L^p spaces in Isabelle

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Abstract

L^p is the space of functions whose p-th power is integrable. It is one of the most fundamental Banach spaces that is used in analysis and probability. We develop a framework for function spaces, and then implement the L^p spaces in this framework using the existing integration theory in Isabelle/HOL. Our development contains most fundamental properties of L^p spaces, notably the Hölder and Minkowski inequalities, completeness of L^p, duality, stability under almost sure convergence, multiplication of functions in L^p and L^q, stability under conditional expectation.

Contents

theory *Functional-Spaces*
imports
HOL−Analysis.Analysis
HOL−Library.Function-Algebras
Ergodic-Theory.SG-Library-Complement
begin

1 Functions as a real vector space

Many functional spaces are spaces of functions. To be able to use the following framework, spaces of functions thus need to be endowed with a vector space structure, coming from pointwise addition and multiplication.

Some instantiations for `fun` are already given in `Function_Algebras.thy` and `Lattices.thy`, we add several.

`minus_fun` is already defined, in `Lattices.thy`, but under the strange name `fun_Compl_def`. We restate the definition so that `unfolding minus_fun_def` works. Same thing for `minus_fun_def`. A better solution would be to have a coherent naming scheme in `Lattices.thy`.

lemmas *uminus-fun-def* = *fun-Compl-def*
lemmas *minus-fun-def* = *fun-diff-def*

lemma *fun-sum-apply*:
fixes $u::'i \Rightarrow 'a \Rightarrow ('b::comm\text{-}monoid\text{-}add)$
shows $(sum\ u\ I)\ x = sum\ (\lambda i.\ u\ i\ x)\ I$
⟨*proof*⟩

instantiation *fun* :: (*type*, *real-vector*) *real-vector*
begin

definition $scaleR\text{-}fun::real \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
where $scaleR\text{-}fun = (\lambda c\ f.\ (\lambda x.\ c *_R f\ x))$

lemma *scaleR-apply* [*simp*, *code*]: $(c *_R f)\ x = c *_R (f\ x)$
⟨*proof*⟩

instance ⟨*proof*⟩
end

lemmas *divideR-apply* = *scaleR-apply*

lemma [*measurable*]:

0 $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *borel-measurable-const-scaleR'* [*measurable* (*raw*)]:
(f::('a $\Rightarrow$ 'b::*real-normed-vector*)) $\in$ *borel-measurable M* $\Longrightarrow$ $c *_R f$ $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *borel-measurable-add'*[*measurable* (*raw*)]:
fixes f g :: 'a $\Rightarrow$ 'b::{*second-countable-topology*, *real-normed-vector*}
assumes f: f $\in$ *borel-measurable M*
assumes g: g $\in$ *borel-measurable M*
shows $f + g$ $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *borel-measurable-uminus'*[*measurable* (*raw*)]:
fixes f g :: 'a $\Rightarrow$ 'b::{*second-countable-topology*, *real-normed-vector*}
assumes f: f $\in$ *borel-measurable M*
shows $-f$ $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *borel-measurable-diff'*[*measurable* (*raw*)]:
fixes f g :: 'a $\Rightarrow$ 'b::{*second-countable-topology*, *real-normed-vector*}
assumes f: f $\in$ *borel-measurable M*
assumes g: g $\in$ *borel-measurable M*
shows $f - g$ $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *borel-measurable-sum'*[*measurable* (*raw*)]:
fixes f::'i $\Rightarrow$ 'a $\Rightarrow$ 'b::{*second-countable-topology*, *real-normed-vector*}
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i$ $\in$ *borel-measurable M*
shows $(\sum i{\in}I.\ f\ i)$ $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *zero-applied-to* [*simp*]:
(*0*::('a $\Rightarrow$ ('b::*real-vector*))) x = *0*
⟨*proof*⟩

2 Quasinorms on function spaces

A central feature of modern analysis is the use of various functional spaces, and of results of functional analysis on them. Think for instance of L^p spaces, of Sobolev or Besov spaces, or variations around them. Here are several relevant facts about this point of view:

- These spaces typically depend on one or several parameters. This makes it difficult to play with type classes in a system without dependent types.

- The L^p spaces are not spaces of functions (their elements are equivalence classes of functions, where two functions are identified if they coincide almost everywhere). However, in usual analysis proofs, one takes a definite representative and works with it, never going to the equivalence class point of view (which only becomes relevant when one wants to use the fact that one has a Banach space at our disposal, to apply functional analytic tools).
- It is important to describe how the spaces are related to each other, with respect to inclusions or compact inclusions. For instance, one of the most important theorems in analysis is Sobolev embedding theorem, describing when one Sobolev space is included in another one. One also needs to be able to take intersections or sums of Banach spaces, for instance to develop interpolation theory.
- Some other spaces play an important role in analysis, for instance the weak L^1 space. This space only has a quasi-norm (i.e., its norm satisfies the triangular inequality up to a fixed multiplicative constant). A general enough setting should also encompass this kind of space. (One could argue that one should also consider more general topologies such as Frechet spaces, to deal with Gevrey or analytic functions. This is true, but considering quasi-norms already gives a wealth of applications).

Given these points, it seems that the most effective way of formalizing this kind of question in Isabelle/HOL is to think of such a functional space not as an abstract space or type, but as a subset of the space of all functions or of all distributions. Functions that do not belong to the functional space under consideration will then have infinite norm. Then inclusions, intersections, and so on, become trivial to implement. Since the same object contains both the information about the norm and the space where the norm is finite, it conforms to the customary habit in mathematics of identifying the two of them, talking for instance about the L^p space and the L^p norm.

All in all, this approach seems quite promising for "real life analysis".

2.1 Definition of quasinorms

typedef (**overloaded**) $('a::\textit{real-vector})\ \textit{quasinorm} = \{(C::\textit{real}, N::('a \Rightarrow \textit{ennreal})).\ (C \geq 1)$
$\wedge\ (\forall\ x\ c.\ N\ (c *_R x) = \textit{ennreal}\ |c| * N(x)) \wedge (\forall\ x\ y.\ N(x{+}y) \leq C * N\ x + C * N\ y)\}$
morphisms *Rep-quasinorm quasinorm-of*
$\langle \textit{proof} \rangle$

definition $\textit{eNorm}::'a\ \textit{quasinorm} \Rightarrow ('a::\textit{real-vector}) \Rightarrow \textit{ennreal}$
  where $\textit{eNorm}\ N\ x = (\textit{snd}\ (\textit{Rep-quasinorm}\ N))\ x$

definition *defect*::(*'a*::*real-vector*) *quasinorm* ⇒ *real*
  where *defect N* = *fst* (*Rep-quasinorm N*)

lemma *eNorm-triangular-ineq*:
  $eNorm\ N\ (x + y) \leq defect\ N * eNorm\ N\ x + defect\ N * eNorm\ N\ y$
⟨*proof*⟩

lemma *defect-ge-1*:
  $defect\ N \geq 1$
⟨*proof*⟩

lemma *eNorm-cmult*:
  $eNorm\ N\ (c *_R x) = ennreal\ |c| * eNorm\ N\ x$
⟨*proof*⟩

lemma *eNorm-zero* [*simp*]:
  $eNorm\ N\ 0 = 0$
⟨*proof*⟩

lemma *eNorm-uminus* [*simp*]:
  $eNorm\ N\ (-x) = eNorm\ N\ x$
⟨*proof*⟩

lemma *eNorm-sum*:
  $eNorm\ N\ (\sum i{\in}\{..{<}n\}.\ u\ i) \leq (\sum i{\in}\{..{<}n\}.\ (defect\ N)\hat{}(Suc\ i) * eNorm\ N\ (u\ i))$
⟨*proof*⟩

Quasinorms are often defined by taking a meaningful formula on a vector subspace, and then extending by infinity elsewhere. Let us show that this results in a quasinorm on the whole space.

definition *quasinorm-on*::(*'a set*) ⇒ *real* ⇒ ((*'a*::*real-vector*) ⇒ *ennreal*) ⇒ *bool*
  where *quasinorm-on F C N* = (
    $(\forall x\ y.\ (x \in F \wedge y \in F) \longrightarrow (x + y \in F) \wedge N\ (x{+}y) \leq C * N\ x + C * N\ y)$
    $\wedge\ (\forall c\ x.\ x \in F \longrightarrow c *_R x \in F \wedge N(c *_R x) = |c| * N\ x)$
    $\wedge\ C \geq 1 \wedge 0 \in F)$

lemma *quasinorm-of*:
  fixes *N*::(*'a*::*real-vector*) ⇒ *ennreal* **and** *C*::*real*
  assumes *quasinorm-on UNIV C N*
  shows *eNorm* (*quasinorm-of* (*C*,*N*)) *x* = *N x*
    defect (*quasinorm-of* (*C*,*N*)) = *C*
⟨*proof*⟩

lemma *quasinorm-onI*:
  fixes *N*::(*'a*::*real-vector*) ⇒ *ennreal* **and** *C*::*real* **and** *F*::*'a set*
  assumes $\bigwedge x\ y.\ x \in F \Longrightarrow y \in F \Longrightarrow x + y \in F$
    $\bigwedge x\ y.\ x \in F \Longrightarrow y \in F \Longrightarrow N\ (x + y) \leq C * N\ x + C * N\ y$

$\bigwedge c\ x.\ c \neq 0 \Longrightarrow x \in F \Longrightarrow c *_R x \in F$
$\bigwedge c\ x.\ c \neq 0 \Longrightarrow x \in F \Longrightarrow N\ (c *_R x) \leq ennreal\ |c| * N\ x$
$0 \in F\ N(0) = 0\ C \geq 1$
shows *quasinorm-on F C N*
⟨*proof*⟩

lemma *extend-quasinorm*:
assumes *quasinorm-on F C N*
shows *quasinorm-on UNIV C* $(\lambda x.\ if\ x \in F\ then\ N\ x\ else\ \infty)$
⟨*proof*⟩

2.2 The space and the zero space of a quasinorm

The space of a quasinorm is the vector subspace where it is meaningful, i.e., finite.

definition $space_N$::('a::real-vector) quasinorm ⇒ 'a set
where $space_N\ N = \{f.\ eNorm\ N\ f < \infty\}$

lemma *spaceN-iff*:
$x \in space_N\ N \longleftrightarrow eNorm\ N\ x < \infty$
⟨*proof*⟩

lemma *spaceN-cmult* [*simp*]:
assumes $x \in space_N\ N$
shows $c *_R x \in space_N\ N$
⟨*proof*⟩

lemma *spaceN-add* [*simp*]:
assumes $x \in space_N\ N\ y \in space_N\ N$
shows $x + y \in space_N\ N$
⟨*proof*⟩

lemma *spaceN-diff* [*simp*]:
assumes $x \in space_N\ N\ y \in space_N\ N$
shows $x - y \in space_N\ N$
⟨*proof*⟩

lemma *spaceN-contains-zero* [*simp*]:
$0 \in space_N\ N$
⟨*proof*⟩

lemma *spaceN-sum* [*simp*]:
assumes $\bigwedge i.\ i \in I \Longrightarrow x\ i \in space_N\ N$
shows $(\sum i{\in}I.\ x\ i) \in space_N\ N$
⟨*proof*⟩

The zero space of a quasinorm is the vector subspace of vectors with zero norm. If one wants to get a true metric space, one should quotient the space by the zero space.

definition $\mathit{zero\text{-}space}_N$::('$a$::$\mathit{real\text{-}vector}$) $\mathit{quasinorm} \Rightarrow$ 'a set
where $\mathit{zero\text{-}space}_N\ N = \{f.\ \mathit{eNorm}\ N\, f = 0\}$

lemma *zero-spaceN-iff*:
$x \in \mathit{zero\text{-}space}_N\ N \longleftrightarrow \mathit{eNorm}\ N\ x = 0$
⟨*proof*⟩

lemma *zero-spaceN-cmult*:
assumes $x \in \mathit{zero\text{-}space}_N\ N$
shows $c *_R x \in \mathit{zero\text{-}space}_N\ N$
⟨*proof*⟩

lemma *zero-spaceN-add*:
assumes $x \in \mathit{zero\text{-}space}_N\ N\ y \in \mathit{zero\text{-}space}_N\ N$
shows $x + y \in \mathit{zero\text{-}space}_N\ N$
⟨*proof*⟩

lemma *zero-spaceN-diff*:
assumes $x \in \mathit{zero\text{-}space}_N\ N\ y \in \mathit{zero\text{-}space}_N\ N$
shows $x - y \in \mathit{zero\text{-}space}_N\ N$
⟨*proof*⟩

lemma *zero-spaceN-subset-spaceN*:
$\mathit{zero\text{-}space}_N\ N \subseteq \mathit{space}_N\ N$
⟨*proof*⟩

On the space, the norms are finite. Hence, it is much more convenient to work there with a real valued version of the norm. We use Norm with a capital N to distinguish it from norms in a (type class) banach space.

definition *Norm*::'a $\mathit{quasinorm} \Rightarrow$ ('a::$\mathit{real\text{-}vector}$) $\Rightarrow \mathit{real}$
where $\mathit{Norm}\ N\ x = \mathit{enn2real}\ (\mathit{eNorm}\ N\ x)$

lemma *Norm-nonneg* [*simp*]:
$\mathit{Norm}\ N\ x \geq 0$
⟨*proof*⟩

lemma *Norm-zero* [*simp*]:
$\mathit{Norm}\ N\ 0 = 0$
⟨*proof*⟩

lemma *Norm-uminus* [*simp*]:
$\mathit{Norm}\ N\ (-x) = \mathit{Norm}\ N\ x$
⟨*proof*⟩

lemma *eNorm-Norm*:
assumes $x \in \mathit{space}_N\ N$
shows $\mathit{eNorm}\ N\ x = \mathit{ennreal}\ (\mathit{Norm}\ N\ x)$
⟨*proof*⟩

lemma *eNorm-Norm′*:
  assumes $x \notin space_N\ N$
  shows $Norm\ N\ x = 0$
⟨*proof*⟩

lemma *Norm-cmult*:
  $Norm\ N\ (c *_R x) = abs\ c * Norm\ N\ x$
⟨*proof*⟩

lemma *Norm-triangular-ineq*:
  assumes $x \in space_N\ N$
  shows $Norm\ N\ (x + y) \leq defect\ N * Norm\ N\ x + defect\ N * Norm\ N\ y$
⟨*proof*⟩

lemma *Norm-triangular-ineq-diff*:
  assumes $x \in space_N\ N$
  shows $Norm\ N\ (x - y) \leq defect\ N * Norm\ N\ x + defect\ N * Norm\ N\ y$
⟨*proof*⟩

lemma *zero-spaceN-iff′*:
  $x \in zero\text{-}space_N\ N \longleftrightarrow (x \in space_N\ N \wedge Norm\ N\ x = 0)$
⟨*proof*⟩

lemma *Norm-sum*:
  assumes $\bigwedge i.\ i < n \Longrightarrow u\ i \in space_N\ N$
  shows $Norm\ N\ (\sum i\in\{..<n\}.\ u\ i) \leq (\sum i\in\{..<n\}.\ (defect\ N)\widehat{\ }(Suc\ i) * Norm\ N\ (u\ i))$
⟨*proof*⟩

2.3 An example: the ambient norm in a normed vector space

definition *N-of-norm*::′*a*::*real-normed-vector quasinorm*
  where $N\text{-}of\text{-}norm = quasinorm\text{-}of\ (1,\ \lambda f.\ norm\ f)$

lemma *N-of-norm*:
  $eNorm\ N\text{-}of\text{-}norm\ f = ennreal\ (norm\ f)$
  $Norm\ N\text{-}of\text{-}norm\ f = norm\ f$
  $defect\ (N\text{-}of\text{-}norm) = 1$
⟨*proof*⟩

lemma *N-of-norm-space* [*simp*]:
  $space_N\ N\text{-}of\text{-}norm = UNIV$
⟨*proof*⟩

lemma *N-of-norm-zero-space* [*simp*]:
  $zero\text{-}space_N\ N\text{-}of\text{-}norm = \{0\}$
⟨*proof*⟩

2.4 An example: the space of bounded continuous functions from a topological space to a normed real vector space

The Banach space of bounded continuous functions is defined in `Bounded_Continuous_Function.thy`, as a type `bcontfun`. We import very quickly the results proved in this file to the current framework.

definition $bcontfun_N$::('a::topological-space ⇒ 'b::real-normed-vector) quasinorm
where $bcontfun_N$ = quasinorm-of (1, λf. if f ∈ bcontfun then norm(Bcontfun f) else (∞::ennreal))

lemma $bcontfun_N$:
fixes f::('a::topological-space ⇒ 'b::real-normed-vector)
shows eNorm $bcontfun_N$ f = (if f ∈ bcontfun then norm(Bcontfun f) else (∞::ennreal))
Norm $bcontfun_N$ f = (if f ∈ bcontfun then norm(Bcontfun f) else 0)
defect ($bcontfun_N$::(('a ⇒ 'b) quasinorm)) = 1
⟨proof⟩

lemma $bcontfun_N$-space:
$space_N$ $bcontfun_N$ = bcontfun
⟨proof⟩

lemma $bcontfun_N$-zero-space:
zero-$space_N$ $bcontfun_N$ = {0}
⟨proof⟩

lemma $bcontfun_N D$:
assumes f ∈ $space_N$ $bcontfun_N$
shows continuous-on UNIV f
⋀x. norm(f x) ≤ Norm $bcontfun_N$ f
⟨proof⟩

lemma $bcontfun_N I$:
assumes continuous-on UNIV f
⋀x. norm(f x) ≤ C
shows f ∈ $space_N$ $bcontfun_N$
Norm $bcontfun_N$ f ≤ C
⟨proof⟩

2.5 Continuous inclusions between functional spaces

Continuous inclusions between functional spaces are now defined

instantiation quasinorm:: (real-vector) preorder
begin

definition less-eq-quasinorm::'a quasinorm ⇒ 'a quasinorm ⇒ bool
where less-eq-quasinorm N1 N2 = (∃ C≥(0::real). ∀ f. eNorm N2 f ≤ C ∗ eNorm N1 f)

definition *less-quasinorm*::$'a$ *quasinorm* $\Rightarrow$ $'a$ *quasinorm* $\Rightarrow$ *bool*
where *less-quasinorm N1 N2* = (*less-eq N1 N2* $\wedge$ ($\neg$ *less-eq N2 N1*))

instance $\langle$*proof*$\rangle$
end

abbreviation *quasinorm-subset* :: ($'a$::*real-vector*) *quasinorm* $\Rightarrow$ $'a$ *quasinorm* $\Rightarrow$ *bool*
where *quasinorm-subset* $\equiv$ *less*

abbreviation *quasinorm-subset-eq* :: ($'a$::*real-vector*) *quasinorm* $\Rightarrow$ $'a$ *quasinorm* $\Rightarrow$ *bool*
where *quasinorm-subset-eq* $\equiv$ *less-eq*

notation
quasinorm-subset (‹'($\subset_N$')›) **and**
quasinorm-subset (‹(-/ $\subset_N$ -)› [51, 51] 50) **and**
quasinorm-subset-eq (‹'($\subseteq_N$')›) **and**
quasinorm-subset-eq (‹(-/ $\subseteq_N$ -)› [51, 51] 50)

lemma *quasinorm-subsetD*:
assumes $N1 \subseteq_N N2$
shows $\exists\, C \geq (0::real).\ \forall f.\ eNorm\ N2\ f \leq C * eNorm\ N1\ f$
$\langle$*proof*$\rangle$

lemma *quasinorm-subsetI*:
assumes $\bigwedge f.\ f \in space_N\ N1 \Longrightarrow eNorm\ N2\ f \leq ennreal\ C * eNorm\ N1\ f$
shows $N1 \subseteq_N N2$
$\langle$*proof*$\rangle$

lemma *quasinorm-subsetI'*:
assumes $\bigwedge f.\ f \in space_N\ N1 \Longrightarrow f \in space_N\ N2$
$\bigwedge f.\ f \in space_N\ N1 \Longrightarrow Norm\ N2\ f \leq C * Norm\ N1\ f$
shows $N1 \subseteq_N N2$
$\langle$*proof*$\rangle$

lemma *quasinorm-subset-space*:
assumes $N1 \subseteq_N N2$
shows $space_N\ N1 \subseteq space_N\ N2$
$\langle$*proof*$\rangle$

lemma *quasinorm-subset-Norm-eNorm*:
assumes $f \in space_N\ N1 \Longrightarrow Norm\ N2\ f \leq C * Norm\ N1\ f$
$N1 \subseteq_N N2$
$C > 0$
shows $eNorm\ N2\ f \leq ennreal\ C * eNorm\ N1\ f$
$\langle$*proof*$\rangle$

lemma *quasinorm-subset-zero-space*:
assumes $N1 \subseteq_N N2$
shows $zero\text{-}space_N\ N1 \subseteq zero\text{-}space_N\ N2$
⟨*proof*⟩

We would like to define the equivalence relation associated to the above order, i.e., the equivalence between norms. This is not equality, so we do not have a true order, but nevertheless this is handy, and not standard in a preorder in Isabelle. The file Library/Preorder.thy defines such an equivalence relation, but including it breaks some proofs so we go the naive way.

definition *quasinorm-equivalent*::$('a::real\text{-}vector)$ *quasinorm* $\Rightarrow$ *'a quasinorm* $\Rightarrow$ *bool* (**infix** ‹$=_N$› *60*)
where *quasinorm-equivalent* $N1\ N2 = ((N1 \subseteq_N N2) \wedge (N2 \subseteq_N N1))$

lemma *quasinorm-equivalent-sym* [*sym*]:
assumes $N1 =_N N2$
shows $N2 =_N N1$
⟨*proof*⟩

lemma *quasinorm-equivalent-trans* [*trans*]:
assumes $N1 =_N N2\ N2 =_N N3$
shows $N1 =_N N3$
⟨*proof*⟩

2.6 The intersection and the sum of two functional spaces

In this paragraph, we define the intersection and the sum of two functional spaces. In terms of the order introduced above, this corresponds to the minimum and the maximum. More important, these are the first two examples of interpolation spaces between two functional spaces, and they are central as all the other ones are built using them.

definition *quasinorm-intersection*::$('a::real\text{-}vector)$ *quasinorm* $\Rightarrow$ *'a quasinorm* $\Rightarrow$ *'a quasinorm* (**infix** ‹$\cap_N$› *70*)
where *quasinorm-intersection* $N1\ N2$ = *quasinorm-of* (*max* (*defect* $N1$) (*defect* $N2$), $\lambda f.$ *eNorm* $N1\ f$ + *eNorm* $N2\ f$)

lemma *quasinorm-intersection*:
eNorm $(N1 \cap_N N2)\ f$ = *eNorm* $N1\ f$ + *eNorm* $N2\ f$
defect $(N1 \cap_N N2)$ = *max* (*defect* $N1$) (*defect* $N2$)
⟨*proof*⟩

lemma *quasinorm-intersection-commute*:
$N1 \cap_N N2 = N2 \cap_N N1$
⟨*proof*⟩

lemma *quasinorm-intersection-space*:
$space_N\ (N1 \cap_N N2) = space_N\ N1 \cap space_N\ N2$
⟨*proof*⟩

lemma *quasinorm-intersection-zero-space*:
$zero\text{-}space_N\ (N1 \cap_N N2) = zero\text{-}space_N\ N1 \cap zero\text{-}space_N\ N2$
⟨*proof*⟩

lemma *quasinorm-intersection-subset*:
$N1 \cap_N N2 \subseteq_N N1$ $N1 \cap_N N2 \subseteq_N N2$
⟨*proof*⟩

lemma *quasinorm-intersection-minimum*:
assumes $N \subseteq_N N1$ $N \subseteq_N N2$
shows $N \subseteq_N N1 \cap_N N2$
⟨*proof*⟩

lemma *quasinorm-intersection-assoc*:
$(N1 \cap_N N2) \cap_N N3 =_N N1 \cap_N (N2 \cap_N N3)$
⟨*proof*⟩

definition *quasinorm-sum*::(*'a*::*real-vector*) *quasinorm* ⇒ *'a quasinorm* ⇒ *'a quasinorm* (**infix** ‹$+_N$› *70*)
where $quasinorm\text{-}sum\ N1\ N2 = quasinorm\text{-}of\ (max\ (defect\ N1)\ (defect\ N2), \lambda f.\ Inf\ \{eNorm\ N1\ f1 + eNorm\ N2\ f2 |\ f1\ f2.\ f = f1 + f2\})$

lemma *quasinorm-sum*:
$eNorm\ (N1 +_N N2)\ f = Inf\ \{eNorm\ N1\ f1 + eNorm\ N2\ f2 |\ f1\ f2.\ f = f1 + f2\}$
$defect\ (N1 +_N N2) = max\ (defect\ N1)\ (defect\ N2)$
⟨*proof*⟩

lemma *quasinorm-sum-limit*:
$\exists f1\ f2.\ (\forall n.\ f = f1\ n + f2\ n) \wedge (\lambda n.\ eNorm\ N1\ (f1\ n) + eNorm\ N2\ (f2\ n)) \longrightarrow eNorm\ (N1 +_N N2)\ f$
⟨*proof*⟩

lemma *quasinorm-sum-space*:
$space_N\ (N1 +_N N2) = \{f + g | f\ g.\ f \in space_N\ N1 \wedge g \in space_N\ N2\}$
⟨*proof*⟩

lemma *quasinorm-sum-zerospace*:
$\{f + g\ | f\ g.\ f \in zero\text{-}space_N\ N1 \wedge g \in zero\text{-}space_N\ N2\} \subseteq zero\text{-}space_N\ (N1 +_N N2)$
⟨*proof*⟩

lemma *quasinorm-sum-subset*:

$N1 \subseteq_N N1 +_N N2$ $N2 \subseteq_N N1 +_N N2$
⟨*proof*⟩

lemma *quasinorm-sum-maximum*:
assumes $N1 \subseteq_N N$ $N2 \subseteq_N N$
shows $N1 +_N N2 \subseteq_N N$
⟨*proof*⟩

lemma *quasinorm-sum-assoc*:
$(N1 +_N N2) +_N N3 =_N N1 +_N (N2 +_N N3)$
⟨*proof*⟩

2.7 Topology

definition $topology_N$::('a::real-vector) quasinorm ⇒ 'a topology
where $topology_N\ N = topology\ (\lambda U.\ \forall x{\in}U.\ \exists e{>}0.\ \forall y.\ eNorm\ N\ (y-x) < e \longrightarrow y \in U)$

lemma *istopology-topology*$_N$:
$istopology\ (\lambda U.\ \forall x{\in}U.\ \exists e{>}0.\ \forall y.\ eNorm\ N\ (y-x) < e \longrightarrow y \in U)$
⟨*proof*⟩

lemma *openin-topology*$_N$:
$openin\ (topology_N\ N)\ U \longleftrightarrow (\forall x{\in}U.\ \exists e{>}0.\ \forall y.\ eNorm\ N\ (y-x) < e \longrightarrow y \in U)$
⟨*proof*⟩

lemma *openin-topology*$_N$*-I*:
assumes $\bigwedge x.\ x \in U \Longrightarrow \exists e{>}0.\ \forall y.\ eNorm\ N\ (y-x) < e \longrightarrow y \in U$
shows $openin\ (topology_N\ N)\ U$
⟨*proof*⟩

lemma *openin-topology*$_N$*-D*:
assumes $openin\ (topology_N\ N)\ U$
$x \in U$
shows $\exists e{>}0.\ \forall y.\ eNorm\ N\ (y-x) < e \longrightarrow y \in U$
⟨*proof*⟩

One should then use this topology to define limits and so on. This is not something specific to quasinorms, but to all topologies defined in this way, not using type classes. However, there is no such body of material (yet?) in Isabelle-HOL, where topology is essentially done with type classes. So, we do not go any further for now.

One exception is the notion of completeness, as it is so important in functional analysis. We give a naive definition, which will be sufficient for the proof of completeness of several spaces. Usually, the most convenient criterion to prove completeness of a normed vector space is in terms of converging series. This criterion is the only nontrivial thing we prove here. We will apply it to prove the completeness of L^p spaces.

definition $\mathit{cauchy\text{-}ine}_N$::$('a$::$\mathit{real\text{-}vector})$ $\mathit{quasinorm} \Rightarrow (\mathit{nat} \Rightarrow 'a) \Rightarrow \mathit{bool}$
where $\mathit{cauchy\text{-}ine}_N\ N\ u = (\forall\, e{>}0.\ \exists\, M.\ \forall\, n{\geq}M.\ \forall\, m{\geq}M.\ \mathit{eNorm}\ N\ (u\ n - u\ m) < e)$

definition $\mathit{tendsto\text{-}ine}_N$::$('a$::$\mathit{real\text{-}vector})$ $\mathit{quasinorm} \Rightarrow (\mathit{nat} \Rightarrow 'a) \Rightarrow 'a => \mathit{bool}$
where $\mathit{tendsto\text{-}ine}_N\ N\ u\ x = (\lambda n.\ \mathit{eNorm}\ N\ (u\ n - x)) \longrightarrow 0$

definition $\mathit{complete}_N$::$('a$::$\mathit{real\text{-}vector})$ $\mathit{quasinorm} \Rightarrow \mathit{bool}$
where $\mathit{complete}_N\ N = (\forall\, u.\ \mathit{cauchy\text{-}ine}_N\ N\ u \longrightarrow (\exists\, x.\ \mathit{tendsto\text{-}ine}_N\ N\ u\ x))$

The above definitions are in terms of eNorms, but usually the nice definitions only make sense on the space of the norm, and are expressed in terms of Norms. We formulate the same definitions with norms, they will be more convenient for the proofs.

definition $\mathit{cauchy\text{-}in}_N$::$('a$::$\mathit{real\text{-}vector})$ $\mathit{quasinorm} \Rightarrow (\mathit{nat} \Rightarrow 'a) \Rightarrow \mathit{bool}$
where $\mathit{cauchy\text{-}in}_N\ N\ u = (\forall\, e{>}0.\ \exists\, M.\ \forall\, n{\geq}M.\ \forall\, m{\geq}M.\ \mathit{Norm}\ N\ (u\ n - u\ m) < e)$

definition $\mathit{tendsto\text{-}in}_N$::$('a$::$\mathit{real\text{-}vector})$ $\mathit{quasinorm} \Rightarrow (\mathit{nat} \Rightarrow 'a) \Rightarrow 'a => \mathit{bool}$
where $\mathit{tendsto\text{-}in}_N\ N\ u\ x = (\lambda n.\ \mathit{Norm}\ N\ (u\ n - x)) \longrightarrow 0$

lemma $\mathit{cauchy\text{-}ine}_N\text{-}I$:
assumes $\bigwedge e.\ e > 0 \Longrightarrow (\exists\, M.\ \forall\, n{\geq}M.\ \forall\, m{\geq}M.\ \mathit{eNorm}\ N\ (u\ n - u\ m) < e)$
shows $\mathit{cauchy\text{-}ine}_N\ N\ u$
⟨*proof*⟩

lemma $\mathit{cauchy\text{-}in}_N\text{-}I$:
assumes $\bigwedge e.\ e > 0 \Longrightarrow (\exists\, M.\ \forall\, n{\geq}M.\ \forall\, m{\geq}M.\ \mathit{Norm}\ N\ (u\ n - u\ m) < e)$
shows $\mathit{cauchy\text{-}in}_N\ N\ u$
⟨*proof*⟩

lemma *cauchy-ine-in*:
assumes $\bigwedge n.\ u\ n \in \mathit{space}_N\ N$
shows $\mathit{cauchy\text{-}ine}_N\ N\ u \longleftrightarrow \mathit{cauchy\text{-}in}_N\ N\ u$
⟨*proof*⟩

lemma *tendsto-ine-in*:
assumes $\bigwedge n.\ u\ n \in \mathit{space}_N\ N\ x \in \mathit{space}_N\ N$
shows $\mathit{tendsto\text{-}ine}_N\ N\ u\ x \longleftrightarrow \mathit{tendsto\text{-}in}_N\ N\ u\ x$
⟨*proof*⟩

lemma $\mathit{complete}_N\text{-}I$:
assumes $\bigwedge u.\ \mathit{cauchy\text{-}in}_N\ N\ u \Longrightarrow (\forall\, n.\ u\ n \in \mathit{space}_N\ N) \Longrightarrow (\exists\, x{\in}\ \mathit{space}_N\ N.\ \mathit{tendsto\text{-}in}_N\ N\ u\ x)$
shows $\mathit{complete}_N\ N$
⟨*proof*⟩

lemma *cauchy-tendsto-in-subseq*:
assumes $\bigwedge n.\ u\ n \in space_N\ N$
$cauchy\text{-}in_N\ N\ u$
$strict\text{-}mono\ r$
$tendsto\text{-}in_N\ N\ (u\ o\ r)\ x$
shows $tendsto\text{-}in_N\ N\ u\ x$
⟨*proof*⟩

proposition $complete_N\text{-}I'$:
assumes $\bigwedge n.\ c\ n > 0$
$\bigwedge u.\ (\forall n.\ u\ n \in space_N\ N) \Longrightarrow (\forall n.\ Norm\ N\ (u\ n) \leq c\ n) \Longrightarrow \exists x\in space_N\ N.\ tendsto\text{-}in_N\ N\ (\lambda n.\ (\sum i\in\{0..<n\}.\ u\ i))\ x$
shows $complete_N\ N$
⟨*proof*⟩

Next, we show when the two examples of norms we have introduced before, the ambient norm in a Banach space, and the norm on bounded continuous functions, are complete. We just have to translate in our setting the already known completeness of these spaces.

lemma *complete-N-of-norm*:
$complete_N\ (N\text{-}of\text{-}norm::'a::banach\ quasinorm)$
⟨*proof*⟩

In the next statement, the assumption that `'a` is a metric space is not necessary, a topological space would be enough, but a statement about uniform convergence is not available in this setting. TODO: fix it.

lemma *complete-bcontfunN*:
$complete_N\ (bcontfun_N::('a::metric\text{-}space \Rightarrow 'b::banach)\ quasinorm)$
⟨*proof*⟩

end

theory *Lp*
imports *Functional-Spaces*
begin

The material in this file is essentially of analytic nature. However, one of the central proofs (the proof of Holder inequality below) uses a probability space, and Jensen's inequality there. Hence, we need to import `Probability`. Moreover, we use several lemmas from `SG_Library_Complement`.

3 Conjugate exponents

Two numbers p and q are *conjugate* if $1/p + 1/q = 1$. This relation keeps appearing in the theory of L^p spaces, as the dual of L^p is L^q where q is the conjugate of p. This relation makes sense for real numbers, but also for ennreals (where the case $p = 1$ and $q = \infty$ is most important). Unfortunately, manipulating the previous relation with ennreals is tedious as there is no good simproc involving addition and division there. To mitigate this difficulty, we prove once and for all most useful properties of such conjugates exponents in this paragraph.

lemma *Lp-cases-1-PInf*:
  assumes $p \geq (1::ennreal)$
  obtains (gr) $p2$ **where** $p = ennreal\ p2\ p2 > 1\ p > 1$
    $|$ (one) $p = 1$
    $|$ $(PInf)$ $p = \infty$
⟨*proof*⟩

lemma *Lp-cases*:
  obtains $(real\text{-}pos)$ $p2$ **where** $p = ennreal\ p2\ p2 > 0\ p > 0$
    $|$ $(zero)$ $p = 0$
    $|$ $(PInf)$ $p = \infty$
⟨*proof*⟩

definition
  $conjugate\text{-}exponent\ p = 1 + 1/(p-1)$

lemma *conjugate-exponent-real*:
  assumes $p > (1::real)$
  shows $1/p + 1/(conjugate\text{-}exponent\ p) = 1$
    $conjugate\text{-}exponent\ p > 1$
    $conjugate\text{-}exponent(conjugate\text{-}exponent\ p) = p$
    $(p-1) * conjugate\text{-}exponent\ p = p$
    $p - p\ /\ conjugate\text{-}exponent\ p = 1$
⟨*proof*⟩

lemma *conjugate-exponent-real-iff*:
  assumes $p > (1::real)$
  shows $q = conjugate\text{-}exponent\ p \longleftrightarrow (1/p + 1/q = 1)$
⟨*proof*⟩

lemma *conjugate-exponent-real-2* [*simp*]:
  $conjugate\text{-}exponent\ (2::real) = 2$
  ⟨*proof*⟩

lemma *conjugate-exponent-realI*:
  assumes $p > (0::real)\ q > 0\ 1/p + 1/q = 1$
  shows $p > 1\ q = conjugate\text{-}exponent\ p\ q > 1\ p = conjugate\text{-}exponent\ q$
⟨*proof*⟩

lemma *conjugate-exponent-real-ennreal*:
  assumes $p > (1::real)$

shows $conjugate\text{-}exponent(ennreal\ p) = ennreal(conjugate\text{-}exponent\ p)$
⟨*proof*⟩

lemma *conjugate-exponent-ennreal-1-2-PInf* [*simp*]:
$conjugate\text{-}exponent\ (1::ennreal) = \infty$
$conjugate\text{-}exponent\ (\infty::ennreal) = 1$
$conjugate\text{-}exponent\ (\top::ennreal) = 1$
$conjugate\text{-}exponent\ (2::ennreal) = 2$
⟨*proof*⟩

lemma *conjugate-exponent-ennreal*:
assumes $p \geq (1::ennreal)$
shows $1/p + 1/(conjugate\text{-}exponent\ p) = 1$
$conjugate\text{-}exponent\ p \geq 1$
$conjugate\text{-}exponent(conjugate\text{-}exponent\ p) = p$
⟨*proof*⟩

lemma *conjugate-exponent-ennreal-iff*:
assumes $p \geq (1::ennreal)$
shows $q = conjugate\text{-}exponent\ p \longleftrightarrow (1/p + 1/q = 1)$
⟨*proof*⟩

lemma *conjugate-exponent-ennrealI*:
assumes $1/p + 1/q = (1::ennreal)$
shows $p \geq 1$ $q \geq 1$ $p = conjugate\text{-}exponent\ q$ $q = conjugate\text{-}exponent\ p$
⟨*proof*⟩

4 Convexity inequalities and integration

In this paragraph, we describe the basic inequalities relating the integral of a function and of its p-th power, for $p > 0$. These inequalities imply in particular that the L^p norm satisfies the triangular inequality, a feature we will need when defining the L^p spaces below. In particular, we prove the Hölder and Minkowski inequalities. The Hölder inequality, especially, is the basis of all further inequalities for L^p spaces.

lemma (**in** *prob-space*) *bound-L1-Lp*:
assumes $p \geq (1::real)$
$f \in borel\text{-}measurable\ M$
$integrable\ M\ (\lambda x.\ |f\ x|\ powr\ p)$
shows $integrable\ M\ f$
$abs(\int x.\ f\ x\ \partial M)\ powr\ p \leq (\int x.\ |f\ x|\ powr\ p\ \partial M)$
$abs(\int x.\ f\ x\ \partial M) \leq (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
⟨*proof*⟩

theorem *Holder-inequality*:
assumes $p > (0::real)$ $q > 0$ $1/p + 1/q = 1$

and [*measurable*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
integrable M $(\lambda x.\ |f\ x|$ *powr* $p)$
integrable M $(\lambda x.\ |g\ x|$ *powr* $q)$
shows *integrable M* $(\lambda x.\ f\ x * g\ x)$
$(\int x.\ |f\ x * g\ x|\ \partial M) \leq (\int x.\ |f\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p) * (\int x.\ |g\ x|$ *powr* $q\ \partial M)$ *powr* $(1/q)$
$abs(\int x.\ f\ x * g\ x\ \partial M) \leq (\int x.\ |f\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p) * (\int x.\ |g\ x|$ *powr* $q\ \partial M)$ *powr* $(1/q)$
⟨*proof*⟩

theorem *Minkowski-inequality*:
assumes $p \geq (1::real)$
and [*measurable*, *simp*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
integrable M $(\lambda x.\ |f\ x|$ *powr* $p)$
integrable M $(\lambda x.\ |g\ x|$ *powr* $p)$
shows *integrable M* $(\lambda x.\ |f\ x + g\ x|$ *powr* $p)$
$(\int x.\ |f\ x + g\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p)$
$\leq (\int x.\ |f\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p) + (\int x.\ |g\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p)$
⟨*proof*⟩

When $p < 1$, the function $x \mapsto |x|^p$ is not convex any more. Hence, the L^p "norm" is not a norm any more, but a quasinorm. This is proved using a different convexity argument, as follows.

theorem *Minkowski-inequality-le-1*:
assumes $p > (0::real)$ $p \leq 1$
and [*measurable*, *simp*]: $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
integrable M $(\lambda x.\ |f\ x|$ *powr* $p)$
integrable M $(\lambda x.\ |g\ x|$ *powr* $p)$
shows *integrable M* $(\lambda x.\ |f\ x + g\ x|$ *powr* $p)$
$(\int x.\ |f\ x + g\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p)$
≤ 2 *powr* $(1/p-1) * (\int x.\ |f\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p) + 2$ *powr* $(1/p-1) * (\int x.\ |g\ x|$ *powr* $p\ \partial M)$ *powr* $(1/p)$
⟨*proof*⟩

5 L^p spaces

We define L^p spaces by giving their defining quasinorm. It is a norm for $p \in [1, \infty]$, and a quasinorm for $p \in (0, 1)$. The construction of a quasinorm from a formula only makes sense if this formula is indeed a quasinorm, i.e., it is homogeneous and satisfies the triangular inequality with the given multiplicative defect. Thus, we have to show that this is indeed the case to be able to use the definition.

definition *Lp-space*::*ennreal* $\Rightarrow$ *'a measure* $\Rightarrow$ (*'a* $\Rightarrow$ *real*) *quasinorm*
where *Lp-space p M* = (
if $p = 0$ *then quasinorm-of* $(1, (\lambda f.$ *if* $(f \in$ *borel-measurable M*) *then* 0 *else* $\infty))$
else if $p < \infty$ *then quasinorm-of* (

if p < 1 then 2 powr (1/enn2real p − 1) else 1,
(λf. if (f ∈ borel-measurable M ∧ integrable M (λx. |f x| powr (enn2real p)))
then (∫ x. |f x| powr (enn2real p) ∂M) powr (1/(enn2real p))
else (∞::ennreal)))
else quasinorm-of (1, (λf. if f ∈ borel-measurable M then esssup M (λx. ereal |f x|) else (∞::ennreal))))

abbreviation 𝔏 == *Lp-space*

5.1 L^∞

Let us check that, for L^∞, the above definition makes sense.

lemma *L-infinity*:
eNorm (𝔏 ∞ M) f = (if f ∈ borel-measurable M then esssup M (λx. ereal |f x|) else (∞::ennreal))
defect (𝔏 ∞ M) = 1
⟨*proof*⟩

lemma *L-infinity-space*:
space$_N$ *(𝔏 ∞ M) = {f ∈ borel-measurable M. ∃ C. AE x in M. |f x| ≤ C}*
⟨*proof*⟩

lemma *L-infinity-zero-space*:
zero-space$_N$ *(𝔏 ∞ M) = {f ∈ borel-measurable M. AE x in M. f x = 0}*
⟨*proof*⟩

lemma *L-infinity-AE-ebound*:
AE x in M. ennreal |f x| ≤ eNorm (𝔏 ∞ M) f
⟨*proof*⟩

lemma *L-infinity-AE-bound*:
assumes *f ∈ space*$_N$ *(𝔏 ∞ M)*
shows *AE x in M. |f x| ≤ Norm (𝔏 ∞ M) f*
⟨*proof*⟩

In the next lemma, the assumption $C \geq 0$ that might seem useless is in fact necessary for the second statement when the space has zero measure. Indeed, any function is then almost surely bounded by any constant!

lemma *L-infinity-I*:
assumes *f ∈ borel-measurable M*
AE x in M. |f x| ≤ C
C ≥ 0
shows *f ∈ space*$_N$ *(𝔏 ∞ M)*
Norm (𝔏 ∞ M) f ≤ C
⟨*proof*⟩

lemma *L-infinity-I'*:

assumes [*measurable*]: $f \in$ *borel-measurable* M
and *AE* x *in* M. *ennreal* $|f\ x| \leq C$
shows *eNorm* $(\mathfrak{L}\ \infty\ M)\ f \leq C$
$\langle proof \rangle$

lemma *L-infinity-pos-measure*:
assumes [*measurable*]: $f \in$ *borel-measurable* M
and *eNorm* $(\mathfrak{L}\ \infty\ M)\ f > (C{::}real)$
shows *emeasure* $M\ \{x \in space\ M.\ |f\ x| > C\} > 0$
$\langle proof \rangle$

lemma *L-infinity-tendsto-AE*:
assumes $tendsto\text{-}in_N\ (\mathfrak{L}\ \infty\ M)\ f\ g$
$\bigwedge n.\ f\ n \in space_N\ (\mathfrak{L}\ \infty\ M)$
$g \in space_N\ (\mathfrak{L}\ \infty\ M)$
shows *AE* x *in* M. $(\lambda n.\ f\ n\ x) \longrightarrow g\ x$
$\langle proof \rangle$

As an illustration of the mechanism of spaces inclusion, let us show that bounded continuous functions belong to L^∞.

lemma *bcontfun-subset-L-infinity*:
assumes *sets* M = *sets borel*
shows $space_N\ bcontfun_N \subseteq space_N\ (\mathfrak{L}\ \infty\ M)$
$\bigwedge f.\ f \in space_N\ bcontfun_N \Longrightarrow Norm\ (\mathfrak{L}\ \infty\ M)\ f \leq Norm\ bcontfun_N\ f$
$\bigwedge f.\ eNorm\ (\mathfrak{L}\ \infty\ M)\ f \leq eNorm\ bcontfun_N\ f$
$bcontfun_N \subseteq_N \mathfrak{L}\ \infty\ M$
$\langle proof \rangle$

5.2 L^p for $0 < p < \infty$

lemma *Lp*:
assumes $p \geq (1{::}real)$
shows *eNorm* $(\mathfrak{L}\ p\ M)\ f = (if\ (f \in borel\text{-}measurable\ M \wedge integrable\ M\ (\lambda x.\ |f\ x|\ powr\ p))$
$then\ (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
$else\ (\infty{::}ennreal))$
$defect\ (\mathfrak{L}\ p\ M) = 1$
$\langle proof \rangle$

lemma *Lp-le-1*:
assumes $p > 0\ p \leq (1{::}real)$
shows *eNorm* $(\mathfrak{L}\ p\ M)\ f = (if\ (f \in borel\text{-}measurable\ M \wedge integrable\ M\ (\lambda x.\ |f\ x|\ powr\ p))$
$then\ (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
$else\ (\infty{::}ennreal))$
$defect\ (\mathfrak{L}\ p\ M) = 2\ powr\ (1/p - 1)$
$\langle proof \rangle$

lemma *Lp-space*:

assumes $p > (0::real)$
shows $space_N$ $(\mathfrak{L}\ p\ M) = \{f \in$ *borel-measurable* $M.$ *integrable* $M\ (\lambda x.\ |f\ x|\ powr\ p)\}$
⟨*proof*⟩

lemma *Lp-I*:
assumes $p > (0::real)$
$f \in$ *borel-measurable* M *integrable* $M\ (\lambda x.\ |f\ x|\ powr\ p)$
shows $f \in space_N\ (\mathfrak{L}\ p\ M)$
$Norm\ (\mathfrak{L}\ p\ M)\ f = (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
$eNorm\ (\mathfrak{L}\ p\ M)\ f = (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
⟨*proof*⟩

lemma *Lp-D*:
assumes $p>0$ $f \in space_N\ (\mathfrak{L}\ p\ M)$
shows $f \in$ *borel-measurable* M
integrable $M\ (\lambda x.\ |f\ x|\ powr\ p)$
$Norm\ (\mathfrak{L}\ p\ M)\ f = (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
$eNorm\ (\mathfrak{L}\ p\ M)\ f = (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
⟨*proof*⟩

lemma *Lp-Norm*:
assumes $p > (0::real)$
$f \in$ *borel-measurable* M
shows $Norm\ (\mathfrak{L}\ p\ M)\ f = (\int x.\ |f\ x|\ powr\ p\ \partial M)\ powr\ (1/p)$
$(Norm\ (\mathfrak{L}\ p\ M)\ f)\ powr\ p = (\int x.\ |f\ x|\ powr\ p\ \partial M)$
⟨*proof*⟩

lemma *Lp-zero-space*:
assumes $p > (0::real)$
shows *zero-space*$_N$ $(\mathfrak{L}\ p\ M) = \{f \in$ *borel-measurable* $M.\ AE\ x\ in\ M.\ f\ x = 0\}$
⟨*proof*⟩

lemma *Lp-tendsto-AE-subseq*:
assumes $p>(0::real)$
tendsto-in$_N$ $(\mathfrak{L}\ p\ M)\ f\ g$
$\bigwedge n.\ f\ n \in space_N\ (\mathfrak{L}\ p\ M)$
$g \in space_N\ (\mathfrak{L}\ p\ M)$
shows $\exists r.$ *strict-mono* $r \wedge (AE\ x\ in\ M.\ (\lambda n.\ f\ (r\ n)\ x) \longrightarrow g\ x)$
⟨*proof*⟩

5.3 Specialization to L^1

lemma *L1-space*:
$space_N\ (\mathfrak{L}\ 1\ M) = \{f.\ integrable\ M\ f\}$
⟨*proof*⟩

lemma *L1-I*:
assumes *integrable* $M\ f$

shows $f \in space_N$ ($\mathfrak{L}$ *1 M*)
Norm ($\mathfrak{L}$ *1 M*) $f = (\int x.\ |f\ x|\ \partial M)$
eNorm ($\mathfrak{L}$ *1 M*) $f = (\int x.\ |f\ x|\ \partial M)$
⟨*proof*⟩

lemma *L1-D*:
assumes $f \in space_N$ ($\mathfrak{L}$ *1 M*)
shows $f \in$ *borel-measurable M*
integrable M f
Norm ($\mathfrak{L}$ *1 M*) $f = (\int x.\ |f\ x|\ \partial M)$
eNorm ($\mathfrak{L}$ *1 M*) $f = (\int x.\ |f\ x|\ \partial M)$
⟨*proof*⟩

lemma *L1-int-ineq*:
abs$(\int x.\ f\ x\ \partial M) \leq$ *Norm* ($\mathfrak{L}$ *1 M*) f
⟨*proof*⟩

In L^1, one can give a direct formula for the eNorm of a measurable function, using a nonnegative integral. The same formula holds in L^p for $p > 0$, with additional powers p and $1/p$, but one can not write it down since `powr` is not defined on `ennreal`.

lemma *L1-Norm*:
assumes [*measurable*]: $f \in$ *borel-measurable M*
shows *Norm* ($\mathfrak{L}$ *1 M*) $f = (\int x.\ |f\ x|\ \partial M)$
eNorm ($\mathfrak{L}$ *1 M*) $f = (\int^+ x.\ |f\ x|\ \partial M)$
⟨*proof*⟩

lemma *L1-indicator*:
assumes [*measurable*]: $A \in$ *sets M*
shows *eNorm* ($\mathfrak{L}$ *1 M*) (*indicator A*) = *emeasure M A*
⟨*proof*⟩

lemma *L1-indicator'*:
assumes [*measurable*]: $A \in$ *sets M*
and *emeasure M A* $\neq \infty$
shows *indicator A* $\in space_N$ ($\mathfrak{L}$ *1 M*)
Norm ($\mathfrak{L}$ *1 M*) (*indicator A*) = *measure M A*
⟨*proof*⟩

5.4 L^0

We have defined L^p for all exponents p, although it does not really make sense for $p = 0$. We have chosen a definition in this case (the space of all measurable functions) so that many statements are true for all exponents. In this paragraph, we show the consistency of this definition.

lemma *L-zero*:
eNorm ($\mathfrak{L}$ *0 M*) f = (*if* $f \in$ *borel-measurable M then 0 else* ∞)
defect ($\mathfrak{L}$ *0 M*) = *1*

$\langle proof \rangle$

lemma *L-zero-space* [*simp*]:
  $space_N$ ($\mathfrak{L}$ *0 M*) = *borel-measurable M*
  $zero\text{-}space_N$ ($\mathfrak{L}$ *0 M*) = *borel-measurable M*
$\langle proof \rangle$

5.5 Basic results on L^p for general p

lemma *Lp-measurable-subset*:
  $space_N$ ($\mathfrak{L}$ *p M*) $\subseteq$ *borel-measurable M*
$\langle proof \rangle$

lemma *Lp-measurable*:
  assumes $f \in space_N$ ($\mathfrak{L}$ *p M*)
  shows $f \in$ *borel-measurable M*
$\langle proof \rangle$

lemma *Lp-infinity-zero-space*:
  assumes $p > (0{::}ennreal)$
  shows $zero\text{-}space_N$ ($\mathfrak{L}$ *p M*) = $\{f \in$ *borel-measurable M. AE x in M. f x* $= 0\}$
$\langle proof \rangle$

lemma (**in** *prob-space*) *Lp-subset-Lq*:
  assumes $p \leq q$
  shows $\bigwedge f.$ *eNorm* ($\mathfrak{L}$ *p M*) *f* $\leq$ *eNorm* ($\mathfrak{L}$ *q M*) *f*
    $\mathfrak{L}$ *q M* $\subseteq_N$ $\mathfrak{L}$ *p M*
    $space_N$ ($\mathfrak{L}$ *q M*) $\subseteq$ $space_N$ ($\mathfrak{L}$ *p M*)
    $\bigwedge f.$ $f \in space_N$ ($\mathfrak{L}$ *q M*) $\Longrightarrow$ *Norm* ($\mathfrak{L}$ *p M*) *f* $\leq$ *Norm* ($\mathfrak{L}$ *q M*) *f*
$\langle proof \rangle$

proposition *Lp-domination*:
  assumes [*measurable*]: $g \in$ *borel-measurable M*
    and $f \in space_N$ ($\mathfrak{L}$ *p M*)
      AE x in M. $|g\ x| \leq |f\ x|$
  shows $g \in space_N$ ($\mathfrak{L}$ *p M*)
    Norm ($\mathfrak{L}$ *p M*) *g* $\leq$ *Norm* ($\mathfrak{L}$ *p M*) *f*
$\langle proof \rangle$

lemma *Lp-Banach-lattice*:
  assumes $f \in space_N$ ($\mathfrak{L}$ *p M*)
  shows $(\lambda x.\ |f\ x|) \in space_N$ ($\mathfrak{L}$ *p M*)
    Norm ($\mathfrak{L}$ *p M*) $(\lambda x.\ |f\ x|)$ = *Norm* ($\mathfrak{L}$ *p M*) *f*
$\langle proof \rangle$

lemma *Lp-bounded-bounded-support*:
  assumes [*measurable*]: $f \in$ *borel-measurable M*
    and *AE x in M.* $|f\ x| \leq C$
      emeasure M $\{x \in$ *space M. f x* $\neq 0\} \neq \infty$

shows $f \in space_N(\mathfrak{L}\ p\ M)$
$\langle proof \rangle$

5.6 L^p versions of the main theorems in integration theory

The space L^p is stable under almost sure convergence, for sequence with bounded norm. This is a version of Fatou's lemma (and it indeed follows from this lemma in the only nontrivial situation where $p \in (0, +\infty)$.

proposition *Lp-AE-limit*:
assumes [*measurable*]: $g \in$ *borel-measurable* M
and $AE\ x\ in\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
shows $eNorm\ (\mathfrak{L}\ p\ M)\ g \leq liminf\ (\lambda n.\ eNorm\ (\mathfrak{L}\ p\ M)\ (f\ n))$
$\langle proof \rangle$

lemma *Lp-AE-limit′*:
assumes $g \in$ *borel-measurable* M
$\bigwedge n.\ f\ n \in space_N\ (\mathfrak{L}\ p\ M)$
$AE\ x\ in\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
$(\lambda n.\ Norm\ (\mathfrak{L}\ p\ M)\ (f\ n)) \longrightarrow l$
shows $g \in space_N\ (\mathfrak{L}\ p\ M)$
$Norm\ (\mathfrak{L}\ p\ M)\ g \leq l$
$\langle proof \rangle$

lemma *Lp-AE-limit″*:
assumes $g \in$ *borel-measurable* M
$\bigwedge n.\ f\ n \in space_N\ (\mathfrak{L}\ p\ M)$
$AE\ x\ in\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
$\bigwedge n.\ Norm\ (\mathfrak{L}\ p\ M)\ (f\ n) \leq C$
shows $g \in space_N\ (\mathfrak{L}\ p\ M)$
$Norm\ (\mathfrak{L}\ p\ M)\ g \leq C$
$\langle proof \rangle$

We give the version of Lebesgue dominated convergence theorem in the setting of L^p spaces.

proposition *Lp-domination-limit*:
fixes *p*::*real*
assumes [*measurable*]: $g \in$ *borel-measurable* M
$\bigwedge n.\ f\ n \in$ *borel-measurable* M
and $m \in space_N\ (\mathfrak{L}\ p\ M)$
$AE\ x\ in\ M.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
$\bigwedge n.\ AE\ x\ in\ M.\ |f\ n\ x| \leq m\ x$
shows $g \in space_N\ (\mathfrak{L}\ p\ M)$
$tendsto\text{-}in_N\ (\mathfrak{L}\ p\ M)\ f\ g$
$\langle proof \rangle$

We give the version of the monotone convergence theorem in the setting of L^p spaces.

proposition *Lp-monotone-limit*:

fixes $f::nat \Rightarrow 'a \Rightarrow real$
assumes $p > (0::ennreal)$
$AE\ x\ in\ M.\ incseq\ (\lambda n.\ f\ n\ x)$
$\bigwedge n.\ Norm\ (\mathfrak{L}\ p\ M)\ (f\ n) \leq C$
$\bigwedge n.\ f\ n \in space_N\ (\mathfrak{L}\ p\ M)$
shows $AE\ x\ in\ M.\ convergent\ (\lambda n.\ f\ n\ x)$
$(\lambda x.\ lim\ (\lambda n.\ f\ n\ x)) \in space_N\ (\mathfrak{L}\ p\ M)$
$Norm\ (\mathfrak{L}\ p\ M)\ (\lambda x.\ lim\ (\lambda n.\ f\ n\ x)) \leq C$
⟨*proof*⟩

5.7 Completeness of L^p

We prove the completeness of L^p.

theorem *Lp-complete*:
$complete_N\ (\mathfrak{L}\ p\ M)$
⟨*proof*⟩

5.8 Multiplication of functions, duality

The next theorem asserts that the multiplication of two functions in L^p and L^q belongs to L^r, where r is determined by the equality $1/r = 1/p + 1/q$. This is essentially a case by case analysis, depending on the kind of L^p space we are considering. The only nontrivial case is when p, q (and r) are finite and nonzero. In this case, it reduces to Hölder inequality.

theorem *Lp-Lq-mult*:
fixes $p\ q\ r::ennreal$
assumes $1/p + 1/q = 1/r$
and $f \in space_N\ (\mathfrak{L}\ p\ M)$ $g \in space_N\ (\mathfrak{L}\ q\ M)$
shows $(\lambda x.\ f\ x * g\ x) \in space_N\ (\mathfrak{L}\ r\ M)$
$Norm\ (\mathfrak{L}\ r\ M)\ (\lambda x.\ f\ x * g\ x) \leq Norm\ (\mathfrak{L}\ p\ M)\ f * Norm\ (\mathfrak{L}\ q\ M)\ g$
⟨*proof*⟩

The previous theorem admits an eNorm version in which one does not assume a priori that the functions under consideration belong to L^p or L^q.

theorem *Lp-Lq-emult*:
fixes $p\ q\ r::ennreal$
assumes $1/p + 1/q = 1/r$
$f \in borel\text{-}measurable\ M$ $g \in borel\text{-}measurable\ M$
shows $eNorm\ (\mathfrak{L}\ r\ M)\ (\lambda x.\ f\ x * g\ x) \leq eNorm\ (\mathfrak{L}\ p\ M)\ f * eNorm\ (\mathfrak{L}\ q\ M)\ g$
⟨*proof*⟩

lemma *Lp-Lq-duality-bound*:
fixes $p\ q::ennreal$
assumes $1/p + 1/q = 1$
$f \in space_N\ (\mathfrak{L}\ p\ M)$
$g \in space_N\ (\mathfrak{L}\ q\ M)$
shows $integrable\ M\ (\lambda x.\ f\ x * g\ x)$

$abs(\int x.\ f\ x * g\ x\ \partial M) \leq Norm\ (\mathfrak{L}\ p\ M)\ f * Norm\ (\mathfrak{L}\ q\ M)\ g$
⟨*proof*⟩

The next theorem asserts that the norm of an L^p function f can be obtained by estimating the integrals of fg over all L^q functions g, where $1/p+1/q = 1$. When $p = \infty$, it is necessary to assume that the space is sigma-finite: for instance, if the space is one single atom of infinite mass, then there is no nonzero L^1 function, so taking for f the constant function equal to 1, it has L^∞ norm equal to 1, but $\int fg = 0$ for all L^1 function g.

theorem *Lp-Lq-duality*:
fixes $p\ q$::*ennreal*
assumes $f \in space_N\ (\mathfrak{L}\ p\ M)$
$1/p + 1/q = 1$
$p = \infty \Longrightarrow$ *sigma-finite-measure M*
shows $bdd\text{-}above\ ((\lambda g.\ (\int x.\ f\ x * g\ x\ \partial M))\text{`}\{g \in space_N\ (\mathfrak{L}\ q\ M).\ Norm\ (\mathfrak{L}\ q\ M)\ g \leq 1\})$
$Norm\ (\mathfrak{L}\ p\ M)\ f = (SUP\ g{\in}\{g \in space_N\ (\mathfrak{L}\ q\ M).\ Norm\ (\mathfrak{L}\ q\ M)\ g \leq 1\}.\ (\int x.\ f\ x * g\ x\ \partial M))$
⟨*proof*⟩

The previous theorem admits a version in which one does not assume a priori that the function under consideration belongs to L^p. This gives an efficient criterion to check if a function is indeed in L^p. In this case, it is always necessary to assume that the measure is sigma-finite.

Note that, in the statement, the Bochner integral $\int fg$ vanishes by definition if fg is not integrable. Hence, the statement really says that the eNorm can be estimated using functions g for which fg is integrable. It is precisely the construction of such functions g that requires the space to be sigma-finite.

theorem *Lp-Lq-duality'*:
fixes $p\ q$::*ennreal*
assumes $1/p + 1/q = 1$
sigma-finite-measure M
and [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $eNorm\ (\mathfrak{L}\ p\ M)\ f = (SUP\ g{\in}\{g \in space_N\ (\mathfrak{L}\ q\ M).\ Norm\ (\mathfrak{L}\ q\ M)\ g \leq 1\}.\ ennreal(\int x.\ f\ x * g\ x\ \partial M))$
⟨*proof*⟩

5.9 Conditional expectations and L^p

The L^p space with respect to a subalgebra is included in the whole L^p space.

lemma *Lp-subalgebra*:
assumes *subalgebra M F*
shows $\bigwedge f.\ eNorm\ (\mathfrak{L}\ p\ M)\ f \leq eNorm\ (\mathfrak{L}\ p\ (restr\text{-}to\text{-}subalg\ M\ F))\ f$
$(\mathfrak{L}\ p\ (restr\text{-}to\text{-}subalg\ M\ F)) \subseteq_N \mathfrak{L}\ p\ M$
$space_N\ ((\mathfrak{L}\ p\ (restr\text{-}to\text{-}subalg\ M\ F))) \subseteq space_N\ (\mathfrak{L}\ p\ M)$
$\bigwedge f.\ f \in space_N\ ((\mathfrak{L}\ p\ (restr\text{-}to\text{-}subalg\ M\ F))) \Longrightarrow Norm\ (\mathfrak{L}\ p\ M)\ f = Norm\ (\mathfrak{L}\ p\ (restr\text{-}to\text{-}subalg\ M\ F))\ f$

⟨*proof*⟩

For $p \geq 1$, the conditional expectation of an L^p function still belongs to L^p, with an L^p norm which is bounded by the norm of the original function. This is wrong for $p < 1$. One can prove this separating the cases and using the conditional version of Jensen's inequality, but it is much more efficient to do it with duality arguments, as follows.

proposition *Lp-real-cond-exp*:
  assumes [*simp*]: *subalgebra M F*
    and $p \geq (1::ennreal)$
      sigma-finite-measure (*restr-to-subalg M F*)
      $f \in space_N$ ($\mathfrak{L}$ *p M*)
  shows *real-cond-exp M F f* $\in space_N$ ($\mathfrak{L}$ *p* (*restr-to-subalg M F*))
    Norm ($\mathfrak{L}$ *p* (*restr-to-subalg M F*)) (*real-cond-exp M F f*) $\leq$ *Norm* ($\mathfrak{L}$ *p M*) *f*
⟨*proof*⟩

lemma *Lp-real-cond-exp-eNorm*:
  assumes [*simp*]: *subalgebra M F*
    and $p \geq (1::ennreal)$
      sigma-finite-measure (*restr-to-subalg M F*)
  shows *eNorm* ($\mathfrak{L}$ *p* (*restr-to-subalg M F*)) (*real-cond-exp M F f*) $\leq$ *eNorm* ($\mathfrak{L}$ *p M*) *f*
⟨*proof*⟩

end