *D* ∈ *K s* ⟹ ∃ *w*∈*D*. ((*w*, *s*) ∈ *R* ∧
w ∈ *S1* ∧ *P w*) ∨ *0* < *n w*
shows *0* < *n s*
using ‹*wf R*› ‹*P s*› ‹*s*∈*S1*›
proof (*induction s*)
case (*less s*)
with *S1* **have** [*simp*]: *s* ∈ *S* **by** *auto*
let *?I* = λ*D*::′*s pmf*. ∫⁺*t*. *n t* ∂*D*
have *0* < *Min* (*?I*‘*K s*)
proof (*safe intro*!: *Min-gr-iff* [*THEN iffD2*])
fix *D* **assume** [*simp*]: *D* ∈ *K s*
from *cont*[*OF* ‹*P s*› ‹*s* ∈ *S1*› ‹*D* ∈ *K s*›]
obtain *w* **where** *w*: *w* ∈ *D 0* < *n w*

by (*force intro*: *less.IH*)
have *in-S*: $\bigwedge t.\ t \in D \Longrightarrow t \in S$
using *set-pmf-closed*[*OF* ‹*s* ∈ *S*› ‹*D* ∈ *K s*›] **by** *auto*
from *w* **have** $0 < pmf\ D\ w * n\ w$
by (*simp add*: *pmf-positive ennreal-zero-less-mult-iff*)
also have $\ldots = (\int^+ t.\ n\ w * indicator\ \{w\}\ t\ \partial D)$
by (*subst nn-integral-cmult-indicator*)
(*auto simp*: *ac-simps emeasure-pmf-single in-S* ‹*w* ∈ *D*›)
also have $\ldots \le (\int^+ t.\ n\ t\ \partial D)$
by (*intro nn-integral-mono-AE*) (*auto split*: *split-indicator simp*: *AE-measure-pmf-iff in-S*)
finally show $0 < (\int^+ t.\ n\ t\ \partial D)$ **.**
qed (*insert K-wf K-finite* ‹*s*∈*S*›, *auto*)
also have $\ldots = n\ s$
unfolding *n-S1*[*OF* ‹*s* ∈ *S1*›]
using *K-wf K-finite* ‹*s*∈*S*› **by** (*intro Min-Inf*) *auto*
finally show $0 < n\ s$ **.**
qed

definition *F-inf* :: (*'s* ⇒ *ennreal*) ⇒ (*'s* ⇒ *ennreal*) **where**
F-inf f = (λ*s*∈*S*. *if s* ∈ *S2 then 1 else if s* ∈ *S1 then* (⨅*D*∈*K s*. $\int^+$ *t*. *f t* ∂*measure-pmf D*) *else 0*)

lemma *F-inf-n*: *F-inf n* = *n*
by (*simp add*: *F-inf-def n-nS12 n-S1 fun-eq-iff*)

lemma *F-inf-nS*[*simp*]: *s* ∉ *S* ⟹ *F-inf f s* = *undefined*
by (*simp add*: *F-inf-def*)

lemma *mono-F-inf*: *mono F-inf*
by (*auto intro*!: *INF-superset-mono nn-integral-mono simp*: *mono-def F-inf-def le-fun-def*)

lemma *S1-nS2*: *s* ∈ *S1* ⟹ *s* ∉ *S2*
using *S1-S2* **by** *auto*

lemma *n-eq-lfp-F-inf*: *n* = *lfp F-inf*
proof (*intro antisym lfp-lowerbound le-funI*)
fix *s* **let** *?I* = λ*D*. ($\int^+$*t*. *lfp F-inf t* ∂*measure-pmf D*)
define *ct* **where** *ct s* = (*SOME D*. *D* ∈ *K s* ∧ (*s* ∈ *S1* ⟶ *lfp F-inf s* = *?I D*))
for *s*
{ fix *s* **assume** *s*: *s* ∈ *S*
then have *finite* (*?I* ‘ *K s*)
by (*auto intro*: *K-finite*)
with *s* **obtain** *D* **where** *D* ∈ *K s* ($\int^+$*t*. *lfp F-inf t* ∂*D*) = *Min* (*?I* ‘ *K s*)
by (*auto simp*: *K-wf dest*!: *Min-in*)
note *this*(*2*)
also have ... = (*INF D* ∈ *K s*. *?I D*)
using *s K-wf* **by** (*subst Min-Inf*) (*auto intro*: *K-finite*)

also have *s* ∈ *S1* ⟹ ... = *lfp F-inf s*
using *s S1-S2* **by** (*subst* (*3*) *lfp-unfold*[*OF mono-F-inf*]) (*auto simp add*: *F-inf-def*)
finally have ∃ *D*. *D* ∈ *K s* ∧ (*s* ∈ *S1* ⟶ *lfp F-inf s* = *?I D*)
using ‹*D* ∈ *K s*› **by** *auto*
then have *ct s* ∈ *K s* ∧ (*s* ∈ *S1* ⟶ *lfp F-inf s* = *?I* (*ct s*))
unfolding *ct-def* **by** (*rule someI-ex*)
then have *ct s* ∈ *K s s* ∈ *S1* ⟹ *lfp F-inf s* = *?I* (*ct s*)
by *auto* **}**
note *ct* = *this*
then have *Pi-ct*: *ct* ∈ *Pi S K*
by *auto*
then have *valid-ct*[*simp*]: ⋀*s*. *s* ∈ *S* ⟹ *simple ct s* ∈ *valid-cfg*
by *simp*
let *?F* = λ*P*. *HLD S2 or* (*HLD S1 aand nxt P*)
define *P* **where** *P s n* =
emeasure (*T* (*simple ct s*)) {*x*∈*space* (*T* (*simple ct s*)). (*?F* ^^ *n*) (λ*x*. *False*) (*s* ## *x*)}
for *s n*
{ assume *s* ∈ *S*
with *S1* **have** [*simp*, *measurable*]: *s* ∈ *S* **by** *auto*
then have *n s* ≤ *v* (*simple ct s*)
by (*intro n-le-v*) (*auto intro*: *simple-cfg-on*[*OF Pi-ct*])
also have ... = *emeasure* (*T* (*simple ct s*)) {*x*∈*space* (*T* (*simple ct s*)). *lfp ?F* (*s* ## *x*)}
using *S1-S2*
by (*simp add*: *v-eq*[*OF simple-valid-cfg*[*OF Pi-ct* ‹*s*∈*S*›]])
(*simp add*: *suntil-lfp space-T*[*symmetric*, *of simple ct s*] *del*: *space-T*)
also have ... = (⨆*n*. *P s n*) **unfolding** *P-def*
apply (*rule emeasure-lfp2*[**where** *P*=λ*M*. ∃ *s*. *M* = *T* (*simple ct s*) **and** *M*=*T* (*simple ct s*)])
apply (*intro exI*[*of - s*] *refl*)
apply (*auto simp*: *sup-continuous-def*) []
apply *auto* []
proof *safe*
fix *A s* **assume** ⋀*N*. ∃ *s*. *N* = *T* (*simple ct s*) ⟹ *Measurable.pred N A*
then have ⋀*s*. *Measurable.pred* (*T* (*simple ct s*)) *A*
by *metis*
then have ⋀*s*. *Measurable.pred St A*
by *simp*
then show *Measurable.pred* (*T* (*simple ct s*)) (λ*xs*. *HLD S2 xs* ∨ *HLD S1 xs* ∧ *nxt A xs*)
by *simp*
qed
also have ... ≤ *lfp F-inf s*
proof (*intro SUP-least*)
fix *n* **from** ‹*s*∈*S*› **show** *P s n* ≤ *lfp F-inf s*
proof (*induct n arbitrary*: *s*)
case *0* **with** *S1* **show** *?case*

by (*subst lfp-unfold*[*OF mono-F-inf*]) (*auto simp*: *P-def*)
next
case (*Suc n*)

show *?case*
proof *cases*
assume $s \in S1$ **with** *S1-S2 S1* **have** *s*[*simp*]: $s \notin S2$ $s \in S$ $s \in S1$ **by** *auto*
have $P\ s\ (Suc\ n) = (\int^+ t.\ P\ t\ n\ \partial ct\ s)$
unfolding *P-def space-T*
apply (*subst emeasure-Collect-T*)
apply (*rule measurable-compose*[*OF measurable-Stream*[*OF measurable-const measurable-ident-sets*[*OF refl*]]])
apply (*measurable*, *assumption*)
apply (*auto simp*: *K-cfg-def map-pmf-rep-eq nn-integral-distr*
intro!: *nn-integral-cong-AE AE-pmfI*)
done
also have ... $\leq (\int^+ t.\ lfp\ F\text{-}inf\ t\ \partial ct\ s)$
using *Pi-closed*[*OF Pi-ct* ‹$s \in S$›]
by (*auto intro*!: *nn-integral-mono-AE Suc simp*: *AE-measure-pmf-iff*)
also have ... $= lfp\ F\text{-}inf\ s$
by (*intro ct*(*2*)[*symmetric*]) *auto*
finally show *?thesis* **.**
next
assume $s \notin S1$ **with** *S2* ‹$s \in S$› **show** *?case*
using *T.emeasure-space-1*[*of simple ct s*]
by (*subst lfp-unfold*[*OF mono-F-inf*]) (*auto simp*: *F-inf-def P-def*)
qed
qed
qed
finally have $n\ s \leq lfp\ F\text{-}inf\ s$ **. }**
moreover have $s \notin S \Longrightarrow n\ s \leq lfp\ F\text{-}inf\ s$
by (*subst lfp-unfold*[*OF mono-F-inf*]) (*simp add*: *n-def F-inf-def*)
ultimately show $n\ s \leq lfp\ F\text{-}inf\ s$
by *blast*
qed (*simp add*: *F-inf-n*)

lemma *real-n*: $s \in S \Longrightarrow ennreal\ (enn2real\ (n\ s)) = n\ s$
by (*cases n s*) *simp-all*

lemma *real-p*: $s \in S \Longrightarrow ennreal\ (enn2real\ (p\ s)) = p\ s$
by (*cases p s*) *simp-all*

lemma *p-ub*:
fixes *x*
assumes $s \in S$
assumes *solution*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow (\sum t{\in}S.\ pmf\ D\ t * x\ t) \leq x\ s$
assumes *solution-0*: $\bigwedge s.\ s \in S \Longrightarrow p\ s = 0 \Longrightarrow x\ s = 0$

assumes *solution-S2*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows *enn2real* (*p s*) $\leq$ *x s* (**is** *?y s* $\leq$ -)
proof −
let *?p* = $\lambda s.$ *enn2real* (*p s*)
from *p-v-memoryless* **obtain** *sc* **where** $sc \in Pi_E\ S\ K$ **and** *p-eq*: $p = v \circ simple$ *sc*
by *auto*
then have *sch*: $\bigwedge s.\ s \in S \Longrightarrow sc\ s \in K\ s$ **and** *sc-Pi*: $sc \in Pi\ S\ K$
by (*auto simp*: *PiE-iff*)

interpret *sc*: *MC-syntax sc* **.**

define *N* **where** $N = \{s{\in}S.\ p\ s = 0\} \cup S2$
{ fix *s* **assume** $s \in S$ $s \notin N$
with *p-nS12* **have** $s \in S1$
by (*auto simp add*: *N-def*) **}**
note *N* = *this*

have *N-S*: $N \subseteq S$
using *S2* **by** (*auto simp*: *N-def*)

have *finite-sc*[*intro*]: $s \in S \Longrightarrow$ *finite* (*sc s*) **for** *s*
using ‹$sc \in Pi_E\ S\ K$› **by** (*auto simp*: *PiE-iff intro*: *set-pmf-finite*)

show *?thesis*
proof *cases*
assume $s \in S - N$
then show *?thesis*
proof (*rule mono-les*)
show ($\bigcup x{\in}S - N.$ *set-pmf* (*sc x*)) $\subseteq S - N \cup N$
using *Pi-closed*[*OF sc-Pi*] **by** *auto*
show *finite* (($\lambda s.$ *?p s* − *x s*) ‘ ($S - N \cup N$))
using *N-S* **by** (*intro finite-imageI finite-subset*[*OF - S-finite*]) *auto*
next
fix *s* **assume** $s \in N$ **then show** *?p s* $\leq$ *x s*
by (*auto simp*: *N-def solution-S2 solution-0*)
next
fix *s* **assume** *s*: $s \in S - N$
then show *integrable* (*sc s*) *x integrable* (*sc s*) *?p*
by (*auto intro*!: *integrable-measure-pmf-finite set-pmf-finite sch*)

from *s* **have** $s \in S1$ $s \in S$
using *p-nS12*[*of s*] **by** (*auto simp*: *N-def*)
then show *?p s* $\leq$ ($\int$ *t. ?p t* ∂*sc s*) + *0*
unfolding *p-eq* **using** *real-v-integral-eq*[*of simple sc s*]
by (*auto simp add*: *v-S1 sc-Pi intro*!: *integral-mono-AE integrable-measure-pmf-finite* *AE-pmfI*)
show ($\int$ *t. x t* ∂*sc s*) + *0* $\leq$ *x s*

```
        using solution[OF ‹s ∈ S1› sch[OF ‹s ∈ S›]]
        by (subst integral-measure-pmf[where A=S])
          (auto intro: S-finite Pi-closed[OF sc-Pi] ‹s ∈ S› simp: ac-simps)

      define X where X = (SIGMA x:UNIV. sc x)
      show ∃ t∈N. (s, t) ∈ X*
      proof (rule ccontr)
        assume ¬ ?thesis
        then have *: ∀ t∈N. (s, t) ∉ X*
          by auto
        with ‹s∈S› have v (simple sc s) = 0
        proof (coinduction arbitrary: s rule: v-eq-0-coinduct)
          case (valid t) with sch show ?case
            by auto
        next
          case (nS2 s) then show ?case
            by (auto simp: N-def)
        next
          case (cont cfg s)
          then have (s, state cfg) ∈ X*
            by (auto simp: X-def set-K-cfg)
          with cont show ?case
            by (auto simp: set-K-cfg intro!: exI intro: Pi-closed[OF sc-Pi])
              (blast intro: rtrancl-trans)
        qed
        then have p s = 0
          unfolding p-eq by simp
        with ‹s∈S› have s∈N
          by (auto simp: N-def)
        with * show False
          by auto
      qed
    qed
  next
    assume s ∉ S − N with ‹s ∈ S› show ?p s ≤ x s
      by (auto simp: N-def solution-0 solution-S2)
  qed
qed

lemma n-lb:
  fixes x
  assumes s ∈ S
  assumes solution: ⋀s D. s ∈ S1 ⟹ D ∈ K s ⟹ x s ≤ (∑ t∈S. pmf D t ∗ x t)
  assumes solution-n0: ⋀s. s ∈ S ⟹ n s = 0 ⟹ x s = 0
  assumes solution-S2: ⋀s. s ∈ S2 ⟹ x s = 1
  shows x s ≤ enn2real (n s) (is - ≤ ?y s)
proof −
  let ?I = λD::'s pmf. ∫+x. n x ∂D
  { fix s assume s ∈ S1
```

with *S1 S1-S2* **have** $n\ s = (\prod D{\in}K\ s.\ ?I\ D)$
by (*subst n-eq-lfp-F-inf*, *subst lfp-unfold*[*OF mono-F-inf*])
(*auto simp add*: *F-inf-def n-eq-lfp-F-inf*)
moreover have $(\prod D{\in}K\ s.\ \int^{+}x.\ n\ x\ \partial measure\text{-}pmf\ D) = Min\ (?I`K\ s)$
using ‹$s \in S1$› *S1 K-wf*
by (*intro cInf-eq-Min finite-imageI K-finite*) *auto*
moreover have $Min\ (?I`K\ s) \in ?I`K\ s$
using ‹$s \in S1$› *S1 K-wf* **by** (*intro Min-in finite-imageI K-finite*) *auto*
ultimately have $\exists D{\in}K\ s.\ (\int^{+}x.\ n\ x\ \partial D) = n\ s$
by *auto* **}**
then have $\bigwedge s.\ s \in S \Longrightarrow \exists D{\in}K\ s.\ s \in S1 \longrightarrow (\int^{+}x.\ n\ x\ \partial D) = n\ s$
using *K-wf* **by** *auto*
then obtain *sc* **where** *sch*: $\bigwedge s.\ s \in S \Longrightarrow sc\ s \in K\ s$
and *n-sc*: $\bigwedge s.\ s \in S1 \Longrightarrow (\int^{+}x.\ n\ x\ \partial sc\ s) = n\ s$
by (*metis S1 subsetD*)
then have *sc-Pi*: $sc \in Pi\ S\ K$
by *auto*

define *N* **where** $N = \{s{\in}S.\ n\ s = 0\} \cup S2$
with *S2* **have** *N-S*: $N \subseteq S$
by *auto*
{ fix *s* **assume** $s \in S\ s \notin N$
with *n-nS12* **have** $s \in S1$
by (*auto simp add*: *N-def*) **}**
note *N* = *this*

let $?n = \lambda s.\ enn2real\ (n\ s)$
show *?thesis*
proof *cases*
assume $s \in S - N$
then show *?thesis*
proof (*rule mono-les*)
show $(\bigcup x{\in}S - N.\ set\text{-}pmf\ (sc\ x)) \subseteq S - N \cup N$
using *Pi-closed*[*OF sc-Pi*] **by** *auto*
show *finite* $((\lambda s.\ x\ s - ?n\ s)\ `\ (S - N \cup N))$
using *N-S* **by** (*intro finite-imageI finite-subset*[*OF - S-finite*]) *auto*
next
fix *s* **assume** $s \in N$ **then show** $x\ s \leq ?n\ s$
by (*auto simp*: *N-def solution-S2 solution-n0*)
next
fix *s* **assume** *s*: $s \in S - N$
then show *integrable* (*sc s*) *x integrable* (*sc s*) *?n*
by (*auto intro*!: *integrable-measure-pmf-finite set-pmf-finite sch*)

from *s* **have** $s \in S1\ s \in S$
using *n-nS12*[*of s*] **by** (*auto simp*: *N-def*)
then have $(\int t.\ ?n\ t\ \partial sc\ s) = ?n\ s$
apply (*subst n-sc*[*symmetric*, *of s*])
apply *simp-all*

```
      apply (subst integral-eq-nn-integral)
      apply (auto simp: Pi-closed[OF sc-Pi] AE-measure-pmf-iff
             intro!: arg-cong[where f=enn2real] nn-integral-cong-AE real-n)
      done
    then show (∫ t. ?n t ∂sc s) + 0 ≤ ?n s
      by simp

    show x s ≤ (∫ t. x t ∂sc s) + 0
      using solution[OF ‹s ∈ S1› sch[OF ‹s ∈ S›]]
      by (subst integral-measure-pmf[where A=S])
        (auto intro: S-finite Pi-closed[OF sc-Pi] ‹s ∈ S› simp: ac-simps)

    define X where X = (SIGMA x:UNIV. sc x)
    show ∃ t∈N. (s, t) ∈ X*
    proof (rule ccontr)
      assume ¬ ?thesis
      then have *: ∀ t∈N. (s, t) ∉ X*
        by auto
      with ‹s∈S› have v (simple sc s) = 0
      proof (coinduction arbitrary: s rule: v-eq-0-coinduct)
        case (valid t) with sch show ?case
          by auto
      next
        case (nS2 s) then show ?case
          by (auto simp: N-def)
      next
        case (cont cfg s)
        then have (s, state cfg) ∈ X*
          by (auto simp: X-def set-K-cfg)
        with cont show ?case
          by (auto simp: set-K-cfg intro!: exI intro: Pi-closed[OF sc-Pi])
            (blast intro: rtrancl-trans)
      qed
      from n-eq-0[OF ‹s ∈ S› simple-cfg-on this] have n s = 0
        by (auto simp: sc-Pi)
      with ‹s∈S› have s∈N
        by (auto simp: N-def)
      with * show False
        by auto
    qed
  qed
next
  assume s ∉ S − N with ‹s ∈ S› show x s ≤ ?n s
    by (auto simp: N-def solution-n0 solution-S2)
qed
qed

end
```

end

6 Discrete-time Markov Processes

In this file we construct discrete-time Markov processes, e.g. with arbitrary state spaces.

theory *Discrete-Time-Markov-Process*
  imports *Markov-Models-Auxiliary*
begin

lemma *measure-eqI-PiM-sequence*:
  fixes M :: *nat* $\Rightarrow$ *'a measure*
  assumes $*$[*simp*]: *sets P* = *PiM UNIV M sets Q* = *PiM UNIV M*
  assumes *eq*: $\bigwedge A\ n.\ (\bigwedge i.\ A\ i \in$ *sets* $(M\ i)) \Longrightarrow$
    P (*prod-emb UNIV M* {..*n*} (Pi_E {..*n*} *A*)) = *Q* (*prod-emb UNIV M* {..*n*} (Pi_E {..*n*} *A*))
  assumes *A*: *finite-measure P*
  shows $P = Q$
proof (*rule measure-eqI-PiM-infinite*[*OF* $*$ - *A*])
  fix *J* :: *nat set* **and** F'
  assume *J*: *finite J* $\bigwedge i.\ i \in J \Longrightarrow F'\ i \in$ *sets* (*M i*)

  define *n* **where** *n* = (*if J* = {} *then 0 else Max J*)
  define *F* **where** *F i* = (*if i* $\in$ *J then* F' *i else space* (*M i*)) **for** *i*
  then have *F*[*simp*, *measurable*]: *F i* $\in$ *sets* (*M i*) **for** *i*
    using *J* **by** *auto*
  have *emb-eq*: *prod-emb UNIV M J* (Pi_E *J* F') = *prod-emb UNIV M* {..*n*} (Pi_E {..*n*} *F*)
  proof *cases*
    assume *J* = {} **then show** *?thesis*
      by (*auto simp add*: *n-def F-def*[*abs-def*] *prod-emb-def PiE-def*)
  next
    assume *J* $\neq$ {} **then show** *?thesis*
      by (*auto simp*: *prod-emb-def PiE-iff F-def n-def less-Suc-eq-le* ‹*finite J*› *split*: *if-split-asm*)
  qed

  show *emeasure P* (*prod-emb UNIV M J* (Pi_E *J* F')) = *emeasure Q* (*prod-emb UNIV M J* (Pi_E *J* F'))
    unfolding *emb-eq* **by** (*rule eq*) *fact*
qed

lemma *distr-cong-simp*:
  M = *K* $\Longrightarrow$ *sets N* = *sets L* $\Longrightarrow$ ($\bigwedge x.\ x \in$ *space M* =*simp*=> *f x* = *g x*) $\Longrightarrow$ *distr M N f* = *distr K L g*
  unfolding *simp-implies-def* **by** (*rule distr-cong*)

6.1 Constructing Discrete-Time Markov Processes

locale *discrete-Markov-process* =
fixes M :: $'a$ *measure* **and** K :: $'a \Rightarrow 'a$ *measure*
assumes K[*measurable*]: $K \in M \to_M$ *prob-algebra* M
begin

lemma *space-K*: $x \in$ *space* $M \Longrightarrow$ *space* $(K\ x) =$ *space* M
using K **unfolding** *prob-algebra-def* **unfolding** *measurable-restrict-space2-iff*
by (*auto dest*: *subprob-measurableD*)

lemma *sets-K*[*measurable-cong*]: $x \in$ *space* $M \Longrightarrow$ *sets* $(K\ x) =$ *sets* M
using K **unfolding** *prob-algebra-def* **unfolding** *measurable-restrict-space2-iff*
by (*auto dest*: *subprob-measurableD*)

lemma *prob-space-K*: $x \in$ *space* $M \Longrightarrow$ *prob-space* $(K\ x)$
using *measurable-space*[*OF K*] **by** (*simp add*: *space-prob-algebra*)

definition K' :: $'a \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a$ *measure*
where
$K'\ x\ n'\ \omega' = K$ (*case-nat* $x\ \omega'\ n'$)

lemma *IT-K'*:
assumes x: $x \in$ *space* M **shows** *Ionescu-Tulcea* $(K'\ x)$ $(\lambda$-. $M)$
unfolding *Ionescu-Tulcea-def* K'-*def*[*abs-def*]
proof *safe*
fix i **show** $(\lambda\omega'.\ K$ (*case* i *of* $0 \Rightarrow x$ | *Suc* $x \Rightarrow \omega'\ x)) \in Pi_M\ \{0..<i\}\ (\lambda$-. $M)$ $\to_M$ *subprob-algebra* M
using x **by** (*intro measurable-prob-algebraD measurable-compose*[*OF - K*]) *measurable*
next
fix i :: *nat* **and** ω **assume** ω: $\omega \in$ *space* $(Pi_M\ \{0..<i\}\ (\lambda$-. $M))$
with x **have** (*case* i *of* $0 \Rightarrow x$ | *Suc* $x \Rightarrow \omega\ x) \in$ *space* M
by (*auto simp*: *space-PiM split*: *nat.split*)
then show *prob-space* (K (*case* i *of* $0 \Rightarrow x$ | *Suc* $x \Rightarrow \omega\ x$))
using K **unfolding** *measurable-restrict-space2-iff prob-algebra-def* **by** *auto*
qed

definition *lim-sequence* :: $'a \Rightarrow (nat \Rightarrow 'a)$ *measure*
where
lim-sequence x = *projective-family.lim UNIV* (*Ionescu-Tulcea.CI* $(K'\ x)$ $(\lambda$-. $M))$ $(\lambda$-. $M)$

lemma
assumes x: $x \in$ *space* M
shows *space-lim-sequence*: *space* (*lim-sequence* x) = *space* $(\Pi_M\ i\in UNIV.\ M)$
and *sets-lim-sequence*[*measurable-cong*]: *sets* (*lim-sequence* x) = *sets* $(\Pi_M\ i\in UNIV.\ M)$
and *emeasure-lim-sequence-emb*: $\bigwedge J\ X.$ *finite* $J \Longrightarrow X \in$ *sets* $(\Pi_M\ j\in J.\ M)$ $\Longrightarrow$

emeasure (*lim-sequence x*) (*prod-emb UNIV* (λ-. *M*) *J X*) =
emeasure (*Ionescu-Tulcea.CI* (*K′ x*) (λ-. *M*) *J*) *X*
and *emeasure-lim-sequence-emb-I0o*: ⋀*n X*. *X* ∈ *sets* (Π_M *i* ∈ {*0*..<*n*}. *M*)
⟹
emeasure (*lim-sequence x*) (*prod-emb UNIV* (λ-. *M*) {*0*..<*n*} *X*) =
emeasure (*Ionescu-Tulcea.C* (*K′ x*) (λ-. *M*) *0 n* (λ*x*. *undefined*)) *X*
proof −
interpret *Ionescu-Tulcea K′ x* λ-. *M*
using *x* **by** (*rule IT-K′*)
show *space* (*lim-sequence x*) = *space* (Π_M *i*∈*UNIV*. *M*)
unfolding *lim-sequence-def* **by** *simp*
show *sets* (*lim-sequence x*) = *sets* (Π_M *i*∈*UNIV*. *M*)
unfolding *lim-sequence-def* **by** *simp*

{ fix *J* :: *nat set* **and** *X* **assume** *finite J X* ∈ *sets* (Π_M *j*∈*J*. *M*)
then show *emeasure* (*lim-sequence x*) (*PF.emb UNIV J X*) = *emeasure* (*CI J*) *X*
unfolding *lim-sequence-def* **by** (*rule lim*) **}**
note *emb* = *this*

have *up-to-I0o*[*simp*]: *up-to* {*0*..<*n*} = *n* **for** *n*
unfolding *up-to-def* **by** (*rule Least-equality*) *auto*

{ fix *n* :: *nat* **and** *X* **assume** *X* ∈ *sets* (Π_M *j*∈{*0*..<*n*}. *M*)
then show *emeasure* (*lim-sequence x*) (*PF.emb UNIV* {*0*..<*n*} *X*) = *emeasure* (*C 0 n* (λ*x*. *undefined*)) *X*
by (*simp add*: *space-C emb CI-def space-PiM distr-id2 sets-C cong*: *distr-cong-simp*)
}
qed

lemma *lim-sequence*[*measurable*]: *lim-sequence* ∈ *M* $\rightarrow_M$ *prob-algebra* (Π_M *i*∈*UNIV*. *M*)
proof (*intro measurable-prob-algebra-generated*[*OF sets-PiM Int-stable-prod-algebra prod-algebra-sets-into-space*])
fix *a* **assume** [*simp*]: *a* ∈ *space M*
interpret *Ionescu-Tulcea K′ a* λ-. *M*
by (*rule IT-K′*) *simp*
have *sp*: *space* (*lim-sequence a*) = *prod-emb UNIV* (λ-. *M*) {} (Π_E *j*∈{}. *space M*) *space* (*CI* {}) = {} $\rightarrow_E$ *space M*
by (*auto simp*: *space-lim-sequence space-PiM prod-emb-def PF.space-P*)
show *prob-space* (*lim-sequence a*)
apply *standard*
using *PF.prob-space-P*[*THEN prob-space.emeasure-space-1*, *of* {}]
apply (*simp add*: *sp emeasure-lim-sequence-emb del*: *PiE-empty-domain*)
done
show *sets* (*lim-sequence a*) = *sets* (Pi_M *UNIV* (λ*i*. *M*))
by (*simp add*: *sets-lim-sequence*)
next
fix *X* :: (*nat* ⇒ ′*a*) *set* **assume** *X* ∈ *prod-algebra UNIV* (λ*i*. *M*)

then obtain *J* :: *nat set* **and** *F* **where** *J*: $J \neq \{\}$ *finite J F* $\in$ *J* $\rightarrow$ *sets M*
and *X*: *X* = *prod-emb UNIV* (λ-. *M*) *J* (Pi_E *J F*)
unfolding *prod-algebra-def* **by** *auto*
then have *Pi-F*: *finite J* Pi_E *J F* $\in$ *sets* (Pi_M *J* (λ-. *M*))
by (*auto intro*: *sets-PiM-I-finite*)

define *n* **where** *n* = (*LEAST n*. ∀ *i*≥*n*. *i* ∉ *J*)
have *J-le-n*: *J* ⊆ {*0*..<*n*}
unfolding *n-def*
using ‹*finite J*›
apply −
apply (*rule LeastI2*[*of* - *Suc* (*Max J*)])
apply (*auto simp*: *Suc-le-eq not-le*[*symmetric*])
done

have *C*: (λ*x*. *Ionescu-Tulcea.C* (*K′ x*) (λ-. *M*) *0 n* (λ*x*. *undefined*)) ∈ *M* $\rightarrow_M$ *subprob-algebra* (Pi_M {*0*..<*n*} (λ-. *M*))
apply (*induction n*)
apply (*subst measurable-cong*)
apply (*rule Ionescu-Tulcea.C.simps*[*OF IT-K′*])
apply *assumption*
apply (*rule measurable-compose*[*OF* - *return-measurable*])
apply *simp*
apply (*subst measurable-cong*)
apply (*rule Ionescu-Tulcea.C.simps*[*OF IT-K′*])
apply *assumption*
apply (*rule measurable-bind′*)
apply *assumption*
apply (*subst measurable-cong*)
proof −
fix *n* :: *nat* **and** *w* **assume** *w* ∈ *space* (*M* $\bigotimes_M$ Pi_M {*0*..<*n*} (λ-. *M*))
then show (*case w of* (*x*, *xa*) ⇒ *Ionescu-Tulcea.eP* (*K′ x*) (λ-. *M*) (*0* + *n*) *xa*) =
(*case w of* (*x*, *xa*) ⇒ *distr* (*K′ x n xa*) (Π_M *i*∈{*0*..<*Suc n*}. *M*) (*fun-upd xa n*))
by (*auto simp*: *space-pair-measure Ionescu-Tulcea.eP-def*[*OF IT-K′*] *split*: *prod.split*)
next
fix *n* **show** (λ*w*. *case w of* (*x*, *xa*) ⇒ *distr* (*K′ x n xa*) (Pi_M {*0*..<*Suc n*} (λ*i*. *M*)) (*fun-upd xa n*))
∈ *M* $\bigotimes_M$ Pi_M {*0*..<*n*} (λ-. *M*) $\rightarrow_M$ *subprob-algebra* (Pi_M {*0*..<*Suc n*} (λ-. *M*))
unfolding *K′-def*
apply *measurable*
apply (*rule measurable-distr2*[**where** *M*=*M*])
apply (*rule measurable-PiM-single′*)
apply (*simp add*: *split-beta′*)
subgoal for *i* **by** (*cases i* = *n*) *auto*
subgoal by (*auto simp*: *split-beta′ PiE-iff extensional-def Pi-iff space-pair-measure*

space-PiM)
    apply (*rule measurable-prob-algebraD*)
    apply (*rule measurable-compose*[*OF - K*])
    apply *measurable*
    done
  qed

  have (λa. *emeasure* (*lim-sequence a*) *X*) ∈ *borel-measurable M* ⟷
   (λa. *emeasure* (*Ionescu-Tulcea.CI* (*K′ a*) (λ-. *M*) *J*) (Pi_E *J F*)) ∈ *borel-measurable M*
   unfolding *X* **using** *J Pi-F* **by** (*intro measurable-cong emeasure-lim-sequence-emb*) *auto*
  also have ...
   apply (*intro measurable-compose*[*OF - measurable-emeasure-subprob-algebra*[*OF Pi-F*(*2*)]]])
    apply (*subst measurable-cong*)
    apply (*subst Ionescu-Tulcea.CI-def*[*OF IT-K′*])
    apply *assumption*
    apply (*subst Ionescu-Tulcea.up-to-def*[*OF IT-K′*])
    apply *assumption*
    unfolding *n-def*[*symmetric*]
    apply (*rule refl*)
   apply (*rule measurable-compose*[*OF - measurable-distr*[*OF measurable-restrict-subset*[*OF J-le-n*]]]])
    apply (*rule C*)
    done
  finally show (λa. *emeasure* (*lim-sequence a*) *X*) ∈ *borel-measurable M* **.**
qed

lemma *step-C*:
  assumes *x*: *x* ∈ *space M*
  shows *Ionescu-Tulcea.C* (*K′ x*) (λ-. *M*) *0 1* (λ-. *undefined*) ⤜ *Ionescu-Tulcea.C* (*K′ x*) (λ-. *M*) *1 n* =
    K x ⤜ (λy. *Ionescu-Tulcea.C* (*K′ x*) (λ-. *M*) *1 n* (*case-nat y* (λ-. *undefined*)))
proof −
  interpret *Ionescu-Tulcea K′ x* λ-. *M*
    using *x* **by** (*rule IT-K′*)

  have [*simp*]: *space* (*K x*) ≠ {}
    using *space-K*[*OF x*] *x* **by** *auto*

  have [*simp*]: ((λ-. *undefined*::′*a*)(*0* := *x*)) = *case-nat x* (λ-. *undefined*) **for** *x*
    by (*auto simp*: *fun-eq-iff split*: *nat.split*)

  have *C 0 1* (λ-. *undefined*) ⤜ *C 1 n* = *eP 0* (λ-. *undefined*) ⤜ *C 1 n*
    using *measurable-eP*[*of 0*] *measurable-C*[*of 1 n*, *measurable del*]
    by (*simp add*: *bind-return*[**where** *N*=Pi_M {*0*} (λ-. *M*)])
  also have ... = *K x* ⤜ (λy. *C 1 n* (*case-nat y* (λ-. *undefined*)))
    using *measurable-C*[*of 1 n*, *measurable del*] *x*[*THEN sets-K*]

by (*simp add*: *eP-def K'-def bind-distr cong*: *measurable-cong-sets*)
finally show *C 0 1* (λ-. *undefined*) ⪢= *C 1 n* = *K x* ⪢= (λ*y*. *C 1 n* (*case-nat y* (λ-. *undefined*))) **.**
qed

lemma *lim-sequence-eq*:
assumes *x*: *x* ∈ *space M*
shows *lim-sequence x* = *bind* (*K x*) (λ*y*. *distr* (*lim-sequence y*) (Π_M *j*∈*UNIV*. *M*) (*case-nat y*))
(**is** - = *?B x*)
proof (*rule measure-eqI-PiM-infinite*)
show *sets* (*lim-sequence x*) = *sets* (Π_M *j*∈*UNIV*. *M*)
using *x* **by** (*rule sets-lim-sequence*)
have [*simp*]: *space* (*K x*) ≠ {}
using *space-K*[*OF x*] *x* **by** *auto*
show *sets* (*?B x*) = *sets* (Pi_M *UNIV* (λ*j*. *M*))
using *x* **by** (*subst sets-bind*) *auto*
interpret *lim-sequence*: *prob-space lim-sequence x*
using *lim-sequence x* **by** (*auto simp*: *measurable-restrict-space2-iff prob-algebra-def*)
show *finite-measure* (*lim-sequence x*)
by (*rule lim-sequence.finite-measure*)

interpret *Ionescu-Tulcea K' x* λ-. *M*
using *x* **by** (*rule IT-K'*)

let *?U* = λ-::*nat*. *undefined* :: *'a*

fix *J* :: *nat set* **and** *F'*
assume *J*: *finite J* ⋀*i*. *i* ∈ *J* ⟹ *F' i* ∈ *sets M*

define *n* **where** *n* = (*if J* = {} *then 0 else Max J*)
define *F* **where** *F i* = (*if i* ∈ *J then F' i else space M*) **for** *i*
then have *F*[*simp*, *measurable*]: *F i* ∈ *sets M* **for** *i*
using *J* **by** *auto*
have *emb-eq*: *PF.emb UNIV J* (Pi_E *J F'*) = *PF.emb UNIV* {*0*..<*Suc n*} (Pi_E {*0*..<*Suc n*} *F*)
proof *cases*
assume *J* = {} **then show** *?thesis*
by (*auto simp add*: *n-def F-def*[*abs-def*] *prod-emb-def PiE-def*)
next
assume *J* ≠ {} **then show** *?thesis*
by (*auto simp*: *prod-emb-def PiE-iff F-def n-def less-Suc-eq-le* ‹*finite J*› *split*: *if-split-asm*)
qed

have *emeasure* (*lim-sequence x*) (*PF.emb UNIV J* (Pi_E *J F'*)) = *emeasure* (*C 0* (*Suc n*) *?U*) (Pi_E {*0*..<*Suc n*} *F*)
using *x* **unfolding** *emb-eq* **by** (*rule emeasure-lim-sequence-emb-I0o*) (*auto intro*!: *sets-PiM-I-finite*)

also have *C 0* (*Suc n*) *?U* = *K x* ≫= (λ*y*. *C 1 n* (*case-nat y ?U*))
using *split-C*[*of ?U 0 Suc 0 n*] *step-C*[*OF x*] **by** *simp*
also have *emeasure* (*K x* ≫= (λ*y*. *C 1 n* (*case-nat y ?U*))) (Pi_E {*0*..<*Suc n*} *F*) =
(∫ $^+$*y*. *C 1 n* (*case-nat y ?U*) (Pi_E {*0*..<*Suc n*} *F*) ∂*K x*)
using *measurable-C*[*of 1 n*, *measurable del*] *x*[*THEN sets-K*] *F x*
by (*intro emeasure-bind*[*OF - measurable-compose*[*OF - measurable-C*]])
(*auto cong*: *measurable-cong-sets intro*!: *measurable-PiM-single′ split*: *nat.split-asm*)
also have ... = (∫ $^+$*y*. *distr* (*lim-sequence y*) (Pi_M *UNIV* (λ*j*. *M*)) (*case-nat y*) (*PF.emb UNIV J* (Pi_E *J F′*)) ∂*K x*)
proof (*intro nn-integral-cong*)
fix *y* **assume** *y* ∈ *space* (*K x*)
then have *y*: *y* ∈ *space M*
using *x* **by** (*simp add*: *space-K*)
then interpret *y*: *Ionescu-Tulcea K′ y* λ-. *M*
by (*rule IT-K′*)

let *?y* = *case-nat y*
have [*simp*]: *?y ?U* ∈ *space* (Pi_M {*0*} (λ*i*. *M*))
using *y* **by** (*auto simp*: *space-PiM PiE-iff extensional-def split*: *nat.split*)
have *yM*[*measurable*]: *?y* ∈ Pi_M {*0*..<*m*} (λ-. *M*) $\rightarrow_M$ Pi_M {*0*..<*Suc m*} (λ*i*. *M*) **for** *m*
using *y* **by** (*intro measurable-PiM-single′*) (*auto simp*: *space-PiM PiE-iff extensional-def split*: *nat.split*)

have *y′*: *?y ?U* ∈ *space* (Pi_M {*0*..<*1*} (λ*i*. *M*))
by (*simp add*: *space-PiM PiE-def y extensional-def split*: *nat.split*)

have *eq1*: *?y* −‘ Pi_E {*0*..<*Suc n*} *F* ∩ *space* (Pi_M {*0*..<*n*} (λ-. *M*)) =
(*if y* ∈ *F 0 then* Pi_E {*0*..<*n*} (*F*∘*Suc*) *else* {})
unfolding *set-eq-iff* **using** *y sets.sets-into-space*[*OF F*]
by (*auto simp*: *space-PiM PiE-iff extensional-def Ball-def split*: *nat.split nat.split-asm*)

have *eq2*: *?y* −‘ *PF.emb UNIV* {*0*..<*Suc n*} (Pi_E {*0*..<*Suc n*} *F*) ∩ *space* (Pi_M *UNIV* (λ-. *M*)) =
(*if y* ∈ *F 0 then PF.emb UNIV* {*0*..<*n*} (Pi_E {*0*..<*n*} (*F*∘*Suc*)) *else* {})
unfolding *set-eq-iff* **using** *y sets.sets-into-space*[*OF F*]
by (*auto simp*: *space-PiM PiE-iff prod-emb-def extensional-def Ball-def split*: *nat.split nat.split-asm*)

let *?I* = *indicator* (*F 0*) *y*

have *C 1 n* (*?y ?U*) = *distr* (*y.C 0 n ?U*) (Π_M *i*∈{*0*..<*Suc n*}. *M*) *?y*
proof (*induction n*)
case (*Suc m*)

have *C 1* (*Suc m*) (*?y ?U*) = *distr* (*y.C 0 m ?U*) (Pi_M {*0*..<*Suc m*} (λ*i*. *M*)) *?y* ≫= *eP* (*Suc m*)

using *Suc* **by** *simp*
also have ... = *y.C 0 m ?U* ≫= (λ*x*. *eP* (*Suc m*) (*?y x*))
by (*intro bind-distr*[**where** *K*=Pi_M {*0..<Suc* (*Suc m*)} (λ-. *M*)]) (*simp-all add*: *y y.space-C y.sets-C cong*: *measurable-cong-sets*)
also have ... = *y.C 0 m ?U* ≫= (λ*x*. *distr* (*y.eP m x*) (Pi_M {*0..<Suc* (*Suc m*)} (λ*i*. *M*)) *?y*)
proof (*intro bind-cong refl*)
fix ω′ **assume** ω′: ω′ ∈ *space* (*y.C 0 m ?U*)
moreover have *K′ x* (*Suc m*) (*?y* ω′) = *K′ y m* ω′
by (*auto simp*: *K′-def*)
ultimately show *eP* (*Suc m*) (*?y* ω′) = *distr* (*y.eP m* ω′) (Pi_M {*0..<Suc* (*Suc m*)} (λ*i*. *M*)) *?y*
unfolding *eP-def y.eP-def*
by (*subst distr-distr*)
(*auto simp*: *y.space-C y.sets-P split*: *nat.split cong*: *measurable-cong-sets intro*!: *distr-cong measurable-fun-upd*[**where** *J*={*0..<m*}])
qed
also have ... = *distr* (*y.C 0 m ?U* ≫= *y.eP m*) (Pi_M {*0..<Suc* (*Suc m*)} (λ*i*. *M*)) *?y*
by (*intro distr-bind*[*symmetric*, *OF - - yM*]) (*auto simp*: *y.space-C y.sets-C cong*: *measurable-cong-sets*)
finally show *?case*
by *simp*
qed (*use y* **in** ‹*simp add*: *PiM-empty distr-return*›)
then have *C 1 n* (*case-nat y ?U*) (Pi_E {*0..<Suc n*} *F*) =
(*distr* (*y.C 0 n ?U*) (Π_M *i*∈{*0..<Suc n*}. *M*) *?y*) (Pi_E {*0..<Suc n*} *F*) **by** *simp*
also have ... = *?I* ∗ *y.C 0 n ?U* (Pi_E {*0..<n*} (*F* ∘ *Suc*))
by (*subst emeasure-distr*) (*auto simp*: *y.sets-C y.space-C eq1 cong*: *measurable-cong-sets*)
also have ... = *?I* ∗ *lim-sequence y* (*PF.emb UNIV* {*0..<n*} (Pi_E {*0..<n*} (*F* ∘ *Suc*)))
using *y* **by** (*simp add*: *emeasure-lim-sequence-emb-I0o sets-PiM-I-finite*)
also have ... = *distr* (*lim-sequence y*) (Pi_M *UNIV* (λ*j*. *M*)) *?y* (*PF.emb UNIV* {*0..<Suc n*} (Pi_E {*0..<Suc n*} *F*))
using *y* **by** (*subst emeasure-distr*) (*simp-all add*: *eq2 space-lim-sequence*)
finally show *emeasure* (*C 1 n* (*case-nat y* (λ-. *undefined*))) (Pi_E {*0..<Suc n*} *F*) =
emeasure (*distr* (*lim-sequence y*) (Pi_M *UNIV* (λ*j*. *M*)) (*case-nat y*)) (*PF.emb UNIV J* (Pi_E *J F′*))
unfolding *emb-eq* **.**
qed
also have ... =
emeasure (*K x* ≫= (λ*y*. *distr* (*lim-sequence y*) (Pi_M *UNIV* (λ*j*. *M*)) (*case-nat y*))) (*PF.emb UNIV J* (Pi_E *J F′*))
using *J*
by (*subst emeasure-bind*[**where** *N*=*PiM UNIV* (λ-. *M*)])
(*auto simp*: *sets-K x intro*!: *measurable-distr2*[*OF - measurable-prob-algebraD*[*OF lim-sequence*]] *cong*: *measurable-cong-sets*)

finally show *emeasure* (*lim-sequence x*) (*PF.emb UNIV J* (Pi_E *J F'*)) =
emeasure (*K x* ≫= (λ*y. distr* (*lim-sequence y*) (Pi_M *UNIV* (λ*j. M*)) (*case-nat y*)))
(*PF.emb UNIV J* (Pi_E *J F'*)) **.**
qed

lemma *AE-lim-sequence*:
assumes *x*[*simp*]: *x* ∈ *space M* **and** *P*[*measurable*]: *Measurable.pred* (Π_M *i*∈*UNIV*. *M*) *P*
shows (*AE* ω *in lim-sequence x. P* ω) ⟷ (*AE y in K x. AE* ω *in lim-sequence y. P* (*case-nat y* ω))
apply (*simp add*: *lim-sequence-eq cong del*: *AE-cong*)
apply (*subst AE-bind*)
apply (*rule measurable-prob-algebraD*)
apply *measurable*
apply (*auto intro*!: *AE-cong simp add*: *space-K AE-distr-iff*)
done

definition *lim-stream* :: *'a* ⇒ *'a stream measure*
where
lim-stream x = *distr* (*lim-sequence x*) (*stream-space M*) *to-stream*

lemma *space-lim-stream*: *space* (*lim-stream x*) = *streams* (*space M*)
unfolding *lim-stream-def* **by** (*simp add*: *space-stream-space*)

lemma *sets-lim-stream*[*measurable-cong*]: *sets* (*lim-stream x*) = *sets* (*stream-space M*)
unfolding *lim-stream-def* **by** *simp*

lemma *lim-stream*[*measurable*]: *lim-stream* ∈ *M* $\rightarrow_M$ *prob-algebra* (*stream-space M*)
unfolding *lim-stream-def*[*abs-def*] **by** (*intro measurable-distr-prob-space2*[*OF lim-sequence*]) *auto*

lemma *space-stream-space-M-ne*: *x* ∈ *space M* ⟹ *space* (*stream-space M*) ≠ {}
using *sconst-streams*[*of x space M*] **by** (*auto simp*: *space-stream-space*)

lemma *prob-space-lim-stream*: *x* ∈ *space M* ⟹ *prob-space* (*lim-stream x*)
using *measurable-space*[*OF lim-stream*, *of x*] **by** (*simp add*: *space-prob-algebra*)

lemma *lim-stream-eq*:
assumes *x*: *x* ∈ *space M*
shows *lim-stream x* = *do* { *y* ← *K x*; ω ← *lim-stream y*; *return* (*stream-space M*) (*y* ## ω) }
unfolding *lim-stream-def*
apply (*subst lim-sequence-eq*[*OF x*])
apply (*subst distr-bind*[*OF - - measurable-to-stream*])
subgoal
by (*auto simp*: *sets-K x cong*: *measurable-cong-sets*

intro!: *measurable-prob-algebraD measurable-distr-prob-space2*[**where** M=Pi_M *UNIV* (λj. M)] *lim-sequence*) []
subgoal
using x **by** (*auto simp add*: *space-K*)
apply (*intro bind-cong refl*)
apply (*subst distr-distr*)
apply (*auto simp*: *space-K sets-lim-sequence x cong*: *measurable-cong-sets intro*!: *distr-cong*)
apply (*subst bind-return-distr'*)
apply (*auto simp*: *space-stream-space-M-ne*)
apply (*subst distr-distr*)
apply (*auto simp*: *space-K sets-lim-sequence x to-stream-nat-case cong*: *measurable-cong-sets intro*!: *distr-cong*)
done

lemma *AE-lim-stream*:
assumes x[*simp*]: $x \in$ *space* M **and** P[*measurable*]: *Measurable.pred* (*stream-space* M) P
shows (*AE* ω *in lim-stream* x. P ω) $\longleftrightarrow$ (*AE* y *in* K x. *AE* ω *in lim-stream* y. P (y ## ω))
unfolding *lim-stream-eq*[*OF* x]
by (*simp-all add*: *space-K space-lim-stream space-stream-space AE-return AE-bind*[*OF measurable-prob-algebraD P*] *cong*: *AE-cong-simp*)

lemma *emeasure-lim-stream*:
assumes x[*measurable*, *simp*]: $x \in$ *space* M **and** A[*measurable*, *simp*]: $A \in$ *sets* (*stream-space* M)
shows *lim-stream* x A = ($\int^+ y$. *emeasure* (*lim-stream* y) (((##) y) −' A ∩ *space* (*stream-space* M)) ∂K x)
apply (*subst lim-stream-eq*, *simp*)
apply (*subst emeasure-bind*[*OF* - - A], *simp add*: *prob-space.not-empty prob-space-K*)
apply (*rule measurable-prob-algebraD*)
apply *measurable*
apply (*intro nn-integral-cong*)
apply (*subst bind-return-distr'*)
apply (*auto intro*!: *prob-space.not-empty prob-space-lim-stream simp*: *space-K emeasure-distr*)
apply (*simp add*: *space-lim-stream space-stream-space*)
done

lemma *lim-stream-eq-coinduct*[*case-names in-space step*]:
fixes R :: *'a* ⇒ *'a stream measure* ⇒ *bool*
assumes x: R x B $x \in$ *space* M
assumes R: $\bigwedge x$ B. R x $B \Longrightarrow \exists B' \in M \rightarrow_M$ *prob-algebra* (*stream-space* M).
(*AE* y *in* K x. R y (B' y) ∨ *lim-stream* y = B' y) ∧
B = *do* { $y \leftarrow K$ x; $\omega \leftarrow B'$ y; *return* (*stream-space* M) (y ## ω) }
shows *lim-stream* x = B
using x
proof (*coinduction arbitrary*: x B *rule*: *stream-space-coinduct*[**where** M=M, *case-names*

step])
  case (*step x B*)
 from *R*[*OF* ‹*R x B*›] **obtain** *B′* **where** *B′*: *B′* ∈ *M* $\rightarrow_M$ *prob-algebra* (*stream-space M*)
  and *ae*: *AE y in K x*. *R y* (*B′ y*) ∨ *lim-stream y* = *B′ y*
  and *eq*: *B* = *K x* ≫= (λ*y*. *B′ y* ≫= (λω. *return* (*stream-space M*) (*y* ## ω)))
  by *blast*
 show *?case*
  apply (*rule bexI*[*of - K x*], *rule bexI*[*OF - lim-stream*], *rule bexI*[*OF - B′*])
  apply (*intro conjI*)
  subgoal
   using *ae AE-space* **by** *eventually-elim* (*insert* ‹*x*∈*space M*›, *auto simp*: *space-K*)
  subgoal
   by (*rule lim-stream-eq*) *fact*
  subgoal
   by (*rule eq*)
  subgoal
   using *K* ‹*x* ∈ *space M*› **by** (*rule measurable-space*)
  done
qed

lemma *prob-space-lim-sequence*: *x* ∈ *space M* ⟹ *prob-space* (*lim-sequence x*)
 using *measurable-space*[*OF lim-sequence*, *of x*] **by** (*simp add*: *space-prob-algebra*)

end

6.2 Strong Markov Property for Discrete-Time Markov Processes

The filtration adopted to streams, i.e. to the *n*-th projection.

definition *stream-filtration* :: ′*a measure* ⇒ *enat* ⇒ ′*a stream measure*
 where *stream-filtration M n* = (*SUP i*∈{*i*::*nat*. *i* ≤ *n*}. *vimage-algebra* (*streams* (*space M*)) (λω . ω !! *i*) *M*)

lemma *measurable-stream-filtration1*: *enat i* ≤ *n* ⟹ (λω. ω !! *i*) ∈ *stream-filtration M n* $\rightarrow_M$ *M*
 by (*auto intro*!: *measurable-SUP1 measurable-vimage-algebra1 snth-in simp*: *stream-filtration-def*)

lemma *measurable-stream-filtration2*:
 f ∈ *space N* → *streams* (*space M*) ⟹ (⋀*i*. *enat i* ≤ *n* ⟹ (λ*x*. *f x* !! *i*) ∈ *N* $\rightarrow_M$ *M*) ⟹ *f* ∈ *N* $\rightarrow_M$ *stream-filtration M n*
 by (*auto simp*: *stream-filtration-def enat-0*
   intro!: *measurable-SUP2 measurable-vimage-algebra2 elim*!: *allE*[*of - 0*::*nat*])

lemma *space-stream-filtration*: *space* (*stream-filtration M n*) = *space* (*stream-space M*)
 by (*auto simp add*: *space-stream-space space-Sup-eq-UN stream-filtration-def enat-0 elim*!: *allE*[*of - 0*])

lemma *sets-stream-filteration-le-stream-space*: *sets* (*stream-filtration M n*) ⊆ *sets* (*stream-space M*)
  unfolding *sets-stream-space-eq stream-filtration-def*
  by (*intro SUP-subset-mono le-measureD2*) (*auto simp*: *space-Sup-eq-UN enat-0 elim*!: *allE*[*of - 0*])

interpretation *stream-filtration*: *filtration space* (*stream-space M*) *stream-filtration M*
proof
  show *space* (*stream-filtration M i*) = *space* (*stream-space M*) **for** *i*
    by (*simp add*: *space-stream-filtration*)
  show *sets* (*stream-filtration M i*) ⊆ *sets* (*stream-filtration M j*) **if** *i* ≤ *j* **for** *i j*
  proof (*rule le-measureD2*)
    show *stream-filtration M i* ≤ *stream-filtration M j*
      using ‹*i* ≤ *j*› **unfolding** *stream-filtration-def* **by** (*intro SUP-subset-mono*) *auto*
  qed (*simp add*: *space-stream-filtration*)
qed

lemma *measurable-stopping-time-stream*:
  stopping-time (*stream-filtration M*) *T* ⟹ *T* ∈ *stream-space M* $\rightarrow_M$ *count-space UNIV*
  using *sets-stream-filteration-le-stream-space*
  by (*subst measurable-cong-sets*[*OF refl sets-borel-eq-count-space*[*symmetric*, **where** *'a=enat*]])
     (*auto intro*!: *measurable-stopping-time simp*: *space-stream-filtration*)

lemma *measurable-stopping-time-All-eq-0*:
  assumes *T*: *stopping-time* (*stream-filtration M*) *T*
  shows {*x*∈*space M*. ∀*ω*∈*streams* (*space M*). *T* (*x* ## *ω*) = *0*} ∈ *sets M*
proof −
  have {*ω*∈*streams* (*space M*). *T ω* = *0*} ∈ *vimage-algebra* (*streams* (*space M*)) (*λω. ω* !! *0*) *M*
    using *stopping-timeD*[*OF T*, *of 0*] **by** (*simp add*: *stream-filtration-def pred-def enat-0-iff*)
  then obtain *A*
    where *A*: *A* ∈ *sets M*
      and ∗: {*ω* ∈ *streams* (*space M*). *T ω* = *0*} = (*λω. ω* !! *0*) −‘ *A* ∩ *streams* (*space M*)
    by (*auto simp*: *sets-vimage-algebra2 streams-shd*)
  have *A* = {*x*∈*space M*. ∀*ω*∈*streams* (*space M*). *T* (*x* ## *ω*) = *0*}
  proof *safe*
    fix *x ω* **assume** *x* ∈ *A ω* ∈ *streams* (*space M*)
    then have *x* ## *ω* ∈ {*ω* ∈ *streams* (*space M*). *T ω* = *0*}
      unfolding ∗ **using** *A*[*THEN sets.sets-into-space*] **by** *auto*
    then show *T* (*x* ## *ω*) = *0* **by** *auto*
  next
    fix *x* **assume** *x* ∈ *space M* ∀*ω*∈*streams* (*space M*). *T* (*x* ## *ω*) = *0*

then have ∀*ω*∈*streams* (*space M*). *x* ## *ω* ∈ {*ω* ∈ *streams* (*space M*). *T ω* = *0*}
by *simp*
with ‹*x*∈*space M*› **show** *x* ∈ *A*
unfolding ∗ **by** (*auto simp*: *streams-empty-iff*)
qed (*use A*[*THEN sets.sets-into-space*] **in** *auto*)
with ‹*A* ∈ *sets M*› **show** *?thesis* **by** *auto*
qed

lemma *stopping-time-0*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** *ω*: *ω* ∈ *streams* (*space M*) *T* (*x* ## *ω*) > *0*
and *ω′*: *ω′* ∈ *streams* (*space M*)
shows *T* (*x* ## *ω′*) > *0*
unfolding *zero-less-iff-neq-zero*
proof
assume *T* (*x* ## *ω′*) = *0*
with *x ω′* **have** *x′*: *x* ## *ω′* ∈ {*ω* ∈ *streams* (*space M*). *T ω* = *0*}
by *auto*

have {*ω*∈*streams* (*space M*). *T ω* = *0*} ∈ *vimage-algebra* (*streams* (*space M*)) (*λω*. *ω* !! *0*) *M*
using *stopping-timeD*[*OF T*, *of 0*] **by** (*simp add*: *stream-filtration-def pred-def enat-0-iff*)
then obtain *A*
where *A*: *A* ∈ *sets M*
and ∗: {*ω* ∈ *streams* (*space M*). *T ω* = *0*} = (*λω*. *ω* !! *0*) −‘ *A* ∩ *streams* (*space M*)
by (*auto simp*: *sets-vimage-algebra2 streams-shd*)
with *x′* **have** *x* ∈ *A*
by *auto*
with *ω x* **have** *x* ## *ω* ∈ (*λω*. *ω* !! *0*) −‘ *A* ∩ *streams* (*space M*)
by *auto*
with *ω* **show** *False*
unfolding ∗[*symmetric*] **by** *auto*
qed

lemma *stopping-time-epred-SCons*:
assumes *T*: *stopping-time* (*stream-filtration M*) *T*
and *x*: *x* ∈ *space M* **and** *ω*: *ω* ∈ *streams* (*space M*) *T* (*x* ## *ω*) > *0*
shows *stopping-time* (*stream-filtration M*) (*λω*. *epred* (*T* (*x* ## *ω*)))
proof (*rule stopping-timeI*, *rule measurable-cong*[*THEN iffD2*])
show *ω* ∈ *space* (*stream-filtration M t*) ⟹ (*epred* (*T* (*x* ## *ω*)) ≤ *t*) = (*T* (*x* ## *ω*) ≤ *eSuc t*) **for** *t ω*
by (*cases T* (*x* ## *ω*) *rule*: *enat-coexhaust*)
(*auto simp add*: *space-stream-filtration space-stream-space dest*!: *stopping-time-0*[*OF T x ω*])
show *Measurable.pred* (*stream-filtration M t*) (*λw*. *T* (*x* ## *w*) ≤ *eSuc t*) **for** *t*
proof (*rule measurable-compose*[*of SCons x*])

show (##) *x* ∈ *stream-filtration M t* $\rightarrow_M$ *stream-filtration M* (*eSuc t*)
proof (*intro measurable-stream-filtration2*)
show *enat i* ≤ *eSuc t* ⟹ (λ*xa*. (*x* ## *xa*) !! *i*) ∈ *stream-filtration M t* $\rightarrow_M$ *M* **for** *i*
using ‹*x*∈*space M*›
by (*cases i*) (*auto simp*: *eSuc-enat*[*symmetric*] *intro*!: *measurable-stream-filtration1*)
qed (*auto simp*: *space-stream-filtration space-stream-space* ‹*x*∈*space M*›)
qed (*rule T*[*THEN stopping-timeD*])
qed

context *discrete-Markov-process*
begin

lemma *lim-stream-strong-Markov*:
assumes *x*: *x* ∈ *space M* **and** *T*: *stopping-time* (*stream-filtration M*) *T*
shows *lim-stream x* =
lim-stream x ⤜ (λω. *case T* ω *of*
enat i ⇒ *distr* (*lim-stream* (ω !! *i*)) (*stream-space M*) (λω′. *stake* (*Suc i*) ω @− ω′)
| ∞ ⇒ *return* (*stream-space M*) ω)
(**is** - = *?L T x*)
using *assms*
proof (*coinduction arbitrary*: *x T rule*: *lim-stream-eq-coinduct*)
case (*step x T*)
note *T* = ‹*stopping-time* (*stream-filtration M*) *T*›[*THEN measurable-stopping-time-stream*, *measurable*]
define *L* **where** *L T x* = *?L T x* **for** *T x*
have *L*[*measurable* (*raw*)]:
(λ(*x*, ω). *T x* ω) ∈ *N* $\bigotimes_M$ *stream-space M* $\rightarrow_M$ *count-space UNIV* ⟹
f ∈ *N* $\rightarrow_M$ *M* ⟹ (λ*x*. *L* (*T x*) (*f x*)) ∈ *N* $\rightarrow_M$ *prob-algebra* (*stream-space M*)
for *f* :: ′*a* ⇒ ′*a* **and** *N T*
unfolding *L-def*
by (*intro measurable-bind-prob-space2*[*OF measurable-compose*[*OF* - *lim-stream*]] *measurable-case-enat*
measurable-distr-prob-space2[*OF measurable-compose*[*OF* - *lim-stream*]]
measurable-return-prob-space measurable-stopping-time-stream)
auto

define *S* **where** *S x* = (*if* ∀ω∈*streams* (*space M*). *T* (*x*##ω) = *0 then lim-stream x else L* (λω. *epred* (*T* (*x* ## ω))) *x*) **for** *x*
then have *S-eq*: ∀ω∈*streams* (*space M*). *T* (*x*##ω) = *0* ⟹ *S x* = *lim-stream x*
¬ (∀ω∈*streams* (*space M*). *T* (*x*##ω) = *0*) ⟹ *S x* = *L* (λω. *epred* (*T* (*x* ## ω))) *x* **for** *x*
by *auto*
have [*measurable*]: *S* ∈ *M* $\rightarrow_M$ *prob-algebra* (*stream-space M*)
unfolding *S-def*[*abs-def*]
by (*subst measurable-If-restrict-space-iff*, *safe intro*!: *L*)
(*auto intro*!: *measurable-stopping-time-All-eq-0 step measurable-restrict-space1*

lim-stream
measurable-compose[*OF* - *measurable-epred*] *measurable-compose*[*OF* - *T*]
measurable-Stream measurable-compose[*OF measurable-fst*]
simp: *measurable-split-conv*)

show *?case*
unfolding *L-def*[*symmetric*]
proof (*intro bexI*[*of* - *S*] *conjI AE-I2*)
fix *y* **assume** $y \in$ *space* (*K x*)
then show ($\exists x$ *T*. $y = x \wedge S\ y = L\ T\ x \wedge x \in$ *space M* $\wedge$ *stopping-time* (*stream-filtration M*) *T*) $\vee$
lim-stream y = *S y*
using ‹*x*∈*space M*›
by (*cases* ∀ ω∈*streams* (*space M*). *T* (*y*##ω) = *0*)
(*auto simp add*: *S-eq space-K intro*!: *exI*[*of* - λω. *epred* (*T* (*y* ## ω))] *stopping-time-epred-SCons step*)
next
note ‹*x*∈*space M*›[*simp*]
have *L T x* = *K x* ⤜
(λ*y*. *lim-stream y* ⤜ (λω. *case T* (*y*##ω) *of*
enat i ⇒ *distr* (*lim-stream* ((*y*##ω) !! *i*)) (*stream-space M*) (λω′. *stake* (*Suc i*) (*y*##ω) @− ω′)
| ∞ ⇒ *return* (*stream-space M*) (*y*##ω))) (**is** - = *K x* ⤜ *?L*′)
unfolding *L-def*
apply (*subst lim-stream-eq*[*OF* ‹*x*∈*space M*›])
apply (*subst bind-assoc*[**where** *N*=*stream-space M* **and** *R*=*stream-space M*, *OF measurable-prob-algebraD measurable-prob-algebraD*];
measurable)
apply (*rule bind-cong*[*OF refl*])
apply (*simp add*: *space-K*)
apply (*subst bind-assoc*[**where** *N*=*stream-space M* **and** *R*=*stream-space M*, *OF measurable-prob-algebraD measurable-prob-algebraD*];
measurable)
apply (*rule bind-cong*[*OF refl*])
apply (*simp add*: *space-lim-stream*)
apply (*subst bind-return*[**where** *N*=*stream-space M*, *OF measurable-prob-algebraD*])
apply (*measurable*; *fail*) []
apply (*simp add*: *space-stream-space*)
apply *rule*
done
also have ... = *K x* ⤜ (λ*y*. *S y* ⤜ (λω. *return* (*stream-space M*) (*y* ## ω)))
proof (*intro bind-cong*[*of K x*] *refl*)
fix *y* **assume** $y \in$ *space* (*K x*)
then have [*simp*]: $y \in$ *space M*
by (*simp add*: *space-K*)
show *?L*′ *y* = *S y* ⤜ (λω. *return* (*stream-space M*) (*y* ## ω))
proof *cases*

assume ∀ω∈*streams* (*space M*). *T* (*y*##ω) = *0*
with *x* **show** *?thesis*
by (*auto simp*: *S-eq space-lim-stream shift.simps*[*abs-def*] *streams-empty-iff bind-const'*[*OF - prob-space-imp-subprob-space*] *prob-space-lim-stream prob-space.prob-space-distr*
intro!: *bind-return-distr'*[*symmetric*]
cong: *bind-cong-simp*)
next
assume ∗: ¬ (∀ω∈*streams* (*space M*). *T* (*y*##ω) = *0*)
then have *T-pos*: ω ∈ *streams* (*space M*) ⟹ *T* (*y* ## ω) ≠ *0* **for** ω
using *stopping-time-0*[*OF* ‹*stopping-time* (*stream-filtration M*) *T*›, *of y* - ω] **by** *auto*
show *?thesis*
apply (*simp add*: *S-eq*(*2*)[*OF* ∗] *L-def*)
apply (*subst bind-assoc*[**where** *N*=*stream-space M* **and** *R*=*stream-space M*, *OF measurable-prob-algebraD measurable-prob-algebraD*];
measurable)
apply (*intro bind-cong refl*)
apply (*auto simp*: *T-pos enat-0 space-lim-stream shift.simps*[*abs-def*] *diff-Suc space-stream-space*
intro!: *bind-return*[**where** *N*=*stream-space M*, *OF measurable-prob-algebraD*, *symmetric*]
bind-distr-return[*symmetric*]
split: *nat.split enat.split*)
done
qed
qed
finally show *L T x* = *K x* ≫= (λ*y*. *S y* ≫= (λω. *return* (*stream-space M*) (*y* ## ω))) **.**
qed *fact*
qed *fact*

end

end

7 Continuous-time Markov chains

theory *Continuous-Time-Markov-Chain*
imports *Discrete-Time-Markov-Process Discrete-Time-Markov-Chain*
begin

7.1 Trace Operations: relate ('*a* × *real*) *stream* and *real* ⇒ '*a*

partial-function (*tailrec*) *trace-at* :: '*a* ⇒ (*real* × '*a*) *stream* ⇒ *real* ⇒ '*a*
where
trace-at s ω *j* = (*case* ω *of* (*t'*, *s'*)##ω ⇒ *if t'* ≤ *j then trace-at s'* ω *j else s*)

lemma *trace-at-simp*[*simp*]: *trace-at s* ((*t'*, *s'*)##ω) *j* = (*if t'* ≤ *j then trace-at s'*

ω j else s)
by (*subst trace-at.simps*) *simp*

lemma *trace-at-eq*:
trace-at s ω j = (*case sfirst* (*λx. j* < *fst* (*shd x*)) *ω of* ∞ ⇒ *undefined* | *enat i* ⇒ (*s* ## *smap snd ω*) !! *i*)
proof (*split enat.split*; *safe*)
assume *sfirst* (*λx. j* < *fst* (*shd x*)) *ω* = ∞
with *sfirst-finite*[*of λx. j* < *fst* (*shd x*) *ω*]
have *alw* (*λx. fst* (*shd x*) ≤ *j*) *ω*
by (*simp add*: *not-ev-iff not-less*)
then show *trace-at s ω j* = *undefined*
by (*induction arbitrary*: *s ω rule*: *trace-at.fixp-induct*) (*auto split*: *stream.split*)
next
show *sfirst* (*λx. j* < *fst* (*shd x*)) *ω* = *enat n* ⟹ *trace-at s ω j* = (*s* ## *smap snd ω*) !! *n* **for** *n*
proof (*induction n arbitrary*: *s ω*)
case *0* **then show** *?case*
by (*subst trace-at.simps*) (*auto simp add*: *enat-0 sfirst-eq-0 split*: *stream.split*)
next
case (*Suc n*) **show** *?case*
using *sfirst.simps*[*of λx. j* < *fst* (*shd x*) *ω*] *Suc.prems Suc.IH*[*of stl ω snd* (*shd ω*)]
by (*cases ω*) (*auto simp add*: *eSuc-enat*[*symmetric*] *split*: *stream.split if-split-asm*)
qed
qed

lemma *trace-at-shift*: *trace-at s* (*smap* (*λ*(*t*, *s'*). (*t* + *t'*, *s'*)) *ω*) *t* = *trace-at s ω* (*t* − *t'*)
by (*induction arbitrary*: *s ω rule*: *trace-at.fixp-induct*) (*auto split*: *stream.split*)

primcorec *merge-at* :: (*real* × *'a*) *stream* ⇒ *real* ⇒ (*real* × *'a*) *stream* ⇒ (*real* × *'a*) *stream*
where
merge-at ω j ω' = (*case ω of* (*t*, *s*) ## *ω* ⇒ *if t* ≤ *j then* (*t*, *s*)##*merge-at ω j ω' else ω'*)

lemma *merge-at-simp*[*simp*]: *merge-at* (*x*##*ω*) *j ω'* = (*if fst x* ≤ *j then x*##*merge-at ω j ω' else ω'*)
by (*cases x*) (*subst merge-at.code*; *simp*)

7.2 Exponential Distribution

definition *exponential* :: *real* ⇒ *real measure*
where
exponential l = *density lborel* (*exponential-density l*)

lemma *space-exponential*: *space* (*exponential l*) = *UNIV*
by (*simp add*: *exponential-def*)

lemma *sets-exponential*[*measurable-cong*]: *sets* (*exponential l*) = *sets borel*
  by (*simp add*: *exponential-def*)

lemma *prob-space-exponential*: *0* < *l* ⟹ *prob-space* (*exponential l*)
  unfolding *exponential-def* **by** (*intro prob-space-exponential-density*)

lemma *AE-exponential*: *0* < *l* ⟹ *AE x in exponential l*. *0* < *x*
  unfolding *exponential-def* **using** *AE-lborel-singleton*[*of 0*] **by** (*auto simp add*: *AE-density exponential-density-def*)

lemma *emeasure-exponential-Ioi-cutoff*:
  assumes *0* < *l*
  shows *emeasure* (*exponential l*) {*x* <..} = *exp* (− (*max 0 x*) ∗ *l*)
proof −
  interpret *prob-space exponential l*
   unfolding *exponential-def* **using** ‹*0*<*l*› **by** (*rule prob-space-exponential-density*)
  have ∗: *prob* {*xa* ∈ *space* (*exponential l*). *max 0 x* < *xa*} = *exp* (− *max 0 x* ∗ *l*)
    apply (*rule exponential-distributedD-gt*[*OF* - - ‹*0*<*l*›])
    apply (*auto simp*: *exponential-def distributed-def*)
    apply (*subst* (*6*) *distr-id*[*symmetric*])
    apply (*subst* (*2*) *distr-cong*)
    apply *simp-all*
    done
  have *emeasure* (*exponential l*) {*x* <..} = *emeasure* (*exponential l*) {*max 0 x* <..}
    using *AE-exponential*[*OF* ‹*0*<*l*›] **by** (*intro emeasure-eq-AE*) *auto*
  also have ... = *exp* (− (*max 0 x*) ∗ *l*)
      using ∗ **unfolding** *emeasure-eq-measure* **by** (*simp add*: *space-exponential greaterThan-def*)
  finally show *?thesis* **.**
qed

lemma *emeasure-exponential-Ioi*:
  0 < *l* ⟹ *0* ≤ *x* ⟹ *emeasure* (*exponential l*) {*x* <..} = *exp* (− *x* ∗ *l*)
  using *emeasure-exponential-Ioi-cutoff*[*of l x*] **by** *simp*

lemma *exponential-eq-stretch*:
  assumes *0* < *l*
  shows *exponential l* = *distr* (*exponential 1*) *borel* (λ*x*. (*1*/*l*) ∗ *x*)
proof (*intro measure-eqI*)
  fix *A* **assume** *A* ∈ *sets* (*exponential l*)
  then have [*measurable*]: *A* ∈ *sets borel*
    by (*simp add*: *sets-exponential*)
  then have [*measurable*]: (λ*x*. *x* / *l*) −‘ *A* ∈ *sets borel*
    by (*rule measurable-sets-borel*[*rotated*]) *simp*
  have *emeasure* (*exponential l*) *A* =
    (∫⁺*x*. *ennreal l* ∗ (*indicator* (((∗) (*1*/*l*) −‘ *A*) ∩ {*0* ..}) (*l* ∗ *x*) ∗ *ennreal* (*exp* (− (*l* ∗ *x*)))) ∂*lborel*)
    using ‹*0* < *l*›

by (*auto simp*: *ac-simps emeasure-distr exponential-def emeasure-density exponential-density-def*
ennreal-mult zero-le-mult-iff
intro!: *nn-integral-cong split*: *split-indicator*)
also have ... = ($\int^+ x.$ *indicator* (((∗) (*1*/*l*) −' *A*) ∩ {*0* ..}) *x* ∗ *ennreal* (*exp* (− *x*)) ∂*lborel*)
using ‹*0*<*l*›
apply (*subst nn-integral-stretch*)
apply (*auto simp*: *nn-integral-cmult*)
apply (*simp add*: *ennreal-mult*[*symmetric*] *mult.assoc*[*symmetric*])
done
also have ... = *emeasure* (*distr* (*exponential 1*) *borel* (λ*x*. (*1*/*l*) ∗ *x*)) *A*
by (*auto simp add*: *emeasure-distr exponential-def emeasure-density exponential-density-def*
intro!: *nn-integral-cong split*: *split-indicator*)
finally show *emeasure* (*exponential l*) *A* = *emeasure* (*distr* (*exponential 1*) *borel* (λ*x*. (*1*/*l*) ∗ *x*)) *A* **.**
qed (*simp add*: *sets-exponential*)

lemma *uniform-measure-exponential*:
assumes *0* < *l 0* ≤ *t*
shows *uniform-measure* (*exponential l*) {*t* <..} = *distr* (*exponential l*) *borel* ((+) *t*) (**is** *?L* = *?R*)
proof (*rule measure-eqI-lessThan*)
fix *x*
have *0* < *emeasure* (*exponential l*) {*t*<..}
unfolding *emeasure-exponential-Ioi*[*OF assms*] **by** *simp*
with *assms* **show** *?L* {*x*<..} < ∞
by (*simp add*: *ennreal-divide-eq-top-iff less-top*[*symmetric*] *lessThan-Int-lessThan emeasure-exponential-Ioi*)
have ∗: ((+) *t* −' {*x*<..} ∩ *space* (*exponential l*)) = {*x* − *t* <..}
by (*auto simp*: *space-exponential*)
show *?L* {*x*<..} = *?R* {*x*<..}
using *assms* **by** (*simp add*: *lessThan-Int-lessThan emeasure-exponential-Ioi divide-ennreal*
emeasure-distr ∗ *emeasure-exponential-Ioi-cutoff exp-diff*[*symmetric*] *field-simps split*: *split-max*)
qed (*auto simp*: *sets-exponential*)

lemma *emeasure-PiM-exponential-Ioi-finite*:
assumes *J* ⊆ *I finite J* $\bigwedge$*i*. *i* ∈ *I* ⟹ *0* < *R i 0* ≤ *x*
shows *emeasure* (Π_M *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π_E *j*∈*J*. {*x*<..})) = *exp* (− *x* ∗ ($\sum$*i*∈*J*. *R i*))
proof (*subst emeasure-PiM-emb*)
from *assms* **show** ($\prod$*i*∈*J*. *emeasure* (*exponential* (*R i*)) {*x*<..}) = *ennreal* (*exp* (− *x* ∗ *sum R J*))
by (*subst prod.cong*[*OF refl emeasure-exponential-Ioi*])
(*auto simp add*: *prod-ennreal exp-sum sum-negf*[*symmetric*] *sum-distrib-left*)
qed (*insert assms*, *auto intro*!: *prob-space-exponential*)

lemma *emeasure-PiM-exponential-Ioi-sequence*:
assumes *R*: *summable R* $\bigwedge i.\ 0 < R\ i\ 0 \leq x$
shows *emeasure* (Π_M *i*∈*UNIV*. *exponential* (*R i*)) (Π *i*∈*UNIV*. {*x*<..}) = *exp* (− *x* ∗ *suminf R*)
proof −
let *?R* = λ*i*. *exponential* (*R i*) **let** *?P* = Π_M *i*∈*UNIV*. *?R i*
let *?N* = λ*n*::*nat*. *prod-emb UNIV ?R* {..<*n*} (Π_E *i*∈{..<*n*}. {*x*<..})
interpret *prob-space ?P*
by (*intro prob-space-PiM prob-space-exponential R*)
have (Π_M *i*∈*UNIV*. *exponential* (*R i*)) (⋂*n*. *?N n*) = (*INF n*. (Π_M *i*∈*UNIV*. *exponential* (*R i*)) (*?N n*))
by (*intro INF-emeasure-decseq*[*symmetric*] *decseq-emb-PiE*) (*auto simp*: *incseq-def*)
also have . . . = (*INF n*. *ennreal* (*exp* (− *x* ∗ (∑ *i*<*n*. *R i*))))
using *R* **by** (*intro INF-cong emeasure-PiM-exponential-Ioi-finite*) *auto*
also have . . . = *ennreal* (*exp* (− *x* ∗ (*SUP n*. (∑ *i*<*n*. *R i*))))
using *R*
by (*subst continuous-at-Sup-antimono*[**where** *f*=λ*r*. *ennreal* (*exp* (− *x* ∗ *r*))])
(*auto intro*!: *bdd-aboveI2*[**where** *M*=∑ *i*. *R i*] *sum-le-suminf summable-mult mult-left-mono*
continuous-mult continuous-at-ennreal continuous-within-exp[*THEN continuous-within-compose3*] *continuous-minus*
simp: *less-imp-le antimono-def image-comp*)
also have . . . = *ennreal* (*exp* (− *x* ∗ (∑ *i*. *R i*)))
using *R* **by** (*subst suminf-eq-SUP-real*) (*auto simp*: *less-imp-le*)
also have (⋂*n*. *?N n*) = (Π *i*∈*UNIV*. {*x*<..})
by (*fastforce simp*: *prod-emb-def Pi-iff PiE-iff space-exponential*)
finally show *?thesis*
using *R* **by** *simp*
qed

lemma *emeasure-PiM-exponential-Ioi-countable*:
assumes *R*: *J* ⊆ *I countable J* $\bigwedge i.\ i \in I \Longrightarrow 0 < R\ i\ 0 \leq x$ **and** *finite*: *integrable* (*count-space J*) *R*
shows *emeasure* (Π_M *i*∈*I*. *exponential* (*R i*)) (*prod-emb I* (λ*i*. *exponential* (*R i*)) *J* (Π_E *j*∈*J*. {*x*<..})) =
exp (− *x* ∗ (*LINT i*|*count-space J*. *R i*))
proof *cases*
assume *finite J* **with** *assms* **show** *?thesis*
by (*subst emeasure-PiM-exponential-Ioi-finite*)
(*auto simp*: *lebesgue-integral-count-space-finite*)
next
assume *infinite J*
let *?R* = λ*i*. *exponential* (*R i*) **let** *?P* = Π_M *i*∈*I*. *?R i*
define *f* **where** *f* = *from-nat-into J*
have *J-eq*: *J* = *range f* **and** *f*: *inj f f* ∈ *UNIV* → *I*
using *from-nat-into-inj-infinite*[*of J*] *range-from-nat-into*[*of J*] ‹*countable J*› ‹*infinite J*› ‹*J* ⊆ *I*›

by (*auto simp*: *inj-on-def f-def simp del*: *range-from-nat-into*)
have *Bf*: *bij-betw f UNIV J*
unfolding *J-eq* **using** *inj-on-imp-bij-betw*[*OF f*(*1*)] **.**

have *summable-R*: *summable* ($\lambda i.\ R\ (f\ i)$)
using *finite* **unfolding** *integrable-bij-count-space*[*OF Bf*, *symmetric*] *integrable-count-space-nat-iff*
by (*rule summable-norm-cancel*)

have *emeasure* ($\Pi_M\ i{\in}UNIV.\ exponential\ (R\ (f\ i))$) ($\Pi\ i{\in}UNIV.\ \{x{<}..\}$) = *exp* ($-\ x * (\sum i.\ R\ (f\ i))$)
using *finite assms* **unfolding** *J-eq* **by** (*intro emeasure-PiM-exponential-Ioi-sequence*[*OF summable-R*]) *auto*
also have ($\Pi_M\ i{\in}UNIV.\ exponential\ (R\ (f\ i))$) = *distr ?P* ($\Pi_M\ i{\in}UNIV.\ exponential\ (R\ (f\ i))$) ($\lambda\omega.\ \lambda i{\in}UNIV.\ \omega\ (f\ i)$)
using *R* **by** (*intro distr-PiM-reindex*[*symmetric*, *OF - f*] *prob-space-exponential*) *auto*
also have ... ($\Pi\ i{\in}UNIV.\ \{x{<}..\}$) = *?P* (($\lambda\omega.\ \lambda i{\in}UNIV.\ \omega\ (f\ i)$) −‘ ($\Pi\ i{\in}UNIV.\ \{x{<}..\}$) ∩ *space ?P*)
using *f*(*2*) **by** (*intro emeasure-distr infprod-in-sets*) (*auto simp*: *Pi-iff*)
also have ($\lambda\omega.\ \lambda i{\in}UNIV.\ \omega\ (f\ i)$) −‘ ($\Pi\ i{\in}UNIV.\ \{x{<}..\}$) ∩ *space ?P* = *prod-emb I ?R J* ($\Pi_E\ j{\in}J.\ \{x{<}..\}$)
by (*auto simp*: *prod-emb-def space-PiM space-exponential Pi-iff J-eq*)
also have ($\sum i.\ R\ (f\ i)$) = (*LINT i*|*count-space J*. *R i*)
using *finite*
by (*subst integral-count-space-nat*[*symmetric*])
(*auto simp*: *integrable-bij-count-space*[*OF Bf*] *integral-bij-count-space*[*OF Bf*])
finally show *?thesis* **.**
qed

lemma *AE-PiM-exponential-suminf-infty*:
fixes *R* :: *nat* ⇒ *real*
assumes *R*: $\bigwedge n.\ 0 < R\ n$ **and** *finite*: ($\sum n.\ ennreal\ (1\ /\ R\ n)$) = *top*
shows *AE* ω *in* $\Pi_M\ n{\in}UNIV.\ exponential\ (R\ n)$. ($\sum n.\ ereal\ (\omega\ n)$) = ∞
proof −
let *?P* = $\Pi_M\ n{\in}UNIV.\ exponential\ (R\ n)$
interpret *prob-space exponential* (*R n*) **for** *n*
by (*intro prob-space-exponential R*)
interpret *product-prob-space* $\lambda n.\ exponential\ (R\ n)$ *UNIV*
proof qed

have *AE-pos*: *AE* ω *in ?P*. $\forall i.\ 0 < \omega\ i$
unfolding *AE-all-countable* **by** (*intro AE-PiM-component allI prob-space-exponential R AE-exponential*) *simp*

have *indep*: *indep-vars* ($\lambda i.\ borel$) ($\lambda i\ x.\ x\ i$) *UNIV*
using *PiM-component*
apply (*subst P.indep-vars-iff-distr-eq-PiM*)
apply (*auto simp*: *restrict-UNIV distr-id2*)

```
  apply (subst distr-id2)
   apply (intro sets-PiM-cong)
    apply (auto simp: sets-exponential cong: distr-cong)
  done

have [simp]: 0 ≤ x + x ∗ R i ⟷ 0 ≤ x for x i
  using zero-le-mult-iff[of x 1 + R i] R[of i] by (simp add: field-simps)

have (∫+ω. eexp (∑n. − ereal (ω n)) ∂?P) = (∫+ω. (INF n. ∏i<n. eexp (−
ereal (ω i))) ∂?P)
proof (intro nn-integral-cong-AE, use AE-pos in eventually-elim)
  fix ω :: nat ⇒ real assume ω: ∀ i. 0 < ω i
  show eexp (∑n. − ereal (ω n)) = (∏n. ∏i<n. eexp (− ereal (ω i)))
  proof (rule LIMSEQ-unique[OF - LIMSEQ-INF])
    show (λi. ∏i<i. eexp (− ereal (ω i))) ⟶ eexp (∑n. − ereal (ω n))
       using ω by (intro eexp-suminf summable-minus-ereal summable-ereal-pos)
(auto intro: less-imp-le)
    show decseq (λn. ∏i<n. eexp (− ereal (ω i)))
      using ω by (auto simp: decseq-def intro!: prod-mono3 intro: less-imp-le)
  qed
qed
also have ... = (INF n. (∫+ω. (∏i<n. eexp (− ereal (ω i))) ∂?P))
proof (intro nn-integral-monotone-convergence-INF-AE')
  show AE ω in ?P. (∏i<Suc n. eexp (− ereal (ω i))) ≤ (∏i<n. eexp (− ereal
(ω i))) for n
    using AE-pos
  proof eventually-elim
    case (elim ω)
    show ?case
      by (rule prod-mono3) (auto simp: elim le-less)
  qed
qed (auto simp: less-top[symmetric])
also have ... = (INF n. (∏i<n. (∫+ω. eexp (− ereal (ω i)) ∂?P)))
proof (intro INF-cong refl indep-vars-nn-integral)
  show indep-vars (λ-. borel) (λi ω. eexp (− ereal (ω i))) {..<n} for n
  proof (rule indep-vars-compose2[of - - - λi x. eexp(− ereal x)])
    show indep-vars (λi. borel) (λi x. x i) {..<n}
      by (rule indep-vars-subset[OF indep]) auto
  qed auto
qed auto
also have ... = (INF n. (∏i<n. R i ∗ (∫+x. indicator {0 ..} ((1 + R i) ∗ x)
∗ ennreal (exp (− ((1 + R i) ∗ x))) ∂lborel)))
  by (subst product-nn-integral-component)
   (auto simp: field-simps exponential-def nn-integral-density ennreal-mult'[symmetric]
ennreal-mult''[symmetric]
           exponential-density-def exp-diff exp-minus nn-integral-cmult[symmetric]
          intro!: INF-cong prod.cong nn-integral-cong split: split-indicator)
also have ... = (INF n. (∏i<n. ennreal (R i / (1 + R i))))
proof (intro INF-cong prod.cong refl)
```

show *R i* ∗ ($\int^+$ *x. indicator* {*0..*} ((*1* + *R i*) ∗ *x*) ∗ *ennreal* (*exp* (− ((*1* + *R i*) ∗ *x*))) ∂*lborel*) =
ennreal (*R i* / (*1* + *R i*)) **for** *i*
using *nn-intergal-power-times-exp-Ici*[*of 0*] ‹*0* < *R i*›
by (*subst nn-integral-stretch*[**where** *c*=*1* + *R i*])
(*auto simp*: *mult.assoc*[*symmetric*] *ennreal-mult''*[*symmetric*] *less-imp-le mult.commute*)
qed
also have ... = (*INF n. ennreal* ($\prod$ *i*<*n. R i* / (*1* + *R i*)))
using *R* **by** (*intro INF-cong refl prod-ennreal divide-nonneg-nonneg*) (*auto simp*: *less-imp-le*)
also have ... = (*INF n. ennreal* (*inverse* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*)))
by (*subst prod-inversef*[*symmetric*]) *simp-all*
also have ... = (*INF n. inverse* (*ennreal* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*)))
using *R* **by** (*subst inverse-ennreal*) (*auto intro*!: *prod-pos divide-pos-pos simp*: *add-pos-pos*)
also have ... = *inverse* (*SUP n. ennreal* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*))
by (*subst continuous-at-Sup-antimono* [**where** *f* = *inverse*])
(*auto simp*: *antimono-def image-comp intro*!: *continuous-on-imp-continuous-within*[*OF continuous-on-inverse-ennreal'*])
also have (*SUP n. ennreal* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*)) = *top*
proof (*cases SUP n. ennreal* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*))
case (*real r*)
have (λ*n. ennreal* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*)) ⟶ *r*
using *R* **unfolding** *real*(*2*)[*symmetric*]
by (*intro LIMSEQ-SUP monoI ennreal-leI prod-mono2*) (*auto intro*!: *divide-nonneg-nonneg add-nonneg-nonneg intro*: *less-imp-le*)
then have (λ*n.* ($\prod$ *i*<*n.* (*1* + *R i*) / *R i*)) ⟶ *r*
by (*rule tendsto-ennrealD*)
(*use R real* **in** ‹*auto intro*!: *always-eventually prod-nonneg divide-nonneg-nonneg add-nonneg-nonneg intro*: *less-imp-le*›)
moreover have (*1* + *R i*) / *R i* = *1* / *R i* + *1* **for** *i*
using ‹*0* < *R i*› **by** (*auto simp*: *field-simps*)
ultimately have *convergent* (λ*n.* $\prod$ *i*<*n. 1* / *R i* + *1*)
by (*auto simp*: *convergent-def*)
then have *summable* (λ*i. 1* / *R i*)
using *R* **by** (*subst summable-iff-convergent-prod*) (*auto intro*: *less-imp-le*)
moreover have *0* ≤ *1* / *R i* **for** *i*
using *R* **by** (*auto simp*: *less-imp-le*)
ultimately show *?thesis*
using *finite ennreal-suminf-neq-top*[*of* λ*i. 1* / *R i*] **by** *blast*
qed
finally have ($\int^+$ω. *eexp* ($\sum$ *n.* − *ereal* (ω *n*)) ∂*?P*) = *0*
by *simp*
then have *AE* ω *in ?P. eexp* ($\sum$ *n.* − *ereal* (ω *n*)) = *0*
by (*subst* (*asm*) *nn-integral-0-iff-AE*) *auto*
then show *?thesis*
using *AE-pos*
proof *eventually-elim*

show ($\forall i.\ 0 < \omega\ i$) $\Longrightarrow$ *eexp* ($\sum n. -$ *ereal* ($\omega\ n$)) $= 0 \Longrightarrow$ ($\sum n.$ *ereal* ($\omega\ n$)) $= \infty$ **for** ω
  apply (*auto simp del*: *uminus-ereal.simps simp add*: *uminus-ereal.simps*[*symmetric*] *intro*!: *summable-iff-suminf-neq-top intro*: *less-imp-le*)
    apply (*subst* (*asm*) *suminf-minus-ereal*)
    apply (*auto intro*!: *summable-ereal-pos intro*: *less-imp-le*)
    done
  qed
qed

7.3 Transition Rates

locale *transition-rates* =
  fixes R :: $'a \Rightarrow 'a \Rightarrow$ *real*
  assumes *R-nonneg*[*simp*]: $\bigwedge x\ y.\ 0 \le R\ x\ y$
  assumes *R-diagonal-0*[*simp*]: $\bigwedge x.\ R\ x\ x = 0$
  assumes *finite-weight*: $\bigwedge x.$ ($\int^+ y.\ R\ x\ y\ \partial$*count-space UNIV*) $< \infty$
  assumes *positive-weight*: $\bigwedge x.\ 0 <$ ($\int^+ y.\ R\ x\ y\ \partial$*count-space UNIV*)
begin

abbreviation S :: (*real* $\times$ $'a$) *measure*
where $S \equiv$ (*borel* $\bigotimes_M$ *count-space UNIV*)

abbreviation T :: (*real* $\times$ $'a$) *stream measure*
where $T \equiv$ *stream-space* S

abbreviation I :: $'a \Rightarrow 'a$ *set*
where $I\ x \equiv \{y.\ 0 < R\ x\ y\}$

lemma *I-countable*: *countable* ($I\ x$)
proof −
  let *?P* = *point-measure UNIV* ($R\ x$)
  interpret *finite-measure ?P*
  proof
    show *emeasure ?P* (*space ?P*) $\neq \infty$
      using *finite-weight*
      by (*simp add*: *emeasure-density point-measure-def less-top*)
  qed
  from *countable-support emeasure-point-measure-finite2*[*of* {-} *UNIV R x*]
  show *?thesis*
    by (*simp add*: *emeasure-eq-measure less-le*)
qed

definition *escape-rate* :: $'a \Rightarrow$ *real* **where**
  escape-rate $x = \int y.\ R\ x\ y\ \partial$*count-space UNIV*

lemma *ennreal-escape-rate*: *ennreal* (*escape-rate* x) $=$ ($\int^+ y.\ R\ x\ y\ \partial$*count-space UNIV*)
  using *finite-weight*[*of* x] **unfolding** *escape-rate-def*

by (*intro nn-integral-eq-integral*[*symmetric*]) (*auto simp*: *integrable-iff-bounded*)

lemma *escape-rate-pos*: *0* < *escape-rate x*
using *positive-weight* **unfolding** *ennreal-escape-rate*[*symmetric*] **by** *simp*

lemma *nonneg-escape-rate*[*simp*]: *0* ≤ *escape-rate x*
using *escape-rate-pos*[*THEN less-imp-le*] **.**

lemma *prob-space-exponential-escape-rate*: *prob-space* (*exponential* (*escape-rate x*))
using *escape-rate-pos* **by** (*rule prob-space-exponential*)

lemma *measurable-escape-rate*[*measurable*]: *escape-rate* ∈ *count-space UNIV* $\rightarrow_M$ *borel*
by *auto*

lemma *measurable-exponential-escape-rate*[*measurable*]: (λ*x*. *exponential* (*escape-rate x*)) ∈ *count-space UNIV* $\rightarrow_M$ *prob-algebra borel*
by (*auto simp*: *space-prob-algebra sets-exponential prob-space-exponential-escape-rate*)

interpretation *pmf-as-function* **.**

lift-definition *J* :: ′*a* ⇒ ′*a pmf* **is** λ*x y*. *R x y* / *escape-rate x*
proof *safe*
show *0* ≤ *R x y* / *escape-rate x* **for** *x y*
by (*auto intro*!: *integral-nonneg-AE divide-nonneg-nonneg R-nonneg simp*: *escape-rate-def*)
show (∫ $^+$*y*. *R x y* / *escape-rate x* ∂*count-space UNIV*) = *1* **for** *x*
using *escape-rate-pos*[*of x*]
by (*auto simp add*: *divide-ennreal*[*symmetric*] *nn-integral-divide ennreal-escape-rate*[*symmetric*] *intro*!: *ennreal-divide-self*)
qed

lemma *set-pmf-J*: *set-pmf* (*J x*) = *I x*
using *escape-rate-pos*[*of x*] **by** (*auto simp*: *set-pmf-iff J.rep-eq less-le*)

interpretation *exp-esc*: *pair-prob-space distr* (*exponential* (*escape-rate x*)) *borel* ((+) *t*) *J x* **for** *x*
proof −
interpret *prob-space distr* (*exponential* (*escape-rate x*)) *borel* ((+) *t*)
by (*intro prob-space.prob-space-distr prob-space-exponential-escape-rate*) *simp*
show *pair-prob-space* (*distr* (*exponential* (*escape-rate x*)) *borel* ((+) *t*)) (*measure-pmf* (*J x*))
by *standard*
qed

7.4 Continuous-time Kernel

definition *K* :: (*real* × ′*a*) ⇒ (*real* × ′*a*) *measure* **where**
K = (λ(*t*, *x*). (*distr* (*exponential* (*escape-rate x*)) *borel* ((+) *t*)) $\bigotimes_M$ *J x*)

interpretation *K*: *discrete-Markov-process borel* $\bigotimes_M$ *count-space UNIV K*
proof
show $K \in$ *borel* $\bigotimes_M$ *count-space UNIV* $\rightarrow_M$ *prob-algebra* (*borel* $\bigotimes_M$ *count-space UNIV*)
unfolding *K-def*
apply *measurable*
apply (*rule measurable-snd*[*THEN measurable-compose*])
apply (*auto simp*: *space-prob-algebra prob-space-measure-pmf*)
done
qed

interpretation *DTMC*: *MC-syntax J* **.**

lemma *in-space-S*[*simp*]: $x \in$ *space S*
by (*simp add*: *space-pair-measure*)

lemma *in-space-T*[*simp*]: $x \in$ *space T*
by (*simp add*: *space-pair-measure space-stream-space*)

lemma *in-space-lim-stream*: $\omega \in$ *space* (*K.lim-stream x*)
unfolding *K.space-lim-stream space-stream-space*[*symmetric*] **by** *simp*

lemma *prob-space-K-lim*: *prob-space* (*K.lim-stream x*)
using *K.lim-stream*[*THEN measurable-space*] **by** (*simp add*: *space-prob-algebra*)

definition *select-first* :: $'a \Rightarrow ('a \Rightarrow real) \Rightarrow 'a \Rightarrow bool$
where *select-first x p y* = $(y \in I\ x \wedge (\forall y' \in I\ x - \{y\}.\ p\ y < p\ y'))$

lemma *select-firstD1*: *select-first x p y* $\Longrightarrow y \in I\ x$
by (*simp add*: *select-first-def*)

lemma *select-first-unique*:
assumes *y*: *select-first x p y1 select-first x p y2* **shows** $y1 = y2$
proof −
have $y1 \neq y2 \Longrightarrow p\ y1 < p\ y2$ $y1 \neq y2 \Longrightarrow p\ y2 < p\ y1$
using *y* **by** (*auto simp*: *select-first-def*)
then show $y1 = y2$
by (*rule-tac ccontr*) *auto*
qed

lemma *The-select-first*[*simp*]: *select-first x p y* $\Longrightarrow$ *The* (*select-first x p*) = *y*
by (*intro the-equality select-first-unique*)

lemma *select-first-INF*:
select-first x p y $\Longrightarrow$ (*INF* $x \in I\ x.\ p\ x) = p\ y$
by (*intro antisym cINF-greatest cINF-lower bdd-belowI2*[**where** *m=p y*])
(*auto simp*: *select-first-def le-less*)

lemma *measurable-select-first*[*measurable*]:
(λp. *select-first x p y*) $\in$ (Π_M *y*$\in$*I x*. *borel*) $\to_M$ *count-space UNIV*
using *I-countable* **unfolding** *select-first-def* **by** (*intro measurable-pred-countable pred-intros-conj1'*) *measurable*

lemma *measurable-THE-select-first*[*measurable*]:
(λp. *The* (*select-first x p*)) $\in$ (Π_M *y*$\in$*I x*. *borel*) $\to_M$ *count-space UNIV*
by (*rule measurable-THE*) (*auto intro*: *select-first-unique I-countable dest*: *select-firstD1*)

lemma *sets-S-eq*: *sets S* = *sigma-sets UNIV* { {*t* ..} × *A* | *t A*. *A* ⊆ − *I x* ∨ (∃ *s*∈*I x*. *A* = {*s*}) }
proof (*subst sets-pair-eq*)
let *?CI* = λ*a*::*real*. {*a* ..} **let** *?Ea* = *range ?CI*

show *?Ea* ⊆ *Pow* (*space borel*) *sets borel* = *sigma-sets* (*space borel*) *?Ea*
unfolding *borel-Ici* **by** *auto*
show *?CI*‘*Rats* ⊆ *?Ea* (⋃ *i*∈*Rats*. *?CI i*) = *space borel*
using *Rats-dense-in-real*[*of x* − *1 x* **for** *x*] **by** (*auto intro*: *less-imp-le*)

let *?Eb* = *Pow* (− *I x*) ∪ (λ*s*. {*s*}) ‘ *I x*
have *b* ∈ *sigma-sets UNIV* (*Pow* (− *I x*) ∪ (λ*s*. {*s*}) ‘ *I x*) **for** *b*
proof −
have *b* = (*b* − *I x*) ∪ (⋃ *x*∈*b* ∩ *I x*. {*x*})
by *auto*
also have ... ∈ *sigma UNIV* (*Pow* (− *I x*) ∪ (λ*s*. {*s*}) ‘ *I x*)
using *I-countable* **by** (*intro sets.Un sets.countable-UN'*) *auto*
finally show *?thesis*
by *simp*
qed
then show *sets* (*count-space UNIV*) = *sigma-sets* (*space* (*count-space UNIV*)) *?Eb*
by *auto*
show *countable* ({− *I x*} ∪ (⋃ *s*∈*I x*. {{*s*}}))
using *I-countable* **by** *auto*
show *sets* (*sigma* (*space borel* × *space* (*count-space UNIV*)) {*a* × *b* |*a b*. *a* ∈ *?Ea* ∧ *b* ∈ *?Eb*}) =
sigma-sets UNIV {{*t* ..} × *A* |*t A*. *A* ⊆ − *I x* ∨ (∃ *s*∈*I x*. *A* = {*s*})}
apply *simp*
apply (*intro arg-cong*[**where** *f*=*sigma-sets* -])
apply *auto*
done
qed (*auto intro*: *countable-rat*)

7.5 Kernel equals Parallel Choice

abbreviation *PAR* :: *'a* ⇒ (*'a* ⇒ *real*) *measure*
where
PAR x ≡ (Π_M *y*∈*I x*. *exponential* (*R x y*))

lemma *PAR-least*:
  assumes *y*: $y \in I\ x$
  shows $PAR\ x\ \{p{\in}space\ (PAR\ x).\ t \le p\ y \wedge select\text{-}first\ x\ p\ y\} =$
    $emeasure\ (exponential\ (escape\text{-}rate\ x))\ \{t\ ..\} * ennreal\ (pmf\ (J\ x)\ y)$
proof −
  let $?E = \lambda y.\ exponential\ (R\ x\ y)$ **let** $?P' = \Pi_M\ y{\in}I\ x - \{y\}.\ ?E\ y$
  interpret P': *prob-space* $?P'$
    by (*intro prob-space-PiM prob-space-exponential*) *simp*
  have ∗: $PAR\ x = (\Pi_M\ y{\in}insert\ y\ (I\ x - \{y\}).\ ?E\ y)$
    using *y* **by** (*intro PiM-cong*) *auto*
  have $0 < R\ x\ y$
    using *y* **by** *simp*
  have ∗∗: $(\lambda(x, X).\ X(y := x)) \in exponential\ (R\ x\ y) \bigotimes_M Pi_M\ (I\ x - \{y\})\ (\lambda i.$
$exponential\ (R\ x\ i)) \rightarrow_M PAR\ x$
    using *y*
    apply (*subst measurable-cong-sets*[*OF sets-pair-measure-cong*[*OF sets-exponential*
sets-PiM-cong[*OF refl sets-exponential*]] *sets-PiM-cong*[*OF refl sets-exponential*]])
    apply *measurable*
    apply (*rule measurable-fun-upd*[**where** $J{=}I\ x - \{y\}$])
    apply *auto*
    done
  have $PAR\ x\ \{p{\in}space\ (PAR\ x).\ t \le p\ y \wedge (\forall y'{\in}I\ x{-}\{y\}.\ p\ y < p\ y')\} =$
    $(\int^+ ty.\ indicator\ \{t..\}\ ty * ?P'\ \{p{\in}space\ ?P'.\ \forall y'{\in}I\ x{-}\{y\}.\ ty < p\ y'\}\ \partial ?E\ y)$
    unfolding ∗ **using** ‹$y \in I\ x$›
    apply (*subst distr-pair-PiM-eq-PiM*[*symmetric*])
    apply (*auto intro*!: *prob-space-exponential simp*: *emeasure-distr insert-absorb*)
    apply (*subst emeasure-distr*[*OF* ∗∗])
    subgoal
      using *I-countable* **by** (*auto simp*: *pred-def*[*symmetric*])
    apply (*subst* P'.*emeasure-pair-measure-alt*)
    subgoal
      using *I-countable*[*of x*]
      apply (*intro measurable-sets*[*OF* ∗∗])
      apply (*auto simp*: *pred-def*[*symmetric*])
      done
    apply (*auto intro*!: *nn-integral-cong arg-cong2*[**where** *f*=*emeasure*] *split*: *split-indicator*
if-split-asm
      simp: *space-exponential space-PiM space-pair-measure PiE-iff extensional-def*)
    done
  also have $\ldots = (\int^+ ty.\ indicator\ \{t..\}\ ty * ennreal\ (exp\ (-\ ty * (escape\text{-}rate\ x$
$-\ R\ x\ y)))\ \partial ?E\ y)$
    apply (*intro nn-integral-cong-AE*)
    using *AE-exponential*[*OF* ‹$0 < R\ x\ y$›]
  proof *eventually-elim*
    fix *ty* :: *real* **assume** $0 < ty$
    have $escape\text{-}rate\ x =$
      $(\int^+ y'.\ R\ x\ y' * indicator\ \{y\}\ y'\ \partial count\text{-}space\ UNIV) + (\int^+ y'.\ R\ x\ y' *$
$indicator\ (I\ x - \{y\})\ y'\ \partial count\text{-}space\ UNIV)$

unfolding *ennreal-escape-rate* **by** (*subst nn-integral-add*[*symmetric*]) (*auto simp*: *less-le split*: *split-indicator intro*!: *nn-integral-cong*)
also have ... = *R x y* + ($\int^+$*y'*. *R x y'* ∂*count-space* (*I x* − {*y*}))
by (*auto simp add*: *nn-integral-count-space-indicator less-le simp del*: *nn-integral-indicator-singleton*
intro!: *arg-cong2*[**where** *f*=(+)] *nn-integral-cong split*: *split-indicator*)
finally have ($\int^+$*y'*. *R x y'* ∂*count-space* (*I x* − {*y*})) = *escape-rate x* − *R x y* ∧ *R x y* ≤ *escape-rate x*
using *escape-rate-pos*[*THEN less-imp-le*]
by (*cases* ($\int^+$*y'*. *R x y'* ∂*count-space* (*I x* − {*y*})))
(*auto simp*: *add-top ennreal-plus*[*symmetric*] *simp del*: *ennreal-plus*)
then have *integrable* (*count-space* (*I x* − {*y*})) (*R x*) (*LINT y'*|*count-space* (*I x* − {*y*}). *R x y'*) = *escape-rate x* − *R x y*
by (*auto simp*: *nn-integral-eq-integrable*)
then have *?P'* (*prod-emb* (*I x*−{*y*}) *?E* (*I x*−{*y*}) (Π_E *j*∈(*I x*−{*y*}). {*ty*<..})) = *exp* (− *ty* ∗ (*escape-rate x* − *R x y*))
using *I-countable* ‹*0* < *ty*› **by** (*subst emeasure-PiM-exponential-Ioi-countable*) *auto*
also have *prod-emb* (*I x*−{*y*}) *?E* (*I x*−{*y*}) (Π_E *j*∈(*I x*−{*y*}). {*ty*<..}) = {*p*∈*space ?P'*. ∀ *y'*∈*I x*−{*y*}. *ty* < *p y'*}
by (*simp add*: *set-eq-iff prod-emb-def space-PiM space-exponential ac-simps Pi-iff*)
finally show *indicator* {*t*..} *ty* ∗ *?P'* {*p*∈*space ?P'*. ∀ *y'*∈*I x*−{*y*}. *ty* < *p y'*} =
indicator {*t*..} *ty* ∗ *ennreal* (*exp* (− *ty* ∗ (*escape-rate x* − *R x y*)))
by *simp*
qed
also have ... = ($\int^+$*ty*. *ennreal* (*R x y*) ∗ (*ennreal* (*exp* (− *ty* ∗ *escape-rate x*)) ∗ *indicator* {*max 0 t*..} *ty*) ∂*lborel*)
by (*auto simp add*: *exponential-def exponential-density-def nn-integral-density ennreal-mult*[*symmetric*] *exp-add*[*symmetric*] *field-simps*
intro!: *nn-integral-cong split*: *split-indicator*)
also have ... = (*R x y* / *escape-rate x*) ∗ *emeasure* (*exponential* (*escape-rate x*)) {*max 0 t*..}
using *escape-rate-pos*[*of x*]
by (*auto simp*: *exponential-def exponential-density-def emeasure-density nn-integral-cmult*[*symmetric*] *ennreal-mult*[*symmetric*]
split: *split-indicator intro*!: *nn-integral-cong*)
also have ... = *pmf* (*J x*) *y* ∗ *emeasure* (*exponential* (*escape-rate x*)) {*t*..}
using *AE-exponential*[*OF escape-rate-pos*[*of x*]]
by (*intro arg-cong2*[**where** *f*=(∗)] *emeasure-eq-AE*) (*auto simp*: *J.rep-eq*)
finally show *?thesis*
using *assms* **by** (*simp add*: *mult-ac select-first-def*)
qed

lemma *AE-PAR-least*: *AE p in PAR x*. ∃ *y*∈*I x*. *select-first x p y*
proof −
have *D*: *disjoint-family-on* (λ*y*. {*p* ∈ *space* (*PAR x*). *select-first x p y*}) (*I x*)
by (*auto simp*: *disjoint-family-on-def dest*: *select-first-unique*)
have *PAR x* {*p*∈*space* (*PAR x*). ∃ *y*∈*I x*. *select-first x p y*} =

PAR x (⋃*y*∈*I x*. {*p*∈*space* (*PAR x*). *select-first x p y*})
by (*auto intro*!: *arg-cong2*[**where** *f*=*emeasure*])
also have ... = (∫⁺*y*. *PAR x* {*p*∈*space* (*PAR x*). *select-first x p y*} ∂*count-space* (*I x*))
using *I-countable* **by** (*intro emeasure-UN-countable D*) *auto*
also have ... = (∫⁺*y*. *PAR x* {*p*∈*space* (*PAR x*). *0* ≤ *p y* ∧ *select-first x p y*} ∂*count-space* (*I x*))
proof (*intro nn-integral-cong emeasure-eq-AE*, *goal-cases*)
case (*1 y*) **with** *AE-PiM-component*[*of I x* λ*y*. *exponential* (*R x y*) *y* (<) *0*] *AE-exponential*[*of R x y*] **show** *?case*
by (*auto simp*: *prob-space-exponential*)
qed (*insert I-countable*, *auto*)
also have ... = (∫⁺*y*. *emeasure* (*exponential* (*escape-rate x*)) {*0* ..} ∗ *ennreal* (*pmf* (*J x*) *y*) ∂*count-space* (*I x*))
by (*auto simp add*: *PAR-least intro*!: *nn-integral-cong*)
also have ... = (∫⁺*y*. *emeasure* (*exponential* (*escape-rate x*)) {*0* ..} ∂*J x*)
by (*auto simp*: *nn-integral-measure-pmf nn-integral-count-space-indicator ac-simps pmf-eq-0-set-pmf set-pmf-J*
simp del: *nn-integral-const intro*!: *nn-integral-cong split*: *split-indicator*)
also have ... = *1*
using *AE-exponential*[*of escape-rate x*]
by (*auto intro*!: *prob-space.emeasure-eq-1-AE prob-space-exponential simp*: *escape-rate-pos less-imp-le*)
finally show *?thesis*
using *I-countable*
by (*subst prob-space.AE-iff-emeasure-eq-1 prob-space-PiM prob-space-exponential*)
(*auto intro*!: *prob-space-PiM prob-space-exponential simp del*: *Set.bex-simps*(*6*))
qed

lemma *K-alt*: *K* (*t*, *x*) = *distr* (Π_M *y*∈*I x*. *exponential* (*R x y*)) *S* (λ*p*. (*t* + (*INF y*∈*I x*. *p y*), *The* (*select-first x p*))) (**is** - = *?R*)
proof (*rule measure-eqI-generator-eq-countable*)
let *?E* = { {*t* ..} × *A* | (*t*::*real*) *A*. *A* ⊆ − *I x* ∨ (∃ *s*∈*I x*. *A* = {*s*}) }
show *Int-stable ?E*
apply (*auto simp*: *Int-stable-def*)
subgoal for *t1 A1 t2 A2*
by (*intro exI*[*of - max t1 t2*] *exI*[*of - A1* ∩ *A2*]) *auto*
subgoal for *t1 t2 y1 y2*
by (*intro exI*[*of - max t1 t2*] *exI*[*of -* {*y1*} ∩ {*y2*}]) *auto*
done
show *sets* (*K* (*t*, *x*)) = *sigma-sets UNIV ?E*
unfolding *K.sets-K*[*OF in-space-S*] **by** (*subst sets-S-eq*) *rule*
show *sets ?R* = *sigma-sets UNIV ?E*
using *sets-S-eq* **by** *simp*
show *countable* ((λ(*t*, *A*). {*t* ..} × *A*) ‘ (ℚ × ({− *I x*} ∪ (λ*s*. {*s*}) ‘ *I x*)))
by (*intro countable-image countable-SIGMA countable-rat countable-Un I-countable*) *auto*

have ∗: (+) *t* −‘ {*t*′..} ∩ *space* (*exponential* (*escape-rate x*)) = {*t*′ − *t*..} **for** *t*′

by (*auto simp*: *space-exponential*)
{ fix X **assume** $X \in ?E$
then consider
t' s **where** $s \in I\ x\ X = \{t'\ ..\} \times \{s\}$
| t' A **where** $A \subseteq -\ I\ x\ X = \{t'\ ..\} \times A$
by *auto*
then show $K\ (t,\ x)\ X = ?R\ X$
proof *cases*
case *1*
have *AE p in PAR x*. $(t' - t \le p\ s \wedge$ *select-first x p s*$) =$
$(t' \le t + (\prod x{\in}I\ x.\ p\ x) \wedge$ *The* (*select-first x p*) $= s)$
using *AE-PAR-least* **by** *eventually-elim* (*auto dest*: *select-first-unique simp*: *select-first-INF*)
with *1 I-countable* **show** *?thesis*
by (*auto simp add*: *K-def measure-pmf.emeasure-pair-measure-Times emeasure-distr emeasure-pmf-single* ∗
PAR-least[*symmetric*] *intro*!: *emeasure-eq-AE*)
next
case *2*
moreover
then have *emeasure* (*measure-pmf* (*J x*)) $A = 0$
by (*subst AE-iff-measurable*[*symmetric*, **where** $P{=}\lambda x.\ x \notin A$])
(*auto simp*: *AE-measure-pmf-iff set-pmf-J subset-eq*)
moreover
have *PAR x* $((\lambda p.\ (t + \prod(p\ `\ (I\ x)),$ *The* (*select-first x p*))) $-`\ (\{t'..\} \times A) \cap$ *space* (*PAR x*)) $= 0$
using ‹$A \subseteq -\ I\ x$› *AE-PAR-least*[*of x*] *I-countable*
by (*subst AE-iff-measurable*[*symmetric*, **where** $P{=}\lambda p.\ (t + \prod(p\ `\ (I\ x)),$ *The* (*select-first x p*)) $\notin \{t'..\} \times A$])
(*auto simp del*: *all-simps*(*5*) *simp add*: *imp-ex imp-conjL subset-eq*)
ultimately show *?thesis*
using *I-countable*
by (*simp add*: *K-def measure-pmf.emeasure-pair-measure-Times emeasure-distr* ∗)
qed }

interpret *prob-space K ts* **for** *ts*
by (*rule K.prob-space-K*) *simp*
show *emeasure* $(K\ (t,\ x))\ a \neq \infty$ **for** a
using *emeasure-finite* **by** *simp*
qed (*insert Rats-dense-in-real*[*of* $x - 1\ x$ **for** x], *auto*, *blast intro*: *less-imp-le*)

lemma *AE-K*: *AE y in K x*. *fst x* < *fst y* ∧ *snd y* ∈ *J* (*snd x*)
unfolding *K-def split-beta*
apply (*subst exp-esc.AE-pair-iff*[*symmetric*])
apply *measurable*
apply (*simp-all add*: *AE-distr-iff AE-measure-pmf-iff exponential-def AE-density exponential-density-def cong del*: *AE-cong*)
using *AE-lborel-singleton*[*of 0*]

apply *eventually-elim*
apply *simp*
done

lemma *AE-lim-stream*:
AE ω in K.lim-stream x. ∀ *i. snd* ((*x* ## *ω*) !! *i*) ∈ *DTMC.acc*‘‘{*snd x*} ∧ *snd* (*ω* !! *i*) ∈ *J* (*snd* ((*x* ## *ω*) !! *i*)) ∧ *fst* ((*x* ## *ω*) !! *i*) < *fst* (*ω* !! *i*)
(**is** *AE ω in K.lim-stream x.* ∀ *i. ?P ω i*)
unfolding *AE-all-countable*
proof
let *?F* = *λi x ω. fst* ((*x* ## *ω*) !! *i*) **and** *?S* = *λi x ω. snd* ((*x* ## *ω*) !! *i*)
fix *i* **show** *AE ω in K.lim-stream x. ?P ω i*
proof (*induction i arbitrary*: *x*)
case *0* **with** *AE-K*[*of x*] **show** *?case*
by (*subst K.AE-lim-stream*) (*auto simp add*: *space-pair-measure cong del*: *AE-cong*)
next
case (*Suc i*)
show *?case*
proof (*subst K.AE-lim-stream*, *goal-cases*)
case *2* **show** *?case*
using *DTMC.countable-reachable*
by (*intro measurable-compose-countable-restrict*[**where** *f*=*?S* (*Suc i*) *x*])
(*simp-all del*: *Image-singleton-iff*)
next
case *3* **show** *?case*
apply (*simp del*: *AE-conj-iff cong del*: *AE-cong*)
using *AE-K*[*of x*]
apply *eventually-elim*
subgoal premises *K-prems* **for** *y*
using *Suc*
by *eventually-elim* (*insert K-prems*, *auto intro*: *converse-rtrancl-into-rtrancl*)
done
qed (*simp add*: *space-pair-measure*)
qed
qed

lemma *measurable-merge-at*[*measurable*]: (*λ*(*ω*, *ω′*). *merge-at ω j ω′*) ∈ (*T* $\bigotimes_M$ *T*) $\to_M$ *T*
proof (*rule measurable-stream-space2*)
define *F* **where** *F x n* = (*case x of* (*ω*::(*real* × *′a*) *stream*, *ω′*) ⇒ *merge-at ω j ω′*) !! *n* **for** *x n*
fix *n*
have (*λx. F x n*) ∈ *stream-space S* $\bigotimes_M$ *stream-space S* $\to_M$ *S*
proof (*induction n*)
case *0* **then show** *?case*
by (*simp add*: *F-def split-beta′ stream.case-eq-if*)
next
case (*Suc n*)

from *Suc*[*measurable*]
have *eq*: $F\ x\ (Suc\ n) = (case\ fst\ x\ of\ (t,\ s)\ \#\#\ \omega \Rightarrow if\ t \le j\ then\ F\ (\omega,\ snd\ x)\ n\ else\ snd\ x\ !!\ Suc\ n)$ **for** *x*
by (*auto simp*: *F-def split*: *prod.split stream.split*)
show *?case*
unfolding *eq stream.case-eq-if* **by** *measurable*
qed
then show $(\lambda x.\ (case\ x\ of\ (\omega,\ \omega') \Rightarrow merge\text{-}at\ \omega\ j\ \omega')\ !!\ n) \in stream\text{-}space\ S \bigotimes_M stream\text{-}space\ S \rightarrow_M S$
unfolding *F-def* **by** *auto*
qed

lemma *measurable-trace-at*[*measurable*]: $(\lambda(s,\ \omega).\ trace\text{-}at\ s\ \omega\ j) \in (count\text{-}space\ UNIV \bigotimes_M T) \rightarrow_M count\text{-}space\ UNIV$
unfolding *trace-at-eq* **by** *measurable*

lemma *measurable-trace-at′*: $(\lambda((s,\ j),\ \omega).\ trace\text{-}at\ s\ \omega\ j) \in ((count\text{-}space\ UNIV \bigotimes_M borel) \bigotimes_M T) \rightarrow_M count\text{-}space\ UNIV$
unfolding *trace-at-eq split-beta′* **by** *measurable*

lemma *K-time-split*:
assumes $t \le j$ **and** [*measurable*]: $f \in S \rightarrow_M borel$
shows $(\int^+ x.\ f\ x * indicator\ \{j <..\}\ (fst\ x)\ \partial K\ (t,\ s)) = (\int^+ x.\ f\ x\ \partial K\ (j,\ s)) * exponential\ (escape\text{-}rate\ s)\ \{j - t <..\}$
proof −
have $(\int^+ y.\ \int^+ x.\ f\ (t + x,\ y) * indicator\ \{j<..\}\ (t + x)\ \partial exponential\ (escape\text{-}rate\ s)\ \partial J\ s) =$
$(\int^+ y.\ \int^+ x.\ f\ (t + x,\ y) * indicator\ \{j - t<..\}\ x\ \partial exponential\ (escape\text{-}rate\ s)\ \partial J\ s)$
by (*intro nn-integral-cong*) (*auto split*: *split-indicator*)
also have $\ldots = (\int^+ y.\ \int^+ x.\ f\ (t + x,\ y)\ \partial uniform\text{-}measure\ (exponential\ (escape\text{-}rate\ s))\ \{j{-}t <..\}\ \partial J\ s) *$
$emeasure\ (exponential\ (escape\text{-}rate\ s))\ \{j - t <..\}$
using ‹$t \le j$› *escape-rate-pos*
by (*subst nn-integral-uniform-measure*)
(*auto simp*: *nn-integral-divide ennreal-divide-times emeasure-exponential-Ioi*)
also have $\ldots = (\int^+ y.\ \int^+ x.\ f\ (j + x,\ y)\ \partial exponential\ (escape\text{-}rate\ s)\ \partial J\ s) *$
$emeasure\ (exponential\ (escape\text{-}rate\ s))\ \{j - t <..\}$
using ‹$t \le j$› *escape-rate-pos* **by** (*simp add*: *uniform-measure-exponential nn-integral-distr*)
finally show *?thesis*
by (*simp add*: *K-def exp-esc.nn-integral-snd*[*symmetric*] *nn-integral-distr*)
qed

lemma *K-in-space*[*simp*]: $K\ x \in space\ (prob\text{-}algebra\ S)$
by (*rule measurable-space* [*OF K.K*]) *simp*

lemma *L-in-space*[*simp*]: $K.lim\text{-}stream\ x \in space\ (prob\text{-}algebra\ T)$
by (*rule measurable-space* [*OF K.lim-stream*]) *simp*

7.6 Markov Chain Property

lemma *lim-time-split*:
$t \le j \Longrightarrow K.lim\text{-}stream\ (t, s) = do\ \{\ \omega \leftarrow K.lim\text{-}stream\ (t, s)\ ;\ \omega' \leftarrow K.lim\text{-}stream\ (j,\ trace\text{-}at\ s\ \omega\ j)\ ;\ return\ T\ (merge\text{-}at\ \omega\ j\ \omega')\}$
(**is** - $\Longrightarrow$ - = *?DO t s*)
proof (*coinduction arbitrary*: *t s rule*: *K.lim-stream-eq-coinduct*)
case *step* **let** *?L* = *K.lim-stream*

note *measurable-compose*[*OF measurable-prob-algebraD measurable-emeasure-subprob-algebra*, *measurable* (*raw*)]

define B' **where** $B' = (\lambda(t', s).\ if\ t' \le j\ then\ ?DO\ t'\ s\ else\ ?L\ (t', s))$
show *?case*
proof (*intro bexI conjI AE-I2*)
show [*measurable*]: $B' \in S \rightarrow_M prob\text{-}algebra\ T$
unfolding *B′-def* **by** *measurable*
show $(\exists\, t\ s.\ y = (t, s) \wedge B'\ y = ?DO\ t\ s \wedge t \le j) \vee ?L\ y = B'\ y$ **for** *y*
by (*cases y*; *cases fst y* $\le$ *j*) (*auto simp*: *B′-def*)
let $?C = \lambda x.\ do\ \{\ \omega \leftarrow ?L\ x;\ \omega' \leftarrow ?L\ (j,\ trace\text{-}at\ s\ (x\#\#\omega)\ j);\ return\ T\ (merge\text{-}at\ (x\#\#\omega)\ j\ \omega')\ \}$
have $?DO\ t\ s = do\ \{\ x \leftarrow K\ (t, s);\ ?C\ x\ \}$
apply (*subst K.lim-stream-eq*[*OF in-space-S*])
apply (*subst bind-assoc*[*OF measurable-prob-algebraD measurable-prob-algebraD*])
apply (*subst measurable-cong-sets*[*OF K.sets-K*[*OF in-space-S*] *refl*])
apply *measurable*
apply (*subst bind-assoc*[*OF measurable-prob-algebraD measurable-prob-algebraD*])
apply *measurable*
apply (*subst bind-cong*[*OF refl bind-cong*[*OF refl bind-return*[*OF measurable-prob-algebraD*]]]])
apply *measurable*
done
also have $\ldots = K\ (t, s) \gg\!= (\lambda y.\ B'\ y \gg\!= (\lambda\omega.\ return\ T\ (y\ \#\#\ \omega)))$ (**is** *?DO′* = *?R*)
proof (*rule measure-eqI*)
have *sets ?DO′ = sets T*
by (*intro sets-bind′*[*OF K-in-space*]) *measurable*
moreover have *sets ?R = sets T*
by (*intro sets-bind′*[*OF K-in-space*]) *measurable*
ultimately show *sets ?DO′ = sets ?R*
by *simp*
fix *A* **assume** $A \in sets\ ?DO'$
then have *A*[*measurable*]: $A \in T$
unfolding ‹*sets ?DO′ = sets T*› **.**
have $?DO'\ A = (\int^+ x.\ ?C\ x\ A\ \partial K\ (t, s))$
by (*subst emeasure-bind-prob-algebra*[*OF K-in-space*]) *measurable*
also have $\ldots = (\int^+ x.\ ?C\ x\ A * indicator\ \{..\ j\}\ (fst\ x)\ \partial K\ (t, s)) + (\int^+ x.\ ?C\ x\ A * indicator\ \{j <..\}\ (fst\ x)\ \partial K\ (t, s))$
by (*subst nn-integral-add*[*symmetric*]) (*auto intro*!: *nn-integral-cong split*: *split-indicator*)

also have ($\int^+ x.$ *?C x A* $*$ *indicator* {.. *j*} (*fst x*) ∂K (*t*, *s*)) =
($\int^+ y.$ *emeasure* (*B′ y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {.. *j*} (*fst y*) ∂K (*t*, *s*))
proof (*intro nn-integral-cong ennreal-mult-right-cong refl arg-cong2*[**where** *f*=*emeasure*])
fix *x* :: *real* × *′a* **assume** *indicator* {..*j*} (*fst x*) ≠ (*0*::*ennreal*)
then have *fst x* ≤ *j*
by (*auto split*: *split-indicator-asm*)
then show *?C x* = (*B′ x* $\gg=$ ($\lambda\omega$. *return T* (*x* ## ω)))
apply (*cases x*)
apply (*simp add*: *B′-def*)
apply (*subst bind-assoc*[*OF measurable-prob-algebraD measurable-prob-algebraD*])
apply *measurable*
apply (*subst bind-assoc*[*OF measurable-prob-algebraD measurable-prob-algebraD*])
apply *measurable*
apply (*subst bind-return*)
apply *measurable*
done
qed
also have ($\int^+ x.$ *?C x A* $*$ *indicator* {*j* <..} (*fst x*) ∂K (*t*, *s*)) =
($\int^+ y.$ *emeasure* (*B′ y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {*j* <..} (*fst y*) ∂K (*t*, *s*))
proof −
have $*$: (+) *t* −‘ {*j*<..} = {*j* − *t* <..}
by *auto*

have ($\int^+ x.$ *?C x A* $*$ *indicator* {*j* <..} (*fst x*) ∂K (*t*, *s*)) =
($\int^+ x.$ *?L* (*j*, *s*) *A* $*$ *indicator* {*j* <..} (*fst x*) ∂K (*t*, *s*))
by (*intro nn-integral-cong ennreal-mult-right-cong refl arg-cong2*[**where** *f*=*emeasure*])
(*auto simp*: *K.sets-lim-stream bind-return′′ bind-const′ prob-space-K-lim prob-space-imp-subprob-space split*: *split-indicator-asm*)
also have … = *?L* (*j*, *s*) *A* $*$ *exponential* (*escape-rate s*) {*j* − *t* <..}
by (*subst nn-integral-cmult*) (*simp-all add*: *K-def exp-esc.nn-integral-snd*[*symmetric*] *emeasure-distr space-exponential* $*$)
also have … = ($\int^+ x.$ *emeasure* (*?L x* $\gg=$ ($\lambda\omega$. *return T* (*x* ## ω))) *A* ∂K (*j*, *s*)) $*$ *exponential* (*escape-rate s*) {*j* − *t* <..}
by (*subst K.lim-stream-eq*) (*auto simp*: *emeasure-bind-prob-algebra*[*OF K-in-space - A*])
also have … = ($\int^+ y.$ *emeasure* (*?L y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {*j* <..} (*fst y*) ∂K (*t*, *s*))
using ‹*t* ≤ *j*› **by** (*rule K-time-split*[*symmetric*]) *measurable*
also have … = ($\int^+ y.$ *emeasure* (*B′ y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {*j* <..} (*fst y*) ∂K (*t*, *s*))
by (*intro nn-integral-cong ennreal-mult-right-cong refl arg-cong2*[**where** *f*=*emeasure*])
(*auto simp add*: *B′-def split*: *split-indicator-asm*)
finally show *?thesis* **.**
qed

also have ($\int^+ y$. *emeasure* (B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {.. *j*} (*fst y*) ∂K (*t*, *s*)) +
($\int^+ y$. *emeasure* (B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* $*$ *indicator* {*j* <..} (*fst y*) ∂K (*t*, *s*)) =
($\int^+ y$. *emeasure* (B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω))) *A* ∂K (*t*, *s*))
by (*subst nn-integral-add*[*symmetric*]) (*auto intro*!: *nn-integral-cong split*: *split-indicator*)
also have ... = *emeasure* (*K* (*t*, *s*) $\gg=$ (λy. B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω)))) *A*
by (*rule emeasure-bind-prob-algebra*[*symmetric*, *OF K-in-space - A*]) *auto*
finally show *?DO'* *A* = *emeasure* (*K* (*t*, *s*) $\gg=$ (λy. B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω)))) *A* **.**
qed
finally show *?DO t s* = *K* (*t*, *s*) $\gg=$ (λy. B' *y* $\gg=$ ($\lambda\omega$. *return T* (*y* ## ω)))
.
qed
qed (*simp add*: *space-pair-measure*)

lemma *K-eq*: *K* (*t*, *s*) = *distr* (*exponential* (*escape-rate s*) $\bigotimes_M$ *J s*) *S* (λ(t', *s*). (*t* + t', *s*))
proof −
have *distr* (*exponential* (*escape-rate s*)) *borel* ((+) *t*) $\bigotimes_M$ *distr* (*J s*) (*J s*) (λx. *x*) =
distr (*exponential* (*escape-rate s*) $\bigotimes_M$ *J s*) (*borel* $\bigotimes_M$ *J s*) (λ(*x*, *y*). (*t* + *x*, *y*))
proof (*intro pair-measure-distr*)
interpret *prob-space distr* (*measure-pmf* (*J s*)) (*measure-pmf* (*J s*)) (λx. *x*)
by (*intro measure-pmf.prob-space-distr*) *simp*
show *sigma-finite-measure* (*distr* (*measure-pmf* (*J s*)) (*measure-pmf* (*J s*)) (λx. *x*))
by *unfold-locales*
qed *auto*
also have ... = *distr* (*exponential* (*escape-rate s*) $\bigotimes_M$ *J s*) *S* (λ(*x*, *y*). (*t* + *x*, *y*))
by (*intro distr-cong refl sets-pair-measure-cong*) *simp*
finally show *?thesis*
by (*simp add*: *K-def*)
qed

lemma *K-shift*: *K* (*t* + t', *s*) = *distr* (*K* (*t*, *s*)) *S* (λ(*t*, *s*). (*t* + t', *s*))
unfolding *K-eq* **by** (*subst distr-distr*) (*auto simp*: *comp-def split-beta'* *ac-simps*)

lemma *K-not-empty*: *space* (*K x*) $\neq$ {}
by (*simp add*: *K-def space-pair-measure split*: *prod.split*)

lemma *lim-stream-not-empty*: *space* (*K.lim-stream x*) $\neq$ {}
by (*simp add*: *K.space-lim-stream space-pair-measure split*: *prod.split*)

lemma *lim-shift*: — Generalize to bijective function on *K.lim-stream* invariant on

K
  K.lim-stream $(t + t', s) = distr\ (K.lim\text{-}stream\ (t, s))\ T\ (smap\ (\lambda(t, s).\ (t + t', s)))$
  (**is** - = *?D t s*)
proof (*coinduction arbitrary*: *t s rule*: *K.lim-stream-eq-coinduct*)
  case *step* **then show** *?case*
  proof (*intro bexI*[*of* - $\lambda(t, s)$. *?D* $(t - t')$ *s*] *conjI*)
    show *?D t s* = *K* $(t + t', s)$ ⤜ $(\lambda y.$ (*case y of* $(t, s) \Rightarrow$ *?D* $(t - t')$ *s*) ⤜ $(\lambda\omega.$ *return T* $(y \#\# \omega)))$
      apply (*subst K.lim-stream-eq*[*OF in-space-S*])
      apply (*subst K-shift*)
      apply (*subst distr-bind*[*OF measurable-prob-algebraD K-not-empty*])
      apply (*measurable*; *fail*)
      apply (*measurable*; *fail*)
      apply (*subst bind-distr*[*OF - measurable-prob-algebraD K-not-empty*])
      apply (*measurable*; *fail*)
      apply (*measurable*; *fail*)
      apply (*intro bind-cong refl*)
      apply (*subst distr-bind*[*OF measurable-prob-algebraD lim-stream-not-empty*])
      apply (*measurable*; *fail*)
      apply (*measurable*; *fail*)
      apply (*simp add*: *distr-return split-beta*)
     apply (*subst bind-distr*[*OF - measurable-prob-algebraD lim-stream-not-empty*])
      apply (*measurable*; *fail*)
      apply (*measurable*; *fail*)
      apply (*simp add*: *split-beta'*)
      done
  qed (*auto cong*: *conj-cong intro*!: *exI*[*of* - - − *t'*])
qed *simp*

lemma *lim-0*: *K.lim-stream* $(t, s) = distr\ (K.lim\text{-}stream\ (0, s))\ T\ (smap\ (\lambda(t', s).\ (t' + t, s)))$
  using *lim-shift*[*of 0 t s*] **by** *simp*

7.7 Explosion time

definition *explosion* :: (*real* × *'a*) *stream* ⇒ *ereal*
  where *explosion* ω = (*SUP i. ereal* (*fst* (ω !! *i*)))

lemma *ball-less-Suc-eq*: (∀ *i*<*Suc n. P i*) ⟷ (*P 0* ∧ (∀ *i*<*n. P* (*Suc i*)))
  using *less-Suc-eq-0-disj* **by** *auto*

lemma *lim-stream-timediff-eq-exponential-1*:
  distr (*K.lim-stream ts*) (*PiM UNIV* (λ-. *borel*))
    (λω *i. escape-rate* (*snd* ((*ts*##ω) !! *i*)) ∗ (*fst* (ω !! *i*) − *fst* ((*ts*##ω) !! *i*))) =
    PiM UNIV (λ-. *exponential 1*)
  (**is** *?D* = *?P*)
proof (*rule measure-eqI-PiM-sequence*)
  show *sets ?D* = *sets* (*PiM UNIV* (λ-. *borel*)) *sets ?P* = *sets* (*PiM UNIV* (λ-.

borel))
  by (*auto intro*!: *sets-PiM-cong simp*: *sets-exponential*)
 have [*measurable*]: *ts* ∈ *space S*
  by *auto*
 { interpret *prob-space ?D*
  by (*intro prob-space.prob-space-distr K.prob-space-lim-stream measurable-abs-UNIV*) *auto*
  show *finite-measure ?D*
   by *unfold-locales* **}**

 interpret *E*: *prob-space exponential 1*
  by (*rule prob-space-exponential*) *simp*
 interpret *P*: *product-prob-space* λ-. *exponential 1 UNIV*
  by *unfold-locales*

 let *distr* - - (*?f ts*) = *?D*

 fix *A* :: *nat* ⇒ *real set* **and** *n* :: *nat* **assume** *A*[*measurable*]: $\bigwedge i$. *A i* ∈ *sets borel*
 define *n′* **where** *n′* = *Suc n*
 have *emeasure ?D* (*prod-emb UNIV* (λ-. *borel*) {..*n*} (Pi_E {..*n*} *A*)) =
  emeasure (*K.lim-stream ts*) {ω∈*space* (*stream-space S*). ∀ *i*<*n′*. *?f ts* ω *i* ∈ *A i*}
  apply (*subst emeasure-distr*)
   apply (*auto intro*!: *measurable-abs-UNIV arg-cong*[**where** *f*=*emeasure* -])
  apply (*auto simp*: *prod-emb-def K.space-lim-stream space-pair-measure n′-def*)
  done
 also have ... = ($\prod$ *i*<*n′*. *emeasure* (*exponential 1*) (*A i*))
  using *A*
 proof (*induction n′ arbitrary*: *A ts*)
  case *0* **then show** *?case*
   using *prob-space.emeasure-space-1*[*OF prob-space-K-lim*]
   by (*simp add*: *K.space-lim-stream space-pair-measure*)
 next
  case (*Suc n A ts*)
  from *Suc.prems*[*measurable*]
  have [*measurable*]: *ts* ∈ *space S*
   by *auto*

  have *emeasure* (*K.lim-stream ts*) {ω ∈ *space* (*stream-space S*). ∀ *i*<*Suc n*. *?f ts* ω *i* ∈ *A i*} =
   ($\int^+$*ts′*. *indicator* (*A 0*) (*escape-rate* (*snd ts*) ∗ (*fst ts′* − *fst ts*)) ∗
    emeasure (*K.lim-stream ts′*) {ω ∈ *space* (*stream-space S*). ∀ *i*<*n*. *?f ts′* ω *i* ∈ *A* (*Suc i*)} ∂*K ts*)
   apply (*subst K.emeasure-lim-stream*)
   apply *simp*
    apply *measurable*
    apply (*auto intro*!: *nn-integral-cong arg-cong2*[**where** *f*=*emeasure*] *split*: *split-indicator*
    simp: *ball-less-Suc-eq*)

done
also have ... $= (\int^+ ts'.$ *indicator* $(A\ 0)$ (*escape-rate* (*snd ts*) $*$ (*fst* ts' $-$ *fst ts*)) ∂K *ts*) $*$
($\prod i{<}n.$ *emeasure* (*exponential 1*) (A (*Suc i*)))
by (*subst Suc.IH*) (*simp-all add*: *nn-integral-multc*)
also have $(\int^+ ts'.$ *indicator* $(A\ 0)$ (*escape-rate* (*snd ts*) $*$ (*fst* ts' $-$ *fst ts*)) ∂K *ts*) $=$
$(\int^+ t.$ *indicator* $(A\ 0)$ (*escape-rate* (*snd ts*) $*$ t) ∂*exponential* (*escape-rate* (*snd ts*)))
by (*simp add*: *K-def exp-esc.nn-integral-snd*[*symmetric*] *nn-integral-distr split*: *prod.split*)
also have ... $=$ *emeasure* (*exponential 1*) $(A\ 0)$
using *escape-rate-pos*[*of snd ts*]
by (*subst exponential-eq-stretch*) (*simp-all add*: *nn-integral-distr*)
also have *emeasure* (*exponential 1*) $(A\ 0)$ $*$ ($\prod i{<}n.$ *emeasure* (*exponential 1*) (A (*Suc i*))) $=$
($\prod i{<}$*Suc n*. *emeasure* (*exponential 1*) (A i))
by (*rule prod.lessThan-Suc-shift*[*symmetric*])
finally show *?case* **.**
qed
also have ... $=$ *emeasure ?P* (*prod-emb UNIV* (λ-. *borel*) $\{..{<}n'\}$ (Pi_E $\{..{<}n'\}$ A))
using *P.emeasure-PiM-emb*[*of* $\{..{<}n'\}$ A] **by** (*simp add*: *prod-emb-def space-exponential*)
finally show *emeasure ?D* (*prod-emb UNIV* (λ-. *borel*) $\{..n\}$ (Pi_E $\{..n\}$ A)) $=$
emeasure ?P (*prod-emb UNIV* (λ-. *borel*) $\{..n\}$ (Pi_E $\{..n\}$ A))
by (*simp add*: n'-*def lessThan-Suc-atMost*)
qed

lemma *AE-explosion-infty*:
assumes *bdd*: *bdd-above* (*range escape-rate*)
shows *AE* ω *in K.lim-stream x*. *explosion* $\omega = \infty$
proof $-$
have *escape-rate undefined* $\leq$ (*SUP x*. *escape-rate x*)
using *bdd* **by** (*intro cSUP-upper*) *auto*
then have *SUP-escape-pos*: $0 <$ (*SUP x*. *escape-rate x*)
using *escape-rate-pos*[*of undefined*] **by** *simp*
then have *SUP-escape-nonneg*: $0 \leq$ (*SUP x*. *escape-rate x*)
by (*rule less-imp-le*)

have [*measurable*]: $x \in$ *space S* **by** *auto*
have ($\sum i.$ *1::ennreal*) $=$ *top*
by (*rule sums-unique*[*symmetric*]) (*auto simp*: *sums-def of-nat-tendsto-top-ennreal*)
then have *AE* ω *in* (*PiM UNIV* (λ-. *exponential 1*)). ($\sum i.$ *ereal* (ω i)) $= \infty$
by (*intro AE-PiM-exponential-suminf-infty*) *auto*
then have *AE* ω *in K.lim-stream x*.
($\sum i.$ *ereal* (*escape-rate* (*snd* ((*x*##ω) !! i)) $*$ (*fst* (ω !! i) $-$ *fst* ((*x*##ω) !! i)))) $= \infty$
apply (*subst* (*asm*) *lim-stream-timediff-eq-exponential-1*[*symmetric*, *of x*])
apply (*subst* (*asm*) *AE-distr-iff*)

apply (*auto intro*!: *measurable-abs-UNIV*)
done
then show *?thesis*
using *AE-lim-stream*
proof *eventually-elim*
case (*elim* ω)
then have *le*: *fst* ((*x*##ω) !! *n*) $\leq$ *fst* ((*x* ## ω) !! *m*) **if** $n \leq m$ **for** *n m*
by (*intro lift-Suc-mono-le*[*OF* - ‹$n \leq m$›, *of* λi. *fst* ((*x* ## ω) !! *i*)]) (*auto intro*: *less-imp-le*)
have [*simp*]: *fst x* $\leq$ *fst* ((*x*##ω) !! *i*) *fst* ((*x*##ω) !! *i*) $\leq$ *fst* (ω !! *i*) **for** *i*
using *le*[*of i Suc i*] *le*[*of 0 i*] **by** *auto*

have ($\sum$ *i*. *ereal* (*escape-rate* (*snd* ((*x* ## ω) !! *i*)) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*x* ## ω) !! *i*)))) =
(*SUP n*. $\sum$ *i*<*n*. *ereal* (*escape-rate* (*snd* ((*x* ## ω) !! *i*)) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*x* ## ω) !! *i*))))
by (*intro suminf-ereal-eq-SUP*) (*auto intro*!: *mult-nonneg-nonneg*)
also have ... $\leq$ (*SUP n*. (*SUP x*. *escape-rate x*) $*$ (*ereal* (*fst* ((*x* ## ω) !! *n*)) $-$ *ereal* (*fst x*)))
proof (*intro SUP-least SUP-upper2*)
fix *n*
have ($\sum$ *i*<*n*. *ereal* (*escape-rate* (*snd* ((*x* ## ω) !! *i*)) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*x* ## ω) !! *i*)))) $\leq$
($\sum$ *i*<*n*. *ereal* ((*SUP i*. *escape-rate i*) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*x* ## ω) !! *i*))))
using *elim bdd* **by** (*intro sum-mono*) (*auto intro*!: *cSUP-upper*)
also have ... = (*SUP i*. *escape-rate i*) $*$ ($\sum$ *i*<*n*. *fst* ((*x* ## ω) !! *Suc i*) $-$ *fst* ((*x* ## ω) !! *i*))
using *elim bdd* **by** (*subst sum-ereal*) (*auto simp*: *sum-distrib-left*)
also have ... = (*SUP i*. *escape-rate i*) $*$ (*fst* ((*x* ## ω) !! *n*) $-$ *fst x*)
by (*subst sum-lessThan-telescope*) *simp*
finally show ($\sum$ *i*<*n*. *ereal* (*escape-rate* (*snd* ((*x* ## ω) !! *i*)) $*$ (*fst* (ω !! *i*) $-$ *fst* ((*x* ## ω) !! *i*))))
$\leq$ (*SUP x*. *escape-rate x*) $*$ (*ereal* (*fst* ((*x* ## ω) !! *n*)) $-$ *ereal* (*fst x*))
by *simp*
qed *simp*
also have ... = (*SUP x*. *escape-rate x*) $*$ ((*SUP n*. *ereal* (*fst* ((*x* ## ω) !! *n*))) $-$ *ereal* (*fst x*))
using *elim SUP-escape-nonneg* **by** (*subst SUP-ereal-mult-left*) (*auto simp*: *SUP-ereal-minus-left*[*symmetric*])
also have (*SUP n*. *ereal* (*fst* ((*x* ## ω) !! *n*))) = *explosion* ω
unfolding *explosion-def*
apply (*intro SUP-eq*)
subgoal for *i* **by** (*intro bexI*[*of* - *i*]) *auto*
subgoal for *i* **by** (*intro bexI*[*of* - *Suc i*]) *auto*
done
finally show *explosion* $\omega = \infty$
using *elim SUP-escape-pos* **by** (*cases explosion* ω) (*auto split*: *if-splits*)
qed
qed

7.8 Transition probability p_t

context
begin

declare [[*inductive-internals* = *true*]]

inductive *trace-in* :: *'a set* ⇒ *real* ⇒ *'a* ⇒ (*real* × *'a*) *stream* ⇒ *bool* **for** *S t*
where
  $t < t' \Longrightarrow s \in S \Longrightarrow$ *trace-in S t s* $((t', s')\#\#\omega)$
| $t \geq t' \Longrightarrow$ *trace-in S t s'* $\omega \Longrightarrow$ *trace-in S t s* $((t', s')\#\#\omega)$

end

lemma *trace-in-simps*[*simp*]:
  trace-in ss t s $(x\#\#\omega)$ = (*if t* < *fst x then s* ∈ *ss else trace-in ss t* (*snd x*) ω)
  by (*cases x*) (*subst trace-in.simps*; *auto*)

lemma *trace-in-eq-lfp*:
  trace-in ss t = *lfp* (λ*F s*. λ(*t'*, *s'*)##ω ⇒ *if t* < *t' then s* ∈ *ss else F s'* ω)
 unfolding *trace-in-def* **by** (*intro arg-cong*[**where** *f*=*lfp*] *ext*) (*auto split*: *stream.splits*)

lemma *trace-in-shiftD*: *trace-in ss t s* ω ⟹ *trace-in ss* (*t* + *t'*) *s* (*smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) ω)
  by (*induction rule*: *trace-in.induct*) *auto*

lemma *trace-in-shift*[*simp*]: *trace-in ss t s* (*smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) ω) ⟷ *trace-in ss* (*t* − *t'*) *s* ω
  using *trace-in-shiftD*[*of ss t s smap* (λ(*t*, *s'*). (*t* + *t'*, *s'*)) ω − *t'*]
    trace-in-shiftD[*of ss t* − *t' s* ω *t'*]
  by (*auto simp add*: *stream.map-comp prod.case-eq-if*)

lemma *measurable-trace-in'*:
  Measurable.pred (*borel* $\bigotimes_M$ *count-space UNIV* $\bigotimes_M$ *T*) (λ(*t*, *s*, ω). *trace-in ss t s* ω)
    (**is** *?M* (λ(*t*, *s*, ω). *trace-in ss t s* ω))
proof −
  let *?F* = λ*F*. λ(*t*, *s*, (*t'*, *s'*)##ω) ⇒ *if t* < *t' then s* ∈ *ss else F* (*t*, *s'*, ω)
  have [*measurable*]: *Measurable.pred* (*count-space UNIV*) (λ*x*. *x* ∈ *ss*)
    by *simp*
  have *trace-in ss* = (λ*t s* ω. *lfp ?F* (*t*, *s*, ω))
    unfolding *trace-in-def*
    apply (*subst lfp-arg*)
    apply (*subst lfp-rolling*[**where** *g*=λ*F t s* ω. *F* (*t*, *s*, ω)])
    subgoal by (*auto simp*: *mono-def le-fun-def split*: *stream.splits*)
    subgoal by (*auto simp*: *mono-def le-fun-def split*: *stream.splits*)
    subgoal
      by (*intro arg-cong*[**where** *f*=*lfp*])
      (*auto simp*: *mono-def le-fun-def split-beta' not-less fun-eq-iff split*: *stream.splits intro*!: *arg-cong*[**where** *f*=*lfp*])

done
then have *eq*: $(\lambda(t, s, \omega).\ trace\text{-}in\ ss\ t\ s\ \omega) = lfp\ ?F$
by *simp*
have *sup-continuous ?F*
by (*auto simp*: *sup-continuous-def fun-eq-iff split*: *stream.splits*)
then show *?thesis*
unfolding *eq*
proof (*rule measurable-lfp*)
fix *F* **assume** *?M F* **then show** *?M* (*?F F*)
by *measurable*
qed
qed

lemma *measurable-trace-in*[*measurable* (*raw*)]:
assumes [*measurable*]: $f \in M \to_M borel\ g \in M \to_M count\text{-}space\ UNIV\ h \in M \to_M T$
shows *Measurable.pred M* $(\lambda x.\ trace\text{-}in\ ss\ (f\ x)\ (g\ x)\ (h\ x))$
using *measurable-compose*[*of* $\lambda x.\ (f\ x, g\ x, h\ x)\ M$, *OF - measurable-trace-in'*[*of ss*]] **by** *simp*

definition $p :: 'a \Rightarrow 'a \Rightarrow real \Rightarrow real$
where $p\ s\ s'\ t = \mathcal{P}(\omega\ in\ K.lim\text{-}stream\ (0, s).\ trace\text{-}in\ \{s'\}\ t\ s\ \omega)$

lemma *p*[*measurable*]: $(\lambda(s, t).\ p\ s\ s'\ t) \in (count\text{-}space\ UNIV \bigotimes_M borel) \to_M borel$
proof −
have ∗: $(SIGMA\ x{:}space\ (count\text{-}space\ UNIV \bigotimes_M borel).\ \{\omega \in streams\ (space\ S).\ trace\text{-}in\ \{s'\}\ (snd\ x)\ (fst\ x)\ \omega\}) =$
$\{x{\in}space\ ((count\text{-}space\ UNIV \bigotimes_M borel) \bigotimes_M T).\ trace\text{-}in\ \{s'\}\ (snd\ (fst\ x))\ (fst\ (fst\ x))\ (snd\ x)\}$
by (*auto simp*: *space-pair-measure*)

note *measurable-trace-at'*[*measurable*]
show *?thesis*
unfolding *p-def*[*abs-def*] *split-beta'*
by (*rule measure-measurable-prob-algebra2*[**where** *N*=*T*])
(*auto simp*: *K.space-lim-stream* ∗ *pred-def*[*symmetric*]
intro!: *pred-count-space-const1 measurable-trace-at'*[*unfolded split-beta'*])
qed

lemma *p-nonpos*: **assumes** $t \leq 0$ **shows** $p\ s\ s'\ t = of\text{-}bool\ (s = s')$
proof −
have $AE\ \omega\ in\ K.lim\text{-}stream\ (0, s).\ trace\text{-}in\ \{s'\}\ t\ s\ \omega = (s = s')$
proof (*subst K.AE-lim-stream*)
show $AE\ y\ in\ K\ (0, s).\ AE\ \omega\ in\ K.lim\text{-}stream\ y.\ trace\text{-}in\ \{s'\}\ t\ s\ (y\ \#\#\ \omega) = (s = s')$
using *AE-K*
proof *eventually-elim*
fix $y :: real \times 'a$ **assume** $fst\ (0, s) < fst\ y \wedge snd\ y \in set\text{-}pmf\ (J\ (snd\ (0,$

s)))
    with ‹*t≤0*› **show** *AE ω in K.lim-stream y. trace-in* {*s*′} *t s* (*y* ## *ω*) = (*s* = *s*′)
      by (*cases y*) *auto*
    qed
  qed *auto*
  then have *p s s*′ *t* = 𝒫(*ω in K.lim-stream* (*0*, *s*). *s* = *s*′)
   unfolding *p-def* **by** (*intro prob-space.prob-eq-AE K.prob-space-lim-stream*) *auto*
  then show *?thesis*
    using *prob-space.prob-space*[*OF K.prob-space-lim-stream*] **by** *simp*
qed

lemma *p-0*: *p s s*′ *0* = *of-bool* (*s* = *s*′)
  using *p-nonpos*[*of 0*] **by** *simp*

lemma *in-sets-T*[*measurable* (*raw*)]: *Measurable.pred T P* ⟹ {*ω*. *P ω*} ∈ *sets T*
  unfolding *pred-def* **by** *simp*

lemma *distr-id*′: *sets M* = *sets N* ⟹ *distr M N* (*λx*. *x*) = *M*
  by (*subst distr-cong*[*of M M N M - λx. x*]) *simp-all*

lemma *p-nonneg*[*simp*]: *0* ≤ *p s s*′ *t*
  by (*simp add*: *p-def*)

lemma *p-le-1*[*simp*]: *p s s*′ *t* ≤ *1*
  unfolding *p-def* **by** (*intro prob-space.prob-le-1 K.prob-space-lim-stream*) *simp*

lemma *p-eq*:
  assumes *0* ≤ *t*
  shows *p s s*′′ *t* = (*of-bool* (*s* = *s*′′) + (*LINT u*:{*0..t*}|*lborel*. *escape-rate s* ∗ *exp* (*escape-rate s* ∗ *u*) ∗ (*LINT s*′|*J s*. *p s*′ *s*′′ *u*))) / *exp* (*t* ∗ *escape-rate s*)
proof −
  have ∗: (+) *0* = (*λx*::*real*. *x*)
    by *auto*
  interpret *L*: *prob-space K.lim-stream x* **for** *x*
    by (*rule K.prob-space-lim-stream*) *simp*
  interpret *E*: *prob-space exponential* (*escape-rate s*) **for** *s*
    by (*intro escape-rate-pos prob-space-exponential*)
  have *p s s*′′ *t* = *emeasure* (*K.lim-stream* (*0*, *s*)) {*ω*∈*space T*. *trace-in* {*s*′′} *t s* *ω*}
   by (*simp add*: *p-def L.emeasure-eq-measure K.space-lim-stream space-stream-space del*: *in-space-T*)
  also have . . . = (∫⁺*y*. *emeasure* (*K.lim-stream y*) {*ω*∈*space T*. *trace-in* {*s*′′} *t s* (*y*##*ω*) } ∂*K* (*0*, *s*))
    apply (*subst K.lim-stream-eq*[*OF in-space-S*])
    apply (*subst emeasure-bind-prob-algebra*[*OF K-in-space*])
    apply (*measurable*; *fail*)
    apply (*measurable*; *fail*)
    apply (*subst bind-return-distr*′[*OF lim-stream-not-empty*])

apply (*measurable*; *fail*)
apply (*simp add*: *emeasure-distr*)
done
also have ... = ($\int^+ y$. *indicator* {*t* <..} (*fst y*) ∗ *of-bool* (*s* = *s''*) + *indicator* {*0*<..*t*} (*fst y*) ∗ *p* (*snd y*) *s''* (*t* − *fst y*) ∂*K* (*0*, *s*))
apply (*intro nn-integral-cong-AE*)
using *AE-K*
apply *eventually-elim*
subgoal for *y*
using *L.emeasure-space-1*
apply (*cases y*)
apply (*auto split*: *split-indicator simp del*: *in-space-T*)
subgoal for *t' s2*
unfolding *p-def L.emeasure-eq-measure*[*symmetric*] *K.space-lim-stream space-stream-space*[*symmetric*]
by (*subst lim-0*) (*simp add*: *emeasure-distr*)
subgoal
by (*auto split*: *split-indicator cong*: *rev-conj-cong simp add*: *K.space-lim-stream space-stream-space simp del*: *in-space-T*)
done
done
also have ... = ($\int^+ u$. $\int^+ s'$. *indicator* {*t* <..} *u* ∗ *of-bool* (*s* = *s''*) +
indicator {*0*<..*t*} *u* ∗ *p s' s''* (*t* − *u*) ∂*J s* ∂*exponential* (*escape-rate s*))
unfolding *K-def*
by (*simp add*: *K-def measure-pmf.nn-integral-fst*[*symmetric*] ∗ *distr-id' sets-exponential*)
also have ... = *ennreal* (*exp* (− *t* ∗ *escape-rate s*) ∗ *of-bool* (*s* = *s''*)) +
($\int^+ u$. *indicator* {*0*<..*t*} *u* ∗ $\int^+ s'$. *p s' s''* (*t* − *u*) ∂*J s* ∂*exponential* (*escape-rate s*))
using ‹*0*≤*t*› **by** (*simp add*: *nn-integral-add nn-integral-cmult ennreal-indicator ennreal-mult emeasure-exponential-Ioi escape-rate-pos*)
also have ($\int^+ u$. *indicator* {*0*<..*t*} *u* ∗ $\int^+ s'$. *p s' s''* (*t* − *u*) ∂*J s* ∂*exponential* (*escape-rate s*)) =
($\int^+ u$. *indicator* {*0*<..*t*} *u* $*_R$ (*LINT s'*|*J s*. *p s' s''* (*t* − *u*)) ∂*exponential* (*escape-rate s*))
by (*simp add*: *measure-pmf.integrable-const-bound*[*of* - *1*] *nn-integral-eq-integral ennreal-mult ennreal-indicator*)
also have ... = (*LINT u*:{*0*<..*t*}|*exponential* (*escape-rate s*). (*LINT s'*|*J s*. *p s' s''* (*t* − *u*)))
unfolding *set-lebesgue-integral-def*
by (*intro nn-integral-eq-integral E.integrable-const-bound*[*of* - *1*] *AE-I2*)
(*auto intro*!: *mult-le-one measure-pmf.integral-le-const measure-pmf.integrable-const-bound*[*of* - *1*])
also have ... = (*LINT u*:{*0*<..*t*}|*lborel*. *escape-rate s* ∗ *exp* (− *escape-rate s* ∗ *u*) ∗ (*LINT s'*|*J s*. *p s' s''* (*t* − *u*)))
unfolding *exponential-def set-lebesgue-integral-def*
by (*subst integral-density*)
(*auto simp*: *ac-simps exponential-density-def fun-eq-iff split*: *split-indicator simp del*: *integral-mult-right integral-mult-right-zero intro*!: *arg-cong2*[**where** *f*=*integral*^{L}])

also have ... = (*LINT u*:{*0..t*}|*lborel*. *escape-rate s* ∗ *exp* (− *escape-rate s* ∗ (*t* − *u*)) ∗ (*LINT s*′|*J s*. *p s*′ *s*′′ *u*))
 using *AE-lborel-singleton*[*of 0*] *AE-lborel-singleton*[*of t*] **unfolding** *set-lebesgue-integral-def*
 by (*subst lborel-integral-real-affine*[**where** *t*=*t* **and** *c*=−*1*])
 (*auto intro*!: *integral-cong-AE split*: *split-indicator*)
also have ... = *exp* (− *t* ∗ *escape-rate s*) ∗ *escape-rate s* ∗ (*LINT u*:{*0..t*}|*lborel*. *exp* (*escape-rate s* ∗ *u*) ∗ (*LINT s*′|*J s*. *p s*′ *s*′′ *u*))
 by (*simp add*: *field-simps exp-diff exp-minus*)
finally show *p s s*′′ *t* = (*of-bool* (*s* = *s*′′) + (*LBINT u*:{*0..t*}. *escape-rate s* ∗ *exp* (*escape-rate s* ∗ *u*) ∗ (*LINT s*′|*J s*. *p s*′ *s*′′ *u*))) / *exp* (*t* ∗ *escape-rate s*)
 unfolding *set-lebesgue-integral-def*
 by (*simp del*: *ennreal-plus add*: *ennreal-plus*[*symmetric*] *exp-minus field-simps*)
qed

lemma *continuous-on-p*: *continuous-on A* (*p s s*′)
proof −
 interpret *E*: *prob-space exponential* (*escape-rate s*′′) **for** *s*′′
 by (*intro escape-rate-pos prob-space-exponential*)
 have *continuous-on* {*..0*} (*p s s*′)
 by (*simp add*: *p-nonpos continuous-on-const cong*: *continuous-on-cong-simp*)
 moreover have *continuous-on* {*0..*} (*p s s*′)
 proof (*subst continuous-on-cong*[*OF refl p-eq*])
 let *?I* = *λt*. *escape-rate s* ∗ *exp* (*escape-rate s* ∗ *t*) ∗ (*LINT s*′′|*J s*. *p s*′′ *s*′ *t*)
 show *continuous-on* {*0..*} (*λt*. (*of-bool* (*s* = *s*′) + (*LBINT u*:{*0..t*}. *?I u*)) / *exp* (*t* ∗ *escape-rate s*))
 proof (*intro continuous-intros continuous-on-LBINT*[*THEN continuous-on-subset*])
 fix *t* :: *real* **assume** *t*: *0* ≤ *t*
 then have *0* ≤ *x* ⟹ *x* ≤ *t* ⟹ *exp* (*x* ∗ *escape-rate s*) ∗ (*LINT s*′′|*J s*. *p s*′′ *s*′ *x*) ≤ *exp* (*t* ∗ *escape-rate s*) ∗ *1* **for** *x*
 by (*intro mult-mono*) (*auto intro*!: *mult-mono measure-pmf.integral-le-const measure-pmf.integrable-const-bound*[*of - 1*])
 with *t* **show** *set-integrable lborel* {*0..t*} *?I*
 using *escape-rate-pos*[*of s*] **unfolding** *set-integrable-def*
 by (*intro integrableI-bounded-set-indicator*[**where** *B*=*escape-rate s* ∗ *exp* (*escape-rate s* ∗ *t*)])
 (*auto simp*: *field-simps*)
 qed *auto*
 qed *simp*
 ultimately have *continuous-on* ({*0..*} ∪ {*..0*}) (*p s s*′)
 by (*intro continuous-on-closed-Un*) *auto*
 also have {*0..*} ∪ {*..0*::*real*} = *UNIV* **by** *auto*
 finally show *?thesis*
 by (*rule continuous-on-subset*) *simp*
qed

lemma *p-vector-derivative*: — Backward equation
 assumes *0* ≤ *t*
 shows (*p s s*′ *has-vector-derivative* (*LINT s*′′|*count-space UNIV*. *R s s*′′ ∗ *p s*′′ *s*′ *t*) − *escape-rate s* ∗ *p s s*′ *t*)

(*at t within* {*0*..})
(**is** (- *has-vector-derivative ?A*) -)
proof −
let *?I* = *λt. escape-rate s* ∗ *exp* (*escape-rate s* ∗ *t*) ∗ (*LINT s''*|*J s. p s'' s' t*)
let *?p* = *λt.* (*of-bool* (*s* = *s'*) + *integral* {*0*..*t*} *?I*) ∗ *exp* (*t* ∗$_R$ − *escape-rate s*)

{ **fix** *t* :: *real* **assume** *0* ≤ *t*
have *p s s' t* = (*of-bool* (*s* = *s'*) + (*LBINT u*:{*0*..*t*}. *?I u*)) ∗ *exp* (− *t* ∗ *escape-rate s*)
using *p-eq*[*OF* ‹*0* ≤ *t*›, *of s s'*] **by** (*simp add*: *exp-minus field-simps*)
also have (*LBINT u*:{*0*..*t*}. *?I u*) = *integral* {*0*..*t*} *?I*
proof (*intro set-borel-integral-eq-integral*)
have *0* ≤ *x* ⟹ *x* ≤ *t* ⟹ *exp* (*x* ∗ *escape-rate s*) ∗ (*LINT s''*|*J s. p s'' s' x*) ≤ *exp* (*t* ∗ *escape-rate s*) ∗ *1* **for** *x*
by (*intro mult-mono*) (*auto intro*!: *mult-mono measure-pmf.integral-le-const measure-pmf.integrable-const-bound*[*of - 1*])
with ‹*0*≤*t*› **show** *set-integrable lborel* {*0*..*t*} *?I*
using *escape-rate-pos*[*of s*] **unfolding** *set-integrable-def*
by (*intro integrableI-bounded-set-indicator*[**where** *B*=*escape-rate s* ∗ *exp* (*escape-rate s* ∗ *t*)])
(*auto simp*: *field-simps*)
qed
finally have *p s s' t* = *?p t*
by *simp* }
note *p-eq* = *this*

have *at-eq*: *at t within* {*0*..} = *at t within* {*0* .. *t* + *1*}
by (*intro at-within-nhd*[**where** *S*={..< *t*+*1*}]) *auto*

have *c-I*: *continuous-on* {*0*..*t* + *1*} *?I*
by (*intro continuous-intros continuous-on-LINT-pmf*[**where** *B*=*1*] *continuous-on-p*) *simp*

show *?thesis*
proof (*subst has-vector-derivative-cong-ev*)
show ∀$_F$ *u in nhds t. u* ∈ {*0*..} ⟶ *p s s' u* = *?p u p s s' t* = *?p t*
using ‹*0*≤*t*› **by** (*simp-all add*: *p-eq*)
have (*?p has-vector-derivative escape-rate s* ∗ ((*LINT s''*|*J s. p s'' s' t*) − *p s s' t*)) (*at t within* {*0*..})
unfolding *at-eq*
apply (*intro refl derivative-eq-intros*)
apply *rule*
apply (*rule integral-has-vector-derivative*[*OF c-I*])
apply (*simp add*: ‹*0* ≤ *t*›)
apply *rule*
apply (*rule exp-scaleR-has-vector-derivative-right*)
apply (*simp add*: *field-simps exp-minus p-eq* ‹*0*≤*t*› *split del*: *split-of-bool*)
done
also have *escape-rate s* ∗ ((*LINT s''*|*J s. p s'' s' t*) − *p s s' t*) =

    (*LINT s''|count-space UNIV. R s s'' * p s'' s' t*) − *escape-rate s * p s s' t*
      using *escape-rate-pos*[*of s*]
      by (*simp add*: *measure-pmf-eq-density integral-density J.rep-eq field-simps*)
    finally show (*?p has-vector-derivative ?A*) (*at t within* {*0..*}) **.**
  qed
qed

coinductive *wf-times* :: *real* ⇒ (*real* × *'a*) *stream* ⇒ *bool*
where
  t < *t'* ⟹ *wf-times t' ω* ⟹ *wf-times t* ((*t'*, *s'*) ## *ω*)

lemma *wf-times-simp*[*simp*]: *wf-times t* (*x* ## *ω*) ⟷ *t* < *fst x* ∧ *wf-times* (*fst x*) *ω*
  by (*cases x*) (*subst wf-times.simps*; *simp*)

lemma *trace-in-merge-at*:
  assumes *ω'*: *wf-times t' ω'*
  shows *trace-in ss t x* (*merge-at ω t' ω'*) ⟷
    (*if t* < *t' then trace-in ss t x ω else* ∃ *y*. *trace-in* {*y*} *t' x ω* ∧ *trace-in ss t y ω'*)
    (**is** *?merge* ⟷ *?cases*)
proof *safe*
  assume *?merge* **from** *this ω'* **show** *?cases*
  proof (*induction ω≡merge-at ω t' ω' arbitrary*: *ω ω'*)
    case (*1 j s' y ω''*) **then show** *?case*
      by (*cases ω*) (*auto split*: *if-splits*)
  next
    case (*2 j x ω' s' ω ω''*) **then show** *?case*
      by (*cases ω*) (*auto split*: *if-splits*)
  qed
next
  assume *?cases* **then show** *?merge*
  proof (*split if-split-asm*)
    assume *trace-in ss t x ω t* < *t'* **from** *this ω'* **show** *?thesis*
    proof *induction*
      case *1* **then show** *?case*
        by (*cases ω'*) *auto*
    qed *auto*
  next
    assume ∃ *y*. *trace-in* {*y*} *t' x ω* ∧ *trace-in ss t y ω'* ¬ *t* < *t'*
    then obtain *y* **where** *trace-in* {*y*} *t' x ω trace-in ss t y ω' t'* ≤ *t*
      by *auto*
    from *this ω'* **show** *?thesis*
      by *induction auto*
  qed
qed

lemma *AE-lim-wf-times*: *AE ω in K.lim-stream* (*t*, *s*). *wf-times t ω*
  using *AE-lim-stream*
proof *eventually-elim*

fix ω **assume** $*$: $\forall i.$ *snd* $(((t, s)\ \#\#\ \omega)\ !!\ i) \in$ *DTMC.acc* `` $\{$*snd* $(t, s)\} \wedge$
snd $(\omega\ !!\ i) \in J$ (*snd* $(((t, s)\ \#\#\ \omega)\ !!\ i)) \wedge$
fst $(((t, s)\ \#\#\ \omega)\ !!\ i) <$ *fst* $(\omega\ !!\ i)$
have $(t\ \#\#$ *smap fst* $\omega)\ !!\ i <$ *fst* $(\omega\ !!\ i)$ **for** i
using $*$[*THEN spec*, *of i*] **by** (*cases i*) *auto*
then show *wf-times t* ω
proof (*coinduction arbitrary*: $t\ \omega$)
case *wf-times* **from** *this*[*THEN spec*, *of 0*] *this*[*THEN spec*, *of Suc i* **for** *i*] **show** *?case*
by (*cases* ω) *auto*
qed
qed

lemma *wf-times-shiftD*: *wf-times* t' (*smap* $(\lambda(t', y).\ (t' + t, y))\ \omega) \Longrightarrow$ *wf-times* $(t' - t)\ \omega$
apply (*coinduction arbitrary*: $t'\ t\ \omega$)
subgoal for $t'\ t\ \omega$
apply (*cases* ω; *cases shd* ω)
apply *auto*
subgoal for $\omega'\ j\ x$
by (*rule exI*[*of - j + t*]) *auto*
done
done

lemma *wf-times-shift*[*simp*]: *wf-times* t' (*smap* $(\lambda(t', y).\ (t' + t, y))\ \omega) =$ *wf-times* $(t' - t)\ \omega$
using *wf-times-shiftD*[*of* $t' - t$ $-t$ *smap* $(\lambda(t', y).\ (t' + t, y))\ \omega$]
by (*auto simp*: *stream.map-comp stream.case-eq-if prod.case-eq-if wf-times-shiftD*)

lemma *trace-in-unique*: *trace-in* $\{y1\}\ t\ x\ \omega \Longrightarrow$ *trace-in* $\{y2\}\ t\ x\ \omega \Longrightarrow y1 = y2$
by (*induction rule*: *trace-in.induct*) *auto*

lemma *trace-at-eq*: *trace-in* $\{z\}\ t\ x\ \omega \Longrightarrow$ *trace-at* $x\ \omega\ t = z$
by (*induction rule*: *trace-in.induct*) *auto*

lemma *AE-lim-acc*: *AE* ω *in K.lim-stream* (t, x). $\forall t\ z.$ *trace-in* $\{z\}\ t\ x\ \omega \longrightarrow (x, z) \in$ *DTMC.acc*
using *AE-lim-stream*
proof (*eventually-elim*, *safe*)
fix $t'\ z\ \omega$ **assume** $*$: $\forall i.$ *snd* $(((t, x)\ \#\#\ \omega)\ !!\ i) \in$ *DTMC.acc* `` $\{$*snd* $(t, x)\} \wedge$
snd $(\omega\ !!\ i) \in J$ (*snd* $(((t, x)\ \#\#\ \omega)\ !!\ i)) \wedge$ *fst* $(((t, x)\ \#\#\ \omega)\ !!\ i) <$ *fst* $(\omega$ $!!\ i)$
and t: *trace-in* $\{z\}\ t'\ x\ \omega$
define X **where** $X =$ *DTMC.acc* `` $\{x\}$
have $(x\ \#\#$ *smap snd* $\omega)\ !!\ i \in X$ **for** i
using $*$[*THEN spec*, *of i*] **by** (*cases i*) (*auto simp*: *X-def*)
from t *this* **have** $z \in X$
proof *induction*
case (*1 j y x* ω) **with** *1.prems*[*of 0*] **show** *?case*

by *simp*
next
case (*2 j y ω x*) **with** *2.prems*[*of Suc i* **for** *i*] **show** *?case*
by *simp*
qed
then show (*x*, *z*) ∈ *DTMC.acc*
by (*simp add*: *X-def*)
qed

lemma *p-add*:
assumes *0* ≤ *t 0* ≤ *t′*
shows *p x y* (*t* + *t′*) = (*LINT z*|*count-space* (*DTMC.acc``*{*x*}). *p x z t* ∗ *p z y t′*)
proof −
interpret *L*: *prob-space K.lim-stream xy* **for** *xy*
by (*rule K.prob-space-lim-stream*) *simp*
interpret *A*: *sigma-finite-measure count-space* (*DTMC.acc``*{*x*})
by (*intro sigma-finite-measure-count-space-countable DTMC.countable-acc*) *simp*
interpret *LA*: *pair-sigma-finite count-space* (*DTMC.acc``*{*x*}) *K.lim-stream xy* **for** *xy*
by *unfold-locales*

have *p x y* (*t* + *t′*) = (∫ + ω. ∫ +ω′. *indicator* {ω∈*space T*. *trace-in* {*y*} (*t* + *t′*) *x* ω} (*merge-at* ω *t* ω′)
∂*K.lim-stream* (*t*, *trace-at x* ω *t*) ∂*K.lim-stream* (*0*, *x*))
unfolding *p-def L.emeasure-eq-measure*[*symmetric*]
apply (*subst lim-time-split*[*OF* ‹*0* ≤ *t*›])
apply (*subst emeasure-bind*[*OF lim-stream-not-empty measurable-prob-algebraD*])
apply (*measurable*; *fail*)
apply (*measurable*; *fail*)
apply (*intro nn-integral-cong*)
apply (*subst emeasure-bind*[*OF lim-stream-not-empty measurable-prob-algebraD*])
apply (*measurable*; *fail*)
apply (*measurable*; *fail*)
apply (*simp add*: *in-space-lim-stream*)
done
also have . . . = (∫ + ω. ∫ +ω′. *indicator* {ω∈*space T*. *trace-in* {*y*} (*t* + *t′*) *x* ω} (*merge-at* ω *t* (*smap* (λ(*t″*, *s*). (*t″* + *t*, *s*)) ω′))
∂*K.lim-stream* (*0*, *trace-at x* ω *t*) ∂*K.lim-stream* (*0*, *x*))
unfolding *lim-0*[*of t*] **by** (*subst nn-integral-distr*) (*measurable*; *fail*)+
also have . . . = (∫ + ω. ∫ +ω′. *of-bool* (∃ *z*∈*DTMC.acc``*{*x*}. *trace-in* {*z*} *t x* ω ∧ *trace-in* {*y*} *t′ z* ω′)
∂*K.lim-stream* (*0*, *trace-at x* ω *t*) ∂*K.lim-stream* (*0*, *x*))
apply (*rule nn-integral-cong-AE*)
using *AE-lim-wf-times AE-lim-acc*
apply *eventually-elim*
subgoal premises ω **for** ω
apply (*rule nn-integral-cong-AE*)
using *AE-lim-wf-times AE-lim-acc*

apply *eventually-elim*
using ω *assms*
apply (*auto simp add*: *trace-in-merge-at indicator-eq-1-iff*)
done
done
also have ... = ($\int^+ \omega.\ \int^+\omega'.\ \int^+z.$ *of-bool* (*trace-in* $\{z\}$ t x $\omega \wedge$ *trace-in* $\{y\}$ t' z ω')
∂*count-space* (*DTMC.acc*‘‘$\{x\}$) ∂*K.lim-stream* (*0*, *trace-at* x ω t) ∂*K.lim-stream* (*0*, x))
by (*intro nn-integral-cong of-bool-Bex-eq-nn-integral*) (*auto dest*: *trace-in-unique*)
also have ... = ($\int^+ \omega.\ \int^+z.\ \int^+\omega'.$ *of-bool* (*trace-in* $\{z\}$ t x $\omega \wedge$ *trace-in* $\{y\}$ t' z ω')
∂*K.lim-stream* (*0*, *trace-at* x ω t) ∂*count-space* (*DTMC.acc*‘‘$\{x\}$) ∂*K.lim-stream* (*0*, x))
apply (*subst LA.Fubini'*)
apply (*subst measurable-split-conv*)
apply (*rule measurable-compose-countable'*[*OF - measurable-fst*])
apply (*auto simp*: *DTMC.countable-acc*)
done
also have ... = ($\int^+z.\ \int^+ \omega.$ *of-bool* (*trace-in* $\{z\}$ t x ω) $*$ $\int^+\omega'.$ *of-bool* (*trace-in* $\{y\}$ t' z ω')
∂*K.lim-stream* (*0*, z) ∂*K.lim-stream* (*0*, x) ∂*count-space* (*DTMC.acc*‘‘$\{x\}$))
apply (*subst LA.Fubini'*)
apply (*subst measurable-split-conv*)
apply (*rule measurable-compose-countable'*[*OF - measurable-fst*])
apply (*rule nn-integral-measurable-subprob-algebra2*)
apply (*measurable*; *fail*)
apply (*rule measurable-prob-algebraD*)
apply (*auto simp*: *DTMC.countable-acc trace-at-eq intro*!: *nn-integral-cong*)
done
also have ... = ($\int^+z.$ ($\int^+ \omega.$ *of-bool* (*trace-in* $\{z\}$ t x ω)∂*K.lim-stream* (*0*, x)) $*$
($\int^+\omega'.$ *of-bool* (*trace-in* $\{y\}$ t' z ω') ∂*K.lim-stream* (*0*, z)) ∂*count-space* (*DTMC.acc*‘‘$\{x\}$))
by (*auto intro*!: *nn-integral-cong simp*: *nn-integral-multc*)
also have ... = ($\int^+z.$ *ennreal* (p x z t) $*$ *ennreal* (p z y t') ∂*count-space* (*DTMC.acc*‘‘$\{x\}$))
unfolding *p-def L.emeasure-eq-measure*[*symmetric*]
by (*auto intro*!: *nn-integral-cong arg-cong2*[**where** f=($*$)]
simp: *nn-integral-indicator*[*symmetric*] *simp del*: *nn-integral-indicator*)
finally have ($\int^+z.$ p x z t $*$ p z y t' ∂*count-space* (*DTMC.acc*‘‘$\{x\}$)) = p x y (t + t')
by (*simp add*: *ennreal-mult*)
then show *?thesis*
by (*subst* (*asm*) *nn-integral-eq-integrable*) *auto*
qed

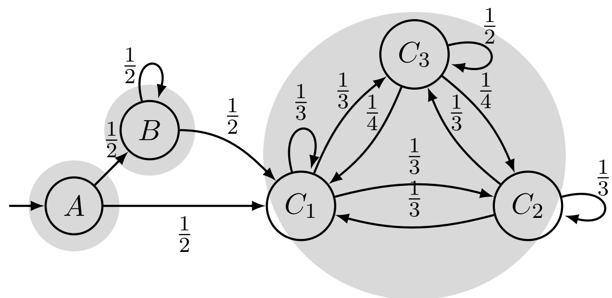
end

end
theory *Markov-Models*
imports
Markov-Models-Auxiliary
Discrete-Time-Markov-Chain
Trace-Space-Equals-Markov-Processes
Classifying-Markov-Chain-States
Markov-Decision-Process
MDP-Reachability-Problem
Discrete-Time-Markov-Process
Continuous-Time-Markov-Chain
begin

end
theory *Example-A*
imports *../Classifying-Markov-Chain-States*
begin

8 Example A

We formalize the following Markov chain:



First we define the state space as its own type:

datatype *state* = *A* | *B* | *C1* | *C2* | *C3*

Now the state space is *UNIV* :: *state set*

lemma *UNIV-state*: *UNIV* = {*A*, *B*, *C1*, *C2*, *C3*}
using *state.nchotomy* **by** *auto*

instance *state* :: *finite*
by *standard* (*simp add*: *UNIV-state*)

The transition function *tau* is easily defined using the case statement, this allows us to give a sparse specification as all *0* cases are collected at the end.

definition *tau* :: *state* ⇒ *state* ⇒ *real* **where**
tau s t = (*case* (*s*, *t*) *of*

$(A, B) \Rightarrow 1 / 2 \mid (A, C1) \Rightarrow 1 / 2$
$\mid (B, B) \Rightarrow 1 / 2 \mid (B, C1) \Rightarrow 1 / 2$
$\mid (C1, C1) \Rightarrow 1 / 3 \mid (C1, C2) \Rightarrow 1 / 3 \mid (C1, C3) \Rightarrow 1 / 3$
$\mid (C2, C1) \Rightarrow 1 / 3 \mid (C2, C2) \Rightarrow 1 / 3 \mid (C2, C3) \Rightarrow 1 / 3$
$\mid (C3, C1) \Rightarrow 1 / 4 \mid (C3, C2) \Rightarrow 1 / 4 \mid (C3, C3) \Rightarrow 1 / 2$
$\mid - \Rightarrow 0)$

lift-definition *K* :: *state* ⇒ *state pmf* **is** *tau*
by (*auto simp*: *tau-def nn-integral-count-space-finite UNIV-state split*: *state.split simp del*: *ennreal-plus*)

We use the *finite-pmf*-locale which introduces the point measure *tau.M*, and provides us with the necessary simplifier setup.

interpretation *A*: *MC-syntax K* **.**

8.1 The essential classs $\{C1, C2, C3\}$

context
begin

interpretation *pmf-as-function* **.**

lemma *A-E-eq*:
set-pmf (*K x*) = (*case x of A* ⇒ {*B*, *C1*} | *B* ⇒ {*B*, *C1*} | - ⇒ {*C1*, *C2*, *C3*})
using *state.nchotomy* **by** *transfer* (*auto simp*: *tau-def split*: *prod.split state.split*)

lemma *A-essential*: *A.essential-class* {*C1*, *C2*, *C3*}
by (*rule A.essential-classI2*) (*auto simp*: *A-E-eq*)

lemma *A-aperiodic*: *A.aperiodic* {*C1*, *C2*, *C3*}
unfolding *A.aperiodic-def*
proof *safe*
have *eq*: $\bigwedge x'$. (*if x′ = C1 then 1 else 0*) = *indicator* {*C1*} *x′* **by** *auto*

show {*C1*, *C2*, *C3*} ∈ *UNIV* // *A.communicating*
using *A-essential* **by** (*simp add*: *A.essential-class-def*)
then have *A.period* {*C1*, *C2*, *C3*} = *Gcd* (*A.period-set C1*)
by (*rule A.period-eq*) *simp*
also have ... = *1*
by (*rule Gcd-nat-eq-one*) (*simp add*: *A-E-eq A.period-set-def A.p-Suc′ A.p-0 eq measure-pmf-single pmf-positive*)
finally show *A.period* {*C1*, *C2*, *C3*} = *1* **.**
qed

8.2 The stationary distribution *n*

Similar to *tau* we introduce *n* using the *finite-pmf*-locale.

lift-definition *n* :: *state pmf* **is** λ*C1* ⇒ *0.3* | *C2* ⇒ *0.3* | *C3* ⇒ *0.4* | - ⇒ *0*
by (*auto simp*: *UNIV-state nn-integral-count-space-finite split*: *state.split*)

lemma *stationary-distribution-N*: *A.stationary-distribution n*
  unfolding *A.stationary-distribution-def*
  apply (*auto intro*!: *pmf-eqI simp*: *pmf-bind integral-measure-pmf*[*of UNIV*])
  apply *transfer*
  apply (*auto simp*: *UNIV-state tau-def split*: *state.split*)
  done

lemma *exclusive-N*[*simp*]: *set-pmf n* = {*C1*, *C2*, *C3*}
  using *state.nchotomy* **by** *transfer* (*auto split*: *state.splits*)

end

lemma *n-is-limit*:
  assumes *x*: *x* ∈ {*C1*, *C2*, *C3*} **and** *y*: *y* ∈ {*C1*, *C2*, *C3*}
  shows (*A.p x y*) ⟶ *pmf n y*
  using *A.stationary-distribution-imp-p-limit*[*OF A-aperiodic A-essential - stationary-distribution-N - x y*]
  by *simp*

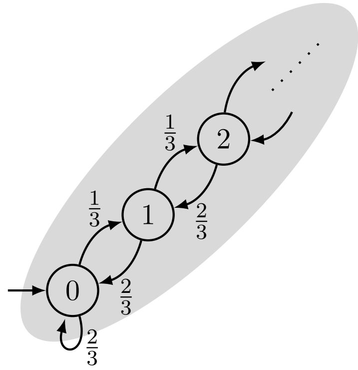
lemma *C-is-pos-recurrent*: *x* ∈ {*C1*, *C2*, *C3*} ⟹ *A.pos-recurrent x*
  using *A.stationary-distributionD*(*1*)[*OF A-essential - stationary-distribution-N*]
by *auto*

lemma *C-recurrence-time*:
  assumes *x*: *x* ∈ {*C1*, *C2*, *C3*}
  shows *A.U′ x x* = *1* / *pmf n x*
proof −
  from *A.stationary-distributionD*(*2*)[*OF A-essential - stationary-distribution-N -*]
  have *A.stat* {*C1*, *C2*, *C3*} = *n* **by** *simp*
  with *x* **have** *1* / *pmf n x* = *1* / *emeasure* (*A.stat* {*C1*, *C2*, *C3*}) {*x*}
   by (*simp add*: *emeasure-pmf-single pmf-positive divide-ennreal ennreal-1*[*symmetric*] *del*: *ennreal-1*)
  also have ... = *A.U′ x x*
    unfolding *A.stat-def* **using** *x*
    by (*subst emeasure-point-measure-finite*) (*simp-all add*: *A.U′-def*)
  finally show *?thesis* **..**
qed

end
theory *Example-B*
  imports ../*Classifying-Markov-Chain-States*
begin

9 Example B

We now formalize the following Markov chain:



As state space we have the set of natural numbers, the transition function *tau* has three cases:

definition K :: *nat* ⇒ *nat pmf* **where**
 $K\ x = map\text{-}pmf\ (\lambda True \Rightarrow x + 1 \mid False \Rightarrow x - 1)\ (bernoulli\text{-}pmf\ (1/3))$

For the special case when $x = 0$ we have $x - 1 = 0$ and hence $tau\ 0\ 0 = (2{::}'a)\ /\ (3{::}'a)$.

We pack this transition function into a discrete Markov kernel.

We call the locale of the Markov chain *B*, hence all constants and theorems from this Markov chain get a *B* prefix.

interpretation *B*: *MC-syntax K* **.**

9.1 Enabled, accessible and communicating states

For each step the predecessor and the successor are enabled (in the *0* case, the predecessor is again *0*. Hence every state is accessible from everywhere and every states is communicating with each other state. Finally we know that the state space is an essential class.

lemma *B-E-eq*: $set\text{-}pmf\ (K\ x) = \{x - 1, x + 1\}$
 by (*auto simp*: *set-pmf-bernoulli K-def split*: *bool.split*)

lemma *B-E-Suc*: $Suc\ x \in set\text{-}pmf\ (K\ x)\ x \in set\text{-}pmf\ (K\ (Suc\ x))$
 unfolding *B-E-eq* **by** *auto*

lemma *B-accessible*[*intro*]: $(i, j) \in B.acc$
proof (*cases i j rule*: *linorder-le-cases*)
 assume $i \le j$ **then show** *?thesis*
  by (*induct rule*: *inc-induct*) (*auto intro*: *B-E-Suc converse-rtrancl-into-rtrancl*)
next
 assume $j \le i$ **then show** *?thesis*
  by (*induct rule*: *dec-induct*) (*auto intro*: *B-E-Suc converse-rtrancl-into-rtrancl*)
qed

lemma *B-communicating*[*intro*]: (*i*, *j*) ∈ *B.communicating*
by (*simp add*: *B.communicating-def B-accessible*)

lemma *B-essential*: *B.essential-class UNIV*
by (*rule B.essential-classI2*) *auto*

9.2 B is aperiodic

lemma *B-aperiodic*: *B.aperiodic UNIV*
unfolding *B.aperiodic-def*
proof *safe*
have *eq*: $\bigwedge x'$. (*if* $x' = 0$ *then 1 else 0*) = *indicator* {*0*} x' **by** *auto*

show *UNIV* ∈ *UNIV* // *B.communicating*
using *B-essential* **by** (*simp add*: *B.essential-class-def*)
then have *B.period UNIV* = *Gcd* (*B.period-set 0*)
by (*rule B.period-eq*) *simp*
also have ... = *1*
by (*rule Gcd-nat-eq-one*) (*simp add*: *B.period-set-def B.p-Suc′ B.p-0 eq measure-pmf-single pmf-positive-iff K-def set-pmf-bernoulli UNIV-bool*)
finally show *B.period UNIV* = *1* **.**
qed

9.3 The stationary distribution *N*

abbreviation *N* :: *nat pmf* **where**
N ≡ *geometric-pmf* (*1* / *2*)

lemma *stationary-distribution-N*: *B.stationary-distribution N*
unfolding *B.stationary-distribution-def*
proof (*rule pmf-eqI*)
fix *a* **show** *pmf N a* = *pmf* (*bind-pmf N K*) *a*
apply (*simp add*: *pmf-bind K-def map-pmf-def*)
apply (*subst integral-measure-pmf*[*of* {*a* − *1*, *a* + *1*}])
apply (*auto split*: *split-indicator-asm nat.splits simp*: *minus-nat.diff-Suc*)
done
qed

9.4 Limit behavior and recurrence times

lemma *limit*: (*B.p i j*) ⟶ (*1*/*2*)^*Suc j*
proof −
have *B.p i j* ⟶ *pmf N j*
by (*rule B.stationary-distribution-imp-p-limit*[*OF B-aperiodic B-essential - stationary-distribution-N*])
auto
then show *?thesis*
by (*simp add*: *ac-simps*)
qed

lemma *pos-recurrent*: *B.pos-recurrent i*
using *B.stationary-distributionD*(*1*)[*OF B-essential - stationary-distribution-N* -]
by *auto*

lemma *recurrence-time*: *B.U′ i i* = *2^Suc i*
proof −
have *B.stat UNIV* = *N*
using *B.stationary-distributionD*(*2*)[*OF B-essential - stationary-distribution-N* -] **by** *simp*
then have *2^Suc i* = *1* / *emeasure* (*B.stat UNIV*) {*i*}
apply (*simp add*: *field-simps emeasure-pmf-single pmf-positive*)
apply (*subst divide-ennreal*[*symmetric*])
apply (*auto simp*: *ennreal-mult ennreal-power*[*symmetric*])
done
also have . . . = *B.U′ i i*
unfolding *B.stat-def*
by (*subst emeasure-point-measure-finite2*)
(*simp-all add*: *B.U′-def*)
finally show *?thesis*
by *simp*
qed

end

theory *PCTL*
imports
../Discrete-Time-Markov-Chain
Gauss−Jordan−Elim−Fun.Gauss-Jordan-Elim-Fun
HOL−Library.While-Combinator
HOL−Library.Monad-Syntax
begin

10 Adapt Gauss-Jordan elimination to DTMCs

locale *Finite-DTMC* =
fixes *K* :: *′s* ⇒ *′s pmf* **and** *S* :: *′s set* **and** *ϱ* :: *′s* ⇒ *real* **and** *ι* :: *′s* ⇒ *′s* ⇒ *real*
assumes *ι-nonneg*[*simp*]: ⋀*s t*. *0* ≤ *ι s t* **and** *ϱ-nonneg*[*simp*]: ⋀*s*. *0* ≤ *ϱ s*
assumes *measurable-ι*: (λ(*a*, *b*). *ι a b*) ∈ *borel-measurable* (*count-space UNIV* ⨂$_M$ *count-space UNIV*)
assumes *finite-S*[*simp*]: *finite S* **and** *S-not-empty*: *S* ≠ {}
assumes *E-closed*: (⋃*s*∈*S*. *set-pmf* (*K s*)) ⊆ *S*
begin

lemma *measurable-ι′*[*measurable* (*raw*)]:
f ∈ *measurable M* (*count-space UNIV*) ⟹ *g* ∈ *measurable M* (*count-space UNIV*) ⟹
(λ*x*. *ι* (*f x*) (*g x*)) ∈ *borel-measurable M*
using *measurable-compose*[*OF - measurable-ι*, *of* λ*x*. (*f x*, *g x*) *M*] **by** *simp*

lemma *measurable-ϱ*[*measurable*]: *ϱ* ∈ *borel-measurable* (*count-space UNIV*)
  by *simp*

sublocale *R?*: *MC-with-rewards K ι ϱ*
  by *standard* (*auto intro*: *ι-nonneg ϱ-nonneg*)

lemma *single-l*:
  fixes *s* **and** *x* :: *real* **assumes** *s* ∈ *S*
  shows ($\sum$ *s'*∈*S*. (*if s'* = *s then 1 else 0*) ∗ *l s'*) = *x* ⟷ *l s* = *x*
  by (*simp add*: *assms if-distrib* [*of* λ*x*. *x* ∗ *a* **for** *a*] *cong*: *if-cong*)

definition *order* = (*SOME f*. *bij-betw f* {..< *card S*} *S*)

lemma
  shows *bij-order*[*simp*]: *bij-betw order* {..< *card S*} *S*
    and *inj-order*[*simp*]: *inj-on order* {..<*card S*}
    and *image-order*[*simp*]: *order* ‘ {..<*card S*} = *S*
    and *order-S*[*simp*, *intro*]: ⋀*i*. *i* < *card S* ⟹ *order i* ∈ *S*
proof −
  from *finite-same-card-bij*[*OF* - *finite-S*] **show** *bij-betw order* {..< *card S*} *S*
    unfolding *order-def* **by** (*rule someI-ex*) *auto*
  then show *inj-on order* {..<*card S*} *order* ‘ {..<*card S*} = *S*
    unfolding *bij-betw-def* **by** *auto*
  then show ⋀*i*. *i* < *card S* ⟹ *order i* ∈ *S*
    by *auto*
qed

lemma *order-Ex*:
  assumes *s* ∈ *S* **obtains** *i* **where** *i* < *card S s* = *order i*
proof −
  from ‹*s* ∈ *S*› **have** *s* ∈ *order* ‘ {..<*card S*}
    by *simp*
  with *that* **show** *thesis*
    by (*auto simp del*: *image-order*)
qed

definition *iorder* = *the-inv-into* {..<*card S*} *order*

lemma *bij-iorder*: *bij-betw iorder S* {..<*card S*}
  unfolding *iorder-def* **by** (*rule bij-betw-the-inv-into bij-order*)+

lemma *iorder-image-eq*: *iorder* ‘ *S* = {..<*card S*}
  and *inj-iorder*: *inj-on iorder S*
  using *bij-iorder* **unfolding** *bij-betw-def* **by** *auto*

lemma *order-iorder*: ⋀*s*. *s* ∈ *S* ⟹ *order* (*iorder s*) = *s*
  unfolding *iorder-def* **using** *bij-order*
  by (*intro f-the-inv-into-f*) (*auto simp*: *bij-betw-def*)

```
definition gauss-jordan' :: ('s ⇒ 's ⇒ real) ⇒ ('s ⇒ real) ⇒ ('s ⇒ real) option
where
  gauss-jordan' M a = do {
     let M' = (λi j. if j = card S then a (order i) else M (order i) (order j)) ;
     sol ← gauss-jordan M' (card S) ;
     Some (λi. sol (iorder i) (card S))
  }

lemma gauss-jordan'-correct:
  assumes gauss-jordan' M a = Some f
  shows ∀ s∈S. (∑ s'∈S. M s s' ∗ f s') = a s
proof −
  note ‹gauss-jordan' M a = Some f›
  moreover define M' where M' = (λi j. if j = card S then
    a (order i) else M (order i) (order j))
  ultimately obtain sol where sol: gauss-jordan M' (card S) = Some sol
    and f: f = (λi. sol (iorder i) (card S))
    by (auto simp: gauss-jordan'-def Let-def split: bind-split-asm)

  from gauss-jordan-correct[OF sol]
  have ∀ i∈{..<card S}. (∑ j<card S. M (order i) (order j) ∗ sol j (card S)) = a
(order i)
    unfolding solution-def M'-def by (simp add: atLeast0LessThan)
  then show ?thesis
    unfolding iorder-image-eq[symmetric] f using inj-iorder
    by (subst (asm) sum.reindex) (auto simp: order-iorder)
qed

lemma gauss-jordan'-complete:
  assumes exists: ∀ s∈S. (∑ s'∈S. M s s' ∗ x s') = a s
  assumes unique: ⋀y. ∀ s∈S. (∑ s'∈S. M s s' ∗ y s') = a s ⟹ ∀ s∈S. y s = x s
  shows ∃ y. gauss-jordan' M a = Some y
proof −
  define M' where M' = (λi j. if j = card S then
    a (order i) else M (order i) (order j))

  { fix x
    have iorder-neq-card-S: ⋀s. s ∈ S ⟹ iorder s ≠ card S
      using iorder-image-eq by (auto simp: set-eq-iff less-le)
    have solution2 M' (card S) (card S) x ⟷
      (∀ s∈{..<card S}. (∑ s'∈{..<card S}. M' s s' ∗ x s') = M' s (card S))
      unfolding solution2-def by (auto simp: atLeast0LessThan)
    also have . . . ⟷ (∀ s∈S. (∑ s'∈S. M s s' ∗ x (iorder s')) = a s)
      unfolding iorder-image-eq[symmetric] M'-def
      using inj-iorder iorder-neq-card-S
      by (simp add: sum.reindex order-iorder)
    finally have solution2 M' (card S) (card S) x ⟷
      (∀ s∈S. (∑ s'∈S. M s s' ∗ x (iorder s')) = a s) . }
  note sol2-eq = this
```

have *usolution M′* (*card S*) (*card S*) (λ*i*. *x* (*order i*))
unfolding *usolution-def*
proof *safe*
from *exists* **show** *solution2 M′* (*card S*) (*card S*) (λ*i*. *x* (*order i*))
by (*simp add*: *sol2-eq order-iorder*)
next
fix *y j* **assume** *y*: *solution2 M′* (*card S*) (*card S*) *y* **and** *j* < *card S*
then have ∀ *s*∈*S*. (∑ *s′*∈*S*. *M s s′* ∗ *y* (*iorder s′*)) = *a s*
by (*simp add*: *sol2-eq*)
from *unique*[*OF this*]
have ∀ *i*∈{..<*card S*}. *y i* = *x* (*order i*)
unfolding *iorder-image-eq*[*symmetric*]
by (*simp add*: *order-iorder*)
with ‹*j* < *card S*› **show** *y j* = *x* (*order j*) **by** *simp*
qed
from *gauss-jordan-complete*[*OF* - *this*]
show *?thesis*
by (*auto simp*: *gauss-jordan′-def simp*: *M′-def*)
qed

end

11 pCTL model checking

11.1 Syntax

datatype *realrel* = *LessEqual* | *Less* | *Greater* | *GreaterEqual* | *Equal*

datatype *′s sform* = *true*
| *Label ′s set*
| *Neg ′s sform*
| *And ′s sform ′s sform*
| *Prob realrel real ′s pform*
| *Exp realrel real ′s eform*
and *′s pform* = *X ′s sform*
| *U nat ′s sform ′s sform*
| *UInfinity ′s sform ′s sform* (‹U^∞›)
and *′s eform* = *Cumm nat* (‹$C^\leq$›)
| *State nat* (‹$I^=$›)
| *Future ′s sform*

primrec *bound-until* **where**
bound-until 0 φ ψ = ψ
| *bound-until* (*Suc n*) φ ψ = ψ *or* (φ *aand nxt* (*bound-until n* φ ψ))

lemma *measurable-bound-until*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (*bound-until n* φ ψ)

by (*induct n*) *simp-all*

11.2 Semantics

primrec *inrealrel* :: *realrel* ⇒ *'a* ⇒ (*'a*::*linorder*) ⇒ *bool* **where**
inrealrel LessEqual r q ⟷ $q \leq r$ |
inrealrel Less r q ⟷ $q < r$ |
inrealrel Greater r q ⟷ $q > r$ |
inrealrel GreaterEqual r q ⟷ $q \geq r$ |
inrealrel Equal r q ⟷ $q = r$

context *Finite-DTMC*
begin

abbreviation *prob s P* ≡ *measure* (*T s*) {*x*∈*space* (*T s*). *P x*}
abbreviation *E s* ≡ *set-pmf* (*K s*)

primrec *svalid* :: *'s sform* ⇒ *'s set*
and *pvalid* :: *'s pform* ⇒ *'s stream* ⇒ *bool*
and *reward* :: *'s eform* ⇒ *'s stream* ⇒ *ennreal* **where**
svalid true = *S* |
svalid (*Label L*) = {*s* ∈ *S*. *s* ∈ *L*} |
svalid (*Neg F*) = *S* − *svalid F* |
svalid (*And F1 F2*) = *svalid F1* ∩ *svalid F2* |
svalid (*Prob rel r F*) = {*s* ∈ *S*. *inrealrel rel r* $\mathcal{P}$(ω *in T s*. *pvalid F* (*s* ## ω)) } |
svalid (*Exp rel r F*) = {*s* ∈ *S*. *inrealrel rel* (*ennreal r*) ($\int^+$ ω. *reward F* (*s* ## ω) ∂*T s*) } |

pvalid (*X F*) = *nxt* (*HLD* (*svalid F*)) |
pvalid (*U k F1 F2*) = *bound-until k* (*HLD* (*svalid F1*)) (*HLD* (*svalid F2*)) |
pvalid (U^∞ *F1 F2*) = *HLD* (*svalid F1*) *suntil HLD* (*svalid F2*) |

reward ($C^\leq$ *k*) = (λω. ($\sum$ *i*<*k*. ϱ (ω !! *i*) + ι (ω !! *i*) (ω !! (*Suc i*)))) |
reward ($I^=$ *k*) = (λω. ϱ (ω !! *k*)) |
reward (*Future F*) = (λω. *if ev* (*HLD* (*svalid F*)) ω *then reward-until* (*svalid F*) (*shd* ω) (*stl* ω) *else* ∞)

lemma *svalid-subset-S*: *svalid F* ⊆ *S*
by (*induct F*) *auto*

lemma *finite-svalid*[*simp*, *intro*]: *finite* (*svalid F*)
using *svalid-subset-S finite-S* **by** (*blast intro*: *finite-subset*)

lemma *svalid-sets*[*measurable*]: *svalid F* ∈ *sets* (*count-space S*)
using *svalid-subset-S* **by** *auto*

lemma *pvalid-sets*[*measurable*]: *Measurable.pred R.S* (*pvalid F*)
by (*cases F*) (*auto intro*!: *svalid-sets*)

lemma *reward-measurable*[*measurable*]: *reward F* ∈ *borel-measurable R.S*
by (*cases F*) *auto*

11.3 Implementation of *Sat*

11.3.1 *Prob0*

definition *Prob0* **where**
Prob0 Φ Ψ = *S* − *while* (λ*R*. ∃ *s*∈Φ. *R* ∩ *E s* ≠ {} ∧ *s* ∉ *R*) (λ*R*. *R* ∪ {*s*∈Φ. *R* ∩ *E s* ≠ {}}) Ψ

lemma *Prob0-subset-S*: *Prob0* Φ Ψ ⊆ *S*
unfolding *Prob0-def* **by** *auto*

lemma *Prob0-iff-reachable*:
assumes Φ ⊆ *S* Ψ ⊆ *S*
shows *Prob0* Φ Ψ = {*s* ∈ *S*. ((*SIGMA x*:Φ. *E x*)* `` {*s*}) ∩ Ψ = {}} (**is** - = *?U*)
unfolding *Prob0-def*
proof (*intro while-rule*[**where** *Q*=λ*R*. *S* − *R* = *?U* **and** *P*=λ*R*. Ψ ⊆ *R* ∧ *R* ⊆ *S* − *?U*] *conjI*)
show *wf* {(*B*, *A*). *A* ⊂ *B* ∧ *B* ⊆ *S*}
by (*rule wf-bounded-set*[**where** *ub*=λ-. *S* **and** *f*=λ*x*. *x*]) *auto*
show Ψ ⊆ *S* − *?U*
using *assms* **by** *auto*

let *?*Δ = λ*R*. {*s*∈Φ. *R* ∩ *E s* ≠ {}}
{ **fix** *R* **assume** *R*: Ψ ⊆ *R* ∧ *R* ⊆ *S* − *?U* **and** ∃ *s*∈Φ. *R* ∩ *E s* ≠ {} ∧ *s* ∉ *R*
with *assms* **show** (*R* ∪ *?*Δ *R*, *R*) ∈ {(*B*, *A*). *A* ⊂ *B* ∧ *B* ⊆ *S*} Ψ ⊆ *R* ∪ *?*Δ *R*
by *auto*

{ **fix** *s s'* **assume** *s*: *s* ∈ Φ *s'* ∈ *R s'* ∈ *E s* **and** *r*: (*Sigma* Φ *E*)* `` {*s*} ∩ Ψ = {}
with *R* **have** (*s*, *s'*) ∈ (*Sigma* Φ *E*)* *s'* ∈ Φ − Ψ
by (*auto elim*: *converse-rtranclE*)
moreover with ‹*s'* ∈ *R*› *R* **obtain** *s''* **where** (*s'*, *s''*) ∈ (*Sigma* Φ *E*)* *s''* ∈ Ψ
by *auto*
ultimately have (*s*, *s''*) ∈ (*Sigma* Φ *E*)* *s''* ∈ Ψ
by *auto*
with *r* **have** *False*
by *auto* }
with ‹Φ ⊆ *S*› *R* **show** *R* ∪ *?*Δ *R* ⊆ *S* − *?U* **by** *auto* }

{ **fix** *R* **assume** *R*: Ψ ⊆ *R* ∧ *R* ⊆ *S* − *?U* **and** *dR*: ¬ (∃ *s*∈Φ. *R* ∩ *E s* ≠ {} ∧ *s* ∉ *R*)
{ **fix** *s t* **assume** *s*: *s* ∈ *S* − *R*
assume *s-t*: (*s*, *t*) ∈ (*Sigma* Φ *E*)* **then have** *t* ∈ *S* − *R*
proof *induct*
case (*step t u*) **with** *R dR E-closed* **show** *?case*
by *auto*

qed *fact*
then have $t \notin \Psi$
using R **by** *auto* **}**
with R **show** $S - R = ?U$
by *auto* **}**
qed *rule*

lemma *Prob0-iff*:
assumes $\Phi \subseteq S$ $\Psi \subseteq S$
shows $Prob0\ \Phi\ \Psi = \{s{\in}S.\ AE\ \omega\ in\ T\ s.\ \neg\ (HLD\ \Phi\ suntil\ HLD\ \Psi)\ (s\ \#\#\ \omega)\}$ (**is** - = *?U*)
unfolding *Prob0-iff-reachable*[*OF assms*]
proof (*intro Collect-cong conj-cong refl iffI*)
fix *s* **assume** *s*: $s \in S$ $(Sigma\ \Phi\ E)^*$ `` $\{s\} \cap \Psi = \{\}$
{ fix ω **assume** $(HLD\ \Phi\ suntil\ HLD\ \Psi)\ \omega$ *enabled* $(shd\ \omega)\ (stl\ \omega)\ (Sigma\ \Phi\ E)^*$ `` $\{shd\ \omega\} \cap \Psi = \{\}$
from *this* **have** *False*
proof *induction*
case $(step\ \omega)$
moreover
then have $(shd\ \omega,\ shd\ (stl\ \omega)) \in (Sigma\ \Phi\ E)^*$
by (*auto simp*: *enabled.simps*[*of* - *stl* ω] *HLD-iff*)
then have $(Sigma\ \Phi\ E)^*$ `` $\{shd\ (stl\ \omega)\} \subseteq (Sigma\ \Phi\ E)^*$ `` $\{shd\ \omega\}$
by *auto*
ultimately show *?case*
by (*auto simp add*: *enabled.simps*[*of* - *stl* ω])
qed (*auto simp*: *HLD-iff*) **}**
from *s this*[*of* $s\ \#\#\ \omega$ **for** ω] **show** $AE\ \omega\ in\ T\ s.\ \neg\ (HLD\ \Phi\ suntil\ HLD\ \Psi)\ (s\ \#\#\ \omega)$
using *AE-T-enabled*[*of s*] **by** *auto*
next
fix *s* **assume** *s*: $AE\ \omega\ in\ T\ s.\ \neg\ (HLD\ \Phi\ suntil\ HLD\ \Psi)\ (s\ \#\#\ \omega)$
{ fix *t* **assume** $(s,\ t) \in (Sigma\ \Phi\ E)^*$ **from** *this s* **have** $t \notin \Psi$
proof (*induction rule*: *converse-rtrancl-induct*)
case (*step s u*) **then show** *?case*
by (*simp add*: *AE-T-iff*[**where** *x=s*] *suntil-Stream*[*of* - - *s*])
qed (*simp add*: *suntil-Stream*) **}**
then show $(Sigma\ \Phi\ E)^*$ `` $\{s\} \cap \Psi = \{\}$
by *auto*
qed

lemma *E-rtrancl-closed*:
assumes $s \in S$ $(s,\ t) \in (SIGMA\ x{:}A.\ B\ x)^*$ $\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq E\ x$ **shows** $t \in S$
using *assms(2,3,1) E-closed* **by** *induction force+*

11.3.2 *Prob1*

definition *Prob1* **where**

```
  Prob1 Y Φ Ψ = Prob0 (Φ − Ψ) Y

lemma Prob1-iff:
  assumes Φ ⊆ S Ψ ⊆ S
  shows Prob1 (Prob0 Φ Ψ) Φ Ψ = {s∈S. AE ω in T s. (HLD Φ suntil HLD Ψ)
(s ## ω)}
    (is Prob1 ?P0 - - = {s∈S. ?pU s})
proof −
  note P0 = Prob0-iff-reachable[OF assms]
  have ∗: Φ − Ψ ⊆ S ?P0 ⊆ S
    using P0 assms by auto

  have P0-subset: S − Φ − Ψ ⊆ ?P0
    unfolding P0 by (auto elim: converse-rtranclE)

  have Prob1 ?P0 Φ Ψ = {s ∈ S. (Sigma (Φ − Ψ) E)∗ `` {s} ∩ ?P0 = {}}
    unfolding Prob0-iff-reachable[OF ∗] Prob1-def ..
  also have . . . = {s∈S. AE ω in T s. (HLD Φ suntil HLD Ψ) (s ## ω)}
  proof (intro Collect-cong conj-cong refl iffI)
    fix s assume s: s ∈ S (Sigma (Φ − Ψ) E)∗ `` {s} ∩ ?P0 = {}
    then have s ∉ ?P0
      by auto
    then have s ∈ Φ − Ψ ∨ s ∈ Ψ
      using P0-subset ‹s ∈ S› by auto
    moreover
    { assume s ∈ Φ − Ψ
      have AE ω in T s. ev (HLD (Ψ ∪ ?P0)) ω
      proof (rule AE-T-ev-HLD)
        fix t assume s-t: (s, t) ∈ acc-on (− (Ψ ∪ ?P0))
        from ‹s ∈ S› s-t have t ∈ S
          by (rule E-rtrancl-closed) auto

        show ∃t′∈Ψ ∪ ?P0. (t, t′) ∈ acc
        proof cases
          assume t ∈ ?P0 then show ?thesis by auto
        next
          assume t ∉ ?P0
          with ‹t∈S› obtain s where t-s: (t, s) ∈ (SIGMA x:Φ. E x)∗ and s ∈ Ψ
            unfolding P0 by auto
          from t-s have (t, s) ∈ acc
            by (rule rev-subsetD) (intro rtrancl-mono Sigma-mono, auto)
          with ‹s ∈ Ψ› show ?thesis by auto
        qed
      next
        have acc-on (− (Ψ ∪ ?P0)) `` {s} ⊆ S
          using ‹s ∈ S› by (auto intro: E-rtrancl-closed)
        then show finite (acc-on (− (Ψ ∪ ?P0)) `` {s})
          using finite-S by (auto dest: finite-subset)
      qed
```

```
      then have AE ω in T s. (HLD Φ suntil HLD Ψ) ω
        using AE-T-enabled
      proof eventually-elim
        fix ω assume ev (HLD (Ψ ∪ ?P0)) ω enabled s ω
        from this s ‹s ∈ Φ − Ψ› show (HLD Φ suntil HLD Ψ) ω
        proof (induction arbitrary: s)
          case (base ω) then show ?case
            by (auto simp: HLD-iff enabled.simps[of s] intro: suntil.intros)
        next
          case (step ω)
          then have (s, shd ω) ∈ (Sigma (Φ − Ψ) E)
            by (auto simp: enabled.simps[of s])
          then have ∗: (Sigma (Φ − Ψ) E)^∗ ‘‘ {shd ω} ∩ ?P0 = {}
            using step.prems by (auto intro: converse-rtrancl-into-rtrancl)
          then have shd ω ∈ Φ − Ψ ∨ shd ω ∈ Ψ shd ω ∈ S
               using P0-subset step.prems(1,2) E-closed by (auto simp add: en-
abled.simps[of s])
          then show ?case
            using step.prems(1) step.IH[OF - - ∗] ‹shd ω ∈ S›
          by (auto simp add: suntil.simps[of - - ω] HLD-iff[abs-def] enabled.simps[of
s ω])
        qed
      qed }
    ultimately show AE ω in T s. (HLD Φ suntil HLD Ψ) (s ## ω)
      by (cases s ∈ Φ − Ψ) (auto simp add: suntil-Stream)
  next
    fix s assume s: s ∈ S AE ω in T s. (HLD Φ suntil HLD Ψ) (s ## ω)
    { fix t assume (s, t) ∈ (SIGMA s:Φ−Ψ. E s)^∗
      from this ‹s ∈ S› have (AE ω in T t. (HLD Φ suntil HLD Ψ) (t ## ω)) ∧
t ∈ S
      proof induction
        case (step t u) with E-closed show ?case
          by (auto simp add: AE-T-iff[of - t] suntil-Stream)
      qed (insert s, auto)
      then have t ∉ ?P0
        unfolding Prob0-iff[OF assms] by (auto dest: T.AE-contr) }
    then show (Sigma (Φ − Ψ) E)^∗ ‘‘ {s} ∩ Prob0 Φ Ψ = {}
      by auto
  qed
  finally show ?thesis .
qed
```

11.3.3 *ProbU*, *ExpCumm*, and *ExpState*

```
abbreviation τ s t ≡ pmf (K s) t

fun ProbU :: 's ⇒ nat ⇒ 's set ⇒ 's set ⇒ real where
ProbU q 0 S1 S2        = (if q ∈ S2 then 1 else 0) |
ProbU q (Suc k) S1 S2 =
```

$(if\ q \in S1\ -\ S2\ then\ (\sum q'{\in}S.\ \tau\ q\ q' * ProbU\ q'\ k\ S1\ S2)$
$\quad else\ if\ q \in S2\ then\ 1\ else\ 0)$

fun $ExpCumm :: 's \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpCumm\ s\ 0 \quad = 0\ |$
$ExpCumm\ s\ (Suc\ k) = \varrho\ s + (\sum s'{\in}S.\ \tau\ s\ s' * (\iota\ s\ s' + ExpCumm\ s'\ k))$

fun $ExpState :: 's \Rightarrow nat \Rightarrow ennreal$ **where**
$ExpState\ s\ 0 \quad = \varrho\ s\ |$
$ExpState\ s\ (Suc\ k) = (\sum s'{\in}S.\ \tau\ s\ s' * ExpState\ s'\ k)$

11.3.4 *LES*

definition $LES :: 's\ set \Rightarrow 's \Rightarrow 's \Rightarrow real$ **where**
$LES\ F\ r\ c =$
$\quad (if\ r \in F\ then\ (if\ c = r\ then\ 1\ else\ 0)$
$\quad\quad else\ (if\ c = r\ then\ \tau\ r\ c - 1\ else\ \tau\ r\ c\ ))$

11.3.5 *ProbUinfty*, compute unbounded until

definition $ProbUinfty :: 's\ set \Rightarrow 's\ set \Rightarrow ('s \Rightarrow real)\ option$ **where**
$ProbUinfty\ S1\ S2 = gauss\text{-}jordan'\ (LES\ (Prob0\ S1\ S2 \cup S2))$
$\quad (\lambda i.\ if\ i \in S2\ then\ 1\ else\ 0)$

11.3.6 *ExpFuture*, compute unbounded reward

definition $ExpFuture :: 's\ set \Rightarrow ('s \Rightarrow ennreal)\ option$ **where**
$ExpFuture\ F = do\ \{$
$\quad let\ N = Prob0\ S\ F\ ;$
$\quad let\ Y = Prob1\ N\ S\ F\ ;$
$\quad sol \leftarrow gauss\text{-}jordan'\ (LES\ (S\ -\ Y \cup F))$
$\quad\quad (\lambda i.\ if\ i \in Y \wedge i \notin F\ then\ -\ \varrho\ i - (\sum s'{\in}S.\ \tau\ i\ s' * \iota\ i\ s')\ else\ 0)\ ;$
$\quad Some\ (\lambda s.\ if\ s \in Y\ then\ ennreal\ (sol\ s)\ else\ \infty)$
$\}$

11.3.7 *Sat*

fun $Sat :: 's\ sform \Rightarrow 's\ set\ option$ **where**
$Sat\ true \quad = Some\ S\ |$
$Sat\ (Label\ L) \quad = Some\ \{s \in S.\ s \in L\}\ |$
$Sat\ (Neg\ F) \quad = do\ \{\ F \leftarrow Sat\ F\ ;\ Some\ (S\ -\ F)\ \}\ |$
$Sat\ (And\ F1\ F2) \quad = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ Some\ (F1 \cap F2)\ \}\ |$

$Sat\ (Prob\ rel\ r\ (X\ F)) \quad = do\ \{\ F \leftarrow Sat\ F\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (\sum q'{\in}F.\ \tau\ q\ q')\}\ \}\ |$
$Sat\ (Prob\ rel\ r\ (U\ k\ F1\ F2)) = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (ProbU\ q\ k\ F1\ F2)\ \}\ \}\ |$
$Sat\ (Prob\ rel\ r\ (U^{\infty}\ F1\ F2)) \quad = do\ \{\ F1 \leftarrow Sat\ F1\ ;\ F2 \leftarrow Sat\ F2\ ;\ P \leftarrow ProbUinfty\ F1\ F2\ ;\ Some\ \{q \in S.\ inrealrel\ rel\ r\ (P\ q)\ \}\ \}\ |$

Sat (*Exp rel r* (*Cumm k*)) = *Some* {*s* ∈ *S*. *inrealrel rel r* (*ExpCumm s k*) } |
Sat (*Exp rel r* (*State k*)) = *Some* {*s* ∈ *S*. *inrealrel rel r* (*ExpState s k*) } |
Sat (*Exp rel r* (*Future F*)) = *do* { *F* ← *Sat F* ; *E* ← *ExpFuture F* ; *Some* {*q* ∈ *S*. *inrealrel rel* (*ennreal r*) (*E q*) } }

lemma *prob-sum*:
s ∈ *S* ⟹ *Measurable.pred R.S P* ⟹ $\mathcal{P}$(ω *in T s*. *P* ω) = (∑ *t*∈*S*. τ *s t* ∗ $\mathcal{P}$(ω *in T t*. *P* (*t* ## ω)))
unfolding *prob-T* **using** *E-closed* **by** (*subst integral-measure-pmf*[*OF finite-S*]) (*auto simp*: *mult.commute*)

lemma *nn-integral-eq-sum*:
s ∈ *S* ⟹ *f* ∈ *borel-measurable R.S* ⟹ (∫⁺*x*. *f x* ∂*T s*) = (∑ *t*∈*S*. τ *s t* ∗ (∫⁺*x*. *f* (*t* ## *x*) ∂*T t*))
unfolding *nn-integral-T* **using** *E-closed*
by (*subst nn-integral-measure-pmf-support*[*OF finite-S*])
(*auto simp*: *mult.commute*)

lemma *T-space*[*simp*]: *measure* (*T s*) (*space R.S*) = *1*
using *T.prob-space* **by** *simp*

lemma *emeasure-T-space*[*simp*]: *emeasure* (*T s*) (*space R.S*) = *1*
using *T.emeasure-space-1* **by** *simp*

lemma τ*-distr*[*simp*]: *s* ∈ *S* ⟹ (∑ *t*∈*S*. τ *s t*) = *1*
using *prob-sum*[*of s* λ-. *True*] **by** *simp*

lemma *ProbU*:
q ∈ *S* ⟹ *ProbU q k* (*svalid F1*) (*svalid F2*) = $\mathcal{P}$(ω *in T q*. *pvalid* (*U k F1 F2*) (*q* ## ω))
proof (*induct k arbitrary*: *q*)
case *0* **with** *T.prob-space* **show** *?case* **by** *simp*
next
case (*Suc k*)

have $\mathcal{P}$(ω *in T q*. *pvalid* (*U* (*Suc k*) *F1 F2*) (*q* ## ω)) =
(*if q* ∈ *svalid F2 then 1 else if q* ∈ *svalid F1 then*
∑ *t*∈*S*. τ *q t* ∗ $\mathcal{P}$(ω *in T t*. *pvalid* (*U k F1 F2*) (*t* ## ω)) *else 0*)
using ‹*q* ∈ *S*› **by** (*subst prob-sum*) *simp-all*
also have . . . = *ProbU q* (*Suc k*) (*svalid F1*) (*svalid F2*)
using *Suc* **by** *simp*
finally show *?case* **..**
qed

lemma *Prob0-imp-not-Psi*:
assumes Φ ⊆ *S* Ψ ⊆ *S s* ∈ *Prob0* Φ Ψ **shows** *s* ∉ Ψ
proof −

have $s \in S$ **using** ‹$s \in Prob0\ \Phi\ \Psi$› *Prob0-subset-S* **by** *auto*
with *assms* **show** *?thesis* **by** (*auto simp add*: *Prob0-iff suntil-Stream*)
qed

lemma *Psi-imp-not-Prob0*:
assumes $\Phi \subseteq S\ \Psi \subseteq S$ **shows** $s \in \Psi \Longrightarrow s \notin Prob0\ \Phi\ \Psi$
using *Prob0-imp-not-Psi*[*OF assms*] **by** *metis*

11.3.8 Finite expected reward

abbreviation $s0 \equiv SOME\ s.\ s \in S$

lemma *s0-in-S*: $s0 \in S$
using *S-not-empty* **by** (*auto intro*!: *someI-ex*[*of* $\lambda x.\ x \in S$])

lemma *nn-integral-reward-finite*:
assumes $s \in S$
assumes *until*: $AE\ \omega\ in\ T\ s.\ (HLD\ S\ suntil\ HLD\ (svalid\ F))\ (s\ \#\#\ \omega)$
shows $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) \neq \infty$
proof −
have $(\int^+ \omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) = (\int^+ \omega.\ reward\text{-}until\ (svalid$
$F)\ s\ \omega\ \partial T\ s)$
using *until* **by** (*auto intro*!: *nn-integral-cong-AE ev-suntil*)
also have $\ldots \neq \infty$
proof *cases*
assume $s \notin svalid\ F$
show *?thesis*
proof (*rule nn-integral-reward-until-finite*)
have $acc\ ``\ \{s\} \subseteq S$
using *E-rtrancl-closed*[*of s - - E*] ‹$s \in S$› **by** *auto*
then show *finite* $(acc\ ``\ \{s\})$
using *finite-S* **by** (*auto dest*: *finite-subset*)
show $AE\ \omega\ in\ T\ s.\ (ev\ (HLD\ (svalid\ F)))\ \omega$
using *until* **by** (*auto simp add*: *suntil-Stream* ‹$s \notin svalid\ F$› *intro*: *ev-suntil*)
qed *auto*
qed *simp*
finally show *?thesis* **.**
qed

lemma *unique*:
assumes *in-S*: $\Phi \subseteq S\ \Psi \subseteq S\ N \subseteq S\ Prob0\ \Phi\ \Psi \subseteq N\ \Psi \subseteq N$
assumes *l1*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l1\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l1\ s')$
assumes *l2*: $\bigwedge s.\ s \in S \Longrightarrow s \notin N \Longrightarrow l2\ s - c\ s = (\sum s' \in S.\ \tau\ s\ s' * l2\ s')$
assumes *eq*: $\bigwedge s.\ s \in N \Longrightarrow l1\ s = l2\ s$
shows $\forall s \in S.\ l1\ s = l2\ s$
proof
fix s **assume** $s \in S$
show $l1\ s = l2\ s$
proof *cases*

assume $s \in N$ **then show** *?thesis*
by (*rule eq*)
next
assume $s \notin N$
show *?thesis*
proof (*rule unique-les*[*of* - $S - N$ *K N*])
show *finite* $((\lambda x.\ l1\ x - l2\ x)\ `\ (S - N \cup N))$ $(\bigcup x{\in}S - N.\ E\ x) \subseteq S - N \cup N$
using *E-closed finite-S* ‹$N \subseteq S$› **by** (*auto dest*: *finite-subset*)
show $\bigwedge s.\ s \in N \Longrightarrow l1\ s = l2\ s$ **by** *fact*
{ fix *s* **assume** $s \in S - N$ **with** *E-closed finite-S* **show** *integrable* (*K s*) *l1* *integrable* (*K s*) *l2*
by (*auto intro*!: *integrable-measure-pmf-finite dest*: *finite-subset*)
obtain *t* **where** $(t \in \Psi \wedge (s, t) \in (Sigma\ \Phi\ E)^*) \vee s \in N$
using ‹$s \in S - N$› *in-S*(*4*) **unfolding** *Prob0-iff-reachable*[*OF in-S*(*1*,*2*)] **by** *auto*
moreover have $(Sigma\ \Phi\ E)^* \subseteq acc$
by (*intro rtrancl-mono Sigma-mono*) *auto*
ultimately show $\exists\, t{\in}N.\ (s, t) \in acc$
using ‹$\Psi \subseteq N$› **by** *auto*
show $l1\ s = integral^L\ (K\ s)\ l1 + c\ s$
using *E-closed l1* ‹$s \in S - N$›
by (*subst integral-measure-pmf*[*OF finite-S*]) (*auto simp*: *subset-eq field-simps*)
show $l2\ s = integral^L\ (K\ s)\ l2 + c\ s$
using *E-closed l2* ‹$s \in S - N$›
by (*subst integral-measure-pmf*[*OF finite-S*]) (*auto simp*: *subset-eq field-simps*)
}
qed (*insert* ‹$s \notin N$› ‹$s \in S$›, *auto*)
qed
qed

lemma *uniqueness-of-ProbU*:
assumes *sol*:
$\forall s{\in}S.\ (\sum s'{\in}S.\ LES\ (Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2)\ s\ s' * l\ s') =$
$(if\ s \in svalid\ F2\ then\ 1\ else\ 0)$
shows $\forall s{\in}S.\ l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$
proof (*rule unique*)
show $svalid\ F1 \subseteq S$ $svalid\ F2 \subseteq S$
$Prob0\ (svalid\ F1)\ (svalid\ F2) \subseteq Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2$
$svalid\ F2 \subseteq Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2$
$Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2 \subseteq S$
using *svalid-subset-S* **by** (*auto simp*: *Prob0-def*)
next
fix *s* **assume** *s*: $s \in S$ $s \notin Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2$
have $(\sum s'{\in}S.\ (if\ s' = s\ then\ \tau\ s\ s' - 1\ else\ \tau\ s\ s') * l\ s') =$
$(\sum s'{\in}S.\ \tau\ s\ s' * l\ s' - (if\ s' = s\ then\ 1\ else\ 0) * l\ s')$
by (*auto intro*!: *sum.cong simp*: *field-simps*)
also have $\ldots = (\sum s'{\in}S.\ \tau\ s\ s' * l\ s') - l\ s$
using ‹$s \in S$› **by** (*simp add*: *sum-subtractf single-l*)

finally show $l\ s - 0 = (\sum s'{\in}S.\ \tau\ s\ s' * l\ s')$
using *sol*[*THEN bspec*, *of s*] *s* **by** (*simp add*: *LES-def*)
next
fix *s* **assume** *s*: $s \in S\ s \notin Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2$
then show $\mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega)) - 0 =$
$(\sum t{\in}S.\ \tau\ s\ t * \mathcal{P}(\omega\ in\ T\ t.\ pvalid\ (U^\infty\ F1\ F2)\ (t\ \#\#\ \omega)))$
unfolding *Prob0-iff*[*OF svalid-subset-S svalid-subset-S*]
by (*subst prob-sum*) (*auto simp add*: *suntil-Stream*)
next
fix *s* **assume** $s \in Prob0\ (svalid\ F1)\ (svalid\ F2) \cup svalid\ F2$
then show $l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$
proof
assume *P0*: $s \in Prob0\ (svalid\ F1)\ (svalid\ F2)$
then have $s \in S\ AE\ \omega\ in\ T\ s.\ \neg\ (HLD\ (svalid\ F1)\ suntil\ HLD\ (svalid\ F2))$ $(s\ \#\#\ \omega)$
unfolding *Prob0-iff*[*OF svalid-subset-S svalid-subset-S*] **by** *auto*
then have $\mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega)) = 0$
by (*intro T.prob-eq-0-AE*) *simp*
moreover have $l\ s = 0$
using ‹$s \in S$› *P0 sol*[*THEN bspec*, *of s*] *Prob0-subset-S*
Prob0-imp-not-Psi[*OF svalid-subset-S svalid-subset-S P0*]
by (*auto simp*: *LES-def single-l split*: *if-split-asm*)
ultimately show $l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$ **by** *simp*
next
assume *s*: $s \in svalid\ F2$
moreover with *svalid-subset-S* **have** $s \in S$ **by** *auto*
moreover note *Psi-imp-not-Prob0*[*OF svalid-subset-S svalid-subset-S s*]
ultimately have $l\ s = 1$
using *sol*[*THEN bspec*, *of s*]
by (*auto simp*: *LES-def single-l dest*: *Psi-imp-not-Prob0*[*OF svalid-subset-S svalid-subset-S*])
then show $l\ s = \mathcal{P}(\omega\ in\ T\ s.\ pvalid\ (U^\infty\ F1\ F2)\ (s\ \#\#\ \omega))$
using *s* **by** (*simp add*: *suntil-Stream*)
qed
qed

lemma *infinite-reward*:
fixes *s F*
defines $N \equiv Prob0\ S\ (svalid\ F)$ (**is** - $\equiv Prob0\ S\ ?F$)
defines $Y \equiv Prob1\ N\ S\ (svalid\ F)$
assumes *s*: $s \in S\ s \notin Y$
shows $(\int^+\omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s) = \infty$
proof −
{ assume $(AE\ \omega\ in\ T\ s.\ ev\ (HLD\ ?F)\ \omega)$
with *AE-T-enabled* **have** $(AE\ \omega\ in\ T\ s.\ (HLD\ S\ suntil\ HLD\ ?F)\ \omega)$
proof *eventually-elim*
fix ω **assume** $ev\ (HLD\ ?F)\ \omega\ enabled\ s\ \omega$
from *this* ‹$s \in S$› **show** $(HLD\ S\ suntil\ HLD\ ?F)\ \omega$
proof (*induction arbitrary*: *s*)

case (*step ω*) **show** *?case*
using *E-closed step.IH*[*of shd ω*] *step.prems*
by (*auto simp*: *subset-eq enabled.simps*[*of s*] *suntil.simps*[*of - - ω*] *HLD-iff*)
qed (*auto intro*: *suntil.intros*)
qed }
moreover have ¬ (*AE ω in T s*. (*HLD S suntil HLD ?F*) (*s ## ω*))
using *s svalid-subset-S* **unfolding** *N-def Y-def* **by** (*simp add*: *Prob1-iff*)
ultimately have ∗: ¬ (*AE ω in T s*. *ev* (*HLD ?F*) (*s ## ω*))
using ‹*s* ∈ *S*› **by** (*cases s* ∈ *?F*) (*auto simp add*: *suntil-Stream ev-Stream*)

show *?thesis*
proof (*rule ccontr*)
assume ¬ *?thesis*
from *nn-integral-PInf-AE*[*OF - this*] ‹*s*∈*S*›
have *AE ω in T s*. *ev* (*HLD ?F*) (*s ## ω*)
by (*simp split*: *if-split-asm*)
with ∗ **show** *False* **..**
qed
qed

11.3.9 The expected reward implies a unique LES

lemma *existence-of-ExpFuture*:
fixes *s F*
assumes *N-def*: *N* ≡ *Prob0 S* (*svalid F*) (**is** - ≡ *Prob0 S ?F*)
assumes *Y-def*: *Y* ≡ *Prob1 N S* (*svalid F*)
assumes *s*: *s* ∈ *S s* ∉ *S* − (*Y* − *?F*)
shows *enn2real* ($\int^+$ω. *reward* (*Future F*) (*s ## ω*) ∂*T s*) − (ϱ *s* + ($\sum$ *s'*∈*S*. τ *s s'* ∗ ι *s s'*)) =
($\sum$ *s'*∈*S*. τ *s s'* ∗ *enn2real* ($\int^+$ω. *reward* (*Future F*) (*s' ## ω*) ∂*T s'*))
proof −
let *?R* = *reward* (*Future F*)

from *s* **have** *s* ∈ *Prob1* (*Prob0 S ?F*) *S ?F*
unfolding *Y-def N-def* **by** *auto*
then have *AE-until*: *AE ω in T s*. (*HLD S suntil HLD* (*svalid F*)) (*s ## ω*)
using *Prob1-iff*[*of S ?F*] *svalid-subset-S* **by** *auto*

from *s* **have** *s* ∉ *?F* **by** *auto*

let *?E* = λ*s'*. $\int^+$ ω. *reward* (*Future F*) (*s' ## ω*) ∂*T s'*
have ∗: ($\sum$ *s'*∈*S*. τ *s s'* ∗ *?E s'*) = ($\sum$ *s'*∈*S*. *ennreal* (τ *s s'* ∗ *enn2real* (*?E s'*)))
proof (*rule sum.cong*)
fix *s'* **assume** *s'* ∈ *S*
show τ *s s'* ∗ *?E s'* = *ennreal* (τ *s s'* ∗ *enn2real* (*?E s'*))
proof *cases*
assume τ *s s'* ≠ *0*
with ‹*s* ∈ *S*› ‹*s'* ∈ *S*› **have** *s'* ∈ *E s* **by** (*simp add*: *set-pmf-iff*)
from ‹*s* ∉ *?F*› *AE-until* **have** *AE ω in T s*. (*HLD S suntil HLD ?F*) (*s ##*

ω)
    using *svalid-subset-S* ‹s ∈ S› **by** *simp*
   with *nn-integral-reward-finite*[*OF* ‹s' ∈ S›, *of F*] ‹s ∈ S› ‹s' ∈ E s› ‹s ∉ ?F›
   have ?E s' ≠ ∞
    by (*simp add*: *AE-T-iff*[*of - s*] *AE-measure-pmf-iff suntil-Stream*
       del: *reward.simps*)
   then show *?thesis* **by** (*cases ?E s'*) (*auto simp*: *ennreal-mult*)
  qed *simp*
 qed *simp*

 have *AE* ω *in T s*. ?R (s ## ω) = ϱ s + ι s (*shd* ω) + ?R ω
  using ‹s ∉ *svalid F*› **by** (*auto simp*: *ev-Stream*)
 then have (∫⁺ω. ?R (s ## ω) ∂T s) = (∫⁺ω. (ϱ s + ι s (*shd* ω)) + ?R ω ∂T
s)
  by (*rule nn-integral-cong-AE*)
 also have ... = (∫⁺ω. ϱ s + ι s (*shd* ω) ∂T s) +
  (∫⁺ω. ?R ω ∂T s)
  using ‹s ∈ S›
  by (*subst nn-integral-add*)
   (*auto simp add*: *space-PiM PiE-iff simp del*: *reward.simps*)
 also have ... = *ennreal* (ϱ s + (∑s'∈S. τ s s' ∗ ι s s')) + (∫⁺ω. ?R ω ∂T s)
  using ‹s ∈ S›
  by (*subst nn-integral-eq-sum*)
   (*auto simp*: *field-simps sum.distrib sum-distrib-left*[*symmetric*] *ennreal-mult*[*symmetric*]
sum-nonneg)
 finally show *?thesis*
  apply (*simp del*: *reward.simps*)
  apply (*subst nn-integral-eq-sum*[*OF* ‹s ∈ S› *reward-measurable*])
   apply (*simp del*: *reward.simps ennreal-plus add*: ∗ *ennreal-plus*[*symmetric*]
sum-nonneg)
  done
qed

lemma *uniqueness-of-ExpFuture*:
 fixes *F*
 assumes *N-def*: N ≡ *Prob0* S (*svalid F*) (**is** - ≡ *Prob0* S ?F)
 assumes *Y-def*: Y ≡ *Prob1* N S (*svalid F*)
 assumes *const-def*: *const* ≡ λs. *if* s ∈ Y ∧ s ∉ *svalid F then* − ϱ s − (∑s'∈S.
τ s s' ∗ ι s s') *else 0*
 assumes *sol*: ⋀s. s∈S ⟹ (∑s'∈S. *LES* (S − Y ∪ ?F) s s' ∗ l s') = *const* s
 shows ∀s∈S. l s = *enn2real* (∫⁺ω. *reward* (*Future F*) (s ## ω) ∂T s)
  (**is** ∀s∈S. l s = *enn2real* (∫⁺ω. ?R (s ## ω) ∂T s))
proof (*rule unique*)
 show S ⊆ S ?F ⊆ S **using** *svalid-subset-S* **by** *auto*
 show S − (Y − ?F) ⊆ S *Prob0* S ?F ⊆ S − (Y − ?F) ?F ⊆ S − (Y − ?F)
  using *svalid-subset-S*
  by (*auto simp add*: *Y-def N-def Prob1-iff*)
   (*auto simp add*: *Prob0-iff dest*!: *T.AE-contr*)
next

fix *s* **assume** $s \in S$ $s \notin S - (Y - ?F)$
then show *enn2real* $(\int^+\omega.\ ?R\ (s\ \#\#\ \omega)\ \partial T\ s) - (\varrho\ s + (\sum s'{\in}S.\ \tau\ s\ s' * \iota\ s\ s')) =$
$(\sum s'{\in}S.\ \tau\ s\ s' * enn2real\ (\int^+\omega.\ ?R\ (s'\ \#\#\ \omega)\ \partial T\ s'))$
by (*rule existence-of-ExpFuture*[*OF N-def Y-def*])
next
fix *s* **assume** $s \in S$ $s \notin S - (Y - ?F)$
then have $s \in Y$ $s \notin ?F$ **by** *auto*
have $(\sum s'{\in}S.\ (if\ s' = s\ then\ \tau\ s\ s' - 1\ else\ \tau\ s\ s') * l\ s') =$
$(\sum s'{\in}S.\ \tau\ s\ s' * l\ s' - (if\ s' = s\ then\ 1\ else\ 0) * l\ s')$
by (*auto intro*!: *sum.cong simp*: *field-simps*)
also have ... $= (\sum s'{\in}S.\ \tau\ s\ s' * l\ s') - l\ s$
using ‹$s \in S$› **by** (*simp add*: *sum-subtractf single-l*)
finally have $l\ s = (\sum s'{\in}S.\ \tau\ s\ s' * l\ s') - (\sum s'{\in}S.\ (if\ s' = s\ then\ \tau\ s\ s' - 1\ else\ \tau\ s\ s') * l\ s')$
by (*simp add*: *field-simps*)
then show $l\ s - (\varrho\ s + (\sum s'{\in}S.\ \tau\ s\ s' * \iota\ s\ s')) = (\sum s'{\in}S.\ \tau\ s\ s' * l\ s')$
using *sol*[*OF* ‹$s \in S$›] ‹$s \in Y$› ‹$s \notin ?F$› **by** (*simp add*: *const-def LES-def*)
next
fix *s* **assume** *s*: $s \in S - (Y - ?F)$
with *sol*[*of s*] **have** $l\ s = 0$
by (*cases* $s \in ?F$) (*simp-all add*: *const-def LES-def single-l*)
also have $0 = enn2real\ (\int^+\omega.\ reward\ (Future\ F)\ (s\ \#\#\ \omega)\ \partial T\ s)$
proof *cases*
assume $s \in ?F$ **then show** *?thesis*
by (*simp add*: *HLD-iff ev-Stream*)
next
assume $s \notin ?F$
with *s* **have** $s \in S - Y$ **by** *auto*
with *infinite-reward*[*of s F*] **show** *?thesis*
by (*simp add*: *Y-def N-def del*: *reward.simps*)
qed
finally show $l\ s = enn2real\ (\int^+\omega.\ ?R\ (s\ \#\#\ \omega)\ \partial T\ s)$ **.**
qed

11.4 Soundness of *Sat*

theorem *Sat-sound*:
$Sat\ F \neq None \Longrightarrow Sat\ F = Some\ (svalid\ F)$
proof (*induct F rule*: *Sat.induct*)
case (*5 rel r F*)
{ fix *q* **assume** $q \in S$
with *svalid-subset-S* **have** $sum\ (\tau\ q)\ (svalid\ F) = \mathcal{P}(\omega\ in\ T\ q.\ HLD\ (svalid\ F)\ \omega)$
by (*subst prob-sum*[*OF* ‹$q{\in}S$›]) (*auto intro*!: *sum.mono-neutral-cong-left*) **}**
with *5* **show** *?case*
by (*auto split*: *bind-split-asm*)

next

case (*6 rel r k F1 F2*)
then show *?case*
by (*simp add*: *ProbU cong*: *conj-cong split*: *bind-split-asm*)

next
case (*7 rel r F1 F2*)
moreover
define *constants* :: *'s* ⇒ *real* **where** *constants* = (λ*s*. *if s* ∈ (*svalid F2*) *then 1 else 0*)
moreover define *distr* **where** *distr* = *LES* (*Prob0* (*svalid F1*) (*svalid F2*) ∪ *svalid F2*)
ultimately obtain *l* **where** *eq*: *Sat F1* = *Some* (*svalid F1*) *Sat F2* = *Some* (*svalid F2*)
and *l*: *gauss-jordan′ distr constants* = *Some l*
by *atomize-elim* (*simp add*: *ProbUinfty-def split*: *bind-split-asm*)

from *l* **have** *P*: *ProbUinfty* (*svalid F1*) (*svalid F2*) = *Some l*
unfolding *ProbUinfty-def constants-def distr-def* **by** *simp*

have ∀ *s*∈*S*. *l s* = $\mathcal{P}$(ω *in T s*. *pvalid* (U^∞ *F1 F2*) (*s* ## ω))
proof (*rule uniqueness-of-ProbU*)
show ∀ *s*∈*S*. (∑ *s′*∈*S*. *LES* (*Prob0* (*svalid F1*) (*svalid F2*) ∪ *svalid F2*) *s s′* ∗ *l s′*) =
(*if s* ∈ *svalid F2 then 1 else 0*)
using *gauss-jordan′-correct*[*OF l*]
unfolding *distr-def constants-def* **by** *simp*
qed
then show *?case*
by (*auto simp add*: *eq P*)
next
case (*8 rel r k*)
{ fix *s* **assume** *s* ∈ *S*
then have *ExpCumm s k* = (∫ ⁺ *x*. *ennreal* (∑ *i*<*k*. ϱ ((*s* ## *x*) !! *i*) + ι ((*s* ## *x*) !! *i*) (*x* !! *i*)) ∂*T s*)
proof (*induct k arbitrary*: *s*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc k*)
have (∫ ⁺ω. *ennreal* (∑ *i*<*Suc k*. ϱ ((*s* ## ω) !! *i*) + ι ((*s* ## ω) !! *i*) (ω !! *i*)) ∂*T s*)
= (∫ ⁺ω. *ennreal* (ϱ *s* + ι *s* (ω !! *0*)) + *ennreal* (∑ *i*<*k*. ϱ (ω !! *i*) + ι (ω !! *i*) (ω !! (*Suc i*))) ∂*T s*)
by (*auto intro*!: *nn-integral-cong*
simp del: *ennreal-plus*
simp: *ennreal-plus*[*symmetric*] *sum-nonneg sum.reindex lessThan-Suc-eq-insert-0 zero-notin-Suc-image*)
also have . . . = (∫ ⁺ω. ϱ *s* + ι *s* (ω !! *0*) ∂*T s*) +
(∫ ⁺ω. (∑ *i*<*k*. ϱ (ω !! *i*) + ι (ω !! *i*) (ω !! (*Suc i*))) ∂*T s*)
using ‹*s* ∈ *S*›

by (*intro nn-integral-add AE-I2*) (*auto simp*: *sum-nonneg*)
also have … = ($\sum s'\in S.\ \tau\ s\ s' * (\varrho\ s + \iota\ s\ s')$) +
($\int^+\omega.\ (\sum i<k.\ \varrho\ (\omega\ !!\ i) + \iota\ (\omega\ !!\ i)\ (\omega\ !!\ (Suc\ i)))\ \partial T\ s$)
using ‹$s \in S$› **by** (*subst nn-integral-eq-sum*)
(*auto simp del*: *ennreal-plus simp*: *ennreal-plus*[*symmetric*] *ennreal-mult*[*symmetric*] *sum-nonneg*)
also have … = ($\sum s'\in S.\ \tau\ s\ s' * (\varrho\ s + \iota\ s\ s')$) +
($\sum s'\in S.\ \tau\ s\ s' * ExpCumm\ s'\ k$)
using ‹$s \in S$› **by** (*subst nn-integral-eq-sum*) (*auto simp*: *Suc*)
also have … = *ExpCumm s* (*Suc k*)
using ‹$s \in S$›
by (*simp add*: *field-simps sum.distrib sum-distrib-left*[*symmetric*] *ennreal-mult*[*symmetric*] *ennreal-plus*[*symmetric*] *sum-nonneg del*: *ennreal-plus*)
finally show *?case* **by** *simp*
qed **}**
then show *?case* **by** *auto*

next
case (*9 rel r k*)
{ fix *s* **assume** $s \in S$
then have *ExpState s k* = ($\int^+ x.\ ennreal\ (\varrho\ ((s\ \#\#\ x)\ !!\ k))\ \partial T\ s$)
proof (*induct k arbitrary*: *s*)
case (*Suc k*) **then show** *?case* **by** (*simp add*: *nn-integral-eq-sum*[*of s*])
qed *simp* **}**
then show *?case* **by** *auto*

next
case (*10 rel r F*)
moreover
let *?F* = *svalid F*
define *N* **where** $N \equiv$ *Prob0 S ?F*
moreover define *Y* **where** $Y \equiv$ *Prob1 N S ?F*
moreover define *const* **where** *const* $\equiv$ ($\lambda s.$ *if* $s \in Y \wedge s \notin$ *?F then* $-\ \varrho\ s\ -$ ($\sum s'\in S.\ \tau\ s\ s' * \iota\ s\ s'$) *else 0*)
ultimately obtain *l*
where *l*: *gauss-jordan'* (*LES* ($S - Y \cup$ *?F*)) *const* = *Some l*
and *F*: *Sat F* = *Some ?F*
by (*auto simp*: *ExpFuture-def Let-def split*: *bind-split-asm*)

from *l* **have** *EF*: *ExpFuture ?F* =
Some ($\lambda s.$ *if* $s \in Y$ *then ennreal* (*l s*) *else* ∞)
unfolding *ExpFuture-def N-def Y-def const-def* **by** *auto*

let *?R* = *reward* (*Future F*)
have *l-eq*: $\forall s\in S.\ l\ s = enn2real\ (\int^+\omega.\ ?R\ (s\ \#\#\ \omega)\ \partial T\ s)$
proof (*rule uniqueness-of-ExpFuture*[*OF N-def Y-def const-def*])
fix *s* **assume** $s \in S$
show $\bigwedge s.\ s\in S \Longrightarrow (\sum s'\in S.\ LES\ (S - Y \cup ?F)\ s\ s' * l\ s') = const\ s$
using *gauss-jordan'-correct*[*OF l*] **by** *auto*

qed
{ **fix** *s* **assume** [*simp*]: *s* ∈ *S* *s* ∈ *Y*
then have *s* ∈ *Prob1* (*Prob0 S ?F*) *S ?F*
unfolding *Y-def N-def* **by** *auto*
then have *AE ω in T s*. (*HLD S suntil HLD ?F*) (*s* ## *ω*)
using *svalid-subset-S* **by** (*auto simp add*: *Prob1-iff*)
from *nn-integral-reward-finite*[*OF* ‹*s* ∈ *S*›] *this*
have ($\int^+$*ω*. *reward* (*Future F*) (*s* ## *ω*) ∂*T s*) ≠ ∞
by *simp*
with *l-eq* ‹*s* ∈ *S*› **have** ($\int^+$*ω*. *reward* (*Future F*) (*s* ## *ω*) ∂*T s*) = *ennreal* (*l s*)
by (*auto simp*: *less-top*) }
moreover
{ **fix** *s* **assume** *s* ∈ *S* *s* ∉ *Y*
with *infinite-reward*[*of s F*]
have ($\int^+$*ω*. *reward* (*Future F*) (*s* ## *ω*) ∂*T s*) = ∞
by (*simp add*: *Y-def N-def*) }
ultimately show *?case*
apply (*auto simp add*: *EF F simp del*: *reward.simps*)
apply (*case-tac x* ∈ *Y*)
apply *auto*
done
qed (*auto split*: *bind-split-asm*)

11.5 Completeness of *Sat*

theorem *Sat-complete*:
Sat F ≠ *None*
proof (*induct F rule*: *Sat.induct*)
case (*7 r rel* Φ Ψ)
then have *F*: *Sat* Φ = *Some* (*svalid* Φ) *Sat* Ψ = *Some* (*svalid* Ψ)
by (*auto intro*!: *Sat-sound*)

define *constants* :: *'s* ⇒ *real* **where** *constants* = (*λs*. *if s* ∈ *svalid* Ψ *then 1 else 0*)
define *distr* **where** *distr* = *LES* (*Prob0* (*svalid* Φ) (*svalid* Ψ) ∪ *svalid* Ψ)
have ∃ *l*. *gauss-jordan' distr constants* = *Some l*
proof (*rule gauss-jordan'-complete*[*OF* - *uniqueness-of-ProbU*])
show ∀ *s*∈*S*. (∑ *s'*∈*S*. *distr s s'* ∗ 𝒫(*ω in T s'*. *pvalid* (U^∞ Φ Ψ) (*s'* ## *ω*))) = *constants s*
apply (*simp add*: *distr-def constants-def LES-def del*: *pvalid.simps space-T*)
proof *safe*
fix *s* **assume** *s* ∈ *svalid* Ψ *s* ∈ *S*
then show (∑ *s'*∈*S*. (*if s'* = *s then 1 else 0*) ∗ 𝒫(*ω in T s'*. *pvalid* (U^∞ Φ Ψ) (*s'* ## *ω*))) = *1*
by (*simp add*: *single-l suntil-Stream*)
next
fix *s* **assume** *s*: *s* ∉ *svalid* Ψ *s* ∈ *S*
let *?x* = *λs'*. 𝒫(*ω in T s'*. *pvalid* (U^∞ Φ Ψ) (*s'* ## *ω*))

show ($\sum s'{\in}S$. (*if* $s \in$ *Prob0* (*svalid* Φ) (*svalid* Ψ) *then if* $s' = s$ *then 1 else 0 else if* $s' = s$ *then* τ s s' $-$ *1 else* τ s s') $*$ *?x* s') $=$ *0*
proof *cases*
assume $s \in$ *Prob0* (*svalid* Φ) (*svalid* Ψ)
with *s* **show** *?thesis*
by (*simp add*: *single-l Prob0-iff svalid-subset-S T.prob-eq-0-AE del*: *space-T*)
next
assume *s-not-0*: $s \notin$ *Prob0* (*svalid* Φ) (*svalid* Ψ)
with *s* **have** $*$:$\bigwedge s'$ ω. $s' \in S \Longrightarrow$ *pvalid* (U^∞ Φ Ψ) (s ## s' ## ω) $=$ *pvalid* (U^∞ Φ Ψ) (s' ## ω)
by (*auto simp*: *suntil-Stream Prob0-iff svalid-subset-S*)

have ($\sum s'{\in}S$. (*if* $s' = s$ *then* τ s s' $-$ *1 else* τ s s') $*$ *?x* s') $=$
($\sum s'{\in}S$. τ s s' $*$ *?x* s' $-$ (*if* $s' = s$ *then 1 else 0*) $*$ *?x* s')
by (*auto intro*!: *sum.cong simp*: *field-simps*)
also have ... $=$ ($\sum s'{\in}S$. τ s s' $*$ *?x* s') $-$ *?x* s
using *s* **by** (*simp add*: *single-l sum-subtractf*)
finally show *?thesis*
using $*$ *prob-sum*[*OF* ‹$s \in S$›] *s-not-0* **by** (*simp del*: *pvalid.simps*)
qed
qed
qed (*simp add*: *distr-def constants-def*)
then have *P*: $\exists$ *l*. *ProbUinfty* (*svalid* Φ) (*svalid* Ψ) $=$ *Some l*
unfolding *ProbUinfty-def constants-def distr-def* **by** *simp*
with *F* **show** *?case*
by *auto*
next
case (*10 rel r* Φ)
then have *F*: *Sat* Φ $=$ *Some* (*svalid* Φ)
by (*auto intro*!: *Sat-sound*)

let *?F* $=$ *svalid* Φ
define *N* **where** $N \equiv$ *Prob0 S ?F*
define *Y* **where** $Y \equiv$ *Prob1 N S ?F*
define *const* **where** *const* $\equiv$ (λs. *if* $s \in Y \wedge s \notin$ *?F then* $-$ ϱ s $-$ ($\sum s'{\in}S$. τ s s' $*$ ι s s') *else 0*)
let *?E* $=$ $\lambda s'$. $\int^+ \omega$. *reward* (*Future* Φ) (s' ## ω) ∂T s'
have $\exists$ *l*. *gauss-jordan'* (*LES* (S $-$ $Y \cup$ *?F*)) *const* $=$ *Some l*
proof (*rule gauss-jordan'-complete*[*OF - uniqueness-of-ExpFuture*[*OF N-def Y-def const-def*]])
show $\forall s{\in}S$. ($\sum s'{\in}S$. *LES* (S $-$ $Y \cup$ *svalid* Φ) s s' $*$ *enn2real* (*?E* s')) $=$ *const s*
proof
fix *s* **assume** $s \in S$
show ($\sum s'{\in}S$. *LES* (S $-$ $Y \cup$ *svalid* Φ) s s' $*$ *enn2real* (*?E* s')) $=$ *const s*
proof *cases*
assume *s*: $s \in S - (Y - \textit{svalid}\ \Phi)$
show *?thesis*
proof *cases*

assume $s \in Y$
with ‹$s \in S$› s ‹$s \in Y$› **show** *?thesis*
by (*simp add*: *LES-def const-def single-l ev-Stream*)
next
assume $s \notin Y$
with *infinite-reward*[*of s* Φ] *Y-def N-def s* ‹$s \in S$›
show *?thesis* **by** (*simp add*: *const-def LES-def single-l del*: *reward.simps*)
qed
next
assume *s*: $s \notin S - (Y - svalid\ \Phi)$

have $(\sum s' {\in} S.\ (if\ s' = s\ then\ \tau\ s\ s' - 1\ else\ \tau\ s\ s') * enn2real\ (?E\ s')) =$
$(\sum s' {\in} S.\ \tau\ s\ s' * enn2real\ (?E\ s') - (if\ s' = s\ then\ 1\ else\ 0) * enn2real$
$(?E\ s'))$
by (*auto intro*!: *sum.cong simp*: *field-simps*)
also have $\ldots = (\sum s' {\in} S.\ \tau\ s\ s' * enn2real\ (?E\ s')) - enn2real\ (?E\ s)$
using ‹$s \in S$› **by** (*simp add*: *sum-subtractf single-l*)
finally show *?thesis*
using *s* ‹$s \in S$› *existence-of-ExpFuture*[*OF N-def Y-def* ‹$s \in S$› *s*]
by (*simp add*: *LES-def const-def del*: *reward.simps*)
qed
qed
qed *simp*
then have *P*: $\exists l.\ ExpFuture\ (svalid\ \Phi) = Some\ l$
unfolding *ExpFuture-def const-def N-def Y-def* **by** *auto*
with *F* **show** *?case*
by *auto*
qed (*force split*: *bind-split*)+

11.6 Completeness and Soundness *Sat*

corollary *Sat*: *Sat* Φ = *Some* (*svalid* Φ)
using *Sat-sound Sat-complete* **by** *auto*

end

end

12 Probabilistic Guarded Command Language (pGCL)

theory *PGCL*
imports *../Markov-Decision-Process*
begin

12.1 Syntax

datatype *′s pgcl* =
Skip
| *Abort*

| *Assign 's ⇒ 's*
| *Seq 's pgcl 's pgcl*
| *Par 's pgcl 's pgcl*
| *If 's ⇒ bool 's pgcl 's pgcl*
| *Prob bool pmf 's pgcl 's pgcl*
| *While 's ⇒ bool 's pgcl*

12.2 Denotational Semantics

primrec *wp* :: *'s pgcl* ⇒ (*'s* ⇒ *ennreal*) ⇒ (*'s* ⇒ *ennreal*) **where**
wp Skip f = *f*
| *wp Abort f* = (λ-. *0*)
| *wp* (*Assign u*) *f* = *f* ∘ *u*
| *wp* (*Seq* c_1 c_2) *f* = *wp* c_1 (*wp* c_2 *f*)
| *wp* (*If b* c_1 c_2) *f* = (λ*s*. *if b s then wp* c_1 *f s else wp* c_2 *f s*)
| *wp* (*Par* c_1 c_2) *f* = *wp* c_1 *f* ⊓ *wp* c_2 *f*
| *wp* (*Prob p* c_1 c_2) *f* = (λ*s*. *pmf p True* ∗ *wp* c_1 *f s* + *pmf p False* ∗ *wp* c_2 *f s*)
| *wp* (*While b c*) *f* = *lfp* (λ*X s*. *if b s then wp c X s else f s*)

lemma *wp-mono*: *mono* (*wp c*)
by (*induction c*)
(*auto simp*: *monotone-def le-fun-def intro*: *order-trans le-infI1 le-infI2*
intro!: *add-mono mult-left-mono lfp-mono*[*THEN le-funD*])

abbreviation *det* :: *'s pgcl* ⇒ *'s* ⇒ (*'s pgcl* × *'s*) *pmf set* (‹≪ -, - ≫›) **where**
det c s ≡ {*return-pmf* (*c*, *s*)}

12.3 Operational Semantics

fun *step* :: (*'s pgcl* × *'s*) ⇒ (*'s pgcl* × *'s*) *pmf set* **where**
step (*Skip*, *s*) = ≪*Skip*, *s*≫
| *step* (*Abort*, *s*) = ≪*Abort*, *s*≫
| *step* (*Assign u*, *s*) = ≪*Skip*, *u s*≫
| *step* (*Seq* c_1 c_2, *s*) = (*map-pmf* (λ(*p1'*, *s'*). (*if p1'* = *Skip then* c_2 *else Seq p1'* c_2, *s'*))) ‘ *step* (c_1, *s*)
| *step* (*If b* c_1 c_2, *s*) = (*if b s then* ≪c_1, *s*≫ *else* ≪c_2, *s*≫)
| *step* (*Par* c_1 c_2, *s*) = ≪c_1, *s*≫ ∪ ≪c_2, *s*≫
| *step* (*Prob p* c_1 c_2, *s*) = {*map-pmf* (λ*b*. *if b then* (c_1, *s*) *else* (c_2, *s*)) *p*}
| *step* (*While b c*, *s*) = (*if b s then* ≪*Seq c* (*While b c*), *s*≫ *else* ≪*Skip*, *s*≫)

lemma *step-finite*: *finite* (*step x*)
by (*induction x rule*: *step.induct*) *simp-all*

lemma *step-non-empty*: *step x* ≠ {}
by (*induction x rule*: *step.induct*) *simp-all*

interpretation *step*: *Markov-Decision-Process step*
proof qed (*rule step-non-empty*)

definition *rF* :: ('*s* ⇒ *ennreal*) ⇒ (('*s pgcl* × '*s*) *stream* ⇒ *ennreal*) ⇒ ('*s pgcl* × '*s*) *stream* ⇒ *ennreal* **where**
rF f F ω = (*if fst* (*shd ω*) = *Skip then f* (*snd* (*shd ω*)) *else F* (*stl ω*))

abbreviation *r* :: ('*s* ⇒ *ennreal*) ⇒ ('*s pgcl* × '*s*) *stream* ⇒ *ennreal* **where**
r f ≡ *lfp* (*rF f*)

lemma *continuous-rF*: *sup-continuous* (*rF f*)
unfolding *rF-def*[*abs-def*]
by (*auto simp*: *sup-continuous-def fun-eq-iff SUP-sup-distrib* [*symmetric*] *image-comp*
split: *prod.splits pgcl.splits*)

lemma *mono-rF*: *mono* (*rF f*)
using *continuous-rF* **by** (*rule sup-continuous-mono*)

lemma *r-unfold*: *r f ω* = (*if fst* (*shd ω*) = *Skip then f* (*snd* (*shd ω*)) *else r f* (*stl ω*))
by (*subst lfp-unfold*[*OF mono-rF*]) (*simp add*: *rF-def*)

lemma *mono-r*: *F* ≤ *G* ⟹ *r F ω* ≤ *r G ω*
by (*rule le-funD*[*of - - ω*], *rule lfp-mono*)
(*auto intro*!: *lfp-mono simp*: *rF-def le-fun-def max.coboundedI2*)

lemma *measurable-rF*:
assumes *F*[*measurable*]: *F* ∈ *borel-measurable step.St*
shows *rF f F* ∈ *borel-measurable step.St*
unfolding *rF-def*[*abs-def*]
apply *measurable*
apply (*rule measurable-compose*[*OF measurable-shd*])
apply *measurable* []
apply (*rule measurable-compose*[*OF measurable-stl*])
apply *measurable* []
apply (*rule predE*)
apply (*rule measurable-compose*[*OF measurable-shd*])
apply *measurable*
done

lemma *measurable-r*[*measurable*]: *r f* ∈ *borel-measurable step.St*
using *continuous-rF measurable-rF* **by** (*rule borel-measurable-lfp*)

lemma *mono-r'*: *mono* (*λF s.* ⨅*D*∈*step s.* ∫⁺ *t.* (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*)
by (*auto intro*!: *monoI le-funI INF-mono*[*OF bexI*] *nn-integral-mono simp*: *le-fun-def*)

lemma *E-inf-r*:
step.E-inf s (*r f*) =
lfp (*λF s.* ⨅*D*∈*step s.* ∫⁺ *t.* (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*) *s*

proof –
  have *step.E-inf s* (*r f*) =
   lfp (λ*F s*. ⨅ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*) *s*
    unfolding *rF-def*[*abs-def*]
  proof (*rule step.E-inf-lfp*[*THEN fun-cong*])
    let *?F* = λ*t x*. (*if fst t* = *Skip then f* (*snd t*) *else x*)
    show (λ(*s*, *x*). *?F s x*) ∈ *borel-measurable* (*count-space UNIV* $\bigotimes_M$ *borel*)
      apply (*simp add*: *measurable-split-conv split-beta'*)
      apply (*intro borel-measurable-max borel-measurable-const measurable-If predE measurable-compose*[*OF measurable-snd*] *measurable-compose*[*OF measurable-fst*])
      apply *measurable*
      done
    show ⋀*s*. *sup-continuous* (*?F s*)
      by (*auto simp*: *sup-continuous-def SUP-sup-distrib*[*symmetric*] *split*: *prod.split pgcl.split*)
    show ⋀*F cfg*. (∫⁺ω. *?F* (*state cfg*) (*F* ω) ∂*step.T cfg*) =
      ?F (*state cfg*) (*nn-integral* (*step.T cfg*) *F*)
      by (*auto simp*: *split*: *pgcl.split prod.split*)
  qed (*rule step-finite*)
  then show *?thesis*
    by *simp*
qed

lemma *E-inf-r-unfold*:
  step.E-inf s (*r f*) = (⨅ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else step.E-inf t* (*r f*)) ∂*measure-pmf D*)
  unfolding *E-inf-r* **by** (*simp add*: *lfp-unfold*[*OF mono-r'*])

lemma *E-inf-r-induct*[*consumes 1*, *case-names step*]:
  assumes *P s y*
  assumes ∗: ⋀*F s y*. *P s y* ⟹
    (⋀*s y*. *P s y* ⟹ *F s* ≤ *y*) ⟹ (⋀*s*. *F s* ≤ *step.E-inf s* (*r f*)) ⟹
    (⨅ *D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then f* (*snd t*) *else F t*) ∂*measure-pmf D*) ≤ *y*
  shows *step.E-inf s* (*r f*) ≤ *y*
  using ‹*P s y*›
  unfolding *E-inf-r*
proof (*induction arbitrary*: *s y rule*: *lfp-ordinal-induct*[*OF mono-r'*[**where** *f*=*f*]])
  case (*1 F*) **with** ∗[*of s y F*] **show** *?case*
    unfolding *le-fun-def E-inf-r*[**where** *f*=*f*, *symmetric*] **by** *simp*
qed (*auto intro*: *SUP-least*)

lemma *E-inf-Skip*: *step.E-inf* (*Skip*, *s*) (*r f*) = *f s*
  by (*subst E-inf-r-unfold*) *simp*

lemma *E-inf-Seq*:
  assumes [*simp*]: ⋀*x*. *0* ≤ *f x*

shows *step.E-inf* (*Seq a b*, *s*) (*r f*) = *step.E-inf* (*a*, *s*) (*r* (λ*s*. *step.E-inf* (*b*, *s*) (*r f*)))
proof (*rule antisym*)
show *step.E-inf* (*Seq a b*, *s*) (*r f*) ≤ *step.E-inf* (*a*, *s*) (*r* (λ*s*. *step.E-inf* (*b*, *s*) (*r f*)))
proof (*coinduction arbitrary*: *a s rule*: *E-inf-r-induct*)
case *step* **then show** *?case*
by (*rewrite* **in** - ≤ ⌑ *E-inf-r-unfold*)
(*force intro*!: *INF-mono*[*OF bexI*] *nn-integral-mono intro*: *le-infI2*
simp: *E-inf-Skip image-comp*)
qed
show *step.E-inf* (*a*, *s*) (*r* (λ*s*. *step.E-inf* (*b*, *s*) (*r f*))) ≤ *step.E-inf* (*Seq a b*, *s*) (*r f*)
proof (*coinduction arbitrary*: *a s rule*: *E-inf-r-induct*)
case *step* **then show** *?case*
by (*rewrite* **in** - ≤ ⌑ *E-inf-r-unfold*)
(*force intro*!: *INF-mono*[*OF bexI*] *nn-integral-mono intro*: *le-infI2*
simp: *E-inf-Skip image-comp*)
qed
qed

lemma *E-inf-While*:
step.E-inf (*While g c*, *s*) (*r f*) =
lfp (λ*F s*. *if g s then step.E-inf* (*c*, *s*) (*r F*) *else f s*) *s*
proof (*rule antisym*)
have *E-inf-While-step*: *step.E-inf* (*While g c*, *s*) (*r f*) =
(*if g s then step.E-inf* (*c*, *s*) (*r* (λ*s*. *step.E-inf* (*While g c*, *s*) (*r f*))) *else f s*)
for *f s*
by (*rewrite E-inf-r-unfold*) (*simp add*: *min-absorb1 E-inf-Seq*)

have *mono* (λ*F s*. *if g s then step.E-inf* (*c*, *s*) (*r F*) *else f s*) (**is** *mono ?F*)
by (*auto intro*!: *mono-r step.E-inf-mono simp*: *mono-def le-fun-def max.coboundedI2*)
then show *lfp ?F s* ≤ *step.E-inf* (*While g c*, *s*) (*r f*)
proof (*induction arbitrary*: *s rule*: *lfp-ordinal-induct*[*consumes 1*])
case *mono* **then show** *?case*
by (*rewrite E-inf-While-step*) (*auto intro*!: *step.E-inf-mono mono-r le-funI*)
qed (*auto intro*: *SUP-least*)

define *w* **where** *w F s* = (⨅*D*∈*step s*. ∫⁺ *t*. (*if fst t* = *Skip then if g* (*snd t*) *then F* (*c*, *snd t*) *else f* (*snd t*) *else F t*) ∂*measure-pmf D*)
for *F s*
have *mono w*
by (*auto simp*: *w-def mono-def le-fun-def intro*!: *INF-mono*[*OF bexI*] *nn-integral-mono*)
[]

define *d* **where** *d* = *c*
define *t* **where** *t* = *Seq d* (*While g c*)
then have (*t* = *While g c* ∧ *d* = *c* ∧ *g s*) ∨ *t* = *Seq d* (*While g c*)
by *auto*

then have *step.E-inf* (t, s) $(r\ f) \leq$ *lfp* w (d, s)
proof (*coinduction arbitrary*: t d s *rule*: *E-inf-r-induct*)
case (*step F t d s*)
from *step*(*1*)
show *?case*
proof (*elim conjE disjE*)
{ **fix** s **have** $\neg$ g $s \Longrightarrow F$ (*While g c*, s) $\leq f\ s$
using *step*(*3*)[*of* (*While g c*, *s*)] **by** (*simp add*: *E-inf-While-step*) }
note [*simp*] = *this*
assume t = *Seq d* (*While g c*) **then show** *?thesis*
by (*rewrite lfp-unfold*[*OF* ‹*mono w*›])
(*auto simp*: *max.absorb2 w-def intro*!: *INF-mono*[*OF bexI*] *nn-integral-mono*
step)
qed (*auto intro*!: *step*)
qed
also have *lfp* w = *lfp* (λF s. *step.E-inf* s (r (λs. *if* g s *then* F (c, s) *else* $f\ s$)))
unfolding *E-inf-r w-def*
by (*rule lfp-lfp*[*symmetric*]) (*auto simp*: *le-fun-def intro*!: *INF-mono*[*OF bexI*]
nn-integral-mono)
finally have *step.E-inf* (*While g c*, s) $(r\ f) \leq$ (*if* g s *then* $\ldots$ (c, s) *else* $f\ s$)
unfolding *t-def d-def* **by** (*rewrite E-inf-r-unfold*) *simp*
also have $\ldots$ = *lfp ?F s*
by (*rewrite lfp-rolling*[*symmetric*, *of* λF s. *if* g s *then* F (c, s) *else* $f\ s$ λF s.
step.E-inf s (r F)])
(*auto simp*: *mono-def le-fun-def sup-apply*[*abs-def*] *if-distrib*[*of max 0*] *max.coboundedI2*
max.absorb2
intro!: *step.E-inf-mono mono-r cong del*: *if-weak-cong*)
finally show *step.E-inf* (*While g c*, s) $(r\ f) \leq \ldots$
.
qed

12.4 Equate Both Semantics

lemma *E-inf-r-eq-wp*: *step.E-inf* (c, s) $(r\ f)$ = *wp* c f s
proof (*induction c arbitrary*: *f s*)
case *Skip* **then show** *?case*
by (*simp add*: *E-inf-Skip*)
next
case *Abort* **then show** *?case*
proof (*intro antisym*)
have *lfp* (λF s. $\prod D{\in}$*step* s. $\int^+$ t. (*if fst* t = *Skip then* f (*snd* t) *else* F t)
∂*measure-pmf* D) $\leq$
(λs. *if* $\exists\, t$. s = (*Abort*, t) *then 0 else* $\top$)
by (*intro lfp-lowerbound*) (*auto simp*: *le-fun-def*)
then show *step.E-inf* (*Abort*, s) $(r\ f) \leq$ *wp Abort f s*
by (*auto simp*: *E-inf-r le-fun-def split*: *if-split-asm*)
qed *simp*
next
case *Assign* **then show** *?case*

```
    by (rewrite E-inf-r-unfold) (simp add: min-absorb1)
next
  case (If b c1 c2) then show ?case
    by (rewrite E-inf-r-unfold) auto
next
  case (Prob p c1 c2) then show ?case
    apply (rewrite E-inf-r-unfold)
    apply auto
    apply (rewrite nn-integral-measure-pmf-support[of UNIV::bool set])
    apply (auto simp: UNIV-bool ac-simps)
    done
next
  case (Par c1 c2) then show ?case
    by (rewrite E-inf-r-unfold) (auto intro: inf.commute)
next
  case (Seq c1 c2) then show ?case
    by (simp add: E-inf-Seq)
next
  case (While g c) then show ?case
    apply (simp add: E-inf-While)
    apply (rewrite While)
    apply auto
    done
qed

end
```

13 Formalization of the Crowds-Protocol

```
theory Crowds-Protocol
  imports ../Discrete-Time-Markov-Chain
begin

lemma cond-prob-nonneg[simp]: 0 ≤ cond-prob M A B
  by (auto simp: cond-prob-def)

lemma (in MC-syntax) emeasure-suntil-geometric:
  assumes [measurable]: Measurable.pred S P
  assumes s ∈ X and *[simp]: 0 ≤ p 0 ≤ r
  assumes r: ⋀s. s ∈ X ⟹ emeasure (T s) {ω∈space (T s). P ω} = ennreal r
  assumes p: ⋀s. s ∈ X ⟹ emeasure (K s) X = ennreal p p < 1
  assumes ⋀t. AE ω in T t. ¬ (P ⊓ (HLD X ⊓ nxt (HLD X suntil P))) ω
  shows emeasure (T s) {ω∈space (T s). (HLD X suntil P) ω} = r / (1 − p)
proof (subst emeasure-suntil-disj)
  let ?F = λF s. emeasure (T s) {ω ∈ space (T s). P ω} + ∫⁺ t. F t * indicator
X t ∂K s
  let ?f = λx. ennreal r + ennreal p * x

  have mono ?F mono ?f
```

by (*auto intro*!: *monoI max.mono add-mono nn-integral-mono mult-left-mono mult-right-mono simp*: *le-fun-def*)

have *1*: *lfp ?f* ≤ *lfp ?F s*
using ‹*s* ∈ *X*›
proof (*induction arbitrary*: *s rule*: *lfp-ordinal-induct*[*OF* ‹*mono ?f*›])
case *step*: (*1 x*)
then have *?f x* ≤ *?F* (λ-. *x*) *s*
by (*auto simp*: *p r*[*simplified*] *nn-integral-cmult mult.commute*[*of - x*]
intro!: *add-mono mult-right-mono*)
also have *?F* (λ-. *x*) ≤ *?F* (*lfp ?F*)
using *step*
by (*intro le-funI add-mono order-refl nn-integral-mono*) (*auto simp*: *split*: *split-indicator*)
finally show *?case*
by (*subst lfp-unfold*[*OF* ‹*mono ?F*›]) (*auto simp*: *le-fun-def*)
qed (*auto intro*!: *Sup-least*)
also have *2*: *lfp ?F s* ≤ *r* / (*1* − *p*)
using ‹*s* ∈ *X*›
proof (*induction arbitrary*: *s rule*: *lfp-ordinal-induct*[*OF* ‹*mono ?F*›])
case (*1 S*)
with *r* **have** *?F S s* ≤ *ennreal r* + (∫⁺*x*. *ennreal* (*r* / (*1* − *p*)) ∗ *indicator X x* ∂*K s*)
by (*intro add-mono nn-integral-mono*) (*auto split*: *split-indicator*)
also have ... ≤ *ennreal r* + *ennreal* (*r* ∗ *p* / (*1* − *p*))
using ‹*s* ∈ *X*› **by** (*simp add*: *nn-integral-cmult-indicator p ennreal-mult''*[*symmetric*])
also have ... = *ennreal* (*r* / (*1* − *p*))
using ‹*p* < *1*› **by** (*simp add*: *field-simps ennreal-plus*[*symmetric*] *del*: *ennreal-plus*)
finally show *?case* **.**
qed (*auto intro*!: *SUP-least*)
finally obtain *x* **where** *x*: *lfp ?f* = *ennreal x* **and** [*simp*]: *0* ≤ *x*
by (*cases lfp ?f*) (*auto simp*: *top-unique*)
from ‹*p* < *1*› **have** ⋀*x*. *x* = *r* + *p* ∗ *x* ⟹ *x* = *r* / (*1* − *p*)
by (*auto simp*: *field-simps*)
with *lfp-unfold*[*OF* ‹*mono ?f*›] ‹*p* < *1*› **have** *lfp ?f* = *r* / (*1* − *p*)
unfolding *x* **by** (*auto simp add*: *ennreal-plus*[*symmetric*] *ennreal-mult*[*symmetric*] *simp del*: *ennreal-plus*)
with *1 2* **show** *lfp ?F s* = *ennreal* (*r* / (*1* − *p*))
by *auto*
qed *fact+*

13.1 Definition of the Crowds-Protocol

datatype *'a state* = *Start* | *Init 'a* | *Mix 'a* | *End*

lemma *inj-Mix*[*simp*]: *inj-on Mix A*
by (*auto intro*: *inj-onI*)

lemma *inj-Init*[*simp*]: *inj-on Init A*
by (*auto intro*: *inj-onI*)

lemma *distinct-state-image*[*simp*]:
Start ∉ *Mix* ‘ *A Init j* ∉ *Mix* ‘ *A End* ∉ *Mix* ‘ *A Mix j* ∈ *Mix* ‘ *A* ⟷ *j* ∈ *A*
Start ∉ *Init* ‘ *A Mix j* ∉ *Init* ‘ *A End* ∉ *Init* ‘ *A Init j* ∈ *Init* ‘ *A* ⟷ *j* ∈ *A*
by *auto*

lemma *Init-cut-Mix*[*simp*]:
Init ‘ *H* ∩ *Mix* ‘ *J* = {}
by *auto*

abbreviation *Jondo B* ≡ *Init*‘*B* ∪ *Mix*‘*B*

locale *Crowds-Protocol* =
fixes *J* :: ′*a set* **and** *C* :: ′*a set* **and** *p-f* :: *real* **and** *p-i* :: ′*a* ⇒ *real*
assumes *J-not-empty*: *J* ≠ {} **and** *finite-J*[*simp*]: *finite J*
assumes *C-smaller*: *C* ⊂ *J* **and** *C-non-empty*: *C* ≠ {}
assumes *p-f*: *0* < *p-f p-f* < *1*
assumes *p-i-nonneg*[*simp*]: ⋀*j*. *j* ∈ *J* ⟹ *0* ≤ *p-i j*
assumes *p-i-distr*: (∑*j*∈*J*. *p-i j*) = *1*
assumes *p-i-C*: ⋀*j*. *j* ∈ *C* ⟹ *p-i j* = *0*
begin

abbreviation *H* :: ′*a set* **where**
H ≡ *J* − *C*

definition *p-j* = *1* / *card J*

lemma *p-f-nonneg*[*simp*]: *0* ≤ *p-f p-f* ≤ *1*
using *p-f* **by** *simp-all*

lemma *p-j-nonneg*[*simp*]: *0* ≤ *p-j*
by (*simp add*: *p-j-def*)

definition *p-H* = *card H* / *card J*

lemma *p-H-nonneg*[*simp*]: *0* ≤ *p-H p-H* ≤ *1*
by (*auto simp*: *p-H-def divide-le-eq-1 card-gt-0-iff intro*!: *card-mono*)

definition *next-prob* :: ′*a state* ⇒ ′*a state* ⇒ *real* **where**
next-prob s t = (*case* (*s*, *t*) *of* (*Start*, *Init j*) ⇒ *if j* ∈ *H then p-i j else 0*
| (*Init j*, *Mix j*′) ⇒ *if j*′ ∈ *J then p-j else 0*
| (*Mix j*, *Mix j*′) ⇒ *if j*′ ∈ *J then p-f* ∗ *p-j else 0*
| (*Mix j*, *End*) ⇒ *1* − *p-f*
| (*End*, *End*) ⇒ *1*
| - ⇒ *0*)

definition *N s* = *embed-pmf* (*next-prob s*)

interpretation *MC-syntax N* **.**

abbreviation $\mathfrak{P} \equiv$ *T Start*

abbreviation *E s* $\equiv$ *set-pmf* (*N s*)

lemma *finite-C*[*simp*]: *finite C*
  using *C-smaller finite-J* **by** (*blast intro*: *finite-subset*)

lemma *sum-p-i-C*[*simp*]: *sum p-i C* = *0*
  by (*auto intro*: *sum.neutral p-i-C*)

lemma *sum-p-i-H*[*simp*]: *sum p-i H* = *1*
  using *C-smaller* **by** (*simp add*: *sum-diff p-i-distr*)

lemma *possible-jondo*:
  obtains *j* **where** $j \in J$ $j \notin C$ *p-i j* $\neq$ *0*
proof (*atomize-elim*, *rule ccontr*)
  assume $\neg$ ($\exists$ *j*. $j \in J \land j \notin C \land$ *p-i j* $\neq$ *0*)
  with *p-i-C* **have** $\forall j \in J$. *p-i j* = *0*
    by *auto*
  with *p-i-distr* **show** *False*
    by *simp*
qed

lemma *C-le-J*[*simp*]: *card C* < *card J*
  using *C-smaller*
  by (*intro psubset-card-mono*) *auto*

lemma *p-H*: *0* < *p-H p-H* < *1*
  using *J-not-empty C-smaller C-non-empty*
  by (*simp-all add*: *p-H-def card-Diff-subset card-mono field-simps zero-less-divide-iff card-gt-0-iff*)

lemma *p-H-p-f-pos*: *0* < *p-H* ∗ *p-f*
  using *p-f p-H* **by** (*simp add*: *zero-less-mult-iff*)

lemma *p-H-p-f-less-1*: *p-H* ∗ *p-f* < *1*
proof −
  have *p-H* ∗ *p-f* < *1* ∗ *1*
    using *p-H p-f* **by** (*intro mult-strict-mono*) *auto*
  then show *p-H* ∗ *p-f* < *1* **by** *simp*
qed

lemma *p-j-pos*: *0* < *p-j*
  unfolding *p-j-def* **using** *J-not-empty* **by** *auto*

lemma *H-compl*: *1* − *p-H* = *real* (*card C*) / *real* (*card J*)

using *C-non-empty J-not-empty C-smaller*
by (*simp add*: *p-H-def card-Diff-subset card-mono of-nat-diff divide-eq-eq field-simps*)

lemma *H-compl2*: *1* − *p-H* = *card C* ∗ *p-j*
unfolding *H-compl p-j-def* **by** *simp*

lemma *H-eq2*: *card H* ∗ *p-j* = *p-H*
unfolding *p-j-def p-H-def* **by** *simp*

lemma *pmf-next-pmf*[*simp*]: *pmf* (*N s*) *t* = *next-prob s t*
unfolding *N-def*
proof (*rule pmf-embed-pmf*)
show ⋀*x*. *0* ≤ *next-prob s x*
using *p-j-pos p-f* **by** (*auto simp*: *next-prob-def intro*: *p-i-nonneg split*: *state.split*)
show (∫⁺ *x*. *ennreal* (*next-prob s x*) ∂*count-space UNIV*) = *1*
using *p-f J-not-empty*
by (*subst nn-integral-count-space′*[**where** *A*=*Init*‘*H* ∪ *Mix*‘*J* ∪ {*End*}])
(*auto simp*: *next-prob-def sum.reindex sum.union-disjoint p-i-distr p-j-def*
split: *state.split*)
qed

lemma *next-prob-Start*[*simp*]: *next-prob Start* (*Init j*) = (*if j* ∈ *H then p-i j else*
0)
by (*auto simp*: *next-prob-def*)

lemma *next-prob-to-Init*[*simp*]: *j* ∈ *H* ⟹ *next-prob s* (*Init j*) =
(*case s of Start* ⇒ *p-i j* | *-* ⇒ *0*)
by (*cases s*) (*auto simp*: *next-prob-def*)

lemma *next-prob-to-Mix*[*simp*]: *j* ∈ *J* ⟹ *next-prob s* (*Mix j*) =
(*case s of Init j* ⇒ *p-j* | *Mix j* ⇒ *p-f* ∗ *p-j* | *-* ⇒ *0*)
by (*cases s*) (*auto simp*: *next-prob-def*)

lemma *next-prob-to-End*[*simp*]: *next-prob s End* =
(*case s of Mix j* ⇒ *1* − *p-f* | *End* ⇒ *1* | *-* ⇒ *0*)
by (*cases s*) (*auto simp*: *next-prob-def*)

lemma *next-prob-from-End*[*simp*]: *next-prob End s* = *0* ⟷ *s* ≠ *End*
by (*cases s*) (*auto simp*: *next-prob-def*)

lemma *next-prob-Mix-MixI*: ∃ *j*. *s* = *Mix j* ⟹ ∃ *j*∈*J*. *s′* = *Mix j* ⟹ *next-prob s*
s′ = *p-f* ∗ *p-j*
by (*cases s*) *auto*

lemma *E-Start*: *E Start* = {*Init j* | *j*. *j* ∈ *H* ∧ *p-i j* ≠ *0* }
using *p-i-C* **by** (*auto simp*: *set-pmf-iff next-prob-def split*: *state.splits if-split-asm*)

lemma *E-Init*: *E* (*Init j*) = {*Mix j* | *j*. *j* ∈ *J* }

using *p-j-pos C-smaller* **by** (*auto simp*: *set-pmf-iff next-prob-def split*: *state.splits if-split-asm*)

lemma *E-Mix*: *E* (*Mix j*) = {*Mix j* | *j*. *j* ∈ *J* } ∪ {*End*}
using *p-j-pos p-f* **by** (*auto simp*: *set-pmf-iff next-prob-def split*: *state.splits if-split-asm*)

lemma *E-End*: *E End* = {*End*}
by (*auto simp*: *set-pmf-iff next-prob-def split*: *state.splits if-split-asm*)

lemma *enabled-End*:
enabled End ω ⟷ ω = *sconst End*
proof *safe*
assume *enabled End* ω **then show** ω = *sconst End*
proof (*coinduction arbitrary*: ω)
case *Eq-stream* **then show** *?case*
by (*auto simp*: *enabled.simps*[*of* - ω] *E-End*)
qed
next
show *enabled End* (*sconst End*)
by *coinduction* (*simp add*: *E-End*)
qed

lemma *AE-End*: (*AE* ω *in T End*. *P* ω) ⟷ *P* (*sconst End*)
proof −
have (*AE* ω *in T End*. *P* ω) ⟷ (*AE* ω *in T End*. *P* ω ∧ ω = *sconst End*)
using *AE-T-enabled*[*of End*] **by** (*simp add*: *enabled-End*)
also have ... = (*AE* ω *in T End*. *P* (*sconst End*) ∧ ω = *sconst End*)
by (*simp add*: *enabled-End del*: *AE-conj-iff cong*: *rev-conj-cong*)
also have ... = (*AE* ω *in T End*. *P* (*sconst End*))
using *AE-T-enabled*[*of End*] **by** (*simp add*: *enabled-End*)
finally show *?thesis*
by *simp*
qed

lemma *emeasure-Init-eq-Mix*:
assumes [*measurable*]: *Measurable.pred S P*
assumes *AE-End*: *AE x in T End*. ¬ *P* (*End* ## *x*)
shows *emeasure* (*T* (*Init j*)) {*x*∈*space* (*T* (*Init j*)). *P x*} =
emeasure (*T* (*Mix j*)) {*x*∈*space* (*T* (*Mix j*)). *P x*} / *p-f*
proof −
have ∗: {*Mix j* | *j*. *j* ∈ *J* } = *Mix* ‘ *J*
by *auto*
show *?thesis*
using *emeasure-eq-0-AE*[*OF AE-End*] *p-f*
apply (*subst* (*1 2*) *emeasure-Collect-T*)
apply *simp*
apply (*subst* (*1 2*) *nn-integral-measure-pmf-finite*)
apply (*auto simp*: *E-Mix E-Init* ∗ *sum.reindex sum-distrib-right*[*symmetric*] *divide-ennreal*

ennreal-times-divide[*symmetric*])
done
qed

What is the probability that the server sees a specific jondo (including the initiator) as sender.

definition *visit* :: *'a set* ⇒ *'a set* ⇒ *'a state stream* ⇒ *bool* **where**
visit I L = *Init'*(*I* ∩ *H*) · (*HLD* (*Mix'J*) *suntil* (*Mix'*(*L* ∩ *J*) · *HLD* {*End*}))

lemma *visit-unique1*:
visit I1 L1 ω ⟹ *visit I2 L2 ω* ⟹ *I1* ∩ *I2* ≠ {}
by (*auto simp*: *visit-def HLD-iff*)

lemma *visit-unique2*:
assumes *visit I1 L1 ω visit I2 L2 ω*
shows *L1* ∩ *L2* ≠ {}
proof −
let *?U* = λ*L ω*. (*HLD* (*Mix'J*) *suntil* ((*Mix'*(*L*∩*J*)) · *HLD* {*End*})) *ω*
have *?U L1* (*stl ω*) *?U L2* (*stl ω*)
using *assms* **by** (*auto simp*: *visit-def*)
then show *L1* ∩ *L2* ≠ {}
proof (*induction stl ω arbitrary*: *ω rule*: *suntil-induct-strong*)
case *base* **then show** *?case*
by (*auto simp add*: *suntil.simps*[*of - - stl* (*stl ω*)] *suntil.simps*[*of - - stl ω*] *HLD-iff*)
next
case *step*
show *?case*
proof *cases*
assume ((*Mix'*(*L2*∩*J*)) · *HLD* {*End*}) (*stl ω*)
with *step.hyps* **show** *?thesis*
by (*auto simp*: *inj-Mix HLD-iff elim*: *suntil.cases*)
next
assume ¬ ((*Mix'*(*L2*∩*J*)) · *HLD* {*End*}) (*stl ω*)
with *step.prems* **have** *?U L2* (*stl* (*stl ω*))
by (*auto elim*: *suntil.cases*)
then show *?thesis*
by (*rule step.hyps*(*4*)[*OF refl*])
qed
qed
qed

lemma *visit-imp-in-H*: *visit* {*i*} *J ω* ⟹ *i* ∈ *H*
by (*auto simp*: *visit-def HLD-iff*)

lemma *emeasure-visit*:
assumes *I*: *I* ⊆ *H* **and** *L*: *L* ⊆ *J*
shows *emeasure* 𝔓 {*ω*∈*space* 𝔓. *visit I L ω*} = (∑ *i*∈*I*. *p-i i*) ∗ (*card L* ∗ *p-j*)
proof −

let *?J* = *HLD* (*Mix*‘*J*) **and** *?E* = (*Mix*‘*L*) · *HLD* {*End*}
let *?φ* = *?J aand not ?E*
let *?P* = *λx P. emeasure* (*T x*) {*ω*∈*space* (*T x*). *P ω*}

have [*intro*]: *finite L*
  using *finite-J* ‹*L* ⊆ *J*› **by** (*blast intro*: *finite-subset*)
have [*simp*, *intro*]: *finite I*
  using *finite-J* ‹*I* ⊆ *H*› **by** (*blast intro*: *finite-subset*)

{ fix *j* **assume** *j*: *j* ∈ *H*
  have *?P* (*Mix j*) (*?J suntil ?E*) = (*p-f* ∗ *p-j* ∗ (*1* − *p-f*) ∗ *card L*) / (*1* − *p-f*)
  proof (*rule emeasure-suntil-geometric*)
    fix *s* **assume** *s*: *s* ∈ *Mix* ‘ *J*
    then have *?P s ?E* = (∫ ⁺*x. ennreal* (*1* − *p-f*) ∗ *indicator* (*Mix*‘*L*) *x* ∂*N s*)
      by (*auto simp add*: *emeasure-HLD-nxt emeasure-HLD AE-measure-pmf-iff emeasure-pmf-single*
        split: *state.split split-indicator simp del*: *space-T nxt.simps*
        intro!: *nn-integral-cong-AE*)
    also have . . . = *ennreal* (*1* − *p-f*) ∗ *emeasure* (*N s*) (*Mix*‘*L*)
      using *p-f* **by** (*intro nn-integral-cmult-indicator*) *auto*
    also have . . . = *ennreal* ((*1* − *p-f*) ∗ *card L* ∗ *p-j* ∗ *p-f*)
      using *s assms*
      by (*subst emeasure-measure-pmf-finite*)
        (*auto simp*: *sum.reindex subset-eq ennreal-mult mult-ac*)
    finally show *?P s ?E* = *p-f* ∗ *p-j* ∗ (*1* − *p-f*) ∗ *card L*
      by *simp*
  next
    show ⋀*t. AE ω in T t.* ¬ (*?E* ⊓ (*?J* ⊓ *nxt* (*?J suntil ?E*))) *ω*
      by (*intro AE-I2*) (*auto simp*: *HLD-iff elim*: *suntil.cases*)
  qed (*insert p-f j*, *auto simp*: *emeasure-measure-pmf-finite sum.reindex p-j-def*)
  then have *?P* (*Init j*) (*?J suntil ?E*) = (*p-f* ∗ *p-j* ∗ (*1* − *p-f*) ∗ *card L*) / (*1* − *p-f*) / *p-f*
    by (*subst emeasure-Init-eq-Mix*) (*simp-all add*: *suntil.simps*[*of* - - *x* ## *s* **for** *x s*] *divide-ennreal p-f*)
  then have *?P* (*Init j*) (*?J suntil ?E*) = *p-j* ∗ *card L*
    using *p-f* **by** *simp* **}**
note *J-suntil-E* = *this*

have *?P Start* (*visit I L*) = (∫ ⁺*x. ?P x* (*?J suntil ?E*) ∗ *indicator* (*Init*‘*I*) *x* ∂*N Start*)
    unfolding *visit-def* **using** *I L* **by** (*subst emeasure-HLD-nxt*) (*auto simp*: *Int-absorb2*)
also have . . . = (∫ ⁺*x. ennreal* (*p-j* ∗ *card L*) ∗ *indicator* (*Init*‘*I*) *x* ∂*N Start*)
  using *I J-suntil-E*
  by (*intro nn-integral-cong ennreal-mult-right-cong*)
    (*auto split*: *split-indicator-asm*)
also have . . . = *ennreal* ((∑ *i*∈*I*. *p-i i*) ∗ *card L* ∗ *p-j*)
  using *p-j-pos assms*
  by (*subst nn-integral-cmult-indicator*)

(*auto simp*: *emeasure-measure-pmf-finite sum.reindex subset-eq ennreal-mult*[*symmetric*] *sum-nonneg*)
finally show *?thesis* **by** (*simp add*: *ac-simps*)
qed

lemma *measurable-visit*[*measurable*]: *Measurable.pred S* (*visit I L*)
by (*simp add*: *visit-def*)

lemma *AE-visit*: *AE* ω *in* $\mathfrak{P}$. *visit H J* ω
proof (*rule T.AE-I-eq-1*)
show *emeasure* $\mathfrak{P}$ $\{\omega \in$ *space* $\mathfrak{P}$. *visit H J* $\omega\} = 1$
using *J-not-empty* **by** (*subst emeasure-visit*) (*simp-all add*: *p-j-def*)
qed *simp*

13.2 Server gets no information

lemma *server-view1*: $j \in J \Longrightarrow \mathcal{P}(\omega$ *in* $\mathfrak{P}$. *visit H* $\{j\}$ $\omega) =$ *p-j*
unfolding *measure-def* **by** (*subst emeasure-visit*) *simp-all*

lemma *server-view-indep*:
$L \subseteq J \Longrightarrow I \subseteq H \Longrightarrow \mathcal{P}(\omega$ *in* $\mathfrak{P}$. *visit I L* $\omega) = \mathcal{P}(\omega$ *in* $\mathfrak{P}$. *visit H L* $\omega) * \mathcal{P}(\omega$ *in* $\mathfrak{P}$. *visit I J* $\omega)$
unfolding *measure-def*
by (*subst* (*1 2 3*) *emeasure-visit*) (*auto simp*: *p-j-def sum-nonneg subset-eq*)

lemma *server-view*: $\mathcal{P}(\omega$ *in* $\mathfrak{P}$. $\exists j \in H$. *visit* $\{j\}$ $\{j\}$ $\omega) =$ *p-j*
using *finite-J*
proof (*subst T.prob-sum*[**where** $I=H$ **and** $P=\lambda j$. *visit* $\{j\}$ $\{j\}$])
show $(\sum j \in H. \mathcal{P}(\omega$ *in* $\mathfrak{P}$. *visit* $\{j\}$ $\{j\}$ $\omega)) =$ *p-j*
by (*auto simp*: *measure-def emeasure-visit sum-distrib-right*[*symmetric*] *simp del*: *space-T sets-T*)
show *AE x in* $\mathfrak{P}$. $(\forall n \in H$. *visit* $\{n\}$ $\{n\}$ $x \longrightarrow (\exists j \in H$. *visit* $\{j\}$ $\{j\}$ $x)) \wedge$
$((\exists j \in H$. *visit* $\{j\}$ $\{j\}$ $x) \longrightarrow (\exists! n.\ n \in H \wedge$ *visit* $\{n\}$ $\{n\}$ $x))$
by (*auto dest*: *visit-unique1*)
qed *simp-all*

13.3 Probability that collaborators gain information

definition *hit-C* = *Init‘H* · *ev* (*HLD* (*Mix‘C*))
definition *before-C B* = (*HLD* (*Jondo H*)) *suntil* ((*Jondo* ($B \cap H$)) · *HLD* (*Mix* ‘ *C*))

lemma *measurable-hit-C*[*measurable*]: *Measurable.pred S hit-C*
by (*simp add*: *hit-C-def*)

lemma *measurable-before-C*[*measurable*]: *Measurable.pred S* (*before-C B*)
by (*simp add*: *before-C-def*)

lemma *before-C*:
assumes ω: *enabled Start* ω

shows *before-C B ω* ⟷
((*Init‘H* · (*HLD* (*Mix‘H*) *suntil* (*Mix‘*(*B* ∩ *H*) · *HLD* (*Mix‘C*)))) *or* (*Init‘*(*B* ∩ *H*) · *HLD* (*Mix‘C*))) *ω*
proof −
{ **fix** *ω s* **assume** ((*HLD* (*Jondo H*)) *suntil* (*Jondo* (*B* ∩ *H*) · *HLD* (*Mix ‘ C*))) *ω*
enabled s ω s ∈ *Jondo H*
then have (*HLD* (*Mix ‘ H*) *suntil* (*Mix ‘* (*B* ∩ *H*) · (*HLD* (*Mix ‘ C*)))) *ω*
proof (*induction arbitrary*: *s*)
case (*base ω*) **then show** *?case*
by (*auto simp*: *HLD-iff enabled.simps*[*of - ω*] *E-Init E-Mix intro*!: *suntil.intros*(*1*))
next
case (*step ω*) **from** *step.prems step.hyps step.IH*[*of shd ω*] **show** *?case*
by (*auto simp*: *HLD-iff enabled.simps*[*of - ω*] *E-Init E-Mix*
suntil.simps[*of - - ω*] *enabled-End suntil-sconst*)
qed }
note *this*[*of stl ω shd ω*]
moreover
{ **fix** *ω s* **assume** (*HLD* (*Mix ‘ H*) *suntil* (*Mix ‘* (*B* ∩ *H*) · (*HLD* (*Mix ‘ C*)))) *ω*
enabled s ω s ∈ *Jondo H*
then have ((*HLD* (*Jondo H*)) *suntil* ((*Jondo* (*B* ∩ *H*)) · *HLD* (*Mix ‘ C*))) *ω*
proof (*induction arbitrary*: *s*)
case (*step ω*) **from** *step.prems step.hyps step.IH*[*of shd ω*] **show** *?case*
by (*auto simp*: *HLD-iff enabled.simps*[*of - ω*] *E-Init E-Mix*
suntil.simps[*of - - ω*] *enabled-End suntil-sconst*)
qed (*auto intro*: *suntil.intros simp*: *HLD-iff*) }
note *this*[*of stl ω shd ω*]
ultimately show *?thesis*
using *assms*
using ‹*enabled Start ω*›
unfolding *before-C-def suntil.simps*[*of - - ω*] *enabled.simps*[*of - ω*]
by (*auto simp*: *E-Start HLD-iff*)
qed

lemma *before-C-unique*:
assumes *ω*: *before-C I1 ω before-C I2 ω* **shows** *I1* ∩ *I2* ≠ {}
using *ω* **unfolding** *before-C-def*
proof *induction*
case (*base ω*) **then show** *?case*
by (*auto simp add*: *suntil.simps*[*of - - ω*] *suntil.simps*[*of - - stl ω*] *HLD-iff*)
next
case (*step ω*) **then show** *?case*
by (*auto simp add*: *suntil.simps*[*of - - ω*] *suntil.simps*[*of - - stl ω*] *HLD-iff*)
qed

lemma *hit-C-imp-before-C*:
assumes *enabled Start ω hit-C ω* **shows** *before-C H ω*
proof −

let *?X* = *Init'H* ∪ *Mix'H*
{ fix *ω s* **assume** *ev* (*HLD* (*Mix'C*)) *ω s*∈*?X enabled s ω*
then have ((*HLD* (*Jondo H*)) *suntil* (*?X* · *HLD* (*Mix ' C*))) (*s ## ω*)
proof (*induction arbitrary*: *s rule*: *ev-induct-strong*)
case (*step ω s*) **from** *step.IH*[*of shd ω*] *step.prems step.hyps* **show** *?case*
by (*auto simp*: *enabled.simps*[*of - ω*] *suntil-Stream E-Init E-Mix HLD-iff enabled-End ev-sconst*)
qed (*auto simp*: *suntil-Stream*) **}**
from *this*[*of stl ω shd ω*] *assms* **show** *?thesis*
by (*auto simp*: *before-C-def hit-C-def enabled.simps*[*of - ω*] *E-Start*)
qed

lemma *before-C-single*:
assumes *before-C I ω* **shows** ∃ *i*∈*I* ∩ *H*. *before-C* {*i*} *ω*
using *assms* **unfolding** *before-C-def* **by** *induction* (*auto simp*: *HLD-iff intro*: *suntil.intros*)

lemma *before-C-imp-in-H*: *before-C* {*i*} *ω* ⟹ *i* ∈ *H*
by (*auto dest*: *before-C-single*)

13.4 The probability that the sender hits a collaborator

lemma *Pr-hit-C*: 𝒫(*ω in* 𝔓. *hit-C ω*) = (*1* − *p-H*) / (*1* − *p-H* ∗ *p-f*)
proof −
let *?P* = λ*x P*. *emeasure* (*T x*) {*ω*∈*space* (*T x*). *P ω*}
let *?M* = *HLD* (*Mix ' C*) **and** *?I* = *Init'H* **and** *?J* = *Mix'H*
let *?φ* = (*HLD ?J*) *aand not ?M*

{ fix *s* **assume** *s*: *s* ∈ *Jondo J*
have *AE ω in T s*. *ev ?M ω* ⟷ (*HLD ?J suntil ?M*) *ω*
using *AE-T-enabled*
proof *eventually-elim*
fix *ω* **assume** *ω*: *enabled s ω*
show *ev ?M ω* ⟷ (*HLD ?J suntil ?M*) *ω*
proof
assume *ev ?M ω*
from *this ω s* **show** (*HLD ?J suntil ?M*) *ω*
proof (*induct arbitrary*: *s rule*: *ev-induct-strong*)
case (*step ω*) **then show** *?case*
by (*auto simp*: *HLD-iff enabled.simps*[*of - ω*] *suntil.simps*[*of - - ω*] *E-End E-Init E-Mix*
enabled-End ev-sconst)
qed (*auto simp*: *HLD-iff E-Init intro*: *suntil.intros*)
qed (*rule ev-suntil*)
qed }
note *ev-eq-suntil* = *this*

have *?P Start hit-C* = ($\int^+$*x*. *?P x* (*ev ?M*) ∗ *indicator ?I x* ∂*N Start*)
unfolding *hit-C-def* **by** (*rule emeasure-HLD-nxt*) *measurable*

also have ... = ($\int^+ x$. *ennreal* ((*1* − *p-H*) / (*1* − *p-f* ∗ *p-H*)) ∗ *indicator ?I x* ∂*N Start*)
proof (*intro nn-integral-cong ennreal-mult-right-cong refl*)
fix *x* **assume** *indicator* (*Init* ' *H*) *x* ≠ *0*
then have *x* ∈ *?I*
by (*auto split*: *split-indicator-asm*)
{ **fix** *j* **assume** *j*: *j* ∈ *H*
with *ev-eq-suntil*[*of Mix j*] **have** *?P* (*Mix j*) (*ev ?M*) = *?P* (*Mix j*) ((*HLD ?J*) *suntil ?M*)
by (*intro emeasure-eq-AE*) *auto*
also have ... = (((*1* − *p-H*) ∗ *p-f*)) / (*1* − *p-H* ∗ *p-f*)
proof (*rule emeasure-suntil-geometric*)
fix *s* **assume** *s*: *s* ∈ *Mix* ' *H*
from *s C-smaller* **show** *?P s ?M* = *ennreal* ((*1* − *p-H*) ∗ *p-f*)
by (*subst emeasure-HLD*)
(*auto simp add*: *emeasure-measure-pmf-finite sum.reindex subset-eq p-j-def H-compl*)
from *s* **show** *emeasure* (*N s*) (*Mix*'*H*) = *p-H* ∗ *p-f*
by (*auto simp*: *emeasure-measure-pmf-finite sum.reindex p-H-def p-j-def*)
qed (*insert j*, *auto simp*: *HLD-iff p-H-p-f-less-1*)
finally have *?P* (*Init j*) (*ev ?M*) = (*1* − *p-H*) / (*1* − *p-H* ∗ *p-f*)
using *p-f*
by (*subst emeasure-Init-eq-Mix*)
(*auto simp*: *ev-Stream AE-End ev-sconst HLD-iff mult-le-one divide-ennreal*)
}
then show *?P x* (*ev ?M*) = (*1* − *p-H*) / (*1* − *p-f* ∗ *p-H*)
using ‹*x* ∈ *?I*› **by** (*auto simp*: *mult-ac*)
qed
also have ... = *ennreal* ((*1* − *p-H*) / (*1* − *p-H* ∗ *p-f*))
using *p-j-pos p-H p-H-p-f-less-1*
by (*subst nn-integral-cmult-indicator*)
(*auto simp*: *emeasure-measure-pmf-finite sum.reindex subset-eq mult-ac intro*!: *divide-nonneg-nonneg*)
finally show *?thesis*
by (*simp add*: *measure-def mult-le-one*)
qed

lemma *before-C-imp-hit-C*:
assumes *enabled Start ω before-C B ω*
shows *hit-C ω*
proof −
{ **fix** *ω j* **assume** ((*HLD* (*Jondo H*)) *suntil* (*Jondo* (*B* ∩ *H*) · *HLD* (*Mix* ' *C*))) *ω*
j ∈ *H enabled* (*Mix j*) *ω*
then have *ev* (*HLD* (*Mix*'*C*)) *ω*
proof (*induction arbitrary*: *j rule*: *suntil-induct-strong*)
case (*step ω*) **then show** *?case*
by (*auto simp*: *enabled.simps*[*of* - *ω*] *E-Mix enabled-End ev-sconst suntil-sconst HLD-iff*)

```
    qed auto }
  from this[of stl (stl ω)] assms show hit-C ω
    by (force simp: before-C-def hit-C-def E-Start HLD-iff E-Init
      enabled.simps[of - ω] ev.simps[of - ω] suntil.simps[of - - ω]
      enabled.simps[of - stl ω] ev.simps[of - stl ω] suntil.simps[of - - stl ω])
qed

lemma negE: ¬ P ⟹ P ⟹ False
  by blast

lemma Pr-visit-before-C:
  assumes L: L ⊆ H and I: I ⊆ H
  shows 𝒫(ω in 𝔓. visit I J ω ∧ before-C L ω | hit-C ω ) =
    (∑ i∈I. p-i i) ∗ card L ∗ p-j ∗ p-f + (∑ i∈I ∩ L. p-i i) ∗ (1 − p-H ∗ p-f)
proof −
  let ?M = Mix‘H
  let ?P = λx P. emeasure (T x) {ω∈space (T x). P ω}
  let ?V = (visit I J aand before-C L) aand hit-C
  let ?U = HLD ?M suntil (Mix‘L · HLD (Mix‘C))
  let ?L = HLD (Mix‘C)

  have IJ: x ∈ I ⟹ x ∈ J for x
    using I by auto

  have [simp, intro]: finite I finite L
    using L I by (auto dest: finite-subset)

  have ?P Start ?V = ?P Start ((Init‘I · ?U) or (Init‘(I ∩ L) · ?L))
  proof (rule emeasure-Collect-eq-AE)
    show AE ω in 𝔓. ?V ω ⟷ ((Init‘I · ?U) or (Init‘(I ∩ L) · ?L)) ω
      using AE-T-enabled AE-visit
    proof eventually-elim
      case (elim ω)
      then show ?case
        using before-C-imp-hit-C[of ω L]  before-C[of ω L] I L
        by (auto simp: visit-def HLD-iff Int-absorb2)
    qed
    show Measurable.pred 𝔓 ((Init‘I · ?U) or (Init‘(I ∩ L) · ?L))
      by measurable
  qed measurable
  also have ... = ?P Start (Init‘I · ?U) + ?P Start (Init‘(I ∩ L) · ?L)
    using L I
    apply (subst plus-emeasure)
    apply (auto intro!: arg-cong2[where f=emeasure])
    apply (subst (asm) suntil.simps)
    apply (auto simp add: HLD-iff[abs-def] elim: suntil.cases)
    done
  also have ?P Start (Init‘(I ∩ L) · ?L) = (∑ i∈I∩L. p-i i ∗ (1 − p-H))
    using L I C-smaller p-j-pos
```

apply (*subst emeasure-HLD-nxt emeasure-HLD*, *simp*)+
apply (*subst nn-integral-indicator-finite*)
apply (*auto simp*: *emeasure-measure-pmf-finite sum.reindex next-prob-def sum.If-cases Int-absorb2 H-compl2 ennreal-mult*[*symmetric*] *sum-nonneg sum-distrib-left*[*symmetric*] *sum-distrib-right*[*symmetric*] *intro*!: *sum.cong sum-nonneg*)
apply (*subst* (*asm*) *ennreal-inj*)
apply (*auto intro*!: *mult-nonneg-nonneg sum-nonneg sum.mono-neutral-left elim*!: *negE*)
done
also have *?P Start* (*Init'I* · *?U*) = ($\sum$ *i*∈*I*. *?P* (*Init i*) *?U* ∗ *p-i i*)
using *I*
by (*subst emeasure-HLD-nxt*, *simp*)
(*auto simp*: *nn-integral-indicator-finite sum.reindex emeasure-measure-pmf-finite intro*!: *sum.cong*[*OF refl*])
also have ... = ($\sum$ *i*∈*I*. *ennreal* (*p-f* ∗ (*1* − *p-H*) ∗ *p-j* ∗ *card L* / (*1* − *p-H* ∗ *p-f*)) ∗ *p-i i*)
proof (*intro sum.cong refl arg-cong2*[**where** *f*=(∗)])
fix *i* **assume** *i* ∈ *I*
with *I* **have** *i*: *i* ∈ *H*
by *auto*
have *?P* (*Mix i*) *?U* = (*p-f* ∗ *p-f* ∗ (*1* − *p-H*) ∗ *p-j* ∗ *card L* / (*1* − *p-H* ∗ *p-f*))
unfolding *before-C-def*
proof (*rule emeasure-suntil-geometric*[**where** *X*=*?M*])
show *Mix i* ∈ *?M*
using *i* **by** *auto*
next
fix *s* **assume** *s* ∈ *?M*
with *p-f p-j-pos L C-smaller*[*THEN less-imp-le*]
show *?P s* (*Mix'L* · (*HLD* (*Mix ' C*))) = *ennreal* (*p-f* ∗ *p-f* ∗ (*1* − *p-H*) ∗ *p-j* ∗ *card L*)
apply (*simp add*: *emeasure-HLD emeasure-HLD-nxt del*: *nxt.simps space-T*)
apply (*subst nn-integral-measure-pmf-support*[*of Mix'L*])
apply (*auto simp add*: *subset-eq emeasure-measure-pmf-finite sum.reindex H-compl p-j-def ennreal-mult*[*symmetric*] *ennreal-of-nat-eq-real-of-nat*)
done
next
fix *s* **assume** *s* ∈ *?M* **then show** *emeasure* (*N s*) *?M* = *ennreal* (*p-H* ∗ *p-f*)
by (*auto simp add*: *emeasure-measure-pmf-finite sum.reindex H-eq2*)
next
show *AE ω in T t*. ¬ ((*Mix ' L* · *?L*) ⊓ (*HLD* (*Mix ' H*) ⊓ *nxt ?U*)) *ω* **for** *t*
using *L*
apply (*simp add*: *AE-T-iff*[*of - t*])
apply (*subst AE-T-iff*; *simp*)
apply (*auto simp*: *HLD-iff suntil-Stream*)
done
qed (*insert L*, *auto simp*: *p-H-p-f-less-1 E-Mix*)

then show *?P* (*Init i*) *?U* = *p-f* ∗ (*1* − *p-H*) ∗ *p-j* ∗ *card L* / (*1* − *p-H* ∗ *p-f*)
by (*subst emeasure-Init-eq-Mix*)
(*auto simp*: *AE-End suntil-Stream divide-ennreal mult-le-one p-f*)
qed
finally have ∗: $\mathcal{P}$(*ω in T Start. ?V ω*) =
(*p-f* ∗ (*1* − *p-H*) ∗ *p-j* ∗ (*card L*) / (*1* − *p-H* ∗ *p-f*)) ∗ ($\sum$ *i*∈*I*. *p-i i*) +
($\sum$ *i*∈*I* ∩ *L*. *p-i i*) ∗ (*1* − *p-H*)
using *sum-nonneg* [*of I* ∩ *L p-i*] *sum-nonneg* [*of I p-i*]
by (*simp add*: *mult-ac measure-def sum-distrib-right*[*symmetric*] *sum-distrib-left*[*symmetric*]
sum-divide-distrib[*symmetric*] *IJ ennreal-mult*[*symmetric*]
mult-le-one ennreal-plus[*symmetric*]
del: *ennreal-plus*)
show *?thesis*
unfolding *cond-prob-def Pr-hit-C* ∗
using ∗
using *p-f p-H p-j-pos p-H-p-f-less-1* **by** (*simp add*: *divide-simps*) (*simp add*: *field-simps*)
qed

lemma *Pr-visit-eq-before-C*:
$\mathcal{P}$(*ω in* $\mathfrak{P}$. ∃*j*∈*H*. *visit* {*j*} *J ω* ∧ *before-C* {*j*} *ω* | *hit-C ω*) = *1* − (*p-H* − *p-j*) ∗ *p-f*
proof −
let *?V* = λ*j*. *visit* {*j*} *J aand before-C* {*j*} **and** *?H* = *hit-C*
let *?J* = *H*
have $\mathcal{P}$(*ω in* $\mathfrak{P}$. (∃*j*∈*?J*. *?V j ω*) ∧ *?H ω*) = ($\sum$ *j*∈*?J*. $\mathcal{P}$(*ω in* $\mathfrak{P}$. (*?V j aand ?H*) *ω*))
proof (*rule T.prob-sum*)
show *AE ω in* $\mathfrak{P}$. (∀*j*∈*?J*. (*?V j aand ?H*) *ω* ⟶ ((∃*j*∈*?J*. *?V j ω*) ∧ *?H ω*)) ∧
(((∃*j*∈*?J*. *?V j ω*) ∧ *?H ω*) ⟶ (∃!*j*. *j*∈*?J* ∧ (*?V j aand ?H*) *ω*))
by (*auto intro*!: *AE-I2 dest*: *visit-unique1*)
qed *auto*
then have $\mathcal{P}$(*ω in* $\mathfrak{P}$. (∃*j*∈*?J*. *?V j ω*) | *?H ω*) = ($\sum$ *j*∈*?J*. $\mathcal{P}$(*ω in* $\mathfrak{P}$. *?V j ω* | *?H ω*))
by (*simp add*: *cond-prob-def sum-divide-distrib*)
also have ... = *p-j* ∗ *p-f* + (*1* − *p-H* ∗ *p-f*)
by (*simp add*: *Pr-visit-before-C sum-distrib-right*[*symmetric*] *sum.distrib*)
finally show *?thesis*
by (*simp add*: *field-simps*)
qed

lemma *probably-innocent*:
assumes *approx*: *1* / (*2* ∗ (*p-H* − *p-j*)) ≤ *p-f* **and** *p-H* ≠ *p-j*
shows $\mathcal{P}$(*ω in* $\mathfrak{P}$. ∃*j*∈*H*. *visit* {*j*} *J ω* ∧ *before-C* {*j*} *ω* | *hit-C ω*) ≤ *1* / *2*
unfolding *Pr-visit-eq-before-C*
proof −
have [*simp*]: ⋀*n* :: *nat*. *1* ≤ *real n* ⟷ *1* ≤ *n* **by** *auto*

have *0* ≤ *p-j* **unfolding** *p-j-def* **by** *auto*
then have *1* ∗ *p-j* ≤ *p-H*
unfolding *H-eq2*[*symmetric*] **using** *C-smaller*
by (*intro mult-mono*) (*auto simp*: *Suc-le-eq card-Diff-subset not-le*)
with ‹*p-H* ≠ *p-j*› **have** *p-j* < *p-H* **by** *auto*
with *approx* **show** *1* − (*p-H* − *p-j*) ∗ *p-f* ≤ *1* / *2*
by (*auto simp add*: *field-simps divide-le-eq split*: *if-split-asm*)
qed

lemma *Pr-before-C*:
assumes *L*: *L* ⊆ *H*
shows 𝒫(*ω in* 𝔓. *before-C L ω* | *hit-C ω*) =
card L ∗ *p-j* ∗ *p-f* + (∑ *l*∈*L*. *p-i l*) ∗ (*1* − *p-H* ∗ *p-f*)
proof −
have 𝒫(*ω in* 𝔓. *before-C L ω* | *hit-C ω*) =
𝒫(*ω in* 𝔓. *visit H J ω* ∧ *before-C L ω* | *hit-C ω*)
using *AE-visit* **by** (*auto intro*!: *T.cond-prob-eq-AE*)
also have . . . = *card L* ∗ *p-j* ∗ *p-f* + (∑ *i*∈*L*. *p-i i*) ∗ (*1* − *p-H* ∗ *p-f*)
using *L* **by** (*subst Pr-visit-before-C*[*OF L order-refl*]) (*auto simp*: *Int-absorb1*)
finally show *?thesis* **.**
qed

lemma *P-visit*:
assumes *I*: *I* ⊆ *H*
shows 𝒫(*ω in* 𝔓. *visit I J ω* | *hit-C ω*) = (∑ *i*∈*I*. *p-i i*)
proof −
have 𝒫(*ω in* 𝔓. *visit I J ω* | *hit-C ω*) =
𝒫(*ω in* 𝔓. *visit I J ω* ∧ *before-C H ω* | *hit-C ω*)
proof (*rule T.cond-prob-eq-AE*)
show *AE x in* 𝔓. *hit-C x* ⟶
visit I J x = (*visit I J x* ∧ *before-C H x*)
using *AE-T-enabled* **by** *eventually-elim* (*auto intro*: *hit-C-imp-before-C*)
qed *auto*
also have . . . = *sum p-i I*
using *I* **by** (*subst Pr-visit-before-C*[*OF order-refl*]) (*auto simp*: *Int-absorb2 field-simps p-H-def p-j-def*)
finally show *?thesis* **.**
qed

13.5 Probability space of hitting a collaborator

definition *hC* = *uniform-measure* 𝔓 {*ω*∈*space* 𝔓. *hit-C ω*}

lemma *emeasure-hit-C-not-0*: *emeasure* 𝔓 {*ω* ∈ *space* 𝔓. *hit-C ω*} ≠ *0*
using *p-H p-H-p-f-less-1* **unfolding** *Pr-hit-C T.emeasure-eq-measure* **by** *auto*

lemma *measurable-hC*[*measurable* (*raw*)]:
A ∈ *sets S* ⟹ *A* ∈ *sets hC*
f ∈ *measurable M S* ⟹ *f* ∈ *measurable M hC*

$g \in measurable\ S\ M \Longrightarrow g \in measurable\ hC\ M$
$A \cap space\ S \in sets\ S \Longrightarrow A \cap space\ hC \in sets\ S$
unfolding *hC-def uniform-measure-def*
by *simp-all*

lemma *vimage-Int-space-C*[*simp*]:
$f -` \{x\} \cap space\ hC = \{\omega\in space\ S.\ f\ \omega = x\}$
by (*auto simp*: *hC-def*)

sublocale *hC*: *information-space hC 2*
proof −
interpret *hC*: *prob-space hC*
unfolding *hC-def*
using *emeasure-hit-C-not-0*
by (*intro prob-space-uniform-measure*) *auto*
show *information-space hC 2*
by *standard simp*
qed

abbreviation
mutual-information-Pow-CP (‹$\mathcal{I}$'(- ; -')›) **where**
$\mathcal{I}(X\ ;\ Y) \equiv hC.mutual\text{-}information\ 2\ (count\text{-}space\ (X`space\ hC))\ (count\text{-}space\ (Y`space\ hC))\ X\ Y$

lemma *simple-functionI*:
assumes *finite* (*range f*)
assumes [*measurable*]: $\bigwedge x.\ \{\omega\in space\ S.\ f\ \omega = x\} \in sets\ S$
shows *simple-function hC f*
using *assms* **unfolding** *simple-function-def hC-def*
by (*simp add*: *vimage-def space-stream-space*)

13.6 Estimate the information to the collaborators

lemma *measure-hC*[*simp*]:
assumes *A*[*measurable*]: $A \in sets\ S$
shows $measure\ hC\ A = \mathcal{P}(\omega\ in\ \mathfrak{P}.\ \omega \in A\ |\ hit\text{-}C\ \omega\ )$
unfolding *hC-def cond-prob-def*
using *emeasure-hit-C-not-0 A*
by (*subst measure-uniform-measure*) (*simp-all add*: *T.emeasure-eq-measure Int-def conj-ac*)

13.6.1 Setup random variables for mutual information

definition $first\text{-}J\ \omega = (THE\ i.\ visit\ \{i\}\ J\ \omega)$

lemma *first-J-eq*:
$visit\ \{i\}\ J\ \omega \Longrightarrow first\text{-}J\ \omega = i$
unfolding *first-J-def* **by** (*intro the-equality*) (*auto dest*: *visit-unique1*)

lemma *AE-first-J*:

AE ω in 𝔓. visit {i} J ω ⟷ first-J ω = i
using *AE-visit*
proof *eventually-elim*
fix *ω* **assume** *visit H J ω*
then obtain *j* **where** *visit {j} J ω j ∈ H*
by (*auto simp*: *visit-def HLD-iff*)
then show *visit {i} J ω ⟷ first-J ω = i*
by (*auto dest*: *visit-unique1 first-J-eq*)
qed

lemma *measurbale-first-J*[*measurable*]: *first-J ∈ measurable S* (*count-space UNIV*)
unfolding *first-J-def*[*abs-def*]
by (*intro measurable-THE*[**where** *I=H*])
(*auto dest*: *visit-imp-in-H visit-unique1 intro*: *countable-finite*)

definition *last-H ω =* (*THE i. before-C {i} ω*)

lemma *measurbale-last-H*[*measurable*]: *last-H ∈ measurable S* (*count-space UNIV*)
unfolding *last-H-def*[*abs-def*]
by (*intro measurable-THE*[**where** *I=H*])
(*auto dest*: *before-C-single before-C-unique intro*: *countable-finite*)

lemma *last-H-eq*:
before-C {i} ω ⟹ last-H ω = i
unfolding *last-H-def* **by** (*intro the-equality*) (*auto dest*: *before-C-unique*)

lemma *last-H*:
assumes *enabled Start ω hit-C ω*
shows *before-C {last-H ω} ω last-H ω ∈ H*
by (*metis before-C-single hit-C-imp-before-C last-H-eq Int-iff assms*)+

lemma *AE-last-H*:
AE ω in 𝔓. hit-C ω ⟶ before-C {i} ω ⟷ last-H ω = i
using *AE-T-enabled*
proof *eventually-elim*
fix *ω* **assume** *enabled Start ω* **then show** *hit-C ω ⟶ before-C {i} ω =* (*last-H ω = i*)
by (*auto dest*: *last-H last-H-eq*)
qed

lemma *information-flow*:
defines *h ≡ real* (*card H*)
assumes *init-uniform*: *⋀i. i ∈ H ⟹ p-i i = 1 / h*
shows *ℐ*(*first-J* ; *last-H*) *≤* (*1 −* (*h − 1*) *∗ p-j ∗ p-f*) *∗ log 2 h*
proof −
let *?il = λi l. 𝒫*(*ω in 𝔓. visit {i} J ω ∧ before-C {l} ω | hit-C ω*)
let *?i = λi. 𝒫*(*ω in 𝔓. visit {i} J ω | hit-C ω*)
let *?l = λl. 𝒫*(*ω in 𝔓. before-C {l} ω | hit-C ω*)

```
from init-uniform have init-H: ⋀i. i ∈ H ⟹ p-i i = p-j / p-H
  by (simp add: p-j-def p-H-def h-def)

from h-def have 1/h = p-j/p-H h = p-H / p-j p-H = h ∗ p-j
  by (auto simp: p-H-def p-j-def field-simps)
from C-smaller have h-pos: 0 < h
  by (auto simp add: card-gt-0-iff h-def)

let ?s = (h − 1) ∗ p-j
let ?f = ?s ∗ p-f

from psubset-card-mono[OF - C-smaller]
have 1 ≤ card J − card C
  by (simp del: C-le-J)
then have 1 ≤ h
  using C-smaller
  by (simp add: h-def card-Diff-subset card-mono field-simps del: C-le-J)

have log-le-0: ?f ∗ log 2 (p-H ∗ p-f) ≤ ?f ∗ log 2 1
  using p-H-p-f-less-1 p-H-p-f-pos p-j-pos p-f ‹1 ≤ h›
  by (intro mult-left-mono log-mono mult-nonneg-nonneg) auto

have (h − 1) ∗ p-j < 1
  using ‹1 ≤ h› C-smaller
  by (auto simp: h-def p-j-def divide-less-eq card-Diff-subset card-mono)
then have 1: (h − 1) ∗ p-j ∗ p-f < 1 ∗ 1
  using p-f by (intro mult-strict-mono) auto

{ fix ω have first-J ω ∈ H ∨ first-J ω = (THE x. False)
    apply (cases ∀ i. ¬ visit {i} J ω)
    apply (simp add: first-J-def)
    apply (auto dest: visit-imp-in-H first-J-eq)
    done }
then have range-fj: range first-J ⊆ H ∪ {THE x. False}
  by auto

have sf-fj: simple-function hC first-J
  by (rule simple-functionI) (auto intro: finite-subset[OF range-fj])

have sd-fj: simple-distributed hC first-J ?i
  apply (rule hC.simple-distributedI[OF sf-fj])
  apply (auto intro!: T.cond-prob-eq-AE)
  apply (auto simp: space-stream-space)
  using AE-first-J
  apply eventually-elim
  apply auto
  done

{ fix ω have last-H ω ∈ H ∨ last-H ω = (THE x. False)
```

apply (*cases* ∀ *i*. ¬ *before-C* {*i*} *ω*)
apply (*simp add*: *last-H-def*)
apply (*auto dest*: *before-C-imp-in-H last-H-eq*)
done }
then have *range-lnc*: *range last-H* ⊆ *H* ∪ {*THE x. False*}
by *auto*

have *sf-lnc*: *simple-function hC last-H*
by (*rule simple-functionI*) (*auto intro*: *finite-subset*[*OF range-lnc*])

have *sd-lnc*: *simple-distributed hC last-H ?l*
apply (*rule hC.simple-distributedI*[*OF sf-lnc*])
apply (*auto intro*!: *T.cond-prob-eq-AE*)
apply (*auto simp*: *space-stream-space*)
using *AE-last-H*
apply *eventually-elim*
apply *auto*
done

have *sd-fj-lnc*: *simple-distributed hC* (*λω.* (*first-J ω*, *last-H ω*)) (*λ*(*i*, *l*). *?il i l*)
apply (*rule hC.simple-distributedI*)
apply (*rule simple-function-Pair*[*OF sf-fj sf-lnc*])
apply (*auto intro*!: *T.cond-prob-eq-AE*)
apply (*auto simp*: *space-stream-space*)
using *AE-last-H AE-first-J*
apply *eventually-elim*
apply *auto*
done

define *c* **where** *c* = (*SOME j. j* ∈ *C*)
have *c*: *c* ∈ *C*
using *C-non-empty* **unfolding** *ex-in-conv*[*symmetric*] *c-def* **by** (*rule someI-ex*)

let *?inner* = *λi.* ∑ *l*∈*H*. *?il i l* ∗ *log 2* (*?il i l* / (*?i i* ∗ *?l l*))
{ **fix** *i* **assume** *i*: *i* ∈ *H*
with *h-pos* **have** *card-idx*: *real-of-nat* (*card* (*H* − {*i*})) = *p-H* / *p-j* − *1*
by (*auto simp add*: *p-j-def p-H-def h-def*)

have *neq0*: *p-j* ≠ *0 p-H* ≠ *0*
unfolding *p-j-def p-H-def*
using *C-smaller i* **by** *auto*

from *i* **have** *?inner i* =
(∑ *l*∈*H* − {*i*}. *?il i l* ∗ *log 2* (*?il i l* / (*?i i* ∗ *?l l*))) +
?il i i ∗ *log 2* (*?il i i* / (*?i i* ∗ *?l i*))
by (*simp add*: *sum-diff*)
also have . . . =
(∑ *l*∈*H* − {*i*}. *p-j*/*p-H* ∗ *p-j* ∗ *p-f* ∗ *log 2* (*p-j* ∗ *p-f* / (*p-j* ∗ *p-f* + *p-j*/*p-H* ∗ (*1* − *p-H* ∗ *p-f*)))) +

p-j/*p-H* ∗ (*p-j* ∗ *p-f* + (*1* − *p-H* ∗ *p-f*)) ∗ *log 2* ((*p-j* ∗ *p-f* + (*1* − *p-H* ∗ *p-f*)) / (*p-j* ∗ *p-f* + *p-j*/*p-H* ∗ (*1* − *p-H* ∗ *p-f*)))
using *i p-f p-j-pos p-H*
apply (*simp add*: *Pr-visit-before-C P-visit init-H Pr-before-C*
del: *sum-constant*)
apply (*simp add*: *divide-simps distrib-left*)
apply (*intro arg-cong2*[**where** *f*=(∗)] *refl arg-cong2*[**where** *f*=*log*])
apply (*auto simp*: *field-simps*)
done
also have ... = (*?f* ∗ *log 2* (*h* ∗ *p-j* ∗ *p-f*) + (*1* − *?f*) ∗ *log 2* ((*1* − *?f*) ∗ *h*)) / *h*
using *neq0 p-f* **by** (*simp add*: *card-idx field-simps* ‹*p-H* = *h* ∗ *p-j*›)
finally have *?inner i* = (*?f* ∗ *log 2* (*h* ∗ *p-j* ∗ *p-f*) + (*1* − *?f*) ∗ *log 2* ((*1* − *?f*) ∗ *h*)) / *h* **.** **}**
then have (∑ *i*∈*H*. *?inner i*) = *?f* ∗ *log 2* (*h* ∗ *p-j* ∗ *p-f*) + (*1* − *?f*) ∗ *log 2* ((*1* − *?f*) ∗ *h*)
using *h-pos* **by** (*simp add*: *h-def*[*symmetric*])
also have ... = *?f* ∗ *log 2* (*p-H* ∗ *p-f*) + (*1* − *?f*) ∗ *log 2* ((*1* − *?f*) ∗ *h*)
by (*simp add*: ‹*h* = *p-H* / *p-j*›)
also have ... ≤ (*1* − *?f*) ∗ *log 2* ((*1* − *?f*) ∗ *h*)
using *log-le-0* **by** *simp*
also have ... ≤ (*1* − *?f*) ∗ *log 2 h*
using *h-pos* ‹*1* ≤ *h*› *1 p-j-pos p-f*
by (*intro mult-left-mono log-mono mult-pos-pos mult-nonneg-nonneg*) *auto*
finally have (∑ *i*∈*H*. *?inner i*) ≤ (*1* − *?f*) ∗ *log 2 h* **.**
also have (∑ *i*∈*H*. *?inner i*) =
(∑ (*i*, *l*)∈(*first-J*‘*space S*) × (*last-H*‘*space S*). *?il i l* ∗ *log 2* (*?il i l* / (*?i i* ∗ *?l l*)))
unfolding *sum.cartesian-product*
proof (*safe intro*!: *sum.mono-neutral-cong-left del*: *DiffE DiffI*)
show *finite* ((*first-J* ‘ *space S*) × (*last-H* ‘ *space S*))
using *sf-fj sf-lnc* **by** (*auto simp add*: *hC-def dest*!: *simple-functionD*(*1*))
next
fix *i* **assume** *i* ∈ *H*
then have *visit* {*i*} *J* (*Init i* ## *Mix i* ## *sconst End*)
before-C {*i*} (*Init i* ## *Mix c* ## *sconst End*)
by (*auto simp*: *before-C-def visit-def suntil-Stream HLD-iff c*)
then show *i* ∈ *first-J* ‘ *space S i* ∈ *last-H* ‘ *space S*
by (*auto simp*: *space-stream-space image-iff eq-commute dest*!: *first-J-eq last-H-eq*)
next
fix *i l* **assume** (*i*, *l*) ∈ *first-J* ‘ *space S* × *last-H* ‘ *space S* − *H* × *H*
then have *H*: *i* ∉ *H* ∨ *l* ∉ *H*
by *auto*
have 𝒫(*ω in* 𝔓. (*visit* {*i*} *J ω* ∧ *before-C* {*l*} *ω*) ∧ *hit-C ω*) = *0*
using *H* **by** (*intro T.prob-eq-0-AE*) (*auto dest*: *visit-imp-in-H before-C-imp-in-H*)
then show *?il i l* ∗ *log 2* (*?il i l* / (*?i i* ∗ *?l l*)) = *0*
by (*simp add*: *cond-prob-def*)
qed

```
  also have ... = ℐ(first-J ; last-H)
    unfolding sum.cartesian-product
   apply (subst hC.mutual-information-simple-distributed[OF sd-fj sd-lnc sd-fj-lnc])
    apply (simp add: hC-def)
  proof (safe intro!: sum.mono-neutral-right imageI)
    show finite ((first-J ‘ space S) × (last-H ‘ space S))
      using sf-fj sf-lnc by (auto simp add: hC-def dest!: simple-functionD(1))
  next
    fix i l assume (first-J i, last-H l) ∉ (λx. (first-J x, last-H x)) ‘ space S
    moreover
    { fix i l assume i ∈ H l ∈ H
      then have visit {i} J (Init i ## Mix l ## Mix c ## sconst End)
        before-C {l} (Init i ## Mix l ## Mix c ## sconst End)
      using c C-smaller by (auto simp: before-C-def visit-def HLD-iff suntil-Stream)
      then have first-J (Init i ## Mix l ## Mix c ## sconst End) = i
        last-H (Init i ## Mix l ## Mix c ## sconst End) = l
        by (auto intro!: first-J-eq last-H-eq) }
    note this[of first-J i last-H l]
    ultimately have (first-J i, last-H l) ∉ H×H
      by (auto simp: space-stream-space image-iff eq-commute) metis
     then have 𝒫(ω in 𝔓. (visit {first-J i} J ω ∧ before-C {last-H l} ω) ∧ hit-C
ω) = 0
      by (intro T.prob-eq-0-AE) (auto dest: visit-imp-in-H before-C-imp-in-H)
    then show ?il (first-J i) (last-H l) *
      log 2 (?il (first-J i) (last-H l) / (?i (first-J i) * ?l (last-H l))) = 0
      by (simp add: cond-prob-def)
  qed
  finally show ?thesis by simp
qed

end

end
```

14 Formalizing the IPv4-address allocation in ZeroConf

```
theory Zeroconf-Analysis
  imports ../Discrete-Time-Markov-Chain
begin

declare UNIV-bool[simp]
```

14.1 Definition of a ZeroConf allocation run

```
datatype zc-state = start
                  | probe nat
                  | ok
```

| *error*

lemma *inj-probe*: *inj-on probe X*
by (*auto simp*: *inj-on-def*)

Countability of *zc-state* simplifies measurability of functions on *zc-state*.

instance *zc-state* :: *countable*
proof
have *countable* ({*start*, *ok*, *error*} ∪ *probe'UNIV*)
by *auto*
also have {*start*, *ok*, *error*} ∪ *probe'UNIV* = *UNIV*
using *zc-state.nchotomy* **by** *auto*
finally show ∃*f*::*zc-state* ⇒ *nat*. *inj f*
using *inj-on-to-nat-on*[*of UNIV* :: *zc-state set*] **by** *auto*
qed

locale *Zeroconf-Analysis* =
fixes *N* :: *nat* **and** *p q r e* :: *real*
assumes *p*: $0 < p$ $p < 1$ **and** *q*: $0 < q$ $q < 1$
assumes *r*[*simp*]: $0 \leq r$ **and** *e*[*simp*]: $0 \leq e$
begin

lemma *p-bounds*[*simp*]: $0 \leq p$ $p \leq 1$
using *p* **by** *auto*

lemma *q-bounds*[*simp*]: $0 \leq q$ $q \leq 1$
using *q* **by** *auto*

abbreviation *states* **where**
states ≡ *probe* ' {.. *N*} ∪ {*start*, *ok*, *error*}

primrec τ :: *zc-state* ⇒ *zc-state pmf* **where**
τ *start* = *map-pmf* (λ*True* ⇒ *probe 0* | *False* ⇒ *ok*) (*bernoulli-pmf q*)
| τ (*probe n*) = *map-pmf* (λ*True* ⇒ (*if n* < *N then probe* (*Suc n*) *else error*) | *False* ⇒ *start*) (*bernoulli-pmf p*)
| τ *ok* = *return-pmf ok*
| τ *error* = *return-pmf error*

primrec ϱ :: *zc-state* ⇒ *zc-state* ⇒ *real* **where**
ϱ *start* = (λ-. *0*) (*probe 0* := *r*, *ok* := *r* ∗ (*N* + *1*))
| ϱ (*probe n*) = (*if n* < *N then* (λ-. *0*) (*probe* (*Suc n*) := *r*) *else* (λ-. *0*) (*error* := *e*))
| ϱ *ok* = (λ-. *0*) (*ok* := *0*)
| ϱ *error* = (λ-. *0*) (*error* := *0*)

lemma *ϱ-nonneg'*[*simp*]: *0* ≤ ϱ *s t*
using *r e* **by** (*cases s*) *auto*

sublocale *MC-with-rewards* τ ϱ λ*s*. *0*

proof qed (*simp-all add*: *pair-measure-countable*)

14.2 The allocation run is a rewarded DTMC

abbreviation $E\ s \equiv set\text{-}pmf\ (\tau\ s)$

lemma *enabled-ok*: $enabled\ ok\ \omega \longleftrightarrow \omega = sconst\ ok$
  by (*simp add*: *enabled-iff-sconst*)

lemma *finite-E*[*intro*, *simp*]: *finite* ($E\ s$)
  by (*cases s*) *auto*

lemma *E-closed*: $s \in states \Longrightarrow E\ s \subseteq states$
  using *p q* **by** (*cases s*) (*auto split*: *bool.splits*)

lemma *enabled-error*: $enabled\ error\ \omega \longleftrightarrow \omega = sconst\ error$
  by (*simp add*: *enabled-iff-sconst*)

lemma *pos-neg-q-pn*: $0 < 1 - q * (1 - p \widehat{\ } Suc\ N)$
proof −
  have $p \widehat{\ } Suc\ N \leq 1 \widehat{\ } Suc\ N$
    using *p* **by** (*intro power-mono*) *auto*
  with *p q* **have** $q * (1 - p \widehat{\ } Suc\ N) < 1 * 1$
    by (*intro mult-strict-mono*) (*auto simp*: *field-simps simp del*: *power-Suc*)
  then show *?thesis* **by** *simp*
qed

lemma *to-error*: **assumes** $n \leq N$ **shows** (*probe n*, *error*) $\in$ *acc*
  using ‹$n \leq N$›
proof (*induction rule*: *inc-induct*)
  case (*step n'*) **with** *p* **show** *?case*
    by (*intro rtrancl-trans*[*OF r-into-rtrancl step.IH*]) *auto*
qed (*insert p*, *auto*)

14.3 Probability of a erroneous allocation

definition $P\text{-}err\ s = \mathcal{P}(\omega\ in\ T\ s.\ ev\ (HLD\ \{error\})\ (s\ \#\#\ \omega))$

lemma *P-err*:
  defines $p\text{-}start == (q * p \widehat{\ } Suc\ N)\ /\ (1 - q * (1 - p \widehat{\ } Suc\ N))$
  defines $p\text{-}probe == (\lambda n.\ p \widehat{\ } Suc\ (N - n) + (1 - p \widehat{\ } Suc\ (N - n)) * p\text{-}start)$
  assumes *s*: $s \in states - \{ok, error\}$
  shows $P\text{-}err\ s = (case\ s\ of\ ok \Rightarrow 0\ |\ error \Rightarrow 1\ |\ probe\ n \Rightarrow p\text{-}probe\ n\ |\ start$
$\Rightarrow p\text{-}start)$
    (**is** $\ldots = ?E\ s$)
  using *s*
proof (*rule unique-les*)
  have [*arith*]: $0 \leq p * (q * p \widehat{\ } N)$
    using *p q* **by** *simp*
  have *p-eq*: $p\text{-}start = p\text{-}probe\ 0 * q$

$\bigwedge n.\ n < N \Longrightarrow p\text{-}probe\ n = p\text{-}probe\ (Suc\ n) * p + p\text{-}start * (1 - p)$
p-probe N = *p* + *p-start* ∗ (*1* − *p*)
using *p q*
by (*auto simp*: *p-probe-def p-start-def power-Suc*[*symmetric*] *Suc-diff-Suc divide-simps*
simp del: *power-Suc*)
(*auto simp*: *field-simps*)
fix *s* **assume** *s*: *s* ∈ *states* − {*ok*, *error*}
then show *?E s* = (∫ *t*. *?E t* ∂τ *s*) + *0*
using *p q* **by** (*auto intro*: *p-eq*)
show ∃ *t*∈{*ok*, *error*}. (*s*, *t*) ∈ *acc*
using *s q to-error* **by** *auto*
from *s* **show** *P-err s* = *integral*^{L} (*measure-pmf* (τ *s*)) *P-err* + *0*
unfolding *P-err-def*[*abs-def*] **by** (*subst prob-T*) (*auto simp*: *ev-Stream simp del*: *UNIV-bool*)
next
fix *s* **assume** *s* ∈ {*ok*, *error*} **then show** *P-err s* = *?E s*
by (*auto intro*!: *T.prob-eq-0-AE T.prob-Collect-eq-1*[*THEN iffD2*]
simp: *P-err-def AE-sconst ev-sconst HLD-iff ev-Stream T.prob-space*
simp del: *space-T sets-T*)
qed (*insert p q*, *auto intro*!: *integrable-measure-pmf-finite split*: *if-split-asm*)

lemma *P-err-start*: *P-err start* = (*q* ∗ *p* ^ *Suc N*) / (*1* − *q* ∗ (*1* − *p* ^ *Suc N*))
by (*simp add*: *P-err*)

14.4 An allocation run terminates almost surely

lemma *states-closed*:
assumes *s* ∈ *states*
assumes (*s*, *t*) ∈ *acc-on* (− {*error*, *ok*})
shows *t* ∈ *states*
using *assms*(*2*,*1*) *p q* **by** *induction* (*auto split*: *if-split-asm*)

lemma *finite-reached*:
assumes *s*: *s* ∈ *states* **shows** *finite* (*acc-on* (− {*error*, *ok*}) ‘‘ {*s*})
using *states-closed*[*OF s*]
by (*rule-tac finite-subset*[*of - states*]) *auto*

lemma *AE-reaches-error-or-ok*:
assumes *s*: *s* ∈ *states*
shows *AE* ω *in T s*. *ev* (*HLD* {*error*, *ok*}) ω
proof (*rule AE-T-ev-HLD*)
{ fix *t* **assume** *t*: (*s*, *t*) ∈ *acc-on* (− {*error*, *ok*})
with *states-closed*[*OF s t*] *to-error p q* **show** ∃ *t'*∈{*error*, *ok*}. (*t*, *t'*) ∈ *acc*
by *auto* **}**
qed (*rule finite-reached*[*OF s*])

14.5 Expected runtime of an allocation run

definition *R s* = (∫$^+$ ω. *reward-until* {*error*, *ok*} *s* ω ∂*T s*)

definition R' *s* = *enn2real* (*R s*)

lemma *R-iter*: $s \neq error \Longrightarrow s \neq ok \Longrightarrow R\ s = (\int^+ t.\ ennreal\ (\varrho\ s\ t) + R\ t\ \partial\tau\ s)$
  unfolding *R-def* **using** *T.emeasure-space-1*
  by (*subst nn-integral-T*)
    (*auto simp del*: *τ.simps ϱ.simps simp add*: *AE-measure-pmf-iff nn-integral-add intro*!: *nn-integral-cong-AE*)

lemma *R-finite*:
  assumes *s*: $s \in states$
  shows $R\ s \neq \infty$
  unfolding *R-def*
proof (*rule nn-integral-reward-until-finite*)
  { fix *t* **assume** $(s, t) \in acc$ **from** *this s p q* **have** $t \in states$
    by *induction* (*auto split*: *if-split-asm*) **}**
  then have acc `` $\{s\} \subseteq states$
   by *auto*
  then show *finite* (acc `` $\{s\}$)
   by (*auto dest*: *finite-subset*)
qed (*auto simp*: *AE-reaches-error-or-ok*[*OF s*])

lemma *R-less-top*: $s \in states \Longrightarrow R\ s < top$
  using *R-finite*[*of s*] **by** (*subst less-top*[*symmetric*]) *simp*

lemma R'*-iter*: **assumes** *s*: $s \in states$ $s \neq error$ $s \neq ok$ **shows** $R'\ s = (\int t.\ \varrho\ s\ t + R'\ t\ \partial\tau\ s)$
  unfolding R'*-def R-iter*[*OF s*(*2*,*3*)]
proof (*rule enn2real-nn-integral-eq-integral*)
  have $t \in E\ s \Longrightarrow R\ t < top$ **for** *t*
   using ‹*s*∈*states*› *E-closed*[*of s*] **by** (*intro R-less-top*) *auto*
  then show *AE t in τ s. ennreal* (*ϱ s t*) + *R t* = *ennreal* (*ϱ s t* + *enn2real* (*R t*))
   by (*auto simp*: *AE-measure-pmf-iff intro*!: *ennreal-enn2real*[*symmetric*])
qed *auto*

lemma *cost-from-start*:
  R' *start* =
   $(q * (r + p\hat{\ }Suc\ N * e + r * p * (1 - p\hat{\ }N) / (1 - p)) + (1 - q) * (r * Suc\ N)) /$
   $(1 - q + q * p\hat{\ }Suc\ N)$
proof −
  have *ok-error*: $R'\ ok = 0 \wedge R'\ error = 0$
   unfolding R'*-def R-def* **by** (*subst* (*1 2*) *reward-until-unfold*[*abs-def*]) *simp*

  then have *R-start*: $R'\ start = q * (r + R'\ (probe\ 0)) + (1 - q) * (r * (N + 1))$
   using *q r* **by** (*subst* R'*-iter*) (*simp-all add*: *field-simps*)

```
  have R-probe: ⋀n. n < N ⟹ R' (probe n) = p * R' (probe (Suc n)) + p * r +
(1 − p) * R' start
    using p r by (subst R'-iter) (simp-all add: field-simps distrib-right)

  have R-N: R' (probe N) = p * e + (1 − p) * R' start
    using p e ok-error by (subst R'-iter) (auto simp: mult.commute )

  { fix n
    assume n ≤ N
    then have R' (probe (N − n)) =
      p ^ Suc n * e + (1 − p^n) * r * p / (1 − p) + (1 − p^Suc n) * R' start
    proof (induct n)
      case 0 with R-N show ?case by simp
    next
      case (Suc n)
      moreover then have Suc (N − Suc n) = N − n by simp
      ultimately show ?case
        using R-probe[of N − Suc n] p by (simp-all add: field-simps Suc)
    qed }
  from this[of N]
  have [simp]: R' (probe 0) = p ^ Suc N * e + (1 − p^N) * r * p / (1 − p) +
(1 − p^Suc N) * R' start
    by simp
  have R' start − q * (1 − p^Suc N) * R' start =
     q * (r + p^Suc N * e + (1 − p^N) * r * p / (1 − p)) + (1 − q) * (r * (N
+ 1))
    by (subst R-start) (simp-all add: field-simps)
  then have R' start = (q * (r + p^Suc N * e + (1 − p^N) * r * p / (1 − p))
+ (1 − q) * (r * Suc N)) /
    (1 − q * (1 − p^Suc N))
    using pos-neg-q-pn by (simp-all add: field-simps)
  then show ?thesis
    by (simp add: field-simps)
qed

end

interpretation ZC: Zeroconf-Analysis 2 16 / 65024 :: real 0.01 0.002 3600
  by standard auto

lemma ZC.P-err start ≤ 1 / 10^12
 unfolding ZC.P-err-start by (simp add: power-divide power-one-over[symmetric])

lemma ZC.R' start ≤ 0.007
 unfolding ZC.cost-from-start by (simp add: power-divide power-one-over[symmetric])

end
```

15 Formalization of the Gossip-Broadcast

theory *Gossip-Broadcast*
  imports *../Discrete-Time-Markov-Chain*
begin

lemma *inj-on-upd-PiE*:
  assumes $i \notin I$ **shows** *inj-on* $(\lambda(x,f).\ f(i := x))\ (M \times (\Pi_E\ i{\in}I.\ A\ i))$
  unfolding *PiE-def*
proof (*safe intro*!: *inj-onI ext*)
  fix $f\ g :: 'a \Rightarrow 'b$ **and** $x\ y :: 'b$
  assume $*$: $f(i := x) = g(i := y)\ f \in \mathit{extensional}\ I\ g \in \mathit{extensional}\ I$
  then show $x = y$ **by** (*auto simp*: *fun-eq-iff split*: *if-split-asm*)
  fix i' **from** $*$ ‹$i \notin I$› **show** $f\ i' = g\ i'$
    by (*cases* $i' = i$) (*auto simp*: *fun-eq-iff extensional-def split*: *if-split-asm*)
qed

lemma *sum-folded-product*:
  fixes $I :: 'i\ set$ **and** $f :: 's \Rightarrow 'i \Rightarrow 'a$::{*semiring-0*, *comm-monoid-mult*}
  assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow$ *finite* $(S\ i)$
  shows $(\sum x{\in}Pi_E\ I\ S.\ \prod i{\in}I.\ f\ (x\ i)\ i) = (\prod i{\in}I.\ \sum s{\in}S\ i.\ f\ s\ i)$
using *assms* **proof** (*induct I*)
  case *empty* **then show** *?case* **by** *simp*
next
  case (*insert i I*)
  have $*$: Pi_E (*insert i I*) $S = (\lambda(x, f).\ f(i := x))$ ‘ $(S\ i \times Pi_E\ I\ S)$
    by (*auto simp*: *PiE-def intro*!: *image-eqI ext dest*: *extensional-arb*)
  have $(\sum x{\in}Pi_E$ (*insert i I*) $S.\ \prod i{\in}$*insert i I*. $f\ (x\ i)\ i) =$
    sum $((\lambda x.\ \prod i{\in}$*insert i I*. $f\ (x\ i)\ i) \circ ((\lambda(x, f).\ f(i := x))))\ (S\ i \times Pi_E\ I\ S)$
    unfolding $*$ **using** *insert* **by** (*intro sum.reindex*) (*auto intro*!: *inj-on-upd-PiE*)
  also have $\ldots = (\sum (a, x){\in}(S\ i \times Pi_E\ I\ S).\ f\ a\ i * (\prod i{\in}I.\ f\ (x\ i)\ i))$
    using *insert* **by** (*force intro*!: *sum.cong prod.cong arg-cong2*[**where** $f{=}(*)$])
  also have $\ldots = (\sum a{\in}S\ i.\ f\ a\ i * (\sum x{\in}Pi_E\ I\ S.\ \prod i{\in}I.\ f\ (x\ i)\ i))$
    by (*simp add*: *sum.cartesian-product sum-distrib-left*)
  finally show *?case*
    using *insert* **by** (*simp add*: *sum-distrib-right*)
qed

15.1 Definition of the Gossip-Broadcast

datatype *state* = *listening* | *sending* | *sleeping*

type-synonym *sys-state* = (*nat* × *nat*) ⇒ *state*

lemma *state-UNIV*: *UNIV* = {*listening*, *sending*, *sleeping*}
  by (*auto intro*: *state.exhaust*)

locale *gossip-broadcast* =
  fixes *size* :: *nat* **and** *p* :: *real*
  assumes *size*: $0 <$ *size*

assumes *p*: *0* < *p* *p* < *1*
begin

interpretation *pmf-as-function* **.**

definition *states* :: *sys-state set* **where**
states = ({..< *size*} × {..< *size*}) $\rightarrow_E$ {*listening*, *sending*, *sleeping*}

definition *start* :: *sys-state* **where**
start = (λ*x*∈{..< *size*}×{..< *size*}. *listening*)((*0*, *0*) := *sending*)

definition *neighbour-sending* **where**
neighbour-sending s = (λ(*x*,*y*).
(*x* > *0* ∧ *s* (*x* − *1*, *y*) = *sending*) ∨
(*x* < *size* ∧ *s* (*x* + *1*, *y*) = *sending*) ∨
(*y* > *0* ∧ *s* (*x*, *y* − *1*) = *sending*) ∨
(*y* < *size* ∧ *s* (*x*, *y* + *1*) = *sending*))

definition *node-trans* :: *sys-state* ⇒ (*nat* × *nat*) ⇒ *state* ⇒ *state* ⇒ *real* **where**
node-trans g x s = (*case s of*
listening ⇒ (*if neighbour-sending g x*
then (λ-.*0*) (*sending* := *p*, *sleeping* := *1* − *p*)
else (λ-.*0*) (*listening* := *1*))
| *sending* ⇒ (λ-.*0*) (*sleeping* := *1*)
| *sleeping* ⇒ (λ-.*0*) (*sleeping* := *1*))

lemma *node-trans-sum-eq-1* [*simp*]:
node-trans g x s′ listening + (*node-trans g x s′ sending* + *node-trans g x s′ sleeping*) = *1*
by (*simp add*: *node-trans-def split*: *state.split*)

lemma *node-trans-nonneg*[*simp*]: *0* ≤ *node-trans s x i j*
using *p* **by** (*auto simp*: *node-trans-def split*: *state.split*)

lift-definition *proto-trans* :: *sys-state* ⇒ *sys-state pmf* **is**
λ*s s′*. *if s′* ∈ *states then* (∏*x*∈{..< *size*}×{..< *size*}. *node-trans s x* (*s x*) (*s′ x*)) *else 0*
proof
let *?f* = λ*s s′*. *if s′* ∈ *states then* (∏*x*∈{..< *size*}×{..< *size*}. *node-trans s x* (*s x*) (*s′ x*)) *else 0*
fix *s* **show** ∀*t*. *0* ≤ *?f s t*
using *p* **by** (*auto intro*!: *prod-nonneg simp*: *node-trans-def split*: *state.split*)
show (∫⁺*t*. *?f s t* ∂*count-space UNIV*) = *1*
apply (*subst nn-integral-count-space′*[*of states*])
apply (*simp-all add*: *prod-nonneg*)
proof −
show (∑*x*∈*states*. ∏*xa*∈{..<*size*} × {..<*size*}. *node-trans s xa* (*s xa*) (*x xa*)) = *1*
unfolding *states-def* **by** (*subst sum-folded-product*) *simp-all*

show *finite states*
by (*auto simp*: *states-def intro*!: *finite-PiE*)
qed
qed

end

15.2 The Gossip-Broadcast forms a DTMC

sublocale *gossip-broadcast* $\subseteq$ *MC-syntax proto-trans* **.**

end

16 Certification of Reachability Problems on MDPs

theory *MDP-RP-Certification*
imports
../MDP-Reachability-Problem
HOL−Library.IArray
HOL−Library.Code-Target-Numeral
begin

context *Reachability-Problem*
begin

lemma *p-ub′*:
fixes *x*
assumes *1*: $s \in S$ $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow (\sum t{\in}S.\ pmf\ D\ t * x\ t) \leq x\ s$
assumes *2*: $\bigwedge s.\ s \in S1 \Longrightarrow x\ s \neq 0 \Longrightarrow (\exists\, t{\in}S2.\ (s,\ t) \in (SIGMA\ s{:}S1.\ \bigcup D{\in}K\ s.\ set\text{-}pmf\ D)^*)$
assumes *3*: $\bigwedge s.\ s \in S - S1 - S2 \Longrightarrow x\ s = 0$
assumes *4*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows *enn2real* ($p\ s$) $\leq x\ s$
proof (*rule p-ub*[*OF 1 - 4*])
fix *s* **assume** $s \in S$ $p\ s = 0$ **with** *2*[*of s*] *p-pos*[*of s*] *p-S2*[*of s*] *3*[*of s*] **show** $x\ s = 0$
by (*cases* $x\ s = 0$) *auto*
qed

lemma *n-lb′*:
fixes *x*
assumes *wf R*
assumes *1*: $s \in S$ $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \leq (\sum t{\in}S.\ pmf\ D\ t * x\ t)$
assumes *2*: $\bigwedge s\ D.\ s \in S1 \Longrightarrow D \in K\ s \Longrightarrow x\ s \neq 0 \Longrightarrow \exists\, t{\in}D.\ ((t,\ s) \in R \wedge t \in S1 \wedge x\ t \neq 0) \vee t \in S2$
assumes *3*: $\bigwedge s.\ s \in S - S1 - S2 \Longrightarrow x\ s = 0$
assumes *4*: $\bigwedge s.\ s \in S2 \Longrightarrow x\ s = 1$
shows $x\ s \leq$ *enn2real* ($n\ s$)
proof (*rule n-lb*[*OF 1 - 4*])

fix *s* **assume** ∗: $s \in S\ n\ s = 0$
show $x\ s = 0$
proof (*rule ccontr*)
assume $x\ s \neq 0$
with ∗ *n-S2*[*of s*] *n-nS12*[*of s*] *3*[*of s*] **have** $s \in S1$
by (*metis DiffI zero-neq-one*)
have $0 < n\ s$
by (*intro n-pos*[*of* $\lambda s.\ x\ s \neq 0$, *OF* ‹$x\ s \neq 0$› ‹$s \in S1$› ‹*wf R*›])
(*metis zero-less-one n-S2 2*)
with ‹$n\ s = 0$› **show** *False* **by** *auto*
qed
qed

end

no-notation *Stream.snth* (**infixl** ‹!!› *100*) — we use !! for IArray

16.1 Computable representation

record *mdp-reachability-problem* =
state-count :: *nat*
distrs :: (*nat* × *rat*) *list list iarray*
states1 :: *bool iarray*
states2 :: *bool iarray*

record *'a RP-sub-cert* =
solution :: *rat iarray*
witness :: (*'a* × *nat*) *iarray*

record *RP-cert* =
pos-cert :: (*nat* × *nat*) *RP-sub-cert*
neg-cert :: *nat list RP-sub-cert*

definition *sparse-mult sx y* = *sum-list* (*map* (λ(*n*, *x*). *x* ∗ *y* !! *n*) *sx*)

primrec *lookup* **where**
lookup d [] *x* = *d*
| *lookup d* (*y*#*ys*) *x* = (*if fst y* = *x then snd y else lookup d ys x*)

lemma *lookup-eq-map-of*: *lookup d xs x* = (*case map-of xs x of Some x* ⇒ *x* | *None* ⇒ *d*)
by (*induct xs*) *simp-all*

lemma *lookup-in-set*:
distinct (*map fst xs*) ⟹ $x \in$ *set xs* ⟹ *lookup d xs* (*fst x*) = *snd x*
unfolding *lookup-eq-map-of* **by** (*subst map-of-is-SomeI*[**where** *y*=*snd x*]) *simp-all*

lemma *lookup-not-in-set*:
$x \notin$ *fst* ‘ *set xs* ⟹ *lookup d xs x* = *d*

unfolding *lookup-eq-map-of*
by (*subst map-of-eq-None-iff*[*of xs x*, *THEN iffD2*]) *auto*

lemma *lookup-nonneg*:
(⋀*x v*. (*x*, *v*) ∈ *set xs* ⟹ *0* ≤ *v*) ⟹ (*0*::′*a*::*ordered-comm-monoid-add*) ≤ *lookup 0 xs x*
apply (*induction xs*)
apply *simp*
apply *force*
done

lemma *sparse-mult-eq-sum-lookup*:
fixes *xs* :: (*nat* × ′*a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)
shows *sparse-mult xs y* = (∑ *i*<*M*. *lookup 0 xs i* ∗ *y* !! *i*)
proof −
from ‹*distinct* (*map fst xs*)› **have** *distinct xs inj-on fst* (*set xs*)
by (*simp-all add*: *distinct-map*)
then have *sparse-mult xs y* = (∑ *x*∈*set xs*. *snd x* ∗ *y* !! *fst x*)
by (*auto intro*!: *sum.cong simp add*: *sparse-mult-def sum-list-distinct-conv-sum-set*)
also have . . . = (∑ *x*∈*set xs*. *lookup 0 xs* (*fst x*) ∗ *y* !! *fst x*)
by (*intro sum.cong refl arg-cong2*[**where** *f*=(∗)]) (*simp add*: *lookup-in-set assms*)
also have . . . = (∑ *x*∈*fst* ‘ *set xs*. *lookup 0 xs x* ∗ *y* !! *x*)
using ‹*inj-on fst* (*set xs*)› **by** (*simp add*: *sum.reindex*)
also have . . . = (∑ *x*<*M*. *lookup 0 xs x* ∗ *y* !! *x*)
using *assms*(*1*)
by (*intro sum.mono-neutral-cong-left*)
(*auto simp*: *list-all-iff lookup-eq-map-of map-of-eq-None-iff*[*THEN iffD2*])
finally show *?thesis* **.**
qed

lemma *sum-list-eq-sum-lookup*:
fixes *xs* :: (*nat* × ′*a*::*comm-semiring-1*) *list*
assumes *list-all* (λ(*n*, *x*). *n* < *M*) *xs distinct* (*map fst xs*)
shows *sum-list* (*map snd xs*) = (∑ *i*<*M*. *lookup 0 xs i*)
proof −
from ‹*distinct* (*map fst xs*)› **have** *distinct xs inj-on fst* (*set xs*)
by (*simp-all add*: *distinct-map*)
then have *sum-list* (*map snd xs*) = (∑ *x*∈*set xs*. *snd x*)
by (*auto intro*!: *sum.cong simp add*: *sparse-mult-def sum-list-distinct-conv-sum-set*)
also have . . . = (∑ *x*∈*set xs*. *lookup 0 xs* (*fst x*))
by (*intro sum.cong refl arg-cong2*[**where** *f*=(∗)]) (*simp add*: *lookup-in-set assms*)
also have . . . = (∑ *x*∈*fst* ‘ *set xs*. *lookup 0 xs x*)
using ‹*inj-on fst* (*set xs*)› **by** (*simp add*: *sum.reindex*)
also have . . . = (∑ *x*<*M*. *lookup 0 xs x*)
using *assms*(*1*)
by (*intro sum.mono-neutral-cong-left*)

(*auto simp*: *list-all-iff lookup-eq-map-of map-of-eq-None-iff* [*THEN iffD2*])
finally show *?thesis* **.**
qed

definition
valid-mdp-rp mdp ⟷
$0 <$ *state-count mdp* ∧
IArray.length (*distrs mdp*) = *state-count mdp* ∧
IArray.length (*states1 mdp*) = *state-count mdp* ∧
IArray.length (*states2 mdp*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*. ¬ (*states1 mdp* !! *i* ∧ *states2 mdp* !! *i*) ∧
list-all (λ*ds*. *distinct* (*map fst ds*) ∧ *list-all* (λ(*n*, *x*). $0 \leq x \wedge n <$ *state-count mdp*) *ds* ∧
sum-list (*map snd ds*) = *1*) (*distrs mdp* !! *i*) ∧
¬ *List.null* (*distrs mdp* !! *i*))

definition
valid-sub-cert mdp c ord check ⟷
IArray.length (*witness c*) = *state-count mdp* ∧
IArray.length (*solution c*) = *state-count mdp* ∧
(∀ *i*<*state-count mdp*.
if states2 mdp !! *i then solution c* !! *i* = *1*
else if states1 mdp !! *i then* *0* ≤ *solution c* !! *i* ∧
(*list-all* (λ*ds*. *ord* (*sparse-mult ds* (*solution c*)) (*solution c* !! *i*)) (*distrs mdp* !! *i*)) ∧
(*0* < *solution c* !! *i* ⟶ *check* (*distrs mdp* !! *i*) (*witness c* !! *i*))
else solution c !! *i* = *0*)

definition
valid-pos-cert mdp c ⟷
valid-sub-cert mdp c (≤)
(λ*D* ((*j*, *a*), *n*). *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧ *0* < *solution c* !! *j* ∧
a < *length D* ∧ *lookup 0* (*D* ! *a*) *j* ≠ *0*)

definition
valid-neg-cert mdp c ⟷
valid-sub-cert mdp c (≥)
(λ*D* (*J*, *n*). *list-all2* (λ*j d*. *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧
lookup 0 d j ≠ *0* ∧ *0* < *solution c* !! *j*) *J D*)

definition
valid-cert mdp c ⟷ *valid-pos-cert mdp* (*pos-cert c*) ∧ *valid-neg-cert mdp* (*neg-cert c*)

lemma *valid-mdp-rpD-length*:
assumes *valid-mdp-rp mdp*
shows *0* < *state-count mdp IArray.length* (*distrs mdp*) = *state-count mdp*
IArray.length (*states1 mdp*) = *state-count mdp IArray.length* (*states2 mdp*) =

state-count mdp
using *assms* **by** (*auto simp*: *valid-mdp-rp-def*)

lemma *valid-mdp-rpD*:
assumes *valid-mdp-rp mdp i* < *state-count mdp*
shows ¬ (*states1 mdp* !! *i* ∧ *states2 mdp* !! *i*)
and ⋀*ds n x*. *ds* ∈ *set* (*distrs mdp* !! *i*) ⟹ (*n*, *x*) ∈ *set ds* ⟹ *n* < *state-count mdp*
and ⋀*ds n x*. *ds* ∈ *set* (*distrs mdp* !! *i*) ⟹ (*n*, *x*) ∈ *set ds* ⟹ *0* ≤ *x*
and ⋀*ds*. *ds* ∈ *set* (*distrs mdp* !! *i*) ⟹ *sum-list* (*map snd ds*) = *1*
and ⋀*ds*. *ds* ∈ *set* (*distrs mdp* !! *i*) ⟹ *distinct* (*map fst ds*)
and *distrs mdp* !! *i* ≠ []
using *assms* **by** (*auto simp*: *valid-mdp-rp-def list-all-iff List.null-def elim*!: *allE*[*of* - *i*])

lemma *valid-mdp-rp-sparse-mult*:
assumes *valid-mdp-rp mdp i* < *state-count mdp ds* ∈ *set* (*distrs mdp* !! *i*)
shows *sparse-mult ds y* = (∑ *i*<*state-count mdp*. *lookup 0 ds i* ∗ *y* !! *i*)
using *valid-mdp-rpD*(*2*,*5*)[*OF assms*] **by** (*intro sparse-mult-eq-sum-lookup*) (*auto simp*: *list-all-iff*)

lemma *valid-sub-certD*:
assumes *valid-mdp-rp mdp valid-sub-cert mdp c ord check i* < *state-count mdp*
shows ¬ *states1 mdp* !! *i* ⟹ ¬ *states2 mdp* !! *i* ⟹ *solution c* !! *i* = *0*
and *states2 mdp* !! *i* ⟹ *solution c* !! *i* = *1*
and *states1 mdp* !! *i* ⟹ *0* ≤ *solution c* !! *i*
and ⋀*ds*. *states1 mdp* !! *i* ⟹ *ds* ∈ *set* (*distrs mdp* !! *i*) ⟹ *ord* (*sparse-mult ds* (*solution c*)) (*solution c* !! *i*)
and ⋀*ds*. *states1 mdp* !! *i* ⟹ *0* < *solution c* !! *i* ⟶ *check* (*distrs mdp* !! *i*) (*witness c* !! *i*)
using *assms*(*2*,*3*) *valid-mdp-rpD*(*1*)[*OF assms*(*1*,*3*)]
by (*auto simp add*: *valid-sub-cert-def list-all-iff*)

lemma *valid-pos-certD*:
assumes *valid-mdp-rp mdp valid-pos-cert mdp c i* < *state-count mdp states1 mdp* !! *i*
0 < *solution c* !! *i witness c* !! *i* = ((*j*, *a*), *n*)
shows *snd* (*witness c* !! *j*) < *n* ∧ *j* < *state-count mdp* ∧ *a* < *length* (*distrs mdp* !! *i*) ∧
lookup 0 ((*distrs mdp* !! *i*) ! *a*) *j* ≠ *0* ∧ *0* < *solution c* !! *j*
using *valid-sub-certD*(*5*)[*OF assms*(*1*) *assms*(*2*)[*unfolded valid-pos-cert-def*] *assms*(*3*,*4*)] *assms*(*5*−) **by** *auto*

lemma *valid-neg-certD*:
assumes *valid-mdp-rp mdp valid-neg-cert mdp c i* < *state-count mdp states1 mdp* !! *i*
0 < *solution c* !! *i witness c* !! *i* = (*js*, *n*)
shows *list-all2* (λ*j ds*. *j* < *state-count mdp* ∧ *snd* (*witness c* !! *j*) < *n* ∧ *lookup 0 ds j* ≠ *0* ∧ *0* < *solution c* !! *j*) *js* (*distrs mdp* !! *i*)

using *valid-sub-certD*(*5*)[*OF assms*(*1*) *assms*(*2*)[*unfolded valid-neg-cert-def*] *assms*(*3*)] *assms*(*4* −) **by** *auto*

context
  fixes *mdp c*
  assumes *rp*: *valid-mdp-rp mdp*
  assumes *cert*: *valid-cert mdp c*
begin

interpretation *pmf-as-function* **.**

abbreviation $S \equiv \{..< \textit{state-count mdp}\}$
abbreviation $S1 \equiv \{i.\ i < \textit{state-count mdp} \wedge (\textit{states1 mdp})\ !!\ i\}$
abbreviation $S2 \equiv \{i.\ i < \textit{state-count mdp} \wedge (\textit{states2 mdp})\ !!\ i\}$

lift-definition *K* :: *nat* ⇒ *nat pmf set* **is**
  $\lambda i.$ *if* $i <$ *state-count mdp then*
    { (λ*j*. *of-rat* (*lookup 0 D j*) :: *real*) | *D*. *D* ∈ *set* (*distrs mdp* !! *i*) }
    else { *indicator* {*0*} }
proof (*auto split*: *if-split-asm simp del*: *IArray.sub-def*)
  fix *n D* **assume** *n*: *n* < *state-count mdp* **and** *D*: *D* ∈ *set* (*distrs mdp* !! *n*)
  from *valid-mdp-rpD*(*3*)[*OF rp this*] **show** *nn*: ⋀*i*. *0* ≤ *lookup 0 D i*
    by (*auto simp add*: *lookup-eq-map-of split*: *option.split dest*: *map-of-SomeD*)
  show ($\int^+$ *x*. *ennreal* (*real-of-rat* (*lookup 0 D x*)) ∂*count-space UNIV*) = *1*
    using *valid-mdp-rpD*(*2*,*3*,*4*,*5*)[*OF rp n D*]
    apply (*subst nn-integral-count-space'*[*of* {..< *state-count mdp*}])
    apply (*auto intro*: *nn lookup-not-in-set simp*: *of-rat-sum*[*symmetric*] *lookup-nonneg*)
    apply (*subst sum-list-eq-sum-lookup*[*symmetric*])
    apply (*auto simp*: *list-all-iff lookup-eq-map-of split*: *option.split*)
    done
next
  show ($\int^+$ *x*. *ennreal* (*indicator* {*0*} *x*) ∂*count-space UNIV*) = *1*
    by (*subst nn-integral-count-space'*[*of* {*0*}]) *auto*
qed

interpretation *MDP*: *Reachability-Problem K S S1 S2*
proof
  show *S1* ∩ *S2* = {} *S1* ⊆ *S S2* ⊆ *S*
    using *valid-mdp-rpD*(*1*)[*OF rp*] **by** *auto*
  show *finite S S* ≠ {}
    using ‹*valid-mdp-rp mdp*› **by** (*auto simp add*: *valid-mdp-rp-def*)
  show ⋀*s*. *K s* ≠ {}
    using *valid-mdp-rpD*(*6*)[*OF rp*] **by** *transfer simp*
  show ⋀*s*. *finite* (*K s*)
    by *transfer simp*

  fix *s* **assume** *s* ∈ *S* **then show** (⋃*D*∈*K s*. *set-pmf D*) ⊆ *S*
    using *valid-mdp-rpD*(*2*)[*OF rp*]
    by *transfer* (*auto simp*: *lookup-eq-map-of split*: *option.splits dest*!: *map-of-SomeD*)

qed

definition *P-max s* = *enn2real* (*MDP.p s*)
definition *P-min s* = *enn2real* (*MDP.n s*)

lemma
  assumes *i* < *state-count mdp*
  shows *P-max*: *P-max i* ≤ *real-of-rat* (*solution* (*pos-cert c*) !! *i*) (**is** *?max*)
    and *P-min*: *P-min i* ≥ *real-of-rat* (*solution* (*neg-cert c*) !! *i*) (**is** *?min*)
proof −
  have *valid-pos-cert mdp* (*pos-cert c*) *valid-neg-cert mdp* (*neg-cert c*)
    using ‹*valid-cert mdp c*› **by** (*auto simp*: *valid-cert-def*)
  note *pos* = *this*(*1*)[*unfolded valid-pos-cert-def*] **and** *neg* = *this*(*2*)[*unfolded valid-neg-cert-def*]

  let *?x* = λ*s*. *real-of-rat* (*solution* (*pos-cert c*) !! *s*)
  have *enn2real* (*MDP.p i*) ≤ *?x i*
  proof (*rule MDP.p-ub′*)
    show *i* ∈ *S* **using** *assms* **by** *simp*
  next
    fix *s D* **assume** *s* ∈ *S1 D* ∈ *K s*
    then obtain *j* **where** *j*: *j* < *length* (*distrs mdp* !! *s*)
      ⋀*i*. *i* < *state-count mdp* ⟹ *pmf D i* = *real-of-rat* (*lookup 0* (*distrs mdp* !! *s*
! *j*) *i*)
      by *transfer* (*auto simp*: *in-set-conv-nth*)
    with *valid-sub-certD*(*4*)[*OF* ‹*valid-mdp-rp mdp*› *pos*, *of s distrs mdp* !! *s* ! *j*] ‹*s*
∈ *S1*›
         valid-mdp-rp-sparse-mult[*OF* ‹*valid-mdp-rp mdp*›, *of s distrs mdp* !! *s* ! *j*
solution (*pos-cert c*)]
    show ($\sum$ *t*∈*S*. *pmf D t* ∗ *?x t*) ≤ *?x s*
      by (*simp add*: *of-rat-mult*[*symmetric*] *of-rat-sum*[*symmetric*] *of-rat-less-eq j*)
  next
    fix *s a* **assume** *s* ∈ *S2* **then show** *?x s* = *1*
      using *valid-sub-certD*[*OF* ‹*valid-mdp-rp mdp*› *pos*] **by** *simp*
  next
    fix *s* **define** *X* **where** *X* = (*SIGMA s*:*S1*. ⋃*D*∈*K s*. *set-pmf D*)
    assume *s* ∈ *S1 ?x s* ≠ *0*
    with *valid-sub-certD*(*3*)[*OF rp pos*, *of s*]
    have *0* < *?x s*
      by *simp*
    with ‹*s*∈*S1*› **show** ∃ *t*∈*S2*. (*s*, *t*) ∈ X^*
    proof (*induction n*≡*snd* (*witness* (*pos-cert c*) !! *s*) *arbitrary*: *s rule*: *less-induct*)
      case (*less s*)
      obtain *t a n* **where** *eq*: *witness* (*pos-cert c*) !! *s* = ((*t*, *a*), *n*)
        by (*metis prod.exhaust*)
         from *valid-pos-certD*[*OF rp* ‹*valid-pos-cert mdp* (*pos-cert c*)› - - - *this*]
less.prems
      have *ord*: *snd* (*witness* (*pos-cert c*) !! *t*) < *snd* (*witness* (*pos-cert c*) !! *s*)
        and *t*: *lookup 0* (*distrs mdp* !! *s* ! *a*) *t* ≠ *0 0* < *?x t t*∈*S a* < *length* (*distrs*
mdp !! *s*)

```
        unfolding eq by auto
      with ‹s∈S1› have X: (s, t) ∈ X
        unfolding X-def
        by (transfer fixing: s t a c)
          (auto simp: X-def in-set-conv-nth
                intro!: exI[of - λj. real-of-rat (lookup 0 (distrs mdp !! s ! a) j)]
                        exI[of - distrs mdp !! s ! a] exI[of - a])
      show ?case
      proof cases
        assume t ∈ S1
        with less.hyps[OF ord - ‹0 < ?x t›] X show ?thesis
          by auto
      next
        assume t ∉ S1
        with valid-sub-certD[OF ‹valid-mdp-rp mdp› pos, of t] ‹0 < ?x t› ‹t∈S›
        have t ∈ S2
          by auto
        with X show ?thesis
          by auto
      qed
    qed
  next
    fix s assume s ∈ S − S1 − S2 then show ?x s = 0
      using valid-sub-certD(1)[OF ‹valid-mdp-rp mdp› pos, of s] by simp
  qed
  then show ?max
    by (simp add: P-max-def)

  let ?x = λs. real-of-rat (solution (neg-cert c) !! s)
  have ?x i ≤ enn2real (MDP.n i)
  proof (rule MDP.n-lb′)
    show i ∈ S using assms by simp
  next
    fix s D assume s ∈ S1 D ∈ K s
    then obtain j where j: j < length (distrs mdp !! s)
      ⋀i. i < state-count mdp ⟹ pmf D i = real-of-rat (lookup 0 (distrs mdp !! s
! j) i)
      by transfer (auto simp: in-set-conv-nth)
    with valid-sub-certD(4)[OF ‹valid-mdp-rp mdp› neg, of s distrs mdp !! s ! j] ‹s
∈ S1›
        valid-mdp-rp-sparse-mult[OF ‹valid-mdp-rp mdp›, of s distrs mdp !! s ! j
solution (neg-cert c)]
    show ?x s ≤ (∑t∈S. pmf D t ∗ ?x t)
      by (simp add: of-rat-mult[symmetric] of-rat-sum[symmetric] of-rat-less-eq j)
  next
    fix s a assume s ∈ S2 then show ?x s = 1
      using valid-sub-certD[OF ‹valid-mdp-rp mdp› neg] by simp
  next
     show wf ((S × S ∩ {(s, t). snd (witness (neg-cert c) !! t) < snd (witness
```

(*neg-cert c*) !! *s*)})$^{-1}$) (**is** *wf ?F*)
using *MDP.S-finite*
by (*intro finite-acyclic-wf-converse acyclicI-order*[**where** *f*=λ*s*. *snd* (*witness* (*neg-cert c*) !! *s*)]) *auto*

fix *s D* **assume** *2*: *s* ∈ *S1 D* ∈ *K s* **and** *?x s* ≠ *0*
then have *0* < *?x s*
using *valid-sub-certD*(*3*)[*OF* ‹*valid-mdp-rp mdp*› *neg*, *of s*] **by** *auto*

from *2* **obtain** *a* **where** *a*: *a* < *length* (*distrs mdp* !! *s*)
⋀*i*. *i* < *state-count mdp* ⟹ *pmf D i* = *real-of-rat* (*lookup 0* (*distrs mdp* !! *s* ! *a*) *i*)
by *transfer* (*auto simp*: *in-set-conv-nth*)

obtain *js n* **where** *eq*: *witness* (*neg-cert c*) !! *s* = (*js*, *n*)
by (*metis prod.exhaust*)
from *valid-neg-certD*[*OF* ‹*valid-mdp-rp mdp*› ‹*valid-neg-cert mdp* (*neg-cert c*)› - - - *eq*] *a* ‹*s* ∈ *S1*› ‹*0* < *?x s*›
have ∗: *length js* = *length* (*distrs mdp* !! *s*) *js* ! *a* ∈ *S*
snd (*witness* (*neg-cert c*) !! (*js* ! *a*)) < *snd* (*witness* (*neg-cert c*) !! *s*)
lookup 0 (*distrs mdp* !! *s* ! *a*) (*js* ! *a*) ≠ *0*
0 < *?x* (*js* ! *a*)
unfolding *eq* **by** (*auto dest*: *list-all2-nthD2 list-all2-lengthD*)
with *a* ‹*s* ∈ *S1*› **have** *js-a*: *js* ! *a* ∈ *D* (*js* ! *a*, *s*) ∈ *?F*
by (*auto simp*: *set-pmf-iff*)

show ∃ *t*∈*D*. (*t*, *s*) ∈ *?F* ∧ *t* ∈ *S1* ∧ *?x t* ≠ *0* ∨ *t* ∈ *S2*
proof *cases*
assume *js* ! *a* ∈ *S1* **with** *js-a* ‹*0* < *?x* (*js* ! *a*)› **show** *?thesis* **by** *auto*
next
assume *js* ! *a* ∉ *S1*
with ‹*0* < *?x* (*js* ! *a*)› ‹*js*!*a* ∈ *S*› *valid-sub-certD*[*OF rp neg*, *of js* ! *a*]
have *js* ! *a* ∈ *S2*
by (*auto simp*: *less-le*)
with ‹*js* ! *a* ∈ *D*› **show** *?thesis*
by *auto*
qed
next
fix *s* **assume** *s* ∈ *S* − *S1* − *S2* **then show** *?x s* = *0*
using *valid-sub-certD*(*1*)[*OF* ‹*valid-mdp-rp mdp*› *neg*, *of s*] **by** *simp*
qed
then show *?min*
by (*simp add*: *P-min-def*)
qed

end

end

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