Local Lexing

Steven Obua

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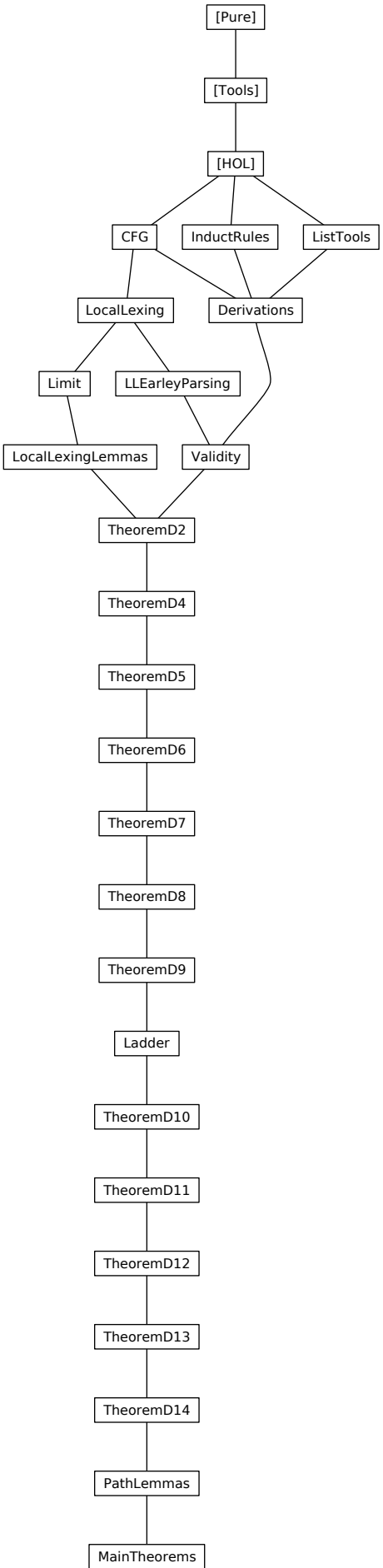
Abstract

This formalisation accompanies the paper Local Lexing[1], which introduces a novel parsing concept of the same name. The paper also gives a high-level algorithm for local lexing as an extension of Earley's algorithm. This formalisation proves the algorithm to be correct with respect to its local lexing semantics. As a special case, this formalisation thus also contains a proof of the correctness of Earley's algorithm. The paper contains a short outline of how this formalisation is organised.

Contents

[1] https://arxiv.org/abs/1702.03277


[Pure]
[Tools]
[HOL]
CFG
InductRules
ListTools
LocalLexing
Derivations
Limit
LLEarleyParsing
LocalLexingLemmas
Validity
TheoremD2
TheoremD4
TheoremD5
TheoremD6
TheoremD7
TheoremD8
TheoremD9
Ladder
TheoremD10
TheoremD11
TheoremD12
TheoremD13
TheoremD14
PathLemmas
MainTheorems

theory *CFG*
imports *Main*
begin

typedecl *symbol*

type-synonym *rule* = *symbol* × *symbol list*

type-synonym *sentence* = *symbol list*

locale *CFG* =
  fixes $\mathfrak{N}$ :: *symbol set*
  fixes $\mathfrak{T}$ :: *symbol set*
  fixes $\mathfrak{R}$ :: *rule set*
  fixes $\mathfrak{S}$ :: *symbol*
  assumes *disjunct-symbols*: $\mathfrak{N} \cap \mathfrak{T} = \{\}$
  assumes *startsymbol-dom*: $\mathfrak{S} \in \mathfrak{N}$
  assumes *validRules*: $\forall\ (N, \alpha) \in \mathfrak{R}.\ N \in \mathfrak{N} \wedge (\forall\ s \in set\ \alpha.\ s \in \mathfrak{N} \cup \mathfrak{T})$
begin

definition *is-terminal* :: *symbol* ⇒ *bool*
where
  is-terminal s = $(s \in \mathfrak{T})$

definition *is-nonterminal* :: *symbol* ⇒ *bool*
where
  is-nonterminal s = $(s \in \mathfrak{N})$

lemma *is-nonterminal-startsymbol*:*is-nonterminal* $\mathfrak{S}$
  ⟨*proof*⟩

definition *is-symbol* :: *symbol* ⇒ *bool*
where
  is-symbol s = (*is-terminal s* ∨ *is-nonterminal s*)

definition *is-sentence* :: *sentence* ⇒ *bool*
where
  is-sentence s = *list-all is-symbol s*

definition *is-word* :: *sentence* ⇒ *bool*
where
  is-word s = *list-all is-terminal s*

definition *derives1* :: *sentence* ⇒ *sentence* ⇒ *bool*
where
  derives1 u v =
    (∃ *x y N* α.
        u = *x* @ [*N*] @ *y*
      ∧ *v* = *x* @ α @ *y*

$\wedge$ *is-sentence x*
$\wedge$ *is-sentence y*
$\wedge$ $(N, \alpha) \in \mathfrak{R}$)

definition *derivations1* :: (*sentence* $\times$ *sentence*) *set*
where
derivations1 = { (*u*,*v*) | *u v*. *derives1 u v* }

definition *derivations* :: (*sentence* $\times$ *sentence*) *set*
where
derivations = *derivations1*$\widehat{}*$

definition *derives* :: *sentence* $\Rightarrow$ *sentence* $\Rightarrow$ *bool*
where
derives u v = ((*u*, *v*) $\in$ *derivations*)

definition *is-derivation* :: *sentence* $\Rightarrow$ *bool*
where
is-derivation u = *derives* [$\mathfrak{S}$] *u*

definition $\mathcal{L}$ :: *sentence set*
where
$\mathcal{L}$ = { *v* | *v*. *is-word v* $\wedge$ *is-derivation v*}

definition $\mathcal{L}_P$ :: *sentence set*
where
$\mathcal{L}_P$ = { *u* | *u v*. *is-word u* $\wedge$ *is-derivation* (*u*@*v*) }

end

end
theory *LocalLexing*
imports *CFG*
begin

typedecl *character*

type-synonym *lexer* = *character list* $\Rightarrow$ *nat* $\Rightarrow$ *nat set*

type-synonym *token* = *symbol* $\times$ *character list*

type-synonym *tokens* = *token list*

definition *terminal-of-token* :: *token* $\Rightarrow$ *symbol*
where
terminal-of-token t = *fst t*

definition *terminals* :: *tokens* $\Rightarrow$ *sentence*
where

```
  terminals ts = map terminal-of-token ts

definition chars-of-token :: token ⇒ character list
where
  chars-of-token t = snd t

fun chars :: tokens ⇒ character list
where
  chars [] = []
| chars (t#ts) = (chars-of-token t) @ (chars ts)

fun charslength :: tokens ⇒ nat
where
  charslength cs = length (chars cs)

definition is-lexer :: lexer ⇒ bool
where
  is-lexer lexer =
    (∀ D p l. (p ≤ length D ∧ l ∈ lexer D p ⟶ p + l ≤ length D) ∧
          (p > length D ⟶ lexer D p = {}))

type-synonym selector = token set ⇒ token set ⇒ token set

definition is-selector :: selector ⇒ bool
where
  is-selector sel = (∀ A B. A ⊆ B ⟶ (A ⊆ sel A B ∧ sel A B ⊆ B))

fun by-length :: nat ⇒ tokens set ⇒ tokens set
where
  by-length l tss = { ts . ts ∈ tss ∧ length (chars ts) = l }

fun funpower :: ('a ⇒ 'a) ⇒ nat ⇒ ('a ⇒ 'a)
where
  funpower f 0 x = x
| funpower f (Suc n) x = f (funpower f n x)

definition natUnion :: (nat ⇒ 'a set) ⇒ 'a set
where
  natUnion f = ⋃ { f n | n. True }

definition limit  :: ('a set ⇒ 'a set) ⇒ 'a set ⇒ 'a set
where
  limit f x = natUnion (λ n. funpower f n x)

locale LocalLexing = CFG +
  fixes Lex :: symbol ⇒ lexer
  fixes Sel :: selector
  assumes Lex-is-lexer: ∀ t ∈ 𝔗. is-lexer (Lex t)
  assumes Sel-is-selector: is-selector Sel
```

```
  fixes Doc :: character list
begin

definition admissible :: tokens ⇒ bool
where
  admissible ts = (terminals ts ∈ ℒ_P)

definition Append :: token set ⇒ nat ⇒ tokens set ⇒ tokens set
where
  Append Z k P = P ∪
    { p @ [t] | p t. p ∈ by-length k P ∧ t ∈ Z ∧ admissible (p @ [t])}

definition 𝒳 :: nat ⇒ token set
where
  𝒳 k = {(t, ω) | t l ω. t ∈ 𝔗 ∧ l ∈ Lex t Doc k ∧ ω = take l (drop k Doc)}

definition 𝒲 :: tokens set ⇒ nat ⇒ token set
where
  𝒲 P k =  { u. u ∈ 𝒳 k ∧ (∃ p ∈ by-length k P. admissible (p@[u])) }

definition 𝒴 :: token set ⇒ tokens set ⇒ nat ⇒ token set
where
  𝒴 T P k = Sel T (𝒲 P k)

fun 𝒫 :: nat ⇒ nat ⇒ tokens set
and 𝒬 :: nat ⇒ tokens set
and 𝒵 :: nat ⇒ nat ⇒ token set
where
  𝒫 0 0 = {[]}
| 𝒫 k (Suc u) = limit (Append (𝒵 k (Suc u)) k) (𝒫 k u)
| 𝒫 (Suc k) 0 = 𝒬 k
| 𝒵 k 0 = {}
| 𝒵 k (Suc u) = 𝒴 (𝒵 k u) (𝒫 k u) k
| 𝒬 k = natUnion (𝒫 k)

definition 𝔓 :: tokens set
where
  𝔓 = 𝒬 (length Doc)

definition ll :: tokens set
where
  ll = { p . p ∈ 𝔓 ∧ charslength p = length Doc ∧ terminals p ∈ ℒ }

end

end
theory LLEarleyParsing
imports LocalLexing
begin
```

datatype *item* =
  Item
    (*item-rule*: *rule*)
    (*item-dot* : *nat*)
    (*item-origin* : *nat*)
    (*item-end* : *nat*)

type-synonym *items* = *item set*

definition *item-nonterminal* :: *item* ⇒ *symbol*
where
  item-nonterminal x = *fst* (*item-rule x*)

definition *item-rhs* :: *item* ⇒ *sentence*
where
  item-rhs x = *snd* (*item-rule x*)

definition *item-α* :: *item* ⇒ *sentence*
where
  item-α x = *take* (*item-dot x*) (*item-rhs x*)

definition *item-β* :: *item* ⇒ *sentence*
where
  item-β x = *drop* (*item-dot x*) (*item-rhs x*)

definition *init-item* :: *rule* ⇒ *nat* ⇒ *item*
where
  init-item r k = *Item r 0 k k*

definition *is-complete* :: *item* ⇒ *bool*
where
  is-complete x = (*item-dot x* ≥ *length* (*item-rhs x*))

definition *next-symbol* :: *item* ⇒ *symbol option*
where
  next-symbol x = (*if is-complete x then None else Some* ((*item-rhs x*) ! (*item-dot x*)))

definition *inc-item* :: *item* ⇒ *nat* ⇒ *item*
where
  inc-item x k = *Item* (*item-rule x*) (*item-dot x* + *1*) (*item-origin x*) *k*

definition *bin* :: *items* ⇒ *nat* ⇒ *items*
where
  bin I k = { *x* . *x* ∈ *I* ∧ *item-end x* = *k* }

context *LocalLexing* **begin**

definition *Init* :: *items*
where
Init = { *init-item r 0* | *r*. *r* ∈ ℜ ∧ *fst r* = 𝔖 }

definition *Predict* :: *nat* ⇒ *items* ⇒ *items*
where
Predict k I = *I* ∪
{ *init-item r k* | *r x*. *r* ∈ ℜ ∧ *x* ∈ *bin I k* ∧
next-symbol x = *Some*(*fst r*) }

definition *Complete* :: *nat* ⇒ *items* ⇒ *items*
where
Complete k I = *I* ∪ { *inc-item x k* | *x y*.
x ∈ *bin I* (*item-origin y*) ∧ *y* ∈ *bin I k* ∧ *is-complete y* ∧
next-symbol x = *Some* (*item-nonterminal y*) }

definition *TokensAt* :: *nat* ⇒ *items* ⇒ *token set*
where
TokensAt k I = { (*t*, *s*) | *t s x l*. *x* ∈ *bin I k* ∧
next-symbol x = *Some t* ∧ *is-terminal t* ∧
l ∈ *Lex t Doc k* ∧ *s* = *take l* (*drop k Doc*) }

definition *Tokens* :: *nat* ⇒ *token set* ⇒ *items* ⇒ *token set*
where
Tokens k T I = *Sel T* (*TokensAt k I*)

definition *Scan* :: *token set* ⇒ *nat* ⇒ *items* ⇒ *items*
where
Scan T k I = *I* ∪
{ *inc-item x* (*k* + *length c*) | *x t c*. *x* ∈ *bin I k* ∧ (*t*, *c*) ∈ *T* ∧
next-symbol x = *Some t* }

definition π :: *nat* ⇒ *token set* ⇒ *items* ⇒ *items*
where
π *k T I* =
limit (λ *I*. *Scan T k* (*Complete k* (*Predict k I*))) *I*

fun 𝒥 :: *nat* ⇒ *nat* ⇒ *items*
and ℐ :: *nat* ⇒ *items*
and 𝒯 :: *nat* ⇒ *nat* ⇒ *token set*
where
𝒥 *0 0* = π *0* {} *Init*
| 𝒥 *k* (*Suc u*) = π *k* (𝒯 *k* (*Suc u*)) (𝒥 *k u*)
| 𝒥 (*Suc k*) *0* = π (*Suc k*) {} (ℐ *k*)
| 𝒯 *k 0* = {}
| 𝒯 *k* (*Suc u*) = *Tokens k* (𝒯 *k u*) (𝒥 *k u*)
| ℐ *k* = *natUnion* (𝒥 *k*)

definition ℑ :: *items*

where
$\mathfrak{I} = \mathcal{I}\ (\mathit{length}\ \mathit{Doc})$

definition *is-finished* :: *item* ⇒ *bool* **where**
is-finished $x = (\mathit{item\text{-}nonterminal}\ x = \mathfrak{S} \wedge \mathit{item\text{-}origin}\ x = 0 \wedge \mathit{item\text{-}end}\ x = \mathit{length}\ \mathit{Doc} \wedge$
is-complete $x)$

definition *earley-recognised* :: *bool*
where
earley-recognised $= (\exists\ x \in \mathfrak{I}.\ \mathit{is\text{-}finished}\ x)$

end

end
theory *Limit*
imports *LocalLexing*
begin

definition *setmonotone* :: $('a\ \mathit{set} \Rightarrow 'a\ \mathit{set}) \Rightarrow \mathit{bool}$
where
setmonotone $f = (\forall\ X.\ X \subseteq f\ X)$

lemma *setmonotone-funpower*: *setmonotone* $f \Longrightarrow$ *setmonotone* $(\mathit{funpower}\ f\ n)$
⟨*proof*⟩

lemma *subset-setmonotone*: *setmonotone* $f \Longrightarrow X \subseteq f\ X$
⟨*proof*⟩

lemma *elem-setmonotone*: *setmonotone* $f \Longrightarrow x \in X \Longrightarrow x \in f\ X$
⟨*proof*⟩

lemma *elem-natUnion*: $(\forall\ n.\ x \in f\ n) \Longrightarrow x \in \mathit{natUnion}\ f$
⟨*proof*⟩

lemma *subset-natUnion*: $(\forall\ n.\ X \subseteq f\ n) \Longrightarrow X \subseteq \mathit{natUnion}\ f$
⟨*proof*⟩

lemma *setmonotone-limit*:
assumes *fmono*: *setmonotone* f
shows *setmonotone* $(\mathit{limit}\ f)$
⟨*proof*⟩

lemma[*simp*]: $\mathit{funpower}\ \mathit{id}\ n = \mathit{id}$
⟨*proof*⟩

lemma[*simp*]: $\mathit{limit}\ \mathit{id} = \mathit{id}$
⟨*proof*⟩

lemma *natUnion-decompose*[*consumes 1*, *case-names Decompose*]:
assumes *p*: $p \in natUnion\ S$
assumes *decompose*: $\bigwedge n\ p.\ p \in S\ n \Longrightarrow P\ p$
shows $P\ p$
⟨*proof*⟩

lemma *limit-induct*[*consumes 1*, *case-names Init Iterate*]:
assumes *p*: $(p :: 'a) \in limit\ f\ X$
assumes *init*: $\bigwedge p.\ p \in X \Longrightarrow P\ p$
assumes *iterate*: $\bigwedge p\ Y.\ (\bigwedge q\ .\ q \in Y \Longrightarrow P\ q) \Longrightarrow p \in f\ Y \Longrightarrow P\ p$
shows $P\ p$
⟨*proof*⟩

definition *chain* :: $(nat \Rightarrow 'a\ set) \Rightarrow bool$
where
$chain\ C = (\forall\ i.\ C\ i \subseteq C\ (i + 1))$

definition *continuous* :: $('a\ set \Rightarrow 'b\ set) \Rightarrow bool$
where
$continuous\ f = (\forall\ C.\ chain\ C \longrightarrow (chain\ (f\ o\ C) \wedge f\ (natUnion\ C) = natUnion\ (f\ o\ C)))$

lemma *continuous-apply*:
$continuous\ f \Longrightarrow chain\ C \Longrightarrow f\ (natUnion\ C) = natUnion\ (f\ o\ C)$
⟨*proof*⟩

lemma *continuous-imp-mono*:
assumes *continuous*: *continuous f*
shows *mono f*
⟨*proof*⟩

lemma *mono-maps-chain-to-chain*:
assumes *f*: *mono f*
assumes *C*: *chain C*
shows *chain* $(f\ o\ C)$
⟨*proof*⟩

lemma *natUnion-upperbound*:
$(\bigwedge n.\ f\ n \subseteq G) \Longrightarrow (natUnion\ f) \subseteq G$
⟨*proof*⟩

lemma *funpower-upperbound*:
$(\bigwedge I.\ I \subseteq G \Longrightarrow f\ I \subseteq G) \Longrightarrow I \subseteq G \Longrightarrow funpower\ f\ n\ I \subseteq G$
⟨*proof*⟩

lemma *limit-upperbound*:
$(\bigwedge I.\ I \subseteq G \Longrightarrow f\ I \subseteq G) \Longrightarrow I \subseteq G \Longrightarrow limit\ f\ I \subseteq G$
⟨*proof*⟩

lemma *elem-limit-simp*: $x \in \mathit{limit}\ f\ X = (\exists\ n.\ x \in \mathit{funpower}\ f\ n\ X)$
⟨*proof*⟩

definition *pointwise* :: $('a\ \mathit{set} \Rightarrow 'b\ \mathit{set}) \Rightarrow \mathit{bool}$ **where**
pointwise $f = (\forall\ X.\ f\ X = \bigcup\ \{\ f\ \{x\} \mid x.\ x \in X\})$

lemma *pointwise-simp*:
assumes f: *pointwise* f
shows $f\ X = \bigcup\ \{\ f\ \{x\} \mid x.\ x \in X\}$
⟨*proof*⟩

lemma *natUnion-elem*: $x \in f\ n \Longrightarrow x \in \mathit{natUnion}\ f$
⟨*proof*⟩

lemma *limit-elem*: $x \in \mathit{funpower}\ f\ n\ X \Longrightarrow x \in \mathit{limit}\ f\ X$
⟨*proof*⟩

lemma *limit-step-pointwise*:
assumes x: $x \in \mathit{limit}\ f\ X$
assumes f: *pointwise* f
assumes y: $y \in f\ \{x\}$
shows $y \in \mathit{limit}\ f\ X$
⟨*proof*⟩

definition *pointbase* :: $('a\ \mathit{set} \Rightarrow 'b\ \mathit{set}) \Rightarrow 'a\ \mathit{set} \Rightarrow 'b\ \mathit{set}$ **where**
pointbase $F\ I = \bigcup\ \{\ F\ X \mid X.\ \mathit{finite}\ X \wedge X \subseteq I\ \}$

definition *pointbased* :: $('a\ \mathit{set} \Rightarrow 'b\ \mathit{set}) \Rightarrow \mathit{bool}$ **where**
pointbased $f = (\exists\ F.\ f = \mathit{pointbase}\ F)$

lemma *pointwise-implies-pointbased*:
assumes *pointwise*: *pointwise* f
shows *pointbased* f
⟨*proof*⟩

lemma *pointbase-is-mono*:
mono (*pointbase* f)
⟨*proof*⟩

lemma *chain-implies-mono*: *chain* $C \Longrightarrow \mathit{mono}\ C$
⟨*proof*⟩

lemma *chain-cover-witness*: *finite* $X \Longrightarrow \mathit{chain}\ C \Longrightarrow X \subseteq \mathit{natUnion}\ C \Longrightarrow \exists\ n.$
$X \subseteq C\ n$
⟨*proof*⟩

lemma *pointbase-is-continuous*:
continuous (*pointbase* f)
⟨*proof*⟩

lemma *pointbased-implies-continuous*:
pointbased f $\Longrightarrow$ *continuous f*
⟨*proof*⟩

lemma *setmonotone-implies-chain-funpower*:
assumes *setmonotone*: *setmonotone f*
shows *chain* (λ *n*. *funpower f n I*)
⟨*proof*⟩

lemma *natUnion-subset*: (⋀ *n*. ∃ *m*. *f n* ⊆ *g m*) $\Longrightarrow$ *natUnion f* ⊆ *natUnion g*
⟨*proof*⟩

lemma *natUnion-eq*[*case-names Subset Superset*]:
(⋀ *n*. ∃ *m*. *f n* ⊆ *g m*) $\Longrightarrow$ (⋀ *n*. ∃ *m*. *g n* ⊆ *f m*) $\Longrightarrow$ *natUnion f* = *natUnion g*
⟨*proof*⟩

lemma *natUnion-shift*[*symmetric*]:
assumes *chain*: *chain C*
shows *natUnion C* = *natUnion* (λ *n*. *C* (*n* + *m*))
⟨*proof*⟩

definition *regular* :: ('*a set* ⇒ '*a set*) ⇒ *bool*
where
regular f = (*setmonotone f* ∧ *continuous f*)

lemma *regular-fixpoint*:
assumes *regular*: *regular f*
shows *f* (*limit f I*) = *limit f I*
⟨*proof*⟩

lemma *fix-is-fix-of-limit*:
assumes *fixpoint*: *f I* = *I*
shows *limit f I* = *I*
⟨*proof*⟩

lemma *limit-is-idempotent*: *regular f* $\Longrightarrow$ *limit f* (*limit f I*) = *limit f I*
⟨*proof*⟩

definition *mk-regular1* :: ('*b* ⇒ '*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ '*a*) ⇒ '*a set* ⇒ '*a set*
where
mk-regular1 P F I = *I* ∪ { *F q x* | *q x*. *x* ∈ *I* ∧ *P q x* }

definition *mk-regular2* :: ('*b* ⇒ '*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ '*a* ⇒ '*a*) ⇒ '*a set* ⇒ '*a set* **where**
mk-regular2 P F I = *I* ∪ { *F q x y* | *q x y*. *x* ∈ *I* ∧ *y* ∈ *I* ∧ *P q x y* }

lemma *setmonotone-mk-regular1*: *setmonotone* (*mk-regular1 P F*)

⟨*proof*⟩

lemma *setmonotone-mk-regular2*: *setmonotone* (*mk-regular2 P F*)
⟨*proof*⟩

lemma *pointbased-mk-regular1*: *pointbased* (*mk-regular1 P F*)
⟨*proof*⟩

lemma *pointbased-mk-regular2*: *pointbased* (*mk-regular2 P F*)
⟨*proof*⟩

lemma *regular1*:*regular* (*mk-regular1 P F*)
⟨*proof*⟩

lemma *regular2*: *regular* (*mk-regular2 P F*)
⟨*proof*⟩

lemma *continuous-comp*:
assumes *f*: *continuous f*
assumes *g*: *continuous g*
shows *continuous* (*g o f*)
⟨*proof*⟩

lemma *setmonotone-comp*:
assumes *f*: *setmonotone f*
assumes *g*: *setmonotone g*
shows *setmonotone* (*g o f*)
⟨*proof*⟩

lemma *regular-comp*:
assumes *f*: *regular f*
assumes *g*: *regular g*
shows *regular* (*g o f*)
⟨*proof*⟩

lemma *setmonotone-id*[*simp*]: *setmonotone id*
⟨*proof*⟩

lemma *continuous-id*[*simp*]: *continuous id*
⟨*proof*⟩

lemma *regular-id*[*simp*]: *regular id*
⟨*proof*⟩

lemma *regular-funpower*: *regular f* ⟹ *regular* (*funpower f n*)
⟨*proof*⟩

lemma *mono-id*[*simp*]: *mono id*
⟨*proof*⟩

lemma *mono-funpower*:
  assumes *mono*: *mono f*
  shows *mono* (*funpower f n*)
⟨*proof*⟩

lemma *mono-limit*:
  assumes *mono*: *mono f*
  shows *mono* (*limit f*)
⟨*proof*⟩

lemma *continuous-funpower*:
  assumes *continuous*: *continuous f*
  shows *continuous* (*funpower f n*)
⟨*proof*⟩

lemma *natUnion-swap*:
  natUnion (λ *i*. *natUnion* (λ *j*. *f i j*)) = *natUnion* (λ *j*. *natUnion* (λ *i*. *f i j*))
⟨*proof*⟩

lemma *continuous-limit*:
  assumes *continuous*: *continuous f*
  shows *continuous* (*limit f*)
⟨*proof*⟩

lemma *regular-limit*: *regular f* ⟹ *regular* (*limit f*)
⟨*proof*⟩

lemma *regular-implies-mono*: *regular f* ⟹ *mono f*
⟨*proof*⟩

lemma *regular-implies-setmonotone*: *regular f* ⟹ *setmonotone f*
⟨*proof*⟩

lemma *regular-implies-continuous*: *regular f* ⟹ *continuous f*
⟨*proof*⟩

end
theory *LocalLexingLemmas*
imports *LocalLexing Limit*
begin

context *LocalLexing* **begin**

lemma[*simp*]: *setmonotone* (*Append Z k*) ⟨*proof*⟩

lemma *subset-PSuc*: $\mathcal{P}\ k\ u \subseteq \mathcal{P}\ k$ (*Suc u*)
  ⟨*proof*⟩

lemma *subset-fSuc-strict*:
assumes f: $\bigwedge u.\ f\ u \subseteq f\ (Suc\ u)$
shows $u < v \Longrightarrow f\ u \subseteq f\ v$
⟨*proof*⟩

lemma *subset-fSuc*:
assumes f: $\bigwedge u.\ f\ u \subseteq f\ (Suc\ u)$
shows $u \leq v \Longrightarrow f\ u \subseteq f\ v$
⟨*proof*⟩

lemma *subset-$\mathcal{P}$k*: $u \leq v \Longrightarrow \mathcal{P}\ k\ u \subseteq \mathcal{P}\ k\ v$
⟨*proof*⟩

lemma *subset-$\mathcal{P}\mathcal{Q}$k*: $\mathcal{P}\ k\ u \subseteq \mathcal{Q}\ k$ ⟨*proof*⟩

lemma *subset-$\mathcal{Q}\mathcal{P}$Suc*: $\mathcal{Q}\ k \subseteq \mathcal{P}\ (Suc\ k)\ u$
⟨*proof*⟩

lemma *subset-$\mathcal{Q}$Suc*: $\mathcal{Q}\ k \subseteq \mathcal{Q}\ (Suc\ k)$
⟨*proof*⟩

lemma *subset-$\mathcal{Q}$*: $i \leq j \Longrightarrow \mathcal{Q}\ i \subseteq \mathcal{Q}\ j$
⟨*proof*⟩

lemma *empty-$\mathcal{X}$*[*simp*]: $k > length\ Doc \Longrightarrow \mathcal{X}\ k = \{\}$
⟨*proof*⟩

lemma *Sel-empty*[*simp*]: $Sel\ \{\}\ \{\} = \{\}$
⟨*proof*⟩

lemma *empty-$\mathcal{Z}$*[*simp*]: $k > length\ Doc \Longrightarrow \mathcal{Z}\ k\ u = \{\}$
⟨*proof*⟩

lemma[*simp*]: $Append\ \{\}\ k = id$ ⟨*proof*⟩

lemma[*simp*]: $k > length\ Doc \Longrightarrow \mathcal{P}\ k\ v = \mathcal{P}\ k\ 0$
⟨*proof*⟩

lemma $\mathcal{Q}$*SucEq*: $k \geq length\ Doc \Longrightarrow \mathcal{Q}\ (Suc\ k) = \mathcal{Q}\ k$
⟨*proof*⟩

lemma $\mathcal{Q}$*-converges*:
assumes k: $k \geq length\ Doc$
shows $\mathcal{Q}\ k = \mathfrak{P}$
⟨*proof*⟩

lemma $\mathfrak{P}$*-covers-*$\mathcal{Q}$: $\mathcal{Q}\ k \subseteq \mathfrak{P}$
⟨*proof*⟩

lemma *Sel-upper-bound*: $A \subseteq B \Longrightarrow Sel\ A\ B \subseteq B$
⟨*proof*⟩

lemma *Sel-lower-bound*: $A \subseteq B \Longrightarrow A \subseteq Sel\ A\ B$
⟨*proof*⟩

lemma $\mathfrak{P}$-*covers-*$\mathcal{P}$: $\mathcal{P}\ k\ u \subseteq \mathfrak{P}$
⟨*proof*⟩

lemma $\mathcal{W}$-*montone*: $P \subseteq Q \Longrightarrow \mathcal{W}\ P\ k \subseteq \mathcal{W}\ Q\ k$
⟨*proof*⟩

lemma *Sel-precondition*:
$\mathcal{Z}\ k\ u \subseteq \mathcal{W}\ (\mathcal{P}\ k\ u)\ k$
⟨*proof*⟩

lemma $\mathcal{W}$-*bounded-by-*$\mathcal{X}$: $\mathcal{W}\ P\ k \subseteq \mathcal{X}\ k$
⟨*proof*⟩

lemma $\mathcal{Z}$-*subset-*$\mathcal{X}$: $\mathcal{Z}\ k\ n \subseteq \mathcal{X}\ k$
⟨*proof*⟩

lemma $\mathcal{Z}$-*subset-Suc*: $\mathcal{Z}\ k\ n \subseteq \mathcal{Z}\ k\ (Suc\ n)$
⟨*proof*⟩

lemma $\mathcal{Y}$-*upper-bound*: $\mathcal{Y}\ (\mathcal{Z}\ k\ u)\ (\mathcal{P}\ k\ u)\ k \subseteq \mathcal{W}\ \ (\mathcal{P}\ k\ u)\ k$
⟨*proof*⟩

lemma $\mathfrak{P}$-*induct*[*consumes 1*, *case-names Base Induct*]:
assumes *p*: $p \in \mathfrak{P}$
assumes *base*: $P\ []$
assumes *induct*: $\bigwedge\ p\ k\ u.\ (\bigwedge\ q.\ q \in \mathcal{P}\ k\ u \Longrightarrow P\ q) \Longrightarrow p \in \mathcal{P}\ k\ (Suc\ u) \Longrightarrow P\ p$
shows $P\ p$
⟨*proof*⟩

lemma *Append-mono*: $U \subseteq V \Longrightarrow P \subseteq Q \Longrightarrow Append\ U\ k\ P \subseteq Append\ V\ k\ Q$
⟨*proof*⟩

lemma *pointwise-Append*: *pointwise* $(Append\ T\ k)$
⟨*proof*⟩

lemma *regular-Append*: *regular* $(Append\ T\ k)$
⟨*proof*⟩

end

end
theory *InductRules*

imports *Main*
begin

lemma *disjCases2*[*consumes 1*, *case-names 1 2*]:
  assumes *AB*: $A \vee B$
  and *AP*: $A \Longrightarrow P$
  and *BP*: $B \Longrightarrow P$
  shows P
⟨*proof*⟩

lemma *disjCases3*[*consumes 1*, *case-names 1 2 3*]:
  assumes *AB*: $A \vee B \vee C$
  and *AP*: $A \Longrightarrow P$
  and *BP*: $B \Longrightarrow P$
  and *CP*: $C \Longrightarrow P$
  shows P
⟨*proof*⟩

lemma *disjCases4*[*consumes 1*, *case-names 1 2 3 4*]:
  assumes *AB*: $A \vee B \vee C \vee D$
  and *AP*: $A \Longrightarrow P$
  and *BP*: $B \Longrightarrow P$
  and *CP*: $C \Longrightarrow P$
  and *DP*: $D \Longrightarrow P$
  shows P
⟨*proof*⟩

lemma *disjCases5*[*consumes 1*, *case-names 1 2 3 4 5*]:
  assumes *AB*: $A \vee B \vee C \vee D \vee E$
  and *AP*: $A \Longrightarrow P$
  and *BP*: $B \Longrightarrow P$
  and *CP*: $C \Longrightarrow P$
  and *DP*: $D \Longrightarrow P$
  and *EP*: $E \Longrightarrow P$
  shows P
⟨*proof*⟩

lemma *minimal-witness-ex*:
  assumes *k*: $P\ (k::nat)$
  shows $\exists\ k0.\ k0 \leq k \wedge P\ k0 \wedge (\forall\ k.\ k < k0 \longrightarrow \neg\ (P\ k))$
⟨*proof*⟩

lemma *minimal-witness*[*consumes 1*, *case-names Minimal*]:
  assumes $P\ (k::nat)$
  and $\bigwedge K.\ K \leq k \Longrightarrow P\ K \Longrightarrow (\bigwedge k.\ k < K \Longrightarrow \neg\ (P\ k)) \Longrightarrow Q$
  shows Q
⟨*proof*⟩

lemma *ex-minimal-witness*[*consumes 1*, *case-names Minimal*]:

assumes $\exists\ k.\ P\ (k{::}nat)$
and $\bigwedge K.\ P\ K \Longrightarrow (\bigwedge k.\ k < K \Longrightarrow \neg\ (P\ k)) \Longrightarrow Q$
shows Q
⟨*proof*⟩

end
theory *ListTools*
imports *Main*
begin

definition *is-first* :: $'a \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}first\ x\ u = (\exists\ v.\ u = [x]@v)$

definition *is-last* :: $'a \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}last\ x\ u = (\exists\ v.\ u = v@[x])$

definition *is-prefix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}prefix\ u\ v = (\exists\ w.\ u@w = v)$

definition *is-proper-prefix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}proper\text{-}prefix\ u\ v = (\exists\ w.\ w \neq [] \wedge u@w = v)$

lemma *is-prefix-eq-proper-prefix*: $is\text{-}prefix\ a\ b = (a = b \vee is\text{-}proper\text{-}prefix\ a\ b)$
⟨*proof*⟩

lemma *is-proper-prefix-eq-prefix*: $is\text{-}proper\text{-}prefix\ a\ b = (a \neq b \wedge is\text{-}prefix\ a\ b)$
⟨*proof*⟩

definition *is-suffix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}suffix\ u\ v = (\exists\ w.\ w@u = v)$

definition *is-proper-suffix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}proper\text{-}suffix\ u\ v = (\exists\ w.\ w \neq [] \wedge w@u = v)$

lemma *is-suffix-eq-proper-suffix*: $is\text{-}suffix\ a\ b = (a = b \vee is\text{-}proper\text{-}suffix\ a\ b)$
⟨*proof*⟩

lemma *is-proper-suffix-eq-suffix*: $is\text{-}proper\text{-}suffix\ a\ b = (a \neq b \wedge is\text{-}suffix\ a\ b)$
⟨*proof*⟩

lemma *is-prefix-unsplit*: $is\text{-}prefix\ u\ a \Longrightarrow u\ @\ (drop\ (length\ u)\ a) = a$
⟨*proof*⟩

lemma *le-take-same*: $i \leq j \Longrightarrow take\ j\ a = take\ j\ b \Longrightarrow take\ i\ a = take\ i\ b$
⟨*proof*⟩

lemma *is-first-drop-length*:
assumes $k \leq length\ a$
and $k > length\ u$
and $v = X\#w$
and $take\ k\ a = take\ k\ (u@v)$
shows *is-first* $X\ (drop\ (length\ u)\ a)$
⟨*proof*⟩

lemma *is-first-cons*: *is-first* $x\ (y\#ys) = (x = y)$
⟨*proof*⟩

lemma *list-all-pos-neg-ex*: *list-all* $P\ D \Longrightarrow \neg\ (list\text{-}all\ Q\ D) \Longrightarrow$
$\exists\ k.\ k < length\ D \wedge P(D\ !\ k) \wedge \neg(Q(D\ !\ k))$
⟨*proof*⟩

lemma *split-list-at*: $k < length\ D \Longrightarrow D = (take\ k\ D)@[D\ !\ k]@(drop\ (Suc\ k)\ D)$
⟨*proof*⟩

lemma *take-eq-take-append*: $i \leq j \Longrightarrow j \leq length\ a \Longrightarrow \exists\ u.\ take\ j\ a = take\ i\ a$
$@\ u$
⟨*proof*⟩

lemma *is-proper-suffix-length-cmp*: *is-proper-suffix* $a\ b \Longrightarrow length\ a < length\ b$
⟨*proof*⟩

end
theory *Derivations*
imports *CFG ListTools InductRules*
begin

context *CFG* **begin**

lemma [*simp*]: *is-terminal* $t \Longrightarrow$ *is-symbol* t
⟨*proof*⟩

lemma [*simp*]: *is-sentence* [] ⟨*proof*⟩

lemma [*simp*]: *is-word* [] ⟨*proof*⟩

lemma [*simp*]: *is-word* $u \Longrightarrow$ *is-sentence* u
⟨*proof*⟩

definition *leftderives1* :: *sentence* ⇒ *sentence* ⇒ *bool*
where

leftderives1 u v =
(∃ *x y N α*.
u = *x* @ [*N*] @ *y*
∧ *v* = *x* @ *α* @ *y*
∧ *is-word x*
∧ *is-sentence y*
∧ (*N*, *α*) ∈ ℜ)

lemma *leftderives1-implies-derives1*[*simp*]: *leftderives1 u v* ⟹ *derives1 u v*
⟨*proof*⟩

definition *leftderivations1* :: (*sentence* × *sentence*) *set*
where
leftderivations1 = { (*u*,*v*) | *u v*. *leftderives1 u v* }

lemma [*simp*]: *leftderivations1* ⊆ *derivations1*
⟨*proof*⟩

definition *leftderivations* :: (*sentence* × *sentence*) *set*
where
leftderivations = *leftderivations1*^*

lemma *rtrancl-subset-implies*: *a* ⊆ *b* ⟹ *a* ⊆ *b*^* ⟨*proof*⟩

lemma *leftderivations-subset-derivations*[*simp*]: *leftderivations* ⊆ *derivations*
⟨*proof*⟩

definition *leftderives* :: *sentence* ⇒ *sentence* ⇒ *bool*
where
leftderives u v = ((*u*, *v*) ∈ *leftderivations*)

lemma *leftderives-implies-derives*[*simp*]: *leftderives u v* ⟹ *derives u v*
⟨*proof*⟩

definition *is-leftderivation* :: *sentence* ⇒ *bool*
where
is-leftderivation u = *leftderives* [𝔖] *u*

lemma *leftderivation-implies-derivation*[*simp*]:
is-leftderivation u ⟹ *is-derivation u*
⟨*proof*⟩

lemma *leftderives-refl*[*simp*]: *leftderives u u*
⟨*proof*⟩

lemma *leftderives1-implies-leftderives*[*simp*]:*leftderives1 a b* ⟹ *leftderives a b*
⟨*proof*⟩

lemma *leftderives-trans*: *leftderives a b* ⟹ *leftderives b c* ⟹ *leftderives a c*

⟨*proof*⟩

lemma *leftderives1-eq-leftderivations1*: *leftderives1 x y* = ((*x*, *y*) ∈ *leftderivations1*)
⟨*proof*⟩

lemma *leftderives-induct*[*consumes 1*, *case-names Base Step*]:
assumes *derives*: *leftderives a b*
assumes *Pa*: *P a*
assumes *induct*: ⋀*y z*. *leftderives a y* ⟹ *leftderives1 y z* ⟹ *P y* ⟹ *P z*
shows *P b*
⟨*proof*⟩

end

context *CFG* **begin**

lemma *derives1-implies-derives*[*simp*]:*derives1 a b* ⟹ *derives a b*
⟨*proof*⟩

lemma *derives-trans*: *derives a b* ⟹ *derives b c* ⟹ *derives a c*
⟨*proof*⟩

lemma *derives1-eq-derivations1*: *derives1 x y* = ((*x*, *y*) ∈ *derivations1*)
⟨*proof*⟩

lemma *derives-induct*[*consumes 1*, *case-names Base Step*]:
assumes *derives*: *derives a b*
assumes *Pa*: *P a*
assumes *induct*: ⋀*y z*. *derives a y* ⟹ *derives1 y z* ⟹ *P y* ⟹ *P z*
shows *P b*
⟨*proof*⟩

end

context *CFG* **begin**

definition *Derives1* :: *sentence* ⇒ *nat* ⇒ *rule* ⇒ *sentence* ⇒ *bool*
where
Derives1 u i r v =
(∃ *x y N α*.
u = *x* @ [*N*] @ *y*
∧ *v* = *x* @ *α* @ *y*
∧ *is-sentence x*
∧ *is-sentence y*
∧ (*N*, *α*) ∈ ℜ
∧ *r* = (*N*, *α*) ∧ *i* = *length x*)

lemma *Derives1-split*:
Derives1 u i r v $\Longrightarrow \exists\ x\ y.\ u = x\ @\ [fst\ r]\ @\ y \wedge v = x\ @\ (snd\ r)\ @\ y \wedge length\ x = i$
⟨*proof*⟩

lemma *Derives1-implies-derives1*: *Derives1 u i r v* $\Longrightarrow$ *derives1 u v*
⟨*proof*⟩

lemma *derives1-implies-Derives1*: *derives1 u v* $\Longrightarrow \exists\ i\ r.$ *Derives1 u i r v*
⟨*proof*⟩

lemma *Derives1-unique-dest*: *Derives1 u i r v* $\Longrightarrow$ *Derives1 u i r w* $\Longrightarrow v = w$
⟨*proof*⟩

lemma *Derives1-unique-src*: *Derives1 u i r w* $\Longrightarrow$ *Derives1 v i r w* $\Longrightarrow u = v$
⟨*proof*⟩

type-synonym *derivation* = (*nat* × *rule*) *list*

fun *Derivation* :: *sentence* ⇒ *derivation* ⇒ *sentence* ⇒ *bool*
where
Derivation a [] *b* = (*a* = *b*)
| *Derivation a* (*d*#*D*) *b* = (∃ *x*. *Derives1 a* (*fst d*) (*snd d*) *x* ∧ *Derivation x D b*)

lemma *Derivation-implies-derives*: *Derivation a D b* $\Longrightarrow$ *derives a b*
⟨*proof*⟩

lemma *Derivation-Derives1*: *Derivation a S y* $\Longrightarrow$ *Derives1 y i r z* $\Longrightarrow$ *Derivation a* (*S*@[(*i*,*r*)]) *z*
⟨*proof*⟩

lemma *derives-implies-Derivation*: *derives a b* $\Longrightarrow \exists\ D.$ *Derivation a D b*
⟨*proof*⟩

lemma *Derives1-take*: *Derives1 a i r b* $\Longrightarrow$ *take i a* = *take i b*
⟨*proof*⟩

lemma *Derives1-drop*: *Derives1 a i r b* $\Longrightarrow$ *drop* (*Suc i*) *a* = *drop* (*i* + *length* (*snd r*)) *b*
⟨*proof*⟩

lemma *Derives1-bound*: *Derives1 a i r b* $\Longrightarrow$ *i* < *length a*
⟨*proof*⟩

lemma *Derives1-length*: *Derives1 a i r b* $\Longrightarrow$ *length b* = *length a* + *length* (*snd r*) − *1*
⟨*proof*⟩

definition *leftmost* :: *nat* ⇒ *sentence* ⇒ *bool*
where
leftmost i s = (*i* < *length s* ∧ *is-word* (*take i s*) ∧ *is-nonterminal* (*s* ! *i*))

lemma *set-take*: *set* (*take n s*) = { *s* ! *i* | *i*. *i* < *n* ∧ *i* < *length s*}
⟨*proof*⟩

lemma *list-all-take*: *list-all P* (*take n s*) = (∀ *i*. *i* < *n* ∧ *i* < *length s* ⟶ *P* (*s* ! *i*))
⟨*proof*⟩

lemma *is-sentence-concat*: *is-sentence* (*x*@*y*) = (*is-sentence x* ∧ *is-sentence y*)
⟨*proof*⟩

lemma *is-sentence-cons*: *is-sentence* (*x*#*xs*) = (*is-symbol x* ∧ *is-sentence xs*)
⟨*proof*⟩

lemma *rule-nonterminal-type*[*simp*]: (*N*, *α*) ∈ ℜ ⟹ *is-nonterminal N*
⟨*proof*⟩

lemma *rule-α-type*[*simp*]: (*N*, *α*) ∈ ℜ ⟹ *is-sentence α*
⟨*proof*⟩

lemma [*simp*]: *is-nonterminal N* ⟹ *is-symbol N*
⟨*proof*⟩

lemma *Derives1-sentence1*[*elim*]: *Derives1 a i r b* ⟹ *is-sentence a*
⟨*proof*⟩

lemma *Derives1-sentence2*[*elim*]: *Derives1 a i r b* ⟹ *is-sentence b*
⟨*proof*⟩

lemma [*elim*]: *Derives1 a i r b* ⟹ *r* ∈ ℜ
⟨*proof*⟩

lemma *is-sentence-symbol*: *is-sentence a* ⟹ *i* < *length a* ⟹ *is-symbol* (*a* ! *i*)
⟨*proof*⟩

lemma *is-symbol-distinct*: *is-symbol x* ⟹ *is-terminal x* ≠ *is-nonterminal x*
⟨*proof*⟩

lemma *is-terminal-nonterminal*: *is-terminal x* ⟹ *is-nonterminal x* ⟹ *False*
⟨*proof*⟩

lemma *Derives1-leftmost*:
assumes *Derives1 a i r b*
shows ∃ *j*. *leftmost j a* ∧ *j* ≤ *i*
⟨*proof*⟩

lemma *Derivation-leftmost*: $D \neq [] \Longrightarrow$ *Derivation a D b* $\Longrightarrow \exists\ i.$ *leftmost i a*
⟨*proof*⟩

lemma *nonword-has-nonterminal*:
is-sentence a $\Longrightarrow\ \neg$ (*is-word a*) $\Longrightarrow \exists\ k.\ k <$ *length a* $\wedge$ *is-nonterminal* (*a* ! *k*)
⟨*proof*⟩

lemma *leftmost-cons-nonterminal*:
is-nonterminal x $\Longrightarrow$ *leftmost 0* (*x*#*xs*)
⟨*proof*⟩

lemma *leftmost-cons-terminal*:
is-terminal x $\Longrightarrow$ *leftmost i* (*x*#*xs*) = (*i* > *0* $\wedge$ *leftmost* (*i* − *1*) *xs*)
⟨*proof*⟩

lemma *is-nonterminal-cons-terminal*:
is-terminal x $\Longrightarrow$ *k* < *length* (*x* # *a*) $\Longrightarrow$ *is-nonterminal* ((*x* # *a*) ! *k*) $\Longrightarrow$
k > *0* $\wedge$ *k* − *1* < *length a* $\wedge$ *is-nonterminal* (*a* ! (*k* − *1*))
⟨*proof*⟩

lemma *leftmost-exists*:
is-sentence a $\Longrightarrow$ *k* < *length a* $\Longrightarrow$ *is-nonterminal* (*a* ! *k*) $\Longrightarrow$
$\exists\ i.$ *leftmost i a* $\wedge$ *i* ≤ *k*
⟨*proof*⟩

lemma *nonword-leftmost-exists*:
is-sentence a $\Longrightarrow\ \neg$ (*is-word a*) $\Longrightarrow \exists\ i.$ *leftmost i a*
⟨*proof*⟩

lemma *leftmost-unaffected-Derives1*: *leftmost j a* $\Longrightarrow$ *j* < *i* $\Longrightarrow$ *Derives1 a i r b*
$\Longrightarrow$ *leftmost j b*
⟨*proof*⟩

definition *derivation-ge* :: *derivation* ⇒ *nat* ⇒ *bool*
where
derivation-ge D i = ($\forall\ d \in$ *set D*. *fst d* ≥ *i*)

lemma *derivation-ge-cons*: *derivation-ge* (*d*#*D*) *i* = (*fst d* ≥ *i* $\wedge$ *derivation-ge D i*)
⟨*proof*⟩

lemma *derivation-ge-append*:
derivation-ge (*D*@*E*) *i* = (*derivation-ge D i* $\wedge$ *derivation-ge E i*)
⟨*proof*⟩

lemma *leftmost-unaffected-Derivation*:
derivation-ge D (*Suc i*) $\Longrightarrow$ *leftmost i a* $\Longrightarrow$ *Derivation a D b* $\Longrightarrow$ *leftmost i b*
⟨*proof*⟩

lemma *le-Derives1-take*:
assumes *le*: $i \leq j$
and *D*: *Derives1 a j r b*
shows *take i a* = *take i b*
⟨*proof*⟩

lemma *Derivation-take*: *derivation-ge D i* ⟹ *Derivation a D b* ⟹ *take i a* = *take i b*
⟨*proof*⟩

lemma *leftmost-cons-less*: $i <$ *length u* ⟹ *leftmost i* (*u*@*v*) = *leftmost i u*
⟨*proof*⟩

lemma *leftmost-is-nonterminal*: *leftmost i u* ⟹ *is-nonterminal* (*u* ! *i*)
⟨*proof*⟩

lemma *is-word-is-terminal*: $i <$ *length u* ⟹ *is-word u* ⟹ *is-terminal* (*u* ! *i*)
⟨*proof*⟩

lemma *leftmost-append*:
assumes *leftmost*: *leftmost i* (*u*@*v*)
and *is-word*: *is-word u*
shows *length u* $\leq i$
⟨*proof*⟩

lemma *derivation-ge-empty*[*simp*]: *derivation-ge* [] *i*
⟨*proof*⟩

lemma *leftmost-notword*: *leftmost i a* ⟹ $j > i$ ⟹ ¬ (*is-word* (*take j a*))
⟨*proof*⟩

lemma *leftmost-unique*: *leftmost i a* ⟹ *leftmost j a* ⟹ $i = j$
⟨*proof*⟩

lemma *leftmost-Derives1*: *leftmost i a* ⟹ *Derives1 a j r b* ⟹ $i \leq j$
⟨*proof*⟩

lemma *leftmost-Derives1-propagate*:
assumes *leftmost*: *leftmost i a*
and *Derives1*: *Derives1 a j r b*
shows (*is-word b* ∧ $i = j$) ∨ (∃ *k*. *leftmost k b* ∧ $i \leq k$)
⟨*proof*⟩

lemma *is-word-Derives1*[*elim*]: *is-word a* ⟹ *Derives1 a i r b* ⟹ *False*
⟨*proof*⟩

lemma *is-word-Derivation*[*elim*]: *is-word a* ⟹ *Derivation a D b* ⟹ *D* = []
⟨*proof*⟩

lemma *leftmost-Derivation*:
leftmost i a $\Longrightarrow$ *Derivation a D b* $\Longrightarrow$ $j \leq i$ $\Longrightarrow$ *derivation-ge D j*
⟨*proof*⟩

lemma *derivation-ge-list-all*: *derivation-ge D i* = *list-all* (λ *d*. *fst d* ≥ *i*) *D*
⟨*proof*⟩

lemma *split-derivation-leftmost*:
assumes *derivation-ge D i*
and ¬ (*derivation-ge D* (*Suc i*))
shows ∃ *E F r*. *D* = *E*@[(*i*, *r*)]@*F* ∧ *derivation-ge E* (*Suc i*)
⟨*proof*⟩

lemma *Derives1-Derives1-swap*:
assumes *i* < *j*
and *Derives1 a j p b*
and *Derives1 b i q c*
shows ∃ *b'*. *Derives1 a i q b'* ∧ *Derives1 b'* (*j* − *1* + *length* (*snd q*)) *p c*
⟨*proof*⟩

definition *derivation-shift* :: *derivation* ⇒ *nat* ⇒ *nat* ⇒ *derivation*
where
derivation-shift D left right = *map* (λ *d*. (*fst d* − *left* + *right*, *snd d*)) *D*

lemma *derivation-shift-empty*[*simp*]: *derivation-shift* [] *left right* = []
⟨*proof*⟩

lemma *derivation-shift-cons*[*simp*]:
derivation-shift (*d*#*D*) *left right* = ((*fst d* − *left* + *right*, *snd d*)#(*derivation-shift D left right*))
⟨*proof*⟩

lemma *Derivation-append*: *Derivation a* (*D*@*E*) *c* = (∃ *b*. *Derivation a D b* ∧ *Derivation b E c*)
⟨*proof*⟩

lemma *Derivation-implies-append*:
Derivation a D b $\Longrightarrow$ *Derivation b E c* $\Longrightarrow$ *Derivation a* (*D*@*E*) *c*
⟨*proof*⟩

lemma *Derivation-swap-single-end-to-front*:
i < *j* $\Longrightarrow$ *derivation-ge D j* $\Longrightarrow$ *Derivation a* (*D*@[(*i*,*r*)]) *b* $\Longrightarrow$
Derivation a ((*i*,*r*)#(*derivation-shift D 1* (*length* (*snd r*)))) *b*
⟨*proof*⟩

lemma *Derivation-swap-single-mid-to-front*:
assumes *i* < *j*
and *derivation-ge D j*
and *Derivation a* (*D*@[(*i*,*r*)]@*E*) *b*

shows *Derivation a ((i,r)#((derivation-shift D 1 (length (snd r)))@E)) b*
⟨*proof*⟩

lemma *length-derivation-shift*[*simp*]:
length(*derivation-shift D left right*) = *length D*
⟨*proof*⟩

definition *LeftDerives1* :: *sentence* ⇒ *nat* ⇒ *rule* ⇒ *sentence* ⇒ *bool*
where
LeftDerives1 u i r v = (*leftmost i u* ∧ *Derives1 u i r v*)

lemma *LeftDerives1-implies-leftderives1*: *LeftDerives1 u i r v* ⟹ *leftderives1 u v*
⟨*proof*⟩

lemma *leftmost-Derives1-leftderives*:
leftmost i a ⟹ *Derives1 a i r b* ⟹ *leftderives b c* ⟹ *leftderives a c*
⟨*proof*⟩

theorem *Derivation-implies-leftderives-gen*:
Derivation a D (*u*@*v*) ⟹ *is-word u* ⟹ (∃ *w*.
leftderives a (*u*@*w*) ∧
(*v* = [] ⟶ *w* = []) ∧
(∀ *X*. *is-first X v* ⟶ *is-first X w*))
⟨*proof*⟩

lemma *derives-implies-leftderives-gen*: *derives a* (*u*@*v*) ⟹ *is-word u* ⟹ (∃ *w*.
leftderives a (*u*@*w*) ∧
(*v* = [] ⟶ *w* = []) ∧
(∀ *X*. *is-first X v* ⟶ *is-first X w*))
⟨*proof*⟩

lemma *derives-implies-leftderives*: *derives a b* ⟹ *is-word b* ⟹ *leftderives a b*
⟨*proof*⟩

fun *LeftDerivation* :: *sentence* ⇒ *derivation* ⇒ *sentence* ⇒ *bool*
where
LeftDerivation a [] *b* = (*a* = *b*)
| *LeftDerivation a* (*d*#*D*) *b* = (∃ *x*. *LeftDerives1 a* (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x D b*)

lemma *LeftDerives1-implies-Derives1*: *LeftDerives1 a i r b* ⟹ *Derives1 a i r b*
⟨*proof*⟩

lemma *LeftDerivation-implies-Derivation*:
LeftDerivation a D b ⟹ *Derivation a D b*
⟨*proof*⟩

lemma *LeftDerivation-implies-leftderives*: *LeftDerivation a D b* ⟹ *leftderives a b*
⟨*proof*⟩

lemma *leftmost-witness*[*simp*]: *leftmost* (*length x*) (*x*@(*N*#*y*)) = (*is-word x* ∧ *is-nonterminal N*)
⟨*proof*⟩

lemma *leftderives1-implies-LeftDerives1*:
assumes *leftderives1*: *leftderives1 u v*
shows ∃ *i r*. *LeftDerives1 u i r v*
⟨*proof*⟩

lemma *LeftDerivation-LeftDerives1*:
LeftDerivation a S y ⟹ *LeftDerives1 y i r z* ⟹ *LeftDerivation a* (*S*@[(*i*,*r*)]) *z*
⟨*proof*⟩

lemma *leftderives-implies-LeftDerivation*: *leftderives a b* ⟹ ∃ *D*. *LeftDerivation a D b*
⟨*proof*⟩

lemma *LeftDerivation-append*:
LeftDerivation a (*D*@*E*) *c* = (∃ *b*. *LeftDerivation a D b* ∧ *LeftDerivation b E c*)
⟨*proof*⟩

lemma *LeftDerivation-implies-append*:
LeftDerivation a D b ⟹ *LeftDerivation b E c* ⟹ *LeftDerivation a* (*D*@*E*) *c*
⟨*proof*⟩

lemma *Derivation-unique-dest*: *Derivation a D b* ⟹ *Derivation a D c* ⟹ *b* = *c*
⟨*proof*⟩

lemma *Derivation-unique-src*: *Derivation a D c* ⟹ *Derivation b D c* ⟹ *a* = *b*
⟨*proof*⟩

lemma *LeftDerives1-unique*: *LeftDerives1 a i r b* ⟹ *LeftDerives1 a j s b* ⟹ *i* = *j* ∧ *r* = *s*
⟨*proof*⟩

lemma *leftlang*: $\mathcal{L}$ = { *v* | *v*. *is-word v* ∧ *is-leftderivation v* }
⟨*proof*⟩

lemma *leftprefixlang*: $\mathcal{L}_P$ = { *u* | *u v*. *is-word u* ∧ *is-leftderivation* (*u*@*v*) }
⟨*proof*⟩

lemma *derives-implies-leftderives-cons*:
is-word a ⟹ *derives u* (*a*@*X*#*b*) ⟹ ∃ *c*. *leftderives u* (*a*@*X*#*c*)
⟨*proof*⟩

lemma *is-word-append*[*simp*]: *is-word* (*a*@*b*) = (*is-word a* ∧ *is-word b*)
⟨*proof*⟩

lemma $\mathcal{L}_P$*-split*: $a@b \in \mathcal{L}_P \Longrightarrow a \in \mathcal{L}_P$
⟨*proof*⟩

lemma $\mathcal{L}_P$*-is-word*: $a \in \mathcal{L}_P \Longrightarrow$ *is-word a*
⟨*proof*⟩

definition *Derive* :: *sentence* ⇒ *derivation* ⇒ *sentence*
where
Derive a D = (*THE b*. *Derivation a D b*)

lemma *Derivation-dest-ex-unique*: *Derivation a D b* ⟹ ∃! *x*. *Derivation a D x*
⟨*proof*⟩

lemma *Derive*:
assumes *ab*: *Derivation a D b*
shows *Derive a D* = *b*
⟨*proof*⟩

end

end
theory *Validity*
imports *LLEarleyParsing Derivations*
begin

context *LocalLexing* **begin**

definition *wellformed-token* :: *token* ⇒ *bool*
where
wellformed-token t = *is-terminal* (*terminal-of-token t*)

definition *wellformed-tokens* :: *tokens* ⇒ *bool*
where
wellformed-tokens ts = *list-all wellformed-token ts*

definition *doc-tokens* :: *tokens* ⇒ *bool*
where
doc-tokens p = (*wellformed-tokens p* ∧ *is-prefix* (*chars p*) *Doc*)

definition *wellformed-item* :: *item* ⇒ *bool*
where
wellformed-item x = (
item-rule x ∈ ℜ ∧
item-origin x ≤ *item-end x* ∧
item-end x ≤ *length Doc* ∧
item-dot x ≤ *length* (*item-rhs x*))

definition *wellformed-items* :: *items* ⇒ *bool*
where

wellformed-items X = (∀ *x* ∈ *X*. *wellformed-item x*)

lemma *is-word-terminals*: *wellformed-tokens p* ⟹ *is-word* (*terminals p*)
⟨*proof*⟩

lemma *is-word-subset*: *is-word x* ⟹ *set y* ⊆ *set x* ⟹ *is-word y*
⟨*proof*⟩

lemma *is-word-terminals-take*: *wellformed-tokens p* ⟹ *is-word*(*terminals* (*take n p*))
⟨*proof*⟩

lemma *is-word-terminals-drop*: *wellformed-tokens p* ⟹ *is-word*(*terminals* (*drop n p*))
⟨*proof*⟩

definition *pvalid* :: *tokens* ⇒ *item* ⇒ *bool*
where
pvalid p x = (∃ *u γ*.
wellformed-tokens p ∧
wellformed-item x ∧
u ≤ *length p* ∧
charslength p = *item-end x* ∧
charslength (*take u p*) = *item-origin x* ∧
is-derivation (*terminals* (*take u p*) @ [*item-nonterminal x*] @ *γ*) ∧
derives (*item-α x*) (*terminals* (*drop u p*)))

definition *Gen* :: *tokens set* ⇒ *items*
where
Gen P = { *x* | *x p*. *p* ∈ *P* ∧ *pvalid p x* }

lemma *wellformed-items* (*Gen P*)
⟨*proof*⟩

lemma *wellformed-items* (*Init*)
⟨*proof*⟩

definition *pvalid-left* :: *tokens* ⇒ *item* ⇒ *bool*
where
pvalid-left p x = (∃ *u γ*.
wellformed-tokens p ∧
wellformed-item x ∧
u ≤ *length p* ∧
charslength p = *item-end x* ∧
charslength (*take u p*) = *item-origin x* ∧
is-leftderivation (*terminals* (*take u p*) @ [*item-nonterminal x*] @ *γ*) ∧
leftderives (*item-α x*) (*terminals* (*drop u p*)))

lemma *pvalid-left*: *pvalid p x* = *pvalid-left p x*

⟨*proof*⟩

lemma $\mathcal{L}_P$*-wellformed-tokens*: *terminals* $p \in \mathcal{L}_P \Longrightarrow$ *wellformed-tokens* p
⟨*proof*⟩

end

end
theory *TheoremD2*
imports *LocalLexingLemmas Validity Derivations*
begin

context *LocalLexing* **begin**

definition *splits-at* :: *sentence* ⇒ *nat* ⇒ *sentence* ⇒ *symbol* ⇒ *sentence* ⇒ *bool*
where
splits-at δ i α N β = ($i <$ *length* $\delta \wedge \alpha =$ *take* i $\delta \wedge N = \delta$! $i \wedge \beta =$ *drop* (*Suc* i) δ)

lemma *splits-at-combine*: *splits-at* δ i α N $\beta \Longrightarrow \delta = \alpha$ @ [N] @ β
⟨*proof*⟩

lemma *splits-at-combine-dest*: *Derives1* a i r $b \Longrightarrow$ *splits-at* a i α N $\beta \Longrightarrow b = \alpha$ @ (*snd* r) @ β
⟨*proof*⟩

lemma *Derives1-nonterminal*:
assumes *Derives1* a i r b
assumes *splits-at* a i α N β
shows *fst* $r = N \wedge$ *is-nonterminal* N
⟨*proof*⟩

lemma *splits-at-ex*: *Derives1* δ i r $s \Longrightarrow \exists$ α N β. *splits-at* δ i α N β
⟨*proof*⟩

lemma *splits-at-α*: *Derives1* δ i r $s \Longrightarrow$ *splits-at* δ i α N $\beta \Longrightarrow$
$\alpha =$ *take* i $\delta \wedge \alpha =$ *take* i $s \wedge$ *length* $\alpha = i$
⟨*proof*⟩

lemma *LeftDerives1-splits-at-is-word*: *LeftDerives1* δ i r $s \Longrightarrow$ *splits-at* δ i α N β $\Longrightarrow$ *is-word* α
⟨*proof*⟩

lemma *splits-at-β*: *Derives1* δ i r $s \Longrightarrow$ *splits-at* δ i α N $\beta \Longrightarrow$
$\beta =$ *drop* (*Suc* i) $\delta \wedge \beta =$ *drop* ($i +$ *length* (*snd* r)) $s \wedge$ *length* $\beta =$ *length* $\delta - i - 1$
⟨*proof*⟩

lemma *Derives1-prefix*:
assumes *ab*: *Derives1* δ *i r* (*a*@*b*)
assumes *split*: *splits-at* δ *i* α *N* β
shows *is-prefix* α *a* $\vee$ *is-prefix a* α
⟨*proof*⟩

lemma *Derives1-suffix*:
assumes *ab*: *Derives1* δ *i r* (*a*@*b*)
assumes *split*: *splits-at* δ *i* α *N* β
shows *is-suffix* β *b* $\vee$ *is-suffix b* β
⟨*proof*⟩

lemma *Derives1-skip-prefix*:
length a $\leq$ *i* $\Longrightarrow$ *Derives1* (*a*@*b*) *i r* (*a*@*c*) $\Longrightarrow$ *Derives1 b* (*i* − *length a*) *r c*
⟨*proof*⟩

lemma *cancel-suffix*:
assumes *a* @ *c* = *b* @ *d*
assumes *length c* $\leq$ *length d*
shows *a* = *b* @ (*take* (*length d* − *length c*) *d*)
⟨*proof*⟩

lemma *is-sentence-take*:
is-sentence y $\Longrightarrow$ *is-sentence* (*take n y*)
⟨*proof*⟩

lemma *Derives1-skip-suffix*:
assumes *i*: *i* < *length a*
assumes *D*: *Derives1* (*a*@*c*) *i r* (*b*@*c*)
shows *Derives1 a i r b*
⟨*proof*⟩

lemma *drop-cancel-suffix*: *a*@*c* = *drop n* (*b*@*c*) $\Longrightarrow$ *a* = *drop n b*
⟨*proof*⟩

lemma *drop-keep-last*: *u* ≠ [] $\Longrightarrow$ *u* = *drop n* (*a*@[*X*]) $\Longrightarrow$ *u* = *drop n a* @ [*X*]
⟨*proof*⟩

lemma *Derives1-X-is-part-of-rule*[*consumes 2*, *case-names Suffix Prefix*]:
assumes *aXb*: *Derives1* δ *i r* (*a*@[*X*]@*b*)
assumes *split*: *splits-at* δ *i* α *N* β
assumes *prefix*: $\bigwedge$ β. δ = *a* @ [*X*] @ β $\Longrightarrow$ *length a* < *i* $\Longrightarrow$
Derives1 β (*i* − *length a* − *1*) *r b* $\Longrightarrow$ *False*
assumes *suffix*: $\bigwedge$ α. δ = α @ [*X*] @ *b* $\Longrightarrow$ *Derives1* α *i r a* $\Longrightarrow$ *False*
shows $\exists$ *u v*. *a* = α @ *u* $\wedge$ *b* = *v* @ β $\wedge$ (*snd r*) = *u*@[*X*]@*v*
⟨*proof*⟩

lemma $\mathcal{L}_P$*-derives*: *a* $\in$ $\mathcal{L}_P$ $\Longrightarrow$ $\exists$ *b*. *derives* [$\mathfrak{S}$] (*a*@*b*)
⟨*proof*⟩

lemma $\mathcal{L}_P$-*leftderives*: $a \in \mathcal{L}_P \Longrightarrow \exists\ b.\ leftderives\ [\mathfrak{S}]\ (a@b)$
⟨*proof*⟩

lemma *Derives1-rule*: $Derives1\ a\ i\ r\ b \Longrightarrow r \in \mathfrak{R}$
⟨*proof*⟩

lemma *is-prefix-empty*[*simp*]: $is\text{-}prefix\ []\ a$
⟨*proof*⟩

lemma *is-prefix-cons*: $is\text{-}prefix\ (x\ \#\ a)\ b = (\exists\ c.\ b = x\ \#\ c \wedge is\text{-}prefix\ a\ c)$
⟨*proof*⟩

lemma *is-prefix-cancel*[*simp*]: $is\text{-}prefix\ (a@b)\ (a@c) = is\text{-}prefix\ b\ c$
⟨*proof*⟩

lemma *is-prefix-chars*: $is\text{-}prefix\ a\ b \Longrightarrow is\text{-}prefix\ (chars\ a)\ (chars\ b)$
⟨*proof*⟩

lemma *is-prefix-length*: $is\text{-}prefix\ a\ b \Longrightarrow length\ a \leq length\ b$
⟨*proof*⟩

lemma *is-prefix-take*[*simp*]: $is\text{-}prefix\ (take\ n\ a)\ a$
⟨*proof*⟩

lemma *doc-tokens-length*: $doc\text{-}tokens\ p \Longrightarrow length\ (chars\ p) \leq length\ Doc$
⟨*proof*⟩

fun *count-terminals* :: *sentence* ⇒ *nat* **where**
count-terminals [] = *0*
| *count-terminals* (*x*#*xs*) = (*if* (*is-terminal x*) *then Suc* (*count-terminals xs*) *else* (*count-terminals xs*))

lemma *count-terminals-upper-bound*: $count\text{-}terminals\ p \leq length\ p$
⟨*proof*⟩

lemma *count-terminals-append*[*simp*]: $count\text{-}terminals\ (a@b) = count\text{-}terminals\ a + count\text{-}terminals\ b$
⟨*proof*⟩

lemma *Derives1-count-terminals*:
assumes *D*: *Derives1 a i r b*
shows $count\text{-}terminals\ b = count\text{-}terminals\ a + count\text{-}terminals\ (snd\ r)$
⟨*proof*⟩

lemma *Derives1-count-terminals-leq*:
assumes *D*: *Derives1 a i r b*
shows $count\text{-}terminals\ a \leq count\text{-}terminals\ b$
⟨*proof*⟩

lemma *Derivation-count-terminals-leq*:
Derivation a E b $\Longrightarrow$ *count-terminals a* $\leq$ *count-terminals b*
⟨*proof*⟩

lemma *derives-count-terminals-leq*: *derives a b* $\Longrightarrow$ *count-terminals a* $\leq$ *count-terminals b*
⟨*proof*⟩

lemma *is-word-cons*[*simp*]: *is-word* (*x*#*xs*) = (*is-terminal x* $\wedge$ *is-word xs*)
⟨*proof*⟩

lemma *count-terminals-of-word*: *is-word w* $\Longrightarrow$ *count-terminals w* = *length w*
⟨*proof*⟩

lemma *length-terminals*[*simp*]: *length* (*terminals p*) = *length p*
⟨*proof*⟩

lemma *path-length-is-upper-bound*:
assumes *p*: *wellformed-tokens p*
assumes α: *is-word* α
assumes *derives*: *derives* (α@*u*) (*terminals p*)
shows *length* α $\leq$ *length p*
⟨*proof*⟩

lemma *is-word-Derives1-index*:
assumes *w*: *is-word w*
assumes *derives1*: *Derives1* (*w*@*a*) *i r b*
shows *i* $\geq$ *length w*
⟨*proof*⟩

lemma *is-word-Derivation-derivation-ge*:
assumes *w*: *is-word w*
assumes *D*: *Derivation* (*w*@*a*) *D b*
shows *derivation-ge D* (*length w*)
⟨*proof*⟩

lemma *derives-word-is-prefix*:
assumes *w*: *is-word w*
assumes *derives*: *derives* (*w*@*a*) *b*
shows *is-prefix w b*
⟨*proof*⟩

lemma *terminals-take*[*simp*]: *terminals* (*take n p*) = *take n* (*terminals p*)
⟨*proof*⟩

lemma *terminals-drop*[*simp*]: *terminals* (*drop n p*) = *drop n* (*terminals p*)
⟨*proof*⟩

lemma *take-prefix*[*simp*]: *is-prefix a b* $\Longrightarrow$ *take* (*length a*) *b* = *a*
⟨*proof*⟩

lemma *Derives1-drop-prefixword*:
assumes *w*: *is-word w*
assumes *wa-b*: *Derives1* (*w*@*a*) *i r b*
shows *Derives1 a* (*i* − *length w*) *r* (*drop* (*length w*) *b*)
⟨*proof*⟩

lemma *derives1-drop-prefixword*:
assumes *w*: *is-word w*
assumes *wa-b*: *derives1* (*w*@*a*) *b*
shows *derives1 a* (*drop* (*length w*) *b*)
⟨*proof*⟩

lemma *derives1-is-word-is-prefix-drop*:
assumes *w*: *is-word w*
assumes *w-a*: *is-prefix w a*
assumes *ab*: *derives1 a b*
shows *derives1* (*drop* (*length w*) *a*) (*drop* (*length w*) *b*)
⟨*proof*⟩

lemma *derives-drop-prefixword-helper*:
derives a b $\Longrightarrow$ *is-word w* $\Longrightarrow$ *is-prefix w a* $\Longrightarrow$ *derives* (*drop* (*length w*) *a*) (*drop* (*length w*) *b*)
⟨*proof*⟩

lemma *derive-drop-prefixword*:
is-word w $\Longrightarrow$ *derives* (*w*@*a*) *b* $\Longrightarrow$ *derives a* (*drop* (*length w*) *b*)
⟨*proof*⟩

lemma *thmD2′*:
assumes *X*: *is-terminal X*
assumes *p*: *doc-tokens p*
assumes *pX*: (*terminals p*)@[*X*] ∈ $\mathcal{L}_P$
shows ∃ *x*. *pvalid p x* ∧ *next-symbol x* = *Some X*
⟨*proof*⟩

lemma *admissible-wellformed-tokens*: *admissible p* $\Longrightarrow$ *wellformed-tokens p*
⟨*proof*⟩

lemma *chars-append*[*simp*]: *chars* (*a*@*b*) = (*chars a*)@(*chars b*)
⟨*proof*⟩

lemma *chars-of-token-simp*[*simp*]: *chars-of-token* (*a*, *b*) = *b*
⟨*proof*⟩

lemma $\mathcal{X}$*-is-prefix*: *t* ∈ $\mathcal{X}$ *k* $\Longrightarrow$ *is-prefix* (*snd t*) (*drop k Doc*)
⟨*proof*⟩

lemma *is-prefix-append*: *is-prefix* (*a*@*b*) *D* = (*is-prefix a D* ∧ *is-prefix b* (*drop* (*length a*) *D*))
⟨*proof*⟩

lemma 𝔓*-are-doc-tokens*: *p* ∈ 𝔓 ⟹ *doc-tokens p*
⟨*proof*⟩

theorem *thmD2*:
assumes *X*: *is-terminal X*
assumes *p*: *p* ∈ 𝔓
assumes *pX*: (*terminals p*)@[*X*] ∈ $\mathcal{L}_P$
shows ∃ *x*. *pvalid p x* ∧ *next-symbol x* = *Some X*
⟨*proof*⟩

end

end
theory *TheoremD4*
imports *TheoremD2*
begin

context *LocalLexing* **begin**

lemma 𝒳*-are-terminals*: *u* ∈ 𝒳 *k* ⟹ *is-terminal* (*terminal-of-token u*)
⟨*proof*⟩

lemma *terminals-append*[*simp*]: *terminals* (*a*@*b*) = ((*terminals a*) @ (*terminals b*))
⟨*proof*⟩

lemma *terminals-singleton*[*simp*]: *terminals* [*u*] = [*terminal-of-token u*]
⟨*proof*⟩

lemma *terminal-of-token-simp*[*simp*]: *terminal-of-token* (*a*, *b*) = *a*
⟨*proof*⟩

lemma *pvalid-item-end*: *pvalid p x* ⟹ *item-end x* = *charslength p*
⟨*proof*⟩

lemma 𝒲*-elem-in-TokensAt*:
assumes *P*: *P* ⊆ 𝔓
assumes *u-in-*𝒲: *u* ∈ 𝒲 *P k*
shows *u* ∈ *TokensAt k* (*Gen P*)
⟨*proof*⟩

lemma *is-derivation-is-sentence*: *is-derivation s* ⟹ *is-sentence s*
⟨*proof*⟩

lemma *is-sentence-cons*: *is-sentence* $(N\#s) = (\textit{is-symbol}\ N \wedge \textit{is-sentence}\ s)$
⟨*proof*⟩

lemma *is-derivation-step*:
assumes *uNv*: *is-derivation* $(u@[N]@v)$
assumes $N\alpha$: $(N, \alpha) \in \mathfrak{R}$
shows *is-derivation* $(u@\alpha@v)$
⟨*proof*⟩

lemma *is-derivation-derives*:
derives $\alpha\ \beta \Longrightarrow$ *is-derivation* $(u@\alpha@v) \Longrightarrow$ *is-derivation* $(u@\beta@v)$
⟨*proof*⟩

lemma *item-rhs-split*: *item-rhs* $x = (\textit{item-}\alpha\ x)@(\textit{item-}\beta\ x)$
⟨*proof*⟩

lemma *pvalid-is-derivation-terminals-item-*β:
assumes *pvalid*: *pvalid* $p\ x$
shows $\exists\ \delta.$ *is-derivation* $((\textit{terminals}\ p)@(\textit{item-}\beta\ x)@\delta)$
⟨*proof*⟩

lemma *next-symbol-not-complete*: *next-symbol* $x = \textit{Some}\ t \Longrightarrow \neg\ (\textit{is-complete}\ x)$
⟨*proof*⟩

lemma *next-symbol-starts-item-*β:
assumes *wf*: *wellformed-item* x
assumes *next-symbol*: *next-symbol* $x = \textit{Some}\ t$
shows $\exists\ \delta.\ \textit{item-}\beta\ x = t\#\delta$
⟨*proof*⟩

lemma *pvalid-prefixlang*:
assumes *pvalid*: *pvalid* $p\ x$
assumes *is-terminal*: *is-terminal* t
assumes *next-symbol*: *next-symbol* $x = \textit{Some}\ t$
shows $(\textit{terminals}\ p)\ @\ [t] \in \mathcal{L}_P$
⟨*proof*⟩

lemma *TokensAt-elem-in-*$\mathcal{W}$:
assumes P: $P \subseteq \mathfrak{P}$
assumes *u-in-Tokens-at*: $u \in \textit{TokensAt}\ k\ (\textit{Gen}\ P)$
shows $u \in \mathcal{W}\ P\ k$
⟨*proof*⟩

theorem *thmD4*:
assumes P: $P \subseteq \mathfrak{P}$
shows $\mathcal{W}\ P\ k = \textit{TokensAt}\ k\ (\textit{Gen}\ P)$
⟨*proof*⟩

end

end
theory *TheoremD5*
imports *TheoremD4*
begin

context *LocalLexing* **begin**

lemma *Scan-empty*: *Scan* {} *k I* = *I*
⟨*proof*⟩

lemma *π-no-tokens*: *π k* {} *I* = *limit* (λ *I*. *Complete k* (*Predict k I*)) *I*
⟨*proof*⟩

lemma *bin-elem*: *x* ∈ *bin I k* ⟹ *x* ∈ *I*
⟨*proof*⟩

lemma *Gen-implies-pvalid*: *x* ∈ *Gen P* ⟹ ∃ *p* ∈ *P*. *pvalid p x*
⟨*proof*⟩

lemma *wellformed-init-item*[*simp*]: *r* ∈ ℜ ⟹ *k* ≤ *length Doc* ⟹ *wellformed-item* (*init-item r k*)
⟨*proof*⟩

lemma *init-item-origin*[*simp*]: *item-origin* (*init-item r k*) = *k*
⟨*proof*⟩

lemma *init-item-end*[*simp*]: *item-end* (*init-item r k*) = *k*
⟨*proof*⟩

lemma *init-item-nonterminal*[*simp*]: *item-nonterminal* (*init-item r k*) = *fst r*
⟨*proof*⟩

lemma *init-item-α*[*simp*]: *item-α* (*init-item r k*) = []
⟨*proof*⟩

lemma *Predict-elem-in-Gen*:
assumes *I-in-Gen-P*: *I* ⊆ *Gen P*
assumes *k*: *k* ≤ *length Doc*
assumes *x-in-Predict*: *x* ∈ *Predict k I*
shows *x* ∈ *Gen P*
⟨*proof*⟩

lemma *Predict-subset-Gen*:
assumes *I* ⊆ *Gen P*
assumes *k* ≤ *length Doc*
shows *Predict k I* ⊆ *Gen P*
⟨*proof*⟩

lemma *nth-superfluous-append*[*simp*]: $i < \mathit{length}\ a \Longrightarrow (a@b)!i = a!i$
⟨*proof*⟩

lemma *tokens-nth-in-*$\mathcal{Z}$:
$p \in \mathfrak{P} \Longrightarrow \forall\ i.\ i < \mathit{length}\ p \longrightarrow (\exists\ u.\ p\ !\ i \in \mathcal{Z}\ (\mathit{charslength}\ (\mathit{take}\ i\ p))\ u)$
⟨*proof*⟩

lemma *path-append-token*:
assumes *p*: $p \in \mathcal{P}\ k\ u$
assumes *t*: $t \in \mathcal{Z}\ k\ (\mathit{Suc}\ u)$
assumes *pt*: $\mathit{admissible}\ (p@[t])$
assumes *k*: $\mathit{charslength}\ p = k$
shows $p@[t] \in \mathcal{P}\ k\ (\mathit{Suc}\ u)$
⟨*proof*⟩

definition *indexlt-rel* :: $((\mathit{nat} \times \mathit{nat}) \times (\mathit{nat} \times \mathit{nat}))\ \mathit{set}$ **where**
$\mathit{indexlt\text{-}rel} = \mathit{less\text{-}than} <\!*\mathit{lex}*\!>\ \mathit{less\text{-}than}$

definition *indexlt* :: $\mathit{nat} \Rightarrow \mathit{nat} \Rightarrow \mathit{nat} \Rightarrow \mathit{nat} \Rightarrow \mathit{bool}$ **where**
$\mathit{indexlt}\ k'\ u'\ k\ u = (((k',\ u'),\ (k,\ u)) \in \mathit{indexlt\text{-}rel})$

lemma *indexlt-simp*: $\mathit{indexlt}\ k'\ u'\ k\ u = (k' < k \vee (k' = k \wedge u' < u))$
⟨*proof*⟩

lemma *wf-indexlt-rel*: *wf indexlt-rel*
⟨*proof*⟩

lemma $\mathcal{P}$*-induct*[*consumes 1*, *case-names Induct*]:
assumes $p \in \mathcal{P}\ k\ u$
assumes *induct*: $\bigwedge\ p\ k\ u\ .\ (\bigwedge\ p'\ k'\ u'.\ p' \in \mathcal{P}\ k'\ u' \Longrightarrow \mathit{indexlt}\ k'\ u'\ k\ u \Longrightarrow P\ p'\ k'\ u')$
$\Longrightarrow p \in \mathcal{P}\ k\ u \Longrightarrow P\ p\ k\ u$
shows $P\ p\ k\ u$
⟨*proof*⟩

lemma *nonempty-path-indices*:
assumes *p*: $p \in \mathcal{P}\ k\ u$
assumes *nonempty*: $p \neq []$
shows $k > 0 \vee u > 0$
⟨*proof*⟩

lemma *base-paths*:
assumes *p*: $p \in \mathcal{P}\ k\ 0$
assumes *k*: $k > 0$
shows $\exists\ u.\ p \in \mathcal{P}\ (k - 1)\ u$
⟨*proof*⟩

lemma *indexlt-trans*: $\mathit{indexlt}\ k''\ u''\ k'\ u' \Longrightarrow \mathit{indexlt}\ k'\ u'\ k\ u \Longrightarrow \mathit{indexlt}\ k''\ u''\ k\ u$

⟨*proof*⟩

definition *is-continuation* :: *nat* ⇒ *nat* ⇒ *tokens* ⇒ *tokens* ⇒ *bool* **where**
is-continuation k u q ts = (*q* ∈ $\mathcal{P}$ *k u* ∧ *charslength q* = *k* ∧ *admissible* (*q*@*ts*)
∧
(∀ *t* ∈ *set ts*. *t* ∈ $\mathcal{Z}$ *k* (*Suc u*)) ∧ (∀ *t* ∈ *set* (*butlast ts*). *chars-of-token t* = []))

lemma *limit-Append-path-nonelem-split*: *p* ∈ *limit* (*Append T k*) ($\mathcal{P}$ *k u*) ⟹ *p* ∉
$\mathcal{P}$ *k u* ⟹
∃ *q ts*. *p* = *q*@*ts* ∧ *q* ∈ $\mathcal{P}$ *k u* ∧ *charslength q* = *k* ∧ *admissible* (*q*@*ts*) ∧ (∀ *t*
∈ *set ts*. *t* ∈ *T*) ∧
(∀ *t* ∈ *set* (*butlast ts*). *chars-of-token t* = [])
⟨*proof*⟩

lemma *limit-Append-path-nonelem-split′*:
p ∈ *limit* (*Append* ($\mathcal{Z}$ *k* (*Suc u*)) *k*) ($\mathcal{P}$ *k u*) ⟹ *p* ∉ $\mathcal{P}$ *k u* ⟹
∃ *q ts*. *p* = *q*@*ts* ∧ *is-continuation k u q ts*
⟨*proof*⟩

lemma *final-step-of-path*: *p* ∈ $\mathcal{P}$ *k u* ⟹ *p* ≠ [] ⟹ (∃ *q ts k′ u′*. *p* = *q*@*ts* ∧
indexlt k′ u′ k u
∧ *is-continuation k′ u′ q ts*)
⟨*proof*⟩

lemma *terminals-empty*[*simp*]: *terminals* [] = []
⟨*proof*⟩

lemma *empty-in-*$\mathcal{L}_P$[*simp*]: [] ∈ $\mathcal{L}_P$
⟨*proof*⟩

lemma *admissible-empty*[*simp*]: *admissible* []
⟨*proof*⟩

lemma $\mathfrak{P}$*-are-admissible*: *p* ∈ $\mathfrak{P}$ ⟹ *admissible p*
⟨*proof*⟩

lemma *prefix-of-empty-is-empty*: *is-prefix q* [] ⟹ *q* = []
⟨*proof*⟩

lemma *subset-*$\mathcal{P}$:
assumes *leq*: *k′* < *k* ∨ (*k′* = *k* ∧ *u′* ≤ *u*)
shows $\mathcal{P}$ *k′ u′* ⊆ $\mathcal{P}$ *k u*
⟨*proof*⟩

lemma *empty-path-is-elem*[*simp*]: [] ∈ $\mathcal{P}$ *k u*
⟨*proof*⟩

lemma *is-prefix-of-append*:
assumes *is-prefix p* (*a*@*b*)

shows *is-prefix p a* $\vee$ ($\exists$ *b'*. *b'* $\neq$ [] $\wedge$ *is-prefix b' b* $\wedge$ *p* = *a*@*b'*)
⟨*proof*⟩

lemma *prefix-is-continuation*: *is-continuation k u p ts* $\Longrightarrow$ *is-prefix ts' ts* $\Longrightarrow$
is-continuation k u p ts'
⟨*proof*⟩

lemma *charslength-0*: ($\forall$ *t* $\in$ *set ts*. *chars-of-token t* = []) = (*charslength ts* = *0*)
⟨*proof*⟩

lemma *is-continuation-in-*$\mathcal{P}$: *is-continuation k u p ts* $\Longrightarrow$ *p*@*ts* $\in$ $\mathcal{P}$ *k* (*Suc u*)
⟨*proof*⟩

lemma *indexlt-subset-*$\mathcal{P}$: *indexlt k' u' k u* $\Longrightarrow$ $\mathcal{P}$ *k'* (*Suc u'*) $\subseteq$ $\mathcal{P}$ *k u*
⟨*proof*⟩

lemma *prefixes-are-paths*: *p* $\in$ $\mathcal{P}$ *k u* $\Longrightarrow$ *is-prefix x p* $\Longrightarrow$ *x* $\in$ $\mathcal{P}$ *k u*
⟨*proof*⟩

lemma *empty-or-last-of-suffix*:
assumes *q* = *q'* @ [*t*]
assumes *q* = *p* @ *ts*
shows *ts* = [] $\vee$ ($\exists$ *ts'*. *q'* = *p* @ *ts'* $\wedge$ *ts'*@[*t*] = *ts*)
⟨*proof*⟩

lemma *is-prefix-butlast*: *is-prefix q* (*butlast p*) $\Longrightarrow$ *is-prefix q p*
⟨*proof*⟩

lemma *last-step-of-path*:
q $\in$ $\mathcal{P}$ *k u* $\Longrightarrow$ *q* = *q'*@[*t*] $\Longrightarrow$
$\exists$ *k' u'*. *indexlt k' u' k u* $\wedge$ *q* $\in$ $\mathcal{P}$ *k'* (*Suc u'*) $\wedge$ *charslength q'* = *k'* $\wedge$ *t* $\in$ $\mathcal{Z}$ *k'*
(*Suc u'*)
⟨*proof*⟩

lemma *charslength-of-butlast-0*: *p* $\in$ $\mathcal{P}$ *k 0* $\Longrightarrow$ *p* = *q*@[*t*] $\Longrightarrow$ *charslength q* < *k*
⟨*proof*⟩

lemma *charslength-of-butlast*: *p* $\in$ $\mathcal{P}$ *k u* $\Longrightarrow$ *p* = *q*@[*t*] $\Longrightarrow$ *charslength q* $\leq$ *k*
⟨*proof*⟩

lemma *last-token-of-path*:
assumes *q* $\in$ $\mathcal{P}$ *k u*
assumes *q* = *q'*@[*t*]
assumes *charslength q'* = *k*
shows *t* $\in$ $\mathcal{Z}$ *k u*
⟨*proof*⟩

lemma *final-step-of-path'*: *p* $\in$ $\mathcal{P}$ *k u* $\Longrightarrow$ *p* $\notin$ $\mathcal{P}$ *k* (*u* − *1*) $\Longrightarrow$
$\exists$ *q ts*. *u* > *0* $\wedge$ *p* = *q*@*ts* $\wedge$ *is-continuation k* (*u* − *1*) *q ts*

⟨*proof*⟩

lemma *is-continuation-continue*:
assumes *is-continuation k u q ts*
assumes *charslength ts = 0*
assumes $t \in \mathcal{Z}\ k\ (Suc\ u)$
assumes *admissible* $(q\ @\ ts\ @\ [t])$
shows *is-continuation k u q* $(ts@[t])$
⟨*proof*⟩

theorem *compatibility-def*:
assumes *p-in-dom*: $p \in \mathcal{P}\ k\ u$
assumes *q-in-dom*: $q \in \mathcal{P}\ k\ u$
assumes *p-charslength*: *charslength p = k*
assumes *q-split*: $q = q'@[t]$
assumes $q'len$: *charslength* $q' = k$
assumes *admissible*: *admissible* $(p\ @\ [t])$
shows $p\ @\ [t] \in \mathcal{P}\ k\ u$
⟨*proof*⟩

lemma *is-prefix-admissible*:
assumes *is-prefix a b*
assumes *admissible b*
shows *admissible a*
⟨*proof*⟩

lemma *butlast-split*: $n < length\ q \Longrightarrow butlast\ q = (take\ n\ q)@(drop\ n\ (butlast\ q))$
⟨*proof*⟩

lemma *in-$\mathcal{P}$-charslength*:
assumes *p-dom*: $p \in \mathcal{P}\ k\ u$
shows $\exists\ v.\ p \in \mathcal{P}\ (charslength\ p)\ v$
⟨*proof*⟩

theorem *general-compatibility*:
$p \in \mathcal{P}\ k\ u \Longrightarrow q \in \mathcal{P}\ k\ u \Longrightarrow charslength\ p = charslength\ (take\ n\ q)$
$\Longrightarrow charslength\ p \leq k \Longrightarrow admissible\ (p\ @\ (drop\ n\ q)) \Longrightarrow p\ @\ (drop\ n\ q) \in \mathcal{P}\ k\ u$
⟨*proof*⟩

lemma *wellformed-item-derives*:
assumes *wellformed*: *wellformed-item x*
shows *derives* [*item-nonterminal x*] (*item-rhs x*)
⟨*proof*⟩

lemma *wellformed-complete-item-β*:
assumes *wellformed*: *wellformed-item x*

assumes *complete*: *is-complete x*
shows *item-β x* = []
⟨*proof*⟩

lemma *wellformed-complete-item-derives*:
assumes *wellformed*: *wellformed-item x*
assumes *complete*: *is-complete x*
shows *derives* [*item-nonterminal x*] (*item-α x*)
⟨*proof*⟩

lemma *is-derivation-implies-admissible*:
is-derivation (*terminals p* @ *δ*) ⟹ *is-word* (*terminals p*) ⟹ *admissible p*
⟨*proof*⟩

lemma *item-rhs-of-inc-item*[*simp*]: *item-rhs* (*inc-item x k*) = *item-rhs x*
⟨*proof*⟩

lemma *item-rule-of-inc-item*[*simp*]: *item-rule* (*inc-item x k*) = *item-rule x*
⟨*proof*⟩

lemma *item-origin-of-inc-item*[*simp*]: *item-origin* (*inc-item x k*) = *item-origin x*
⟨*proof*⟩

lemma *item-end-of-inc-item*[*simp*]: *item-end* (*inc-item x k*) = *k*
⟨*proof*⟩

lemma *item-dot-of-inc-item*[*simp*]: *item-dot* (*inc-item x k*) = (*item-dot x*) + *1*
⟨*proof*⟩

lemma *item-nonterminal-of-inc-item*[*simp*]: *item-nonterminal* (*inc-item x k*) = *item-nonterminal x*
⟨*proof*⟩

lemma *wellformed-inc-item*:
assumes *wellformed*: *wellformed-item x*
assumes *next-symbol*: *next-symbol x* = *Some s*
assumes *k-upper-bound*: *k* ≤ *length Doc*
assumes *k-lower-bound*: *k* ≥ *item-end x*
shows *wellformed-item* (*inc-item x k*)
⟨*proof*⟩

lemma *item-α-of-inc-item*:
assumes *wellformed*: *wellformed-item x*
assumes *next-symbol*: *next-symbol x* = *Some s*
shows *item-α* (*inc-item x k*) = *item-α x* @ [*s*]
⟨*proof*⟩

lemma *derives1-pad*:
assumes *derives1*: *derives1 α β*

assumes *u*: *is-sentence u*
assumes *v*: *is-sentence v*
shows *derives1* (*u*@*α*@*v*) (*u*@*β*@*v*)
⟨*proof*⟩

lemma *derives-pad*:
derives α β ⟹ *is-sentence u* ⟹ *is-sentence v* ⟹ *derives* (*u*@*α*@*v*) (*u*@*β*@*v*)
⟨*proof*⟩

lemma *derives1-is-sentence*: *derives1 α β* ⟹ *is-sentence α* ∧ *is-sentence β*
⟨*proof*⟩

lemma *derives-is-sentence*: *derives α β* ⟹ (*α* = *β*) ∨ (*is-sentence α* ∧ *is-sentence β*)
⟨*proof*⟩

lemma *derives-append*:
assumes *au*: *derives a u*
assumes *bv*: *derives b v*
assumes *is-sentence-a*: *is-sentence a*
assumes *is-sentence-b*: *is-sentence b*
shows *derives* (*a*@*b*) (*u*@*v*)
⟨*proof*⟩

lemma *is-sentence-item-α*: *wellformed-item x* ⟹ *is-sentence* (*item-α x*)
⟨*proof*⟩

lemma *is-nonterminal-item-nonterminal*: *wellformed-item x* ⟹ *is-nonterminal* (*item-nonterminal x*)
⟨*proof*⟩

lemma *Complete-elem-in-Gen*:
assumes *I-in-Gen*: *I* ⊆ *Gen* (*P k u*)
assumes *k*: *k* ≤ *length Doc*
assumes *x-in-Complete*: *x* ∈ *Complete k I*
shows *x* ∈ *Gen* (*P k u*)
⟨*proof*⟩

lemma *Complete-subset-Gen*:
assumes *I-in-Gen-P*: *I* ⊆ *Gen* (*P k u*)
assumes *k*: *k* ≤ *length Doc*
shows *Complete k I* ⊆ *Gen* (*P k u*)
⟨*proof*⟩

lemma *P-are-admissible*: *p* ∈ *P k u* ⟹ *admissible p*
⟨*proof*⟩

lemma *is-continuation-base*:
assumes *p-dom*: *p* ∈ *P k u*

assumes *charslength-p*: $charslength\ p = k$
shows *is-continuation* $k\ u\ p\ []$
⟨*proof*⟩

lemma *is-continuation-empty-chars*:
$is\text{-}continuation\ k\ u\ q\ ts \Longrightarrow charslength\ (q@ts) = k \Longrightarrow chars\ ts = []$
⟨*proof*⟩

lemma $\mathcal{Z}$*-subset*: $u \leq v \Longrightarrow \mathcal{Z}\ k\ u \subseteq \mathcal{Z}\ k\ v$
⟨*proof*⟩

lemma *is-continuation-increase-u*:
assumes *cont*: $is\text{-}continuation\ k\ u\ q\ ts$
assumes *uv*: $u \leq v$
shows $is\text{-}continuation\ k\ v\ q\ ts$
⟨*proof*⟩

lemma *pvalid-next-symbol-derivable*:
assumes *pvalid*: $pvalid\ p\ x$
assumes *next-symbol*: $next\text{-}symbol\ x = Some\ s$
shows $\exists\ \delta.\ is\text{-}derivation((terminals\ p)@[s]@\delta)$
⟨*proof*⟩

lemma *pvalid-admissible*:
assumes *pvalid*: $pvalid\ p\ x$
shows $admissible\ p$
⟨*proof*⟩

lemma *pvalid-next-terminal-admissible*:
assumes *pvalid*: $pvalid\ p\ x$
assumes *next-symbol*: $next\text{-}symbol\ x = Some\ t$
assumes *terminal*: $is\text{-}terminal\ t$
shows $admissible\ (p@[(t,\ c)])$
⟨*proof*⟩

lemma $\mathcal{X}$*-wellformed*: $t \in \mathcal{X}\ k \Longrightarrow wellformed\text{-}token\ t$
⟨*proof*⟩

lemma $\mathcal{Z}$*-wellformed*: $t \in \mathcal{Z}\ k\ u \Longrightarrow wellformed\text{-}token\ t$
⟨*proof*⟩

lemma *Scan-elem-in-Gen*:
assumes *I-in-Gen*: $I \subseteq Gen\ (\mathcal{P}\ k\ u)$
assumes *k*: $k \leq length\ Doc$
assumes *T*: $T \subseteq \mathcal{Z}\ k\ u$
assumes *x-in-Scan*: $x \in Scan\ T\ k\ I$
shows $x \in Gen\ (\mathcal{P}\ k\ u)$
⟨*proof*⟩

lemma *Scan-subset-Gen*:
assumes *I-in-Gen*: $I \subseteq Gen\ (\mathcal{P}\ k\ u)$
assumes *k*: $k \leq length\ Doc$
assumes *T*: $T \subseteq \mathcal{Z}\ k\ u$
shows $Scan\ T\ k\ I \subseteq Gen\ (\mathcal{P}\ k\ u)$
⟨*proof*⟩

theorem *thmD5*:
assumes *I*: $I \subseteq Gen\ (\mathcal{P}\ k\ u)$
assumes *k*: $k \leq length\ Doc$
assumes *T*: $T \subseteq \mathcal{Z}\ k\ u$
shows $\pi\ k\ T\ I \subseteq Gen\ (\mathcal{P}\ k\ u)$
⟨*proof*⟩

end

end
theory *TheoremD6*
imports *TheoremD5*
begin

context *LocalLexing* **begin**

definition *inc-dot* :: $nat \Rightarrow item \Rightarrow item$
where
$inc\text{-}dot\ d\ x = Item\ (item\text{-}rule\ x)\ (item\text{-}dot\ x + d)\ (item\text{-}origin\ x)\ (item\text{-}end\ x)$

lemma *inc-dot-0*[*simp*]: $inc\text{-}dot\ 0\ x = x$
⟨*proof*⟩

lemma *Predict-mk-regular1*:
$\exists\ (P :: rule \Rightarrow item \Rightarrow bool)\ F.\ Predict\ k = mk\text{-}regular1\ P\ F$
⟨*proof*⟩

lemma *Complete-mk-regular2*:
$\exists\ (P :: dummy \Rightarrow item \Rightarrow item \Rightarrow bool)\ F.\ Complete\ k = mk\text{-}regular2\ P\ F$
⟨*proof*⟩

lemma *Scan-mk-regular1*:
$\exists\ (P :: token \Rightarrow item \Rightarrow bool)\ F.\ Scan\ T\ k = mk\text{-}regular1\ P\ F$
⟨*proof*⟩

lemma *Predict-regular*: $regular\ (Predict\ k)$
⟨*proof*⟩

lemma *Complete-regular*: $regular\ (Complete\ k)$
⟨*proof*⟩

lemma *Scan-regular*: $regular\ (Scan\ T\ k)$

⟨*proof*⟩

lemma *π-functional*: $\pi\ k\ T = \mathit{limit}\ ((\mathit{Scan}\ T\ k)\ o\ (\mathit{Complete}\ k)\ o\ (\mathit{Predict}\ k))$
⟨*proof*⟩

lemma *π-step-regular*: $\mathit{regular}\ ((\mathit{Scan}\ T\ k)\ o\ (\mathit{Complete}\ k)\ o\ (\mathit{Predict}\ k))$
⟨*proof*⟩

lemma *π-regular*: $\mathit{regular}\ (\pi\ k\ T)$
⟨*proof*⟩

lemma *π-fix*: $\mathit{Scan}\ T\ k\ (\mathit{Complete}\ k\ (\mathit{Predict}\ k\ (\pi\ k\ T\ I))) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *π-fix′*: $((\mathit{Scan}\ T\ k)\ o\ (\mathit{Complete}\ k)\ o\ (\mathit{Predict}\ k))\ (\pi\ k\ T\ I) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *setmonotone-cases*:
assumes *setmonotone f*
shows $f\ X = X \vee X \subset f\ X$
⟨*proof*⟩

lemma *distribute-fixpoint-over-setmonotone-comp*:
assumes *f*: *setmonotone f*
assumes *g*: *setmonotone g*
assumes *fixpoint*: $(f\ o\ g)\ I = I$
shows $f\ I = I \wedge g\ I = I$
⟨*proof*⟩

lemma *distribute-fixpoint-over-setmonotone-comp-3*:
assumes *f*: *setmonotone f*
assumes *g*: *setmonotone g*
assumes *h*: *setmonotone h*
assumes *fixpoint*: $(f\ o\ g\ o\ h)\ I = I$
shows $f\ I = I \wedge g\ I = I \wedge h\ I = I$
⟨*proof*⟩

lemma *Predict-π-fix*: $\mathit{Predict}\ k\ (\pi\ k\ T\ I) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *Scan-π-fix*: $\mathit{Scan}\ T\ k\ (\pi\ k\ T\ I) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *Complete-π-fix*: $\mathit{Complete}\ k\ (\pi\ k\ T\ I) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *π-idempotent*: $\pi\ k\ T\ (\pi\ k\ T\ I) = \pi\ k\ T\ I$
⟨*proof*⟩

lemma *derivation-shift-identity*[*simp*]: *derivation-shift D 0 0 = D*
⟨*proof*⟩

lemma *Derivation-skip-prefix*: *Derivation* (*u*@*v*) *D w* ⟹ *derivation-ge D* (*length u*) ⟹
Derivation v (*derivation-shift D* (*length u*) *0*) (*drop* (*length u*) *w*)
⟨*proof*⟩

lemma *leftmost-skip-prefix*: *leftmost i* (*u*@*v*) ⟹ *i* ≥ *length u* ⟹ *leftmost* (*i* − *length u*) *v*
⟨*proof*⟩

lemma *LeftDerivation-skip-prefix*: *LeftDerivation* (*u*@*v*) *D w* ⟹ *derivation-ge D* (*length u*) ⟹
LeftDerivation v (*derivation-shift D* (*length u*) *0*) (*drop* (*length u*) *w*)
⟨*proof*⟩

lemma *splits-at-append*: *splits-at u i u1 N u2* ⟹ *splits-at* (*u*@*v*) *i u1 N* (*u2*@*v*)
⟨*proof*⟩

lemma *LeftDerives1-append-leftmost-unique*: *LeftDerives1* (*a*@*b*) *i r c* ⟹ *leftmost j a* ⟹ *i* = *j*
⟨*proof*⟩

lemma *drop-derivation-shift*:
drop n (*derivation-shift D left right*) = *derivation-shift* (*drop n D*) *left right*
⟨*proof*⟩

lemma *take-derivation-shift*:
take n (*derivation-shift D left right*) = *derivation-shift* (*take n D*) *left right*
⟨*proof*⟩

lemma *derivation-shift-0-shift*: *derivation-shift* (*derivation-shift D left1 0*) *left2 right2* =
derivation-shift D (*left1* + *left2*) *right2*
⟨*proof*⟩

lemma *splits-at-append-prefix*:
splits-at v i α *N* β ⟹ *splits-at* (*u*@*v*) (*i* + *length u*) (*u*@α) *N* β
⟨*proof*⟩

lemma *splits-at-implies-Derives1*: *splits-at* δ *i* α *N* β ⟹ *is-sentence* δ ⟹ *r*∈ ℜ
⟹ *fst r* = *N*
⟹ *Derives1* δ *i r* (α@(*snd r*)@β)
⟨*proof*⟩

lemma *Derives1-append-prefix*:
assumes *Derives1*: *Derives1 v i r w*
assumes *u*: *is-sentence u*

shows *Derives1* $(u@v)$ $(i + length\ u)$ r $(u@w)$
⟨*proof*⟩

lemma *leftmost-prepend-word*: *leftmost i v* $\Longrightarrow$ *is-word u* $\Longrightarrow$ *leftmost* $(i + length\ u)$ $(u@v)$
⟨*proof*⟩

lemma *LeftDerives1-append-prefix*:
assumes *Derives1*: *LeftDerives1 v i r w*
assumes *u*: *is-word u*
shows *LeftDerives1* $(u@v)$ $(i + length\ u)$ r $(u@w)$
⟨*proof*⟩

lemma *Derivation-append-prefix*: *Derivation v D w* $\Longrightarrow$ *is-sentence u* $\Longrightarrow$
Derivation $(u@v)$ $(derivation\text{-}shift\ D\ 0\ (length\ u))$ $(u@w)$
⟨*proof*⟩

lemma *LeftDerivation-append-prefix*: *LeftDerivation v D w* $\Longrightarrow$ *is-word u* $\Longrightarrow$
LeftDerivation $(u@v)$ $(derivation\text{-}shift\ D\ 0\ (length\ u))$ $(u@w)$
⟨*proof*⟩

lemma *derivation-ge-shift-simp*: *derivation-ge D i* $\Longrightarrow$ $i \geq l$ $\Longrightarrow$ $r \geq l$ $\Longrightarrow$
derivation-shift D l r = *derivation-shift D 0* $(r - l)$
⟨*proof*⟩

lemma *append-dropped-prefix*: *is-prefix u v* $\Longrightarrow$ *drop* $(length\ u)$ $v = w$ $\Longrightarrow$ $u@w = v$
⟨*proof*⟩

lemma *derivation-ge-shift-plus*:
assumes *derivation-ge D u*
assumes *derivation-ge* $(derivation\text{-}shift\ D\ u\ 0)$ *v*
shows *derivation-ge D* $(u + v)$
⟨*proof*⟩

lemma *LeftDerivation-breakdown*:
LeftDerivation $(u@v)$ *D w* $\Longrightarrow$ $\exists\ n\ w1\ w2.\ w = w1\ @\ w2\ \wedge$
LeftDerivation u $(take\ n\ D)$ *w1* $\wedge$
derivation-ge $(drop\ n\ D)$ $(length\ w1)$ $\wedge$
LeftDerivation v $(derivation\text{-}shift\ (drop\ n\ D)\ (length\ w1)\ 0)$ *w2*
⟨*proof*⟩

lemma *Derives1-terminals-stay*:
assumes *Derives1*: *Derives1 u i r v*
assumes *t-dom*: $t \in set\ u$
assumes *terminal*: *is-terminal t*
shows $t \in set\ v$
⟨*proof*⟩

lemma *Derivation-terminals-stay*: *Derivation u D v* $\Longrightarrow$ *t* $\in$ *set u* $\Longrightarrow$ *is-terminal t* $\Longrightarrow$ *t* $\in$ *set v*
⟨*proof*⟩

lemma *Derivation-empty-no-terminals*: *Derivation u D* [] $\Longrightarrow$ *t* $\in$ *set u* $\Longrightarrow$ *is-nonterminal t*
⟨*proof*⟩

lemma *mono-subset-elem*: *mono f* $\Longrightarrow$ *A* $\subseteq$ *B* $\Longrightarrow$ *x* $\in$ *f A* $\Longrightarrow$ *x* $\in$ *f B* ⟨*proof*⟩

lemma *wellformed-inc-dot*: *wellformed-item x* $\Longrightarrow$ *item-dot x* + *d* $\leq$ *length* (*item-rhs x*) $\Longrightarrow$
wellformed-item(*inc-dot d x*)
⟨*proof*⟩

lemma *init-item-dot*[*simp*]: *item-dot* (*init-item r k*) = *0*
⟨*proof*⟩

lemma *init-item-rhs*[*simp*]: *item-rhs* (*init-item r k*) = *snd r*
⟨*proof*⟩

lemma *init-item-β*[*simp*]: *item-β* (*init-item r k*) = *snd r*
⟨*proof*⟩

lemma *mono-π*: *mono* (*π k T*)
⟨*proof*⟩

lemma *π-subset-elem-trans*:
assumes *Y*: *Y* $\subseteq$ *π k T X*
assumes *z*: *z* $\in$ *π k T Y*
shows *z* $\in$ *π k T X*
⟨*proof*⟩

lemma *inc-dot-origin*[*simp*]: *item-origin* (*inc-dot d x*) = *item-origin x*
⟨*proof*⟩

lemma *inc-dot-end*[*simp*]: *item-end* (*inc-dot d x*) = *item-end x*
⟨*proof*⟩

lemma *inc-dot-rhs*[*simp*]: *item-rhs* (*inc-dot d x*) = *item-rhs x*
⟨*proof*⟩

lemma *inc-dot-dot*[*simp*]: *item-dot* (*inc-dot d x*) = *item-dot x* + *d*
⟨*proof*⟩

lemma *inc-dot-nonterminal*[*simp*]: *item-nonterminal* (*inc-dot d x*) = *item-nonterminal x*
⟨*proof*⟩

lemma *Predict-subset-π*: *Predict k X* ⊆ *π k T X*
⟨*proof*⟩

lemma *Complete-subset-π*: *Complete k X* ⊆ *π k T X*
⟨*proof*⟩

lemma *inc-inc-dot*[*simp*]: *inc-dot a* (*inc-dot b x*) = *inc-dot* (*a* + *b*) *x*
⟨*proof*⟩

lemma *thmD6-Left*: *wellformed-item x* ⟹ *item-β x* = *δ* @ *ω* ⟹ *item-end x* = *k* ⟹
LeftDerivation δ D [] ⟹ *inc-dot* (*length δ*) *x* ∈ *π k* {} {*x*}
⟨*proof*⟩

lemma *derives-empty-implies-LeftDerivation*: *derives δ* [] ⟹ ∃ *D*. *LeftDerivation δ D* []
⟨*proof*⟩

lemma *thmD6*: *wellformed-item x* ⟹ *item-β x* = *δ* @ *ω* ⟹ *item-end x* = *k* ⟹

derives δ [] ⟹ *inc-dot* (*length δ*) *x* ∈ *π k* {} {*x*}
⟨*proof*⟩

end

end
theory *TheoremD7*
imports *TheoremD6*
begin

context *LocalLexing* **begin**

lemma *Derives1-keep-first-terminal*: *Derives1* (*x*#*u*) *i r* (*y*#*v*) ⟹ *is-terminal x* ⟹ *x* = *y*
⟨*proof*⟩

lemma *Derives1-nonterminal-head*:
assumes *Derives1 u i r* (*N*#*v*)
assumes *is-nonterminal N*
shows ∃ *u′ M*. *u* = *M*#*u′* ∧ *is-nonterminal M*
⟨*proof*⟩

lemma *sentence-starts-with-nonterminal*:
assumes *is-nonterminal N*
assumes *derives u* []
shows ∃ *X r*. *u*@[*N*] = *X*#*r* ∧ *is-nonterminal X*
⟨*proof*⟩

lemma *Derives1-nonterminal-head′*:

assumes *Derives1 u i r* (*v1*@[*N*]@*v2*)
assumes *is-nonterminal N*
assumes *derives v1* []
shows ∃ *u′ M*. *u* = *M*#*u′* ∧ *is-nonterminal M*
⟨*proof*⟩

lemma *thmD7-helper*:
assumes *LeftDerivation* [𝔖] *D* (*N*#*v*)
assumes *is-nonterminal N*
assumes 𝔖 ≠ *N*
shows ∃ *n M a a1 a2 w*. *n* < *length D* ∧ (*M*, *a*) ∈ ℜ ∧ *LeftDerivation* [𝔖] (*take n D*) (*M*#*w*) ∧
a = *a1* @ [*N*] @ *a2* ∧ *derives a1* []
⟨*proof*⟩

lemma *head-of-item-β-is-next-symbol*:
wellformed-item x ⟹ *item-β x* = *t*#*δ* ⟹ *next-symbol x* = *Some t*
⟨*proof*⟩

lemma *next-symbol-predicts*: *next-symbol x* = *Some N* ⟹ (*N*, *a*) ∈ ℜ ⟹ *k* = *item-end x* ⟹
init-item (*N*, *a*) *k* ∈ *Predict k* {*x*}
⟨*proof*⟩

lemma *thmD7-LeftDerivation*: *LeftDerivation* [𝔖] *D* (*N*#*γ*) ⟹ *is-nonterminal N* ⟹ (*N*, *α*) ∈ ℜ ⟹
init-item (*N*, *α*) *0* ∈ *π 0* {} *Init*
⟨*proof*⟩

theorem *thmD7*: *is-derivation* (*N*#*γ*) ⟹ *is-nonterminal N* ⟹ (*N*, *α*) ∈ ℜ ⟹

init-item (*N*, *α*) *0* ∈ *π 0* {} *Init*
⟨*proof*⟩

end

end
theory *TheoremD8*
imports *TheoremD7*
begin

context *LocalLexing* **begin**

lemma *wellformed-tokens-empty-path*[*simp*]: *wellformed-tokens* []
⟨*proof*⟩

lemma *P-0-0-Gen*: *Gen* (*P 0 0*) = { *x* . *wellformed-item x* ∧ *item-origin x* = *0* ∧ *item-end x* = *0* ∧
derives (*item-α x*) [] ∧ (∃ *γ*. *is-derivation* ([*item-nonterminal x*] @ *γ*)) }

⟨*proof*⟩

lemma *Init-subset-Gen*: *Init* ⊆ *Gen* (*P* *0* *0*)
⟨*proof*⟩

lemma *J-0-0-subset-Gen*: *J* *0* *0* ⊆ *Gen* (*P* *0* *0*)
⟨*proof*⟩

lemma *inc-dot-rule*[*simp*]: *item-rule* (*inc-dot* *d* *x*) = *item-rule* *x*
⟨*proof*⟩

lemma *init-item-rule*[*simp*]: *item-rule* (*init-item* *r* *k*) = *r*
⟨*proof*⟩

lemma *item-dot-is-α-length*: *wellformed-item* *x* ⟹ *item-dot* *x* = *length* (*item-α* *x*)
⟨*proof*⟩

lemma *Gen-subset-J-0-0-helper*:
assumes *wellformed-item* *x*
assumes *item-origin* *x* = *0*
assumes *item-end* *x* = *0*
assumes *derives* (*item-α* *x*) []
assumes *is-derivation* (*item-nonterminal* *x* # *γ*)
shows *x* ∈ *π* *0* {} *Init*
⟨*proof*⟩

lemma *Gen-subset-J-0-0*: *Gen* (*P* *0* *0*) ⊆ *J* *0* *0*
⟨*proof*⟩

theorem *thmD8*: *J* *0* *0* = *Gen* (*P* *0* *0*)
⟨*proof*⟩

end

end
theory *TheoremD9*
imports *TheoremD8*
begin

context *LocalLexing* **begin**

definition *items-le* :: *nat* ⇒ *items* ⇒ *items*
where
items-le *k* *I* = { *x* . *x* ∈ *I* ∧ *item-end* *x* ≤ *k* }

definition *items-eq* :: *nat* ⇒ *items* ⇒ *items*
where
items-eq *k* *I* = { *x* . *x* ∈ *I* ∧ *item-end* *x* = *k* }

definition *paths-le* :: *nat* ⇒ *tokens set* ⇒ *tokens set*
where
paths-le k P = { *p* . *p* ∈ *P* ∧ *charslength p* ≤ *k* }

definition *paths-eq* :: *nat* ⇒ *tokens set* ⇒ *tokens set*
where
paths-eq k P = { *p* . *p* ∈ *P* ∧ *charslength p* = *k* }

lemma *items-le-pointwise*: *pointwise* (*items-le k*)
⟨*proof*⟩

lemma *items-le-is-filter*: *items-le k I* ⊆ *I*
⟨*proof*⟩

lemma *items-eq-pointwise*: *pointwise* (*items-eq k*)
⟨*proof*⟩

lemma *items-eq-is-filter*: *items-eq k I* ⊆ *I*
⟨*proof*⟩

lemma *paths-le-pointwise*: *pointwise* (*paths-le k*)
⟨*proof*⟩

lemma *paths-le-continuous*: *continuous* (*paths-le k*)
⟨*proof*⟩

lemma *paths-le-mono*: *mono* (*paths-le k*)
⟨*proof*⟩

lemma *paths-le-is-filter*: *paths-le k P* ⊆ *P*
⟨*proof*⟩

lemma *paths-eq-pointwise*: *pointwise* (*paths-eq k*)
⟨*proof*⟩

lemma *paths-eq-is-filter*: *paths-eq k P* ⊆ *P*
⟨*proof*⟩

lemma *Predict-item-end*: *x* ∈ *Predict k Y* ⟹ *item-end x* = *k* ∨ *x* ∈ *Y*
⟨*proof*⟩

lemma *Complete-item-end*: *x* ∈ *Complete k Y* ⟹ *item-end x* = *k* ∨ *x* ∈ *Y*
⟨*proof*⟩

lemma *𝒥-0-0-item-end*: *x* ∈ *𝒥 0 0* ⟹ *item-end x* = *0*
⟨*proof*⟩

lemma *items-le-𝒥-0-0*: *items-le 0* (*𝒥 0 0*) = *𝒥 0 0*

⟨*proof*⟩

lemma *paths-le-$\mathcal{P}$-0-0*: *paths-le 0* ($\mathcal{P}$ *0 0*) = $\mathcal{P}$ *0 0*
⟨*proof*⟩

definition *empty-tokens* :: *token set* ⇒ *token set*
where
empty-tokens T = { *t* . *t* ∈ *T* ∧ *chars-of-token t* = [] }

lemma *items-le-Predict*: *items-le k* (*Predict k I*) = *Predict k* (*items-le k I*)
⟨*proof*⟩

lemma *items-le-Complete*:
wellformed-items I ⟹ *items-le k* (*Complete k I*) = *Complete k* (*items-le k I*)
⟨*proof*⟩

lemma *items-le-Scan*:
items-le k (*Scan T k I*) = *Scan* (*empty-tokens T*) *k* (*items-le k I*)
⟨*proof*⟩

lemma *wellformed-items-Gen*: *wellformed-items* (*Gen P*)
⟨*proof*⟩

lemma *wellformed-$\mathcal{J}$-0-0*: *wellformed-items* ($\mathcal{J}$ *0 0*)
⟨*proof*⟩

lemma *wellformed-items-Predict*:
wellformed-items I ⟹ *wellformed-items* (*Predict k I*)
⟨*proof*⟩

lemma *wellformed-items-Complete*:
wellformed-items I ⟹ *wellformed-items* (*Complete k I*)
⟨*proof*⟩

lemma *$\mathcal{X}$-length-bound*: (*t*, *c*) ∈ $\mathcal{X}$ *k* ⟹ *k* + *length c* ≤ *length Doc*
⟨*proof*⟩

lemma *wellformed-items-Scan*:
wellformed-items I ⟹ *T* ⊆ $\mathcal{X}$ *k* ⟹ *wellformed-items* (*Scan T k I*)
⟨*proof*⟩

lemma *wellformed-items-π*:
assumes *wellformed-items I*
assumes *T* ⊆ $\mathcal{X}$ *k*
shows *wellformed-items* (π *k T I*)
⟨*proof*⟩

lemma *$\mathcal{J}$-subset-Suc-u*: $\mathcal{J}$ *k u* ⊆ $\mathcal{J}$ *k* (*Suc u*)
⟨*proof*⟩

lemma *mono-TokensAt*: *mono* (*TokensAt k*)
⟨*proof*⟩

lemma *𝒯-subset-TokensAt*: $\mathcal{T}\ k\ u \subseteq$ *TokensAt k* ($\mathcal{J}\ k\ u$)
⟨*proof*⟩

lemma *TokensAt-subset-𝒳*: *TokensAt k I* $\subseteq \mathcal{X}\ k$
⟨*proof*⟩

lemma *wellformed-items-𝒥-induct-u*:
assumes *wellformed-items* ($\mathcal{J}\ k\ u$)
shows *wellformed-items* ($\mathcal{J}\ k$ (*Suc u*))
⟨*proof*⟩

lemma *wellformed-items-𝒥-k-u-if-0*: *wellformed-items* ($\mathcal{J}\ k\ 0$) $\Longrightarrow$ *wellformed-items* ($\mathcal{J}\ k\ u$)
⟨*proof*⟩

lemma *wellformed-items-natUnion*: ($\bigwedge k$. *wellformed-items* (*I k*)) $\Longrightarrow$ *wellformed-items* (*natUnion I*)
⟨*proof*⟩

lemma *wellformed-items-ℐ-k-if-0*: *wellformed-items* ($\mathcal{J}\ k\ 0$) $\Longrightarrow$ *wellformed-items* ($\mathcal{I}\ k$)
⟨*proof*⟩

lemma *wellformed-items-𝒥-ℐ*: *wellformed-items* ($\mathcal{J}\ k\ u$) $\wedge$ *wellformed-items* ($\mathcal{I}\ k$)
⟨*proof*⟩

lemma *wellformed-items-𝒥*: *wellformed-items* ($\mathcal{J}\ k\ u$)
⟨*proof*⟩

lemma *wellformed-items-ℐ*: *wellformed-items* ($\mathcal{I}\ k$)
⟨*proof*⟩

lemma *funpower-consume-function*:
assumes *law*: $\bigwedge X.\ P\ X \Longrightarrow f\ (g\ X) = h\ (f\ X) \wedge P\ (g\ X)$
shows $P\ I \Longrightarrow P$ (*funpower g n I*) $\wedge\ f$ (*funpower g n I*) = *funpower h n* (*f I*)
⟨*proof*⟩

lemma *limit-consume-function*:
assumes *continuous*: *continuous f*
assumes *law*: $\bigwedge X.\ P\ X \Longrightarrow f\ (g\ X) = h\ (f\ X) \wedge P\ (g\ X)$
assumes *setmonotone*: *setmonotone g*
shows $P\ I \Longrightarrow f$ (*limit g I*) = *limit h* (*f I*)
⟨*proof*⟩

lemma *items-le-π-swap*:

assumes *wellformed-I*: *wellformed-items I*
assumes *T*: $T \subseteq \mathcal{X}\ k$
shows *items-le k* (π *k T I*) = π *k* (*empty-tokens T*) (*items-le k I*)
⟨*proof*⟩

lemma *items-le-idempotent*: *items-le k* (*items-le k I*) = *items-le k I*
⟨*proof*⟩

lemma *paths-le-idempotent*: *paths-le k* (*paths-le k P*) = *paths-le k P*
⟨*proof*⟩

lemma *items-le-fix-D*:
assumes *items-le-fix*: *items-le k I* = *I*
assumes *x-dom*: $x \in I$
shows *item-end x* ≤ *k*
⟨*proof*⟩

lemma *remove-paths-le-in-subset-Gen*:
assumes *items-le k I* = *I*
assumes $I \subseteq$ *Gen P*
shows $I \subseteq$ *Gen* (*paths-le k P*)
⟨*proof*⟩

lemma *mono-Gen*: *mono Gen*
⟨*proof*⟩

lemma *empty-tokens-idempotent*: *empty-tokens* (*empty-tokens T*) = *empty-tokens T*
⟨*proof*⟩

lemma *empty-tokens-is-filter*: *empty-tokens T* ⊆ *T*
⟨*proof*⟩

lemma *items-le-paths-le*: *items-le k* (*Gen P*) = *Gen* (*paths-le k P*)
⟨*proof*⟩

lemma *bin-items-le*[*symmetric*]: *bin I k* = *bin* (*items-le k I*) *k*
⟨*proof*⟩

lemma *TokensAt-items-le*[*symmetric*]: *TokensAt k I* = *TokensAt k* (*items-le k I*)
⟨*proof*⟩

lemma *by-length-paths-le*[*symmetric*]: *by-length k P* = *by-length k* (*paths-le k P*)
⟨*proof*⟩

lemma $\mathcal{W}$*-paths-le*[*symmetric*]: $\mathcal{W}$ *P k* = $\mathcal{W}$ (*paths-le k P*) *k*
⟨*proof*⟩

theorem $\mathcal{T}$*-equals-*$\mathcal{Z}$*-induct-step*:

assumes *induct*: *items-le k* ($\mathcal{J}$ *k u*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k u*))
assumes *induct-tokens*: $\mathcal{T}$ *k u* = $\mathcal{Z}$ *k u*
shows $\mathcal{T}$ *k* (*Suc u*) = $\mathcal{Z}$ *k* (*Suc u*)
⟨*proof*⟩

theorem *thmD9*:
assumes *induct*: *items-le k* ($\mathcal{J}$ *k u*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k u*))
assumes *induct-tokens*: $\mathcal{T}$ *k u* = $\mathcal{Z}$ *k u*
assumes *k*: *k* ≤ *length Doc*
shows *items-le k* ($\mathcal{J}$ *k* (*Suc u*)) ⊆ *Gen* (*paths-le k* ($\mathcal{P}$ *k* (*Suc u*)))
⟨*proof*⟩

end

end
theory *Ladder*
imports *TheoremD9*
begin

context *LocalLexing* **begin**

definition *LeftDerivationFix* :: *sentence* ⇒ *nat* ⇒ *derivation* ⇒ *nat* ⇒ *sentence* ⇒ *bool*
where
LeftDerivationFix α *i D j* β = (*is-sentence* α ∧ *is-sentence* β
∧ *LeftDerivation* α *D* β ∧ *i* < *length* α ∧ *j* < *length* β
∧ α ! *i* = β ! *j* ∧ (∃ *E F*. *D* = *E*@(*derivation-shift F 0* (*Suc j*)) ∧
LeftDerivation (*take i* α) *E* (*take j* β) ∧
LeftDerivation (*drop* (*Suc i*) α) *F* (*drop* (*Suc j*) β)))

definition *LeftDerivationIntro* ::
sentence ⇒ *nat* ⇒ *rule* ⇒ *nat* ⇒ *derivation* ⇒ *nat* ⇒ *sentence* ⇒ *bool*
where
LeftDerivationIntro α *i r ix D j* γ = (∃ β. *LeftDerives1* α *i r* β ∧
ix < *length* (*snd r*) ∧ (*snd r*) ! *ix* = γ ! *j* ∧
LeftDerivationFix β (*i* + *ix*) *D j* γ)

lemma *LeftDerivationFix-empty*[*simp*]: *is-sentence* α ⟹ *i* < *length* α ⟹ *LeftDerivationFix* α *i* [] *i* α
⟨*proof*⟩

lemma *Derive-empty*[*simp*]: *Derive a* [] = *a*
⟨*proof*⟩

lemma *LeftDerivation-append1*: *LeftDerivation a* (*D*@[(*i*, *r*)]) *c* ⟹ ∃ *b*. *LeftDerivation a D b*
∧ *LeftDerives1 b i r c*
⟨*proof*⟩

lemma *Derivation-append1*: *Derivation a* (*D*@[(*i*, *r*)]) *c* ⟹ ∃ *b*. *Derivation a D b*
∧ *Derives1 b i r c*
⟨*proof*⟩

lemma *Derivation-take-derive*:
assumes *Derivation a D b*
shows *Derivation a* (*take n D*) (*Derive a* (*take n D*))
⟨*proof*⟩

lemma *LeftDerivation-take-derive*:
assumes *LeftDerivation a D b*
shows *LeftDerivation a* (*take n D*) (*Derive a* (*take n D*))
⟨*proof*⟩

lemma *Derivation-Derive-take-Derives1*:
assumes *N* ≠ *0*
assumes *N* ≤ *length D*
assumes *Derivation a D b*
assumes α: α = *Derive a* (*take* (*N* − *1*) *D*)
assumes β = *Derive a* (*take N D*)
shows *Derives1* α (*fst* (*D* ! (*N* − *1*))) (*snd* (*D* ! (*N* − *1*))) β
⟨*proof*⟩

lemma *LeftDerivation-Derive-take-LeftDerives1*:
assumes *N* ≠ *0*
assumes *N* ≤ *length D*
assumes *LeftDerivation a D b*
assumes α: α = *Derive a* (*take* (*N* − *1*) *D*)
assumes β = *Derive a* (*take N D*)
shows *LeftDerives1* α (*fst* (*D* ! (*N* − *1*))) (*snd* (*D* ! (*N* − *1*))) β
⟨*proof*⟩

lemma *LeftDerives1-skip-prefix*:
length a ≤ *i* ⟹ *LeftDerives1* (*a*@*b*) *i r* (*a*@*c*) ⟹ *LeftDerives1 b* (*i* − *length a*) *r c*
⟨*proof*⟩

lemma *LeftDerives1-skip-suffix*:
assumes *i*: *i* < *length a*
assumes *D*: *LeftDerives1* (*a*@*c*) *i r* (*b*@*c*)
shows *LeftDerives1 a i r b*
⟨*proof*⟩

lemma *LeftDerives1-X-is-part-of-rule*[*consumes 2*, *case-names Suffix Prefix*]:
assumes *aXb*: *LeftDerives1* δ *i r* (*a*@[*X*]@*b*)
assumes *split*: *splits-at* δ *i* α *N* β
assumes *prefix*: ⋀ β. δ = *a* @ [*X*] @ β ⟹ *length a* < *i* ⟹ *is-word* (*a* @ [*X*])
⟹

LeftDerives1 β $(i - length\ a - 1)$ r $b \Longrightarrow$ *False*
assumes *suffix*: $\bigwedge \alpha.\ \delta = \alpha\ @\ [X]\ @\ b \Longrightarrow$ *LeftDerives1* α i r $a \Longrightarrow$ *False*
shows $\exists\ u\ v.\ a = \alpha\ @\ u \wedge b = v\ @\ \beta \wedge (snd\ r) = u@[X]@v$
⟨*proof*⟩

lemma *LeftDerivationFix-grow-suffix*:
assumes *LDF*: *LeftDerivationFix* $(b1@[X]@b2)$ $(length\ b1)$ D j c
assumes *suffix-b2*: *LeftDerives1 suffix e r b2*
assumes *is-word-b1X*: *is-word* $(b1@[X])$
shows *LeftDerivationFix* $(b1@[X]@suffix)$ $(length\ b1)$ $((e + length\ (b1@[X]), r)\#D)$ j c
⟨*proof*⟩

lemma *Derives1-append-suffix*:
assumes *Derives1*: *Derives1 v i r w*
assumes *u*: *is-sentence u*
shows *Derives1* $(v@u)$ i r $(w@u)$
⟨*proof*⟩

lemma *leftmost-append-suffix*: *leftmost i v* $\Longrightarrow$ *leftmost i* $(v@u)$
⟨*proof*⟩

lemma *LeftDerives1-append-suffix*:
assumes *Derives1*: *LeftDerives1 v i r w*
assumes *u*: *is-sentence u*
shows *LeftDerives1* $(v@u)$ i r $(w@u)$
⟨*proof*⟩

lemma *LeftDerivationFix-is-sentence*:
LeftDerivationFix a i D j b $\Longrightarrow$ *is-sentence a* $\wedge$ *is-sentence b*
⟨*proof*⟩

lemma *LeftDerivationIntro-is-sentence*:
LeftDerivationIntro α i r *ix D j* $\gamma \Longrightarrow$ *is-sentence* $\alpha \wedge$ *is-sentence* γ
⟨*proof*⟩

lemma *LeftDerivationFix-grow-prefix*:
assumes *LDF*: *LeftDerivationFix* $(b1@[X]@b2)$ $(length\ b1)$ D j c
assumes *prefix-b1*: *LeftDerives1 prefix e r b1*
shows *LeftDerivationFix* $(prefix@[X]@b2)$ $(length\ prefix)$ $((e,\ r)\#D)$ j c
⟨*proof*⟩

lemma *LeftDerivationFixOrIntro*:
LeftDerivation a D $\gamma \Longrightarrow$ *is-sentence* $\gamma \Longrightarrow j <$ *length* $\gamma \Longrightarrow$
$(\exists\ i.$ *LeftDerivationFix a i D j* $\gamma) \vee$
$(\exists\ d\ \alpha\ ix.\ d <$ *length D* $\wedge$ *LeftDerivation a* $(take\ d\ D)$ $\alpha \wedge$
LeftDerivationIntro α $(fst\ (D\ !\ d))$ $(snd\ (D\ !\ d))$ *ix* $(drop\ (Suc\ d)\ D)$ j $\gamma)$
⟨*proof*⟩

type-synonym *deriv* = *nat* × *nat* × *nat*
type-synonym *ladder* = *deriv list*

definition *deriv-n* :: *deriv* ⇒ *nat* **where**
deriv-n d = *fst d*

definition *deriv-j* :: *deriv* ⇒ *nat* **where**
deriv-j d = *fst* (*snd d*)

definition *deriv-ix* :: *deriv* ⇒ *nat* **where**
deriv-ix d = *snd* (*snd d*)

definition *deriv-i* :: *deriv* ⇒ *nat* **where**
deriv-i d = *snd* (*snd d*)

definition *ladder-j* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-j L index = *deriv-j* (*L* ! *index*)

definition *ladder-i* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-i L index = (*if index* = *0 then deriv-i* (*hd L*) *else ladder-j L* (*index* − *1*))

definition *ladder-n* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-n L index = *deriv-n* (*L* ! *index*)

definition *ladder-prev-n* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-prev-n L index = (*if index* = *0 then 0 else* (*ladder-n L* (*index* − *1*)))

definition *ladder-ix* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-ix L index = (*if index* = *0 then undefined else deriv-ix* (*L* ! *index*))

definition *ladder-last-j* :: *ladder* ⇒ *nat* **where**
ladder-last-j L = *ladder-j L* (*length L* − *1*)

definition *ladder-last-n* :: *ladder* ⇒ *nat* **where**
ladder-last-n L = *ladder-n L* (*length L* − *1*)

definition *is-ladder* :: *derivation* ⇒ *ladder* ⇒ *bool* **where**
is-ladder D L = (*L* ≠ [] ∧
(∀ *u*. *u* < *length L* ⟶ *ladder-n L u* ≤ *length D*) ∧
(∀ *u v*. *u* < *v* ∧ *v* < *length L* ⟶ *ladder-n L u* < *ladder-n L v*) ∧
ladder-last-n L = *length D*)

definition *ladder-γ* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒ *sentence* **where**
ladder-γ a D L index = *Derive a* (*take* (*ladder-n L index*) *D*)

definition *ladder-α* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒ *sentence* **where**
ladder-α a D L index = (*if index* = *0 then a else ladder-γ a D L* (*index* − *1*))

definition *LeftDerivationIntrosAt* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒

bool **where**
LeftDerivationIntrosAt a D L index = (
let α = *ladder-*α *a D L index in*
let i = *ladder-i L index in*
let j = *ladder-j L index in*
let ix = *ladder-ix L index in*
let γ = *ladder-*γ *a D L index in*
let n = *ladder-n L* (*index* − *1*) *in*
let m = *ladder-n L index in*
let e = *D* ! *n in*
let E = *drop* (*Suc n*) (*take m D*) *in*
i = *fst e* ∧
LeftDerivationIntro α *i* (*snd e*) *ix E j* γ)

definition *LeftDerivationIntros* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *bool* **where**
LeftDerivationIntros a D L = (
∀ *index*. *1* ≤ *index* ∧ *index* < *length L* ⟶ *LeftDerivationIntrosAt a D L index*)

definition *LeftDerivationLadder* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *sentence* ⇒ *bool* **where**
LeftDerivationLadder a D L b = (
LeftDerivation a D b ∧
is-ladder D L ∧
LeftDerivationFix a (*ladder-i L 0*) (*take* (*ladder-n L 0*) *D*) (*ladder-j L 0*) (*ladder-*γ *a D L 0*) ∧
LeftDerivationIntros a D L)

definition *mk-deriv-fix* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *deriv* **where**
mk-deriv-fix i n j = (*n*, *j*, *i*)

definition *mk-deriv-intro* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *deriv* **where**
mk-deriv-intro ix n j = (*n*, *j*, *ix*)

lemma *mk-deriv-fix-i*[*simp*]: *deriv-i* (*mk-deriv-fix i n j*) = *i*
⟨*proof*⟩

lemma *mk-deriv-fix-j*[*simp*]: *deriv-j* (*mk-deriv-fix i n j*) = *j*
⟨*proof*⟩

lemma *mk-deriv-fix-n*[*simp*]: *deriv-n* (*mk-deriv-fix i n j*) = *n*
⟨*proof*⟩

lemma *mk-deriv-intro-i*[*simp*]: *deriv-i* (*mk-deriv-intro i n j*) = *i*
⟨*proof*⟩

lemma *mk-deriv-intro-ix*[*simp*]: *deriv-ix* (*mk-deriv-intro ix n j*) = *ix*
⟨*proof*⟩

lemma *mk-deriv-intro-j*[*simp*]: *deriv-j* (*mk-deriv-intro i n j*) = *j*
⟨*proof*⟩

lemma *mk-deriv-intro-n*[*simp*]: *deriv-n* (*mk-deriv-intro i n j*) = *n*
⟨*proof*⟩

lemma *LeftDerivationFix-implies-ex-ladder*:
LeftDerivationFix a i D j γ ⟹ ∃ *L*. *LeftDerivationLadder a D L γ* ∧
ladder-last-j L = *j* ∧ *ladder-last-n L* = *length D*
⟨*proof*⟩

lemma *trivP*[*case-names prems*]: *P* ⟹ *P* ⟨*proof*⟩

lemma *LeftDerivationLadder-ladder-n-bound*:
assumes *LeftDerivationLadder a D L b*
assumes *index* < *length L*
shows *ladder-n L index* ≤ *length D*
⟨*proof*⟩

lemma *LeftDerivationLadder-deriv-n-bound*:
assumes *LeftDerivationLadder a D L b*
assumes *index* < *length L*
shows *deriv-n* (*L* ! *index*) ≤ *length D*
⟨*proof*⟩

lemma *ladder-n-simp1*[*simp*]: *u* < *length L* ⟹ *ladder-n* (*L*@*L'*) *u* = *ladder-n L u*
⟨*proof*⟩

lemma *ladder-n-simp2*[*simp*]: *ladder-n* (*L*@[*d*]) (*length L*) = *deriv-n d*
⟨*proof*⟩

lemma *ladder-j-simp1*[*simp*]: *u* < *length L* ⟹ *ladder-j* (*L*@*L'*) *u* = *ladder-j L u*
⟨*proof*⟩

lemma *ladder-j-simp2*[*simp*]: *ladder-j* (*L*@[*d*]) (*length L*) = *deriv-j d*
⟨*proof*⟩

lemma *ladder-i-simp1*[*simp*]: *u* < *length L* ⟹ *ladder-i* (*L*@*L'*) *u* = *ladder-i L u*
⟨*proof*⟩

lemma *ladder-ix-simp1*[*simp*]: *u* < *length L* ⟹ *ladder-ix* (*L*@*L'*) *u* = *ladder-ix L u*
⟨*proof*⟩

lemma *ladder-ix-simp2*[*simp*]: *L* ≠ [] ⟹ *ladder-ix* (*L*@[*d*]) (*length L*) = *deriv-ix d*
⟨*proof*⟩

lemma *ladder-γ-simp1*[*simp*]: $u < \mathit{length}\ L \Longrightarrow \mathit{ladder\text{-}\gamma}\ a\ D\ (L@L')\ u = \mathit{ladder\text{-}\gamma}\ a\ D\ L\ u$
⟨*proof*⟩

lemma *ladder-γ-simp2*[*simp*]: $u < \mathit{length}\ L \Longrightarrow \mathit{is\text{-}ladder}\ D\ L \Longrightarrow$
$\mathit{ladder\text{-}\gamma}\ a\ (D@D')\ L\ u = \mathit{ladder\text{-}\gamma}\ a\ D\ L\ u$
⟨*proof*⟩

lemma *ladder-α-simp1*[*simp*]: $u < \mathit{length}\ L \Longrightarrow \mathit{ladder\text{-}\alpha}\ a\ D\ (L@L')\ u = \mathit{ladder\text{-}\alpha}\ a\ D\ L\ u$
⟨*proof*⟩

lemma *ladder-α-simp2*[*simp*]: $u < \mathit{length}\ L \Longrightarrow \mathit{is\text{-}ladder}\ D\ L \Longrightarrow$
$\mathit{ladder\text{-}\alpha}\ a\ (D@D')\ L\ u = \mathit{ladder\text{-}\alpha}\ a\ D\ L\ u$
⟨*proof*⟩

lemma *ladder-n-minus-1-bound*: $\mathit{is\text{-}ladder}\ D\ L \Longrightarrow \mathit{index} \geq 1 \Longrightarrow \mathit{index} < \mathit{length}\ L \Longrightarrow$
$\mathit{ladder\text{-}n}\ L\ (\mathit{index} - \mathit{Suc}\ 0) < \mathit{length}\ D$
⟨*proof*⟩

lemma *LeftDerivationIntrosAt-ignore-appendix*:
assumes *is-ladder*: $\mathit{is\text{-}ladder}\ D\ L$
assumes *hyp*: $\mathit{LeftDerivationIntrosAt}\ a\ D\ L\ \mathit{index}$
assumes *index-ge*: $\mathit{index} \geq 1$
assumes *index-less*: $\mathit{index} < \mathit{length}\ L$
shows $\mathit{LeftDerivationIntrosAt}\ a\ (D\ @\ D')\ (L\ @\ L')\ \mathit{index}$
⟨*proof*⟩

lemma *ladder-i-eq-last-j*: $L \neq [] \Longrightarrow \mathit{ladder\text{-}i}\ (L\ @\ L')\ (\mathit{length}\ L) = \mathit{ladder\text{-}last\text{-}j}\ L$
⟨*proof*⟩

lemma *ladder-last-n-intro*: $L \neq [] \Longrightarrow \mathit{ladder\text{-}n}\ L\ (\mathit{length}\ L - \mathit{Suc}\ 0) = \mathit{ladder\text{-}last\text{-}n}\ L$
⟨*proof*⟩

lemma *is-ladder-not-empty*: $\mathit{is\text{-}ladder}\ D\ L \Longrightarrow L \neq []$
⟨*proof*⟩

lemma *last-ladder-γ*:
assumes *is-ladder*: $\mathit{is\text{-}ladder}\ D\ L$
assumes *ladder-last-n*: $\mathit{ladder\text{-}last\text{-}n}\ L = \mathit{length}\ D$
shows $\mathit{ladder\text{-}\gamma}\ a\ D\ L\ (\mathit{length}\ L - \mathit{Suc}\ 0) = \mathit{Derive}\ a\ D$
⟨*proof*⟩

lemma *ladder-α-full*:
assumes *is-ladder*: $\mathit{is\text{-}ladder}\ D\ L$
assumes *ladder-last-n*: $\mathit{ladder\text{-}last\text{-}n}\ L = \mathit{length}\ D$

shows *ladder-α a (D @ D′) (L @ L′) (length L) = Derive a D*
⟨*proof*⟩

lemma *LeftDerivationIntro-implies-LeftDerivation*:
LeftDerivationIntro α i r ix D j γ ⟹ LeftDerivation α ((i,r)#D) γ
⟨*proof*⟩

lemma *LeftDerivationLadder-grow*:
LeftDerivationLadder a D L α ⟹ ladder-last-j L = i ⟹
LeftDerivationIntro α i r ix E j γ ⟹
LeftDerivationLadder a (D@[(i, r)]@E) (L@[mk-deriv-intro ix (Suc(length D + length E)) j]) γ
⟨*proof*⟩

lemma *LeftDerivationIntro-bounds-ij*:
LeftDerivationIntro α i r ix D j β ⟹ i < length α ∧ j < length β
⟨*proof*⟩

theorem *LeftDerivationLadder-exists*: *LeftDerivation a D γ ⟹ is-sentence γ ⟹ j < length γ ⟹*
∃ L. LeftDerivationLadder a D L γ ∧ ladder-last-j L = j
⟨*proof*⟩

lemma *LeftDerivationLadder-L-0*:
assumes *LeftDerivationLadder α D L β*
assumes *length L = 1*
shows *∃ i. LeftDerivationFix α i D (ladder-last-j L) β*
⟨*proof*⟩

lemma *LeftDerivationFix-splits-at-derives*:
assumes *LeftDerivationFix a i D j b*
shows *∃ U a1 a2 b1 b2. splits-at a i a1 U a2 ∧ splits-at b j b1 U b2 ∧*
derives a1 b1 ∧ derives a2 b2
⟨*proof*⟩

lemma *LeftDerivation-append-suffix*:
LeftDerivation a D b ⟹ is-sentence c ⟹ LeftDerivation (a@c) D (b@c)
⟨*proof*⟩

lemma *LeftDerivation-impossible*: *LeftDerivation a D b ⟹ i < length a ⟹*
is-nonterminal (a ! i) ⟹ derivation-ge D (Suc i) ⟹ D = []
⟨*proof*⟩

lemma *derivation-ge-shift*: *derivation-ge (derivation-shift F 0 j) j*
⟨*proof*⟩

lemma *LeftDerivationFix-splits-at-nonterminal*:
assumes *LeftDerivationFix a i D j b*
assumes *is-nonterminal (a ! i)*

shows $\exists$ *U a1 a2 b1*. *splits-at a i a1 U a2* $\wedge$ *splits-at b j b1 U a2* $\wedge$ *LeftDerivation a1 D b1*
⟨*proof*⟩

lemma *LeftDerivationIntro-implies-nonterminal*:
LeftDerivationIntro α *i* (*snd e*) *ix E j* γ $\Longrightarrow$ *is-nonterminal* (α ! *i*)
⟨*proof*⟩

lemma *LeftDerivationIntrosAt-implies-nonterminal*:
LeftDerivationIntrosAt a D L index $\Longrightarrow$ *is-nonterminal*((*ladder-*α *a D L index*) ! (*ladder-i L index*))
⟨*proof*⟩

lemma *LeftDerivationIntro-examine-rule*:
LeftDerivationIntro α *i r ix D j* γ $\Longrightarrow$ *splits-at* α *i* α*1 M* α*2* $\Longrightarrow$
$\exists$ η. *M* = *fst r* $\wedge$ η = *snd r* $\wedge$ (*M*, η) $\in$ $\mathfrak{R}$
⟨*proof*⟩

lemma *LeftDerivation-skip-prefixword-ex*:
assumes *LeftDerivation* (*u*@*v*) *D w*
assumes *is-word u*
shows $\exists$ *w'*. *w* = *u*@*w'* $\wedge$ *LeftDerivation v* (*derivation-shift D* (*length u*) *0*) *w'*
⟨*proof*⟩

definition *ladder-cut* :: *ladder* $\Rightarrow$ *nat* $\Rightarrow$ *ladder*
where *ladder-cut L n* = (*let i* = *length L* − *1 in L*[*i* := (*n*, *snd* (*L* ! *i*))])

fun *deriv-shift* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *deriv* $\Rightarrow$ *deriv*
where *deriv-shift dn dj* (*n*, *j*, *i*) = (*n* − *dn*, *j* − *dj*, *i*)

definition *ladder-shift* :: *ladder* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\Rightarrow$ *ladder*
where *ladder-shift L dn dj* = *map* (*deriv-shift dn dj*) *L*

lemma *splits-at-append-suffix-prevails*:
assumes *splits-at* (*a*@*b*) *i u N v*
assumes *i* < *length a*
shows $\exists$ *v'*. *v* = *v'*@*b* $\wedge$ *a*=*u*@[*N*]@*v'*
⟨*proof*⟩

lemma *derivation-shift-right-left-cancel*:
derivation-shift (*derivation-shift D 0 r*) *r 0* = *D*
⟨*proof*⟩

lemma *derivation-shift-left-right-cancel*:
assumes *derivation-ge D r*
shows *derivation-shift* (*derivation-shift D r 0*) *0 r* = *D*
⟨*proof*⟩

lemma *LeftDerivation-ge-take*:

assumes *derivation-ge D k*
assumes *LeftDerivation a D b*
assumes $D \neq []$
shows *take k a = take k b* $\wedge$ *is-word* (*take k a*)
⟨*proof*⟩

lemma *LeftDerivationFix-splits-at-symbol*:
assumes *LeftDerivationFix a i D j b*
shows $\exists$ *U a1 a2 b1 b2 n. splits-at a i a1 U a2* $\wedge$ *splits-at b j b1 U b2* $\wedge$
n $\leq$ *length D* $\wedge$ *LeftDerivation a1* (*take n D*) *b1* $\wedge$ *derivation-ge* (*drop n D*)
(*Suc*(*length b1*)) $\wedge$
LeftDerivation a2 (*derivation-shift* (*drop n D*) (*Suc*(*length b1*)) *0*) *b2* $\wedge$
(*n = length D* $\vee$ (*n < length D* $\wedge$ *is-word* (*b1*@[*U*])))
⟨*proof*⟩

lemma *LeftDerivation-breakdown'*: *LeftDerivation* (*u* @ *v*) *D w* $\Longrightarrow$
$\exists$ *n w1 w2*.
n $\leq$ *length D* $\wedge$
w = w1 @ *w2* $\wedge$
LeftDerivation u (*take n D*) *w1* $\wedge$
derivation-ge (*drop n D*) (*length w1*) $\wedge$
LeftDerivation v (*derivation-shift* (*drop n D*) (*length w1*) *0*) *w2*
⟨*proof*⟩

lemma *LeftDerives1-append-replace-in-left*:
assumes *ld1*: *LeftDerives1* (*α*@*δ*) *i r β*
assumes *i-bound*: *i < length α*
shows $\exists$ *α'*. *β = α'*@*δ* $\wedge$ *LeftDerives1 α i r α'* $\wedge$ *i* + *length* (*snd r*) $\leq$ *length α'*
⟨*proof*⟩

lemma *LeftDerivationIntro-propagate*:
assumes *intro*: *LeftDerivationIntro* (*α*@*δ*) *i r ix D j γ*
assumes *i-α*: *i < length α*
assumes *non*: *is-nonterminal* (*γ* ! *j*)
shows $\exists$ *ω*. *LeftDerivation α* ((*i*,*r*)#*D*) *ω* $\wedge$ *γ = ω*@*δ* $\wedge$ *j < length ω*
⟨*proof*⟩

lemma *LeftDerivationIntro-finish*:
assumes *intro*: *LeftDerivationIntro* (*α*@*δ*) *i r ix D j γ*
assumes *i-α*: *i < length α*
shows $\exists$ *k ω δ'*.
k $\leq$ *length D* $\wedge$
LeftDerivation α ((*i*, *r*)#(*take k D*)) *ω* $\wedge$
LeftDerivation (*α* @ *δ*) ((*i*, *r*)#(*take k D*)) (*ω* @ *δ*) $\wedge$
derivation-ge (*drop k D*) (*length ω*) $\wedge$
LeftDerivation δ (*derivation-shift* (*drop k D*) (*length ω*) *0*) *δ'* $\wedge$
γ = ω @ *δ'* $\wedge$ *j < length ω*
⟨*proof*⟩

lemma *LeftDerivationLadder-propagate*:
LeftDerivationLadder $(\alpha@\delta)$ *D L* $\gamma \Longrightarrow$ *ladder-i L 0* $<$ *length* $\alpha \Longrightarrow n =$ *ladder-n L index*
$\Longrightarrow$ *index* $<$ *length L* $\Longrightarrow$
if (*index* + *1* $<$ *length L*) *then*
$(\exists\ \beta.$ *LeftDerivation* α (*take n D*) $\beta \wedge$ *ladder-*γ $(\alpha@\delta)$ *D L index* $= \beta@\delta \wedge$
ladder-j L index $<$ *length* $\beta)$
else
$(\exists\ n'\ \beta\ \delta'.$ (*index* $= 0 \vee$ *ladder-prev-n L index* $< n') \wedge n' \leq n \wedge$ *LeftDerivation* α (*take* n' *D*) $\beta \wedge$
LeftDerivation $(\alpha@\delta)$ (*take* n' *D*) $(\beta@\delta) \wedge$
derivation-ge (*drop* n' *D*) (*length* β) $\wedge$
LeftDerivation δ (*derivation-shift* (*drop* n' *D*) (*length* β) *0*) $\delta' \wedge$
ladder-γ $(\alpha@\delta)$ *D L index* $= \beta@\delta' \wedge$ *ladder-j L index* $<$ *length* $\beta)$
⟨*proof*⟩

lemma *ladder-i-of-cut-at-0*:
assumes *L-non-empty*: $L \neq []$
shows *ladder-i* (*ladder-cut L n*) *0* $=$ *ladder-i L 0*
⟨*proof*⟩

lemma *ladder-last-j-of-cut*:
assumes *L-non-empty*: $L \neq []$
shows *ladder-last-j* (*ladder-cut L n*) $=$ *ladder-last-j L*
⟨*proof*⟩

lemma *length-ladder-cut*:
assumes *L-non-empty*: $L \neq []$
shows *length* (*ladder-cut L n*) $=$ *length L*
⟨*proof*⟩

lemma *ladder-last-n-of-cut*:
assumes *L-non-empty*: $L \neq []$
shows *ladder-last-n* (*ladder-cut L n*) $= n$
⟨*proof*⟩

lemma *ladder-n-of-cut*:
assumes *L-non-empty*: $L \neq []$
assumes *index* $<$ *length L* $-$ *1*
shows *ladder-n* (*ladder-cut L n*) *index* $=$ *ladder-n L index*
⟨*proof*⟩

lemma *ladder-n-prev-bound*:
assumes *ladder*: *is-ladder D L*
assumes *u-bound*: *u* $<$ *length L* $-$ *1*
shows *ladder-n L u* $\leq$ *ladder-prev-n L* (*length L* $-$ *1*)
⟨*proof*⟩

lemma *ladder-n-last-is-length*:

assumes *is-ladder D L*
shows *ladder-n L (length L − 1) = length D*
⟨*proof*⟩

lemma *derivation-ge-shift-implies-derivation-ge*:
assumes *dge*: *derivation-ge (derivation-shift F 0 j) k*
shows *derivation-ge F (k − j)*
⟨*proof*⟩

lemma *Derives1-bound′*: *Derives1 a i r b ⟹ i ≤ length b*
⟨*proof*⟩

lemma *LeftDerivation-Derives1-last*:
assumes *LeftDerivation a D b*
assumes *D ≠ []*
shows *Derives1 (Derive a (take (length D − 1) D)) (fst (last D)) (snd (last D)) b*
⟨*proof*⟩

lemma *last-of-prefix-in-set*:
assumes *n < length E*
assumes *D = E@F*
shows *last E ∈ set (drop n D)*
⟨*proof*⟩

lemma *LeftDerivationFix-cut-appendix*:
assumes *ldfix*: *LeftDerivationFix (α@δ) i D j (β@δ′)*
assumes *α-β*: *LeftDerivation α (take n D) β*
assumes *n-bound*: *n ≤ length D*
assumes *dge*: *derivation-ge (drop n D) (length β)*
assumes *i-in*: *i < length α*
assumes *j-in*: *j < length β*
shows *LeftDerivationFix α i (take n D) j β*
⟨*proof*⟩

lemma *LeftDerivationFix-cut-appendix′*:
assumes *ldfix*: *LeftDerivationFix (α@δ) i D j (β@δ′)*
assumes *α-β*: *LeftDerivation α D β*
assumes *i-in*: *i < length α*
assumes *j-in*: *j < length β*
shows *LeftDerivationFix α i D j β*
⟨*proof*⟩

lemma *LeftDerivationIntro-cut-appendix*:
assumes *ldfix*: *LeftDerivationIntro (α@δ) i r ix D j (β@δ′)*
assumes *α-β*: *LeftDerivation α ((i,r)#(take n D)) β*
assumes *n-bound*: *n ≤ length D*
assumes *dge*: *derivation-ge (drop n D) (length β)*
assumes *i-in*: *i < length α*

assumes *j-in*: $j < \mathit{length}\ \beta$
shows *LeftDerivationIntro* $\alpha\ i\ r\ ix\ (\mathit{take}\ n\ D)\ j\ \beta$
⟨*proof*⟩

lemma *LeftDerivationIntro-cut-appendix'*:
assumes *ldfix*: *LeftDerivationIntro* $(\alpha@\delta)\ i\ r\ ix\ D\ j\ (\beta@\delta')$
assumes α-β: *LeftDerivation* $\alpha\ ((i,r)\#D)\ \beta$
assumes *i-in*: $i < \mathit{length}\ \alpha$
assumes *j-in*: $j < \mathit{length}\ \beta$
shows *LeftDerivationIntro* $\alpha\ i\ r\ ix\ D\ j\ \beta$
⟨*proof*⟩

lemma *ladder-n-monotone*: *is-ladder* $D\ L \Longrightarrow u \leq v \Longrightarrow v < \mathit{length}\ L \Longrightarrow$ *ladder-n* $L\ u \leq$ *ladder-n* $L\ v$
⟨*proof*⟩

lemma *ladder-i-cut*:
assumes *index-bound*: $\mathit{index} < \mathit{length}\ L$
shows *ladder-i* $(\mathit{ladder\text{-}cut}\ L\ n)\ \mathit{index} = \mathit{ladder\text{-}i}\ L\ \mathit{index}$
⟨*proof*⟩

lemma *ladder-j-cut*:
assumes *index-bound*: $\mathit{index} < \mathit{length}\ L$
shows *ladder-j* $(\mathit{ladder\text{-}cut}\ L\ n)\ \mathit{index} = \mathit{ladder\text{-}j}\ L\ \mathit{index}$
⟨*proof*⟩

lemma *ladder-ix-cut*:
assumes *index-lower-bound*: $\mathit{index} > 0$
assumes *index-upper-bound*: $\mathit{index} < \mathit{length}\ L$
shows *ladder-ix* $(\mathit{ladder\text{-}cut}\ L\ n)\ \mathit{index} = \mathit{ladder\text{-}ix}\ L\ \mathit{index}$
⟨*proof*⟩

lemma *LeftDerivation-from-in-between*:
assumes α-β: *LeftDerivation* $\alpha\ (\mathit{take}\ u\ D)\ \beta$
assumes α-γ: *LeftDerivation* $\alpha\ (\mathit{take}\ v\ D)\ \gamma$
assumes *u-le-v*: $u \leq v$
shows *LeftDerivation* $\beta\ (\mathit{drop}\ u\ (\mathit{take}\ v\ D))\ \gamma$
⟨*proof*⟩

lemma *LeftDerivationLadder-cut-appendix-helper*:
assumes *LDLadder*: *LeftDerivationLadder* $(\alpha@\delta)\ D\ L\ \gamma$
assumes *ladder-i-in-α*: *ladder-i* $L\ 0 < \mathit{length}\ \alpha$
shows $\exists\ E\ F\ \gamma 1\ \gamma 2\ L'.\ D = E@F \wedge$
$\gamma = \gamma 1\ @\ \gamma 2 \wedge$
LeftDerivationLadder $\alpha\ E\ L'\ \gamma 1 \wedge$
derivation-ge $F\ (\mathit{length}\ \gamma 1) \wedge$
LeftDerivation $\delta\ (\mathit{derivation\text{-}shift}\ F\ (\mathit{length}\ \gamma 1)\ 0)\ \gamma 2 \wedge$
$L' = \mathit{ladder\text{-}cut}\ L\ (\mathit{length}\ E)$
⟨*proof*⟩

theorem *LeftDerivationLadder-cut-appendix*:
assumes *LDLadder*: *LeftDerivationLadder* $(\alpha@\delta)$ *D L* γ
assumes *ladder-i-in-α*: *ladder-i L 0* $<$ *length* α
shows $\exists$ *E F* $\gamma 1$ $\gamma 2$ L'. *D* = *E*@*F* $\wedge$
$\gamma = \gamma 1$ @ $\gamma 2$ $\wedge$
LeftDerivationLadder α *E* L' $\gamma 1$ $\wedge$
derivation-ge F (*length* $\gamma 1$) $\wedge$
LeftDerivation δ (*derivation-shift F* (*length* $\gamma 1$) *0*) $\gamma 2$ $\wedge$
length L' = *length L* $\wedge$ *ladder-i* L' *0* = *ladder-i L 0* $\wedge$
ladder-last-j L' = *ladder-last-j L*
⟨*proof*⟩

definition *ladder-stepdown-diff* :: *ladder* ⇒ *nat* **where**
ladder-stepdown-diff L = *Suc* (*ladder-n L 0*)

definition *ladder-stepdown-α-0* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *sentence* **where**
ladder-stepdown-α-0 a D L = *Derive a* (*take* (*ladder-stepdown-diff L*) *D*)

lemma *LeftDerivationIntro-LeftDerives1*:
assumes *LeftDerivationIntro* α *i r ix D j* γ
assumes *splits-at* α *i a1 A a2*
shows *LeftDerives1* α *i r* (*a1*@(*snd r*)@*a2*)
⟨*proof*⟩

lemma *LeftDerives1-Derive*:
assumes *LeftDerives1* α *i r* γ
shows *Derive* α [(*i*, *r*)] = γ
⟨*proof*⟩

lemma *ladder-stepdown-α-0-altdef*:
assumes *ladder*: *LeftDerivationLadder* α *D L* γ
assumes *length-L*: *length L* > *1*
assumes *split*: *splits-at* (*ladder-α* α *D L 1*) (*ladder-i L 1*) *a1 A a2*
shows *ladder-stepdown-α-0* α *D L* = *a1* @ (*snd* (*snd* (*D* ! (*ladder-n L 0*)))) @ *a2*
⟨*proof*⟩

lemma *ladder-i-0-bound*:
assumes *ld*: *LeftDerivationLadder* α *D L* γ
shows *ladder-i L 0* $<$ *length* α
⟨*proof*⟩

lemma *ladder-j-bound*:
assumes *ld*: *LeftDerivationLadder* α *D L* γ
assumes *index-bound*: *index* $<$ *length L*
shows *ladder-j L index* $<$ *length* (*ladder-*γ α *D L index*)
⟨*proof*⟩

lemma *ladder-last-j-bound*:
assumes *ld*: *LeftDerivationLadder* α *D L* γ
shows *ladder-last-j L* < *length* γ
⟨*proof*⟩

fun *ladder-shift-n* :: *nat* ⇒ *ladder* ⇒ *ladder* **where**
ladder-shift-n N [] = []
| *ladder-shift-n N* ((*n*, *j*, *i*)#*L*) = ((*n* − *N*, *j*, *i*)#(*ladder-shift-n N L*))

fun *ladder-stepdown* :: *ladder* ⇒ *ladder*
where
ladder-stepdown [] = *undefined*
| *ladder-stepdown* [*v*] = *undefined*
| *ladder-stepdown* ((*n0*, *j0*, *i0*)#(*n1*, *j1*, *ix1*)#*L*) =
(*n1* − *Suc n0*, *j1*, *j0* + *ix1*) # (*ladder-shift-n* (*Suc n0*) *L*)

lemma *ladder-shift-n-length*:
length (*ladder-shift-n N L*) = *length L*
⟨*proof*⟩

lemma *ladder-stepdown-prepare*:
assumes *length L* > *1*
shows *L* = (*ladder-n L 0*, *ladder-j L 0*, *ladder-i L 0*)#
(*ladder-n L 1*, *ladder-j L 1*, *ladder-ix L 1*)#(*drop 2 L*)
⟨*proof*⟩

lemma *ladder-stepdown-length*:
assumes *length L* > *1*
shows *length* (*ladder-stepdown L*) = *length L* − *1*
⟨*proof*⟩

lemma *ladder-stepdown-i-0*:
assumes *length L* > *1*
shows *ladder-i* (*ladder-stepdown L*) *0* = *ladder-i L 1* + *ladder-ix L 1*
⟨*proof*⟩

lemma *ladder-shift-n-cons*: *ladder-shift-n N* (*x*#*L*) = (*fst x* − *N*, *snd x*)#(*ladder-shift-n N L*)
⟨*proof*⟩

lemma *ladder-shift-n-drop*: *ladder-shift-n N* (*drop n L*) = *drop n* (*ladder-shift-n N L*)
⟨*proof*⟩

lemma *drop-2-shift*:
assumes *index* > *0*
assumes *length L* > *1*
shows *drop 2 L* ! (*index* − *Suc 0*) = *L* ! *Suc index*

⟨*proof*⟩

lemma *ladder-shift-n-at*:
index < *length L* ⟹ (*ladder-shift-n N L*) ! *index* = (*fst* (*L* ! *index*) − *N*, *snd* (*L* ! *index*))
⟨*proof*⟩

lemma *ladder-stepdown-j*:
assumes *length-L-greater-1*: *length L* > *1*
assumes *L'*: *L'* = *ladder-stepdown L*
assumes *index-bound*: *index* < *length L'*
shows *ladder-j L' index* = *ladder-j L* (*Suc index*)
⟨*proof*⟩

lemma *ladder-stepdown-last-j*:
assumes *length-L-greater-1*: *length L* > *1*
shows *ladder-last-j* (*ladder-stepdown L*) = *ladder-last-j L*
⟨*proof*⟩

lemma *ladder-stepdown-n*:
assumes *length-L-greater-1*: *length L* > *1*
assumes *L'*: *L'* = *ladder-stepdown L*
assumes *index-bound*: *index* < *length L'*
shows *ladder-n L' index* = *ladder-n L* (*Suc index*) − *ladder-stepdown-diff L*
⟨*proof*⟩

lemma *ladder-stepdown-ix*:
assumes *length-L-greater-1*: *length L* > *1*
assumes *L'*: *L'* = *ladder-stepdown L*
assumes *index-lower-bound*: *0* < *index*
assumes *index-upper-bound*: *index* < *length L'*
shows *ladder-ix L' index* = *ladder-ix L* (*Suc index*)
⟨*proof*⟩

lemma *Derive-Derive*:
assumes *Derivation* α (*D*@*E*) γ
shows *Derive* (*Derive* α *D*) *E* = *Derive* α (*D*@*E*)
⟨*proof*⟩

lemma *drop-at-shift*:
assumes *n* ≤ *index*
assumes *index* < *length D*
shows *drop n D* ! (*index* − *n*) = *D* ! *index*
⟨*proof*⟩

theorem *LeftDerivationLadder-stepdown*:
assumes *ldl*: *LeftDerivationLadder* α *D L* γ
assumes *length-L*: *length L* > *1*
shows ∃ *L'*. *LeftDerivationLadder* (*ladder-stepdown-α-0* α *D L*) (*drop* (*ladder-stepdown-diff*

L) *D*)
L′ γ ∧ *length L′* = *length L* − *1* ∧ *ladder-i L′ 0* = *ladder-i L 1* + *ladder-ix L 1* ∧
ladder-last-j L′ = *ladder-last-j L*
⟨*proof*⟩

fun *ladder-shift-j* :: *nat* ⇒ *ladder* ⇒ *ladder* **where**
ladder-shift-j d [] = []
| *ladder-shift-j d* ((*n*, *j*, *i*)#*L*) = ((*n*, *j* − *d*, *i*)#(*ladder-shift-j d L*))

definition *ladder-cut-prefix* :: *nat* ⇒ *ladder* ⇒ *ladder*
where
ladder-cut-prefix d L =
(*ladder-shift-j d L*)[*0* := (*ladder-n L 0*, *ladder-j L 0* − *d*, *ladder-i L 0* − *d*)]

lemma *ladder-shift-j-length*:
length (*ladder-shift-j d L*) = *length L*
⟨*proof*⟩

lemma *ladder-cut-prefix-length*:
shows *length* (*ladder-cut-prefix d L*) = *length L*
⟨*proof*⟩

lemma *ladder-shift-j-cons*: *ladder-shift-j d* (*x*#*L*) = (*fst x*, *fst* (*snd x*) − *d*, *snd*(*snd x*))#
(*ladder-shift-j d L*)
⟨*proof*⟩

lemma *deriv-j-ladder-shift-j*:
index < *length L* ⟹ *deriv-j* (*ladder-shift-j d L* ! *index*) = *deriv-j* (*L* ! *index*) − *d*
⟨*proof*⟩

lemma *deriv-n-ladder-shift-j*:
index < *length L* ⟹ *deriv-n* (*ladder-shift-j d L* ! *index*) = *deriv-n* (*L* ! *index*)
⟨*proof*⟩

lemma *deriv-ix-ladder-shift-j*:
index < *length L* ⟹ *deriv-ix* (*ladder-shift-j d L* ! *index*) = *deriv-ix* (*L* ! *index*)
⟨*proof*⟩

lemma *ladder-cut-prefix-j*:
assumes *index-bound*: *index* < *length L*
assumes *length-L*: *length L* > *0*
shows *ladder-j* (*ladder-cut-prefix d L*) *index* = *ladder-j L index* − *d*
⟨*proof*⟩

lemma *hd-0-subst*: *length L* > *0* ⟹ *hd* (*L* [*0* := *x*]) = *x*
⟨*proof*⟩

lemma *ladder-cut-prefix-i*:
assumes *index-bound*: $index < length\ L$
assumes *length-L*: $length\ L > 0$
shows $ladder\text{-}i\ (ladder\text{-}cut\text{-}prefix\ d\ L)\ index = ladder\text{-}i\ L\ index - d$
⟨*proof*⟩

lemma *ladder-cut-prefix-n*:
assumes *index-bound*: $index < length\ L$
assumes *length-L*: $length\ L > 0$
shows $ladder\text{-}n\ (ladder\text{-}cut\text{-}prefix\ d\ L)\ index = ladder\text{-}n\ L\ index$
⟨*proof*⟩

lemma *ladder-cut-prefix-ix*:
assumes *index-bound*: $index < length\ L$
assumes *length-L*: $length\ L > 0$
shows $ladder\text{-}ix\ (ladder\text{-}cut\text{-}prefix\ d\ L)\ index = ladder\text{-}ix\ L\ index$
⟨*proof*⟩

lemma *LeftDerivationFix-derivation-ge-is-nonterminal*:
assumes *ldfix*: $LeftDerivationFix\ \alpha\ i\ D\ j\ \gamma$
assumes *derivation-ge-d*: $derivation\text{-}ge\ D\ d$
assumes *is-nonterminal*: $is\text{-}nonterminal\ (\gamma\ !\ j)$
shows $(D = [] \wedge \alpha = \gamma \wedge i = j) \vee (i > d \wedge j \geq d)$
⟨*proof*⟩

lemma *LeftDerivationFix-derivation-ge*:
assumes *ldfix*: $LeftDerivationFix\ \alpha\ i\ D\ j\ \gamma$
assumes *derivation-ge-d*: $derivation\text{-}ge\ D\ d$
shows $i = j \vee (i > d \wedge j \geq d)$
⟨*proof*⟩

lemma *LeftDerivationIntro-derivation-ge*:
assumes *ldintro*: $LeftDerivationIntro\ \alpha\ i\ r\ ix\ D\ j\ \gamma$
assumes *i-ge-d*: $i \geq d$
assumes *derivation-ge-d*: $derivation\text{-}ge\ D\ d$
shows $j \geq d$
⟨*proof*⟩

lemma *derivation-ge-LeftDerivationLadder*:
assumes *derivation-ge-d*: $derivation\text{-}ge\ D\ d$
assumes *ladder*: $LeftDerivationLadder\ \alpha\ D\ L\ \gamma$
assumes *ladder-i-0*: $ladder\text{-}i\ L\ 0 \geq d$
shows $index < length\ L \Longrightarrow ladder\text{-}i\ L\ index \geq d \wedge ladder\text{-}j\ L\ index \geq d$
⟨*proof*⟩

lemma *derivation-shift-append*:
$derivation\text{-}shift\ (A@B)\ left\ right =$
$(derivation\text{-}shift\ A\ left\ right)\ @\ (derivation\text{-}shift\ B\ left\ right)$

⟨*proof*⟩

lemma *derivation-shift-right-left-subtract*:
right ≥ *left* ⟹ *derivation-shift* (*derivation-shift L 0 right*) *left 0* =
derivation-shift L 0 (*right* − *left*)
⟨*proof*⟩

lemma *LeftDerivationFix-cut-prefix*:
assumes *LeftDerivationFix* (*δ*@*α*) *i D j γ*
assumes *derivation-ge D* (*length δ*)
assumes *i* ≥ *length δ*
assumes *is-word-δ*: *is-word δ*
shows ∃ *γ′*. *γ* = *δ* @ *γ′* ∧
LeftDerivationFix α (*i* − *length δ*) (*derivation-shift D* (*length δ*) *0*) (*j* − *length δ*) *γ′*
⟨*proof*⟩

lemma *LeftDerives1-propagate-prefix*:
LeftDerives1 (*δ* @ *α*) *i r β* ⟹ *i* ≥ *length δ* ⟹ *is-prefix δ β*
⟨*proof*⟩

lemma *LeftDerivationIntro-cut-prefix*:
assumes *LeftDerivationIntro* (*δ*@*α*) *i r ix D j γ*
assumes *derivation-ge D* (*length δ*)
assumes *i* ≥ *length δ*
assumes *is-word-δ*: *is-word δ*
shows ∃ *γ′*. *γ* = *δ* @ *γ′* ∧
LeftDerivationIntro α (*i* − *length δ*) *r ix* (*derivation-shift D* (*length δ*) *0*) (*j* − *length δ*) *γ′*
⟨*proof*⟩

lemma *LeftDerivationLadder-implies-LeftDerivation-at-index*:
assumes *LeftDerivationLadder α D L γ*
assumes *index* < *length L*
shows *LeftDerivation α* (*take* (*ladder-n L index*) *D*) (*ladder-γ α D L index*)
⟨*proof*⟩

lemma *LeftDerivationLadder-cut-prefix-propagate*:
assumes *ladder*: *LeftDerivationLadder* (*δ*@*α*) *D L γ*
assumes *is-word-δ*: *is-word δ*
assumes *derivation-ge-δ*: *derivation-ge D* (*length δ*)
assumes *ladder-i-0*: *ladder-i L 0* ≥ *length δ*
assumes *L′*: *L′* = *ladder-cut-prefix* (*length δ*) *L*
assumes *D′*: *D′* = *derivation-shift D* (*length δ*) *0*
shows *index* < *length L* ⟹
LeftDerivation α (*take* (*ladder-n L′ index*) *D′*) (*ladder-γ α D′ L′ index*) ∧
ladder-α (*δ*@*α*) *D L index* = *δ*@(*ladder-α α D′ L′ index*) ∧
ladder-γ (*δ*@*α*) *D L index* = *δ*@(*ladder-γ α D′ L′ index*)
⟨*proof*⟩

theorem *LeftDerivationLadder-cut-prefix*:
assumes *ladder*: *LeftDerivationLadder* $(\delta @ \alpha)$ *D L* γ
assumes *is-word-δ*: *is-word* δ
assumes *ladder-i-0*: *ladder-i L 0* $\geq$ *length* δ
shows $\exists$ D' L' γ'. $\gamma = \delta$ @ $\gamma' \wedge$
LeftDerivationLadder α D' L' $\gamma' \wedge$
D' = *derivation-shift D* (*length* δ) *0* $\wedge$
length L' = *length L* $\wedge$ *ladder-i* L' *0* + *length* δ = *ladder-i L 0* $\wedge$
ladder-last-j L' + *length* δ = *ladder-last-j L*
⟨*proof*⟩

end

end
theory *TheoremD10*
imports *TheoremD9 Ladder*
begin

context *LocalLexing* **begin**

lemma $\mathcal{P}$*-wellformed*: $p \in \mathcal{P}$ *k u* $\Longrightarrow$ *wellformed-tokens p*
⟨*proof*⟩

lemma $\mathcal{X}$*-token-length*: $t \in \mathcal{X}$ *k* $\Longrightarrow$ *k* + *length* (*chars-of-token t*) $\leq$ *length Doc*
⟨*proof*⟩

lemma *mono-Scan*: *mono* (*Scan T k*)
⟨*proof*⟩

lemma π*-apply-setmonotone*: $x \in I \Longrightarrow x \in \pi$ *k T I*
⟨*proof*⟩

lemma *Scan-apply-setmonotone*: $x \in I \Longrightarrow x \in$ *Scan T k I*
⟨*proof*⟩

lemma *leftderives-padfront*:
assumes *leftderives* α β
assumes *is-word u*
shows *leftderives* $(u @ \alpha)$ $(u @ \beta)$
⟨*proof*⟩

lemma *leftderives-padback*:
assumes *leftderives* α β
assumes *is-sentence u*
shows *leftderives* $(\alpha @ u)$ $(\beta @ u)$
⟨*proof*⟩

lemma *leftderives-pad*:
  assumes α-β: *leftderives* α β
  assumes *is-word*: *is-word u*
  assumes *is-sentence*: *is-sentence v*
  shows *leftderives* $(u@\alpha@v)$ $(u@\beta@v)$
⟨*proof*⟩

lemma *leftderives-rule*:
  assumes $(N, w) \in \mathfrak{R}$
  shows *leftderives* $[N]$ *w*
⟨*proof*⟩

lemma *leftderives-rule-step*:
  assumes *ld*: *leftderives a* $(u@[N]@v)$
  assumes *rule*: $(N, w) \in \mathfrak{R}$
  assumes *is-word*: *is-word u*
  assumes *is-sentence*: *is-sentence v*
  shows *leftderives a* $(u@w@v)$
⟨*proof*⟩

lemma *leftderives-trans-step*:
  assumes *ld*: *leftderives a* $(u@b@v)$
  assumes *rule*: *leftderives b c*
  assumes *is-word*: *is-word u*
  assumes *is-sentence*: *is-sentence v*
  shows *leftderives a* $(u@c@v)$
⟨*proof*⟩

lemma *charslength-of-prefix*:
  assumes *is-prefix a b*
  shows *charslength a* $\leq$ *charslength b*
⟨*proof*⟩

lemma *item-rhs-simp*[*simp*]: *item-rhs* $(Item\ (N, \alpha)\ d\ i\ j) = \alpha$
  ⟨*proof*⟩

definition *Prefixes* :: *'a list* $\Rightarrow$ *'a list set*
where
  Prefixes p = $\{$ *q* . *is-prefix q p* $\}$

lemma $\mathfrak{P}$-*wellformed*: $p \in \mathfrak{P} \Longrightarrow$ *wellformed-tokens p*
  ⟨*proof*⟩

lemma *Prefixes-reflexive*[*simp*]: *p* $\in$ *Prefixes p*
  ⟨*proof*⟩

lemma *Prefixes-is-prefix*: *q* $\in$ *Prefixes p* = *is-prefix q p*
  ⟨*proof*⟩

lemma *prefixes-are-paths'*: $p \in \mathfrak{P} \Longrightarrow$ *is-prefix* $q\ p \Longrightarrow q \in \mathfrak{P}$
⟨*proof*⟩

lemma *thmD10-ladder*:
$p \in \mathfrak{P} \Longrightarrow$
charslength $p = k \Longrightarrow$
$X \in T \Longrightarrow$
$T \subseteq \mathcal{X}\ k \Longrightarrow$
$(N, \alpha @ \beta) \in \mathfrak{R} \Longrightarrow$
$r \leq$ *length* $p \Longrightarrow$
leftderives $[\mathfrak{S}]$ $((terminals\ (take\ r\ p)) @ [N] @ \gamma) \Longrightarrow$
LeftDerivationLadder $\alpha\ D\ L$ $(terminals\ ((drop\ r\ p) @ [X])) \Longrightarrow$
ladder-last-j $L =$ *length* $(drop\ r\ p) \Longrightarrow$
$k' = k +$ *length* $(chars\text{-}of\text{-}token\ X) \Longrightarrow$
$x =$ *Item* $(N, \alpha @ \beta)$ $(length\ \alpha)$ $(charslength\ (take\ r\ p))\ k' \Longrightarrow$
$I =$ *items-le* k' $(\pi\ k'\ \{\})$ $(Scan\ T\ k\ (Gen\ (Prefixes\ p))))$
$\Longrightarrow x \in I$
⟨*proof*⟩

theorem *thmD10*:
assumes *p-dom*: $p \in \mathfrak{P}$
assumes *p-charslength*: *charslength* $p = k$
assumes *X-dom*: $X \in T$
assumes *T-dom*: $T \subseteq \mathcal{X}\ k$
assumes *rule-dom*: $(N, \alpha @ \beta) \in \mathfrak{R}$
assumes *r*: $r \leq$ *length* p
assumes *leftderives-start*: *leftderives* $[\mathfrak{S}]$ $((terminals\ (take\ r\ p)) @ [N] @ \gamma)$
assumes *leftderives-α*: *leftderives* α $(terminals\ ((drop\ r\ p) @ [X]))$
assumes *k'*: $k' = k +$ *length* $(chars\text{-}of\text{-}token\ X)$
assumes *item-def*: $x =$ *Item* $(N, \alpha @ \beta)$ $(length\ \alpha)$ $(charslength\ (take\ r\ p))\ k'$
assumes *I*: $I =$ *items-le* k' $(\pi\ k'\ \{\})$ $(Scan\ T\ k\ (Gen\ (Prefixes\ p))))$
shows $x \in I$
⟨*proof*⟩

end

end
theory *TheoremD11*
imports *TheoremD10*
begin

context *LocalLexing* **begin**

lemma *LeftDerivationLadder-length-1*:
assumes *ladder*: *LeftDerivationLadder* $\alpha\ D\ L\ \gamma$
assumes *singleton-L*: *length* $L = 1$
shows *LeftDerivationFix* α $(ladder\text{-}i\ L\ 0)\ D$ $(ladder\text{-}last\text{-}j\ L)\ \gamma$
⟨*proof*⟩

lemma *LeftDerivationFix-from-singleton-helper*:
assumes *LeftDerivationFix* [*A*] *0 D* (*length u*) (*u* @ [*B*] @ *v*)
shows *D* = []
⟨*proof*⟩

lemma *LeftDerivationFix-from-singleton*:
assumes *LeftDerivationFix* [*A*] *i D j* γ
shows *D* = []
⟨*proof*⟩

lemma *LeftDerivationLadder-ladder-γ-last*:
assumes *LeftDerivationLadder* α *D L* γ
shows γ = *ladder-γ* α *D L* (*length L* − *1*)
⟨*proof*⟩

theorem *thmD11-helper*:
p ∈ 𝔓 ⟹
charslength p = *k* ⟹
X ∈ *T* ⟹
T ⊆ 𝒳 *k* ⟹
q = *p* @ [*X*] ⟹
(*N*, α@β) ∈ ℜ ⟹
r ≤ *length q* ⟹
LeftDerivation [𝔖] *D* ((*terminals* (*take r q*))@[*N*]@γ) ⟹
leftderives α (*terminals* (*drop r q*)) ⟹
k′ = *k* + *length* (*chars-of-token X*) ⟹
x = *Item* (*N*, α@β) (*length* α) (*charslength* (*take r q*)) *k′* ⟹
I = *items-le k′* (π *k′* {} (*Scan T k* (*Gen* (*Prefixes p*)))) ⟹
x ∈ *I*
⟨*proof*⟩

theorem *thmD11*:
assumes *p-dom*: *p* ∈ 𝔓
assumes *p-charslength*: *charslength p* = *k*
assumes *X-dom*: *X* ∈ *T*
assumes *T-dom*: *T* ⊆ 𝒳 *k*
assumes *q-def*: *q* = *p* @ [*X*]
assumes *rule-dom*: (*N*, α@β) ∈ ℜ
assumes *r*: *r* ≤ *length q*
assumes *leftderives-start*: *leftderives* [𝔖] ((*terminals* (*take r q*))@[*N*]@γ)
assumes *leftderives-α*: *leftderives* α (*terminals* (*drop r q*))
assumes *k′*: *k′* = *k* + *length* (*chars-of-token X*)
assumes *item-def*: *x* = *Item* (*N*, α@β) (*length* α) (*charslength* (*take r q*)) *k′*
assumes *I*: *I* = *items-le k′* (π *k′* {} (*Scan T k* (*Gen* (*Prefixes p*))))
shows *x* ∈ *I*
⟨*proof*⟩

end

end
theory *TheoremD12*
imports *TheoremD11*
begin

context *LocalLexing* **begin**

lemma *charslength-appendix-is-empty*:
charslength (*p*@*ts*) = *charslength p* ⟹ (⋀ *t*. *t* ∈ *set ts* ⟹ *chars-of-token t* = [])
⟨*proof*⟩

lemma *empty-tokens-have-charslength-0*:
(⋀ *t*. *t* ∈ *set ts* ⟹ *chars-of-token t* = []) ⟹ *charslength ts* = *0*
⟨*proof*⟩

lemma *π-idempotent'*: *π k* {} (*π k T I*) = *π k T I*
⟨*proof*⟩

theorem *thmD12*:
assumes *induct*: *items-le k* (*𝒥 k u*) = *Gen* (*paths-le k* (*𝒫 k u*))
assumes *induct-tokens*: *𝒯 k u* = *𝒵 k u*
shows *items-le k* (*𝒥 k* (*Suc u*)) ⊇ *Gen* (*paths-le k* (*𝒫 k* (*Suc u*)))
⟨*proof*⟩

end

end
theory *TheoremD13*
imports *TheoremD12*
begin

context *LocalLexing* **begin**

lemma *pointwise-natUnion-swap*:
assumes *pointwise-f*: *pointwise f*
shows *f* (*natUnion G*) = *natUnion* (λ *u*. *f* (*G u*))
⟨*proof*⟩

lemma *pointwise-Gen*: *pointwise Gen*
⟨*proof*⟩

lemma *thmD13-part1*:
assumes *start*: *items-le k* (*𝒥 k 0*) = *Gen* (*paths-le k* (*𝒫 k 0*))
assumes *valid-k*: *k* ≤ *length Doc*
shows *items-le k* (*𝒥 k u*) = *Gen* (*paths-le k* (*𝒫 k u*)) ∧ *𝒯 k u* = *𝒵 k u*
⟨*proof*⟩

lemma *thmD13-part2*:

assumes *start*: *items-le k* ($\mathcal{J}$ *k 0*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k 0*))
assumes *valid-k*: $k \leq$ *length Doc*
shows *items-le k* ($\mathcal{I}$ *k*) = *Gen* (*paths-le k* ($\mathcal{Q}$ *k*))
⟨*proof*⟩

theorem *thmD13*:
assumes *start*: *items-le k* ($\mathcal{J}$ *k 0*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k 0*))
assumes *valid-k*: $k \leq$ *length Doc*
shows *items-le k* ($\mathcal{J}$ *k u*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k u*)) ∧ $\mathcal{T}$ *k u* = $\mathcal{Z}$ *k u*
∧ *items-le k* ($\mathcal{I}$ *k*) = *Gen* (*paths-le k* ($\mathcal{Q}$ *k*))
⟨*proof*⟩

end

end
theory *TheoremD14*
imports *TheoremD13*
begin

context *LocalLexing* **begin**

lemma *empty-tokens-of-empty*[*simp*]: *empty-tokens* {} = {}
⟨*proof*⟩

lemma *items-le-split-via-eq*: *items-le* (*Suc k*) *J* = *items-le k J* ∪ *items-eq* (*Suc k*) *J*
⟨*proof*⟩

lemma *paths-le-split-via-eq*: *paths-le* (*Suc k*) *P* = *paths-le k P* ∪ *paths-eq* (*Suc k*) *P*
⟨*proof*⟩

lemma *natUnion-superset*:
shows *g i* ⊆ *natUnion g*
⟨*proof*⟩

definition *indexle* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
indexle k′ u′ k u = ((*indexlt k′ u′ k u*) ∨ (*k′* = *k* ∧ *u′* = *u*))

definition *produced-by-scan-step* :: *item* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
produced-by-scan-step x k u = (∃ *k′ u′ y X*. *indexle k′ u′ k u* ∧ *y* ∈ $\mathcal{J}$ *k′ u′* ∧
item-end y = *k′* ∧ *X* ∈ ($\mathcal{T}$ *k′ u′*) ∧ *x* = *inc-item y* (*k′* + *length* (*chars-of-token X*)) ∧
next-symbol y = *Some* (*terminal-of-token X*))

lemma *indexle-trans*: *indexle k″ u″ k′ u′* ⟹ *indexle k′ u′ k u* ⟹ *indexle k″ u″ k u*
⟨*proof*⟩

lemma *produced-by-scan-step-trans*:
assumes *indexle* k' u' k u
assumes *produced-by-scan-step* x k' u'
shows *produced-by-scan-step* x k u
⟨*proof*⟩

lemma $\mathcal{J}$*-induct*[*consumes 1*, *case-names Induct*]:
assumes $x \in \mathcal{J}\ k\ u$
assumes *induct*: $\bigwedge x\ k\ u\ .\ (\bigwedge x'\ k'\ u'.\ x' \in \mathcal{J}\ k'\ u' \Longrightarrow \mathit{indexlt}\ k'\ u'\ k\ u \Longrightarrow P\ x'\ k'\ u') \Longrightarrow x \in \mathcal{J}\ k\ u \Longrightarrow P\ x\ k\ u$
shows $P\ x\ k\ u$
⟨*proof*⟩

lemma π*-no-tokens-item-end*:
assumes *x-in-*π: $x \in \pi\ k\ \{\}\ I$
shows *item-end* $x = k \vee x \in I$
⟨*proof*⟩

lemma *natUnion-ex*: $x \in \mathit{natUnion}\ f \Longrightarrow \exists\ i.\ x \in f\ i$
⟨*proof*⟩

lemma *locate-in-limit*:
assumes *x-in-limit*: $x \in \mathit{limit}\ f\ X$
assumes *x-notin-X*: $x \notin X$
shows $\exists\ n.\ x \in \mathit{funpower}\ f\ (\mathit{Suc}\ n)\ X \wedge x \notin \mathit{funpower}\ f\ n\ X$
⟨*proof*⟩

lemma *produced-by-scan-step*:
$x \in \mathcal{J}\ k\ u \Longrightarrow \mathit{item\text{-}end}\ x > k \Longrightarrow \mathit{produced\text{-}by\text{-}scan\text{-}step}\ x\ k\ u$
⟨*proof*⟩

lemma *limit-single-step*:
assumes $x \in f\ X$
shows $x \in \mathit{limit}\ f\ X$
⟨*proof*⟩

lemma *Gen-union*: $\mathit{Gen}\ (A \cup B) = \mathit{Gen}\ A \cup \mathit{Gen}\ B$
⟨*proof*⟩

lemma *is-prefix-Prefixes-subset*:
assumes *is-prefix* q p
shows *Prefixes* $q \subseteq$ *Prefixes* p
⟨*proof*⟩

lemma *Prefixes-subset-*$\mathcal{P}$:
assumes $p \in \mathcal{P}\ k\ u$
shows *Prefixes* $p \subseteq \mathcal{P}\ k\ u$
⟨*proof*⟩

lemma *Prefixes-subset-paths-le*:
assumes *Prefixes p* $\subseteq$ *P*
shows *Prefixes p* $\subseteq$ *paths-le* (*charslength p*) *P*
⟨*proof*⟩

lemma *Scan-*$\mathcal{J}$*-subset-*$\mathcal{J}$:
Scan ($\mathcal{T}$ *k* (*Suc u*)) *k* ($\mathcal{J}$ *k u*) $\subseteq$ $\mathcal{J}$ *k* (*Suc u*)
⟨*proof*⟩

lemma *subset-*$\mathcal{J}$*k*: *u* $\leq$ *v* $\Longrightarrow$ $\mathcal{J}$ *k u* $\subseteq$ $\mathcal{J}$ *k v*
thm $\mathcal{J}$*-subset-Suc-u*
⟨*proof*⟩

lemma *subset-*$\mathcal{J}\mathcal{I}$*k*: $\mathcal{J}$ *k u* $\subseteq$ $\mathcal{I}$ *k* ⟨*proof*⟩

lemma *subset-*$\mathcal{I}\mathcal{J}$*Suc*: $\mathcal{I}$ *k* $\subseteq$ $\mathcal{J}$ (*Suc k*) *u*
⟨*proof*⟩

lemma *subset-*$\mathcal{I}$*Suc*: $\mathcal{I}$ *k* $\subseteq$ $\mathcal{I}$ (*Suc k*)
⟨*proof*⟩

lemma *subset-*$\mathcal{I}$: *i* $\leq$ *j* $\Longrightarrow$ $\mathcal{I}$ *i* $\subseteq$ $\mathcal{I}$ *j*
⟨*proof*⟩

lemma *subset-*$\mathcal{J}$:
assumes *leq*: *k'* < *k* $\vee$ (*k'* = *k* $\wedge$ *u'* $\leq$ *u*)
shows $\mathcal{J}$ *k' u'* $\subseteq$ $\mathcal{J}$ *k u*
⟨*proof*⟩

lemma $\mathcal{J}$*-subset*:
assumes *indexle k' u' k u*
shows $\mathcal{J}$ *k' u'* $\subseteq$ $\mathcal{J}$ *k u*
⟨*proof*⟩

lemma *Scan-items-le*:
assumes *bounded-T*: $\bigwedge$ *t* . *t* $\in$ *T* $\Longrightarrow$ *length* (*chars-of-token t*) $\leq$ *l*
shows *Scan T k* (*items-le k P*) $\subseteq$ *items-le* (*k* + *l*) (*Scan T k P*)
⟨*proof*⟩

lemma *Scan-mono-tokens*:
P $\subseteq$ *Q* $\Longrightarrow$ *Scan P k I* $\subseteq$ *Scan Q k I*
⟨*proof*⟩

theorem *thmD14*: *k* $\leq$ *length Doc* $\Longrightarrow$ *items-le k* ($\mathcal{J}$ *k u*) = *Gen* (*paths-le k* ($\mathcal{P}$ *k u*)) $\wedge$ $\mathcal{T}$ *k u* = $\mathcal{Z}$ *k u*
$\wedge$ *items-le k* ($\mathcal{I}$ *k*) = *Gen* (*paths-le k* ($\mathcal{Q}$ *k*))
⟨*proof*⟩

end

end
theory *PathLemmas*
imports *TheoremD14*
begin

context *LocalLexing* **begin**

lemma *characterize-*$\mathcal{P}$:
  $(\forall\ i < \mathit{length}\ p.\ \exists\, u.\ p\ !\ i \in \mathcal{Z}\ (\mathit{charslength}\ (\mathit{take}\ i\ p))\ u) \Longrightarrow \mathit{admissible}\ p \Longrightarrow$
  $\exists\ u.\ p \in \mathcal{P}\ (\mathit{charslength}\ p)\ u$
⟨*proof*⟩

lemma *drop-empty-tokens*:
  assumes *p*: $p \in \mathfrak{P}$
  assumes *r*: $r \leq \mathit{length}\ p$
  assumes *empty*: $\mathit{charslength}\ (\mathit{take}\ r\ p) = 0$
  assumes *admissible*: $\mathit{admissible}\ (\mathit{drop}\ r\ p)$
  shows $\mathit{drop}\ r\ p \in \mathfrak{P}$
⟨*proof*⟩

end

end
theory *MainTheorems*
imports *PathLemmas*
begin

context *LocalLexing* **begin**

theorem $\mathfrak{I}$*-is-generated-by-*$\mathfrak{P}$: $\mathfrak{I} = \mathit{Gen}\ \mathfrak{P}$
⟨*proof*⟩

definition *finished-item* :: *symbol list* ⇒ *item*
where
  $\mathit{finished\text{-}item}\ \alpha = \mathit{Item}\ (\mathfrak{S}, \alpha)\ (\mathit{length}\ \alpha)\ 0\ (\mathit{length}\ \mathit{Doc})$

lemma *item-rule-finished-item*[*simp*]: $\mathit{item\text{-}rule}\ (\mathit{finished\text{-}item}\ \alpha) = (\mathfrak{S}, \alpha)$
  ⟨*proof*⟩

lemma *item-origin-finished-item*[*simp*]: $\mathit{item\text{-}origin}\ (\mathit{finished\text{-}item}\ \alpha) = 0$
  ⟨*proof*⟩

lemma *item-end-finished-item*[*simp*]: $\mathit{item\text{-}end}\ (\mathit{finished\text{-}item}\ \alpha) = \mathit{length}\ \mathit{Doc}$
  ⟨*proof*⟩

lemma *item-dot-finished-item*[*simp*]: $\mathit{item\text{-}dot}\ (\mathit{finished\text{-}item}\ \alpha) = \mathit{length}\ \alpha$
  ⟨*proof*⟩

lemma *item-rhs-finished-item*[*simp*]: *item-rhs* (*finished-item* α) = α
⟨*proof*⟩

lemma *item-α-finished-item*[*simp*]: *item-α* (*finished-item* α) = α
⟨*proof*⟩

lemma *item-nonterminal-finished-item*[*simp*]: *item-nonterminal* (*finished-item* α) = $\mathfrak{S}$
⟨*proof*⟩

lemma *Derives1-of-singleton*:
assumes *Derives1* [*N*] *i r* α
shows *i* = *0* ∧ *r* = (*N*, α)
⟨*proof*⟩

definition *pvalid-with* :: *tokens* ⇒ *item* ⇒ *nat* ⇒ *symbol list* ⇒ *bool*
where
pvalid-with p x u γ =
(*wellformed-tokens p* ∧
wellformed-item x ∧
u ≤ *length p* ∧
charslength p = *item-end x* ∧
charslength (*take u p*) = *item-origin x* ∧
is-derivation (*terminals* (*take u p*) @ [*item-nonterminal x*] @ γ) ∧
derives (*item-α x*) (*terminals* (*drop u p*)))

lemma *pvalid-with*: *pvalid p x* = (∃ *u* γ. *pvalid-with p x u* γ)
⟨*proof*⟩

theorem *Completeness*:
assumes *p-in-ll*: *p* ∈ *ll*
shows ∃ α. *pvalid-with p* (*finished-item* α) *0* [] ∧ *finished-item* α ∈ $\mathfrak{I}$
⟨*proof*⟩

theorem *Soundness*:
assumes *finished-item-α*: *finished-item* α ∈ $\mathfrak{I}$
shows ∃ *p*. *pvalid-with p* (*finished-item* α) *0* [] ∧ *p* ∈ *ll*
⟨*proof*⟩

lemma *is-finished-and-finished-item*:
assumes *wellformed-x*: *wellformed-item x*
shows *is-finished x* = (∃ α. *x* = *finished-item* α)
⟨*proof*⟩

theorem *Correctness*:
shows (*ll* ≠ {}) = *earley-recognised*
⟨*proof*⟩

end

end