Lower Semicontinuous Functions

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Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

Contents

1 Lower semicontinuous functions

theory *Lower-Semicontinuous*
imports *HOL−Analysis.Multivariate-Analysis*
begin

1.1 Relative interior in one dimension

lemma *rel-interior-ereal-semiline*:
  fixes *a* :: *ereal*
  shows *rel-interior* $\{y.\ a \leq ereal\ y\} = \{y.\ a < ereal\ y\}$
⟨*proof*⟩

lemma *closed-ereal-semiline*:
  fixes *a* :: *ereal*
  shows *closed* $\{y.\ a \leq ereal\ y\}$
⟨*proof*⟩

lemma *ereal-semiline-unique*:
fixes $a\ b$ :: *ereal*
shows $\{y.\ a \leq ereal\ y\} = \{y.\ b \leq ereal\ y\} \longleftrightarrow a = b$
⟨*proof*⟩

1.2 Lower and upper semicontinuity

definition
lsc-at :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$lsc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow f\ x0 \leq l)$

definition
usc-at :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$usc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow l \leq f\ x0)$

lemma *lsc-at-mem*:
assumes *lsc-at x0 f*
assumes $x \longrightarrow x0$
assumes $(f \circ x) \longrightarrow A$
shows $f\ x0 \leq A$
⟨*proof*⟩

lemma *usc-at-mem*:
assumes *usc-at x0 f*
assumes $x \longrightarrow x0$
assumes $(f \circ x) \longrightarrow A$
shows $f\ x0 \geq A$
⟨*proof*⟩

lemma *lsc-at-open*:
fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
shows $lsc\text{-}at\ x0\ f \longleftrightarrow$
$(\forall S.\ open\ S \wedge f\ x0 \in S \longrightarrow (\exists T.\ open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ f\ x' \leq f\ x0$
$\longrightarrow f\ x' \in S)))$
(**is** $?lhs \longleftrightarrow ?rhs$)
⟨*proof*⟩

lemma *lsc-at-open-mem*:
fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
assumes *lsc-at x0 f*
assumes $open\ S \wedge f\ x0 : S$
obtains T **where** $open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ (f\ x' \leq f\ x0 \longrightarrow f\ x' \in S))$
⟨*proof*⟩

lemma *lsc-at-MInfty*:
fixes f :: $'a::topological\text{-}space \Rightarrow ereal$

assumes $f\ x0 = -\infty$
shows *lsc-at x0 f*
⟨*proof*⟩

lemma *lsc-at-PInfty*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes $f\ x0 = \infty$
shows *lsc-at x0 f* ⟷ *continuous* (*at x0*) *f*
⟨*proof*⟩

lemma *lsc-at-real*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes $|f\ x0| \neq \infty$
shows *lsc-at x0 f* ⟷ $(\forall e.\ e{>}0 \longrightarrow (\exists T.\ open\ T \wedge x0 : T \wedge (\forall y \in T.\ f\ y > f\ x0 - e)))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-at-ereal*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ $(\forall C{<}f(x0).\ \exists T.\ open\ T \wedge x0 \in T \wedge (\forall y \in T.\ f\ y > C))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lst-at-ball*:
fixes f :: $'a$::*metric-space* => *ereal*
shows *lsc-at x0 f* ⟷ $(\forall C{<}f(x0).\ \exists d{>}0.\ \forall y \in (ball\ x0\ d).\ C{<}f(y))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lst-at-delta*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ $(\forall C{<}f(x0).\ \exists d{>}0.\ \forall y.\ dist\ x0\ y < d \longrightarrow C < f\ y)$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-liminf-at*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ $f\ x0 \leq Liminf\ (at\ x0)\ f$
⟨*proof*⟩

lemma *lsc-liminf-at-eq*:

fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(f\ x0 = min\ (f\ x0)\ (Liminf\ (at\ x0)\ f))$
⟨*proof*⟩

lemma *lsc-imp-liminf*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes *lsc-at x0 f*
assumes $x \longrightarrow x0$
shows $f\ x0 \leq liminf\ (f \circ x)$
⟨*proof*⟩

lemma *lsc-liminf*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall x.\ x \longrightarrow x0 \longrightarrow f\ x0 \leq liminf\ (f \circ x))$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall x\ c.\ x \longrightarrow x0 \wedge (\forall n.\ f(x\ n) \leq c) \longrightarrow f(x0) \leq c)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-gen*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall x\ c\ c0.\ x \longrightarrow x0 \wedge c \longrightarrow c0 \wedge (\forall n.\ f(x\ n) \leq c\ n) \longrightarrow f(x0) \leq c0)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-mem*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes *lsc-at x0 f*
assumes $x \longrightarrow x0\ c \longrightarrow c0$
assumes $\forall n.\ f(x\ n) \leq c\ n$
shows $f(x0) \leq c0$
⟨*proof*⟩

lemma *lsc-uminus*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0* $(\lambda x.\ -f\ x)$ $\longleftrightarrow$ *usc-at x0 f*
⟨*proof*⟩

lemma *usc-limsup*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $usc\text{-}at\ x0\ f \longleftrightarrow (\forall x.\ x \longrightarrow x0 \longrightarrow f\ x0 \geq limsup\ (f \circ x))$
(**is** $?lhs \longleftrightarrow ?rhs$)
$\langle proof \rangle$

lemma *usc-imp-limsup*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
assumes $usc\text{-}at\ x0\ f$
assumes $x \longrightarrow x0$
shows $f\ x0 \geq limsup\ (f \circ x)$
$\langle proof \rangle$

lemma *usc-limsup-at*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $usc\text{-}at\ x0\ f \longleftrightarrow f\ x0 \geq Limsup\ (at\ x0)\ f$
$\langle proof \rangle$

lemma *continuous-iff-lsc-usc*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $continuous\ (at\ x0)\ f \longleftrightarrow (lsc\text{-}at\ x0\ f) \wedge (usc\text{-}at\ x0\ f)$
$\langle proof \rangle$

lemma *continuous-lsc-compose*:
assumes $lsc\text{-}at\ (g\ x0)\ f\ continuous\ (at\ x0)\ g$
shows $lsc\text{-}at\ x0\ (f \circ g)$
$\langle proof \rangle$

lemma *continuous-isCont*:
$continuous\ (at\ x0)\ f \longleftrightarrow isCont\ f\ x0$
$\langle proof \rangle$

lemma *isCont-iff-lsc-usc*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $isCont\ f\ x0 \longleftrightarrow (lsc\text{-}at\ x0\ f) \wedge (usc\text{-}at\ x0\ f)$
$\langle proof \rangle$

definition
$lsc :: ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$lsc\ f \longleftrightarrow (\forall x.\ lsc\text{-}at\ x\ f)$

definition

usc :: ($'a$::*topological-space* ⇒ $'b$::*order-topology*) ⇒ *bool* **where**
usc f ⟷ (∀ *x*. *usc-at x f*)

lemma *continuous-UNIV-iff-lsc-usc*:
fixes *f* :: $'a$::*metric-space* ⇒ *ereal*
shows (∀ *x*. *continuous* (*at x*) *f*) ⟷ (*lsc f*) ∧ (*usc f*)
⟨*proof*⟩

1.3 Epigraphs

definition *Epigraph S* (*f*::- ⇒ *ereal*) = {*xy*. *fst xy* : *S* ∧ *f* (*fst xy*) ≤ *ereal*(*snd xy*)}

lemma *mem-Epigraph*: (*x*, *y*) ∈ *Epigraph S f* ⟷ *x* ∈ *S* ∧ *f x* ≤ *ereal y* ⟨*proof*⟩

lemma *ereal-closed-levels*:
fixes *f* :: $'a$::*metric-space* ⇒ *ereal*
shows (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*}) ⟷ (∀ *r*. *closed* {*x*. *f*(*x*)≤*ereal r*})
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-iff*:
fixes *f* :: $'a$::*metric-space* ⇒ *ereal*
shows (*lsc f* ⟷ (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*})) ∧ (*lsc f* ⟷ *closed* (*Epigraph UNIV f*))
⟨*proof*⟩

definition *lsc-hull* :: ($'a$::*metric-space* ⇒ *ereal*) ⇒ ($'a$::*metric-space* ⇒ *ereal*) **where**
lsc-hull f = (*SOME g*. *Epigraph UNIV g* = *closure*(*Epigraph UNIV f*))

lemma *epigraph-mono*:
fixes *f* :: $'a$::*metric-space* ⇒ *ereal*
shows (*x*,*y*):*Epigraph UNIV f* ∧ *y*≤*z* ⟶ (*x*,*z*):*Epigraph UNIV f*
⟨*proof*⟩

lemma *closed-epigraph-lines*:
fixes *S* :: ($'a$::*metric-space* ∗ $'b$::*metric-space*) *set*
assumes *closed S*
shows *closed* {*z*. (*x*, *z*) : *S*}
⟨*proof*⟩

lemma *mono-epigraph*:
  fixes S :: ($'a$::*metric-space* $*$ *real*) *set*
  assumes *mono*: $\forall\, x\ y\ z.\ (x,y){:}S \wedge y{\le}z \longrightarrow (x,z){:}S$
  assumes *closed S*
  shows $\exists\, g.\ ((Epigraph\ UNIV\ g) = S)$
$\langle proof \rangle$

lemma *lsc-hull-exists*:
  fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
  shows $\exists\, g.\ Epigraph\ UNIV\ g = closure\ (Epigraph\ UNIV\ f)$
$\langle proof \rangle$

lemma *epigraph-invertible*:
  assumes $Epigraph\ UNIV\ f = Epigraph\ UNIV\ g$
  shows $f{=}g$
$\langle proof \rangle$

lemma *lsc-hull-ex-unique*:
  fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
  shows $\exists!g.\ Epigraph\ UNIV\ g = closure\ (Epigraph\ UNIV\ f)$
$\langle proof \rangle$

lemma *epigraph-lsc-hull*:
  fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
  shows $Epigraph\ UNIV\ (lsc\text{-}hull\ f) = closure(Epigraph\ UNIV\ f)$
$\langle proof \rangle$

lemma *lsc-hull-expl*:
  $(g = lsc\text{-}hull\ f) \longleftrightarrow (Epigraph\ UNIV\ g = closure(Epigraph\ UNIV\ f))$
$\langle proof \rangle$

lemma *lsc-lsc-hull*: $lsc\ (lsc\text{-}hull\ f)$
  $\langle proof \rangle$

lemma *epigraph-subset-iff*:
  fixes $f\ g$ :: $'a$::*metric-space* $\Rightarrow$ *ereal*
  shows $Epigraph\ UNIV\ f \le Epigraph\ UNIV\ g \longleftrightarrow (\forall\, x.\ g\ x \le f\ x)$
$\langle proof \rangle$

lemma *lsc-hull-le*: $(lsc\text{-}hull\ f)\ x \le f\ x$

⟨*proof*⟩

lemma *lsc-hull-greatest*:
fixes *f g* :: *'a::metric-space* ⇒ *ereal*
assumes *lsc g* ∀ *x*. *g x* ≤ *f x*
shows ∀ *x*. *g x* ≤ (*lsc-hull f*) *x*
⟨*proof*⟩

lemma *lsc-hull-iff-greatest*:
fixes *f g* :: *'a::metric-space* ⇒ *ereal*
shows (*g* = *lsc-hull f*) ⟷
lsc g ∧ (∀ *x*. *g x* ≤ *f x*) ∧ (∀ *h*. *lsc h* ∧ (∀ *x*. *h x* ≤ *f x*) ⟶ (∀ *x*. *h x* ≤ *g x*))
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-hull-mono*:
fixes *f g* :: *'a::metric-space* ⇒ *ereal*
assumes ∀ *x*. *g x* ≤ *f x*
shows ∀ *x*. (*lsc-hull g*) *x* ≤ (*lsc-hull f*) *x*
⟨*proof*⟩

lemma *lsc-hull-lsc*:
lsc f ⟷ (*f* = *lsc-hull f*)
⟨*proof*⟩

lemma *lsc-hull-liminf-at*:
fixes *f* :: *'a::metric-space* ⇒ *ereal*
shows ∀ *x*. (*lsc-hull f*) *x* = *min* (*f x*) (*Liminf* (*at x*) *f*)
⟨*proof*⟩

lemma *lsc-hull-same-inf*:
fixes *f* :: *'a::metric-space* ⇒ *ereal*
shows (*INF x*. *lsc-hull f x*) = (*INF x*. *f x*)
⟨*proof*⟩

1.4 Convex Functions

definition
convex-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
convex-on s f ⟷
(∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1*
⟶ *f* ($u *_R x + v *_R y$) ≤ *ereal u* ∗ *f x* + *ereal v* ∗ *f y*)

lemma *convex-on-ereal-mem*:
  assumes *convex-on s f*
  assumes *x*:*s y*:*s*
  assumes $u \geq 0$ $v \geq 0$ $u+v=1$
  shows $f\ (u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y$
⟨*proof*⟩

lemma *convex-on-ereal-subset*: *convex-on t f* $\Longrightarrow s \leq t \Longrightarrow$ *convex-on s f*
  ⟨*proof*⟩

lemma *convex-on-ereal-univ*: *convex-on UNIV f* $\longleftrightarrow$ ($\forall$ *S*. *convex-on S f*)
  ⟨*proof*⟩

lemma *ereal-pos-sum-distrib-left*:
  fixes f :: $'a \Rightarrow ereal$
  assumes $r \geq 0$ $r \neq \infty$
  shows $r * sum\ f\ A = sum\ (\lambda n.\ r * f\ n)\ A$
⟨*proof*⟩

lemma *convex-ereal-add*:
  fixes $f\ g$ :: $'a::real\text{-}vector \Rightarrow ereal$
  assumes *convex-on s f convex-on s g*
  shows *convex-on s* $(\lambda x.\ f\ x + g\ x)$
⟨*proof*⟩

lemma *convex-ereal-cmul*:
  assumes $0 \leq (c::ereal)$ *convex-on s f*
  shows *convex-on s* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *convex-ereal-max*:
  fixes $f\ g$ :: $'a::real\text{-}vector \Rightarrow ereal$
  assumes *convex-on s f convex-on s g*
  shows *convex-on s* $(\lambda x.\ max\ (f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *convex-on-ereal-alt*:
  fixes C :: $'a::real\text{-}vector\ set$
  assumes *convex C*
  shows *convex-on C f* =
  $(\forall x \in C.\ \forall y \in C.\ \forall m :: real.\ m \geq 0 \land m \leq 1$
      $\longrightarrow f\ (m *_R x + (1 - m) *_R y) \leq (ereal\ m) * f\ x + (1 - (ereal\ m)) * f\ y)$

⟨*proof*⟩

lemma *convex-on-ereal-alt-mem*:
fixes C :: $'a$::*real-vector set*
assumes *convex C*
assumes *convex-on C f*
assumes $x : C\ y : C$
assumes $(m::real) \geq 0\ m \leq 1$
shows $f\ (m *_R x + (1 - m) *_R y) \leq (ereal\ m) * f\ x + (1 - (ereal\ m)) * f\ y$
⟨*proof*⟩

lemma *ereal-add-right-mono*: $(a::ereal) \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

lemma *convex-on-ereal-sum-aux*:
assumes $1-a>0$
shows $(1 - ereal\ a) * (ereal\ (c\ /\ (1 - a)) * b) = (ereal\ c) * b$
⟨*proof*⟩

lemma *convex-on-ereal-sum*:
fixes a :: $'a \Rightarrow real$
fixes y :: $'a \Rightarrow\ 'b$::*real-vector*
fixes f :: $'b \Rightarrow ereal$
assumes *finite s* $s \neq \{\}$
assumes *convex-on C f*
assumes *convex C*
assumes $(SUM\ i : s.\ a\ i) = 1$
assumes $\forall i.\ i \in s \longrightarrow a\ i \geq 0$
assumes $\forall i.\ i \in s \longrightarrow y\ i \in C$
shows $f\ (SUM\ i : s.\ a\ i *_R y\ i) \leq (SUM\ i : s.\ ereal\ (a\ i) * f\ (y\ i))$
⟨*proof*⟩

lemma *sum-2*: $sum\ u\ \{1::nat..2\} = (u\ 1)+(u\ 2)$
⟨*proof*⟩

lemma *convex-on-ereal-iff*:
assumes *convex s*
shows *convex-on s f* $\longleftrightarrow (\forall k\ u\ x.\ (\forall i{\in}\{1..k::nat\}.\ 0 \leq u\ i \wedge x\ i : s) \wedge sum\ u$
$\{1..k\} = 1 \longrightarrow$
$f\ (sum\ (\lambda i.\ u\ i *_R x\ i)\ \{1..k\}\ ) \leq sum\ (\lambda i.\ (ereal\ (u\ i)) * f(x\ i))\ \{1..k\}\ )$
(**is** *?rhs* $\longleftrightarrow$ *?lhs*)
⟨*proof*⟩

lemma *convex-Epigraph*:
assumes *convex S*
shows $convex(Epigraph\ S\ f) \longleftrightarrow convex\text{-}on\ S\ f$
⟨*proof*⟩

lemma *convex-EpigraphI*:
$convex\text{-}on\ s\ f \Longrightarrow convex\ s \Longrightarrow convex(Epigraph\ s\ f)$
⟨*proof*⟩

definition
$concave\text{-}on :: {'a}::real\text{-}vector\ set \Rightarrow ({'a} \Rightarrow ereal) \Rightarrow bool$ **where**
$concave\text{-}on\ S\ f \longleftrightarrow convex\text{-}on\ S\ (\lambda x.\ -\ f\ x)$

definition
$finite\text{-}on :: {'a}::real\text{-}vector\ set \Rightarrow ({'a} \Rightarrow ereal) \Rightarrow bool$ **where**
$finite\text{-}on\ S\ f \longleftrightarrow (\forall\, x{\in}S.\ (f\ x \neq \infty \wedge f\ x \neq -\infty))$

definition
$affine\text{-}on :: {'a}::real\text{-}vector\ set \Rightarrow ({'a} \Rightarrow ereal) \Rightarrow bool$ **where**
$affine\text{-}on\ S\ f \longleftrightarrow (convex\text{-}on\ S\ f \wedge concave\text{-}on\ S\ f \wedge finite\text{-}on\ S\ f)$

definition
$domain\ (f::\text{-} \Rightarrow ereal) = \{x.\ f\ x < \infty\}$

lemma *domain-Epigraph-aux*:
assumes $x \neq \infty$
shows $\exists\, r.\ x{\leq}ereal\ r$
⟨*proof*⟩

lemma *domain-Epigraph*:
$domain\ f = \{x.\ \exists\, y.\ (x,y) \in Epigraph\ UNIV\ f\}$
⟨*proof*⟩

lemma *domain-Epigraph-fst*:
$domain\ f = fst\ \text{`}\ (Epigraph\ UNIV\ f)$
⟨*proof*⟩

lemma *convex-on-domain*:
$convex\text{-}on\ (domain\ f)\ f \longleftrightarrow convex\text{-}on\ UNIV\ f$
⟨*proof*⟩

lemma *convex-on-domain2*:

$convex\text{-}on\ (domain\ f)\ f \longleftrightarrow (\forall S.\ convex\text{-}on\ S\ f)$
⟨*proof*⟩

lemma *convex-domain*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow ereal$
assumes $convex\text{-}on\ UNIV\ f$
shows $convex\ (domain\ f)$
⟨*proof*⟩

lemma *infinite-convex-domain-iff*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow ereal$
assumes $\forall x.\ (f\ x = \infty \mid f\ x = -\infty)$
shows $convex\text{-}on\ UNIV\ f \longleftrightarrow convex\ (domain\ f)$
⟨*proof*⟩

lemma *convex-PInfty-outside*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow ereal$
assumes $convex\text{-}on\ UNIV\ f\ convex\ S$
shows $convex\text{-}on\ UNIV\ (\lambda x.\ if\ x{:}S\ then\ (f\ x)\ else\ \infty)$
⟨*proof*⟩

definition
$proper\text{-}on :: 'a::real\text{-}vector\ set \Rightarrow ('a \Rightarrow ereal) \Rightarrow bool$ **where**
$proper\text{-}on\ S\ f \longleftrightarrow ((\forall x{\in}S.\ f\ x \neq -\infty) \wedge (\exists x{\in}S.\ f\ x \neq \infty))$

definition
$proper :: ('a::real\text{-}vector \Rightarrow ereal) \Rightarrow bool$ **where**
$proper\ f \longleftrightarrow proper\text{-}on\ UNIV\ f$

lemma *proper-iff*:
$proper\ f \longleftrightarrow ((\forall x.\ f\ x \neq -\infty) \wedge (\exists x.\ f\ x \neq \infty))$
⟨*proof*⟩

lemma *improper-iff*:
$\sim(proper\ f) \longleftrightarrow ((\exists x.\ f\ x = -\infty) \mid (\forall x.\ f\ x = \infty))$
⟨*proof*⟩

lemma *ereal-MInf-plus*[*simp*]: $-\infty + x = (if\ x = \infty\ then\ \infty\ else\ -\infty{::}ereal)$
⟨*proof*⟩

lemma *convex-improper*:
fixes $f :: 'a::euclidean\text{-}space \Rightarrow ereal$
assumes $convex\text{-}on\ UNIV\ f$

assumes $\sim$(*proper f*)
shows $\forall\, x \in$*rel-interior*(*domain f*). $f\, x = -\infty$
⟨*proof*⟩

lemma *convex-improper2*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
shows $f\, x = \infty \mid f\, x = -\infty \mid x$: *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *convex-lsc-improper*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
assumes *lsc f*
shows $f\, x = \infty \mid f\, x = -\infty$
⟨*proof*⟩

lemma *convex-lsc-hull*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
shows *convex-on UNIV* (*lsc-hull f*)
⟨*proof*⟩

lemma *improper-lsc-hull*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes $\sim$(*proper f*)
shows $\sim$(*proper* (*lsc-hull f*))
⟨*proof*⟩

lemma *lsc-hull-convex-improper*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
shows $\forall\, x \in$*rel-interior*(*domain f*). (*lsc-hull f*) $x = f\, x$
⟨*proof*⟩

lemma *convex-with-rel-open-domain*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes *rel-open* (*domain f*)
shows $(\forall\, x.\ f\, x > -\infty) \mid (\forall\, x.\ (f\, x = \infty \mid f\, x = -\infty))$

⟨*proof*⟩

lemma *convex-with-UNIV-domain*:
 fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
 assumes *convex-on UNIV f*
 assumes *domain f* = *UNIV*
 shows $(\forall x.\ f\ x > -\infty) \lor (\forall x.\ f\ x = -\infty)$
⟨*proof*⟩

lemma *rel-interior-Epigraph*:
 fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
 assumes *convex-on UNIV f*
 shows (x,z) : *rel-interior* (*Epigraph UNIV f*) ⟷
  (x : *rel-interior* (*domain f*) ∧ $f\ x <$ *ereal* z)
⟨*proof*⟩

lemma *rel-interior-EpigraphI*:
 fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
 assumes *convex-on UNIV f*
 shows *rel-interior* (*Epigraph UNIV f*) =
  {(x,z) |x z. x : *rel-interior* (*domain f*) ∧ $f\ x <$ *ereal* z}
⟨*proof*⟩

lemma *convex-less-ri-domain*:
 fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
 assumes *convex-on UNIV f*
 assumes $\exists\, x.\ f\ x < a$
 shows ∃ x∈*rel-interior* (*domain f*). $f\ x < a$
⟨*proof*⟩

lemma *rel-interior-eq-between*:
 fixes S T :: ($'m$::*euclidean-space*) *set*
 assumes *convex S convex T*
 shows (*rel-interior S* = *rel-interior T*) ⟷ (*rel-interior S* ≤ T ∧ T ≤ *closure S*)
⟨*proof*⟩

lemma *convex-less-in-riS*:
 fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
 assumes *convex-on UNIV f*
 assumes *convex S rel-interior S* ≤ *domain f*

assumes $\exists\, x \in closure\ S.\ f\ x < a$
shows $\exists\, x \in rel\text{-}interior\ S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-less-inS*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes *convex S* $S \leq$ *domain f*
assumes $\exists\, x \in closure\ S.\ f\ x < a$
shows $\exists\, x \in S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-finite-geq-on-closure*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
assumes *convex S finite-on S f*
assumes $\forall\, x \in S.\ f\ x \geq a$
shows $\forall\, x \in closure\ S.\ f\ x \geq a$
⟨*proof*⟩

lemma *lsc-hull-of-convex-determined*:
fixes $f\ g$ :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f convex-on UNIV g*
assumes *rel-interior* (*domain f*) = *rel-interior* (*domain g*)
assumes $\forall\, x \in$ *rel-interior* (*domain f*). $f\ x = g\ x$
shows *lsc-hull f* = *lsc-hull g*
⟨*proof*⟩

lemma *domain-lsc-hull-between*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
shows *domain f* $\leq$ *domain* (*lsc-hull f*)
$\wedge$ *domain* (*lsc-hull f*) $\leq$ *closure* (*domain f*)
⟨*proof*⟩

lemma *domain-vs-domain-lsc-hull*:
fixes f :: $'a$::*euclidean-space* $\Rightarrow$ *ereal*
assumes *convex-on UNIV f*
shows *rel-interior*(*domain* (*lsc-hull f*)) = *rel-interior*(*domain f*)
$\wedge$ *closure*(*domain* (*lsc-hull f*)) = *closure*(*domain f*)
$\wedge$ *aff-dim*(*domain* (*lsc-hull f*)) = *aff-dim*(*domain f*)
⟨*proof*⟩

lemma *vertical-line-affine*:

fixes x :: $'a$::*euclidean-space*
shows *affine* $\{(x,m{::}real)|m.\ m{:}UNIV\}$
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-onRI*:
fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows $\forall x \in$ *rel-interior* (*domain f*). ($f\ x$ = (*lsc-hull f*) x)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-outside*:
fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows $\forall x.\ x \notin$ *closure* (*domain f*) ⟶ ($f\ x$ = (*lsc-hull f*) x)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees*:
fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows $\forall x.$ ($f\ x$ = (*lsc-hull f*) x) | x : *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex-proper*:
fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *proper* (*lsc-hull f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex*:
fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *lsc* (*lsc-hull f*) ∧ *proper* (*lsc-hull f*) ∧ *convex-on UNIV* (*lsc-hull f*) ∧
($\forall x.$ ($f\ x$ = (*lsc-hull f*) x) | x : *rel-frontier* (*domain f*))
⟨*proof*⟩

lemma *affine-no-rel-frontier*:
fixes S :: ($'n$::*euclidean-space*) *set*
assumes *affine S*
shows *rel-frontier* $S = \{\}$
⟨*proof*⟩

lemma *convex-with-affine-domain-is-lsc*:

fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *affine* (*domain f*)
shows *lsc f*
⟨*proof*⟩

lemma *convex-finite-is-lsc*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *finite-on UNIV f*
shows *lsc f*
⟨*proof*⟩

lemma *always-eventually-within*:
$(\forall x{\in}S.\ P\ x) \Longrightarrow$ *eventually P* (*at x within S*)
⟨*proof*⟩

lemma *ereal-divide-pos*:
assumes (*a::ereal*)>*0 b*>*0*
shows *a*/(*ereal b*)>*0*
⟨*proof*⟩

lemma *real-interval-limpt*:
assumes *a*<*b*
shows (*b::real*) *islimpt* {*a..*<*b*}
⟨*proof*⟩

lemma *lsc-hull-of-convex-aux*:
Limsup (*at 1 within* {*0..*<*1*}) (λm. *ereal* ((*1*−*m*)∗*a*+*m*∗*b*)) ≤ *ereal b*
⟨*proof*⟩

lemma *lsc-hull-of-convex*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *x* : *rel-interior* (*domain f*)
shows ((λm. *f*((*1*−*m*)$*_R$ *x* + *m* $*_R$ *y*)) ⟶ (*lsc-hull f*) *y*) (*at 1 within* {*0..*<*1*})
(**is** (*?g* ⟶ - *y*) -)
⟨*proof*⟩

end