# The Localization of a Commutative Ring

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**Abstract**

We formalize the localization [1, II, §4] of a commutative ring $R$ with respect to a multiplicative subset (i.e. a submonoid of $R$ seen as a multiplicative monoid).
This localization is itself a commutative ring and we build the natural homomorphism of rings from $R$ to its localization.

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**theory** *Localization*
**imports** *Main HOL−Algebra.Group HOL−Algebra.Ring HOL−Algebra.AbelCoset*
**begin**

Contents:

- We define the localization of a commutative ring R with respect to a multiplicative subset, i.e. with respect to a submonoid of R (seen as a multiplicative monoid), cf. [*rec-rng-of-frac*].
- We prove that this localization is a commutative ring (cf. [*crng-rng-of-frac*]) equipped with a homomorphism of rings from R (cf. [*rng-to-rng-of-frac-is-ring-hom*]).

# 1 The Localization of a Commutative Ring

## 1.1 Localization

**locale** *submonoid* = *monoid M* **for** *M* (**structure**) +
**fixes** *S*
**assumes** *subset* : $S \subseteq$ *carrier M*

**and** *m-closed* [*intro*, *simp*] : $[\![x \in S; y \in S]\!] \Longrightarrow x \otimes y \in S$
**and** *one-closed* [*simp*] : $\mathbf{1} \in S$

**lemma** (**in** *submonoid*) *is-submonoid*: *submonoid M S*
⟨*proof*⟩

**locale** *mult-submonoid-of-rng* = *ring R* + *submonoid R S* **for** *R* **and** *S*

**locale** *mult-submonoid-of-crng* = *cring R* + *mult-submonoid-of-rng R S* **for** *R* **and** *S*

**locale** *eq-obj-rng-of-frac* = *cring R* + *mult-submonoid-of-crng R S* **for** *R* (**structure**) **and** *S* +
**fixes** *rel*
**defines** $rel \equiv (\!|carrier = carrier\ R \times S,\ eq = \lambda(r,s)\ (r',s').\ \exists\, t{\in}S.\ t \otimes ((s' \otimes r) \ominus (s \otimes r')) = \mathbf{0}|\!)$

**lemma** (**in** *abelian-group*) *minus-to-eq* :
**assumes** *abelian-group G* **and** $x \in carrier\ G$ **and** $y \in carrier\ G$ **and** $x \ominus y = \mathbf{0}$
**shows** $x = y$
⟨*proof*⟩

**lemma** (**in** *eq-obj-rng-of-frac*) *equiv-obj-rng-of-frac*:
**shows** *equivalence rel*
⟨*proof*⟩

**definition** *eq-class-of-rng-of-frac*:: - ⇒ *'a* ⇒ *'b* ⇒ *-set* (**infix** ‹|ı› *10*)
**where** $r\ |_{rel}\ s \equiv \{(r', s') \in carrier\ rel.\ (r, s) .=_{rel} (r', s')\}$

**lemma** *class-of-to-rel*:
**shows** $class\text{-}of_{rel}\ (r, s) = (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** (**in** *eq-obj-rng-of-frac*) *zero-in-mult-submonoid*:
**assumes** $\mathbf{0} \in S$ **and** $(r, s) \in carrier\ rel$ **and** $(r', s') \in carrier\ rel$
**shows** $(r\ |_{rel}\ s) = (r'\ |_{rel}\ s')$
⟨*proof*⟩

**definition** *set-eq-class-of-rng-of-frac*:: - ⇒ *-set* (‹*set'-class'-of*ı›)
**where** $set\text{-}class\text{-}of_{rel} \equiv \{(r\ |_{rel}\ s)|\ r\ s.\ (r, s) \in carrier\ rel\}$

**lemma** *elem-eq-class*:
**assumes** *equivalence S* **and** $x \in carrier\ S$ **and** $y \in carrier\ S$ **and** $x .=_S y$
**shows** $class\text{-}of_S\ x = class\text{-}of_S\ y$
⟨*proof*⟩

**lemma** (**in** *abelian-group*) *four-elem-comm*:
**assumes** $a \in carrier\ G$ **and** $b \in carrier\ G$ **and** $c \in carrier\ G$ **and** $d \in carrier$

$G$
**shows** $a \ominus c \oplus b \ominus d = a \oplus b \ominus c \ominus d$
⟨*proof*⟩

**lemma** (**in** *abelian-monoid*) *right-add-eq*:
**assumes** $a = b$
**shows** $c \oplus a = c \oplus b$
⟨*proof*⟩

**lemma** (**in** *abelian-monoid*) *right-minus-eq*:
**assumes** $a = b$
**shows** $c \ominus a = c \ominus b$
⟨*proof*⟩

**lemma** (**in** *abelian-group*) *inv-add*:
**assumes** $a \in$ *carrier* $G$ **and** $b \in$ *carrier* $G$
**shows** $\ominus\ (a \oplus b) = \ominus\ a \ominus b$
⟨*proof*⟩

**lemma** (**in** *abelian-group*) *right-inv-add*:
**assumes** $a \in$ *carrier* $G$ **and** $b \in$ *carrier* $G$ **and** $c \in$ *carrier* $G$
**shows** $c \ominus a \ominus b = c \ominus (a \oplus b)$
⟨*proof*⟩

**context** *eq-obj-rng-of-frac*
**begin**

**definition** *carrier-rng-of-frac*:: - *partial-object*
**where** *carrier-rng-of-frac* ≡ (|*carrier* = *set-class-of*$_{rel}$|)

**definition** *mult-rng-of-frac*:: [-*set*, -*set*] ⇒ -*set*
**where** *mult-rng-of-frac* $X$ $Y$ ≡
*let* $x' = (SOME\ x.\ x \in X)$ *in*
*let* $y' = (SOME\ y.\ y \in Y)$ *in*
$(fst\ x' \otimes fst\ y')|_{rel}\ (snd\ x' \otimes snd\ y')$

**definition** *rec-monoid-rng-of-frac*:: - *monoid*
**where** *rec-monoid-rng-of-frac* ≡ (|*carrier* = *set-class-of*$_{rel}$, *mult* = *mult-rng-of-frac*, *one* = $(\mathbf{1}|_{rel}\ \mathbf{1})$|)

**lemma** *member-class-to-carrier*:
**assumes** $x \in (r\ |_{rel}\ s)$ **and** $y \in (r'\ |_{rel}\ s')$
**shows** $(fst\ x \otimes fst\ y,\ snd\ x \otimes snd\ y) \in$ *carrier rel*
⟨*proof*⟩

**lemma** *member-class-to-member-class*:
**assumes** $x \in (r\ |_{rel}\ s)$ **and** $y \in (r'\ |_{rel}\ s')$
**shows** $(fst\ x \otimes fst\ y\ |_{rel}\ snd\ x \otimes snd\ y) \in$ *set-class-of*$_{rel}$
⟨*proof*⟩

**lemma** *closed-mult-rng-of-frac* :
**assumes** $(r,\ s) \in$ *carrier rel* **and** $(t,\ u) \in$ *carrier rel*
**shows** $(r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (t\ |_{rel}\ u) \in set\text{-}class\text{-}of_{rel}$
⟨*proof*⟩

**lemma** *non-empty-class*:
**assumes** $(r,\ s) \in$ *carrier rel*
**shows** $(r\ |_{rel}\ s) \neq \{\}$
⟨*proof*⟩

**lemma** *mult-rng-of-frac-fundamental-lemma*:
**assumes** $(r,\ s) \in$ *carrier rel* **and** $(r',\ s') \in$ *carrier rel*
**shows** $(r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') = (r \otimes r'\ |_{rel}\ s \otimes s')$
⟨*proof*⟩

**lemma** *member-class-to-assoc*:
**assumes** $x \in (r\ |_{rel}\ s)$ **and** $y \in (t\ |_{rel}\ u)$ **and** $z \in (v\ |_{rel}\ w)$
**shows** $((fst\ x \otimes fst\ y) \otimes fst\ z\ |_{rel}\ (snd\ x \otimes snd\ y) \otimes snd\ z) = (fst\ x \otimes (fst\ y \otimes fst\ z)\ |_{rel}\ snd\ x \otimes (snd\ y \otimes snd\ z))$
⟨*proof*⟩

**lemma** *assoc-mult-rng-of-frac*:
**assumes** $(r,\ s) \in$ *carrier rel* **and** $(t,\ u) \in$ *carrier rel* **and** $(v,\ w) \in$ *carrier rel*
**shows** $((r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (t\ |_{rel}\ u)) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (v\ |_{rel}\ w) =$
$(r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} ((t\ |_{rel}\ u) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (v\ |_{rel}\ w))$
⟨*proof*⟩

**lemma** *left-unit-mult-rng-of-frac*:
**assumes** $(r,\ s) \in$ *carrier rel*
**shows** $\mathbf{1}_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s) = (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** *right-unit-mult-rng-of-frac*:
**assumes** $(r,\ s) \in$ *carrier rel*
**shows** $(r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} \mathbf{1}_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} = (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** *monoid-rng-of-frac*:
**shows** *monoid* (*rec-monoid-rng-of-frac*)
⟨*proof*⟩

**lemma** *comm-mult-rng-of-frac*:
**assumes** $(r,\ s) \in$ *carrier rel* **and** $(r',\ s') \in$ *carrier rel*
**shows** $(r\ |_{rel}\ s) \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') = (r'\ |_{rel}\ s') \otimes_{rec\text{-}monoid\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** *comm-monoid-rng-of-frac*:
  **shows** *comm-monoid* (*rec-monoid-rng-of-frac*)
  ⟨*proof*⟩

**definition** *add-rng-of-frac*:: [-*set*, -*set*] ⇒ -*set*
  **where** *add-rng-of-frac X Y* ≡
*let* $x' = (SOME\ x.\ x \in X)$ *in*
*let* $y' = (SOME\ y.\ y \in Y)$ *in*
$(snd\ y' \otimes fst\ x' \oplus snd\ x' \otimes fst\ y')\ |_{rel}\ (snd\ x' \otimes snd\ y')$

**definition** *rec-rng-of-frac*:: - *ring*
  **where** *rec-rng-of-frac* ≡
$(\!|$ $carrier = set\text{-}class\text{-}of_{rel}$, $mult = mult\text{-}rng\text{-}of\text{-}frac$, $one = (\mathbf{1}|_{rel}\ \mathbf{1})$, $zero = (\mathbf{0}\ |_{rel}\ \mathbf{1})$, $add = add\text{-}rng\text{-}of\text{-}frac$ $|\!)$

**lemma** *add-rng-of-frac-fundamental-lemma*:
  **assumes** $(r, s) \in carrier\ rel$ **and** $(r', s') \in carrier\ rel$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') = (s' \otimes r \oplus s \otimes r'\ |_{rel}\ s \otimes s')$
⟨*proof*⟩

**lemma** *closed-add-rng-of-frac*:
  **assumes** $(r, s) \in carrier\ rel$ **and** $(r', s') \in carrier\ rel$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') \in set\text{-}class\text{-}of_{rel}$
⟨*proof*⟩

**lemma** *closed-rel-add*:
  **assumes** $(r, s) \in carrier\ rel$ **and** $(r', s') \in carrier\ rel$
  **shows** $(s' \otimes r \oplus s \otimes r', s \otimes s') \in carrier\ rel$
⟨*proof*⟩

**lemma** *assoc-add-rng-of-frac*:
  **assumes** $(r, s) \in carrier\ rel$ **and** $(r', s') \in carrier\ rel$ **and** $(r'', s'') \in carrier\ rel$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r''|_{rel}\ s'') =$
  $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} ((r'\ |_{rel}\ s') \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r''\ |_{rel}\ s''))$
⟨*proof*⟩

**lemma** *add-rng-of-frac-zero*:
  **shows** $(\mathbf{0}\ |_{rel}\ \mathbf{1}) \in set\text{-}class\text{-}of_{rel}$
  ⟨*proof*⟩

**lemma** *l-unit-add-rng-of-frac*:
  **assumes** $(r, s) \in carrier\ rel$
  **shows** $\mathbf{0}_{rec\text{-}rng\text{-}of\text{-}frac} \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s) = (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** *r-unit-add-rng-of-frac*:
  **assumes** $(r, s) \in carrier\ rel$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} \mathbf{0}_{rec\text{-}rng\text{-}of\text{-}frac} = (r\ |_{rel}\ s)$

⟨*proof*⟩

**lemma** *comm-add-rng-of-frac*:
  **assumes** $(r, s) \in \mathit{carrier\ rel}$ **and** $(r', s') \in \mathit{carrier\ rel}$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') = (r'\ |_{rel}\ s') \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s)$
⟨*proof*⟩

**lemma** *class-of-zero-rng-of-frac*:
  **assumes** $s \in S$
  **shows** $(\mathbf{0}\ |_{rel}\ s) = \mathbf{0}_{rec\text{-}rng\text{-}of\text{-}frac}$
⟨*proof*⟩

**lemma** *r-inv-add-rng-of-frac*:
  **assumes** $(r, s) \in \mathit{carrier\ rel}$
  **shows** $(r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (\ominus\ r\ |_{rel}\ s) = \mathbf{0}_{rec\text{-}rng\text{-}of\text{-}frac}$
⟨*proof*⟩

**lemma** *l-inv-add-rng-of-frac*:
  **assumes** $(r, s) \in \mathit{carrier\ rel}$
  **shows** $(\ominus\ r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s) = \mathbf{0}_{rec\text{-}rng\text{-}of\text{-}frac}$
⟨*proof*⟩

**lemma** *abelian-group-rng-of-frac*:
  **shows** *abelian-group* (*rec-rng-of-frac*)
⟨*proof*⟩

**lemma** *r-distr-rng-of-frac*:
  **assumes** $(r, s) \in \mathit{carrier\ rel}$ **and** $(r', s') \in \mathit{carrier\ rel}$ **and** $(r'', s'') \in \mathit{carrier\ rel}$
  **shows** $((r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s')) \otimes_{rec\text{-}rng\text{-}of\text{-}frac} (r''\ |_{rel}\ s'') =$
    $(r\ |_{rel}\ s) \otimes_{rec\text{-}rng\text{-}of\text{-}frac} (r''\ |_{rel}\ s'') \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s') \otimes_{rec\text{-}rng\text{-}of\text{-}frac}$
$(r''\ |_{rel}\ s'')$
⟨*proof*⟩

**lemma** *l-distr-rng-of-frac*:
  **assumes** $(r, s) \in \mathit{carrier\ rel}$ **and** $(r', s') \in \mathit{carrier\ rel}$ **and** $(r'', s'') \in \mathit{carrier\ rel}$
  **shows** $(r''\ |_{rel}\ s'') \otimes_{rec\text{-}rng\text{-}of\text{-}frac} ((r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r'\ |_{rel}\ s')) =$
    $(r''\ |_{rel}\ s'') \otimes_{rec\text{-}rng\text{-}of\text{-}frac} (r\ |_{rel}\ s) \oplus_{rec\text{-}rng\text{-}of\text{-}frac} (r''\ |_{rel}\ s'') \otimes_{rec\text{-}rng\text{-}of\text{-}frac}$
$(r'\ |_{rel}\ s')$
⟨*proof*⟩

**lemma** *rng-rng-of-frac*:
  **shows** *ring* (*rec-rng-of-frac*)
⟨*proof*⟩

**lemma** *crng-rng-of-frac*:
  **shows** *cring* (*rec-rng-of-frac*)
  ⟨*proof*⟩

**lemma** *simp-in-frac*:

**assumes** $(r, s) \in$ *carrier rel* **and** $s' \in S$
**shows** $(r \mid_{rel} s) = (s' \otimes r \mid_{rel} s' \otimes s)$
$\langle proof \rangle$

### 1.2 The Natural Homomorphism from a Ring to Its Localization

**definition** *rng-to-rng-of-frac* :: $'a \Rightarrow ('a \times 'a)$ *set* **where**
*rng-to-rng-of-frac* $r \equiv (r \mid_{rel} \mathbf{1})$

**lemma** *rng-to-rng-of-frac-is-ring-hom* :
**shows** *rng-to-rng-of-frac* $\in$ *ring-hom R rec-rng-of-frac*
$\langle proof \rangle$

**lemma** *Im-rng-to-rng-of-frac-unit*:
**assumes** $x \in$ *rng-to-rng-of-frac* ‘ $S$
**shows** $x \in$ *Units rec-rng-of-frac*
$\langle proof \rangle$

**lemma** *eq-class-to-rel*:
**assumes** $(r, s) \in$ *carrier R* $\times$ $S$ **and** $(r', s') \in$ *carrier R* $\times$ $S$ **and** $(r \mid_{rel} s) = (r' \mid_{rel} s')$
**shows** $(r, s) \mathbin{.=_{rel}} (r', s')$
$\langle proof \rangle$

**lemma** *rng-to-rng-of-frac-without-zero-div-is-inj*:
**assumes** $\mathbf{0} \notin S$ **and** $\forall\, a \in$ *carrier R*.$\forall\, b \in$ *carrier R*. $a \otimes b = \mathbf{0} \longrightarrow a = \mathbf{0} \vee b = \mathbf{0}$
**shows** *a-kernel R rec-rng-of-frac rng-to-rng-of-frac* $= \{\mathbf{0}\}$
$\langle proof \rangle$

**end**

**end**

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## References

[1] S. Lang. *Algebra*. Springer, revised third edition edition, 2002.