Markov Decision Processes with Rewards

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Abstract

We present a formalization of Markov Decision Processes with rewards. In particular we first build on Hölzl's formalization [1] of MDPs and extend them with rewards. We proceed with an analysis of the expected total discounted reward criterion for infinite horizon MDPs. The central result is the construction of the iteration rule for the Bellman operator. We prove the optimality equations for this operator and show the existence of an optimal stationary deterministic solution. The analysis can be used to obtain dynamic programming algorithms such as value iteration and policy iteration to solve Markov Decision Processes with formal guarantees. Our formalization is based upon chapters 5 and 6 in Puterman's book [2].

Contents

1 Bounded Functions

theory *Bounded-Functions*
imports
HOL.Topological-Spaces
HOL−Analysis.Uniform-Limit
HOL−Probability.Probability
begin

1.1 Definition

definition *bfun* = $\{f.$ *bounded* (*range f*)$\}$

typedef (**overloaded**) ('*a*, '*b*) *bfun* ((- $\Rightarrow_b$ -) [*22*] *21*) =
bfun::('*a* $\Rightarrow$ '*b* :: *metric-space*) *set*
morphisms *apply-bfun Bfun*
⟨*proof*⟩

declare [[*coercion apply-bfun* :: ('*a* $\Rightarrow_b$ ('*b* :: *metric-space*)) $\Rightarrow$ '*a* $\Rightarrow$ '*b*]]

setup-lifting *type-definition-bfun*

lemma *bounded-apply-bfun*[*intro*, *simp*]: *bounded* (*range* (*apply-bfun x*))
⟨*proof*⟩

lemma *bfun-eqI*[*intro*]: ($\bigwedge x.$ *apply-bfun f x* = *apply-bfun g x*) $\Longrightarrow$ *f* = *g*
⟨*proof*⟩

lemma *bfun-eqD*[*dest*]: *f* = *g* $\Longrightarrow$ ($\bigwedge x.$ *apply-bfun f x* = *apply-bfun g x*)
⟨*proof*⟩

lemma *bfunE*:
assumes *f* $\in$ *bfun*
obtains *g* **where** *f* = *apply-bfun g*
⟨*proof*⟩

lemma *const-bfun*: ($\lambda x.$ *b*) $\in$ *bfun*
⟨*proof*⟩

lift-definition *const-bfun*::$'b \Rightarrow ('a \Rightarrow_b ('b :: metric\text{-}space))$ **is** $\lambda(c::'b)$ -. c
⟨*proof*⟩

lemma *bounded-dist-le-SUP-dist*:
$bounded\ (range\ f) \Longrightarrow bounded\ (range\ g) \Longrightarrow dist\ (f\ x)\ (g\ x) \leq (SUP\ x.\ dist\ (f\ x)\ (g\ x))$
⟨*proof*⟩

instantiation *bfun* :: (*type*, *metric-space*) *metric-space*
begin

lift-definition *dist-bfun* :: $('a \Rightarrow_b 'b) \Rightarrow ('a \Rightarrow_b 'b) \Rightarrow real$
is $\lambda f\ g.\ (SUP\ x.\ dist\ (f\ x)\ (g\ x))$ ⟨*proof*⟩

definition *uniformity-bfun* :: $(('a \Rightarrow_b 'b) \times 'a \Rightarrow_b 'b)\ filter$
where *uniformity-bfun* = $(INF\ e{\in}\{0 <..\}.\ principal\ \{(x,\ y).\ dist\ x\ y < e\})$

definition *open-bfun* :: $('a \Rightarrow_b 'b)\ set \Rightarrow bool$
where *open-bfun* $S = (\forall\, x{\in}S.\ \forall_F\ (x',\ y)\ in\ uniformity.\ x' = x \longrightarrow y \in S)$

lemma *dist-bounded*:
fixes $f\ g :: 'a \Rightarrow_b 'b$
shows $dist\ (f\ x)\ (g\ x) \leq dist\ f\ g$
⟨*proof*⟩

lemma *dist-bound*:
fixes $f\ g :: 'a \Rightarrow_b ('b :: metric\text{-}space)$
assumes $\bigwedge x.\ dist\ (f\ x)\ (g\ x) \leq b$
shows $dist\ f\ g \leq b$
⟨*proof*⟩

lemma *dist-fun-lt-imp-dist-val-lt*:
fixes $f\ g :: 'a \Rightarrow_b 'b$
assumes $dist\ f\ g < e$
shows $dist\ (f\ x)\ (g\ x) < e$
⟨*proof*⟩

instance
⟨*proof*⟩

end

lift-definition *PiC*::$'a\ set \Rightarrow ('a \Rightarrow ('b :: metric\text{-}space)\ set) \Rightarrow ('a \Rightarrow_b 'b)\ set$

is $\lambda I\ X.\ Pi\ I\ X \cap bfun$
⟨*proof*⟩

lemma *mem-PiC-iff*: $x \in PiC\ I\ X \longleftrightarrow apply\text{-}bfun\ x \in Pi\ I\ X$
⟨*proof*⟩

lemmas *mem-PiCD* = *mem-PiC-iff*[*THEN iffD1*]
and *mem-PiCI* = *mem-PiC-iff*[*THEN iffD2*]

lemma *tendsto-bfun-uniform-limit*:
fixes $f::'i \Rightarrow 'a \Rightarrow_b ('b :: metric\text{-}space)$
assumes $(f \longrightarrow l)\ F$
shows *uniform-limit UNIV f l F*
⟨*proof*⟩

lemma *uniform-limit-tendsto-bfun*:
fixes $f::'i \Rightarrow 'a \Rightarrow_b ('b :: metric\text{-}space)$
and $l::'a \Rightarrow_b 'b$
assumes *uniform-limit UNIV f l F*
shows $(f \longrightarrow l)\ F$
⟨*proof*⟩

1.2 Supremum Norm

instantiation *bfun* :: (*type*, *real-normed-vector*) *real-vector*
begin

lemma *uminus-cont*: $f \in bfun \Longrightarrow (\lambda x.\ -\ f\ x) \in bfun$ **for** $f::'a \Rightarrow 'b$
⟨*proof*⟩

lemma *plus-cont*: $f \in bfun \Longrightarrow g \in bfun \Longrightarrow (\lambda x.\ f\ x + g\ x) \in bfun$
for $f\ g::'a \Rightarrow 'b$
⟨*proof*⟩

lemma *minus-cont*: $f \in bfun \Longrightarrow g \in bfun \Longrightarrow (\lambda x.\ f\ x - g\ x) \in bfun$
for $f\ g::'a \Rightarrow 'b$
⟨*proof*⟩

lemma *scaleR-cont*: $f \in bfun \Longrightarrow (\lambda x.\ a *_R f\ x) \in bfun$ **for** $f :: 'a \Rightarrow 'b$
⟨*proof*⟩

lemma *bfun-normI*[*intro*]: $(\bigwedge x.\ norm\ (f\ x) \le b) \Longrightarrow f \in bfun$
⟨*proof*⟩

lift-definition *uminus-bfun*::$('a \Rightarrow_b 'b) \Rightarrow ('a \Rightarrow_b 'b)$ **is** $\lambda f\ x.\ -\ f\ x$
⟨*proof*⟩

lift-definition *plus-bfun*::$('a \Rightarrow_b 'b) \Rightarrow ('a \Rightarrow_b 'b) \Rightarrow 'a \Rightarrow_b 'b$ **is** λf

$g\ x.\ f\ x + g\ x$
⟨*proof*⟩

lift-definition *minus-bfun*::$('a \Rightarrow_b 'b) \Rightarrow ('a \Rightarrow_b 'b) \Rightarrow 'a \Rightarrow_b 'b$ **is** $\lambda f\ g\ x.\ f\ x - g\ x$
⟨*proof*⟩

lift-definition *zero-bfun*::$'a \Rightarrow_b 'b$ **is** λ-. *0*
⟨*proof*⟩

lemma *const-bfun-0-eq-0*[*simp*]: *const-bfun 0 = 0*
⟨*proof*⟩

lift-definition *scaleR-bfun*::*real* $\Rightarrow ('a \Rightarrow_b 'b) \Rightarrow 'a \Rightarrow_b 'b$ **is** $\lambda r\ g\ x.\ r *_R g\ x$
⟨*proof*⟩

lemmas [*simp*] =
const-bfun.rep-eq
uminus-bfun.rep-eq
plus-bfun.rep-eq
minus-bfun.rep-eq
zero-bfun.rep-eq
scaleR-bfun.rep-eq

instance
⟨*proof*⟩
end

lemma *scaleR-cont'*: $f \in bfun \Longrightarrow (\lambda x.\ a * f\ x) \in bfun$ **for** $f :: 'a \Rightarrow real$
⟨*proof*⟩

lemma *bfun-norm-le-SUP-norm*:
$f \in bfun \Longrightarrow norm\ (f\ x) \leq (SUP\ x.\ norm\ (f\ x))$
⟨*proof*⟩

instantiation *bfun* :: (*type*, *real-normed-vector*) *real-normed-vector*
begin

definition *norm-bfun* :: $('a, 'b)\ bfun \Rightarrow real$
where *norm-bfun f = dist f 0*

definition *sgn* $(f::('a,'b)\ bfun) = f\ /_R\ norm\ f$

instance
⟨*proof*⟩
end

lemma *norm-bfun-def'*: *norm f* = (⨆*x*. *norm* ((*f* :: *'a* ⇒$_b$ *'b* :: *real-normed-vector*) *x*))
⟨*proof*⟩

lemma *norm-le-norm-bfun*: *norm* (*apply-bfun f x*) ≤ *norm f*
⟨*proof*⟩

lemma *abs-le-norm-bfun*: *abs* (*apply-bfun f x*) ≤ *norm f*
⟨*proof*⟩

lemma *le-norm-bfun*: *apply-bfun f x* ≤ *norm f*
⟨*proof*⟩

1.3 Complete Space

lemma *tendsto-add*: *P* ⟶ (*L* :: *'a* :: *real-normed-vector*) ⟹ (λ*n*. *P n* + *c*) ⟶ *L* + *c*
⟨*proof*⟩

lemma *lim-add*: *convergent P* ⟹ *lim* (λ*n*. *P n* + (*c* :: *'a* ::*real-normed-vector*)) = *lim P* + *c*
⟨*proof*⟩

lemma *complete-bfun*:
assumes *cauchy-f*: *Cauchy* (*f* :: *nat* ⇒ (*'a*, *'b* :: {*complete-space*, *real-normed-vector*}) *bfun*)
shows *convergent f*
⟨*proof*⟩

lemma *norm-bound*:
fixes *f* :: (*'a*, *'b*::*real-normed-vector*) *bfun*
assumes ⋀*x*. *norm* (*apply-bfun f x*) ≤ *b*
shows *norm f* ≤ *b*
⟨*proof*⟩

lemma *bfun-bounded-norm-range*: *bounded* (*range* (λ*s*. *norm* (*apply-bfun v s*)))
⟨*proof*⟩

instance *bfun* :: (*type*, *banach*) *banach*
⟨*proof*⟩

lemma *bfun-prob-space-integrable*:
assumes *prob-space S v* ∈ *borel-measurable S*
assumes (*v* :: *'a* ⇒ *'b* :: {*second-countable-topology*, *banach*}) ∈ *bfun*

shows *integrable S v*
⟨*proof*⟩

lemma *bfun-integral-bound*:
assumes $(v :: 'a \Rightarrow 'c :: \{euclidean\text{-}space\}) \in bfun$
shows $(\lambda S.\ \int x.\ v\ x\ \partial(S :: 'a\ pmf)) \in bfun$
⟨*proof*⟩

lemma *scale-bfun*[*intro*!]: $f \in bfun \Longrightarrow (\lambda x.\ (k::real) * f\ x) \in bfun$
⟨*proof*⟩

lemma *bfun-spec*[*intro*]: $f \in bfun \Longrightarrow (\lambda x.\ f\ (g\ x)) \in bfun$
⟨*proof*⟩

lemma *apply-bfun-bfun*[*simp*]: *apply-bfun f* $\in$ *bfun*
⟨*proof*⟩

lemma *bfun-integral-bound'*[*intro*]: $(v :: 'a \Rightarrow 'c :: \{euclidean\text{-}space\})$
$\in bfun \Longrightarrow$
$(\lambda S.\ \int x.\ v\ x\ \partial((F\ S) :: 'a\ pmf)) \in bfun$
⟨*proof*⟩

lift-definition *bfun-comp* :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow_b 'c::metric\text{-}space) \Rightarrow$
$('a \Rightarrow_b 'c)$ **is**
$\lambda g\ bf\ x.\ bf\ (g\ x)$
⟨*proof*⟩

1.4 Order Instance

class *ordered-real-normed-vector* = *real-normed-vector* + *ordered-real-vector*

instance *real* :: *ordered-real-normed-vector*
⟨*proof*⟩

instantiation *bfun* :: (-, *ordered-real-normed-vector*) *ordered-real-normed-vector*
begin

definition *less-eq-bfun f g* $\equiv \forall x.$ *apply-bfun f x* $\leq$ *apply-bfun g x*
definition *less-bfun f g* $\equiv \forall x.$ *apply-bfun f x* $\leq$ *apply-bfun g x* $\wedge$ $(\exists y.$
f y $<$ *g y*)

instance
⟨*proof*⟩
end

lemma *less-eq-bfunI*[*intro*]: $(\bigwedge x.$ *apply-bfun f x* $\leq$ *apply-bfun g x*$) \Longrightarrow$
f $\leq$ *g*
⟨*proof*⟩

lemma *less-eq-bfunD*[*dest*]: *f* $\leq$ *g* $\Longrightarrow$ $(\bigwedge x.$ *apply-bfun f x* $\leq$ *apply-bfun*
g x)

⟨*proof*⟩

1.5 Miscellaneous

instantiation *bfun* :: (*type*, *one*) *one* **begin**

lift-definition *one-bfun* :: *'s* ⇒$_b$ *real* **is** *λx. 1*
⟨*proof*⟩

instance
⟨*proof*⟩
end

declare *one-bfun.rep-eq* [*simp*]

lemma *apply-bfun-one* [*simp*]: *apply-bfun* (*1* :: - ⇒$_b$ *real*) *x* = *1*
⟨*proof*⟩

lemma *norm-bfun-one*[*simp*]: *norm* (*1* :: *'a* ⇒$_b$ *real*) = *1*
⟨*proof*⟩

lemma *range-bfunI*[*intro*]: *bounded* (*range f*) ⟹ *f* ∈ *bfun*
⟨*proof*⟩

lemma *finite-bfun*[*simp*]: (*λ*(*i*::-::*finite*). *f i*) ∈ *bfun*
⟨*proof*⟩

lemma *bounded-apply-bfun'*:
assumes *bounded* ((*F* :: *'c* ⇒ *'d* ⇒$_b$ *'b*::*real-normed-vector*) ‘ *S*)
shows *bounded* ((*λb*. (*F b*) *x*) ‘ *S*)
⟨*proof*⟩

lemma *bfun-tendsto-apply-bfun*:
assumes *h*: (*F* :: (*nat* ⇒ *'a* ⇒$_b$ *real*)) ⟶ (*y* :: *'a* ⇒$_b$ *real*)
shows (*λn*. *F n x*) ⟶ *y x*
⟨*proof*⟩

1.6 Bounded Functions and Vectors

lemma *vec-bfun*[*simp*, *intro*]: ($) *x* ∈ *bfun*
⟨*proof*⟩

lemma *norm-bfun-le-norm-vec*: *norm* (*bfun.Bfun* (($) (*x* :: *real*^'*c* :: *finite*))) ≤ *norm x*
⟨*proof*⟩

lemma *bounded-linear-bfun-nth*: *bounded-linear f* ⟹ *bounded-linear* (*λv*. *bfun.Bfun* (($) (*f v*)))

⟨*proof*⟩

lemma *norm-vec-le-norm-bfun*:
norm (*vec-lambda* (*apply-bfun* (*x* :: *'d*::*finite* ⇒$_b$ *real*))) ≤ *norm x* ∗ *card* (*UNIV* :: *'d set*)
⟨*proof*⟩

end

2 Bounded Linear Functions

theory *Blinfun-Util*
imports
HOL−Analysis.Bounded-Linear-Function
Bounded-Functions
begin

2.1 Composition

lemma *blinfun-compose-id*[*simp*]:
id-blinfun o_L *f* = *f*
f o_L *id-blinfun* = *f*
⟨*proof*⟩

lemma *blinfun-compose-assoc*: *F* o_L *G* o_L *H* = *F* o_L (*G* o_L *H*)
⟨*proof*⟩

lemma *blinfun-compose-diff-right*: *f* o_L (*g* − *h*) = (*f* o_L *g*) − (*f* o_L *h*)
⟨*proof*⟩

2.2 Power

overloading
blinfunpow ≡ *compow* :: *nat* ⇒ (*'a*::*real-normed-vector* ⇒$_L$ *'a*) ⇒ (*'a* ⇒$_L$ *'a*)
begin

primrec *blinfunpow* :: *nat* ⇒ (*'a*::*real-normed-vector* ⇒$_L$ *'a*) ⇒ (*'a* ⇒$_L$ *'a*)
where
blinfunpow 0 f = *id-blinfun*
| *blinfunpow* (*Suc n*) *f* = *f* o_L *blinfunpow n f*

end

lemma *bounded-pow-blinfun*[*intro*]:
assumes *bounded* (*range* (*F*::*nat* ⇒ *'a*::*real-normed-vector* ⇒$_L$ *'a*))
shows *bounded* (*range* (λ*t*. (*F t*)^^(*Suc n*)))
⟨*proof*⟩

lemma *blincomp-scaleR-right*: $(a *_R (F :: 'a :: real\text{-}normed\text{-}vector \Rightarrow_L 'a)) \hat{\hat{}} t = a \hat{} t *_R F \hat{\hat{}} t$
⟨*proof*⟩
lemma *summable-inv-Q*:
fixes $Q :: 'a :: banach \Rightarrow_L 'a$
assumes *onorm-le*: $norm\ (id\text{-}blinfun - Q) < 1$
shows *summable* $(\lambda n.\ (id\text{-}blinfun - Q) \hat{\hat{}} n)$
⟨*proof*⟩

lemma *blinfunpow-assoc*: $(F :: 'a :: real\text{-}normed\text{-}vector \Rightarrow_L 'a) \hat{\hat{}} (Suc\ n) = (F \hat{\hat{}} n) \circ_L F$
⟨*proof*⟩

lemma *norm-blinfunpow-le*: $norm\ ((f :: 'b :: real\text{-}normed\text{-}vector \Rightarrow_L 'b) \hat{\hat{}} n) \leq norm\ f \hat{} n$
⟨*proof*⟩

lemma *blinfunpow-nonneg*:
assumes $\bigwedge v.\ 0 \leq v \Longrightarrow 0 \leq blinfun\text{-}apply\ (f :: ('b :: \{ord, real\text{-}normed\text{-}vector\} \Rightarrow_L 'b))\ v$
shows $0 \leq v \Longrightarrow 0 \leq (f \hat{\hat{}} n)\ v$
⟨*proof*⟩

lemma *blinfunpow-mono*:
assumes $\bigwedge u\ v.\ u \leq v \Longrightarrow (f :: 'b :: \{ord, real\text{-}normed\text{-}vector\} \Rightarrow_L 'b)\ u \leq f\ v$
shows $u \leq v \Longrightarrow (f \hat{\hat{}} n)\ u \leq (f \hat{\hat{}} n)\ v$
⟨*proof*⟩

lemma *banach-blinfun*:
fixes $C :: 'b :: \{real\text{-}normed\text{-}vector, complete\text{-}space\} \Rightarrow_L 'b$
assumes $norm\ C < 1$
shows $\exists! v.\ C\ v = v \bigwedge v.\ (\lambda n.\ (C \hat{\hat{}} n)\ v) \longrightarrow (THE\ v.\ C\ v = v)$
⟨*proof*⟩

2.3 Geometric Sum

lemma *inv-one-sub-Q*:
fixes $Q :: 'a :: banach \Rightarrow_L 'a$
assumes $onorm\text{-}le: norm\ (id\text{-}blinfun - Q) < 1$
shows $(Q \circ_L (\sum i.\ (id\text{-}blinfun - Q) \hat{\hat{}} i)) = id\text{-}blinfun$
and $(\sum i.\ (id\text{-}blinfun - Q) \hat{\hat{}} i) \circ_L Q = id\text{-}blinfun$
⟨*proof*⟩

lemma *inv-norm-le*:
fixes $Q :: 'a :: banach \Rightarrow_L 'a$
assumes $norm\ Q < 1$
shows $(id\text{-}blinfun - Q) \circ_L (\sum i.\ Q \hat{\hat{}} i) = id\text{-}blinfun$

$(\sum i.\ Q\widehat{\ }\widehat{\ }i)\ o_L\ (id\text{-}blinfun - Q) = id\text{-}blinfun$
⟨*proof*⟩

lemma *inv-norm-le′*:
fixes $Q :: 'a :: banach \Rightarrow_L 'a$
assumes $norm\ Q < 1$
shows $(id\text{-}blinfun - Q)\ ((\sum i.\ Q\widehat{\ }\widehat{\ }i)\ x) = x$
$(\sum i.\ Q\widehat{\ }\widehat{\ }i)\ ((id\text{-}blinfun - Q)\ x) = x$
⟨*proof*⟩

2.4 Inverses

definition $is\text{-}inverse_L\ X\ Y \longleftrightarrow X\ o_L\ Y = id\text{-}blinfun \wedge Y\ o_L\ X = id\text{-}blinfun$

abbreviation $invertible_L\ X \equiv \exists X'.\ is\text{-}inverse_L\ X\ X'$

lemma $is\text{-}inverse_L\text{-}I[intro]$:
assumes $X\ o_L\ Y = id\text{-}blinfun\ Y\ o_L\ X = id\text{-}blinfun$
shows $is\text{-}inverse_L\ X\ Y$
⟨*proof*⟩

lemma $is\text{-}inverse_L\text{-}D[dest]$:
assumes $is\text{-}inverse_L\ X\ Y$
shows $X\ o_L\ Y = id\text{-}blinfun\ Y\ o_L\ X = id\text{-}blinfun$
⟨*proof*⟩

lemma $invertible_L\text{-}D[dest]$:
assumes $invertible_L\ f$
obtains g **where** $f\ o_L\ g = id\text{-}blinfun\ g\ o_L\ f = id\text{-}blinfun$
⟨*proof*⟩

lemma $invertible_L\text{-}I[intro]$:
assumes $f\ o_L\ g = id\text{-}blinfun\ g\ o_L\ f = id\text{-}blinfun$
shows $invertible_L\ f$
⟨*proof*⟩

lemma $is\text{-}inverse_L\text{-}comm$: $is\text{-}inverse_L\ X\ Y \longleftrightarrow is\text{-}inverse_L\ Y\ X$
⟨*proof*⟩

lemma $is\text{-}inverse_L\text{-}unique$: $is\text{-}inverse_L\ f\ g \Longrightarrow is\text{-}inverse_L\ f\ h \Longrightarrow g = h$
⟨*proof*⟩

lemma $is\text{-}inverse_L\text{-}ex1$: $is\text{-}inverse_L\ f\ g \Longrightarrow \exists!h.\ is\text{-}inverse_L\ f\ h$
⟨*proof*⟩

lemma $is\text{-}inverse_L\text{-}ex1'$: $\exists x.\ is\text{-}inverse_L\ f\ x \Longrightarrow \exists!x.\ is\text{-}inverse_L\ f\ x$
⟨*proof*⟩

definition $inv_L\ f = (THE\ g.\ is\text{-}inverse_L\ f\ g)$

lemma inv_L*-eq*:
assumes $is\text{-}inverse_L\ f\ g$
shows $inv_L\ f = g$
⟨*proof*⟩

lemma inv_L*-I*:
assumes $f\ o_L\ g = id\text{-}blinfun\ g\ o_L\ f = id\text{-}blinfun$
shows $g = inv_L\ f$
⟨*proof*⟩

lemma *inv-app1* [*simp*]: $invertible_L\ X \Longrightarrow (inv_L\ X)\ o_L\ X = id\text{-}blinfun$
⟨*proof*⟩

lemma *inv-app2*[*simp*]: $invertible_L\ X \Longrightarrow X\ o_L\ (inv_L\ X)\ = id\text{-}blinfun$
⟨*proof*⟩

lemma *inv-app1'*[*simp*]: $invertible_L\ X \Longrightarrow inv_L\ X\ (X\ v) = v$
⟨*proof*⟩

lemma *inv-app2'*[*simp*]: $invertible_L\ X \Longrightarrow X\ (inv_L\ X\ v) = v$
⟨*proof*⟩

lemma [*simp*]: $invertible_L\ X \Longrightarrow inv_L\ (inv_L\ X) = X$
⟨*proof*⟩

lemma inv_L*-cancel-iff*:
assumes $invertible_L\ f$
shows $f\ x = y \longleftrightarrow x = inv_L\ f\ y$
⟨*proof*⟩

lemma $invertible_L$*-inf-sum*:
assumes $norm\ (X :: 'b :: banach \Rightarrow_L 'b) < 1$
shows $invertible_L\ (id\text{-}blinfun - X)$
⟨*proof*⟩

lemma inv_L*-inf-sum*:
fixes $X :: 'b :: banach \Rightarrow_L -$
assumes $norm\ X < 1$
shows $inv_L\ (id\text{-}blinfun - X) = (\sum i.\ X\ ^{\frown\frown}\ i)$
⟨*proof*⟩

lemma $is\text{-}inverse_L$*-compose*:
assumes $invertible_L\ f\ invertible_L\ g$
shows $is\text{-}inverse_L\ (f\ o_L\ g)\ (inv_L\ g\ o_L\ inv_L\ f)$
⟨*proof*⟩

lemma *invertible$_L$-compose*: *invertible$_L$ f* $\Longrightarrow$ *invertible$_L$ g* $\Longrightarrow$ *invertible$_L$* (*f* o_L *g*)
⟨*proof*⟩

lemma *inv$_L$-compose*:
assumes *invertible$_L$ f invertible$_L$ g*
shows*inv$_L$* (*f* o_L *g*) = (*inv$_L$ g*) o_L (*inv$_L$ f*)
⟨*proof*⟩

lemma *inv$_L$-id-blinfun*[*simp*]: *inv$_L$ id-blinfun* = *id-blinfun*
⟨*proof*⟩

2.5 Norm

lemma *bounded-range-subset*:
bounded (*range f* :: *real set*) $\Longrightarrow$ *bounded* (*f* ‘ *X*′)
⟨*proof*⟩

lemma *bounded-const*: *bounded* ((λ-. *x*) ‘ *X*)
⟨*proof*⟩

lift-definition *bfun-pos* :: (′*d* $\Rightarrow_b$ *real*) $\Rightarrow$ (′*d* $\Rightarrow_b$ *real*) **is** λ*f i*. *if f i* < *0 then* −*f i else f i*
⟨*proof*⟩

lemma *bfun-pos-zero*[*simp*]: *bfun-pos f* = *0* $\longleftrightarrow$ *f* = *0*
⟨*proof*⟩

lift-definition *bfun-nonneg* :: (′*d* $\Rightarrow_b$ *real*) $\Rightarrow$ (′*d* $\Rightarrow_b$ *real*) **is** λ*f i*. *if f i* ≤ *0 then 0 else f i*
⟨*proof*⟩

lemma *bfun-nonneg-split*: *bfun-nonneg x* − *bfun-nonneg* (− *x*) = *x*
⟨*proof*⟩

lemma *blinfun-split*: *blinfun-apply f x* = *f* (*bfun-nonneg x*) − *f* (*bfun-nonneg* (− *x*))
⟨*proof*⟩

lemma *bfun-nonneg-pos*: *bfun-nonneg x* + *bfun-nonneg* (−*x*) = *bfun-pos x*
⟨*proof*⟩

lemma *bfun-nonneg*: *0* ≤ *bfun-nonneg f*
⟨*proof*⟩

lemma *bfun-pos-eq-nonneg*: *bfun-pos n* = *bfun-nonneg n* + *bfun-nonneg* (−*n*)
⟨*proof*⟩

lemma *blinfun-mono-norm-pos*:
fixes $f :: ('c \Rightarrow_b real) \Rightarrow_L ('d \Rightarrow_b real)$
assumes $\bigwedge v :: 'c \Rightarrow_b real.\ v \geq 0 \Longrightarrow f\ v \geq 0$
shows $norm\ (f\ n) \leq norm\ (f\ (bfun\text{-}pos\ n))$
⟨*proof*⟩

lemma *norm-bfun-pos*[*simp*]: $norm\ (bfun\text{-}pos\ f) = norm\ f$
⟨*proof*⟩

lemma *norm-blinfun-mono-eq-nonneg*:
fixes $f :: ('c \Rightarrow_b real) \Rightarrow_L ('d \Rightarrow_b real)$
assumes $\bigwedge v :: 'c \Rightarrow_b real.\ v \geq 0 \Longrightarrow f\ v \geq 0$
shows $norm\ f = (\bigsqcup v \in \{v.\ v \geq 0\}.\ norm\ (f\ v)\ /\ norm\ v)$
⟨*proof*⟩

lemma *norm-blinfun-normalized-le*: $norm\ (blinfun\text{-}apply\ f\ v)\ /\ norm\ v \leq norm\ f$
⟨*proof*⟩

lemma *norm-blinfun-mono-eq-nonneg'*:
fixes $f :: ('c \Rightarrow_b real) \Rightarrow_L ('d \Rightarrow_b real)$
assumes $\bigwedge v :: 'c \Rightarrow_b real.\ 0 \leq v \Longrightarrow 0 \leq f\ v$
shows $norm\ f = (\bigsqcup x \in \{x.\ norm\ x = 1 \wedge x \geq 0\}.\ norm\ (f\ x))$
⟨*proof*⟩

lemma *norm-blinfun-mono-le-norm-one*:
fixes $f :: ('c \Rightarrow_b real) \Rightarrow_L ('d \Rightarrow_b real)$
assumes $\bigwedge v :: 'c \Rightarrow_b real.\ v \geq 0 \Longrightarrow f\ v \geq 0$
assumes $norm\ x = 1\ 0 \leq x$
shows $norm\ (f\ x) \leq norm\ (f\ 1)$
⟨*proof*⟩

lemma *norm-blinfun-mono-eq-one*:
fixes $f :: ('c \Rightarrow_b real) \Rightarrow_L ('d \Rightarrow_b real)$
assumes $\bigwedge v :: 'c \Rightarrow_b real.\ v \geq 0 \Longrightarrow f\ v \geq 0$
shows $norm\ f = norm\ (f\ 1)$
⟨*proof*⟩

2.6 Miscellaneous

lemma *bounded-linear-apply-bfun*: $bounded\text{-}linear\ (\lambda x.\ apply\text{-}bfun\ x\ i)$
⟨*proof*⟩

lemma *lim-blinfun-apply*: $convergent\ X \Longrightarrow (\lambda n.\ blinfun\text{-}apply\ (X\ n)\ u) \longrightarrow lim\ X\ u$
⟨*proof*⟩

lemma *bounded-apply-blinfun*:

assumes *bounded* ((*F* :: *'c* ⇒ *'d*::*real-normed-vector* ⇒$_L$ *'b*::*real-normed-vector*) ‘ *S*)
shows *bounded* ((λ*b*. *blinfun-apply* (*F b*) *x*) ‘ *S*)
⟨*proof*⟩
end
theory *MDP-reward-Util*
imports *Blinfun-Util*
begin

3 Auxiliary Lemmas

3.1 Summability

lemma *summable-powser-const*:
fixes *c* :: *real*
assumes |*c*| < *1*
shows *summable* (λ*n*. *c*^*n* ∗ *x*)
⟨*proof*⟩

3.2 Infinite sums

lemma *suminf-split-head'*:
summable (*f* :: *nat* ⇒ *'x* :: *real-normed-vector*) ⟹ *suminf f* = *f 0* + (∑ *n*. *f* (*Suc n*))
⟨*proof*⟩

lemma *sum-disc-lim*:
assumes |*c* :: *real*| < *1*
shows (∑ *x*. *c*^*x* ∗ *B*) = *B* /(*1*−*c*)
⟨*proof*⟩

3.3 Bounded Functions

lemma *suminf-apply-bfun*:
fixes *f* :: *nat* ⇒ *'c* ⇒$_b$ *real*
assumes *summable f*
shows (∑ *i*. *f i*) *x* = (∑ *i*. *f i x*)
⟨*proof*⟩

lemma *sum-apply-bfun*:
fixes *f* :: *nat* ⇒ *'c* ⇒$_b$ *real*
shows (∑ *i*<*n*. *f i*) *x* = (∑ *i*<*n*. *apply-bfun* (*f i*) *x*)
⟨*proof*⟩

3.4 Push-Forward of a Bounded Function

lemma *integrable-bfun-prob-space* [*simp*]:
integrable (*measure-pmf P*) (λ*t*. *apply-bfun f* (*F t*) :: *real*)
⟨*proof*⟩

lift-definition *push-exp* :: $('b \Rightarrow 'c\ pmf) \Rightarrow ('c \Rightarrow_b real) \Rightarrow ('b \Rightarrow_b real)$ **is**
$\lambda c\ f\ s.\ measure\text{-}pmf.expectation\ (c\ s)\ f$
⟨*proof*⟩

declare *push-exp.rep-eq*[*simp*]

lemma *norm-push-exp-le-norm*: $norm\ (push\text{-}exp\ d\ x) \leq norm\ x$
⟨*proof*⟩

lemma *push-exp-bounded-linear* [*simp*]: *bounded-linear* (*push-exp d*)
⟨*proof*⟩

lemma *onorm-push-exp* [*simp*]: $onorm\ (push\text{-}exp\ d) = 1$
⟨*proof*⟩

lemma *push-exp-return*[*simp*]: $push\text{-}exp\ return\text{-}pmf = id$
⟨*proof*⟩

3.5 Boundedness

lemma *bounded-abs*[*intro*]:
$bounded\ (X' :: real\ set) \Longrightarrow bounded\ (abs\ `\ X')$
⟨*proof*⟩

lemma *bounded-abs-range*[*intro*]:
$bounded\ (range\ f :: real\ set) \Longrightarrow bounded\ (range\ (\lambda x.\ abs\ (f\ x)))$
⟨*proof*⟩

3.6 Probability Theory

lemma *integral-measure-pmf-bind*:
assumes $(\bigwedge x.\ |(f :: 'b \Rightarrow real)\ x| \leq B)$
shows $(\int x.\ f\ x\ \partial((measure\text{-}pmf\ M) \gg\!= (\lambda x.\ measure\text{-}pmf\ (N\ x))))$
$= (\int x.\ \int y.\ f\ y\ \partial N\ x\ \partial M)$
⟨*proof*⟩

lemma *lemma-4-3-1′*:
assumes $set\text{-}pmf\ p \subseteq W$
and $bounded\ ((w :: 'c \Rightarrow real)\ `\ W)$
and $W \neq \{\}$
and $measure\text{-}pmf.expectation\ p\ w = (\bigsqcup p \in \{p.\ set\text{-}pmf\ p \subseteq W\}.\ measure\text{-}pmf.expectation\ p\ w)$
shows $\exists x \in W.\ measure\text{-}pmf.expectation\ p\ w = w\ x$
⟨*proof*⟩

lemma *lemma-4-3-1*:
assumes $set\text{-}pmf\ p \subseteq W\ integrable\ (measure\text{-}pmf\ p)\ w\ bounded\ ((w :: 'c \Rightarrow real)\ `\ W)$

shows *measure-pmf.expectation p w* $\leq \bigsqcup (w \text{ ` } W)$
⟨*proof*⟩

lemma *bounded-integrable*:
assumes *bounded* (*range v*) *v* ∈ *borel-measurable* (*measure-pmf p*)
shows *integrable* (*measure-pmf p*) (*v* :: *'c* ⇒ *real*)
⟨*proof*⟩

3.7 Argmax

lemma *finite-is-arg-max*: *finite X* ⟹ *X* ≠ {} ⟹ ∃ *x*. *is-arg-max* (*f* :: *'c* ⇒ *real*) (λ*x*. *x* ∈ *X*) *x*
⟨*proof*⟩

lemma *finite-arg-max-le*:
assumes *finite* (*X* :: *'c set*) *X* ≠ {}
shows *s* ∈ *X* ⟹ (*f* :: *'c* ⇒ *real*) *s* ≤ *f* (*arg-max-on* (*f* :: *'c* ⇒ *real*) *X*)
⟨*proof*⟩

lemma *arg-max-on-in*:
assumes *finite* (*X* :: *'c set*) *X* ≠ {}
shows (*arg-max-on* (*f* :: *'c* ⇒ *real*) *X*) ∈ *X*
⟨*proof*⟩

lemma *finite-arg-max-eq-Max*:
assumes *finite* (*X* :: *'c set*) *X* ≠ {}
shows (*f* :: *'c* ⇒ *real*) (*arg-max-on f X*) = *Max* (*f* ` *X*)
⟨*proof*⟩

lemma *arg-max-SUP*: *is-arg-max* (*f* :: *'b* ⇒ *real*) (λ*x*. *x* ∈ *X*) *m* ⟹ *f m* = (⨆ (*f* ` *X*))
⟨*proof*⟩

definition *has-max X* ≡ ∃ *x* ∈ *X*. ∀ *x'* ∈ *X*. *x'* ≤ *x*
definition *has-arg-max f X* ≡ ∃ *x*. *is-arg-max f* (λ*x*. *x* ∈ *X*) *x*

lemma *has-max* ((*f* :: *'b* ⇒ *real*) ` *X*) ⟷ *has-arg-max f X*
⟨*proof*⟩

lemma *has-arg-max-is-arg-max*: *has-arg-max f X* ⟹ *is-arg-max f* (λ*x*. *x* ∈ *X*) (*arg-max f* (λ*x*. *x* ∈ *X*))
⟨*proof*⟩

lemma *has-arg-max-arg-max*: *has-arg-max f X* ⟹ (*arg-max f* (λ*x*. *x* ∈ *X*)) ∈ *X*
⟨*proof*⟩

lemma *app-arg-max-ge*: *has-arg-max* $(f :: 'b \Rightarrow real)$ $X \Longrightarrow x \in X \Longrightarrow f\ x \leq f\ (arg\text{-}max\text{-}on\ f\ X)$
⟨*proof*⟩

lemma *app-arg-max-eq-SUP*: *has-arg-max* $(f :: 'b \Rightarrow real)$ $X \Longrightarrow f\ (arg\text{-}max\text{-}on\ f\ X) = \bigsqcup (f\ `\ X)$
⟨*proof*⟩

lemma *SUP-is-arg-max*:
assumes $x \in X$ *bdd-above* $(f\ `\ X)$ $(f :: 'c \Rightarrow real)$ $x = \bigsqcup (f\ `\ X)$
shows *is-arg-max* f $(\lambda x.\ x \in X)$ x
⟨*proof*⟩

lemma *is-arg-max-linorderI*[*intro*]: **fixes** $f :: 'c \Rightarrow 'b :: linorder$
assumes $P\ x$ $\bigwedge y.\ (P\ y \Longrightarrow f\ x \geq f\ y)$
shows *is-arg-max f P x*
⟨*proof*⟩

lemma *is-arg-max-linorderD*[*dest*]: **fixes** $f :: 'c \Rightarrow 'b :: linorder$
assumes *is-arg-max f P x*
shows $P\ x$ $(P\ y \Longrightarrow f\ x \geq f\ y)$
⟨*proof*⟩

lemma *is-arg-max-cong*:
assumes $\bigwedge x.\ P\ x \Longrightarrow f\ x = g\ x$
shows *is-arg-max f P x* $\longleftrightarrow$ *is-arg-max g P x*
⟨*proof*⟩

lemma *is-arg-max-congI*:
assumes *is-arg-max f P x* $\bigwedge x.\ P\ x \Longrightarrow f\ x = g\ x$
shows *is-arg-max g P x*
⟨*proof*⟩

3.8 Contraction Mappings

definition *is-contraction* $C \equiv \exists l.\ 0 \leq l \wedge l < 1 \wedge (\forall v\ u.\ dist\ (C\ v)\ (C\ u) \leq l * dist\ v\ u)$

lemma *banach'*:
fixes $C :: 'b :: complete\text{-}space \Rightarrow 'b$
assumes *is-contraction C*
shows $\exists !v.\ C\ v = v$ $\bigwedge v.\ (\lambda n.\ (C\ \widehat{\ }\widehat{\ }\ n)\ v) \longrightarrow (THE\ v.\ C\ v = v)$
⟨*proof*⟩

lemma *contraction-dist*:
fixes $C :: 'b :: complete\text{-}space \Rightarrow 'b$
assumes $\bigwedge v\ u.\ dist\ (C\ v)\ (C\ u) \leq c\ * dist\ v\ u$
assumes $0 \leq c$ $c < 1$

shows $(1 - c) * dist\ v\ (THE\ v.\ C\ v = v) \leq dist\ v\ (C\ v)$
⟨*proof*⟩

3.9 Limits

lemma *tendsto-bfun-sandwich*:
assumes
$(f :: nat \Rightarrow 'b \Rightarrow_b real) \longrightarrow x\ (g :: nat \Rightarrow 'b \Rightarrow_b real) \longrightarrow x$
eventually $(\lambda n.\ f\ n \leq h\ n)$ *sequentially eventually* $(\lambda n.\ h\ n \leq g\ n)$ *sequentially*
shows $(h :: nat \Rightarrow 'b \Rightarrow_b real) \longrightarrow x$
⟨*proof*⟩

3.10 Supremum

lemma *SUP-add-le*:
assumes $X \neq \{\}$ *bounded* $(B\ `\ X)$ *bounded* $(A'\ `\ X)$
shows $(\bigsqcup c \in X.\ (B :: 'a \Rightarrow real)\ c + A'\ c) \leq (\bigsqcup b \in X.\ B\ b) + (\bigsqcup a \in X.\ A'\ a)$
⟨*proof*⟩

lemma *le-SUP-diff'*:
assumes *ne*: $X \neq \{\}$
and *bdd*: *bounded* $(B\ `\ X)$ *bounded* $(A'\ `\ X)$
and *sup-le*: $(\bigsqcup a \in X.\ (A' :: 'a \Rightarrow real)\ a) \leq (\bigsqcup b \in X.\ B\ b)$
shows $(\bigsqcup b \in X.\ B\ b) - (\bigsqcup a \in X.\ (A' :: 'a \Rightarrow real)\ a) \leq (\bigsqcup c \in X.\ B\ c - A'\ c)$
⟨*proof*⟩

lemma *le-SUP-diff*:
fixes $A' :: 'a \Rightarrow real$
assumes $X \neq \{\}$ *bounded* $(B\ `\ X)$ *bounded* $(A'\ `\ X)$ $(\bigsqcup a \in X.\ A'\ a) \leq (\bigsqcup b \in X.\ B\ b)$
shows $0 \leq (\bigsqcup c \in X.\ B\ c - A'\ c)$
⟨*proof*⟩

lemma *bounded-SUP-mul*[*simp*]:
$X \neq \{\} \Longrightarrow 0 \leq l \Longrightarrow bounded\ (f\ `\ X) \Longrightarrow (\bigsqcup x \in X.\ (l :: real) * f\ x) = (l * (\bigsqcup x \in X.\ f\ x))$
⟨*proof*⟩

lemma *abs-cSUP-le*[*intro*]:
$X \neq \{\} \Longrightarrow bounded\ (F\ `\ X) \Longrightarrow |\bigsqcup x \in X.\ (F\ x) :: real| \leq (\bigsqcup x \in X.\ |F\ x|)$
⟨*proof*⟩

end

4 Discrete-Time Markov Decision Processes with Arbitrary State Spaces

In this file we construct discrete-time Markov decision processes, e.g. with arbitrary state spaces. Proofs and definitions are adapted from Markov_Models.Discrete_Time_Markov_Process.

theory *MDP-cont*
imports *HOL−Probability.Probability*
begin

lemma *Ionescu-Tulcea-C-eq*:
assumes $\bigwedge i\ h.\ h \in space\ (PiM\ \{0..<i\}\ N) \Longrightarrow P\ i\ h = P'\ i\ h$
assumes *h*: *Ionescu-Tulcea P N Ionescu-Tulcea P′ N*
shows *Ionescu-Tulcea.C P N 0 n* $(\lambda x.\ undefined)$ = *Ionescu-Tulcea.C P′ N 0 n* $(\lambda x.\ undefined)$
⟨*proof*⟩

lemma *Ionescu-Tulcea-CI-eq*:
assumes $\bigwedge i\ h.\ h \in space\ (PiM\ \{0..<i\}\ N) \Longrightarrow P\ i\ h = P'\ i\ h$
assumes *h*: *Ionescu-Tulcea P N Ionescu-Tulcea P′ N*
shows *Ionescu-Tulcea.CI P N* = *Ionescu-Tulcea.CI P′ N*
⟨*proof*⟩

lemma *measure-eqI-PiM-sequence*:
fixes *M* :: *nat* ⇒ *′a measure*
assumes ∗[*simp*]: *sets P* = *PiM UNIV M sets Q* = *PiM UNIV M*
assumes *eq*: $\bigwedge A\ n.\ (\bigwedge i.\ A\ i \in sets\ (M\ i)) \Longrightarrow$
P (*prod-emb UNIV M* {..*n*} (Pi_E {..*n*} *A*)) = *Q* (*prod-emb UNIV M* {..*n*} (Pi_E {..*n*} *A*))
assumes *A*: *finite-measure P*
shows *P* = *Q*
⟨*proof*⟩

lemma *distr-cong-simp*:
M = *K* ⟹ *sets N* = *sets L* ⟹ $(\bigwedge x.\ x \in space\ M$ =*simp*=> *f x* = *g x*) ⟹ *distr M N f* = *distr K L g*
⟨*proof*⟩

4.1 Definition and Basic Properties

locale *discrete-MDP* =
fixes *Ms* :: *′s measure*
and *Ma* :: *′a measure*
and *A* :: *′s* ⇒ *′a set*
and *K* :: *′s* × *′a* ⇒ *′s measure*

assumes *A-s*: $\bigwedge s.\ A\ s \in sets\ Ma$

assumes *A-ne*: $\bigwedge s.\ A\ s \neq \{\}$

assumes *ex-pol*: $\exists\, \delta \in Ms \to_M Ma.\ \forall\, s.\ \delta\ s \in A\ s$

assumes *K*[*measurable*]: $K \in Ms \bigotimes_M Ma \to_M$ *prob-algebra Ms*
begin

lemma *space-prodI*[*intro*]: $x \in$ *space* $A' \Longrightarrow y \in$ *space* $B \Longrightarrow (x,y) \in$ *space* $(A' \bigotimes_M B)$
⟨*proof*⟩

abbreviation $M \equiv Ms \bigotimes_M Ma$
abbreviation *Ma-A* $s \equiv$ *restrict-space Ma* $(A\ s)$

lemma *space-ma*[*intro*]: $s \in$ *space Ms* $\Longrightarrow a \in$ *space Ma* $\Longrightarrow (s,a) \in$ *space M*
⟨*proof*⟩

lemma *space-x0*[*simp*]: *x0* $\in$ *space* (*prob-algebra Ms*) $\Longrightarrow$ *space x0* $=$ *space Ms*
⟨*proof*⟩

lemma *A-subs-Ma*: $A\ s \subseteq$ *space Ma*
⟨*proof*⟩

lemma *space-Ma-A-subset*: $s \in$ *space Ms* $\Longrightarrow$ *space* (*Ma-A s*) $\subseteq A\ s$
⟨*proof*⟩

lemma *K-restrict* [*measurable*]: $K \in (Ms \bigotimes_M$ *Ma-A s*$) \to_M$ *prob-algebra Ms*
⟨*proof*⟩

lemma *measurable-K-act*[*measurable*, *intro*]: $s \in$ *space Ms* $\Longrightarrow (\lambda a.\ K\ (s,\ a)) \in Ma \to_M$ *prob-algebra Ms*
⟨*proof*⟩

lemma *measurable-K-st*[*measurable*, *intro*]: $a \in$ *space Ma* $\Longrightarrow (\lambda s.\ K\ (s,\ a)) \in Ms \to_M$ *prob-algebra Ms*
⟨*proof*⟩

lemma *space-K*[*simp*]: *sa* $\in$ *space M* $\Longrightarrow$ *space* $(K\ sa) =$ *space Ms*
⟨*proof*⟩

lemma *space-K2*[*simp*]: $s \in$ *space Ms* $\Longrightarrow a \in$ *space Ma* $\Longrightarrow$ *space* $(K\ (s,\ a)) =$ *space Ms*
⟨*proof*⟩

lemma *space-K-E*: $s' \in$ *space* $(K\ (s,a)) \Longrightarrow s \in$ *space Ms* $\Longrightarrow a \in$ *space Ma* $\Longrightarrow s' \in$ *space Ms*

⟨*proof*⟩

lemma *sets-K*: $sa \in space\ M \Longrightarrow sets\ (K\ sa) = sets\ Ms$
⟨*proof*⟩

lemma *sets-K'*[*measurable-cong*]: $s \in space\ Ms \Longrightarrow a \in space\ Ma \Longrightarrow sets\ (K\ (s,a)) = sets\ Ms$
⟨*proof*⟩

lemma *prob-space-K*[*intro*]: $sa \in space\ M \Longrightarrow prob\text{-}space\ (K\ sa)$
⟨*proof*⟩

lemma *prob-space-K2*[*intro*]: $s \in space\ Ms \Longrightarrow a \in space\ Ma \Longrightarrow prob\text{-}space\ (K\ (s,a))$
⟨*proof*⟩

lemma *K-in-space* [*intro*]: $m \in space\ M \Longrightarrow K\ m \in space\ (prob\text{-}algebra\ Ms)$
⟨*proof*⟩

4.2 Policies

type-synonym $('c,\ 'd)\ pol = nat \Rightarrow ((nat \Rightarrow 'c \times 'd) \times 'c) \Rightarrow 'd\ measure$

abbreviation $H\ i \equiv Pi_M\ \{0..<i\}\ (\lambda\text{-}.\ M)$

abbreviation $Hs\ i \equiv\ H\ i \bigotimes_M Ms$

lemma *space-H1*: $j < (i :: nat) \Longrightarrow \omega \in space\ (H\ i) \Longrightarrow \omega\ j \in space\ M$
⟨*proof*⟩

lemma *space-case-nat*[*intro*]:
assumes $\omega \in space\ (H\ i)\ s \in space\ Ms$
shows $case\text{-}nat\ s\ (fst \circ \omega)\ i \in space\ Ms$
⟨*proof*⟩

lemma *undefined-in-H0*: $(\lambda\text{-}.\ undefined) \in space\ (H\ (0 :: nat))$
⟨*proof*⟩

lemma *sets-K-Suc*[*measurable-cong*]: $h \in space\ (H\ (Suc\ n)) \Longrightarrow sets\ (K\ (h\ n)) = sets\ Ms$
⟨*proof*⟩

A decision rule is a function from states to distributions over enabled actions.

definition $is\text{-}dec0\ d \equiv d \in Ms \rightarrow_M prob\text{-}algebra\ Ma$

definition *is-dec* $(t :: nat)$ $d \equiv (d \in Hs\ t \to_M prob\text{-}algebra\ Ma)$

lemma *is-dec0* $d \Longrightarrow$ *is-dec* t $(\lambda(\text{-},s).\ d\ s)$
⟨*proof*⟩

A policy is a function from histories to valid decision rules.

definition *is-policy* :: $('s, 'a)$ *pol* $\Rightarrow$ *bool* **where**
is-policy $p \equiv \forall i.$ *is-dec* i $(p\ i)$

abbreviation *p0* :: $('s, 'a)$ *pol* $\Rightarrow$ $'s \Rightarrow$ $'a$ *measure* **where**
p0 $p\ s \equiv p\ (0 :: nat)\ (\lambda\text{-}.\ undefined, s)$

context
fixes p **assumes** p[*simp*]: *is-policy* p
begin

lemma *is-policyD*[*measurable*]: $p\ i \in Hs\ i \to_M prob\text{-}algebra\ Ma$
⟨*proof*⟩

lemma *space-policy*[*simp*]: $hs \in space\ (Hs\ i) \Longrightarrow space\ (p\ i\ hs) = space\ Ma$
⟨*proof*⟩

lemma *space-policy'*[*simp*]: $h \in space\ (H\ i) \Longrightarrow s \in space\ Ms \Longrightarrow space\ (p\ i\ (h,s)) = space\ Ma$
⟨*proof*⟩

lemma *space-policyI*[*intro*]:
assumes $s \in space\ Ms$ $h \in space\ (H\ i)$ $a \in space\ Ma$
shows $a \in space\ (p\ i\ (h,s))$
⟨*proof*⟩

lemma *sets-policy*[*simp*]: $hs \in space\ (Hs\ i) \Longrightarrow sets\ (p\ i\ hs) = sets\ Ma$
⟨*proof*⟩

lemma *sets-policy'*[*measurable-cong*, *simp*]:
$h \in space\ (H\ i) \Longrightarrow s \in space\ Ms \Longrightarrow sets\ (p\ i\ (h,s)) = sets\ Ma$
⟨*proof*⟩

lemma *sets-policy''*[*measurable-cong*, *simp*]:
$h \in space\ ((Pi_M\ \{\}\ (\lambda\text{-}.\ M))) \Longrightarrow s \in space\ Ms \Longrightarrow sets\ (p\ 0\ (h,s)) = sets\ Ma$
⟨*proof*⟩

lemma *policy-prob-space*: $hs \in space\ (Hs\ i) \Longrightarrow prob\text{-}space\ (p\ i\ hs)$
⟨*proof*⟩

lemma *policy-prob-space'*: $h \in space\ (H\ i) \Longrightarrow s \in space\ Ms \Longrightarrow$ *prob-space* $(p\ i\ (h,s))$
⟨*proof*⟩

lemma *prob-space-p0*: $x \in space\ Ms \Longrightarrow$ *prob-space* $(p0\ p\ x)$
⟨*proof*⟩

lemma *p0-sets*[*measurable-cong*]: $x \in space\ Ms \Longrightarrow sets\ (p\ 0\ (\lambda\text{-}.\ undefined,x)) = sets\ Ma$
⟨*proof*⟩

lemma *space-p0*[*simp*]: $s \in space\ Ms \Longrightarrow space\ (p0\ p\ s) = space\ Ma$
⟨*proof*⟩

lemma *return-policy-prob-algebra* [*measurable*]:
$h \in space\ (H\ n) \Longrightarrow x \in space\ Ms \Longrightarrow (\lambda a.\ return\ M\ (x,\ a)) \in p\ n\ (h,\ x) \rightarrow_M prob\text{-}algebra\ M$
⟨*proof*⟩
end

4.3 Successor Policy

To shift the policy by one step, we provide a single state-action pair as history

definition *Suc-policy* $p\ sa = (\lambda i\ (h,\ s).\ p\ (Suc\ i)\ (\lambda i'.\ case\text{-}nat\ sa\ h\ i',\ s))$

lemma *p-as-Suc-policy*: $p\ (Suc\ i)\ (h,\ s) = Suc\text{-}policy\ p\ ((h\ 0))\ i\ (\lambda i.\ h\ (Suc\ i),\ s)$
⟨*proof*⟩

lemma *is-policy-Suc-policy*[*intro*]:
assumes *s*: $sa \in space\ M$ **and** *p*: *is-policy p*
shows *is-policy* (*Suc-policy p sa*)
⟨*proof*⟩

lemma *Suc-policy-measurable-step*[*measurable*]:
assumes *is-policy p*
shows $(\lambda x.\ Suc\text{-}policy\ p\ (fst\ (fst\ x))\ n\ (snd\ (fst\ x),\ snd\ x)) \in (M \bigotimes_M Pi_M\ \{0..<n\}\ (\lambda\text{-}.\ M)) \bigotimes_M Ms \rightarrow_M prob\text{-}algebra\ Ma$
⟨*proof*⟩

4.4 Single-Step Distribution

K' takes a policy, a distribution over 's, the epoch, and a history, produces a distribution over the next state-action pair.

definition $K' :: ('s,\ 'a)\ pol \Rightarrow 's\ measure \Rightarrow nat \Rightarrow (nat \Rightarrow ('s \times 'a))$

$\Rightarrow$ ('s $\times$ 'a) *measure*
where
K' p s0 n ω = do {
s ← case-nat s0 (K ∘ ω) n;
a ← p n (ω, s);
return M (s, a)
}

lemma *prob-space-K'*:
assumes *p*: *is-policy p* **and** *x*: *x0 ∈ space (prob-algebra Ms)* **and** *h*: *h ∈ space (H n)*
shows *prob-space (K' p x0 n h)*
⟨*proof*⟩

lemma *measurable-K'*[*measurable*]:
assumes *p*: *is-policy p* **and** *x*: *x ∈ space (prob-algebra Ms)*
shows *K' p x i ∈ H i →$_M$ prob-algebra M*
⟨*proof*⟩

4.5 Initial State-Action Distribution

K0 produces the initial state-action distribution from a state distribution and a policy.

definition *K0 p s0 = K' p s0 0 (λ-. undefined)*

lemma *K0-def'*:
K0 p s0 = do {
s ← s0;
a ← p0 p s;
return M (s, a)}
⟨*proof*⟩

lemma *K0-prob*[*measurable*]: *is-policy p ⟹ K0 p ∈ prob-algebra Ms →$_M$ prob-algebra M*
⟨*proof*⟩

lemma *prob-space-K0*: *is-policy p ⟹ x0 ∈ space (prob-algebra Ms) ⟹ prob-space (K0 p x0)*
⟨*proof*⟩

lemma *space-K0*[*simp*]: *is-policy p ⟹ s ∈ space (prob-algebra Ms) ⟹ space (K0 p s) = space M*
⟨*proof*⟩

lemma *sets-K0*[*measurable-cong*]:
assumes *is-policy p s ∈ space (prob-algebra Ms)*
shows *sets (K0 p s) = sets M*
⟨*proof*⟩

lemma *K0-return-eq-p0*:
assumes *is-policy p s* ∈ *space Ms*
shows *K0 p* (*return Ms s*) = *p0 p s* ⤜ (λ*a*. *return M* (*s*,*a*))
⟨*proof*⟩

lemma *M-ne-policy*[*intro*]: *is-policy p* ⟹ *s* ∈ *space* (*prob-algebra Ms*) ⟹ *space M* ≠ {}
⟨*proof*⟩

4.6 Sequence Space of the MDP

We can instantiate *Ionescu-Tulcea* with *K′*.

lemma *IT-K′*: *is-policy p* ⟹ *x* ∈ *space* (*prob-algebra Ms*) ⟹ *Ionescu-Tulcea* (*K′ p x*) (λ-. *M*)
⟨*proof*⟩

definition *lim-sequence* :: (′*s*, ′*a*) *pol* ⇒ ′*s measure* ⇒ (*nat* ⇒ (′*s* × ′*a*)) *measure*
where
lim-sequence p x = *projective-family.lim UNIV* (*Ionescu-Tulcea.CI* (*K′ p x*) (λ-. *M*)) (λ-. *M*)

lemma
assumes *x*: *x* ∈ *space* (*prob-algebra Ms*) **and** *p*: *is-policy p*
shows *space-lim-sequence*: *space* (*lim-sequence p x*) = *space* (Π$_M$ *i*∈*UNIV*. *M*)
and *sets-lim-sequence*[*measurable-cong*]: *sets* (*lim-sequence p x*) = *sets* (Π$_M$ *i*∈*UNIV*. *M*)
and *emeasure-lim-sequence-emb*: ⋀*J X*. *finite J* ⟹ *X* ∈ *sets* (Π$_M$ *j*∈*J*. *M*) ⟹
emeasure (*lim-sequence p x*) (*prod-emb UNIV* (λ-. *M*) *J X*) =
emeasure (*Ionescu-Tulcea.CI* (*K′ p x*) (λ-. *M*) *J*) *X*
and *emeasure-lim-sequence-emb-I0o*: ⋀*n X*. *X* ∈ *sets* (Π$_M$ *i* ∈ {*0*..<*n*}. *M*) ⟹
emeasure (*lim-sequence p x*) (*prod-emb UNIV* (λ-. *M*) {*0*..<*n*} *X*) =
emeasure (*Ionescu-Tulcea.C* (*K′ p x*) (λ-. *M*) *0 n* (λ*x*. *undefined*)) *X*
⟨*proof*⟩

lemma *lim-sequence-prob-space*:
assumes *is-policy p s* ∈ *space* (*prob-algebra Ms*)
shows *prob-space* (*lim-sequence p s*)
⟨*proof*⟩

4.7 Measurablility of the Sequence Space

lemma *lim-sequence*[*measurable*]:
assumes *p*: *is-policy p*

shows *lim-sequence p ∈ prob-algebra Ms →M prob-algebra* (Π_M *i∈UNIV. M*)
⟨*proof*⟩

lemma *lim-sequence-aux*[*measurable*]:
assumes *p*: *is-policy p*
assumes *f* : $\bigwedge$*x. x ∈ space M ⟹ is-policy* (*f x*)
assumes *f'*: $\bigwedge$*n.* (λ*x. f* (*fst* (*fst x*)) *n* (*snd* (*fst x*), *snd x*)) ∈
(*M* $\bigotimes_M$ *Pi*$_M$ {*0..<n*} (λ-. *M*)) $\bigotimes_M$ *Ms →M prob-algebra Ma*
assumes *gm*: *g ∈ M →M prob-algebra Ms*
shows (λ*x. lim-sequence* (*f x*) (*g x*)) ∈ *M →M prob-algebra* (*Pi*$_M$ *UNIV* (λ-. *M*))
⟨*proof*⟩

lemma *lim-sequence-Suc-return*[*measurable*]:
assumes *p*: *is-policy p*
assumes *s*: *s ∈ space Ms*
shows (λ*x. lim-sequence* (*Suc-policy p* (*s*, *snd x*)) (*return Ms* (*fst x*))) ∈
M →M prob-algebra (*Pi*$_M$ *UNIV* (λ-. *M*))
⟨*proof*⟩

lemma *lim-sequence-Suc-K*[*measurable*]:
assumes *is-policy p*
shows (λ*x. lim-sequence* (*Suc-policy p x*) (*K x*)) ∈ *M →M prob-algebra* (*Pi*$_M$ *UNIV* (λ-. *M*))
⟨*proof*⟩

4.8 Iteration Rule

lemma *step-C*:
assumes *x*: *x ∈ space* (*prob-algebra Ms*) **and** *p*: *is-policy p*
shows *Ionescu-Tulcea.C* (*K' p x*) (λ-. *M*) *0 1* (λ-. *undefined*) ⤜
Ionescu-Tulcea.C (*K' p x*) (λ-. *M*) *1 n* =
K0 p x ⤜ (λ*a. Ionescu-Tulcea.C* (*K' p x*) (λ-. *M*) *1 n* (*case-nat a* (λ-. *undefined*)))
⟨*proof*⟩

lemma *lim-sequence-eq*:
assumes *x*: *x ∈ space* (*prob-algebra Ms*) **assumes** *p*: *is-policy p*
shows *lim-sequence p x* =
K0 p x ⤜ (λ*y. distr* (*lim-sequence* (*Suc-policy p y*) (*K y*)) (Π_M -∈*UNIV. M*) (*case-nat y*))
(**is** - = *?B p x*)
⟨*proof*⟩

4.9 Stream Space of the MDP

definition *lim-stream* :: ('*s*, '*a*) *pol* ⇒ '*s measure* ⇒ ('*s* × '*a*) *stream measure*

where
lim-stream p x = *distr* (*lim-sequence p x*) (*stream-space M*) *to-stream*

lemma *space-lim-stream*: *space* (*lim-stream p x*) = *streams* (*space M*)
⟨*proof*⟩

lemma *sets-lim-stream*[*measurable-cong*]: *sets* (*lim-stream p x*) = *sets* (*stream-space M*)
⟨*proof*⟩

lemma *lim-stream*[*measurable*]:
assumes *is-policy p*
shows *lim-stream p* ∈ *prob-algebra Ms* →$_M$ *prob-algebra* (*stream-space M*)
⟨*proof*⟩

lemma *lim-stream-Suc*[*measurable*]:
assumes *p*: *is-policy p*
shows (λ*a*. *lim-stream* (*Suc-policy p a*) (*K a*)) ∈ *M* →$_M$ *prob-algebra* (*stream-space M*)
⟨*proof*⟩

lemma *space-stream-space-M-ne*: *x* ∈ *space M* ⟹ *space* (*stream-space M*) ≠ {}
⟨*proof*⟩

lemma *prob-space-lim-stream*[*intro*]:
assumes *is-policy p x* ∈ *space* (*prob-algebra Ms*)
shows *prob-space* (*lim-stream p x*)
⟨*proof*⟩

lemma *prob-space-step*:
assumes *is-policy p x* ∈ *space M*
shows *prob-space* (*lim-stream* (*Suc-policy p x*) (*K x*))
⟨*proof*⟩

lemma *lim-stream-eq*:
assumes *p*: *is-policy p*
assumes *x*: *x* ∈ *space* (*prob-algebra Ms*)
shows *lim-stream p x* = *do* {
y ← *K0 p x*;
ω ← *lim-stream* (*Suc-policy p y*) (*K y*);
return (*stream-space M*) (*y* ## *ω*)
}
⟨*proof*⟩

end
end

theory *MDP-disc*
imports
MDP-cont
HOL−Library.Omega-Words-Fun
begin

5 Markov Decision Processes with Discrete State Spaces

lemma (**in** *prob-space*) *integral-stream-space*:
fixes $f :: 'a\ stream \Rightarrow ('b :: \{banach, second\text{-}countable\text{-}topology, real\text{-}normed\text{-}vector\})$
assumes *int-f*: *integrable* (*stream-space M*) *f*
assumes [*measurable*]: $f \in$ *borel-measurable* (*stream-space M*)
shows $(\int X.\ f\ X\ \partial stream\text{-}space\ M) = (\int x.\ (\int X.\ f\ (x\ \#\#\ X)\ \partial stream\text{-}space\ M)\ \partial M)$
⟨*proof*⟩

lemma *prefix-cons*:
Omega-Words-Fun.prefix (*Suc n*) *seq* = *seq 0*#*Omega-Words-Fun.prefix n* (λ*n*. *seq* (*Suc n*))
⟨*proof*⟩

lemma *restrict-Suc*: *restrict y* {*0*..<*Suc i*} (*Suc n*) = (*restrict* (λ*n*. *y* (*Suc n*)) {*0*..<*i*}) *n*
⟨*proof*⟩

lemma *prefix-restrict*: *Omega-Words-Fun.prefix i* (*restrict y* {*0*..<*i*}) = *Omega-Words-Fun.prefix i y*
⟨*proof*⟩

lemma *prefix-measurable*[*measurable*]:
Omega-Words-Fun.prefix i ∈ Pi_M {*0*..<*i*}
(λ-. *count-space* (*UNIV* :: (*'s* ::*countable* × *'a*::*countable*) *set*)) →$_M$ *count-space UNIV*
⟨*proof*⟩

no-notation *Omega-Words-Fun.build* (**infixr** ‹##› *65*)

locale *discrete-MDP* =
fixes *A* :: *'s*::*countable* ⇒ *'a*::*countable set* — enabled actions
and *K* :: *'s* × *'a* ⇒ *'s pmf* — MDP kernel, transition probabilities
assumes
A-ne: ⋀*s*. *A s* ≠ {} — set of enabled actions is nonempty
begin

5.1 Policies

Type synonym for decision rules.

type-synonym $('c, 'd)\ dec = 'c \Rightarrow 'd\ pmf$

definition *is-dec* :: $('s, 'a)\ dec \Rightarrow bool$ **where**
$is\text{-}dec\ d \equiv \forall s.\ d\ s \subseteq A\ s$

lemma *is-decI*[*intro*]:
$(\bigwedge s.\ set\text{-}pmf\ (d\ s) \subseteq A\ s) \Longrightarrow is\text{-}dec\ d$
⟨*proof*⟩

abbreviation $D_R \equiv \{d.\ is\text{-}dec\ d\}$

definition *is-dec-det* :: $('s \Rightarrow 'a) \Rightarrow bool$ **where**
$is\text{-}dec\text{-}det\ d \equiv \forall s.\ d\ s \in A\ s$

abbreviation $D_D \equiv \{d.\ is\text{-}dec\text{-}det\ d\}$

definition $mk\text{-}dec\text{-}det\ d\ s = return\text{-}pmf\ (d\ s)$

lemma *is-dec-mk-dec-det-iff* [*simp*]: $is\text{-}dec\ (mk\text{-}dec\text{-}det\ d) \longleftrightarrow is\text{-}dec\text{-}det\ d$
⟨*proof*⟩

lemma *D-det-to-MR*[*intro*]: $is\text{-}dec\text{-}det\ d \Longrightarrow is\text{-}dec\ (mk\text{-}dec\text{-}det\ d)$
⟨*proof*⟩

Due to the assumption $A\ ?s \neq \{\}$, a deterministic decision rule always exists. It immediately follows via $is\text{-}dec\ (mk\text{-}dec\text{-}det\ ?d) = is\text{-}dec\text{-}det\ ?d$ that a randomized decision rule also exists.

lemma *SOME-is-dec-det*: $is\text{-}dec\text{-}det\ (\lambda s.\ SOME\ a.\ a \in A\ s)$
⟨*proof*⟩

lemma *ex-dec-det* [*simp*]: $\exists d.\ is\text{-}dec\text{-}det\ d$
⟨*proof*⟩

lemma *D-det-ne* [*simp*]: $D_D \neq \{\}$
⟨*proof*⟩

lemma D_R*-ne* [*simp*]: $D_R \neq \{\}$
⟨*proof*⟩

lemma *ex-dec*[*intro*, *simp*]: $\exists d.\ is\text{-}dec\ d$
⟨*proof*⟩

Type synonym for policies.

type-synonym $('c, 'd)\ pol = ('c \times 'd)\ list \Rightarrow ('c, 'd)\ dec$

A policy assigns a decision rule to each observed past.

definition *is-policy* :: $('s, 'a)$ *pol* $\Rightarrow$ *bool* **where**
is-policy p $\equiv \forall$ *hs. is-dec* (*p hs*)

abbreviation $\Pi_{HR} \equiv \{p.$ *is-policy p*$\}$

Deterministic policies

definition *is-deterministic p* $\equiv$ *is-policy p* $\wedge$ $(\forall$ *h s.* $\exists$ *a. p h s* $=$ *return-pmf a*)

definition *mk-det p h s* $\equiv$ *return-pmf* (*p h s*)

abbreviation $\Pi_{HD} \equiv \{p. \forall h.\ p\ h \in D_D\}$

Markovian policies

definition *is-markovian p* $\equiv$ *is-policy p* $\wedge$ $(\forall$ *h h'. length h* $=$ *length h'* $\longrightarrow$ *p h* $=$ *p h'*)

definition *mk-markovian* :: $(nat \Rightarrow ('s, 'a)$ *dec*$) \Rightarrow ('s, 'a)$ *pol* **where**
mk-markovian p $\equiv$ $(\lambda h.$ *p* (*length h*))

lemma *is-markovian-mk-iff*[*simp*]: *is-markovian* (*mk-markovian p*) $\longleftrightarrow$ $(\forall$ *n. is-dec* (*p n*))
⟨*proof*⟩

lemma *is-markovian-mk*[*intro*]: $\forall$ *n. is-dec* (*p n*) $\Longrightarrow$ *is-markovian* (*mk-markovian p*)
⟨*proof*⟩

lemma *mk-markovian-nil* [*simp*]: *mk-markovian p* [] $=$ *p 0*
⟨*proof*⟩

definition *mk-markovian-det p* $\equiv$ $(\lambda h$ *s. return-pmf* (*p* (*length h*) *s*))

abbreviation $\Pi_{MD} \equiv \{p. \forall n.\ p\ n \in D_D\}$
abbreviation $\Pi_{MR} \equiv \{p. \forall n.\ p\ n \in D_R\}$

lemma Π_{MR}*-imp-policies*[*intro*]: $p \in \Pi_{MR} \Longrightarrow$ *mk-markovian p* $\in$ Π_{HR}
⟨*proof*⟩

lemma Π_{MD}*-MR-iff*[*simp*]: $(\lambda n.$ *mk-dec-det* (*p n*)$) \in \Pi_{MR} \longleftrightarrow p \in$ Π_{MD}
⟨*proof*⟩

lemma Π_{MD}*-to-MR*[*intro*]: $p \in \Pi_{MD} \Longrightarrow (\lambda n.$ *mk-dec-det* (*p n*)$) \in$ Π_{MR}
⟨*proof*⟩

lemma Π_{MD}-*ne*[*simp*]: $\Pi_{MD} \neq \{\}$
⟨*proof*⟩

lemma Π_{MR}-*ne*[*simp*]: $\Pi_{MR} \neq \{\}$
⟨*proof*⟩

lemma *policies-ne*[*simp*, *intro*]: $\Pi_{HR} \neq \{\}$
⟨*proof*⟩

Stationary policies

definition *is-stationary p* ≡ *is-policy p* ∧ (∀ *h h'*. *p h* = *p h'*)

lemma *is-stationary-const-iff*[*simp*]: *is-stationary* (λ-. *d*) = *is-dec d*
⟨*proof*⟩

lemma *is-stationary-const*[*intro*]: *is-dec d* ⟹ *is-stationary* (λ-. *d*)
⟨*proof*⟩

abbreviation *mk-stationary p* ≡ *mk-markovian* (λ-. *p*)
abbreviation *mk-stationary-det d* ≡ *mk-markovian* (λ-. *mk-dec-det d*)

5.1.1 Successor Policy

After taking the first step in the MDP, we will know which state and which action got selected during the initial epoch. To obtain a policy that acts as if the current epoch was the initial one, we prepend the observed state-action pair to the history. The result is again a policy, i.e. it satisfies *is-policy*.

definition *π-Suc p sa h* = *p* (*sa*#*h*)

lemma *is-policy-π-Suc* [*intro*]: *is-policy p* ⟹ *is-policy* (*π-Suc p sa*)
⟨*proof*⟩

lemma *Suc-mk-markovian*[*simp*]: *π-Suc* (*mk-markovian p*) *x* = *mk-markovian* (λ*n*. *p* (*Suc n*))
⟨*proof*⟩

5.2 Stream Space of the MDP

5.2.1 Initial State-Action Distribution

If we fix a decision rule *d* and an initial distribution of states *S0*, we obtain a distribution over state-action pairs in the following way: First, the initial state *s* is sampled from *S0*, then an action *a* is selected from *d s*.

definition *K0 d S0* = *do* {

s ← *S0*;
a ← *d s*;
return-pmf (*s*,*a*)
}

notation *K0* (*K*$_0$)

lemma *K0-iff*: *K0 d S0* = *S0* ⤜ (λ*s*. *map-pmf* (λ*a*. (*s*,*a*)) (*d s*))
⟨*proof*⟩

lemma *vimage-pair*[*simp*]: *Pair x* −‘ {*p*} = (*if x* = *fst p then* {*snd p*} *else* {})
⟨*proof*⟩

lemma *pmf-K0* [*simp*]: *pmf* (*K0 d S0*) (*s*,*a*) = *pmf S0 s* ∗ *pmf* (*d s*) *a*
⟨*proof*⟩

lemma *set-pmf-K0*: *set-pmf* (*K0 p S0*) = {(*s*,*a*). *s* ∈ *S0* ∧ *a* ∈ *p s*}
⟨*proof*⟩

lemma *fst-K0*[*simp*]: *map-pmf fst* (*K0 p S0*) = *S0*
⟨*proof*⟩

abbreviation *S* ≡ *stream-space* (*count-space UNIV*)

We inherit the trace space from MDPs with continuous state-action spaces

interpretation *MDP-cont*: *MDP-cont.discrete-MDP count-space UNIV count-space UNIV A K*
⟨*proof*⟩

lemma *count-space-M*[*simp*]: *MDP-cont.M* = *count-space UNIV*
⟨*proof*⟩

lemma *space-M*[*simp*]: *space MDP-cont.M* = *UNIV*
⟨*proof*⟩

We reuse the stream space provided by *MDP-cont.lim-stream*

definition *T* :: (′*s*, ′*a*) *pol* ⇒ ′*s pmf* ⇒ (′*s* × ′*a*) *stream measure*
where *T p* = *MDP-cont.lim-stream* (λ*n* (*h*,*s*). *p* (*Omega-Words-Fun.prefix n h*) *s*)

lemma *sets-T*[*measurable-cong*]:
sets (*T p x*) = *sets S*
⟨*proof*⟩

lemma *space-stream-space-ne*[*simp*]: *space S* ≠ {}
⟨*proof*⟩

lemma *space-T*[*simp*]: *space* (*T p S0*) = *space S*
⟨*proof*⟩

lemma *is-policy-MDP-cont*[*intro*]:
fixes *p* :: ('*s* × '*a*) *list* ⇒ '*s* ⇒ '*a pmf*
shows *MDP-cont.is-policy* (λ*n* (*h*,*s*). *p* (*Omega-Words-Fun.prefix n h*) *s*)
⟨*proof*⟩

lemma *prob-space-T*[*intro*, *simp*]: *prob-space* (*T p x*)
⟨*proof*⟩

lemma *T-subprob*[*simp*]:
T p S0 ∈ *space* (*subprob-algebra S*)
⟨*proof*⟩

lemma *T-subprob-space* [*simp*]: *subprob-space* (*T p S0*)
⟨*proof*⟩

lemma *K0-MDP-cont-eq*:
MDP-cont.K0 (λ*x* (*h*,*s*). *measure-pmf* (*p* (*Omega-Words-Fun.prefix x h*) *s*)) (*measure-pmf S0*) =
K0 (*p* []) *S0*
⟨*proof*⟩

5.2.2 Decomposition of the Stream Space

The distribution of traces/walks the MDP allows should intuitively satisfy the following rule:

1. select the initial state *s* from *S0*
2. pass it to the decision rule *p* [] to determine a distribution over actions
3. select the action *a*

- finally pass the state-action pair (*s*, *a*) to the kernel *K* to get a new distribution over states *s0′*

  Then the iteration repeats with the updated policy *π-Suc p* (*s*, *a*).

  The result carries over from ⟦*MDP-cont.is-policy ?p*; *?x* ∈ *space* (*prob-algebra* (*count-space UNIV*))⟧ ⟹ *MDP-cont.lim-stream ?p ?x* = *MDP-cont.K0 ?p ?x* ⤜ (λ*y*. *MDP-cont.lim-stream* (*MDP-cont.Suc-policy ?p y*) (*measure-pmf* (*K y*)) ⤜ (λω. *return* (*stream-space MDP-cont.M*) (*y* ## ω))).

lemma *T-eq*:

shows $T\ p\ S0 = do\ \{$
$sa \leftarrow measure\text{-}pmf\ (K0\ (p\ [])\ S0);$
$\omega \leftarrow T\ (\pi\text{-}Suc\ p\ sa)\ (K\ sa);$
$return\ S\ (sa\ \#\#\ \omega)$
$\}$
⟨*proof*⟩

lemma *T-eq-distr*:
shows $T\ p\ S0 = measure\text{-}pmf\ (K0\ (p\ [])\ S0) \gg\!= (\lambda sa.\ distr\ (T\ (\pi\text{-}Suc\ p\ sa)\ (K\ sa))\ S\ ((\#\#)\ sa))$
⟨*proof*⟩

The iteration rule lets us nicely decompose integrals (expected values) over functions on traces of the MDP.

lemma *integral-T*:
fixes $f :: ('s \times 'a)\ stream \Rightarrow real$
assumes *f-bounded*: $\bigwedge x.\ |f\ x| \leq B$
assumes *f*: $f \in borel\text{-}measurable\ S$
shows $(\int t.\ f\ t\ \partial T\ p\ x) = \int sa.\ \int t'.\ f\ (sa\#\#t')\ \partial T\ (\pi\text{-}Suc\ p\ sa)\ (K\ sa)\ \partial K0\ (p\ [])\ x$
⟨*proof*⟩

lemma *nn-integral-T*:
assumes *f*: $f \in borel\text{-}measurable\ S$
shows $(\int^+ t.\ f\ t\ \partial T\ p\ x) = (\int^+ sa.\ \int^+ t'.\ f\ (sa\#\#t')\ \partial T\ (\pi\text{-}Suc\ p\ sa)\ (K\ sa)\ \partial K0\ (p\ [])\ x)$
⟨*proof*⟩

5.2.3 A Denotational View on the Stochastic Process

Many definitions on MDPs do not rely on the individual traces but only on the distribution of states and actions at each epoch.

We define this view on the trace space as the repeated iteration of K_0 and K. It conincides with the definition of T.

primrec $Pn :: ('s, 'a)\ pol \Rightarrow 's\ pmf \Rightarrow nat \Rightarrow ('s \times 'a)\ pmf$ **where**
$Pn\ p\ S0\ 0 = K0\ (p\ [])\ S0$
$|\ Pn\ p\ S0\ (Suc\ n) = K0\ (p\ [])\ S0 \gg\!= (\lambda sa.\ Pn\ (\pi\text{-}Suc\ p\ sa)\ (K\ sa)\ n)$
declare *Pn.simps(2)*[*simp del*]

lemma *Pn-eq-T*:
shows $measure\text{-}pmf\ (Pn\ p\ S0\ n) = distr\ (T\ p\ S0)\ (count\text{-}space\ UNIV)\ (\lambda t.\ t\ !!\ n)$
⟨*proof*⟩

The definition of *Pn* also allows us to easily prove that only enabled actions can occur in the traces of the MDP.

lemma *Pn-in-A*: $is\text{-}policy\ p \Longrightarrow (s,\ a) \in Pn\ p\ S0\ n \Longrightarrow a \in A\ s$

⟨*proof*⟩

lemma *T-in-A*:
assumes *is-policy p*
shows *AE t in T p S0. snd* (*t* !! *n*) ∈ *A* (*fst* (*t* !! *n*))
⟨*proof*⟩

5.2.4 State Process

Alongside *Pn*, we also define the state and action distributions as projections.

definition *Xn p S0 n* = *map-pmf fst* (*Pn p S0 n*)

lemma *X0* [*simp*]: *Xn p S0 0* = *S0*
⟨*proof*⟩

lemma *Xn-Suc*: *Xn p S0* (*Suc n*) = *Pn p S0 n* ⤜ *K*
⟨*proof*⟩

lemma *Pn-markovian-eq-Xn-bind*: *Pn* (*mk-markovian p*) *S0 n* = *K0* (*p n*) (*Xn* (*mk-markovian p*) *S0 n*)
⟨*proof*⟩

lemma *Xn-Suc′*: *Xn p S0* (*Suc n*) = *K0* (*p* []) *S0* ⤜ (λ*sa*. *Xn* (*π-Suc p sa*) (*K sa*) *n*)
⟨*proof*⟩

lemma *set-pmf-X0* [*simp*]: *set-pmf* (*Xn p S0 0*) = *S0*
⟨*proof*⟩

lemma *set-pmf-PSuc*: *set-pmf* (*Pn* (*mk-markovian p*) *S0 n*) =
{(*s*, *a*). *s* ∈ *set-pmf* (*Xn* (*mk-markovian p*) *S0 n*) ∧ *a* ∈ *p n s*}
⟨*proof*⟩

5.2.5 The Conditional Distribution of Actions

Actions are selected wrt. the whole history of state-action pairs encountered so far. The following definition defines the expected action selection when only the current state is given.

definition *Y-cond-X p S0 n x* = *map-pmf snd* (*cond-pmf* (*Pn p S0 n*) {(*s*,*a*). *s* = *x*})

lemma *prob-K0-X* [*simp*]: *measure-pmf.prob* (*K0 p S0*) {(*s*, *a*). *s* = *x*} = *pmf S0 x*
⟨*proof*⟩

lemma *prob-Pn-X*[*simp*]: *measure-pmf.prob* (*Pn p S0 n*) {(*s*, *a*). *s* = *x*} = *pmf* (*Xn p S0 n*) *x*

⟨*proof*⟩

lemma *pmf-Pn-pair*:
assumes *sa* ∈ *set-pmf* (*Pn p S0 n*)
shows *pmf* (*Pn p S0 n*) *sa* = *pmf* (*Y-cond-X p S0 n* (*fst sa*)) (*snd sa*) ∗ *pmf* (*Xn p S0 n*) (*fst sa*)
⟨*proof*⟩

lemma *pmf-Pn*:
assumes *x* ∈ *set-pmf* (*Xn p S0 n*)
shows *pmf* (*Pn p S0 n*) (*x*,*a*) = *pmf* (*Y-cond-X p S0 n x*) *a* ∗ *pmf* (*Xn p S0 n*) *x*
⟨*proof*⟩

lemma *pmf-Y-cond-X*:
assumes *x* ∈ *set-pmf* (*Xn p S0 n*)
shows *pmf* (*Y-cond-X p S0 n x*) *a* = *pmf* (*Pn p S0 n*) (*x*,*a*) / *pmf* (*Xn p S0 n*) *x*
⟨*proof*⟩

lemma *Y-cond-X-0*[*simp*]:
assumes *x* ∈ *set-pmf S0*
shows *Y-cond-X p S0 0 x* = *p* [] *x*
⟨*proof*⟩

lemma *Y-cond-X-markovian*[*simp*]:
assumes *h*: *x* ∈ *Xn* (*mk-markovian p*) *S0 n*
shows *Y-cond-X* (*mk-markovian p*) *S0 n x* = *p n x*
⟨*proof*⟩

lemma *Pn-eq-Xn-Y-cond*: *Pn p S0 n* = *Xn p S0 n* ⤜ (λ*x*. *map-pmf* (λ*a*. (*x*, *a*)) (*Y-cond-X p S0 n x*))
⟨*proof*⟩

lemma *Pn-eq-Xn-Y-cond′*:
Pn p S0 n = *Xn p S0 n* ⤜ (λ*s*. *Y-cond-X p S0 n s* ⤜ (λ*a*. *return-pmf* (*s*,*a*)))
⟨*proof*⟩

lemma *Pn-markovian-Suc*: *Pn* (*mk-markovian p*) *S0* (*Suc n*) =
Pn (*mk-markovian p*) *S0 n* ⤜ (λ*sa*. *K0* (*p* (*Suc n*)) (*K sa*))
⟨*proof*⟩

5.2.6 Action Process

The distribution of actions.

definition *Yn p S0 n* = *map-pmf snd* (*Pn p S0 n*)

lemma *Y0*: *Yn p S0 0 = S0 ⤜ p []*
⟨*proof*⟩

For markovian policies, the decision rules at each epoch are independent of each other, hence we may express *Yn* solely in terms of *Xn* and the current decision rule.

lemma *Yn-markovian*: *Yn (mk-markovian p) S0 n = Xn (mk-markovian p) S0 n ⤜ p n*
⟨*proof*⟩

5.3 Restriction to Markovian Policies

abbreviation *as-markovian p S0 n x ≡*
if x ∈ (Xn p S0 n) then Y-cond-X p S0 n x else return-pmf (SOME a. a ∈ A x)

For states which cannot occur we choose an arbitrary enabled action, as in this case we cannot make any statements about *Y-cond-X* (a distribution conditioned on an event with probability 0).

lemma *is-*Π_{MR}*-as-markovian*:
assumes *p*: *is-policy p*
shows *as-markovian p S0* $\in \Pi_{MR}$
⟨*proof*⟩

lemma *is-policy-as-markovian*: *is-policy p ⟹ is-policy (mk-markovian (as-markovian p S0))*
⟨*proof*⟩

theorem *Pn-as-markovian-eq*: *Pn (mk-markovian (as-markovian p S0)) S0 = Pn p S0*
⟨*proof*⟩

5.4 MDPs without Initial Distribution

From now on, we assume a known, deterministic initial state. All results from the previous discussion carry over as we are now in the special case where we the initial state is of the form *return-pmf s*.

definition $\mathcal{T}$ *p s ≡ T p (return-pmf s)*

lemma $\mathcal{T}$*-eq-return-distr*: $\mathcal{T}$ *p s =*
measure-pmf (p [] s) ⤜ (λa. distr (T (π-Suc p (s,a)) (K (s,a))) S ((##) (s,a)))
⟨*proof*⟩

lemma $\mathcal{T}$*-eq-return*:

shows $\mathcal{T}\ p\ s = do\ \{$
$y \leftarrow measure\text{-}pmf\ (p\ []\ s);$
$\omega \leftarrow T\ (\pi\text{-}Suc\ p\ (s,y))\ (K\ (s,y));$
$return\ S\ ((s,y)\ \#\#\ \omega)$
$\}$
⟨*proof*⟩

lemma $\mathcal{T}$*-return*:
shows $T\ p\ S0 = measure\text{-}pmf\ S0 \gg\!= \mathcal{T}\ p$
⟨*proof*⟩

lemma $\mathcal{T}$*-return-eq*:
shows
$\mathcal{T}\ p\ s = do\ \{$
$a \leftarrow measure\text{-}pmf\ (p\ []\ s);$
$s' \leftarrow measure\text{-}pmf\ (K\ (s,a));$
$w \leftarrow T\ (\pi\text{-}Suc\ p\ (s,a))\ (return\text{-}pmf\ s');$
$return\ S\ ((s,a)\#\#w)$
$\}$
⟨*proof*⟩

lemma $\mathcal{T}$*-eq*:
shows $\mathcal{T}\ p\ s = do\ \{$
$a \leftarrow measure\text{-}pmf\ (p\ []\ s);$
$s' \leftarrow measure\text{-}pmf\ (K\ (s,a));$
$w \leftarrow \mathcal{T}\ (\pi\text{-}Suc\ p\ (s,a))\ s';$
$return\ S\ ((s,a)\#\#w)$
$\}$
⟨*proof*⟩

lemma $\mathcal{T}$*-prob-space*[*intro*]: *prob-space* $(\mathcal{T}\ p\ s)$
⟨*proof*⟩

lemma $\mathcal{T}$*-sets*[*measurable-cong*]:
sets $(\mathcal{T}\ p\ s) = sets\ S$
⟨*proof*⟩

lemma *measurable-ident-Suc'*[*measurable*]:
$(\lambda x.\ x) \in \mathcal{T}\ (\pi\text{-}Suc\ p\ sa)\ s' \rightarrow_M S$
⟨*proof*⟩

lemma *nn-integral-*$\mathcal{T}$:
fixes $f :: ('s \times 'a)\ stream \Rightarrow real$
assumes f[*measurable*]: $f \in borel\text{-}measurable\ S$
shows $(\int^+ t.\ f\ t\ \partial\mathcal{T}\ p\ s)$
$= \int^+ a.\ \int^+ s'.\ \int^+ t'.\ f\ ((s,a)\#\#t')\ \partial\mathcal{T}\ (\pi\text{-}Suc\ p\ (s,a))\ s'\ \partial K\ (s,a)$
$\partial p\ []\ s$
⟨*proof*⟩

lemma *integral-$\mathcal{T}$*:
fixes $f :: ('s \times 'a)\ stream \Rightarrow real$
assumes *f-bounded*: $\bigwedge x.\ |f\ x| \leq B$
assumes f[*measurable*]: $f \in borel\text{-}measurable\ S$
shows $(\int t.\ f\ t\ \partial\mathcal{T}\ p\ s)$
$= \int a.\ \int s'.\ \int t'.\ f\ ((s,a)\#\#t')\ \partial\mathcal{T}\ (\pi\text{-}Suc\ p\ (s,a))\ s'\ \partial K\ (s,a)\ \partial p\ []\ s$
⟨*proof*⟩

lemma *integrable-$\mathcal{T}$-bounded*[*intro*]:
fixes $f :: ('s \times 'a)\ stream \Rightarrow 'd :: \{second\text{-}countable\text{-}topology, banach\}$
assumes f[*measurable*]: $f \in borel\text{-}measurable\ S$
assumes *b*: $bounded\ (range\ f)$
shows $integrable\ (\mathcal{T}\ p\ s)\ f$
⟨*proof*⟩

definition $Pn'\ p\ s = Pn\ p\ (return\text{-}pmf\ s)$
definition $Xn'\ p\ s = Xn\ p\ (return\text{-}pmf\ s)$
definition $Yn'\ p\ s = Yn\ p\ (return\text{-}pmf\ s)$
definition $K0'\ d\ s \equiv map\text{-}pmf\ (\lambda a.\ (s,\ a))\ (d\ s)$

definition $K\text{-}st\ d\ s \equiv d\ s \gg\!= (\lambda a.\ K\ (s,a))$

lemma *pmf-K-st*: $pmf\ (K\text{-}st\ d\ s)\ t = \int a.\ pmf\ (K(s,\ a))\ t\ \partial d\ s$
⟨*proof*⟩

K-st defines the distribution over the successor states for a given decision rule and state. It is mostly useful for markovian policies, as the information which action was selected is lost.

lemma *P0'*[*simp*]: $Pn'\ p\ s\ 0 = K0'\ (p\ [])\ s$
⟨*proof*⟩

lemma *X0'*[*simp*]: $Xn'\ p\ s\ 0 = return\text{-}pmf\ s$
⟨*proof*⟩

lemma *Pn-return-pmf*: $S0 \gg\!= (\lambda s'.\ Pn\ p\ (return\text{-}pmf\ s')\ n) = Pn\ p\ S0\ n$
⟨*proof*⟩

lemma *PSuc'*: $Pn'\ p\ s\ (Suc\ n) = K0'\ (p\ [])\ s \gg\!= (\lambda sa.\ K\ sa \gg\!= (\lambda s'.\ Pn'\ (\pi\text{-}Suc\ p\ sa)\ s'\ n))$
⟨*proof*⟩

lemma *PSuc'-markovian*:
$Pn'\ (mk\text{-}markovian\ p)\ s\ (Suc\ n) = K\text{-}st\ (p\ 0)\ s \gg\!= (\lambda s'.\ Pn'\ (mk\text{-}markovian\ (p \circ Suc))\ s'\ n)$
⟨*proof*⟩

lemma *Xn'-Suc*: $Xn'\ p\ s\ (Suc\ n) = Pn'\ p\ s\ n \gg\!= K$

⟨*proof*⟩

lemma *Xn'-Pn'*: *Xn' p s n* = *map-pmf fst* (*Pn' p s n*)
⟨*proof*⟩

lemma *Suc-Xn'*: *Xn' p s* (*Suc n*) = *p* [] *s* ⤜ (λ*a*. *K* (*s*,*a*) ⤜ (λ*s'*. *Xn'* (*π-Suc p* (*s*,*a*)) *s' n*))
⟨*proof*⟩

lemma *Suc-Xn'-markovian*:
Xn' (*mk-markovian p*) *s* (*Suc n*) = *K-st* (*p 0*) *s* ⤜ (λ*s'*. *Xn'* (*mk-markovian* (λ*n*. *p* (*Suc n*))) *s' n*)
⟨*proof*⟩

lemma *Xn'-split*: *Xn'* (*mk-markovian p*) *s* (*n* + *m*) =
Xn' (*mk-markovian p*) *s n* ⤜ (λ*s*. *Xn'* (*mk-markovian* (λ*i*. *p* (*i* + *n*))) *s m*)
⟨*proof*⟩

lemma *Yn'-markovian*: *Yn'* (*mk-markovian p*) *s n* = *Xn'* (*mk-markovian p*) *s n* ⤜ *p n*
⟨*proof*⟩

lemma *Pn'-markovian-eq-Xn'-bind*: *Pn'* (*mk-markovian p*) *s n* = *Xn'* (*mk-markovian p*) *s n* ⤜ *K0'* (*p n*)
⟨*proof*⟩

lemma *Pn'-eq-T*: *measure-pmf* (*Pn' p s n*) = *distr* (*T p s*) (*count-space UNIV*) (λ*t*. *t* !! *n*)
⟨*proof*⟩
end
end

theory *MDP-reward*
imports
Bounded-Functions
MDP-reward-Util
Blinfun-Util
MDP-disc
begin

6 Markov Decision Processes with Rewards

locale *MDP-reward* = *discrete-MDP A K*
for
A **and**
K :: *'s* ::*countable* × *'a* ::*countable* ⇒ *'s pmf* +
fixes

$r :: ('s \times 'a) \Rightarrow real$ **and**
$l :: real$
assumes
zero-le-disc [*simp*]: $0 \leq l$ **and**
disc-lt-one [*simp*]: $l < 1$ **and**
r-bounded: *bounded* (*range* r)
begin

This extension to the basic MDPs is formalized with another locale. It assumes the existence of a reward function r which takes a state-action pair to a real number. We assume that the function is bounded *r-bounded*.

Furthermore, we fix a discounting factor l, where $0 \leq l \wedge l < 1$.

6.1 Basic Properties

lemma *r-bfun*: $r \in bfun$
⟨*proof*⟩

lemma *r-bounded'*: *bounded* (r ‘ X)
⟨*proof*⟩

lemma *abs-disc-eq*[*simp*]: $|l \frown i * x| = l \frown i * |x|$
⟨*proof*⟩

definition $r_M = (\bigsqcup sa.\ |r\ sa|)$

lemma *abs-r-le-r*$_M$: $|r\ sa| \leq r_M$
⟨*proof*⟩

lemma *abs-r*$_M$*-eq-r*$_M$ [*simp*]: $|r_M| = r_M$
⟨*proof*⟩

lemma r_M*-nonneg*: $0 \leq r_M$
⟨*proof*⟩

6.2 Summability

lemma *summable-disc* [*intro*, *simp*]: *summable* ($\lambda i.\ l \frown i * x$)
⟨*proof*⟩

lemma *summable-r-disc*[*intro*, *simp*]:
summable ($\lambda i.\ |l \frown i * r\ (sa\ i)|$)
summable ($\lambda i.\ l \frown i * |r\ (sa\ i)|$)
summable ($\lambda i.\ l \frown i * r\ (sa\ i)$)
⟨*proof*⟩

6.3 Reward over a Trace

abbreviation *ν-trace-fin t N* ≡ $\sum i < N.\ l \frown i * r\ (t\ !!\ i)$
abbreviation *ν-trace t* ≡ $\sum i.\ l \frown i * r\ (t\ !!\ i)$

lemma *abs-ν-trace-le*: |*ν-trace t*| ≤ $(\sum i.\ l \frown i * r_M)$
⟨*proof*⟩

lemma *abs-ν-trace-fin-le*: |*ν-trace-fin t N*| ≤ $(\sum i < N.\ l \frown i * r_M)$
⟨*proof*⟩

lemma *measurable-suminf-reward* [*measurable*]: *ν-trace* ∈ *borel-measurable S*
⟨*proof*⟩

lemma *integrable-ν-trace-fin*: *integrable* ($\mathcal{T}$ *p s*) (λ*t*. *ν-trace-fin t N*)
⟨*proof*⟩

lemma *integrable-ν-trace*: *integrable* ($\mathcal{T}$ *p s*) *ν-trace*
⟨*proof*⟩

6.4 Integrals over Rewards

lemma *measurable-r-nth* [*measurable*]: (λ*t*. *r* (*t* !! *i*)) ∈ *borel-measurable S*
⟨*proof*⟩

lemma *integrable-r-nth* [*simp*]: *integrable* ($\mathcal{T}$ *p s*) (λ*t*. *r* (*t* !! *i*))
⟨*proof*⟩

lemma *expectation-abs-r-le*: *measure-pmf.expectation d* (λ*a*. |*r* (*s*, *a*)|) ≤ r_M
⟨*proof*⟩

6.5 Expected Total Discounted Reward

context
fixes *p* :: ('*s*, '*a*) *pol*
begin

6.6 Expected Finite-Horizon Discounted Reward

definition *ν-fin n s* = $\int$ *t*. *ν-trace-fin t n* $\partial\mathcal{T}$ *p s*

lemma *abs-ν-fin-le*: |*ν-fin N s*| ≤ $(\sum i{<}N.\ l \frown i * r_M)$
⟨*proof*⟩

lemma *ν-fin-Suc*[*simp*]: *ν-fin* (*Suc n*) *s* = *ν-fin n s* + *l* ⌢ *n* * $\int$ *t*. *r* (*t* !! *n*) $\partial\mathcal{T}$ *p s*
⟨*proof*⟩

lemma *ν-fin-zero*[*simp*]: *ν-fin 0 s = 0*
⟨*proof*⟩

lemma *ν-fin-eq-Pn*: *ν-fin n s* = (∑ *i*<*n*. *l*^*i* ∗ *measure-pmf.expectation* (*Pn*′ *p s i*) *r*)
⟨*proof*⟩

definition *ν s* = *lim* (λ*n*. *ν-fin n s*)

lemma *ν-eq-lim*: *ν s* = *lim* (λ*n*. *ν-fin n s*)
⟨*proof*⟩

lemma *ν-eq-ν-trace*: *ν s* = ∫ *t*. *ν-trace t* ∂*T p s*
⟨*proof*⟩

lemma *abs-ν-le*: |*ν s*| ≤ (∑ *i*. *l*^*i* ∗ *r*$_M$)
⟨*proof*⟩

lemma *ν-le*: *ν s* ≤ (∑ *i*. *l*^*i* ∗ *r*$_M$)
⟨*proof*⟩

lemma *ν-eq-Pn*: *ν s* = (∑ *i*. *l*^*i* ∗ *measure-pmf.expectation* (*Pn*′ *p s i*) *r*)
⟨*proof*⟩

lemma *ν-bfun*: *ν* ∈ *bfun*
⟨*proof*⟩

lemma *ν-fin-bfun*: (λ*s*. *ν-fin N s*) ∈ *bfun*
⟨*proof*⟩

lift-definition ν_b :: '*s* ⇒$_b$ *real* **is** *ν*
⟨*proof*⟩

lemma *norm-ν-le*: *norm* ν_b ≤ *r*$_M$ / (*1*−*l*)
⟨*proof*⟩
end

lemma *ν-as-markovian*: *ν* (*mk-markovian* (*as-markovian p* (*return-pmf s*))) *s* = *ν p s*
⟨*proof*⟩

6.6.1 Optimal Reward

definition *ν-MD s* ≡ ⨆ *p* ∈ Π$_{MD}$. *ν* (*mk-markovian-det p*) *s*
definition *ν-opt s* ≡ ⨆ *p* ∈ Π$_{HR}$. *ν p s*

lemma *ν-le-ν-opt* [*intro*]:
assumes *is-policy p*
shows $\nu\ p\ s \leq \nu\text{-}opt\ s$
⟨*proof*⟩

lemma *ν-opt-eq-MR*: $\nu\text{-}opt\ s = (\bigsqcup p \in \Pi_{MR}.\ \nu\ (mk\text{-}markovian\ p)\ s)$
⟨*proof*⟩

lemma *ν-opt-bfun*: $\nu\text{-}opt \in bfun$
⟨*proof*⟩

lift-definition $\nu_b\text{-}opt :: 's \Rightarrow_b real$ **is** *ν-opt*
⟨*proof*⟩

6.7 Reward of a Decision Rule

context
fixes $d :: ('s, 'a)\ dec$
begin
abbreviation $r\text{-}dec\ s \equiv \int a.\ r\ (s, a)\ \partial d\ s$

lemma *abs-r-dec-le*: $|r\text{-}dec\ s| \leq r_M$
⟨*proof*⟩

lemma *r-dec-eq-r-K0*: $r\text{-}dec\ s = measure\text{-}pmf.expectation\ (K0'\ d\ s)\ r$
⟨*proof*⟩

lemma *r-dec-bfun*: $r\text{-}dec \in bfun$
⟨*proof*⟩

lift-definition $r\text{-}dec_b :: 's \Rightarrow_b real$ **is** *r-dec*
⟨*proof*⟩

lemma *norm-r-dec-le*: $norm\ r\text{-}dec_b \leq r_M$
⟨*proof*⟩
end

lemma *r-dec-det* [*simp*]: $r\text{-}dec\ (mk\text{-}dec\text{-}det\ d)\ s = r\ (s, d\ s)$
⟨*proof*⟩

declare $r\text{-}dec_b.rep\text{-}eq$[*simp*] *bfun.Bfun-inverse*[*simp*]

6.8 Push-Forward of a Function Through the MDP

lemma *norm-l-pow-eq*[*simp*]: $norm\ (l\hat{}t *_R F) = l\hat{}t * norm\ F$
⟨*proof*⟩

lemma *summable-norm-disc-I* [*intro*]:
assumes $summable\ (\lambda t.\ (l\hat{}t * norm\ F))$

shows *summable* $(\lambda t.\ norm\ (l\hat{}t *_R F))$
⟨*proof*⟩

lemma *summable-norm-disc-I'*[*intro*]:
assumes *summable* $(\lambda t.\ (l\hat{}t * norm\ (F\ t)))$
shows *summable* $(\lambda t.\ norm\ (l\hat{}t *_R F\ t))$
⟨*proof*⟩

lemma *summable-discI* [*intro*]:
assumes *bounded* (*range F*)
shows *summable* $(\lambda t.\ l\hat{}t * norm\ (F\ t))$
⟨*proof*⟩

lemma *summable-disc-reward* [*intro*]:
assumes *bounded* (*range* $(F :: nat \Rightarrow 'b :: banach)$)
shows *summable* $(\lambda t.\ l\hat{}t *_R (F\ t))$
⟨*proof*⟩

context
fixes $p :: nat \Rightarrow ('s, 'a)\ dec$
begin
definition $\mathcal{P}_X\ n = push\text{-}exp\ (\lambda s.\ Xn'\ (mk\text{-}markovian\ p)\ s\ n)$

lemma $\mathcal{P}_X$*-0*[*simp*]: $\mathcal{P}_X\ 0 = id$
⟨*proof*⟩

lemma $\mathcal{P}_X$*-bounded-linear*[*simp*]: *bounded-linear* $(\mathcal{P}_X\ t)$
⟨*proof*⟩

lemma *norm-*$\mathcal{P}_X$ [*simp*]: *onorm* $(\mathcal{P}_X\ t) = 1$
⟨*proof*⟩

lemma *norm-*$\mathcal{P}_X$*-apply*[*simp*]: *norm* $(\mathcal{P}_X\ n\ x) \leq norm\ x$
⟨*proof*⟩

lemma $\mathcal{P}_X$*-bound-r*: *norm* $(\mathcal{P}_X\ t\ (r\text{-}dec_b\ (p\ t))) \leq r_M$
⟨*proof*⟩

lemma $\mathcal{P}_X$*-bounded-r*: *bounded* (*range* $(\lambda t.\ (\mathcal{P}_X\ t\ (r\text{-}dec_b\ (p\ t)))))$
⟨*proof*⟩

lemma *summable-norm-disc-reward'*[*simp*]:
shows *summable* $(\lambda t.\ l\hat{}t * norm\ (\mathcal{P}_X\ t\ (r\text{-}dec_b\ (p\ t))))$
⟨*proof*⟩

lemma *summable-disc-reward-*$\mathcal{P}_X$ [*simp*]:
shows *summable* $(\lambda t.\ l\hat{}t *_R \mathcal{P}_X\ t\ (r\text{-}dec_b\ (p\ t)))$
⟨*proof*⟩

lemma *disc-reward-tendsto*:
(λn. ∑ t<n. l^t *$_R$* $\mathcal{P}_X$ t (r-dec$_b$ (p t))) ⟶ (∑ t. l^t *$_R$* $\mathcal{P}_X$ t (r-dec$_b$ (p t)))
⟨*proof*⟩

end

lemma $\mathcal{P}_X$*-Suc*: $\mathcal{P}_X$ p (Suc n) v = push-exp (K-st (p 0)) (($\mathcal{P}_X$ (λn. p (Suc n)) n) v)
⟨*proof*⟩

lemma $\mathcal{P}_X$*-Suc′*: $\mathcal{P}_X$ p (Suc n) v = $\mathcal{P}_X$ p n (push-exp (K-st (p n)) v)
⟨*proof*⟩

lemma $\mathcal{P}_X$*-sconst*: $\mathcal{P}_X$ (λ-. p) n = (push-exp (K-st p))^^n
⟨*proof*⟩

lemma *norm-P-n*[*simp*]: onorm (push-exp (K-st p) ^^ n) = 1
⟨*proof*⟩

lemma *summable-norm-bfun-disc*: summable (λt. l^t * norm (apply-bfun f t))
⟨*proof*⟩

lemma *summable-bfun-disc* [*simp*]: summable (λt. l^t * (apply-bfun f t))
⟨*proof*⟩

lemma *summable-disc-norm*: summable (λx. l^x * norm c)
⟨*proof*⟩

lemma *norm-bfun-disc-le*: norm f ≤ B ⟹ (∑ x. l^x * norm (apply-bfun f x)) ≤ (∑ x. l^x * B)
⟨*proof*⟩

lemma *norm-bfun-disc-le′*: norm f ≤ B ⟹ (∑ x. l^x * (apply-bfun f x)) ≤ (∑ x. l^x * B)
⟨*proof*⟩

lemma *sum-disc-lim-l*: (∑ x. l^x * B) = B /(1−l)
⟨*proof*⟩

lemma *sum-disc-bound*: (∑ x. l^x * apply-bfun f x) ≤ (norm f) /(1−l)
⟨*proof*⟩

lemma *sum-disc-bound′*:
fixes f :: nat ⇒ 'b ⇒$_b$ real

assumes h: $\forall\, n.\ norm\ (f\ n) \leq B$
shows $norm\ (\sum x.\ l \widehat{}\ x *_R f\ x) \leq B\ /(1-l)$
⟨*proof*⟩

lemma ν_b-*le-opt* [*intro*]: $p \in \Pi_{HR} \Longrightarrow \nu_b\ p \leq \nu_b$-*opt*
⟨*proof*⟩

lemma ν_b-*le-opt-MD* [*intro*]: $p \in \Pi_{MD} \Longrightarrow \nu_b\ (mk\text{-}markovian\text{-}det\ p) \leq \nu_b$-*opt*
⟨*proof*⟩

lemma ν_b-*le-opt-DD* [*intro*]: *is-dec-det d* $\Longrightarrow \nu_b\ (mk\text{-}stationary\text{-}det\ d) \leq \nu_b$-*opt*
⟨*proof*⟩

lemma ν_b-*le-opt-DR* [*intro*]: *is-dec d* $\Longrightarrow \nu_b\ (mk\text{-}stationary\ d) \leq \nu_b$-*opt*
⟨*proof*⟩

lemma ν_b-*opt-eq-MR*: ν_b-*opt s* $= (\bigsqcup p \in \Pi_{MR}.\ \nu_b\ (mk\text{-}markovian\ p)\ s)$
⟨*proof*⟩

lemma ν_b-*as-markovian*: $\nu_b\ (mk\text{-}markovian\ (as\text{-}markovian\ p\ (return\text{-}pmf\ s)))\ s = \nu_b\ p\ s$
⟨*proof*⟩

lemma ν-*elem*: $\nu\ (mk\text{-}markovian\ p)\ s = (\sum i.\ l \widehat{}\ i * \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i))\ s)$
⟨*proof*⟩

lemma ν_b-*eq-*$\mathcal{P}_X$: $\nu_b\ (mk\text{-}markovian\ p) = (\sum i.\ l \widehat{}\ i *_R \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i)))$
⟨*proof*⟩

lemma ν-*eq-*$\mathcal{P}_X$: $\nu\ (mk\text{-}markovian\ p) = (\sum i.\ l \widehat{}\ i *_R \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i)))$
⟨*proof*⟩

$\mathcal{P}_1\ d\ v$ defines for each state the expected value of v after taking a single step in the MDP according to the decision rule d.

context
fixes $d :: ('s, 'a)\ dec$
begin
lift-definition $\mathcal{P}_1 :: ('s \Rightarrow_b real) \Rightarrow_L ('s \Rightarrow_b real)$ **is** *push-exp* (*K-st d*)
⟨*proof*⟩

lemma $\mathcal{P}_1$-*pow*: *blinfun-apply* $(\mathcal{P}_1\ \widehat{\ }\widehat{\ }\ n) =$ *blinfun-apply* $\mathcal{P}_1\ \widehat{\ }\widehat{\ }\ n$

⟨*proof*⟩

lemma $\mathcal{P}_X$-*const*: $\mathcal{P}_X\ (\lambda\text{-}.\ d)\ n = \mathcal{P}_1\ \widehat{\ }\widehat{\ }\ n$
⟨*proof*⟩

lemma *norm-*$\mathcal{P}_1$ [*simp*]: *norm* $\mathcal{P}_1 = 1$
⟨*proof*⟩

lemma *norm-*$\mathcal{P}_1$*-pow* [*simp*]: *norm* $(\mathcal{P}_1\ \widehat{\ }\widehat{\ }\ t) = 1$
⟨*proof*⟩

lemma *norm-*$\mathcal{P}_1$*-l-less*: *norm* $(l *_R \mathcal{P}_1) < 1$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-pos*: $0 \le u \Longrightarrow 0 \le \mathcal{P}_1\ u$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-n-pos*: $0 \le u \Longrightarrow 0 \le (\mathcal{P}_1\ \widehat{\ }\widehat{\ }\ n)\ u$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-n-disc-pos*: $0 \le u \Longrightarrow 0 \le (l\widehat{\ }n *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }n)\ u$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-sum-pos*: $0 \le u \Longrightarrow 0 \le (\sum t{\le}n.\ l\widehat{\ }t *_R (\mathcal{P}_1\ \widehat{\ }\widehat{\ }\ t))\ u$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-sum-ge*:
assumes $0 \le u$
shows $u \le (\sum t{\le}n.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)\ u$
⟨*proof*⟩

lemma *disc-*$\mathcal{P}_1$*-tendsto*: $(\lambda n.\ (\sum t{\le}n.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)) \longrightarrow (\sum t.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)$
⟨*proof*⟩

lemma *disc-*$\mathcal{P}_1$*-lim*: *lim* $(\lambda n.\ (\sum t{\le}n.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }\ t)) = (\sum t.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)$
⟨*proof*⟩

lemma *convergent-disc-*$\mathcal{P}_1$: *convergent* $(\lambda n.\ (\sum t{\le}n.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t))$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-suminf-ge*:
assumes $0 \le u$ **shows** $u \le (\sum t.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)\ u$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-suminf-pos*:
assumes $0 \le u$
shows $0 \le (\sum t.\ l\widehat{\ }t *_R \mathcal{P}_1\ \widehat{\ }\widehat{\ }t)\ u$

⟨*proof*⟩

lemma *lemma-6-1-2-b*:
assumes $v \leq u$
shows $(\sum t.\ l\hat{}t *_R \mathcal{P}_1\ \hat{}\hat{}t)\ v \leq (\sum t.\ l\hat{}t *_R \mathcal{P}_1\ \hat{}\hat{}t)\ u$
⟨*proof*⟩

6.9 The Bellman Operator L

definition $L\ v \equiv r\text{-}dec_b\ d + l *_R \mathcal{P}_1\ v$

lemma *norm-L-le*: $norm\ (L\ v) \leq r_M + l * norm\ v$
⟨*proof*⟩

lemma *abs-L-le*: $|L\ v\ s| \leq r_M + l * norm\ v$
⟨*proof*⟩

lemma ν*-stationary*: $\nu_b\ (mk\text{-}stationary\ d) = (\sum t.\ l\hat{}t *_R (\mathcal{P}_1\ \hat{}\hat{}\ t))\ (r\text{-}dec_b\ d)$
⟨*proof*⟩
end

lemma $\mathcal{P}_1$*-eq-*$\mathcal{P}_X$*-one*: *blinfun-apply* $(\mathcal{P}_1\ (p\ 0)) = \mathcal{P}_X\ p\ 1$
⟨*proof*⟩

The value of a markovian policy can be expressed in terms of L.

lemma ν*-step*: $\nu_b\ (mk\text{-}markovian\ p) = L\ (p\ 0)\ (\nu_b\ (mk\text{-}markovian\ (\lambda n.\ p\ (Suc\ n))))$
⟨*proof*⟩

lemma *L-*ν*-fix*: $\nu_b\ (mk\text{-}stationary\ d) = L\ d\ (\nu_b\ (mk\text{-}stationary\ d))$
⟨*proof*⟩

lemma *L-fix-*ν:
assumes $L\ p\ v = v$
shows $v = \nu_b\ (mk\text{-}stationary\ p)$
⟨*proof*⟩

lemma *L-*ν*-fix-iff*: $L\ d\ v = v \longleftrightarrow v = \nu_b\ (mk\text{-}stationary\ d)$
⟨*proof*⟩

lemma *apply-bfun-bounded-above* [*simp*, *intro*]:
fixes $f :: 'c \Rightarrow_b real$
shows *bounded* $(f\ `\ X)$
⟨*proof*⟩

lemma *apply-bfun-bdd-above*[*simp*, *intro*]:
fixes $f :: 'c \Rightarrow_b real$
shows *bdd-above* $(f \mathbin{`} X)$
⟨*proof*⟩

lemma *L-bounded*[*simp*, *intro*]: *bounded* (*range* $(\lambda p.\ L\ p\ v\ s)$)
⟨*proof*⟩

lemma *L-bounded'*[*simp*, *intro*]: *bounded* $((\lambda p.\ L\ p\ v\ s) \mathbin{`} X)$
⟨*proof*⟩

lemma *L-bdd-above*[*simp*, *intro*]: *bdd-above* $((\lambda p.\ L\ p\ v\ s) \mathbin{`} X)$
⟨*proof*⟩

6.10 Optimality Equations

definition $\mathcal{L}\ (v :: 's \Rightarrow_b real)\ s = (\bigsqcup d \in D_R.\ L\ d\ v\ s)$

lemma $\mathcal{L}$*-bfun*: $\mathcal{L}\ v \in bfun$
⟨*proof*⟩

lift-definition $\mathcal{L}_b :: ('s \Rightarrow_b real) \Rightarrow 's \Rightarrow_b real$ **is** $\mathcal{L}$
⟨*proof*⟩

lemma *L-le-*$\mathcal{L}_b$: *is-dec* $d \Longrightarrow L\ d\ v \leq \mathcal{L}_b\ v$
⟨*proof*⟩

6.11 Monotonicity

lemma $\mathcal{P}_1$*-mono*[*intro*]: $a \leq b \Longrightarrow \mathcal{P}_1\ p\ a \leq \mathcal{P}_1\ p\ b$
⟨*proof*⟩

lemma $\mathcal{P}_X$*-mono*[*intro*]: $a \leq b \Longrightarrow \mathcal{P}_X\ p\ n\ a \leq \mathcal{P}_X\ p\ n\ b$
⟨*proof*⟩

lemma *L-mono*: $u \leq v \Longrightarrow L\ d\ u \leq L\ d\ v$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-mono*:
assumes $u \leq v$ **shows** $\mathcal{L}_b\ u \leq \mathcal{L}_b\ v$
⟨*proof*⟩

lemma *step-mono*:
assumes $\mathcal{L}_b\ v \leq v\ d \in D_R$
shows $L\ d\ v \leq v$
⟨*proof*⟩

lemma *step-mono-elem*:

assumes $v \leq \mathcal{L}_b\ v\ e > 0$
shows $\exists\, d{\in}D_R.\ v \leq L\ d\ v + e *_R 1$
⟨*proof*⟩

lemma *p-n-π-MD*[*intro*]: $p \in \Pi_{M\,D} \Longrightarrow p\ n \in D_D$
⟨*proof*⟩

lemma *p-n-π-MR*[*intro*]: $p \in \Pi_{M\,R} \Longrightarrow p\ n \in D_R$
⟨*proof*⟩

lemma $\mathcal{P}_X$*-Suc-n-elem*: $\mathcal{P}_X\ p\ n\ (\mathcal{P}_1\ (p\ n)\ v) = \mathcal{P}_X\ p\ (Suc\ n)\ v$
⟨*proof*⟩

lift-definition ν_b*-fin* :: $('s,\ 'a)\ pol \Rightarrow nat \Rightarrow 's \Rightarrow_b real$ **is** ν*-fin*
⟨*proof*⟩

lemma ν*-fin-elem*: ν*-fin* $(mk\text{-}markovian\ p)\ n\ s = (\sum i{<}n.\ l\hat{}i * \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i))\ s)$
⟨*proof*⟩

lemma ν_b*-fin-eq-*$\mathcal{P}_X$: ν_b*-fin* $(mk\text{-}markovian\ p)\ n = (\sum i{<}n.\ l\hat{}i *_R \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i)))$
⟨*proof*⟩

lemma ν*-fin-eq-*$\mathcal{P}_X$: ν*-fin* $(mk\text{-}markovian\ p)\ n = (\sum i{<}n.\ l\hat{}i *_R \mathcal{P}_X\ p\ i\ (r\text{-}dec_b\ (p\ i)))$
⟨*proof*⟩

abbreviation $\mathcal{P}_d\ p\ n\ v \equiv l\hat{}n *_R \mathcal{P}_X\ p\ n\ v$

lemma $\mathcal{P}_X$*-L-le*:
assumes $\mathcal{L}_b\ v \leq v\ p \in \Pi_{M\,R}$
shows $\mathcal{P}_X\ p\ n\ (L\ (p\ n)\ v) \leq \mathcal{P}_X\ p\ n\ v$
⟨*proof*⟩

lemma ν_b*-fin-tendsto-*ν_b: $(\nu_b\text{-}fin\ (mk\text{-}markovian\ p)) \longrightarrow \nu_b\ (mk\text{-}markovian\ p)$
⟨*proof*⟩

lemma $\mathcal{P}_d$*-lim*: $(\lambda n.\ (\mathcal{P}_d\ p\ n\ v)) \longrightarrow 0$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-bfun-one* [*simp*]:$\mathcal{P}_1\ p\ 1 = 1$
⟨*proof*⟩

lemma $\mathcal{P}_1$*-pow-bfun-one* [*simp*]: $((\mathcal{P}_1\ p)\hat{\ }\hat{\ }t)\ 1 = 1$
⟨*proof*⟩

6.12 Properties of Solutions of the Optimality Equations

lemma $\mathcal{L}$-*dec-ge-opt*:
assumes $\mathcal{L}_b\ v \leq v$
shows ν_b-*opt* $\leq v$
⟨*proof*⟩

lemma $\mathcal{L}$-*inc-le-opt*:
assumes $v \leq \mathcal{L}_b\ v$
shows $v \leq \nu_b$-*opt*
⟨*proof*⟩

lemma $\mathcal{L}$-*fix-imp-opt*:
assumes $v = \mathcal{L}_b\ v$
shows $v = \nu_b$-*opt*
⟨*proof*⟩

lemma *bounded-P*: *bounded* ($\mathcal{P}_1$ ‘ X)
⟨*proof*⟩

6.13 Solutions to the Optimality Equation

6.13.1 $\mathcal{L}_b$ and L are Contraction Mappings

declare *bounded-apply-blinfun*[*intro*] *bounded-apply-bfun'*[*intro*]

lemma *contraction-*$\mathcal{L}$: *dist* ($\mathcal{L}_b\ v$) ($\mathcal{L}_b\ u$) $\leq l *$ *dist* $v\ u$
⟨*proof*⟩

lemma *is-contraction-*$\mathcal{L}$: *is-contraction* $\mathcal{L}_b$
⟨*proof*⟩

lemma *contraction-L*: *dist* ($L\ p\ v$) ($L\ p\ u$) $\leq l *$ *dist* $v\ u$
⟨*proof*⟩

lemma *is-contraction-L*: *is-contraction* ($L\ p$)
⟨*proof*⟩

6.13.2 Existence of a Fixpoint of $\mathcal{L}_b$

lemma $\mathcal{L}_b$-*conv*:
$\exists!v.\ \mathcal{L}_b\ v = v$
$(\lambda n.\ (\mathcal{L}_b\ \widehat{}\ n)\ v) \longrightarrow (THE\ v.\ \mathcal{L}_b\ v = v)$
⟨*proof*⟩

lemma $\mathcal{L}_b$-*fix-iff-opt* [*simp*]: $\mathcal{L}_b\ v = v \longleftrightarrow v = \nu_b$-*opt*
⟨*proof*⟩

lemma ν_b*-opt-fix*: $\nu_b\text{-}opt = (THE\ v.\ \mathcal{L}_b\ v = v)$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-opt* [*simp*]: $\mathcal{L}_b\ \nu_b\text{-}opt = \nu_b\text{-}opt$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-lim*: $(\lambda n.\ (\mathcal{L}_b\ \widehat{\ }\widehat{\ }\ n)\ v) \longrightarrow \nu_b\text{-}opt$
⟨*proof*⟩

lemma *thm-6-2-6*: $\nu_b\ p = \nu_b\text{-}opt \longleftrightarrow \mathcal{L}_b\ (\nu_b\ p) = \nu_b\ p$
⟨*proof*⟩

lemma *thm-6-2-6′*: $\nu\ p = \nu\text{-}opt \longleftrightarrow \mathcal{L}_b\ (\nu_b\ p) = \nu_b\ p$
⟨*proof*⟩

6.14 Existence of Optimal Policies

definition ν*-improving* $v\ d \longleftrightarrow (\forall\, s.\ is\text{-}arg\text{-}max\ (\lambda d.\ (L\ d\ v)\ s)\ (\lambda d.\ d \in D_R)\ d)$

lemma ν*-improving-iff*: $\nu\text{-}improving\ v\ d \longleftrightarrow d \in D_R \wedge (\forall\, d' \in D_R.\ \forall\, s.\ L\ d'\ v\ s \leq L\ d\ v\ s)$
⟨*proof*⟩

lemma ν*-improving-D-MR*[*dest*]: $\nu\text{-}improving\ v\ d \Longrightarrow d \in D_R$
⟨*proof*⟩

lemma ν*-improving-ge*: $\nu\text{-}improving\ v\ d \Longrightarrow d' \in D_R \Longrightarrow L\ d'\ v\ s \leq L\ d\ v\ s$
⟨*proof*⟩

lemma ν*-improving-imp-*$\mathcal{L}_b$: $\nu\text{-}improving\ v\ d \Longrightarrow \mathcal{L}_b\ v = L\ d\ v$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-imp-*ν*-improving*:
assumes $d \in D_R\ \mathcal{L}_b\ v = L\ d\ v$
shows $\nu\text{-}improving\ v\ d$
⟨*proof*⟩

lemma ν*-improving-alt*:
assumes $d \in D_R$
shows $\nu\text{-}improving\ v\ d \longleftrightarrow \mathcal{L}_b\ v = L\ d\ v$
⟨*proof*⟩

definition ν*-conserving* $d = \nu\text{-}improving\ (\nu_b\text{-}opt)\ d$

lemma ν*-conserving-iff*: $\nu\text{-}conserving\ d \longleftrightarrow d \in D_R \wedge (\forall\, d' \in D_R.\ \forall\, s.\ L\ d'\ \nu_b\text{-}opt\ s \leq L\ d\ \nu_b\text{-}opt\ s)$

⟨*proof*⟩

lemma *ν-conserving-ge*: *ν-conserving d* ⟹ *d′* ∈ D_R ⟹ *L d′ ν_b-opt s* ≤ *L d ν_b-opt s*
⟨*proof*⟩

lemma *ν-conserving-imp-*$\mathcal{L}_b$ [*simp*]: *ν-conserving d* ⟹ *L d ν_b-opt* = *ν_b-opt*
⟨*proof*⟩

lemma $\mathcal{L}_b$*-imp-ν-conserving*:
assumes *d* ∈ D_R $\mathcal{L}_b$ *ν_b-opt* = *L d ν_b-opt*
shows *ν-conserving d*
⟨*proof*⟩

lemma *ν-conserving-alt*:
assumes *d* ∈ D_R
shows *ν-conserving d* ⟷ $\mathcal{L}_b$ *ν_b-opt* = *L d ν_b-opt*
⟨*proof*⟩

lemma *ν-conserving-alt′*:
assumes *d* ∈ D_R
shows *ν-conserving d* ⟷ *L d ν_b-opt* = *ν_b-opt*
⟨*proof*⟩

6.14.1 Conserving Decision Rules are Optimal

theorem *ex-improving-imp-conserving*:
assumes ⋀*v*. ∃ *d*. *ν-improving v* (*mk-dec-det d*)
shows ∃ *d*. *ν-conserving* (*mk-dec-det d*)
⟨*proof*⟩

theorem *conserving-imp-opt*[*simp*]:
assumes *ν-conserving* (*mk-dec-det d*)
shows *ν_b* (*mk-stationary-det d*) = *ν_b-opt*
⟨*proof*⟩

lemma *conserving-imp-opt′*:
assumes ∃ *d*. *ν-conserving* (*mk-dec-det d*)
shows ∃ *d* ∈ D_D. (*ν_b* (*mk-stationary-det d*)) = *ν_b-opt*
⟨*proof*⟩

theorem *improving-att-imp-det-opt*:
assumes ⋀*v*. ∃ *d*. *ν-improving v* (*mk-dec-det d*)
shows *ν_b-opt s* = (⨆ *d* ∈ D_D. *ν_b* (*mk-stationary-det d*) *s*)
⟨*proof*⟩

6.14.2 Bellman Operator for Single Actions

abbreviation $L_a\ a\ v\ s \equiv r\ (s,\ a) + l * \mathit{measure\text{-}pmf.expectation}\ (K\ (s,a))\ v$

lemma L_a*-le*:
fixes $v :: 's \Rightarrow_b \mathit{real}$
shows $|L_a\ a\ v\ s| \leq r_M + l * \mathit{norm}\ v$
⟨*proof*⟩

lemma L_a*-bounded*:
bounded (*range* ($\lambda a.\ L_a\ a$ (*apply-bfun* v) s))
⟨*proof*⟩

lemma L_a*-int*:
fixes $d :: 'a\ \mathit{pmf}$ **and** $v :: 's \Rightarrow_b \mathit{real}$
shows $(\int a.\ L_a\ a\ v\ s\ \partial d) = (\int a.\ r\ (s,\ a)\ \partial d) + l * \int a.\ \int s'.\ v\ s'\ \partial K\ (s,\ a)\ \partial d$
⟨*proof*⟩

lemma *L-eq-*L_a: $L\ d\ v\ s = \mathit{measure\text{-}pmf.expectation}\ (d\ s)\ (\lambda a.\ L_a\ a\ v\ s)$
⟨*proof*⟩

lemma *L-eq-*L_a*-det*: $L\ (\mathit{mk\text{-}dec\text{-}det}\ d)\ v\ s = L_a\ (d\ s)\ v\ s$
⟨*proof*⟩

6.14.3 Equivalences involving $\mathcal{L}_b$

lemma *SUP-step-MR-eq*:
$(\bigsqcup d \in D_R.\ L\ d\ v\ s) = (\bigsqcup pa \in \{pa.\ \mathit{set\text{-}pmf}\ pa \subseteq A\ s\}.\ (\int a.\ L_a\ a\ v\ s\ \partial \mathit{measure\text{-}pmf}\ pa))$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-sup-att-dec*:
assumes $d \in D_R\ \mathcal{L}_b\ v = L\ d\ v$
shows $\exists\, d' \in D_D.\ \mathcal{L}_b\ v = L\ (\mathit{mk\text{-}dec\text{-}det}\ d')\ v$
⟨*proof*⟩

lemma *SUP-step-det-eq*: $(\bigsqcup d \in D_D.\ L\ (\mathit{mk\text{-}dec\text{-}det}\ d)\ v\ s) = (\bigsqcup a \in A\ s.\ L_a\ a\ v\ s)$
⟨*proof*⟩

lemma *integrable-*L_a: *integrable* (*measure-pmf* x) ($\lambda a.\ L_a\ a$ (*apply-bfun* v) s)
⟨*proof*⟩

lemma *SUP-*L_a*-eq-det*:
fixes $v :: 's \Rightarrow_b \mathit{real}$
shows $(\bigsqcup p \in \{p.\ \mathit{set\text{-}pmf}\ p \subseteq A\ s\}.\ \int a.\ L_a\ a\ v\ s\ \partial \mathit{measure\text{-}pmf}\ p) =$

$(\bigsqcup a{\in}A\ s.\ L_a\ a\ v\ s)$
⟨*proof*⟩

lemma $\mathcal{L}$*-eq-SUP-det*: $\mathcal{L}\ v\ s = (\bigsqcup d \in D_D.\ L\ (mk\text{-}dec\text{-}det\ d)\ v\ s)$
⟨*proof*⟩

lemma $\mathcal{L}_b$*-eq-SUP-det*: $\mathcal{L}_b\ v\ s = (\bigsqcup d \in D_D.\ L\ (mk\text{-}dec\text{-}det\ d)\ v\ s)$
⟨*proof*⟩

6.14.4 Deterministic Decision Rules are Optimal

lemma *opt-imp-opt-dec-det*:
assumes $p \in \Pi_{H\,R}\ \nu_b\ p = \nu_b\text{-}opt$
shows $\exists\, d \in D_D.\ \nu_b\ (mk\text{-}stationary\ (mk\text{-}dec\text{-}det\ d)) = \nu_b\text{-}opt$
⟨*proof*⟩

6.14.5 Optimal Decision Rules for Finite Action Spaces

lemma *thm-6-2-10*:
assumes $\bigwedge s.\ finite\ (A\ s)$
shows $\exists\, d \in D_D.\ \nu_b\text{-}opt = \nu_b\ (mk\text{-}stationary\text{-}det\ d)$
⟨*proof*⟩

6.14.6 Existence of Epsilon-Optimal Policies

lemma *ex-det-eps*:
assumes $0 < e$
shows $\exists\, d \in D_D.\ \mathcal{L}_b\ v \le L\ (mk\text{-}dec\text{-}det\ d)\ v + e *_R 1$
⟨*proof*⟩

lemma *thm-6-2-11*:
assumes $eps > 0$
shows $\exists\, d \in D_D.\ \nu_b\text{-}opt \le \nu_b\ (mk\text{-}stationary\text{-}det\ d) + eps *_R 1$
⟨*proof*⟩

lemma *ex-det-dist-eps*:
assumes $0 < (e :: real)$
shows $\exists\, d \in D_D.\ dist\ (\mathcal{L}_b\ v)\ (L\ (mk\text{-}dec\text{-}det\ d)\ v) \le e$
⟨*proof*⟩

lemma ν_b*-opt-le-det*: $\nu_b\text{-}opt\ s \le (\bigsqcup d \in D_D.\ \nu_b\ (mk\text{-}stationary\text{-}det\ d)\ s)$
⟨*proof*⟩

lemma ν_b*-opt-eq-det*: $\nu_b\text{-}opt\ s = (\bigsqcup d \in D_D.\ \nu_b\ (mk\text{-}stationary\text{-}det\ d)\ s)$
⟨*proof*⟩

lemma *lemma-6-3-1-a*:
assumes *v0* ∈ *bfun*
shows *uniform-limit UNIV* (λ*n*. ((λ*v*. ℒ (*Bfun v*)) ^^ *n*) *v0*) *ν-opt sequentially*
⟨*proof*⟩

lemma *thm-6-3-1-b-aux*: (λ*n*. *dist* ((ℒ$_b$ ^^ *n*) *v*) ((ℒ$_b$ ^^ (*Suc n*)) *v*)) ⟶ *0*
⟨*proof*⟩

definition *max-L-ex s v* ≡ *has-arg-max* (λ*a*. *L$_a$ a v s*) (*A s*)

end

6.15 More Restrictive MDP Locales

locale *MDP-att-ℒ* = *MDP-reward A K r l*
for
A **and**
K :: *'s* ::*countable* × *'a* ::*countable* ⇒ *'s pmf* **and**
r **and** *l* +
assumes *Sup-att*: *max-L-ex* (*s* :: *'s*) *v*
begin

theorem *thm-6-2-10-a-aux'*:
fixes *v* :: *'s* ⇒$_b$ *real*
assumes *is-arg-max* (λ*a*. *L$_a$ a v s*) (λ*a*. *a* ∈ *A s*) *a*
shows ℒ$_b$ *v s* = *L$_a$ a v s*
⟨*proof*⟩

end

locale *MDP-act* = *MDP-att-ℒ A K r l*
for *A* :: *'s* :: *countable* ⇒ (*'a* :: *countable*) *set* **and** *K r l* +
fixes *arb-act* :: *'a set* ⇒ *'a*
assumes *arb-act-in*[*simp*]: *X* ≠ {} ⟹ *arb-act X* ∈ *X*
begin

definition *is-opt-act v s* = *is-arg-max* (λ*a*. *L$_a$ a v s*) (λ*a*. *a* ∈ *A s*)
abbreviation *opt-acts v s* ≡ {*a*. *is-opt-act v s a*}

lemma *is-opt-act-some*: *is-opt-act v s* (*arb-act* (*opt-acts v s*))
⟨*proof*⟩

lemma *some-opt-acts-in-A*: *arb-act* (*opt-acts v s*) ∈ *A s*
⟨*proof*⟩

lemma *ν-improving-opt-acts*: *ν-improving v0* (*mk-dec-det* (λ*s*. *arb-act*

(*opt-acts* (*apply-bfun v0*) *s*)))
⟨*proof*⟩

end

locale *MDP-finite-type* = *MDP-reward A K r l*
for *A* **and** *K* :: *'s* :: *finite* × *'a* :: *finite* ⇒ *'s pmf* **and** *r l*

context *MDP-reward*
begin

lemma ν_b*-fin-zero*[*simp*]: ν_b*-fin p 0* = *0*
⟨*proof*⟩

lemma ν_b*-fin-Suc*[*simp*]: ν_b*-fin* (*mk-stationary d*) (*Suc n*) = ν_b*-fin* (*mk-stationary d*) *n* + ((*l* $*_R$ $\mathcal{P}_1$ *d*) ^^ *n*) (*r-dec*$_b$ *d*)
⟨*proof*⟩

lemma ν_b*-fin-eq*: ν_b*-fin* (*mk-stationary d*) *n* = (∑ *i* < *n*. ((*l* $*_R$ $\mathcal{P}_1$ *d*) ^^ *i*)) (*r-dec*$_b$ *d*)
⟨*proof*⟩

lemma *L-iter*: (*L d* ^^ *m*) *v* = ν_b*-fin* (*mk-stationary d*) *m* + ((*l* $*_R$ $\mathcal{P}_1$ *d*) ^^ *m*) *v*
⟨*proof*⟩

lemma *bounded-stationary-*ν_b*-fin*: *bounded* ((λ*x*. (ν_b*-fin* (*mk-stationary x*) *N*) *s*) ‘ *X*)
⟨*proof*⟩

lemma *bounded-disc-*$\mathcal{P}_1$: *bounded* ((λ*x*. (((*l* $*_R$ $\mathcal{P}_1$ *x*) ^^ *m*) *v*) *s*) ‘ *X*)
⟨*proof*⟩

lemma *bounded-disc-*$\mathcal{P}_1$*'*: *bounded* ((λ*x*. (($\mathcal{P}_1$ *x* ^^ *m*) *v*) *s*) ‘ *X*)
⟨*proof*⟩

lemma *L-iter-le-*$\mathcal{L}_b$: *is-dec d* ⟹ (*L d* ^^ *n*) *v* ≤ ($\mathcal{L}_b$ ^^ *n*) *v*
⟨*proof*⟩
end

context *MDP-att-*$\mathcal{L}$
begin
lemma L_a*-le-arg-max*: *a* ∈ *A s* ⟹ L_a *a v s* ≤ L_a (*arg-max-on* (λ*a*. L_a *a v s*) (*A s*)) *v s*
⟨*proof*⟩

lemma *arg-max-on-in*: *has-arg-max f Q* $\Longrightarrow$ *arg-max-on f Q* $\in$ *Q*
⟨*proof*⟩

lemma $\mathcal{L}_b$*-eq-*L_a*-max*: $\mathcal{L}_b\ v\ s = L_a\ (arg\text{-}max\text{-}on\ (\lambda a.\ L_a\ a\ v\ s)\ (A\ s))\ v\ s$
⟨*proof*⟩

lemma *ex-opt-det*: $\exists\, d \in D_D.\ \mathcal{L}_b\ v = L\ (mk\text{-}dec\text{-}det\ d)\ v$
⟨*proof*⟩

lemma *ex-improving-det*: $\exists\, d \in D_D.\ \nu\text{-}improving\ v\ (mk\text{-}dec\text{-}det\ d)$
⟨*proof*⟩
end

end

References

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