Lower Semicontinuous Functions

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Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

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1 Lower semicontinuous functions

theory Lower-Semicontinuous
imports HOL-Analysis.Analysis
begin

1.1 Relative interior in one dimension

lemma rel-interior-ereal-semiline:
  fixes a :: ereal
  shows rel-interior {y. a ≤ ereal y} = {y. a < ereal y}
 ⟨proof⟩

lemma closed-ereal-semiline:
  fixes a :: ereal
  shows closed {y. a ≤ ereal y}
 ⟨proof⟩
lemma ereal-semilin-eq:
fixes a b :: ereal
shows \{ y. a ≤ ereal y \} = \{ y. b ≤ ereal y \} ←→ a = b
⟨proof⟩

1.2 Lower and upper semicontinuity

definition
lsc-at :: 'a ⇒ ('a::topological-space ⇒ 'b::order-topology) ⇒ bool where
lsc-at x0 f ←→ (∀ X l. X → x0 ∧ (f ◦ X) → l → f x0 ≤ l)
definition
usc-at :: 'a ⇒ ('a::topological-space ⇒ 'b::order-topology) ⇒ bool where
usc-at x0 f ←→ (∀ X l. X → x0 ∧ (f ◦ X) → l → l ≤ f x0)

lemma lsc-at-mem:
assumes lsc-at x0 f
assumes x → x0
assumes (f ◦ x) → A
shows f x0 ≤ A
⟨proof⟩

lemma usc-at-mem:
assumes usc-at x0 f
assumes x → x0
assumes (f ◦ x) → A
shows f x0 ≥ A
⟨proof⟩

lemma lsc-at-open:
fixes f :: 'a::first-countable-topology ⇒ 'b::{complete-linorder, linorder-topology}
shows lsc-at x0 f ←→
(∀ S. open S ∧ f x0 ∈ S → (∃ T. open T ∧ x0 ∈ T ∧ (∀ x'∈T. f x' ≤ f x0
→ f x' ∈ S)))
(is ?lhs ←→ ?rhs)
⟨proof⟩

lemma lsc-at-open-mem:
fixes f :: 'a::first-countable-topology ⇒ 'b::{complete-linorder, linorder-topology}
assumes lsc-at x0 f
assumes open S ∧ f x0 : S
obeys T where open T ∧ x0 ∈ T ∧ (∀ x'∈T. (f x' ≤ f x0 → f x' ∈ S))
⟨proof⟩

lemma lsc-at-MInf:
fixes f :: 'a::topological-space ⇒ ereal
\begin{itemize}
  \item \textbf{assumes} \( f \ x_0 = -\infty \)
  \item \textbf{shows} \( \text{lsc-at} \ x_0 \ f \)
  \end{itemize}

\textbf{lemma} \text{lsc-at-PInfy}:
\textbf{fixes} \( f :: 'a::metric-space \Rightarrow \text{ereal} \)
\textbf{assumes} \( f \ x_0 = \infty \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff \text{continuous} \ (\text{at} \ x_0) \ f \)

\textbf{lemma} \text{lsc-at-real}:
\textbf{fixes} \( f :: 'a::metric-space \Rightarrow \text{ereal} \)
\textbf{assumes} \( f \ x_0 \neq -\infty \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff (\forall \ e. \ e > 0 \longrightarrow (\exists T. \text{open} \ T \land x_0 \in T \land (\forall y \in T. f y > f \ x_0 - e))) \)
\( (\text{is } \?\text{lhs} \longleftrightarrow \?\text{rhs}) \)

\textbf{lemma} \text{lsc-at-ereal}:
\textbf{fixes} \( f :: 'a::metric-space \Rightarrow \text{ereal} \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff (\forall C < f(x_0). \exists T. \text{open} T \land x_0 \in T \land (\forall y \in T. f y > C)) \)
\( (\text{is } \?\text{lhs} \longleftrightarrow \?\text{rhs}) \)

\textbf{lemma} \text{lst-at-ball}:
\textbf{fixes} \( f :: 'a::metric-space => \text{ereal} \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff (\forall C < f(x_0). \exists d > 0. \forall y \in (\text{ball} x_0 d). C < f(y)) \)
\( (\text{is } \?\text{lhs} \longleftrightarrow \?\text{rhs}) \)

\textbf{lemma} \text{lst-at-delta}:
\textbf{fixes} \( f :: 'a::metric-space => \text{ereal} \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff (\forall C < f(x_0). \exists d > 0. \forall y. \text{dist} x_0 y < d \longrightarrow C < f y) \)
\( (\text{is } \?\text{lhs} \longleftrightarrow \?\text{rhs}) \)

\textbf{lemma} \text{lsc-liminf-at}:
\textbf{fixes} \( f :: 'a::metric-space => \text{ereal} \)
\textbf{shows} \( \text{lsc-at} \ x_0 \ f \iff f x_0 \leq \text{Liminf} \ (\text{at} \ x_0) \ f \)
\( (\text{proof}) \)

\textbf{lemma} \text{lsc-liminf-at-eq}:
fixes f :: 'a::metric-space ⇒ ereal
shows lsc-at x0 f ←→ (f x0 = min (f x0) (Liminf (at x0) f))
 ⟨proof⟩

lemma lsc-imp-liminf:
fixes f :: 'a::metric-space ⇒ ereal
assumes lsc-at x0 f
assumes x −−−−→ x0
shows f x0 ≤ liminf (f o x)
 ⟨proof⟩

lemma lsc-liminf:
fixes f :: 'a::metric-space ⇒ ereal
shows lsc-at x0 f ←→ (∀ x. x −−−−→ x0 → f x0 ≤ liminf (f o x))
(is ?lhs ←→ ?rhs)
⟨proof⟩

lemma lsc-sequentially:
fixes f :: 'a::metric-space ⇒ ereal
shows lsc-at x0 f ←→ (∀ x c. x −−−−→ x0 ∧ (∀ n. f x n) ≤ c) −→ f(x0) ≤ c)
(is ?lhs ←→ ?rhs)
⟨proof⟩

lemma lsc-sequentially-gen:
fixes f :: 'a::metric-space ⇒ ereal
shows lsc-at x0 f ←→ (∀ x c. x −−−−→ x0 ∧ (∀ n. f x n) ≤ c) −→ f(x0) ≤ c)
(is ?lhs ←→ ?rhs)
⟨proof⟩

lemma lsc-sequentially-mem:
fixes f :: 'a::metric-space ⇒ ereal
assumes lsc-at x0 f
assumes x −−−−→ x0 c −−−−→ c0
assumes ∀ n. f x n ≤ c n
shows f(x0) ≤ c0
⟨proof⟩

lemma lsc-uminus:
fixes f :: 'a::metric-space ⇒ ereal
shows lsc-at x0 (λx. − f x) ←→ usc-at x0 f
⟨proof⟩
lemma usc-limsup:
fixes f :: 'a::metric-space ⇒ ereal
shows usc-at x0 f ←→ (∀ x. x → x0 → f x0 ≥ limsup (f ∘ x))
⟨proof⟩

lemma usc-imp-limsup:
fixes f :: 'a::metric-space ⇒ ereal
assumes usc-at x0 f
assumes x ----→ x0
shows f x0 ≥ limsup (f ∘ x)
⟨proof⟩

lemma usc-limsup-at:
fixes f :: 'a::metric-space ⇒ ereal
shows usc-at x0 f ←→ f x0 ≥ Limsup (at x0) f
⟨proof⟩

lemma continuous-iff-lsc-usc:
fixes f :: 'a::metric-space ⇒ ereal
shows continuous (at x0) f ←→ (lsc-at x0 f) ∧ (usc-at x0 f)
⟨proof⟩

lemma continuous-lsc-compose:
assumes lsc-at (g x0) f continuous (at x0) g
shows lsc-at x0 (f ∘ g)
⟨proof⟩

lemma continuous-isCont:
continuous (at x0) f ←→ isCont f x0
⟨proof⟩

lemma isCont-iff-lsc-usc:
fixes f :: 'a::metric-space ⇒ ereal
shows isCont f x0 ←→ (lsc-at x0 f) ∧ (usc-at x0 f)
⟨proof⟩

definition
lsc :: ('a::topological-space ⇒ 'b::order-topology) ⇒ bool
where
lsc f ←→ (∀ x. lsc-at x f)
usc :: ('a::topological-space ⇒ 'b::order-topology) ⇒ bool where
  usc f ←→ (∀ x. usc-at x f)

lemma continuous-UNIV-iff-lsc-usc:
  fixes f :: 'a::metric-space ⇒ ereal
  shows (∀ x. continuous (at x) f) ←→ (lsc f) ∧ (usc f)
  ⟨proof⟩

1.3 Epigraphs

definition Epigraph S (f::→ ereal) = {xy. fst xy : S ∧ f (fst xy) ≤ ereal(snd xy)}

lemma mem-Epigraph: (x, y) ∈ Epigraph S f ←→ x ∈ S ∧ f x ≤ ereal y ⟨proof⟩

lemmaereal-closed-levels:
  fixes f :: 'a::metric-space ⇒ ereal
  shows (∀ y. closed {x. f(x)≤y}) ←→ (∀ r. closed {x. f(x)≤ereal r})
  ⟨lsb, r⟩ ←→ ⟨r, r⟩
  ⟨proof⟩

lemma lsc-iff:
  fixes f :: 'a::metric-space ⇒ ereal
  shows (lsc f) ←→ (∀ y. closed {x. f(x)≤y}) ∧ (lsc f) ←→ closed (Epigraph UNIV f)
  ⟨proof⟩

definition lsc-hull :: ('a::metric-space ⇒ ereal) ⇒ ('a::metric-space ⇒ ereal)
  where
  lsc-hull f = (SOME g. Epigraph UNIV g = closure(Epigraph UNIV f))

lemma epigraph-mono:
  fixes f :: 'a::metric-space ⇒ ereal
  shows (x,y):Epigraph UNIV f ∧ y≤z → (x,z):Epigraph UNIV f
  ⟨proof⟩

lemma closed-epigraph-lines:
  fixes f :: ('a::metric-space * 'b::metric-space) set
  assumes closed f
  shows closed {z. (x, z) : S}
  ⟨proof⟩
lemma mono-epigraph:
  fixes $S :: (\text{\texttt{\textasciitilde}a::metric-space} * \text{real}) \text{ set}$
  assumes mono: $\forall x \ y \ z. (x,y) : S \land y \leq z \rightarrow (x,z) : S$
  assumes closed $S$
  shows $\exists g. ((\text{Epigraph UNIV} \ g) = S)$
⟨proof⟩

lemma lsc-hull-exists:
  fixes $f :: \text{\texttt{\textasciitilde}a::metric-space} \Rightarrow \text{ereal}$
  shows $\exists g. \text{Epigraph UNIV} \ g = \text{closure} \ (\text{Epigraph UNIV} \ f)$
⟨proof⟩

lemma epigraph-invertible:
  assumes $\text{Epigraph UNIV} \ f = \text{Epigraph UNIV} \ g$
  shows $f = g$
⟨proof⟩

lemma lsc-hull-ex-unique:
  fixes $f :: \text{\texttt{\textasciitilde}a::metric-space} \Rightarrow \text{ereal}$
  shows $\exists! g. \text{Epigraph UNIV} \ g = \text{closure} \ (\text{Epigraph UNIV} \ f)$
⟨proof⟩

lemma epigraph-lsc-hull:
  fixes $f :: \text{\texttt{\textasciitilde}a::metric-space} \Rightarrow \text{ereal}$
  shows $\text{Epigraph UNIV} \ (\text{lsc-hull} \ f) = \text{closure}(\text{Epigraph UNIV} \ f)$
⟨proof⟩

lemma lsc-hull-expl:
  ($g = \text{lsc-hull} \ f) \longleftrightarrow (\text{Epigraph UNIV} \ g = \text{closure}(\text{Epigraph UNIV} \ f))$
⟨proof⟩

lemma lsc-lsc-hull: $\text{lsc} \ (\text{lsc-hull} \ f)$
⟨proof⟩

lemma epigraph-subset-iff:
  fixes $f \ g :: \text{\texttt{\textasciitilde}a::metric-space} \Rightarrow \text{ereal}$
  shows $\text{Epigraph UNIV} \ f \leq \text{Epigraph UNIV} \ g \longleftrightarrow (\forall x. g \ x \leq f \ x)$
⟨proof⟩
lemma lsc-hull-le: \((\text{lsc-hull } f)\) \(x \leq f x\)
\langle proof \rangle

lemma lsc-hull-greatest:
fixes \(f, g\) :: ‘a::metric-space ⇒ ereal
assumes \(\text{lsc } g \ \forall x. \ g x \leq f x\)
shows \(\forall x. \ g x \leq (\text{lsc-hull } f) x\)
\langle proof \rangle

lemma lsc-hull-iff-greatest:
fixes \(f, g\) :: ‘a::metric-space ⇒ ereal
shows \((g = \text{lsc-hull } f) \longleftrightarrow \text{lsc } g \land (\forall x. \ g x \leq f x) \land (\forall h. \ \text{lsc } h \land (\forall x. \ h x \leq f x) \longrightarrow (\forall x. \ h x \leq g x))\)
\langle proof \rangle

lemma lsc-hull-mono:
fixes \(f, g\) :: ‘a::metric-space ⇒ ereal
assumes \(\forall x. \ g x \leq f x\)
shows \(\forall x. \ (\text{lsc-hull } g) x \leq (\text{lsc-hull } f) x\)
\langle proof \rangle

lemma lsc-hull-lsc:
\(\text{lsc } f \longleftrightarrow (f = \text{lsc-hull } f)\)
\langle proof \rangle

lemma lsc-hull-liminf-at:
fixes \(f\) :: ‘a::metric-space ⇒ ereal
shows \(\forall x. \ (\text{lsc-hull } f) x = \text{min } (f x) (\text{Liminf at } x) f\)
\langle proof \rangle

lemma lsc-hull-same-inf:
fixes \(f\) :: ‘a::metric-space ⇒ ereal
shows \((\text{INF } x. \ \text{lsc-hull } f x) = (\text{INF } x. \ f x)\)
\langle proof \rangle

1.4 Convex Functions

definition
convex-on :: ‘a::real-vector set ⇒ (‘a ⇒ ereal) ⇒ bool where
c\langle proof \rangle
lemma convex-on-ereal-mem:
assumes convex-on s f
assumes x:s y:s
assumes u≥0 v≥0 u+v=1
shows f (u *R x + v *R y) ≤ereal u * f x +ereal v * f y
⟨proof⟩

lemma convex-on-ereal-subset: convex-on t f ⇒ s ≤ t ⇒ convex-on s f
⟨proof⟩

lemma convex-on-ereal-univ: convex-on UNIV f ←→ (∀S. convex-on S f)
⟨proof⟩

lemmaereal-pos-sum-distrib-left:
fixes f :: 'a ⇒ eral
assumes r≥0 r ≠ ∞
shows r * sum f A = sum (λn. r * f n) A
⟨proof⟩

lemma convex-ereal-add:
fixes f g :: 'a::real-vector ⇒ eral
assumes convex-on s f convex-on s g
shows convex-on s (λx. f x + g x)
⟨proof⟩

lemma convex-ereal-cmul:
assumes 0 ≤ (c::ereal) convex-on s f
shows convex-on s (λx. c * f x)
⟨proof⟩

lemma convex-ereal-max:
fixes f g :: 'a::real-vector ⇒ eral
assumes convex-on s f convex-on s g
shows convex-on s (λx. max (f x) (g x))
⟨proof⟩

lemma convex-ereal-alt:
fixes C :: 'a::real-vector set
assumes convex C
shows convex-on C f =
(∀x ∈ C. ∀y ∈ C. ∀m :: real. m ≥ 0 ∧ m ≤ 1

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\[ \rightarrow f \left( m \ast_R x + (1 - m) \ast_R y \right) \leq (ereal m) \ast f x + (1 - (ereal m)) \ast f y \]

\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{convex-on-ereal-alt-mem}:
\textbf{fixes} \( C :: 'a::{real-vector} set \)
\textbf{assumes} \textit{convex C}
\textbf{assumes} \textit{convex-on C f}
\textbf{assumes} \( x : C \ y : C \)
\textbf{assumes} \( (m :: real) \geq 0 \ m \leq 1 \)
\textbf{shows} \( f \left( m \ast_R x + (1 - m) \ast_R y \right) \leq (ereal m) \ast f x + (1 - (ereal m)) \ast f y \)
\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{ereal-add-right-mono}: \( (a ::ereal) \leq b \implies a + c \leq b + c \)
\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{convex-on-ereal-sum-aux}:
\textbf{assumes} \( 1 - a > 0 \)
\textbf{shows} \( (1 - (ereal a)) \ast (ereal (c / (1 - a))) \ast b = (ereal c) \ast b \)
\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{convex-on-ereal-sum}:
\textbf{fixes} \( a :: 'a \Rightarrow real \)
\textbf{fixes} \( y :: 'a \Rightarrow 'b::{real-vector} \)
\textbf{fixes} \( f :: 'b \Rightarrow eroal \)
\textbf{assumes} \textit{finite s s \neq \{\}}
\textbf{assumes} \textit{convex-on C f}
\textbf{assumes} \textit{convex C}
\textbf{assumes} \( (\text{SUM} \ i : s. \ a i) = 1 \)
\textbf{assumes} \( \forall i. \ i \in s \implies a i \geq 0 \)
\textbf{assumes} \( \forall i. \ i \in s \implies y i \in C \)
\textbf{shows} \( f \left( \text{SUM} \ i : s. \ a i \ast_R y i \right) \leq (\text{SUM} \ i : s. \ eroal \ a i) \ast f (y i) \)
\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{sum-2}: \( \text{sum} \ u \ \{1::nat..2\} = (u \ 1)+(u \ 2) \)
\( \langle \text{proof} \rangle \)

\textbf{lemma} \textit{convex-on-ereal-iff}:
\textbf{assumes} \textit{convex s}
\textbf{shows} \( \textit{convex-on s f} \iff (\forall k u x. \ (\forall i \in \{1..k\} ::nat). \ 0 \leq u i \land x i : s) \land \text{sum} \ u \ \{1..k\} = 1 \implies f \left( \text{sum} \ (\lambda i. \ u i \ast_R x i) \ \{1..k\} \right) \leq \text{sum} \ (\lambda i. \ (ereal \ u i)) \ast f (x i) \ \{1..k\} \)
\( \langle \text{proof} \rangle \)
**lemma** convex-Epigraph:
- **assumes** convex $S$
- **shows** convex $(\text{Epigraph } S f) \iff \text{convex-on } S f$
  ⟨proof⟩

**lemma** convex-EpigraphI:
- $\text{convex-on } s f \implies \text{convex } s \implies \text{convex } (\text{Epigraph } s f)$
  ⟨proof⟩

**definition**
concave-on :: 'a::real-vector set ⇒ ('a ⇒ ereal) ⇒ bool where concave-on $S f \leftarrow\rightleftharpoons\text{convex-on } S (\lambda x. - f x)$

**definition**
finite-on :: 'a::real-vector set ⇒ ('a ⇒ ereal) ⇒ bool where finite-on $S f \leftarrow\rightleftharpoons\forall x \in S. (f x \neq \infty \land f x \neq -\infty)$

**definition**
affine-on :: 'a::real-vector set ⇒ ('a ⇒ ereal) ⇒ bool where affine-on $S f \leftarrow\rightleftharpoons\text{convex-on } S f \land \text{concave-on } S f \land \text{finite-on } S f$

**definition**
domain $(f:: \Rightarrow \text{ereal}) = \{x. f x < \infty\}$

**lemma** domain-Epigraph-aux:
- **assumes** $x \neq \infty$
- **shows** $\exists r. \ x \leq \text{ereal } r$
  ⟨proof⟩

**lemma** domain-Epigraph:
- $\text{domain } f = \{x. \exists y. (x,y) \in \text{Epigraph } \text{UNIV } f\}$
  ⟨proof⟩

**lemma** domain-Epigraph-fst:
- $\text{domain } f = \text{fst } ^{-1} (\text{Epigraph } \text{UNIV } f)$
  ⟨proof⟩

**lemma** convex-on-domain:
- $\text{convex-on } (\text{domain } f) f \leftarrow\rightleftharpoons\text{convex-on } \text{UNIV } f$
  ⟨proof⟩
lemma convex-on-domain2:
convex-on (domain f) f \iff (\forall S. \text{convex-on } S \ f)
⟨proof⟩

lemma convex-domain:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
shows convex (domain f)
⟨proof⟩

lemma infinite-convex-domain-iff:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes \forall x. (f x = \infty | f x = -\infty)
shows convex-on UNIV f \iff convex (domain f)
⟨proof⟩

lemma convex-PInfy-outside:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f convex S
shows convex-on UNIV (\lambda x. if x:S then (f x) else \infty)
⟨proof⟩

definition proper-on :: 'a::real-vector set \Rightarrow ('a \Rightarrow ereal) \Rightarrow bool where
proper-on S f \iff ((\forall x \in S. f x \neq -\infty) \land (\exists x \in S. f x \neq \infty))

definition proper :: ('a::real-vector \Rightarrow ereal) \Rightarrow bool where
proper f \iff proper-on UNIV f

lemma proper-iff:
proper f \iff ((\forall x. f x \neq -\infty) \land (\exists x. f x \neq \infty))
⟨proof⟩

lemma improper-iff:
\neg(proper f) \iff ((\exists x. f x = -\infty) \land (\forall x. f x = \infty))
⟨proof⟩

lemma ereal-MInf-plus[simp]: -\infty + x = (if x = \infty then \infty else -\infty::ereal)
⟨proof⟩

lemma convex-improper:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
assumes \( \sim (\text{proper } f) \)
shows $\forall x \in \text{rel-interior}(\text{domain } f). f x = -\infty$
⟨proof⟩

lemma convex-improper2:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
assumes \( \sim (\text{proper } f) \)
shows $f x = \infty | f x = -\infty | x : \text{rel-frontier } (\text{domain } f)$
⟨proof⟩

lemma convex-lsc-improper:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
assumes \( \sim (\text{proper } f) \)
assumes \( lsc \ f \)
shows $f x = \infty | f x = -\infty$
⟨proof⟩

lemma convex-lsc-hull:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
shows convex-on UNIV \((\text{lsc-hull } f)\)$
⟨proof⟩

lemma improper-lsc-hull:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes \( \sim (\text{proper } f) \)
shows \( \sim (\text{proper } (\text{lsc-hull } f)) \)$
⟨proof⟩

lemma lsc-hull-convex-improper:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
assumes \( \sim (\text{proper } f) \)
shows $\forall x \in \text{rel-interior}(\text{domain } f). (\text{lsc-hull } f) x = f x$
⟨proof⟩

lemma convex-with-rel-open-domain:
fixes $f : 'a::euclidean-space ⇒ ereal$
assumes convex-on UNIV $f$
assumes rel-open (domain f)
shows (\(\forall x. f x > -\infty\)) \lor (\(\forall x. (f x = \infty \lor f x = -\infty)\))
(proof)

lemma convex-with-UNIV-domain:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
assumes domain f = UNIV
shows (\(\forall x. f x > -\infty\)) \lor (\(\forall x. f x = -\infty\))
(proof)

lemma rel-interior-Epigraph:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
shows \((x,z) : rel-interior (Epigraph UNIV f) \iff (x : rel-interior (domain f) \land f x < ereal z)\)
(proof)

lemma rel-interior-EpigraphI:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
shows rel-interior (Epigraph UNIV f) = \{ (x,z) | x, z : rel-interior (domain f) \land f x < ereal z \}
(proof)

lemma convex-less-ri-domain:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
assumes \(\exists x. f x < a\)
shows \(\exists x \in rel-interior (domain f). f x < a\)
(proof)

lemma rel-interior-eq-between:
fixes S T :: ('m::euclidean-space) set
assumes convex S convex T
shows (rel-interior S = rel-interior T) \iff (rel-interior S \leq T \land T \leq closure S)
(proof)

lemma convex-less-in-riS:
fixes f :: 'a::euclidean-space \Rightarrow ereal
assumes convex-on UNIV f
assumes convex S rel-interior S ≤ domain f
assumes ∃x∈closure S. f x < a
shows ∃x∈rel-interior S. f x < a

⟨proof⟩

lemma convex-less-inS:
fixes f :: 'a::euclidean-space ⇒ ereal
assumes convex-on UNIV f
assumes convex S S ≤ domain f
assumes ∃x∈closure S. f x < a
shows ∃x∈S. f x < a
⟨proof⟩

lemma convex-finite-geq-on-closure:
fixes f :: 'a::euclidean-space ⇒ ereal
assumes convex-on UNIV f
assumes convex S finite-on S f
assumes ∀x∈S. f x ≥ a
shows ∀x∈closure S. f x ≥ a
⟨proof⟩

lemma lsc-hull-of-convex-determined:
fixes f g :: 'a::euclidean-space ⇒ ereal
assumes convex-on UNIV f convex-on UNIV g
assumes rel-interior (domain f) = rel-interior (domain g)
assumes ∀x∈rel-interior (domain f). f x = g x
shows lsc-hull f = lsc-hull g
⟨proof⟩

lemma domain-lsc-hull-between:
fixes f :: 'a::euclidean-space ⇒ ereal
shows domain f ≤ domain (lsc-hull f)
    ∧ domain (lsc-hull f) ≤ closure (domain f)
⟨proof⟩

lemma domain-vs-domain-lsc-hull:
fixes f :: 'a::euclidean-space ⇒ ereal
assumes convex-on UNIV f
shows rel-interior(domain (lsc-hull f)) = rel-interior(domain f)
    ∧ closure(domain (lsc-hull f)) = closure(domain f)
    ∧ aff-dim(domain (lsc-hull f)) = aff-dim(domain f)
⟨proof⟩
lemma vertical-line-affine:
  fixes x :: 'a::euclidean-space
  shows affine { (x,m::real)| m: UNIV } 
  ⟨proof⟩

lemma lsc-hull-of-convex-agrees-onRI:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  shows ∀ x ∈ rel-interior (domain f). (f x = (lsc-hull f) x) 
  ⟨proof⟩

lemma lsc-hull-of-convex-agrees-outside:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  shows ∀ x, x /∈ closure (domain f) → (f x = (lsc-hull f) x) 
  ⟨proof⟩

lemma lsc-hull-of-convex-agrees:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  shows ∀ x. (f x = (lsc-hull f) x) | x : rel-frontier (domain f) 
  ⟨proof⟩

lemma lsc-hull-of-proper-convex-proper:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f proper f
  shows proper (lsc-hull f) 
  ⟨proof⟩

lemma lsc-hull-of-proper-convex:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f proper f
  shows lsc (lsc-hull f) ∧ proper (lsc-hull f) ∧ convex-on UNIV (lsc-hull f) ∧ 
  (∀ x. (f x = (lsc-hull f) x) | x : rel-frontier (domain f)) 
  ⟨proof⟩

lemma affine-no-rel-frontier:
  fixes S :: ('n::euclidean-space) set
  assumes affine S
  shows rel-frontier S = {} 
  ⟨proof⟩
lemma convex-with-affine-domain-is-lsc:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  assumes affine (domain f)
  shows lsc f
⟨proof⟩

lemma convex-finite-is-lsc:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  assumes finite-on UNIV f
  shows lsc f
⟨proof⟩

lemma always-eventually-within:
  (∀ x∈S. P x) ⟷ eventually P (at x within S)
⟨proof⟩

lemma ereal-divide-pos:
  assumes (a::ereal)>0 b>0
  shows a/(ereal b)>0
⟨proof⟩

lemma real-interval-limpt:
  assumes a<b
  shows (b::real) islimpt {a..<b}
⟨proof⟩

lemma lsc-hull-of-convex-aux:
  Limsup (at 1 within {0..<1}) (λm. ereal ((1−m)*a+m*b)) ≤ ereal b
⟨proof⟩

lemma lsc-hull-of-convex:
  fixes f :: 'a::euclidean-space ⇒ ereal
  assumes convex-on UNIV f
  assumes x : rel-interior (domain f)
  shows ((λm. f((1−m)*R x + m *R y)) ⟷ (lsc-hull f) y) (at 1 within {0..<1})
  ⟨is (?,y ⟷ - y) ·
⟨proof⟩

end