

# Lower Semicontinuous Functions

By Bogdan Grechuk

September 13, 2023

## Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

## Contents

<b>1</b>	<b>Lower semicontinuous functions</b>	<b>1</b>
1.1	Relative interior in one dimension . . . . .	1
1.2	Lower and upper semicontinuity . . . . .	2
1.3	Epigraphs . . . . .	6
1.4	Convex Functions . . . . .	8

## 1 Lower semicontinuous functions

```
theory Lower-Semicontinuous
imports HOL-Analysis.Multivariate-Analysis
begin
```

### 1.1 Relative interior in one dimension

```
lemma rel-interior-ereal-semiline:
  fixes a :: ereal
  shows rel-interior {y. a ≤ ereal y} = {y. a < ereal y}
⟨proof⟩
```

```
lemma closed-ereal-semiline:
  fixes a :: ereal
  shows closed {y. a ≤ ereal y}
⟨proof⟩
```

**lemma** *ereal-semiline-unique*:  
**fixes**  $a\ b :: \text{ereal}$   
**shows**  $\{y. a \leq \text{ereal } y\} = \{y. b \leq \text{ereal } y\} \longleftrightarrow a = b$   
 $\langle \text{proof} \rangle$

## 1.2 Lower and upper semicontinuity

**definition**  
 $\text{lsc-at} :: 'a \Rightarrow ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{lsc-at } x0\ f \longleftrightarrow (\forall X\ l. X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow f\ x0 \leq l)$

**definition**  
 $\text{usc-at} :: 'a \Rightarrow ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{usc-at } x0\ f \longleftrightarrow (\forall X\ l. X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow l \leq f\ x0)$

**lemma** *lsc-at-mem*:  
**assumes**  $\text{lsc-at } x0\ f$   
**assumes**  $x \longrightarrow x0$   
**assumes**  $(f \circ x) \longrightarrow A$   
**shows**  $f\ x0 \leq A$   
 $\langle \text{proof} \rangle$

**lemma** *usc-at-mem*:  
**assumes**  $\text{usc-at } x0\ f$   
**assumes**  $x \longrightarrow x0$   
**assumes**  $(f \circ x) \longrightarrow A$   
**shows**  $f\ x0 \geq A$   
 $\langle \text{proof} \rangle$

**lemma** *lsc-at-open*:  
**fixes**  $f :: 'a::\text{first-countable-topology} \Rightarrow 'b::\{\text{complete-linorder}, \text{linorder-topology}\}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow$   
 $(\forall S. \text{open } S \wedge f\ x0 \in S \longrightarrow (\exists T. \text{open } T \wedge x0 \in T \wedge (\forall x' \in T. f\ x' \leq f\ x0 \longrightarrow f\ x' \in S)))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle \text{proof} \rangle$

**lemma** *lsc-at-open-mem*:  
**fixes**  $f :: 'a::\text{first-countable-topology} \Rightarrow 'b::\{\text{complete-linorder}, \text{linorder-topology}\}$   
**assumes**  $\text{lsc-at } x0\ f$   
**assumes**  $\text{open } S \wedge f\ x0 \in S$   
**obtains**  $T$  **where**  $\text{open } T \wedge x0 \in T \wedge (\forall x' \in T. (f\ x' \leq f\ x0 \longrightarrow f\ x' \in S))$   
 $\langle \text{proof} \rangle$

**lemma** *lsc-at-MInfty*:  
**fixes**  $f :: 'a::\text{topological-space} \Rightarrow \text{ereal}$

**assumes**  $f\ x0 = -\infty$   
**shows**  $\text{lsc-at } x0\ f$   
 $\langle\text{proof}\rangle$

**lemma**  $\text{lsc-at-PIfty}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $f\ x0 = \infty$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow \text{continuous (at } x0) f$   
 $\langle\text{proof}\rangle$

**lemma**  $\text{lsc-at-real}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $|f\ x0| \neq \infty$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall e. e>0 \longrightarrow (\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. f\ y > f\ x0 - e)))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma**  $\text{lsc-at-ereal}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall C < f(x0). \exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. f\ y > C))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma**  $\text{lst-at-ball}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall C < f(x0). \exists d > 0. \forall y \in (\text{ball } x0\ d). C < f(y))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma**  $\text{lst-at-delta}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall C < f(x0). \exists d > 0. \forall y. \text{dist } x0\ y < d \longrightarrow C < f\ y)$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma**  $\text{lsc-liminf-at}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow f\ x0 \leq \text{Liminf (at } x0) f$   
 $\langle\text{proof}\rangle$

**lemma**  $\text{lsc-liminf-at-eq}$ :

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (f\ x0 = \min (f\ x0)\ (\text{Liminf } (\text{at } x0)\ f))$   
 $\langle\text{proof}\rangle$

**lemma** *lsc-imp-liminf*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{lsc-at } x0\ f$   
**assumes**  $x \longrightarrow x0$   
**shows**  $f\ x0 \leq \text{liminf } (f \circ x)$   
 $\langle\text{proof}\rangle$

**lemma** *lsc-liminf*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall x. x \longrightarrow x0 \longrightarrow f\ x0 \leq \text{liminf } (f \circ x))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma** *lsc-sequentially*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall x\ c. x \longrightarrow x0 \wedge (\forall n. f(x\ n) \leq c) \longrightarrow f(x0) \leq c)$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma** *lsc-sequentially-gen*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall x\ c\ c0. x \longrightarrow x0 \wedge c \longrightarrow c0 \wedge (\forall n. f(x\ n) \leq c \longrightarrow f(x0) \leq c0))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle\text{proof}\rangle$

**lemma** *lsc-sequentially-mem*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{lsc-at } x0\ f$   
**assumes**  $x \longrightarrow x0\ c \longrightarrow c0$   
**assumes**  $\forall n. f(x\ n) \leq c\ n$   
**shows**  $f(x0) \leq c0$   
 $\langle\text{proof}\rangle$

**lemma** *lsc-uminus*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ (\lambda x. -f\ x) \longleftrightarrow \text{usc-at } x0\ f$   
 $\langle\text{proof}\rangle$

**lemma** *usc-limsup*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{usc-at } x0\ f \longleftrightarrow (\forall x. x \longrightarrow x0 \longrightarrow f\ x0 \geq \text{limsup } (f \circ x))$   
**(is** *?lhs*  $\longleftrightarrow$  *?rhs*)  
 $\langle \text{proof} \rangle$

**lemma** *usc-imp-limsup*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{usc-at } x0\ f$   
**assumes**  $x \longrightarrow x0$   
**shows**  $f\ x0 \geq \text{limsup } (f \circ x)$   
 $\langle \text{proof} \rangle$

**lemma** *usc-limsup-at*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{usc-at } x0\ f \longleftrightarrow f\ x0 \geq \text{Limsup } (\text{at } x0)\ f$   
 $\langle \text{proof} \rangle$

**lemma** *continuous-iff-lsc-usc*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{continuous } (\text{at } x0)\ f \longleftrightarrow (\text{lsc-at } x0\ f) \wedge (\text{usc-at } x0\ f)$   
 $\langle \text{proof} \rangle$

**lemma** *continuous-lsc-compose*:  
**assumes**  $\text{lsc-at } (g\ x0)\ f$   $\text{continuous } (\text{at } x0)\ g$   
**shows**  $\text{lsc-at } x0\ (f \circ g)$   
 $\langle \text{proof} \rangle$

**lemma** *continuous-isCont*:  
 $\text{continuous } (\text{at } x0)\ f \longleftrightarrow \text{isCont } f\ x0$   
 $\langle \text{proof} \rangle$

**lemma** *isCont-iff-lsc-usc*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{isCont } f\ x0 \longleftrightarrow (\text{lsc-at } x0\ f) \wedge (\text{usc-at } x0\ f)$   
 $\langle \text{proof} \rangle$

**definition**  
 $\text{lsc} :: ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{lsc } f \longleftrightarrow (\forall x. \text{lsc-at } x\ f)$

**definition**

*usc* :: ('a::topological-space  $\Rightarrow$  'b::order-topology)  $\Rightarrow$  bool **where**  
*usc f*  $\longleftrightarrow$  ( $\forall x$ . *usc-at x f*)

**lemma** *continuous-UNIV-iff-lsc-usc*:  
**fixes** *f* :: 'a::metric-space  $\Rightarrow$  ereal  
**shows** ( $\forall x$ . *continuous (at x) f*)  $\longleftrightarrow$  (*lsc f*)  $\wedge$  (*usc f*)  
 $\langle$ proof $\rangle$

### 1.3 Epigraphs

**definition** *Epigraph S* (*f*::-  $\Rightarrow$  ereal) = {*xy*. *fst xy* : *S*  $\wedge$  *f (fst xy)*  $\leq$  ereal(*snd xy*)}

**lemma** *mem-Epigraph*: (*x, y*)  $\in$  *Epigraph S f*  $\longleftrightarrow$  *x*  $\in$  *S*  $\wedge$  *f x*  $\leq$  ereal *y*  $\langle$ proof $\rangle$

**lemma** *ereal-closed-levels*:  
**fixes** *f* :: 'a::metric-space  $\Rightarrow$  ereal  
**shows** ( $\forall y$ . *closed {x. f(x)  $\leq$  y}*)  $\longleftrightarrow$  ( $\forall r$ . *closed {x. f(x)  $\leq$  ereal r}*)  
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
 $\langle$ proof $\rangle$

**lemma** *lsc-iff*:  
**fixes** *f* :: 'a::metric-space  $\Rightarrow$  ereal  
**shows** (*lsc f*  $\longleftrightarrow$  ( $\forall y$ . *closed {x. f(x)  $\leq$  y}*))  $\wedge$  (*lsc f*  $\longleftrightarrow$  *closed (Epigraph UNIV f)*)  
 $\langle$ proof $\rangle$

**definition** *lsc-hull* :: ('a::metric-space  $\Rightarrow$  ereal)  $\Rightarrow$  ('a::metric-space  $\Rightarrow$  ereal) **where**  
*lsc-hull f* = (SOME *g*. *Epigraph UNIV g* = *closure (Epigraph UNIV f)*)

**lemma** *epigraph-mono*:  
**fixes** *f* :: 'a::metric-space  $\Rightarrow$  ereal  
**shows** (*x, y*):*Epigraph UNIV f*  $\wedge$  *y*  $\leq$  *z*  $\longrightarrow$  (*x, z*):*Epigraph UNIV f*  
 $\langle$ proof $\rangle$

**lemma** *closed-epigraph-lines*:  
**fixes** *S* :: ('a::metric-space \* 'b::metric-space) set  
**assumes** *closed S*  
**shows** *closed {z. (x, z) : S}*  
 $\langle$ proof $\rangle$

**lemma** *mono-epigraph*:

**fixes**  $S :: ('a::\text{metric-space} * \text{real}) \text{ set}$

**assumes** *mono*:  $\forall x y z. (x,y):S \wedge y \leq z \longrightarrow (x,z):S$

**assumes** *closed*  $S$

**shows**  $\exists g. ((\text{Epigraph UNIV } g) = S)$

*<proof>*

**lemma** *lsc-hull-exists*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\exists g. \text{Epigraph UNIV } g = \text{closure} (\text{Epigraph UNIV } f)$

*<proof>*

**lemma** *epigraph-invertible*:

**assumes**  $\text{Epigraph UNIV } f = \text{Epigraph UNIV } g$

**shows**  $f=g$

*<proof>*

**lemma** *lsc-hull-ex-unique*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\exists! g. \text{Epigraph UNIV } g = \text{closure} (\text{Epigraph UNIV } f)$

*<proof>*

**lemma** *epigraph-lsc-hull*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\text{Epigraph UNIV } (\text{lsc-hull } f) = \text{closure}(\text{Epigraph UNIV } f)$

*<proof>*

**lemma** *lsc-hull-expl*:

$(g = \text{lsc-hull } f) \longleftrightarrow (\text{Epigraph UNIV } g = \text{closure}(\text{Epigraph UNIV } f))$

*<proof>*

**lemma** *lsc-lsc-hull*:  $\text{lsc} (\text{lsc-hull } f)$

*<proof>*

**lemma** *epigraph-subset-iff*:

**fixes**  $f g :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\text{Epigraph UNIV } f \leq \text{Epigraph UNIV } g \longleftrightarrow (\forall x. g x \leq f x)$

*<proof>*

**lemma** *lsc-hull-le*:  $(\text{lsc-hull } f) x \leq f x$

$\langle proof \rangle$

**lemma** *lsc-hull-greatest*:  
**fixes**  $f g :: 'a::metric-space \Rightarrow ereal$   
**assumes**  $lsc\ g \ \forall x. g\ x \leq f\ x$   
**shows**  $\forall x. g\ x \leq (lsc\text{-}hull\ f)\ x$   
 $\langle proof \rangle$

**lemma** *lsc-hull-iff-greatest*:  
**fixes**  $f g :: 'a::metric-space \Rightarrow ereal$   
**shows**  $(g = lsc\text{-}hull\ f) \longleftrightarrow$   
 $lsc\ g \wedge (\forall x. g\ x \leq f\ x) \wedge (\forall h. lsc\ h \wedge (\forall x. h\ x \leq f\ x) \longrightarrow (\forall x. h\ x \leq g\ x))$   
**(is**  $?lhs \longleftrightarrow ?rhs$   
 $\langle proof \rangle$

**lemma** *lsc-hull-mono*:  
**fixes**  $f g :: 'a::metric-space \Rightarrow ereal$   
**assumes**  $\forall x. g\ x \leq f\ x$   
**shows**  $\forall x. (lsc\text{-}hull\ g)\ x \leq (lsc\text{-}hull\ f)\ x$   
 $\langle proof \rangle$

**lemma** *lsc-hull-lsc*:  
 $lsc\ f \longleftrightarrow (f = lsc\text{-}hull\ f)$   
 $\langle proof \rangle$

**lemma** *lsc-hull-liminf-at*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$   
**shows**  $\forall x. (lsc\text{-}hull\ f)\ x = min\ (f\ x)\ (Liminf\ (at\ x)\ f)$   
 $\langle proof \rangle$

**lemma** *lsc-hull-same-inf*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$   
**shows**  $(INF\ x. lsc\text{-}hull\ f\ x) = (INF\ x. f\ x)$   
 $\langle proof \rangle$

## 1.4 Convex Functions

**definition**  
 $convex\text{-}on :: 'a::real\text{-}vector\ set \Rightarrow ('a \Rightarrow ereal) \Rightarrow bool$  **where**  
 $convex\text{-}on\ s\ f \longleftrightarrow$   
 $(\forall x \in s. \forall y \in s. \forall u \geq 0. \forall v \geq 0. u + v = 1$   
 $\longrightarrow f\ (u\ *_R\ x + v\ *_R\ y) \leq ereal\ u\ * f\ x + ereal\ v\ * f\ y)$



**lemma** *convex-on-ereal-mem*:

**assumes** *convex-on s f*

**assumes**  $x:s y:s$

**assumes**  $u \geq 0 \ v \geq 0 \ u+v=1$

**shows**  $f (u *_R x + v *_R y) \leq \text{ereal } u * f x + \text{ereal } v * f y$

*<proof>*

**lemma** *convex-on-ereal-subset*:  $\text{convex-on } t f \implies s \leq t \implies \text{convex-on } s f$

*<proof>*

**lemma** *convex-on-ereal-univ*:  $\text{convex-on } UNIV f \longleftrightarrow (\forall S. \text{convex-on } S f)$

*<proof>*

**lemma** *ereal-pos-sum-distrib-left*:

**fixes**  $f :: 'a \Rightarrow \text{ereal}$

**assumes**  $r \geq 0 \ r \neq \infty$

**shows**  $r * \text{sum } f A = \text{sum } (\lambda n. r * f n) A$

*<proof>*

**lemma** *convex-ereal-add*:

**fixes**  $f g :: 'a::\text{real-vector} \Rightarrow \text{ereal}$

**assumes**  $\text{convex-on } s f \ \text{convex-on } s g$

**shows**  $\text{convex-on } s (\lambda x. f x + g x)$

*<proof>*

**lemma** *convex-ereal-cmul*:

**assumes**  $0 \leq (c::\text{ereal}) \ \text{convex-on } s f$

**shows**  $\text{convex-on } s (\lambda x. c * f x)$

*<proof>*

**lemma** *convex-ereal-max*:

**fixes**  $f g :: 'a::\text{real-vector} \Rightarrow \text{ereal}$

**assumes**  $\text{convex-on } s f \ \text{convex-on } s g$

**shows**  $\text{convex-on } s (\lambda x. \text{max } (f x) (g x))$

*<proof>*

**lemma** *convex-on-ereal-alt*:

**fixes**  $C :: 'a::\text{real-vector set}$

**assumes**  $\text{convex } C$

**shows**  $\text{convex-on } C f =$

$(\forall x \in C. \forall y \in C. \forall m :: \text{real}. m \geq 0 \wedge m \leq 1$

$\implies f (m *_R x + (1 - m) *_R y) \leq (\text{ereal } m) * f x + (1 - (\text{ereal } m)) * f y)$

*<proof>*

**lemma** *convex-on-ereal-alt-mem*:

**fixes**  $C :: 'a::\text{real-vector set}$

**assumes** *convex*  $C$

**assumes** *convex-on*  $C f$

**assumes**  $x : C \ y : C$

**assumes**  $(m::\text{real}) \geq 0 \ m \leq 1$

**shows**  $f (m *_R x + (1 - m) *_R y) \leq (\text{ereal } m) * f x + (1 - (\text{ereal } m)) * f y$

*<proof>*

**lemma** *ereal-add-right-mono*:  $(a::\text{ereal}) \leq b \implies a + c \leq b + c$

*<proof>*

**lemma** *convex-on-ereal-sum-aux*:

**assumes**  $1 - a > 0$

**shows**  $(1 - \text{ereal } a) * (\text{ereal } (c / (1 - a)) * b) = (\text{ereal } c) * b$

*<proof>*

**lemma** *convex-on-ereal-sum*:

**fixes**  $a :: 'a \Rightarrow \text{real}$

**fixes**  $y :: 'a \Rightarrow 'b::\text{real-vector}$

**fixes**  $f :: 'b \Rightarrow \text{ereal}$

**assumes** *finite*  $s \ s \neq \{\}$

**assumes** *convex-on*  $C f$

**assumes** *convex*  $C$

**assumes**  $(\text{SUM } i : s. a \ i) = 1$

**assumes**  $\forall i. i \in s \longrightarrow a \ i \geq 0$

**assumes**  $\forall i. i \in s \longrightarrow y \ i \in C$

**shows**  $f (\text{SUM } i : s. a \ i *_R y \ i) \leq (\text{SUM } i : s. \text{ereal } (a \ i) * f (y \ i))$

*<proof>*

**lemma** *sum-2*:  $\text{sum } u \ \{1::\text{nat}..2\} = (u \ 1) + (u \ 2)$

*<proof>*

**lemma** *convex-on-ereal-iff*:

**assumes** *convex*  $s$

**shows** *convex-on*  $s \ f \longleftrightarrow (\forall k \ u \ x. (\forall i \in \{1..k::\text{nat}\}. 0 \leq u \ i \wedge x \ i : s) \wedge \text{sum } u \ \{1..k\} = 1 \longrightarrow$

$f (\text{sum } (\lambda i. u \ i *_R x \ i) \ \{1..k\}) \leq \text{sum } (\lambda i. (\text{ereal } (u \ i)) * f(x \ i)) \ \{1..k\})$

(**is** *?rhs*  $\longleftrightarrow$  *?lhs*)

*<proof>*

**lemma** *convex-Epigraph*:  
**assumes** *convex S*  
**shows**  $\text{convex}(\text{Epigraph } S f) \longleftrightarrow \text{convex-on } S f$   
*<proof>*

**lemma** *convex-EpigraphI*:  
 $\text{convex-on } s f \implies \text{convex } s \implies \text{convex}(\text{Epigraph } s f)$   
*<proof>*

**definition**  
*concave-on* ::  $'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
*concave-on*  $S f \longleftrightarrow \text{convex-on } S (\lambda x. - f x)$

**definition**  
*finite-on* ::  $'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
*finite-on*  $S f \longleftrightarrow (\forall x \in S. (f x \neq \infty \wedge f x \neq -\infty))$

**definition**  
*affine-on* ::  $'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
*affine-on*  $S f \longleftrightarrow (\text{convex-on } S f \wedge \text{concave-on } S f \wedge \text{finite-on } S f)$

**definition**  
 $\text{domain } (f::- \Rightarrow \text{ereal}) = \{x. f x < \infty\}$

**lemma** *domain-Epigraph-aux*:  
**assumes**  $x \neq \infty$   
**shows**  $\exists r. x \leq \text{ereal } r$   
*<proof>*

**lemma** *domain-Epigraph*:  
 $\text{domain } f = \{x. \exists y. (x, y) \in \text{Epigraph UNIV } f\}$   
*<proof>*

**lemma** *domain-Epigraph-fst*:  
 $\text{domain } f = \text{fst } ` (\text{Epigraph UNIV } f)$   
*<proof>*

**lemma** *convex-on-domain*:  
 $\text{convex-on } (\text{domain } f) f \longleftrightarrow \text{convex-on UNIV } f$   
*<proof>*

**lemma** *convex-on-domain2*:

*convex-on (domain f) f*  $\longleftrightarrow$   $(\forall S. \text{convex-on } S f)$   
 ⟨proof⟩

**lemma** *convex-domain*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows** *convex (domain f)*  
 ⟨proof⟩

**lemma** *infinite-convex-domain-iff*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\forall x. (f x = \infty \mid f x = -\infty)$   
**shows** *convex-on UNIV f*  $\longleftrightarrow$  *convex (domain f)*  
 ⟨proof⟩

**lemma** *convex-PInfy-outside*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f convex S*  
**shows** *convex-on UNIV*  $(\lambda x. \text{if } x:S \text{ then } (f x) \text{ else } \infty)$   
 ⟨proof⟩

**definition**  
*proper-on*  $:: 'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
*proper-on S f*  $\longleftrightarrow ((\forall x \in S. f x \neq -\infty) \wedge (\exists x \in S. f x \neq \infty))$

**definition**  
*proper*  $:: ('a::\text{real-vector} \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
*proper f*  $\longleftrightarrow$  *proper-on UNIV f*

**lemma** *proper-iff*:  
*proper f*  $\longleftrightarrow ((\forall x. f x \neq -\infty) \wedge (\exists x. f x \neq \infty))$   
 ⟨proof⟩

**lemma** *improper-iff*:  
 $\sim(\text{proper } f) \longleftrightarrow ((\exists x. f x = -\infty) \mid (\forall x. f x = \infty))$   
 ⟨proof⟩

**lemma** *ereal-MInf-plus[simp]*:  $-\infty + x = (\text{if } x = \infty \text{ then } \infty \text{ else } -\infty::\text{ereal})$   
 ⟨proof⟩

**lemma** *convex-improper*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*

**assumes**  $\sim(\text{proper } f)$   
**shows**  $\forall x \in \text{rel-interior}(\text{domain } f). f x = -\infty$   
 <proof>

**lemma** *convex-improper2*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\sim(\text{proper } f)$   
**shows**  $f x = \infty \mid f x = -\infty \mid x : \text{rel-frontier}(\text{domain } f)$   
 <proof>

**lemma** *convex-lsc-improper*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\sim(\text{proper } f)$   
**assumes** *lsc f*  
**shows**  $f x = \infty \mid f x = -\infty$   
 <proof>

**lemma** *convex-lsc-hull*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows** *convex-on UNIV (lsc-hull f)*  
 <proof>

**lemma** *improper-lsc-hull*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\sim(\text{proper } f)$   
**shows**  $\sim(\text{proper } (\text{lsc-hull } f))$   
 <proof>

**lemma** *lsc-hull-convex-improper*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\sim(\text{proper } f)$   
**shows**  $\forall x \in \text{rel-interior}(\text{domain } f). (\text{lsc-hull } f) x = f x$   
 <proof>

**lemma** *convex-with-rel-open-domain*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes** *rel-open (domain f)*  
**shows**  $(\forall x. f x > -\infty) \mid (\forall x. (f x = \infty \mid f x = -\infty))$

*<proof>*

**lemma** *convex-with-UNIV-domain:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$

**assumes** *convex-on UNIV f*

**assumes**  $\text{domain } f = \text{UNIV}$

**shows**  $(\forall x. f x > -\infty) \vee (\forall x. f x = -\infty)$

*<proof>*

**lemma** *rel-interior-Epigraph:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$

**assumes** *convex-on UNIV f*

**shows**  $(x,z) : \text{rel-interior } (\text{Epigraph UNIV } f) \longleftrightarrow$

$(x : \text{rel-interior } (\text{domain } f) \wedge f x < \text{ereal } z)$

*<proof>*

**lemma** *rel-interior-EpigraphI:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$

**assumes** *convex-on UNIV f*

**shows**  $\text{rel-interior } (\text{Epigraph UNIV } f) =$

$\{(x,z) \mid x z. x : \text{rel-interior } (\text{domain } f) \wedge f x < \text{ereal } z\}$

*<proof>*

**lemma** *convex-less-ri-domain:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$

**assumes** *convex-on UNIV f*

**assumes**  $\exists x. f x < a$

**shows**  $\exists x \in \text{rel-interior } (\text{domain } f). f x < a$

*<proof>*

**lemma** *rel-interior-eq-between:*

**fixes**  $S T :: ('m::\text{euclidean-space}) \text{ set}$

**assumes** *convex S convex T*

**shows**  $(\text{rel-interior } S = \text{rel-interior } T) \longleftrightarrow (\text{rel-interior } S \leq T \wedge T \leq \text{closure } S)$

*<proof>*

**lemma** *convex-less-in-riS:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$

**assumes** *convex-on UNIV f*

**assumes**  $\text{convex } S \text{ rel-interior } S \leq \text{domain } f$

**assumes**  $\exists x \in \text{closure } S. f x < a$   
**shows**  $\exists x \in \text{rel-interior } S. f x < a$   
 <proof>

**lemma** *convex-less-inS*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\text{convex } S \ S \leq \text{domain } f$   
**assumes**  $\exists x \in \text{closure } S. f x < a$   
**shows**  $\exists x \in S. f x < a$   
 <proof>

**lemma** *convex-finite-geq-on-closure*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes** *convex S finite-on S f*  
**assumes**  $\forall x \in S. f x \geq a$   
**shows**  $\forall x \in \text{closure } S. f x \geq a$   
 <proof>

**lemma** *lsc-hull-of-convex-determined*:  
**fixes**  $f g :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f convex-on UNIV g*  
**assumes**  $\text{rel-interior } (\text{domain } f) = \text{rel-interior } (\text{domain } g)$   
**assumes**  $\forall x \in \text{rel-interior } (\text{domain } f). f x = g x$   
**shows**  $\text{lsc-hull } f = \text{lsc-hull } g$   
 <proof>

**lemma** *domain-lsc-hull-between*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**shows**  $\text{domain } f \leq \text{domain } (\text{lsc-hull } f)$   
 $\wedge \text{domain } (\text{lsc-hull } f) \leq \text{closure } (\text{domain } f)$   
 <proof>

**lemma** *domain-vs-domain-lsc-hull*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows**  $\text{rel-interior } (\text{domain } (\text{lsc-hull } f)) = \text{rel-interior } (\text{domain } f)$   
 $\wedge \text{closure } (\text{domain } (\text{lsc-hull } f)) = \text{closure } (\text{domain } f)$   
 $\wedge \text{aff-dim } (\text{domain } (\text{lsc-hull } f)) = \text{aff-dim } (\text{domain } f)$   
 <proof>

**lemma** *vertical-line-affine*:

**fixes**  $x :: 'a::\text{euclidean-space}$   
**shows**  $\text{affine } \{(x,m::\text{real}) \mid m. m:UNIV\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-convex-agrees-onRI}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**shows**  $\forall x \in \text{rel-interior } (\text{domain } f). (f\ x = (\text{lsc-hull } f)\ x)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-convex-agrees-outside}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**shows**  $\forall x. x \notin \text{closure } (\text{domain } f) \longrightarrow (f\ x = (\text{lsc-hull } f)\ x)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-convex-agrees}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**shows**  $\forall x. (f\ x = (\text{lsc-hull } f)\ x) \mid x : \text{rel-frontier } (\text{domain } f)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-proper-convex-proper}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   $\text{proper } f$   
**shows**  $\text{proper } (\text{lsc-hull } f)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-proper-convex}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   $\text{proper } f$   
**shows**  $\text{lsc } (\text{lsc-hull } f) \wedge \text{proper } (\text{lsc-hull } f) \wedge \text{convex-on UNIV } (\text{lsc-hull } f) \wedge$   
 $(\forall x. (f\ x = (\text{lsc-hull } f)\ x) \mid x : \text{rel-frontier } (\text{domain } f))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{affine-no-rel-frontier}$ :  
**fixes**  $S :: ('n::\text{euclidean-space}) \text{ set}$   
**assumes**  $\text{affine } S$   
**shows**  $\text{rel-frontier } S = \{\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{convex-with-affine-domain-is-lsc}$ :



**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**assumes**  $\text{affine } (\text{domain } f)$   
**shows**  $\text{lsc } f$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{convex-finite-is-lsc}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**assumes**  $\text{finite-on UNIV } f$   
**shows**  $\text{lsc } f$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{always-eventually-within}$ :  
 $(\forall x \in S. P x) \implies \text{eventually } P \text{ (at } x \text{ within } S)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{ereal-divide-pos}$ :  
**assumes**  $(a::\text{ereal}) > 0 \ b > 0$   
**shows**  $a / (\text{ereal } b) > 0$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{real-interval-limpt}$ :  
**assumes**  $a < b$   
**shows**  $(b::\text{real}) \text{ islimpt } \{a..<b\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-convex-aux}$ :  
 $\text{Limsup (at 1 within } \{0..<1\}) (\lambda m. \text{ereal } ((1-m)*a+m*b)) \leq \text{ereal } b$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lsc-hull-of-convex}$ :  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f$   
**assumes**  $x : \text{rel-interior } (\text{domain } f)$   
**shows**  $((\lambda m. f((1-m)*_R x + m *_R y)) \longrightarrow (\text{lsc-hull } f) y) \text{ (at 1 within } \{0..<1\})$   
 $\text{(is } (?g \longrightarrow - y) \text{ -)}$   
 $\langle \text{proof} \rangle$

**end**