Lower Semicontinuous Functions

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Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

Contents

1 Lower semicontinuous functions

theory *Lower-Semicontinuous*
imports *HOL−Analysis.Multivariate-Analysis*
begin

1.1 Relative interior in one dimension

lemma *rel-interior-ereal-semiline*:
fixes *a* :: *ereal*
shows *rel-interior* $\{y.\ a \le ereal\ y\} = \{y.\ a < ereal\ y\}$
⟨*proof*⟩

lemma *closed-ereal-semiline*:
fixes *a* :: *ereal*
shows *closed* $\{y.\ a \le ereal\ y\}$
⟨*proof*⟩

lemma *ereal-semiline-unique*:
  fixes $a\ b$ :: *ereal*
  shows $\{y.\ a \le ereal\ y\} = \{y.\ b \le ereal\ y\} \longleftrightarrow a = b$
⟨*proof*⟩

1.2 Lower and upper semicontinuity

definition
  $lsc\text{-}at$ :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
  $lsc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow f\ x0 \le l)$

definition
  $usc\text{-}at$ :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
  $usc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow l \le f\ x0)$

lemma *lsc-at-mem*:
  assumes $lsc\text{-}at\ x0\ f$
  assumes $x \longrightarrow x0$
  assumes $(f \circ x) \longrightarrow A$
  shows $f\ x0 \le A$
  ⟨*proof*⟩

lemma *usc-at-mem*:
  assumes $usc\text{-}at\ x0\ f$
  assumes $x \longrightarrow x0$
  assumes $(f \circ x) \longrightarrow A$
  shows $f\ x0 \ge A$
  ⟨*proof*⟩

lemma *lsc-at-open*:
  fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
  shows $lsc\text{-}at\ x0\ f \longleftrightarrow$
    $(\forall S.\ open\ S \wedge f\ x0 \in S \longrightarrow (\exists T.\ open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ f\ x' \le f\ x0$
$\longrightarrow f\ x' \in S)))$
  (**is** $?lhs \longleftrightarrow ?rhs$)
⟨*proof*⟩

lemma *lsc-at-open-mem*:
  fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
  assumes $lsc\text{-}at\ x0\ f$
  assumes $open\ S \wedge f\ x0 : S$
  obtains T **where** $open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ (f\ x' \le f\ x0 \longrightarrow f\ x' \in S))$
  ⟨*proof*⟩

lemma *lsc-at-MInfty*:
  fixes f :: $'a::topological\text{-}space \Rightarrow ereal$

assumes $f\ x0 = -\infty$
shows *lsc-at x0 f*
⟨*proof*⟩

lemma *lsc-at-PInfty*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes $f\ x0 = \infty$
shows *lsc-at x0 f* ⟷ *continuous* (*at x0*) *f*
⟨*proof*⟩

lemma *lsc-at-real*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes $|f\ x0| \neq \infty$
shows *lsc-at x0 f* ⟷ ($\forall e.\ e>0 \longrightarrow (\exists T.\ open\ T \wedge x0 : T \wedge (\forall y \in T.\ f\ y > f\ x0 - e))$)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-at-ereal*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ ($\forall C<f(x0).\ \exists T.\ open\ T \wedge x0 \in T \wedge (\forall y \in T.\ f\ y > C)$)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lst-at-ball*:
fixes f :: $'a$::*metric-space* => *ereal*
shows *lsc-at x0 f* ⟷ ($\forall C<f(x0).\ \exists d>0.\ \forall y \in (ball\ x0\ d).\ C<f(y)$)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lst-at-delta*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ ($\forall C<f(x0).\ \exists d>0.\ \forall y.\ dist\ x0\ y < d \longrightarrow C < f\ y$)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-liminf-at*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ $f\ x0 \leq$ *Liminf* (*at x0*) *f*
⟨*proof*⟩

lemma *lsc-liminf-at-eq*:

fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ (f *x0* = *min* (f *x0*) (*Liminf* (*at x0*) f))
⟨*proof*⟩

lemma *lsc-imp-liminf*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes *lsc-at x0 f*
assumes $x \longrightarrow$ *x0*
shows f *x0* $\leq$ *liminf* ($f \circ x$)
⟨*proof*⟩

lemma *lsc-liminf*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ ($\forall x.\ x \longrightarrow$ *x0* $\longrightarrow f$ *x0* $\leq$ *liminf* ($f \circ x$))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ ($\forall x\ c.\ x \longrightarrow$ *x0* $\wedge$ ($\forall n.\ f(x\ n) \leq c$) $\longrightarrow f(x0) \leq c$)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-gen*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ ($\forall x\ c\ c0.\ x \longrightarrow$ *x0* $\wedge$ $c \longrightarrow$ *c0* $\wedge$ ($\forall n.\ f(x\ n) \leq c$
n) $\longrightarrow f(x0) \leq c0$)
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-mem*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes *lsc-at x0 f*
assumes $x \longrightarrow$ *x0* $c \longrightarrow$ *c0*
assumes $\forall n.\ f(x\ n) \leq c\ n$
shows $f(x0) \leq c0$
⟨*proof*⟩

lemma *lsc-uminus*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0* ($\lambda x.\ -f\ x$) $\longleftrightarrow$ *usc-at x0 f*
⟨*proof*⟩

lemma *usc-limsup*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *usc-at x0 f* ⟷ (∀ *x*. *x* ⟶ *x0* ⟶ *f x0* ≥ *limsup* (*f* ∘ *x*))
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *usc-imp-limsup*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes *usc-at x0 f*
assumes *x* ⟶ *x0*
shows *f x0* ≥ *limsup* (*f* ∘ *x*)
⟨*proof*⟩

lemma *usc-limsup-at*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *usc-at x0 f* ⟷ *f x0* ≥ *Limsup* (*at x0*) *f*
⟨*proof*⟩

lemma *continuous-iff-lsc-usc*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *continuous* (*at x0*) *f* ⟷ (*lsc-at x0 f*) ∧ (*usc-at x0 f*)
⟨*proof*⟩

lemma *continuous-lsc-compose*:
assumes *lsc-at* (*g x0*) *f continuous* (*at x0*) *g*
shows *lsc-at x0* (*f* ∘ *g*)
⟨*proof*⟩

lemma *continuous-isCont*:
continuous (*at x0*) *f* ⟷ *isCont f x0*
⟨*proof*⟩

lemma *isCont-iff-lsc-usc*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *isCont f x0* ⟷ (*lsc-at x0 f*) ∧ (*usc-at x0 f*)
⟨*proof*⟩

definition
lsc :: ($'a$::*topological-space* ⇒ $'b$::*order-topology*) ⇒ *bool* **where**
lsc f ⟷ (∀ *x*. *lsc-at x f*)

definition

usc :: (*'a*::*topological-space* ⇒ *'b*::*order-topology*) ⇒ *bool* **where**
usc f ⟷ (∀ *x*. *usc-at x f*)

lemma *continuous-UNIV-iff-lsc-usc*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (∀ *x*. *continuous* (*at x*) *f*) ⟷ (*lsc f*) ∧ (*usc f*)
⟨*proof*⟩

1.3 Epigraphs

definition *Epigraph S* (*f*::- ⇒ *ereal*) = {*xy*. *fst xy* : *S* ∧ *f* (*fst xy*) ≤ *ereal*(*snd xy*)}

lemma *mem-Epigraph*: (*x*, *y*) ∈ *Epigraph S f* ⟷ *x* ∈ *S* ∧ *f x* ≤ *ereal y* ⟨*proof*⟩

lemma *ereal-closed-levels*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*}) ⟷ (∀ *r*. *closed* {*x*. *f*(*x*)≤*ereal r*})
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-iff*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (*lsc f* ⟷ (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*})) ∧ (*lsc f* ⟷ *closed* (*Epigraph UNIV f*))
⟨*proof*⟩

definition *lsc-hull* :: (*'a*::*metric-space* ⇒ *ereal*) ⇒ (*'a*::*metric-space* ⇒ *ereal*) **where**
lsc-hull f = (*SOME g*. *Epigraph UNIV g* = *closure*(*Epigraph UNIV f*))

lemma *epigraph-mono*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (*x*,*y*):*Epigraph UNIV f* ∧ *y*≤*z* ⟶ (*x*,*z*):*Epigraph UNIV f*
⟨*proof*⟩

lemma *closed-epigraph-lines*:
fixes *S* :: (*'a*::*metric-space* ∗ *'b*::*metric-space*) *set*
assumes *closed S*
shows *closed* {*z*. (*x*, *z*) : *S*}
⟨*proof*⟩

lemma *mono-epigraph*:
  fixes S :: ($'a$::*metric-space* $*$ *real*) *set*
  assumes *mono*: $\forall\, x\ y\ z.\ (x,y){:}S \wedge y{\leq}z \longrightarrow (x,z){:}S$
  assumes *closed S*
  shows $\exists\, g.\ ((Epigraph\ UNIV\ g) = S)$
⟨*proof*⟩

lemma *lsc-hull-exists*:
  fixes f :: $'a$::*metric-space* ⇒ *ereal*
  shows $\exists\, g.$ *Epigraph UNIV g* = *closure* (*Epigraph UNIV f*)
⟨*proof*⟩

lemma *epigraph-invertible*:
  assumes *Epigraph UNIV f* = *Epigraph UNIV g*
  shows $f{=}g$
⟨*proof*⟩

lemma *lsc-hull-ex-unique*:
  fixes f :: $'a$::*metric-space* ⇒ *ereal*
  shows $\exists!g.$ *Epigraph UNIV g* = *closure* (*Epigraph UNIV f*)
⟨*proof*⟩

lemma *epigraph-lsc-hull*:
  fixes f :: $'a$::*metric-space* ⇒ *ereal*
  shows *Epigraph UNIV* (*lsc-hull f*) = *closure*(*Epigraph UNIV f*)
⟨*proof*⟩

lemma *lsc-hull-expl*:
  (g = *lsc-hull f*) ⟷ (*Epigraph UNIV g* = *closure*(*Epigraph UNIV f*))
⟨*proof*⟩

lemma *lsc-lsc-hull*: *lsc* (*lsc-hull f*)
  ⟨*proof*⟩

lemma *epigraph-subset-iff*:
  fixes $f\ g$ :: $'a$::*metric-space* ⇒ *ereal*
  shows *Epigraph UNIV f* ≤ *Epigraph UNIV g* ⟷ ($\forall\, x.\ g\ x \leq f\ x$)
⟨*proof*⟩

lemma *lsc-hull-le*: (*lsc-hull f*) $x \leq f\ x$

⟨*proof*⟩

lemma *lsc-hull-greatest*:
fixes $f\ g$ :: *'a::metric-space* ⇒ *ereal*
assumes $lsc\ g\ \forall x.\ g\ x \leq f\ x$
shows $\forall x.\ g\ x \leq (lsc\text{-}hull\ f)\ x$
⟨*proof*⟩

lemma *lsc-hull-iff-greatest*:
fixes $f\ g$ :: *'a::metric-space* ⇒ *ereal*
shows $(g = lsc\text{-}hull\ f) \longleftrightarrow$
$lsc\ g \wedge (\forall x.\ g\ x \leq f\ x) \wedge (\forall h.\ lsc\ h \wedge (\forall x.\ h\ x \leq f\ x) \longrightarrow (\forall x.\ h\ x \leq g\ x))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-hull-mono*:
fixes $f\ g$ :: *'a::metric-space* ⇒ *ereal*
assumes $\forall x.\ g\ x \leq f\ x$
shows $\forall x.\ (lsc\text{-}hull\ g)\ x \leq (lsc\text{-}hull\ f)\ x$
⟨*proof*⟩

lemma *lsc-hull-lsc*:
$lsc\ f \longleftrightarrow (f = lsc\text{-}hull\ f)$
⟨*proof*⟩

lemma *lsc-hull-liminf-at*:
fixes f :: *'a::metric-space* ⇒ *ereal*
shows $\forall x.\ (lsc\text{-}hull\ f)\ x = min\ (f\ x)\ (Liminf\ (at\ x)\ f)$
⟨*proof*⟩

lemma *lsc-hull-same-inf*:
fixes f :: *'a::metric-space* ⇒ *ereal*
shows $(INF\ x.\ lsc\text{-}hull\ f\ x) = (INF\ x.\ f\ x)$
⟨*proof*⟩

1.4 Convex Functions

definition
convex-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
convex-on $s\ f \longleftrightarrow$
$(\forall x{\in}s.\ \forall y{\in}s.\ \forall u{\geq}0.\ \forall v{\geq}0.\ u + v = 1$
$\longrightarrow f\ (u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y)$

lemma *convex-on-ereal-mem*:
  assumes *convex-on s f*
  assumes *x:s y:s*
  assumes $u \geq 0$ $v \geq 0$ $u+v=1$
  shows $f\ (u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y$
⟨*proof*⟩

lemma *convex-on-ereal-subset*: $convex\text{-}on\ t\ f \Longrightarrow s \leq t \Longrightarrow convex\text{-}on\ s\ f$
  ⟨*proof*⟩

lemma *convex-on-ereal-univ*: $convex\text{-}on\ UNIV\ f \longleftrightarrow (\forall S.\ convex\text{-}on\ S\ f)$
  ⟨*proof*⟩

lemma *ereal-pos-sum-distrib-left*:
  fixes $f :: 'a \Rightarrow ereal$
  assumes $r \geq 0$ $r \neq \infty$
  shows $r * sum\ f\ A = sum\ (\lambda n.\ r * f\ n)\ A$
⟨*proof*⟩

lemma *convex-ereal-add*:
  fixes $f\ g :: 'a::real\text{-}vector \Rightarrow ereal$
  assumes *convex-on s f convex-on s g*
  shows $convex\text{-}on\ s\ (\lambda x.\ f\ x + g\ x)$
⟨*proof*⟩

lemma *convex-ereal-cmul*:
  assumes $0 \leq (c::ereal)$ *convex-on s f*
  shows $convex\text{-}on\ s\ (\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *convex-ereal-max*:
  fixes $f\ g :: 'a::real\text{-}vector \Rightarrow ereal$
  assumes *convex-on s f convex-on s g*
  shows $convex\text{-}on\ s\ (\lambda x.\ max\ (f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *convex-on-ereal-alt*:
  fixes $C :: 'a::real\text{-}vector\ set$
  assumes *convex C*
  shows $convex\text{-}on\ C\ f =$
  $(\forall x \in C.\ \forall y \in C.\ \forall m :: real.\ m \geq 0 \land m \leq 1$
    $\longrightarrow f\ (m *_R x + (1 - m) *_R y) \leq (ereal\ m) * f\ x + (1 - (ereal\ m)) * f\ y)$

⟨*proof*⟩

lemma *convex-on-ereal-alt-mem*:
fixes C :: $'a$::*real-vector set*
assumes *convex* C
assumes *convex-on* C f
assumes $x : C$ $y : C$
assumes (m::*real*) $\geq$ *0* $m \leq$ *1*
shows f ($m *_R x$ + (*1* − m) $*_R y$) $\leq$ (*ereal* m) $*$ f x + (*1* − (*ereal* m)) $*$ f y
⟨*proof*⟩

lemma *ereal-add-right-mono*: (a::*ereal*) $\leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

lemma *convex-on-ereal-sum-aux*:
assumes *1*−a>*0*
shows (*1* − *ereal* a) $*$ (*ereal* (c / (*1* − a)) $*$ b) = (*ereal* c) $*$ b
⟨*proof*⟩

lemma *convex-on-ereal-sum*:
fixes a :: $'a \Rightarrow$ *real*
fixes y :: $'a \Rightarrow$ $'b$::*real-vector*
fixes f :: $'b \Rightarrow$ *ereal*
assumes *finite* s $s \neq \{\}$
assumes *convex-on* C f
assumes *convex* C
assumes (*SUM* i : s. a i) = *1*
assumes $\forall i.\ i \in s \longrightarrow a\ i \geq$ *0*
assumes $\forall i.\ i \in s \longrightarrow y\ i \in C$
shows f (*SUM* i : s. a i $*_R$ y i) $\leq$ (*SUM* i : s. *ereal* (a i) $*$ f (y i))
⟨*proof*⟩

lemma *sum-2*: *sum* u {*1*::*nat*..*2*} = (u *1*)+(u *2*)
⟨*proof*⟩

lemma *convex-on-ereal-iff*:
assumes *convex* s
shows *convex-on* s f $\longleftrightarrow$ ($\forall$ k u x. ($\forall$ i∈{*1*..k::*nat*}. *0* $\leq$ u i $\wedge$ x i : s) $\wedge$ *sum* u
{*1*..k} = *1* $\longrightarrow$
f (*sum* (λi. u i $*_R$ x i) {*1*..k}) $\leq$ *sum* (λi. (*ereal* (u i)) $*$ f(x i)) {*1*..k})
(**is** *?rhs* $\longleftrightarrow$ *?lhs*)
⟨*proof*⟩

lemma *convex-Epigraph*:
  assumes *convex S*
  shows *convex*(*Epigraph S f*) $\longleftrightarrow$ *convex-on S f*
⟨*proof*⟩

lemma *convex-EpigraphI*:
  convex-on s f $\Longrightarrow$ *convex s* $\Longrightarrow$ *convex*(*Epigraph s f*)
⟨*proof*⟩

definition
  concave-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
  concave-on S f $\longleftrightarrow$ *convex-on S* ($\lambda x. - f\ x$)

definition
  finite-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
  finite-on S f $\longleftrightarrow$ ($\forall x \in S$. ($f\ x \neq \infty \wedge f\ x \neq -\infty$))

definition
  affine-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
  affine-on S f $\longleftrightarrow$ (*convex-on S f* ∧ *concave-on S f* ∧ *finite-on S f*)

definition
  domain (*f*::- ⇒ *ereal*) = $\{x.\ f\ x < \infty\}$

lemma *domain-Epigraph-aux*:
  assumes $x \neq \infty$
  shows $\exists r.\ x \leq$ *ereal r*
⟨*proof*⟩

lemma *domain-Epigraph*:
  domain f = {*x*. ∃ *y*. (*x*,*y*) ∈ *Epigraph UNIV f*}
  ⟨*proof*⟩

lemma *domain-Epigraph-fst*:
  domain f = *fst* ‘ (*Epigraph UNIV f*)
⟨*proof*⟩

lemma *convex-on-domain*:
  convex-on (*domain f*) *f* $\longleftrightarrow$ *convex-on UNIV f*
⟨*proof*⟩

lemma *convex-on-domain2*:

convex-on (*domain f*) *f* ⟷ (∀ *S*. *convex-on S f*)
⟨*proof*⟩

lemma *convex-domain*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows *convex* (*domain f*)
⟨*proof*⟩

lemma *infinite-convex-domain-iff*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes ∀ *x*. (*f x* = ∞ | *f x* = −∞)
shows *convex-on UNIV f* ⟷ *convex* (*domain f*)
⟨*proof*⟩

lemma *convex-PInfty-outside*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f convex S*
shows *convex-on UNIV* (λ*x*. *if x*:*S then* (*f x*) *else* ∞)
⟨*proof*⟩

definition
proper-on :: ′*a*::*real-vector set* ⇒ (′*a* ⇒ *ereal*) ⇒ *bool* **where**
proper-on S f ⟷ ((∀ *x*∈*S*. *f x* ≠ −∞) ∧ (∃ *x*∈*S*. *f x* ≠ ∞))

definition
proper :: (′*a*::*real-vector* ⇒ *ereal*) ⇒ *bool* **where**
proper f ⟷ *proper-on UNIV f*

lemma *proper-iff*:
proper f ⟷ ((∀ *x*. *f x* ≠ −∞) ∧ (∃ *x*. *f x* ≠ ∞))
⟨*proof*⟩

lemma *improper-iff*:
~(*proper f*) ⟷ ((∃ *x*. *f x* = −∞) | (∀ *x*. *f x* = ∞))
⟨*proof*⟩

lemma *ereal-MInf-plus*[*simp*]: −∞ + *x* = (*if x* = ∞ *then* ∞ *else* −∞::*ereal*)
⟨*proof*⟩

lemma *convex-improper*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*

assumes $\sim$(*proper f*)
shows $\forall\, x \in$*rel-interior*(*domain f*). *f x* $= -\infty$
⟨*proof*⟩

lemma *convex-improper2*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
shows *f x* $= \infty$ | *f x* $= -\infty$ | *x* : *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *convex-lsc-improper*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
assumes *lsc f*
shows *f x* $= \infty$ | *f x* $= -\infty$
⟨*proof*⟩

lemma *convex-lsc-hull*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows *convex-on UNIV* (*lsc-hull f*)
⟨*proof*⟩

lemma *improper-lsc-hull*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes $\sim$(*proper f*)
shows $\sim$(*proper* (*lsc-hull f*))
⟨*proof*⟩

lemma *lsc-hull-convex-improper*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes $\sim$(*proper f*)
shows $\forall\, x \in$*rel-interior*(*domain f*). (*lsc-hull f*) *x* = *f x*
⟨*proof*⟩

lemma *convex-with-rel-open-domain*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *rel-open* (*domain f*)
shows ($\forall\, x$. *f x* $> -\infty$) | ($\forall\, x$. (*f x* $= \infty$ | *f x* $= -\infty$))

⟨*proof*⟩

lemma *convex-with-UNIV-domain*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes *domain f* = *UNIV*
  shows $(\forall x.\ f\ x > -\infty) \vee (\forall x.\ f\ x = -\infty)$
⟨*proof*⟩

lemma *rel-interior-Epigraph*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  shows (x,z) : *rel-interior* (*Epigraph UNIV f*) ⟷
    (x : *rel-interior* (*domain f*) ∧ $f\ x$ < *ereal z*)
⟨*proof*⟩

lemma *rel-interior-EpigraphI*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  shows *rel-interior* (*Epigraph UNIV f*) =
    {(x,z) |x z. x : *rel-interior* (*domain f*) ∧ $f\ x$ < *ereal z*}
⟨*proof*⟩

lemma *convex-less-ri-domain*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes $\exists x.\ f\ x < a$
  shows ∃ x∈*rel-interior* (*domain f*). $f\ x < a$
⟨*proof*⟩

lemma *rel-interior-eq-between*:
  fixes S T :: ($'m$::*euclidean-space*) *set*
  assumes *convex S convex T*
  shows (*rel-interior S* = *rel-interior T*) ⟷ (*rel-interior S* ≤ T ∧ T ≤ *closure S*)
⟨*proof*⟩

lemma *convex-less-in-riS*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes *convex S rel-interior S* ≤ *domain f*

assumes $\exists\, x{\in}closure\ S.\ f\ x < a$
shows $\exists\, x{\in}rel\text{-}interior\ S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-less-inS*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
assumes *convex S* $S \leq$ *domain f*
assumes $\exists\, x{\in}closure\ S.\ f\ x < a$
shows $\exists\, x{\in}S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-finite-geq-on-closure*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
assumes *convex S finite-on S f*
assumes $\forall\, x{\in}S.\ f\ x \geq a$
shows $\forall\, x{\in}closure\ S.\ f\ x \geq a$
⟨*proof*⟩

lemma *lsc-hull-of-convex-determined*:
fixes $f\ g$ :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f convex-on UNIV g*
assumes *rel-interior* (*domain f*) = *rel-interior* (*domain g*)
assumes $\forall\, x{\in}rel\text{-}interior\ (domain\ f).\ f\ x = g\ x$
shows *lsc-hull f* = *lsc-hull g*
⟨*proof*⟩

lemma *domain-lsc-hull-between*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
shows *domain f* $\leq$ *domain* (*lsc-hull f*)
$\wedge$ *domain* (*lsc-hull f*) $\leq$ *closure* (*domain f*)
⟨*proof*⟩

lemma *domain-vs-domain-lsc-hull*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
shows *rel-interior*(*domain* (*lsc-hull f*)) = *rel-interior*(*domain f*)
$\wedge$ *closure*(*domain* (*lsc-hull f*)) = *closure*(*domain f*)
$\wedge$ *aff-dim*(*domain* (*lsc-hull f*)) = *aff-dim*(*domain f*)
⟨*proof*⟩

lemma *vertical-line-affine*:

fixes *x* :: *'a::euclidean-space*
shows *affine* {(*x*,*m*::*real*)|*m*. *m*:*UNIV*}
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-onRI*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*∈*rel-interior* (*domain f*). (*f x* = (*lsc-hull f*) *x*)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-outside*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*. *x* ∉ *closure* (*domain f*) ⟶ (*f x* = (*lsc-hull f*) *x*)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*. (*f x* = (*lsc-hull f*) *x*) | *x* : *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex-proper*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *proper* (*lsc-hull f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *lsc* (*lsc-hull f*) ∧ *proper* (*lsc-hull f*) ∧ *convex-on UNIV* (*lsc-hull f*) ∧
(∀ *x*. (*f x* = (*lsc-hull f*) *x*) | *x* : *rel-frontier* (*domain f*))
⟨*proof*⟩

lemma *affine-no-rel-frontier*:
fixes *S* :: (*'n::euclidean-space*) *set*
assumes *affine S*
shows *rel-frontier S* = {}
⟨*proof*⟩

lemma *convex-with-affine-domain-is-lsc*:

fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *affine* (*domain f*)
shows *lsc f*
⟨*proof*⟩

lemma *convex-finite-is-lsc*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *finite-on UNIV f*
shows *lsc f*
⟨*proof*⟩

lemma *always-eventually-within*:
(∀ *x*∈*S*. *P x*) ⟹ *eventually P* (*at x within S*)
⟨*proof*⟩

lemma *ereal-divide-pos*:
assumes (*a*::*ereal*)>*0* *b*>*0*
shows *a*/(*ereal b*)>*0*
⟨*proof*⟩

lemma *real-interval-limpt*:
assumes *a*<*b*
shows (*b*::*real*) *islimpt* {*a*..<*b*}
⟨*proof*⟩

lemma *lsc-hull-of-convex-aux*:
Limsup (*at 1 within* {*0*..<*1*}) (λ*m*. *ereal* ((*1*−*m*)∗*a*+*m*∗*b*)) ≤ *ereal b*
⟨*proof*⟩

lemma *lsc-hull-of-convex*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *x* : *rel-interior* (*domain f*)
shows ((λ*m*. *f*((*1*−*m*)$*_R$ *x* + *m* $*_R$ *y*)) ⟶ (*lsc-hull f*) *y*) (*at 1 within* {*0*..<*1*})
(**is** (*?g* ⟶ - *y*) -)
⟨*proof*⟩

end