Local Lexing

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Abstract

This formalisation accompanies the paper Local Lexing[1], which introduces a novel parsing concept of the same name. The paper also gives a high-level algorithm for local lexing as an extension of Earley's algorithm. This formalisation proves the algorithm to be correct with respect to its local lexing semantics. As a special case, this formalisation thus also contains a proof of the correctness of Earley's algorithm. The paper contains a short outline of how this formalisation is organised.

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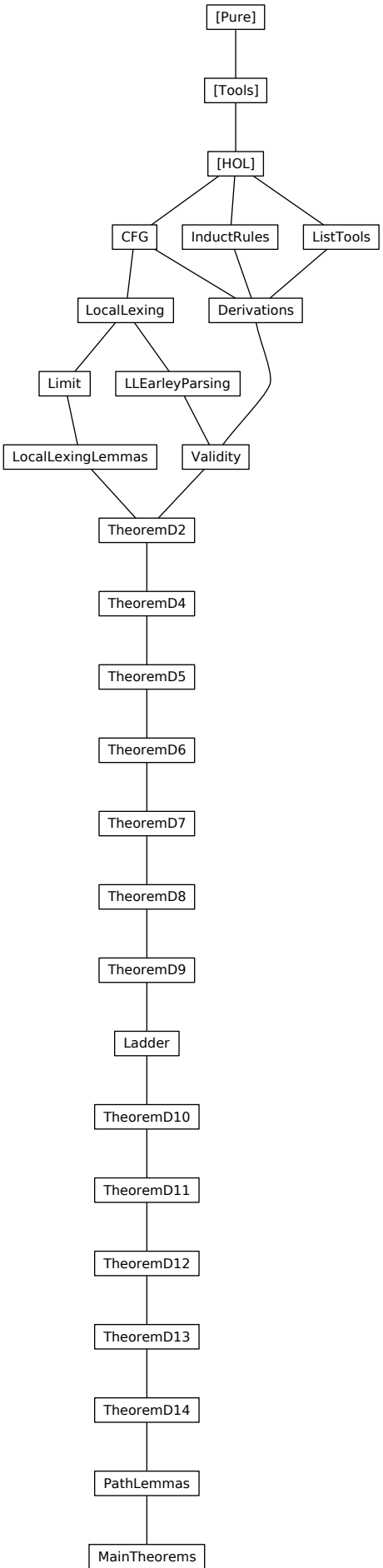
[1] https://arxiv.org/abs/1702.03277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theory *CFG*
imports *Main*
begin

typedecl *symbol*

type-synonym *rule* = *symbol* × *symbol list*

type-synonym *sentence* = *symbol list*

locale *CFG* =
  fixes $\mathfrak{N}$ :: *symbol set*
  fixes $\mathfrak{T}$ :: *symbol set*
  fixes $\mathfrak{R}$ :: *rule set*
  fixes $\mathfrak{S}$ :: *symbol*
  assumes *disjunct-symbols*: $\mathfrak{N} \cap \mathfrak{T} = \{\}$
  assumes *startsymbol-dom*: $\mathfrak{S} \in \mathfrak{N}$
  assumes *validRules*: $\forall\ (N, \alpha) \in \mathfrak{R}.\ N \in \mathfrak{N} \wedge (\forall\ s \in set\ \alpha.\ s \in \mathfrak{N} \cup \mathfrak{T})$
begin

definition *is-terminal* :: *symbol* ⇒ *bool*
where
  is-terminal s = $(s \in \mathfrak{T})$

definition *is-nonterminal* :: *symbol* ⇒ *bool*
where
  is-nonterminal s = $(s \in \mathfrak{N})$

lemma *is-nonterminal-startsymbol*:*is-nonterminal* $\mathfrak{S}$
  by (*simp add*: *is-nonterminal-def startsymbol-dom*)

definition *is-symbol* :: *symbol* ⇒ *bool*
where
  is-symbol s = (*is-terminal s* ∨ *is-nonterminal s*)

definition *is-sentence* :: *sentence* ⇒ *bool*
where
  is-sentence s = *list-all is-symbol s*

definition *is-word* :: *sentence* ⇒ *bool*
where
  is-word s = *list-all is-terminal s*

definition *derives1* :: *sentence* ⇒ *sentence* ⇒ *bool*
where
  derives1 u v =
    (∃ *x y N* α.
        u = *x* @ [*N*] @ *y*
      ∧ *v* = *x* @ α @ *y*

$\wedge$ *is-sentence x*
$\wedge$ *is-sentence y*
$\wedge$ $(N, \alpha) \in \mathfrak{R}$)

definition *derivations1* :: (*sentence* $\times$ *sentence*) *set*
where
derivations1 = { (*u*,*v*) | *u v*. *derives1 u v* }

definition *derivations* :: (*sentence* $\times$ *sentence*) *set*
where
derivations = *derivations1*$\widehat{}*$

definition *derives* :: *sentence* $\Rightarrow$ *sentence* $\Rightarrow$ *bool*
where
derives u v = ((*u*, *v*) $\in$ *derivations*)

definition *is-derivation* :: *sentence* $\Rightarrow$ *bool*
where
is-derivation u = *derives* [$\mathfrak{S}$] *u*

definition $\mathcal{L}$ :: *sentence set*
where
$\mathcal{L}$ = { *v* | *v*. *is-word v* $\wedge$ *is-derivation v*}

definition $\mathcal{L}_P$ :: *sentence set*
where
$\mathcal{L}_P$ = { *u* | *u v*. *is-word u* $\wedge$ *is-derivation* (*u*@*v*) }

end

end
theory *LocalLexing*
imports *CFG*
begin

typedecl *character*

type-synonym *lexer* = *character list* $\Rightarrow$ *nat* $\Rightarrow$ *nat set*

type-synonym *token* = *symbol* $\times$ *character list*

type-synonym *tokens* = *token list*

definition *terminal-of-token* :: *token* $\Rightarrow$ *symbol*
where
terminal-of-token t = *fst t*

definition *terminals* :: *tokens* $\Rightarrow$ *sentence*
where

```
  terminals ts = map terminal-of-token ts

definition chars-of-token :: token ⇒ character list
where
  chars-of-token t = snd t

fun chars :: tokens ⇒ character list
where
  chars [] = []
| chars (t#ts) = (chars-of-token t) @ (chars ts)

fun charslength :: tokens ⇒ nat
where
  charslength cs = length (chars cs)

definition is-lexer :: lexer ⇒ bool
where
  is-lexer lexer =
    (∀ D p l. (p ≤ length D ∧ l ∈ lexer D p ⟶ p + l ≤ length D) ∧
          (p > length D ⟶ lexer D p = {}))

type-synonym selector = token set ⇒ token set ⇒ token set

definition is-selector :: selector ⇒ bool
where
  is-selector sel = (∀ A B. A ⊆ B ⟶ (A ⊆ sel A B ∧ sel A B ⊆ B))

fun by-length :: nat ⇒ tokens set ⇒ tokens set
where
  by-length l tss = { ts . ts ∈ tss ∧ length (chars ts) = l }

fun funpower :: ('a ⇒ 'a) ⇒ nat ⇒ ('a ⇒ 'a)
where
  funpower f 0 x = x
| funpower f (Suc n) x = f (funpower f n x)

definition natUnion :: (nat ⇒ 'a set) ⇒ 'a set
where
  natUnion f = ⋃ { f n | n. True }

definition limit  :: ('a set ⇒ 'a set) ⇒ 'a set ⇒ 'a set
where
  limit f x = natUnion (λ n. funpower f n x)

locale LocalLexing = CFG +
  fixes Lex :: symbol ⇒ lexer
  fixes Sel :: selector
  assumes Lex-is-lexer: ∀ t ∈ 𝔗. is-lexer (Lex t)
  assumes Sel-is-selector: is-selector Sel
```

```
  fixes Doc :: character list
begin

definition admissible :: tokens ⇒ bool
where
  admissible ts = (terminals ts ∈ ℒ_P)

definition Append :: token set ⇒ nat ⇒ tokens set ⇒ tokens set
where
  Append Z k P = P ∪
    { p @ [t] | p t. p ∈ by-length k P ∧ t ∈ Z ∧ admissible (p @ [t])}

definition 𝒳 :: nat ⇒ token set
where
  𝒳 k = {(t, ω) | t l ω. t ∈ 𝔗 ∧ l ∈ Lex t Doc k ∧ ω = take l (drop k Doc)}

definition 𝒲 :: tokens set ⇒ nat ⇒ token set
where
  𝒲 P k =  { u. u ∈ 𝒳 k ∧ (∃ p ∈ by-length k P. admissible (p@[u])) }

definition 𝒴 :: token set ⇒ tokens set ⇒ nat ⇒ token set
where
  𝒴 T P k = Sel T (𝒲 P k)

fun 𝒫 :: nat ⇒ nat ⇒ tokens set
and 𝒬 :: nat ⇒ tokens set
and 𝒵 :: nat ⇒ nat ⇒ token set
where
  𝒫 0 0 = {[]}
| 𝒫 k (Suc u) = limit (Append (𝒵 k (Suc u)) k) (𝒫 k u)
| 𝒫 (Suc k) 0 = 𝒬 k
| 𝒵 k 0 = {}
| 𝒵 k (Suc u) = 𝒴 (𝒵 k u) (𝒫 k u) k
| 𝒬 k = natUnion (𝒫 k)

definition 𝔓 :: tokens set
where
  𝔓 = 𝒬 (length Doc)

definition ll :: tokens set
where
  ll = { p . p ∈ 𝔓 ∧ charslength p = length Doc ∧ terminals p ∈ ℒ }

end

end
theory LLEarleyParsing
imports LocalLexing
begin
```

datatype *item* =
  Item
    (*item-rule*: *rule*)
    (*item-dot* : *nat*)
    (*item-origin* : *nat*)
    (*item-end* : *nat*)

type-synonym *items* = *item set*

definition *item-nonterminal* :: *item* ⇒ *symbol*
where
  item-nonterminal x = *fst* (*item-rule x*)

definition *item-rhs* :: *item* ⇒ *sentence*
where
  item-rhs x = *snd* (*item-rule x*)

definition *item-α* :: *item* ⇒ *sentence*
where
  item-α x = *take* (*item-dot x*) (*item-rhs x*)

definition *item-β* :: *item* ⇒ *sentence*
where
  item-β x = *drop* (*item-dot x*) (*item-rhs x*)

definition *init-item* :: *rule* ⇒ *nat* ⇒ *item*
where
  init-item r k = *Item r 0 k k*

definition *is-complete* :: *item* ⇒ *bool*
where
  is-complete x = (*item-dot x* ≥ *length* (*item-rhs x*))

definition *next-symbol* :: *item* ⇒ *symbol option*
where
  next-symbol x = (*if is-complete x then None else Some* ((*item-rhs x*) ! (*item-dot x*)))

definition *inc-item* :: *item* ⇒ *nat* ⇒ *item*
where
  inc-item x k = *Item* (*item-rule x*) (*item-dot x* + *1*) (*item-origin x*) *k*

definition *bin* :: *items* ⇒ *nat* ⇒ *items*
where
  bin I k = { *x* . *x* ∈ *I* ∧ *item-end x* = *k* }

context *LocalLexing* **begin**

definition *Init* :: *items*
where
Init = { *init-item r 0* | *r*. *r* ∈ ℜ ∧ *fst r* = 𝔖 }

definition *Predict* :: *nat* ⇒ *items* ⇒ *items*
where
Predict k I = *I* ∪
{ *init-item r k* | *r x*. *r* ∈ ℜ ∧ *x* ∈ *bin I k* ∧
next-symbol x = *Some*(*fst r*) }

definition *Complete* :: *nat* ⇒ *items* ⇒ *items*
where
Complete k I = *I* ∪ { *inc-item x k* | *x y*.
x ∈ *bin I* (*item-origin y*) ∧ *y* ∈ *bin I k* ∧ *is-complete y* ∧
next-symbol x = *Some* (*item-nonterminal y*) }

definition *TokensAt* :: *nat* ⇒ *items* ⇒ *token set*
where
TokensAt k I = { (*t*, *s*) | *t s x l*. *x* ∈ *bin I k* ∧
next-symbol x = *Some t* ∧ *is-terminal t* ∧
l ∈ *Lex t Doc k* ∧ *s* = *take l* (*drop k Doc*) }

definition *Tokens* :: *nat* ⇒ *token set* ⇒ *items* ⇒ *token set*
where
Tokens k T I = *Sel T* (*TokensAt k I*)

definition *Scan* :: *token set* ⇒ *nat* ⇒ *items* ⇒ *items*
where
Scan T k I = *I* ∪
{ *inc-item x* (*k* + *length c*) | *x t c*. *x* ∈ *bin I k* ∧ (*t*, *c*) ∈ *T* ∧
next-symbol x = *Some t* }

definition π :: *nat* ⇒ *token set* ⇒ *items* ⇒ *items*
where
π *k T I* =
limit (λ *I*. *Scan T k* (*Complete k* (*Predict k I*))) *I*

fun 𝒥 :: *nat* ⇒ *nat* ⇒ *items*
and ℐ :: *nat* ⇒ *items*
and 𝒯 :: *nat* ⇒ *nat* ⇒ *token set*
where
𝒥 *0 0* = π *0* {} *Init*
| 𝒥 *k* (*Suc u*) = π *k* (𝒯 *k* (*Suc u*)) (𝒥 *k u*)
| 𝒥 (*Suc k*) *0* = π (*Suc k*) {} (ℐ *k*)
| 𝒯 *k 0* = {}
| 𝒯 *k* (*Suc u*) = *Tokens k* (𝒯 *k u*) (𝒥 *k u*)
| ℐ *k* = *natUnion* (𝒥 *k*)

definition ℑ :: *items*

where
$\mathfrak{I} = \mathcal{I}\ (\mathit{length}\ \mathit{Doc})$

definition *is-finished* :: *item* ⇒ *bool* **where**
is-finished $x = (\mathit{item\text{-}nonterminal}\ x = \mathfrak{S} \wedge \mathit{item\text{-}origin}\ x = 0 \wedge \mathit{item\text{-}end}\ x = \mathit{length}\ \mathit{Doc} \wedge$
is-complete $x)$

definition *earley-recognised* :: *bool*
where
earley-recognised $= (\exists\ x \in \mathfrak{I}.\ \mathit{is\text{-}finished}\ x)$

end

end
theory *Limit*
imports *LocalLexing*
begin

definition *setmonotone* :: $('a\ \mathit{set} \Rightarrow 'a\ \mathit{set}) \Rightarrow \mathit{bool}$
where
setmonotone $f = (\forall\ X.\ X \subseteq f\ X)$

lemma *setmonotone-funpower*: *setmonotone* $f \Longrightarrow$ *setmonotone* (*funpower* $f\ n$)
by (*induct n*, *auto simp add*: *setmonotone-def*)

lemma *subset-setmonotone*: *setmonotone* $f \Longrightarrow X \subseteq f\ X$
by (*simp add*: *setmonotone-def*)

lemma *elem-setmonotone*: *setmonotone* $f \Longrightarrow x \in X \Longrightarrow x \in f\ X$
by (*auto simp add*: *setmonotone-def*)

lemma *elem-natUnion*: $(\forall\ n.\ x \in f\ n) \Longrightarrow x \in \mathit{natUnion}\ f$
by (*auto simp add*: *natUnion-def*)

lemma *subset-natUnion*: $(\forall\ n.\ X \subseteq f\ n) \Longrightarrow X \subseteq \mathit{natUnion}\ f$
by (*auto simp add*: *natUnion-def*)

lemma *setmonotone-limit*:
assumes *fmono*: *setmonotone* f
shows *setmonotone* (*limit* f)
proof −
show *setmonotone* (*limit* f)
apply (*auto simp add*: *setmonotone-def limit-def*)
apply (*rule elem-natUnion*, *auto*)
apply (*rule elem-setmonotone*[*OF setmonotone-funpower*])
by (*auto simp add*: *fmono*)
qed

lemma[*simp*]: *funpower id n = id*
by (*rule ext*, *induct n*, *simp-all*)

lemma[*simp*]: *limit id = id*
by (*rule ext*, *auto simp add*: *limit-def natUnion-def*)

lemma *natUnion-decompose*[*consumes 1*, *case-names Decompose*]:
assumes *p*: *p* ∈ *natUnion S*
assumes *decompose*: ⋀ *n p*. *p* ∈ *S n* ⟹ *P p*
shows *P p*
proof −
from *p* **have** ∃ *n*. *p* ∈ *S n*
by (*auto simp add*: *natUnion-def*)
then obtain *n* **where** *p* ∈ *S n* **by** *blast*
from *decompose*[*OF this*] **show** *?thesis* **.**
qed

lemma *limit-induct*[*consumes 1*, *case-names Init Iterate*]:
assumes *p*: (*p* :: *'a*) ∈ *limit f X*
assumes *init*: ⋀ *p*. *p* ∈ *X* ⟹ *P p*
assumes *iterate*: ⋀ *p Y*. (⋀ *q* . *q* ∈ *Y* ⟹ *P q*) ⟹ *p* ∈ *f Y* ⟹ *P p*
shows *P p*
proof −
from *p* **have** *p-in-natUnion*: *p* ∈ *natUnion* (λ *n*. *funpower f n X*)
by (*simp add*: *limit-def*)
{
fix *p* :: *'a*
fix *n* :: *nat*
have *p* ∈ *funpower f n X* ⟹ *P p*
proof (*induct n arbitrary*: *p*)
case *0* **thus** *?case* **using** *init*[*OF 0*[*simplified*]] **by** *simp*
next
case (*Suc n*) **show** *?case*
using *iterate*[*OF Suc*(*1*) *Suc*(*2*)[*simplified*], *simplified*] **by** *simp*
qed
}
with *p-in-natUnion* **show** *?thesis*
by (*induct rule*: *natUnion-decompose*)
qed

definition *chain* :: (*nat* ⇒ *'a set*) ⇒ *bool*
where
chain C = (∀ *i*. *C i* ⊆ *C* (*i* + *1*))

definition *continuous* :: (*'a set* ⇒ *'b set*) ⇒ *bool*
where
continuous f = (∀ *C*. *chain C* ⟶ (*chain* (*f o C*) ∧ *f* (*natUnion C*) = *natUnion* (*f o C*)))

lemma *continuous-apply*:
  continuous f $\Longrightarrow$ *chain C* $\Longrightarrow$ *f* (*natUnion C*) = *natUnion* (*f o C*)
by (*simp add*: *continuous-def*)

lemma *continuous-imp-mono*:
  assumes *continuous*: *continuous f*
  shows *mono f*
proof −
  {
    fix *A* :: *'a set*
    fix *B* :: *'a set*
    assume *sub*: *A* ⊆ *B*
    let *?C* = λ (*i*::*nat*). *if* (*i* = *0*) *then A else B*
    have *chain ?C* **by** (*simp add*: *chain-def sub*)
    then have *fC*: *chain* (*f o ?C*) **using** *continuous continuous-def* **by** *blast*
    then have *f* (*?C 0*) ⊆ *f* (*?C* (*0* + *1*))
    proof −
      have ⋀*f n*. ¬ *chain f* ∨ (*f n*::*'b set*) ⊆ *f* (*Suc n*)
        by (*metis Suc-eq-plus1 chain-def*)
      then show *?thesis* **using** *fC* **by** *fastforce*
    qed
    then have *f A* ⊆ *f B* **by** *auto*
  }
  then show *mono f* **by** (*simp add*: *monoI*)
qed

lemma *mono-maps-chain-to-chain*:
  assumes *f*: *mono f*
  assumes *C*: *chain C*
  shows *chain* (*f o C*)
by (*metis C comp-def f chain-def mono-def*)

lemma *natUnion-upperbound*:
  (⋀ *n*. *f n* ⊆ *G*) $\Longrightarrow$ (*natUnion f*) ⊆ *G*
by (*auto simp add*: *natUnion-def*)

lemma *funpower-upperbound*:
  (⋀ *I*. *I* ⊆ *G* $\Longrightarrow$ *f I* ⊆ *G*) $\Longrightarrow$ *I* ⊆ *G* $\Longrightarrow$ *funpower f n I* ⊆ *G*
proof (*induct n*)
  case *0* **thus** *?case* **by** *simp*
next
  case (*Suc n*) **thus** *?case* **by** *simp*
qed

lemma *limit-upperbound*:
  (⋀ *I*. *I* ⊆ *G* $\Longrightarrow$ *f I* ⊆ *G*) $\Longrightarrow$ *I* ⊆ *G* $\Longrightarrow$ *limit f I* ⊆ *G*
by (*simp add*: *funpower-upperbound limit-def natUnion-upperbound*)

lemma *elem-limit-simp*: *x* ∈ *limit f X* = (∃ *n*. *x* ∈ *funpower f n X*)

by (*auto simp add*: *limit-def natUnion-def*)

definition *pointwise* :: (*'a set* ⇒ *'b set*) ⇒ *bool* **where**
pointwise f = (∀ *X*. *f X* = ⋃ { *f* {*x*} | *x*. *x* ∈ *X*})

lemma *pointwise-simp*:
assumes *f*: *pointwise f*
shows *f X* = ⋃ { *f* {*x*} | *x*. *x* ∈ *X*}
proof −
from *f* **have** ∀ *X*. *f X* = ⋃ { *f* {*x*} | *x*. *x* ∈ *X*}
by (*rule iffD1*[*OF pointwise-def*[**where** *f*=*f*]])
then show *?thesis* **by** *blast*
qed

lemma *natUnion-elem*: *x* ∈ *f n* ⟹ *x* ∈ *natUnion f*
using *natUnion-def* **by** *fastforce*

lemma *limit-elem*: *x* ∈ *funpower f n X* ⟹ *x* ∈ *limit f X*
by (*simp add*: *limit-def natUnion-elem*)

lemma *limit-step-pointwise*:
assumes *x*: *x* ∈ *limit f X*
assumes *f*: *pointwise f*
assumes *y*: *y* ∈ *f* {*x*}
shows *y* ∈ *limit f X*
proof −
from *x* **have** ∃ *n*. *x* ∈ *funpower f n X*
by (*simp add*: *elem-limit-simp*)
then obtain *n* **where** *n*: *x* ∈ *funpower f n X* **by** *blast*
have *y* ∈ *funpower f* (*Suc n*) *X*
apply *simp*
apply (*subst pointwise-simp*[*OF f*])
using *y n* **by** *auto*
then show *y* ∈ *limit f X* **by** (*meson limit-elem*)
qed

definition *pointbase* :: (*'a set* ⇒ *'b set*) ⇒ *'a set* ⇒ *'b set* **where**
pointbase F I = ⋃ { *F X* | *X*. *finite X* ∧ *X* ⊆ *I* }

definition *pointbased* :: (*'a set* ⇒ *'b set*) ⇒ *bool* **where**
pointbased f = (∃ *F*. *f* = *pointbase F*)

lemma *pointwise-implies-pointbased*:
assumes *pointwise*: *pointwise f*
shows *pointbased f*
proof −
let *?F* = λ *X*. *f X*
{
fix *I* :: *'a set*

```
    fix x :: 'b
    have lr: x ∈ pointbase ?F I ⟹ x ∈ f I
    proof −
      assume x: x ∈ pointbase ?F I
      have ∃ X. x ∈ f X ∧ X ⊆ I
        proof −
          have x ∈ ⋃{f A |A. finite A ∧ A ⊆ I}
            by (metis pointbase-def x)
          then show ?thesis
            by blast
        qed
      then obtain X where X:x ∈ f X ∧ X ⊆ I by blast
      have ∃ y. y ∈ I ∧ x ∈ f {y}
        using X apply (simp add: pointwise-simp[OF pointwise, where X=X])
        by blast
      then show x ∈ f I
        apply (simp add: pointwise-simp[OF pointwise, where X=I])
        by blast
    qed
    have rl: x ∈ f I ⟹ x ∈ pointbase ?F I
    proof −
      assume x: x ∈ f I
      have ∃ y. y ∈ I ∧ x ∈ f {y}
        using x apply (simp add: pointwise-simp[OF pointwise, where X=I])
        by blast
      then obtain y where y ∈ I ∧ x ∈ f {y} by blast
      then have ∃ X. x ∈ f X ∧ finite X ∧ X ⊆ I by blast
      then show x ∈ pointbase f I
        apply (simp add: pointbase-def)
        by blast
    qed
    note lr rl
  }
  then have ⋀ I. pointbase f I = f I by blast
  then have pointbase f = f by blast
  then show ?thesis by (metis pointbased-def)
qed

lemma pointbase-is-mono:
  mono (pointbase f)
proof −
  {
    fix A :: 'a set
    fix B :: 'a set
    assume subset: A ⊆ B
    have (pointbase f) A ⊆ (pointbase f) B
      apply (simp add: pointbase-def)
      using subset by fastforce
  }
```

```
  then show ?thesis by (simp add: mono-def)
qed

lemma chain-implies-mono: chain C ⟹ mono C
by (simp add: chain-def mono-iff-le-Suc)

lemma chain-cover-witness: finite X ⟹ chain C ⟹ X ⊆ natUnion C ⟹ ∃ n.
X ⊆ C n
proof (induct rule: finite.induct)
  case emptyI thus ?case by blast
next
  case (insertI X x)
  then have X ⊆ natUnion C by simp
  with insertI have ∃ n. X ⊆ C n by blast
  then obtain n where n: X ⊆ C n by blast
  have x: x ∈ natUnion C using insertI.prems(2) by blast
  then have ∃ m. x ∈ C m
  proof −
    have x ∈ ⋃{A. ∃ n. A = C n} by (metis x natUnion-def)
    then show ?thesis by blast
  qed
  then obtain m where m: x ∈ C m by blast
  have mono-C: ⋀ i j. i ≤ j ⟹ C i ⊆ C j
    using chain-implies-mono insertI(3) mono-def by blast
  show ?case
    apply (rule-tac x=max n m in exI)
    apply auto
    apply (meson contra-subsetD m max.cobounded2 mono-C)
    by (metis max-def mono-C n subsetCE)
qed

lemma pointbase-is-continuous:
  continuous (pointbase f)
proof −
  {
    fix C :: nat ⇒ 'a set
    assume C: chain C
    have mono: chain ((pointbase f) o C)
      by (simp add: C mono-maps-chain-to-chain pointbase-is-mono)
    have subset1: natUnion ((pointbase f) o C) ⊆ (pointbase f) (natUnion C)
    proof (auto)
      fix y :: 'b
      assume y ∈ natUnion ((pointbase f) o C)
      then show y ∈ (pointbase f) (natUnion C)
      proof (induct rule: natUnion-decompose)
        case (Decompose n p)
        thus ?case by (metis comp-apply contra-subsetD mono-def natUnion-elem
          pointbase-is-mono subsetI)
      qed
```

```
    qed
    have subset2: (pointbase f) (natUnion C) ⊆ natUnion ((pointbase f) o C)
    proof (auto)
      fix y :: 'b
      assume y: y ∈ (pointbase f) (natUnion C)
      have ∃ X. finite X ∧ X ⊆ natUnion C ∧ y ∈ f X
      proof −
        have y ∈ ⋃{f A |A. finite A ∧ A ⊆ natUnion C}
          by (metis y pointbase-def)
        then show ?thesis by blast
      qed
      then obtain X where X: finite X ∧ X ⊆ natUnion C ∧ y ∈ f X by blast
      then have ∃ n. X ⊆ C n using chain-cover-witness C by blast
      then obtain n where X-sub-C: X ⊆ C n by blast
      show y ∈ natUnion ((pointbase f) o C)
        apply (rule-tac natUnion-elem[where n=n])
        proof −
          have y ∈ ⋃{f A |A. finite A ∧ A ⊆ C n}
          using X X-sub-C by blast
          then show y ∈ (pointbase f ∘ C) n by (simp add: pointbase-def)
      qed
    qed
    note mono subset1 subset2
  }
  then show ?thesis by (simp add: continuous-def subset-antisym)
qed

lemma pointbased-implies-continuous:
  pointbased f ⟹ continuous f
  using pointbase-is-continuous pointbased-def by force

lemma setmonotone-implies-chain-funpower:
  assumes setmonotone: setmonotone f
  shows chain (λ n. funpower f n I)
by (simp add: chain-def setmonotone subset-setmonotone)

lemma natUnion-subset: (⋀ n. ∃ m. f n ⊆ g m) ⟹ natUnion f ⊆ natUnion g
  by (meson natUnion-elem natUnion-upperbound subset-iff)

lemma natUnion-eq[case-names Subset Superset]:
  (⋀ n. ∃ m. f n ⊆ g m) ⟹ (⋀ n. ∃ m. g n ⊆ f m) ⟹ natUnion f = natUnion
g
by (simp add: natUnion-subset subset-antisym)

lemma natUnion-shift[symmetric]:
  assumes chain: chain C
  shows natUnion C = natUnion (λ n. C (n + m))
proof (induct rule: natUnion-eq)
  case (Subset n)
```

```
    show ?case using chain chain-implies-mono le-add1 mono-def by blast
next
  case (Superset n)
    show ?case by blast
qed

definition regular :: ('a set ⇒ 'a set) ⇒ bool
where
  regular f = (setmonotone f ∧ continuous f)

lemma regular-fixpoint:
  assumes regular: regular f
  shows f (limit f I) = limit f I
proof −
  have setmonotone: setmonotone f using regular regular-def by blast
  have continuous: continuous f using regular regular-def by blast

  let ?C = λ n. funpower f n I
  have chain: chain ?C
    by (simp add: setmonotone setmonotone-implies-chain-funpower)
  have f (limit f I) = f (natUnion ?C)
    using limit-def by metis
  also have f (natUnion ?C) = natUnion (f o ?C)
    by (metis continuous continuous-def chain)
  also have natUnion (f o ?C) = natUnion (λ n. f(funpower f n I))
    by (meson comp-apply)
  also have natUnion (λ n. f(funpower f n I)) = natUnion (λ n. ?C (n + 1))
    by simp
  also have natUnion (λ n. ?C(n + 1)) = natUnion ?C
    apply (subst natUnion-shift)
    using chain by (blast+)
  finally show ?thesis by (simp add: limit-def)
qed

lemma fix-is-fix-of-limit:
  assumes fixpoint: f I = I
  shows limit f I = I
proof −
  have funpower: ⋀ n. funpower f n I = I
  proof −
    fix n :: nat
    from fixpoint show funpower f n I = I
      by (induct n, auto)
  qed
  show ?thesis by (simp add: limit-def funpower natUnion-def)
qed

lemma limit-is-idempotent: regular f ⟹ limit f (limit f I) = limit f I
by (simp add: fix-is-fix-of-limit regular-fixpoint)
```

definition *mk-regular1* :: $('b \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a\ set \Rightarrow 'a\ set$
where
mk-regular1 P F I = $I \cup \{\ F\ q\ x \mid q\ x.\ x \in I \wedge P\ q\ x\ \}$

definition *mk-regular2* :: $('b \Rightarrow 'a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a\ set$
$\Rightarrow 'a\ set$ **where**
mk-regular2 P F I = $I \cup \{\ F\ q\ x\ y \mid q\ x\ y.\ x \in I \wedge y \in I \wedge P\ q\ x\ y\ \}$

lemma *setmonotone-mk-regular1*: *setmonotone* (*mk-regular1 P F*)
by (*simp add*: *mk-regular1-def setmonotone-def*)

lemma *setmonotone-mk-regular2*: *setmonotone* (*mk-regular2 P F*)
by (*simp add*: *mk-regular2-def setmonotone-def*)

lemma *pointbased-mk-regular1*: *pointbased* (*mk-regular1 P F*)
proof −
let *?f* = $\lambda\ X.\ X \cup \{\ F\ q\ x \mid q\ x.\ x \in X \wedge P\ q\ x\ \}$
{
fix *I* :: *'a set*
have *1*: *pointbase ?f I* ⊆ *mk-regular1 P F I*
by (*auto simp add*: *pointbase-def mk-regular1-def*)
have *2*: *mk-regular1 P F I* ⊆ *pointbase ?f I*
apply (*simp add*: *pointbase-def mk-regular1-def*)
apply *blast*
done
from *1 2* **have** *pointbase ?f I* = *mk-regular1 P F I* **by** *blast*
}
then show *?thesis*
apply (*subst pointbased-def*)
apply (*rule-tac x=?f* **in** *exI*)
by *blast*
qed

lemma *pointbased-mk-regular2*: *pointbased* (*mk-regular2 P F*)
proof −
let *?f* = $\lambda\ X.\ X \cup \{\ F\ q\ x\ y \mid q\ x\ y.\ x \in X \wedge y \in X \wedge P\ q\ x\ y\ \}$
{
fix *I* :: *'a set*
have *1*: *pointbase ?f I* ⊆ *mk-regular2 P F I*
by (*auto simp add*: *pointbase-def mk-regular2-def*)
have *2*: *mk-regular2 P F I* ⊆ *pointbase ?f I*
apply (*auto simp add*: *pointbase-def mk-regular2-def*)
apply *blast*
proof −
fix *q x y*
assume *x*: $x \in I$
assume *y*: $y \in I$
assume *P*: *P q x y*

```
      let ?X = {x, y}
      let ?A = ?X ∪ {F q x y |q x y. x ∈ ?X ∧ y ∈ ?X ∧ P q x y}
      show ∃ A. (∃ X. A = X ∪ {F q x y |q x y. x ∈ X ∧ y ∈ X ∧ P q x y} ∧
        finite X ∧ X ⊆ I) ∧ F q x y ∈ A
        apply (rule-tac x=?A in exI)
        apply (rule conjI)
        apply (rule-tac x=?X in exI)
        apply (auto simp add: x y)[1]
        using x y P by blast
    qed
   from 1 2 have pointbase ?f I = mk-regular2 P F I by blast
  }
  then show ?thesis
   apply (subst pointbased-def)
   apply (rule-tac x=?f in exI)
   by blast
qed

lemma regular1:regular (mk-regular1 P F)
by (simp add: pointbased-implies-continuous pointbased-mk-regular1 regular-def
  setmonotone-mk-regular1)

lemma regular2: regular (mk-regular2 P F)
by (simp add: pointbased-implies-continuous pointbased-mk-regular2 regular-def
  setmonotone-mk-regular2)

lemma continuous-comp:
  assumes f: continuous f
  assumes g: continuous g
  shows continuous (g o f)
by (metis (no-types, lifting) comp-assoc comp-def continuous-def f g)

lemma setmonotone-comp:
  assumes f: setmonotone f
  assumes g: setmonotone g
  shows setmonotone (g o f)
by (metis (mono-tags, lifting) comp-def f g rev-subsetD setmonotone-def subsetI)

lemma regular-comp:
  assumes f: regular f
  assumes g: regular g
  shows regular (g o f)
using continuous-comp f g regular-def setmonotone-comp by blast

lemma setmonotone-id[simp]: setmonotone id
  by (simp add: id-def setmonotone-def)

lemma continuous-id[simp]: continuous id
  by (simp add: continuous-def)
```

lemma *regular-id*[*simp*]: *regular id*
by (*simp add*: *regular-def*)

lemma *regular-funpower*: *regular f* $\Longrightarrow$ *regular* (*funpower f n*)
proof (*induct n*)
case *0* **thus** *?case* **by** (*simp add*: *id-def*[*symmetric*])
next
case (*Suc n*)
have *funpower*: *funpower f* (*Suc n*) = *f o* (*funpower f n*)
apply (*rule ext*)
by *simp*
with *Suc* **show** *?case*
by (*auto simp only*: *funpower regular-comp*)
qed

lemma *mono-id*[*simp*]: *mono id*
by (*simp add*: *mono-def id-def*)

lemma *mono-funpower*:
assumes *mono*: *mono f*
shows *mono* (*funpower f n*)
proof (*induct n*)
case *0* **thus** *?case* **by** (*simp add*: *id-def*[*symmetric*])
next
case (*Suc n*)
show *?case* **by** (*simp add*: *Suc.hyps mono monoD monoI*)
qed

lemma *mono-limit*:
assumes *mono*: *mono f*
shows *mono* (*limit f*)
proof(*auto simp add*: *mono-def limit-def*)
fix *A* :: *'a set*
fix *B* :: *'a set*
fix *x*
assume *subset*: *A* $\subseteq$ *B*
assume *x* $\in$ *natUnion* (λ*n. funpower f n A*)
then have $\exists$ *n. x* $\in$ *funpower f n A* **using** *elem-limit-simp limit-def* **by** *fastforce*

then obtain *n* **where** *n*: *x* $\in$ *funpower f n A* **by** *blast*
then have *mono*: *mono* (*funpower f n*) **by** (*simp add*: *mono mono-funpower*)
then have *x* $\in$ *funpower f n B* **by** (*meson contra-subsetD monoD n subset*)
then show *x* $\in$ *natUnion* (λ*n. funpower f n B*) **by** (*simp add*: *natUnion-elem*)
qed

lemma *continuous-funpower*:
assumes *continuous*: *continuous f*
shows *continuous* (*funpower f n*)

```
proof (induct n)
  case 0 thus ?case by (simp add: id-def[symmetric])
next
  case (Suc n)
  have mono: mono (funpower f (Suc n))
    by (simp add: continuous continuous-imp-mono mono-funpower)
  have chain: ∀ C. chain C ⟶ chain ((funpower f (Suc n)) o C)
    by (simp del: funpower.simps add: mono mono-maps-chain-to-chain)
  have limit: ⋀ C. chain C ⟹ (funpower f (Suc n)) (natUnion C) =
      natUnion ((funpower f (Suc n)) o C)
      apply simp
      apply (subst continuous-apply[OF Suc])
      apply simp
      apply (subst continuous-apply[OF continuous])
     apply (simp add: Suc.hyps continuous-imp-mono mono-maps-chain-to-chain)
      apply (rule arg-cong[where f=natUnion])
      apply (rule ext)
      by simp
  from chain limit show ?case using continuous-def by blast
qed

lemma natUnion-swap:
  natUnion (λ i. natUnion (λ j. f i j)) = natUnion (λ j. natUnion (λ i. f i j))
by (metis (no-types, lifting) natUnion-elem natUnion-upperbound subsetI subset-antisym)

lemma continuous-limit:
  assumes continuous: continuous f
  shows continuous (limit f)
proof −
  have mono: mono (limit f)
    by (simp add: continuous continuous-imp-mono mono-limit)
  have chain: ⋀ C. chain C ⟹ chain ((limit f) o C)
    by (simp add: mono mono-maps-chain-to-chain)
  have ⋀ C. chain C ⟹ (limit f) (natUnion C) = natUnion ((limit f) o C)
  proof −
    fix C :: nat ⇒ 'a set
    assume chain-C: chain C
    have contpower: ⋀ n. continuous (funpower f n)
      by (simp add: continuous continuous-funpower)
    have comp: ⋀ n F. F o C = (λ i. F (C i))
      by auto
    have (limit f) (natUnion C) = natUnion (λ n. funpower f n (natUnion C))
      by (simp add: limit-def)
    also have natUnion (λ n. funpower f n (natUnion C)) =
          natUnion (λ n. natUnion ((funpower f n) o C))
      apply (subst continuous-apply[OF contpower])
      apply (simp add: chain-C)+
      done
    also have natUnion (λ n. natUnion ((funpower f n) o C)) =
```

```
    natUnion (λ n. natUnion (λ i. funpower f n (C i)))
    apply (subst comp)
    apply blast
    done
  also have natUnion (λ n. natUnion (λ i. funpower f n (C i))) =
    natUnion (λ i. natUnion (λ n. funpower f n (C i)))
    apply (subst natUnion-swap)
    apply blast
    done
  also have natUnion (λ i. natUnion (λ n. funpower f n (C i))) =
    (natUnion (λ i. limit f (C i)))
    apply (simp add: limit-def)
    done
  also have natUnion (λ i. limit f (C i)) = natUnion ((limit f) o C)
    apply (subst comp)
    apply simp
    done
  finally show (limit f) (natUnion C) = natUnion ((limit f) o C) by blast
 qed
 with chain show ?thesis by (simp add: continuous-def)
qed

lemma regular-limit: regular f ⟹ regular (limit f)
by (simp add: continuous-limit regular-def setmonotone-limit)

lemma regular-implies-mono: regular f ⟹ mono f
by (simp add: continuous-imp-mono regular-def)

lemma regular-implies-setmonotone: regular f ⟹ setmonotone f
by (simp add: regular-def)

lemma regular-implies-continuous: regular f ⟹ continuous f
by (simp add: regular-def)

end
theory LocalLexingLemmas
imports LocalLexing Limit
begin

context LocalLexing begin

lemma[simp]: setmonotone (Append Z k) by (auto simp add: Append-def setmono-
tone-def)

lemma subset-𝒫Suc: 𝒫 k u ⊆ 𝒫 k (Suc u)
 by (simp add: subset-setmonotone[OF setmonotone-limit])

lemma subset-fSuc-strict:
 assumes f: ⋀ u. f u ⊆ f (Suc u)
```

```
  shows u < v ⟹ f u ⊆ f v
proof (induct v)
  show u < 0 ⟹ f u ⊆ f 0
    by auto
next
  fix v
  assume a:(u < v ⟹ f u ⊆ f v)
  assume b:u < Suc v
  from b have c: f u ⊆ f v
    apply (case-tac u < v)
    apply (simp add: a)
    apply (case-tac u = v)
    apply simp
    by auto
  show f u ⊆ f (Suc v)
    apply (rule subset-trans[OF c])
    by (rule f)
qed

lemma subset-fSuc:
  assumes f: ⋀ u. f u ⊆ f (Suc u)
  shows u ≤ v ⟹ f u ⊆ f v
  apply (case-tac u < v)
  apply (rule subset-fSuc-strict[where f=f, OF f])
  by auto

lemma subset-𝒫k: u ≤ v ⟹ 𝒫 k u ⊆ 𝒫 k v
  by (rule subset-fSuc, rule subset-𝒫Suc)

lemma subset-𝒫𝒬k: 𝒫 k u ⊆ 𝒬 k by (auto simp add: natUnion-def)

lemma subset-𝒬𝒫Suc: 𝒬 k ⊆ 𝒫 (Suc k) u
proof −
  have a: 𝒬 k ⊆ 𝒫 (Suc k) 0 by simp
  show ?thesis
    apply (case-tac u = 0)
    apply (simp add: a)
    apply (rule subset-trans[OF a subset-𝒫k])
    by auto
qed

lemma subset-𝒬Suc: 𝒬 k ⊆ 𝒬 (Suc k)
  by (rule subset-trans[OF subset-𝒬𝒫Suc subset-𝒫𝒬k])

lemma subset-𝒬: i ≤ j ⟹ 𝒬 i ⊆ 𝒬 j
  by (rule subset-fSuc[where u=i and v=j and f = 𝒬, OF subset-𝒬Suc])

lemma empty-𝒳[simp]: k > length Doc ⟹ 𝒳 k = {}
  apply (simp add: 𝒳-def)
```

```
  apply (insert Lex-is-lexer)
  by (simp add: is-lexer-def)

lemma Sel-empty[simp]: Sel {} {} = {}
  apply (insert Sel-is-selector)
  by (auto simp add: is-selector-def)

lemma empty-𝒵[simp]: k > length Doc ⟹ 𝒵 k u = {}
  apply (induct u)
  by (simp-all add: 𝒴-def 𝒲-def)

lemma[simp]: Append {} k = id by (auto simp add: Append-def)

lemma[simp]: k > length Doc ⟹ 𝒫 k v = 𝒫 k 0
  by (induct v, simp-all add: 𝒴-def 𝒲-def)

lemma 𝒬SucEq: k ≥ length Doc ⟹ 𝒬 (Suc k) = 𝒬 k
  by (simp add: natUnion-def)

lemma 𝒬-converges:
  assumes k: k ≥ length Doc
  shows 𝒬 k = 𝔓
proof −
  {
    fix n
    have 𝒬 (length Doc + n) = 𝔓
    proof (induct n)
      show 𝒬 (length Doc + 0) = 𝔓 by (simp add: 𝔓-def)
    next
      fix n
      assume hyp: 𝒬 (length Doc + n) = 𝔓
      have 𝒬 (Suc (length Doc + n)) = 𝔓
        by (rule trans[OF 𝒬SucEq hyp], auto)
      then show 𝒬 (length Doc + Suc n) = 𝔓
        by auto
    qed
  }
  note helper = this
  from k have ∃ n. k = length Doc + n by presburger
  then obtain n where n: k = length Doc + n by blast
  then show ?thesis
    apply (simp only: n)
    by (rule helper)
qed

lemma 𝔓-covers-𝒬: 𝒬 k ⊆ 𝔓
proof (case-tac k ≥ length Doc)
  assume k ≥ length Doc
  then have 𝒬: 𝒬 k = 𝔓 by (rule 𝒬-converges)
```

```
  then show 𝒬 k ⊆ 𝔓 by (simp only: 𝒬)
next
  assume ¬ length Doc ≤ k
  then have k < length Doc by auto
  then show ?thesis
    apply (simp only: 𝔓-def)
    apply (rule subset-𝒬)
    by auto
qed

lemma Sel-upper-bound: A ⊆ B ⟹ Sel A B ⊆ B
  by (metis Sel-is-selector is-selector-def)

lemma Sel-lower-bound: A ⊆ B ⟹ A ⊆ Sel A B
  by (metis Sel-is-selector is-selector-def)

lemma 𝔓-covers-𝒫: 𝒫 k u ⊆ 𝔓
  by (rule subset-trans[OF subset-𝒫𝒬k 𝔓-covers-𝒬])

lemma 𝒲-montone: P ⊆ Q ⟹ 𝒲 P k ⊆ 𝒲 Q k
  by (auto simp add: 𝒲-def)

lemma Sel-precondition:
  𝒵 k u ⊆ 𝒲 (𝒫 k u) k
proof (induct u)
  case 0 thus ?case by simp
next
  case (Suc u)
  have 1: 𝒴 (𝒵 k u) (𝒫 k u) k ⊆ 𝒲 (𝒫 k u) k
    apply (simp add: 𝒴-def)
    apply (rule-tac Sel-upper-bound)
    using Suc by simp
  have 2: 𝒲 (𝒫 k u) k ⊆ 𝒲 (𝒫 k (Suc u)) k
    by (metis 𝒲-montone subset-𝒫Suc)
  show ?case
    apply (rule-tac subset-trans[where B=𝒲 (𝒫 k u) k])
    apply (simp add: 1)
    apply (simp only: 2)
    done
qed

lemma 𝒲-bounded-by-𝒳: 𝒲 P k ⊆ 𝒳 k
  by (metis (no-types, lifting) 𝒲-def mem-Collect-eq subsetI)

lemma 𝒵-subset-𝒳: 𝒵 k n ⊆ 𝒳 k
  by (metis Sel-precondition 𝒲-bounded-by-𝒳 rev-subsetD subsetI)

lemma 𝒵-subset-Suc: 𝒵 k n ⊆ 𝒵 k (Suc n)
apply (induct n)
```

```
apply simp
by (metis Sel-lower-bound Sel-precondition 𝒴-def 𝒵.simps(2))

lemma 𝒴-upper-bound: 𝒴 (𝒵 k u) (𝒫 k u) k ⊆ 𝒲 (𝒫 k u) k
  apply (simp add: 𝒴-def)
  by (metis Sel-precondition Sel-upper-bound)

lemma 𝔓-induct[consumes 1, case-names Base Induct]:
  assumes p: p ∈ 𝔓
  assumes base: P []
  assumes induct: ⋀ p k u. (⋀ q. q ∈ 𝒫 k u ⟹ P q) ⟹ p ∈ 𝒫 k (Suc u) ⟹
P p
  shows P p
proof −
  {
    fix p :: tokens
    fix k :: nat
    fix u :: nat
    have p ∈ 𝒫 k u ⟹ P p
    proof (induct k arbitrary: p u)
      case 0
        have p ∈ 𝒫 0 u ⟹ P p
        proof (induct u arbitrary: p)
          case 0 thus ?case using base by simp
        next
          case (Suc u) show ?case
            apply (rule induct[OF - Suc(2)])
            apply (rule Suc(1))
            by simp
        qed
        with 0 show ?case by simp
    next
      case (Suc k)
        have p ∈ 𝒫 (Suc k) u ⟹ P p
        proof (induct u arbitrary: p)
          case 0 thus ?case
            apply simp
            apply (induct rule: natUnion-decompose)
            using Suc by simp
        next
          case (Suc u) show ?case
            apply (rule induct[OF - Suc(2)])
            apply (rule Suc(1))
            by simp
        qed
        with Suc show ?case by simp
    qed
  }
  note all = this
```

from *p* **show** *?thesis*
apply (*simp add*: $\mathfrak{P}$*-def*)
apply (*induct rule*: *natUnion-decompose*)
using *all* **by** *simp*
qed

lemma *Append-mono*: $U \subseteq V \Longrightarrow P \subseteq Q \Longrightarrow$ *Append U k P* $\subseteq$ *Append V k Q*
by (*auto simp add*: *Append-def*)

lemma *pointwise-Append*: *pointwise* (*Append T k*)
by (*auto simp add*: *pointwise-def Append-def*)

lemma *regular-Append*: *regular* (*Append T k*)
proof −
have *pointwise* (*Append T k*) **using** *pointwise-Append* **by** *blast*
then have *pointbased* (*Append T k*) **using** *pointwise-implies-pointbased* **by** *blast*
then have *continuous* (*Append T k*) **using** *pointbased-implies-continuous* **by** *blast*
moreover have *setmonotone* (*Append T k*) **by** (*simp add*: *setmonotone-def Append-def*)
ultimately show *?thesis* **using** *regular-def* **by** *blast*
qed

end

end
theory *InductRules*
imports *Main*
begin

lemma *disjCases2*[*consumes 1*, *case-names 1 2*]:
assumes *AB*: $A \vee B$
and *AP*: $A \Longrightarrow P$
and *BP*: $B \Longrightarrow P$
shows *P*
proof −
from *AB AP BP* **show** *?thesis* **by** *blast*
qed

lemma *disjCases3*[*consumes 1*, *case-names 1 2 3*]:
assumes *AB*: $A \vee B \vee C$
and *AP*: $A \Longrightarrow P$
and *BP*: $B \Longrightarrow P$
and *CP*: $C \Longrightarrow P$
shows *P*
proof −
from *AB AP BP CP* **show** *?thesis* **by** *blast*
qed

lemma *disjCases4*[*consumes 1*, *case-names 1 2 3 4*]:
assumes *AB*: $A \vee B \vee C \vee D$
and *AP*: $A \Longrightarrow P$
and *BP*: $B \Longrightarrow P$
and *CP*: $C \Longrightarrow P$
and *DP*: $D \Longrightarrow P$
shows P
proof −
from *AB AP BP CP DP* **show** *?thesis* **by** *blast*
qed

lemma *disjCases5*[*consumes 1*, *case-names 1 2 3 4 5*]:
assumes *AB*: $A \vee B \vee C \vee D \vee E$
and *AP*: $A \Longrightarrow P$
and *BP*: $B \Longrightarrow P$
and *CP*: $C \Longrightarrow P$
and *DP*: $D \Longrightarrow P$
and *EP*: $E \Longrightarrow P$
shows P
proof −
from *AB AP BP CP DP EP* **show** *?thesis* **by** *blast*
qed

lemma *minimal-witness-ex*:
assumes *k*: $P\ (k::nat)$
shows $\exists\ k0.\ k0 \leq k \wedge P\ k0 \wedge (\forall\ k.\ k < k0 \longrightarrow \neg\ (P\ k))$
proof −
let *?K* = $\{\ h.\ h \leq k \wedge P\ h\ \}$
have *finite-K*: *finite ?K* **by** *auto*
have $k \in ?K$ **by** (*simp add*: *k*)
then have *nonempty-K*: $?K \neq \{\}$ **by** *auto*
let *?k* = *Min ?K*
have *witness*: $?k \leq k \wedge P\ ?k$
by (*metis* (*mono-tags*, *lifting*) *Min-in finite-K mem-Collect-eq nonempty-K*)
have *minimal*: $\forall\ h.\ h < ?k \longrightarrow \neg\ (P\ h)$
by (*metis Min-le witness dual-order.strict-implies-order*
dual-order.trans finite-K leD mem-Collect-eq)
from *witness minimal* **show** *?thesis* **by** *metis*
qed

lemma *minimal-witness*[*consumes 1*, *case-names Minimal*]:
assumes $P\ (k::nat)$
and $\bigwedge\ K.\ K \leq k \Longrightarrow P\ K \Longrightarrow (\bigwedge\ k.\ k < K \Longrightarrow \neg\ (P\ k)) \Longrightarrow Q$
shows Q
proof −
from *assms minimal-witness-ex* **show** *?thesis* **by** *metis*
qed

lemma *ex-minimal-witness*[*consumes 1*, *case-names Minimal*]:

assumes $\exists\ k.\ P\ (k::nat)$
and $\bigwedge K.\ P\ K \Longrightarrow (\bigwedge k.\ k < K \Longrightarrow \neg\ (P\ k)) \Longrightarrow Q$
shows Q
proof −
from *assms minimal-witness-ex* **show** *?thesis* **by** *metis*
qed

end
theory *ListTools*
imports *Main*
begin

definition *is-first* :: $'a \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}first\ x\ u = (\exists\ v.\ u = [x]@v)$

definition *is-last* :: $'a \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}last\ x\ u = (\exists\ v.\ u = v@[x])$

definition *is-prefix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}prefix\ u\ v = (\exists\ w.\ u@w = v)$

definition *is-proper-prefix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}proper\text{-}prefix\ u\ v = (\exists\ w.\ w \neq [] \wedge u@w = v)$

lemma *is-prefix-eq-proper-prefix*: $is\text{-}prefix\ a\ b = (a = b \vee is\text{-}proper\text{-}prefix\ a\ b)$
by (*metis append-Nil2 is-prefix-def is-proper-prefix-def*)

lemma *is-proper-prefix-eq-prefix*: $is\text{-}proper\text{-}prefix\ a\ b = (a \neq b \wedge is\text{-}prefix\ a\ b)$
by (*metis append-self-conv is-prefix-eq-proper-prefix is-proper-prefix-def*)

definition *is-suffix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}suffix\ u\ v = (\exists\ w.\ w@u = v)$

definition *is-proper-suffix* :: $'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$is\text{-}proper\text{-}suffix\ u\ v = (\exists\ w.\ w \neq [] \wedge w@u = v)$

lemma *is-suffix-eq-proper-suffix*: $is\text{-}suffix\ a\ b = (a = b \vee is\text{-}proper\text{-}suffix\ a\ b)$
by (*metis append-Nil is-proper-suffix-def is-suffix-def*)

lemma *is-proper-suffix-eq-suffix*: $is\text{-}proper\text{-}suffix\ a\ b = (a \neq b \wedge is\text{-}suffix\ a\ b)$
by (*metis is-proper-suffix-def is-suffix-eq-proper-suffix self-append-conv2*)

lemma *is-prefix-unsplit*: $is\text{-}prefix\ u\ a \Longrightarrow u\ @\ (drop\ (length\ u)\ a) = a$

by (*metis append-eq-conv-conj is-prefix-def*)

lemma *le-take-same*: $i \leq j \Longrightarrow take\ j\ a = take\ j\ b \Longrightarrow take\ i\ a = take\ i\ b$
by (*metis min.absorb1 take-take*)

lemma *is-first-drop-length*:
assumes $k \leq length\ a$
and $k > length\ u$
and $v = X\#w$
and $take\ k\ a = take\ k\ (u@v)$
shows *is-first X* (*drop* (*length u*) *a*)
proof −
let *?d* = $k - length\ u$
from *assms* **have** *pos-d'*: $?d > 0$ **by** *auto*
from *assms* **have** *take-d'-v*: *take ?d* (*drop* (*length u*) *a*) = *take ?d v*
by (*metis append-eq-conv-conj drop-take*)
then have *take 1* (*drop* (*length u*) *a*) = *take 1 v*
by (*metis One-nat-def Suc-leI le-take-same pos-d'*)
then have *take 1* (*drop* (*length u*) *a*) = [*X*]
by (*simp add*: *assms*)
then show *?thesis*
by (*metis append-take-drop-id is-first-def*)
qed

lemma *is-first-cons*: *is-first x* (*y*#*ys*) = (*x* = *y*)
by (*auto simp add*: *is-first-def*)

lemma *list-all-pos-neg-ex*: *list-all P D* $\Longrightarrow$ ¬ (*list-all Q D*) $\Longrightarrow$
$\exists\ k.\ k < length\ D \wedge P(D\ !\ k) \wedge \neg(Q(D\ !\ k))$
using *list-all-length* **by** *blast*

lemma *split-list-at*: $k < length\ D \Longrightarrow D = (take\ k\ D)@[D\ !\ k]@(drop\ (Suc\ k)\ D)$
by (*metis append-Cons append-Nil id-take-nth-drop*)

lemma *take-eq-take-append*: $i \leq j \Longrightarrow j \leq length\ a \Longrightarrow \exists\ u.\ take\ j\ a = take\ i\ a$
@ *u*
by (*metis le-Suc-ex take-add*)

lemma *is-proper-suffix-length-cmp*: *is-proper-suffix a b* $\Longrightarrow$ *length a* < *length b*
by (*metis add-diff-cancel-right' append-Nil append-eq-append-conv*
diff-is-0-eq is-proper-suffix-def leI length-append list.size(*3*))

end
theory *Derivations*
imports *CFG ListTools InductRules*
begin

context *CFG* **begin**

lemma [*simp*]: *is-terminal t* $\Longrightarrow$ *is-symbol t*
by (*auto simp add*: *is-terminal-def is-symbol-def*)

lemma [*simp*]: *is-sentence* [] **by** (*auto simp add*: *is-sentence-def*)

lemma [*simp*]: *is-word* [] **by** (*auto simp add*: *is-word-def*)

lemma [*simp*]: *is-word u* $\Longrightarrow$ *is-sentence u*
by (*induct u*, *auto simp add*: *is-word-def is-sentence-def*)

definition *leftderives1* :: *sentence* $\Rightarrow$ *sentence* $\Rightarrow$ *bool*
where
leftderives1 u v =
(∃ *x y N* α.
u = *x* @ [*N*] @ *y*
∧ *v* = *x* @ α @ *y*
∧ *is-word x*
∧ *is-sentence y*
∧ (*N*, α) ∈ $\mathfrak{R}$)

lemma *leftderives1-implies-derives1*[*simp*]: *leftderives1 u v* $\Longrightarrow$ *derives1 u v*
apply (*auto simp add*: *leftderives1-def derives1-def*)
apply (*rule-tac x*=*x* **in** *exI*)
apply (*rule-tac x*=*y* **in** *exI*)
apply (*rule-tac x*=*N* **in** *exI*)
by *auto*

definition *leftderivations1* :: (*sentence* × *sentence*) *set*
where
leftderivations1 = { (*u*,*v*) | *u v*. *leftderives1 u v* }

lemma [*simp*]: *leftderivations1* ⊆ *derivations1*
by (*auto simp add*: *leftderivations1-def derivations1-def*)

definition *leftderivations* :: (*sentence* × *sentence*) *set*
where
leftderivations = *leftderivations1*^*

lemma *rtrancl-subset-implies*: *a* ⊆ *b* $\Longrightarrow$ *a* ⊆ *b*^* **by** *blast*

lemma *leftderivations-subset-derivations*[*simp*]: *leftderivations* ⊆ *derivations*
apply (*simp add*: *leftderivations-def derivations-def*)
apply (*rule rtrancl-subset-rtrancl*)
apply (*rule rtrancl-subset-implies*)
by *simp*

definition *leftderives* :: *sentence* $\Rightarrow$ *sentence* $\Rightarrow$ *bool*

where
leftderives u v = (*u*, *v*) ∈ *leftderivations*)

lemma *leftderives-implies-derives*[*simp*]: *leftderives u v* ⟹ *derives u v*
apply (*auto simp add*: *leftderives-def derives-def*)
by (*rule subsetD*[*OF leftderivations-subset-derivations*])

definition *is-leftderivation* :: *sentence* ⇒ *bool*
where
is-leftderivation u = *leftderives* [S] *u*

lemma *leftderivation-implies-derivation*[*simp*]:
is-leftderivation u ⟹ *is-derivation u*
by (*simp add*: *is-leftderivation-def is-derivation-def*)

lemma *leftderives-refl*[*simp*]: *leftderives u u*
by (*auto simp add*: *leftderives-def leftderivations-def*)

lemma *leftderives1-implies-leftderives*[*simp*]:*leftderives1 a b* ⟹ *leftderives a b*
by (*auto simp add*: *leftderives-def leftderivations-def leftderivations1-def*)

lemma *leftderives-trans*: *leftderives a b* ⟹ *leftderives b c* ⟹ *leftderives a c*
by (*auto simp add*: *leftderives-def leftderivations-def*)

lemma *leftderives1-eq-leftderivations1*: *leftderives1 x y* = ((*x*, *y*) ∈ *leftderivations1*)
by (*simp add*: *leftderivations1-def*)

lemma *leftderives-induct*[*consumes 1*, *case-names Base Step*]:
assumes *derives*: *leftderives a b*
assumes *Pa*: *P a*
assumes *induct*: ⋀*y z*. *leftderives a y* ⟹ *leftderives1 y z* ⟹ *P y* ⟹ *P z*
shows *P b*
proof −
note *rtrancl-lemma* = *rtrancl-induct*[**where** *a* = *a* **and** *b* = *b* **and** *r* = *leftderivations1* **and** *P*=*P*]
from *derives Pa induct rtrancl-lemma* **show** *P b*
by (*metis leftderives-def leftderivations-def leftderives1-eq-leftderivations1*)
qed

end

context *CFG* **begin**

lemma *derives1-implies-derives*[*simp*]:*derives1 a b* ⟹ *derives a b*
by (*auto simp add*: *derives-def derivations-def derivations1-def*)

lemma *derives-trans*: *derives a b* ⟹ *derives b c* ⟹ *derives a c*

by (*auto simp add*: *derives-def derivations-def*)

lemma *derives1-eq-derivations1*: *derives1 x y* = ((*x*, *y*) ∈ *derivations1*)
by (*simp add*: *derivations1-def*)

lemma *derives-induct*[*consumes 1*, *case-names Base Step*]:
assumes *derives*: *derives a b*
assumes *Pa*: *P a*
assumes *induct*: ⋀*y z*. *derives a y* ⟹ *derives1 y z* ⟹ *P y* ⟹ *P z*
shows *P b*
proof −
note *rtrancl-lemma* = *rtrancl-induct*[**where** *a* = *a* **and** *b* = *b* **and** *r* = *derivations1* **and** *P*=*P*]
from *derives Pa induct rtrancl-lemma* **show** *P b*
by (*metis derives-def derivations-def derives1-eq-derivations1*)
qed

end

context *CFG* **begin**

definition *Derives1* :: *sentence* ⇒ *nat* ⇒ *rule* ⇒ *sentence* ⇒ *bool*
where
Derives1 u i r v =
(∃ *x y N α*.
u = *x* @ [*N*] @ *y*
∧ *v* = *x* @ *α* @ *y*
∧ *is-sentence x*
∧ *is-sentence y*
∧ (*N*, *α*) ∈ ℜ
∧ *r* = (*N*, *α*) ∧ *i* = *length x*)

lemma *Derives1-split*:
Derives1 u i r v ⟹ ∃ *x y*. *u* = *x* @ [*fst r*] @ *y* ∧ *v* = *x* @ (*snd r*) @ *y* ∧ *length x* = *i*
by (*metis Derives1-def fst-conv snd-conv*)

lemma *Derives1-implies-derives1*: *Derives1 u i r v* ⟹ *derives1 u v*
by (*auto simp add*: *Derives1-def derives1-def*)

lemma *derives1-implies-Derives1*: *derives1 u v* ⟹ ∃ *i r*. *Derives1 u i r v*
by (*auto simp add*: *Derives1-def derives1-def*)

lemma *Derives1-unique-dest*: *Derives1 u i r v* ⟹ *Derives1 u i r w* ⟹ *v* = *w*
by (*auto simp add*: *Derives1-def derives1-def*)

lemma *Derives1-unique-src*: *Derives1 u i r w* ⟹ *Derives1 v i r w* ⟹ *u* = *v*
by (*auto simp add*: *Derives1-def derives1-def*)

type-synonym *derivation* = (*nat* × *rule*) *list*

fun *Derivation* :: *sentence* ⇒ *derivation* ⇒ *sentence* ⇒ *bool*
where
  Derivation a [] *b* = (*a* = *b*)
| *Derivation a* (*d*#*D*) *b* = (∃ *x*. *Derives1 a* (*fst d*) (*snd d*) *x* ∧ *Derivation x D b*)

lemma *Derivation-implies-derives*: *Derivation a D b* ⟹ *derives a b*
proof (*induct D arbitrary*: *a b*)
  case *Nil* **thus** *?case*
    by (*simp add*: *derives-def derivations-def*)
next
  case (*Cons d D*)
    note *ihyps* = *this*
    from *ihyps* **have** ∃ *x*. *Derives1 a* (*fst d*) (*snd d*) *x* ∧ *Derivation x D b* **by** *auto*
    then obtain *x* **where** *Derives1 a* (*fst d*) (*snd d*) *x* **and** *xb*: *Derivation x D b*
by *blast*
    with *Derives1-implies-derives1* **have** *d1*: *derives a x* **by** *auto*
    from *ihyps xb* **have** *d2*:*derives x b* **by** *simp*
    show *derives a b* **by** (*rule derives-trans*[*OF d1 d2*])
qed

lemma *Derivation-Derives1*: *Derivation a S y* ⟹ *Derives1 y i r z* ⟹ *Derivation a* (*S*@[(*i*,*r*)]) *z*
proof (*induct S arbitrary*: *a y z i r*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons s S*) **thus** *?case*
    by (*metis Derivation.simps*(*2*) *append-Cons*)
qed

lemma *derives-implies-Derivation*: *derives a b* ⟹ ∃ *D*. *Derivation a D b*
proof (*induct rule*: *derives-induct*)
  case *Base*
  show *?case* **by** (*rule exI*[**where** *x*=[]], *simp*)
next
  case (*Step y z*)
  note *ihyps* = *this*
  from *ihyps* **obtain** *D* **where** *ay*: *Derivation a D y* **by** *blast*
  from *ihyps derives1-implies-Derives1* **obtain** *i r* **where** *yz*: *Derives1 y i r z* **by**
blast
  from *Derivation-Derives1*[*OF ay yz*] **show** *?case* **by** *auto*
qed

lemma *Derives1-take*: *Derives1 a i r b* ⟹ *take i a* = *take i b*
  by (*auto simp add*: *Derives1-def*)

lemma *Derives1-drop*: *Derives1 a i r b* ⟹ *drop* (*Suc i*) *a* = *drop* (*i* + *length* (*snd*

r)) *b*
by (*auto simp add*: *Derives1-def*)

lemma *Derives1-bound*: *Derives1 a i r b* ⟹ *i* < *length a*
by (*auto simp add*: *Derives1-def*)

lemma *Derives1-length*: *Derives1 a i r b* ⟹ *length b* = *length a* + *length* (*snd r*) − *1*
by (*auto simp add*: *Derives1-def*)

definition *leftmost* :: *nat* ⇒ *sentence* ⇒ *bool*
where
leftmost i s = (*i* < *length s* ∧ *is-word* (*take i s*) ∧ *is-nonterminal* (*s* ! *i*))

lemma *set-take*: *set* (*take n s*) = { *s* ! *i* | *i*. *i* < *n* ∧ *i* < *length s*}
proof (*cases n* ≤ *length s*)
case *True* **thus** *?thesis*
by (*subst List.nth-image*[*symmetric*], *auto*)
next
case *False* **thus** *?thesis*
apply (*subst set-conv-nth*)
by (*metis less-trans linear not-le take-all*)
qed

lemma *list-all-take*: *list-all P* (*take n s*) = (∀ *i*. *i* < *n* ∧ *i* < *length s* ⟶ *P* (*s* ! *i*))
by (*auto simp add*: *set-take list-all-iff*)

lemma *is-sentence-concat*: *is-sentence* (*x*@*y*) = (*is-sentence x* ∧ *is-sentence y*)
by (*auto simp add*: *is-sentence-def*)

lemma *is-sentence-cons*: *is-sentence* (*x*#*xs*) = (*is-symbol x* ∧ *is-sentence xs*)
by (*auto simp add*: *is-sentence-def*)

lemma *rule-nonterminal-type*[*simp*]: (*N*, *α*) ∈ ℜ ⟹ *is-nonterminal N*
apply (*insert validRules*)
by (*auto simp add*: *is-nonterminal-def*)

lemma *rule-α-type*[*simp*]: (*N*, *α*) ∈ ℜ ⟹ *is-sentence α*
apply (*insert validRules*)
by (*auto simp add*: *is-sentence-def is-symbol-def list-all-iff is-terminal-def is-nonterminal-def*)

lemma [*simp*]: *is-nonterminal N* ⟹ *is-symbol N*
by (*simp add*: *is-symbol-def*)

lemma *Derives1-sentence1*[*elim*]: *Derives1 a i r b* ⟹ *is-sentence a*
by (*auto simp add*: *Derives1-def is-sentence-cons is-sentence-concat*)

lemma *Derives1-sentence2*[*elim*]: *Derives1 a i r b* ⟹ *is-sentence b*

```
  by (auto simp add: Derives1-def is-sentence-cons is-sentence-concat)

lemma [elim]: Derives1 a i r b ⟹ r ∈ ℜ
  using Derives1-def by auto

lemma is-sentence-symbol: is-sentence a ⟹ i < length a ⟹ is-symbol (a ! i)
  by (simp add: is-sentence-def list-all-iff)

lemma is-symbol-distinct: is-symbol x ⟹ is-terminal x ≠ is-nonterminal x
  apply (insert disjunct-symbols)
  apply (auto simp add: is-symbol-def is-terminal-def is-nonterminal-def)
  done

lemma is-terminal-nonterminal: is-terminal x ⟹ is-nonterminal x ⟹ False
  by (metis is-symbol-def is-symbol-distinct)

lemma Derives1-leftmost:
  assumes Derives1 a i r b
  shows ∃ j. leftmost j a ∧ j ≤ i
proof −
  let ?J = {j . j < length a ∧ is-nonterminal (a ! j)}
  let ?M = Min ?J
  from assms have J1:i ∈ ?J
    apply (auto simp add: Derives1-def is-nonterminal-def)
    by (metis (mono-tags, lifting) prod.simps(2) validRules)
  have J2:finite ?J by auto
  note J = J1 J2
  from J have M1: ?M ∈ ?J by (rule-tac Min-in, auto)
  {
    fix j
    assume j ∈ ?J
    with J have ?M ≤ j by auto
  }
  note M3 = this[OF J1]
  from assms have a-sentence: is-sentence a by (simp add: Derives1-sentence1)
  have is-word: is-word (take ?M a)
    apply (auto simp add: is-word-def list-all-take)
    proof −
      fix i
      assume i-less-M: i < ?M
      assume i-inbounds: i < length a
      show is-terminal (a ! i)
      proof(cases is-terminal (a ! i))
        case True thus ?thesis by auto
      next
        case False
        then have is-nonterminal (a ! i)
        using i-inbounds a-sentence is-sentence-symbol is-symbol-distinct by blast
        then have i ∈ ?J by (metis i-inbounds mem-Collect-eq)
```

```
      then have ?M < i by (metis J2 Min-le i-less-M leD)
      then have False by (metis i-less-M less-asym')
      then show ?thesis by auto
    qed
  qed
 show ?thesis
  apply (rule-tac exI[where x=?M])
  apply (simp add: leftmost-def)
  by (metis (mono-tags, lifting) M1 M3 is-word mem-Collect-eq)
qed

lemma Derivation-leftmost: D ≠ [] ⟹ Derivation a D b ⟹ ∃ i. leftmost i a
 apply (case-tac D)
 apply (auto)
 apply (metis Derives1-leftmost)
 done

lemma nonword-has-nonterminal:
 is-sentence a ⟹ ¬ (is-word a) ⟹ ∃ k. k < length a ∧ is-nonterminal (a ! k)
 apply (auto simp add: is-sentence-def list-all-iff is-word-def)
 by (metis in-set-conv-nth is-symbol-distinct)

lemma leftmost-cons-nonterminal:
 is-nonterminal x ⟹ leftmost 0 (x#xs)
by (metis CFG.is-word-def CFG-axioms leftmost-def length-greater-0-conv list.distinct(1)

    list-all-simps(2) nth-Cons-0 take-Cons')

lemma leftmost-cons-terminal:
 is-terminal x ⟹ leftmost i (x#xs) = (i > 0 ∧ leftmost (i − 1) xs)
by (metis Suc-diff-1 Suc-less-eq is-terminal-nonterminal is-word-def leftmost-def
length-Cons
    list-all-simps(1) not-gr0 nth-Cons' take-Cons')

lemma is-nonterminal-cons-terminal:
 is-terminal x ⟹ k < length (x # a) ⟹ is-nonterminal ((x # a) ! k) ⟹
  k > 0 ∧ k − 1 < length a ∧ is-nonterminal (a ! (k − 1))
by (metis One-nat-def Suc-leI is-terminal-nonterminal less-diff-conv2
    list.size(4) nth-non-equal-first-eq)

lemma leftmost-exists:
 is-sentence a ⟹ k < length a ⟹ is-nonterminal (a ! k) ⟹
  ∃ i. leftmost i a ∧ i ≤ k
proof (induct a arbitrary: k)
 case Nil thus ?case by auto
next
 case (Cons x a)
 show ?case
 proof(cases is-nonterminal x)
```

```
    case True thus ?thesis
      apply(rule-tac exI[where x=0])
      apply (simp add: leftmost-cons-nonterminal)
      done
  next
    case False
    then have x: is-terminal x
      by (metis Cons.prems(1) is-sentence-cons is-symbol-distinct)
    note k = is-nonterminal-cons-terminal[OF x Cons(3) Cons(4)]
    with Cons have ∃ i. leftmost i a ∧ i ≤ k − 1 by (metis is-sentence-cons)
    then show ?thesis
      apply (auto simp add: leftmost-cons-terminal[OF x])
      by (metis le-diff-conv2 Suc-leI add-Suc-right add-diff-cancel-right' k
          le-0-eq le-imp-less-Suc nat-le-linear)
  qed
qed

lemma nonword-leftmost-exists:
  is-sentence a ⟹ ¬ (is-word a) ⟹ ∃ i. leftmost i a
  by (metis leftmost-exists nonword-has-nonterminal)

lemma leftmost-unaffected-Derives1: leftmost j a ⟹ j < i ⟹ Derives1 a i r b
⟹ leftmost j b
apply (simp add: leftmost-def)
proof −
  assume a1: j < length a ∧ is-word (take j a) ∧ is-nonterminal (a ! j)
  assume a2: j < i
  assume Derives1 a i r b
  then have f3: take i a = take i b
    by (metis Derives1-take)
  have f4: ⋀n ss ssa. take (length (take n (ss::symbol list))) (ssa::symbol list) =
take (length ss) (take n ssa)
    by (metis length-take take-take)
  have f5: ⋀ss. take j (ss::symbol list) = take i (take j ss)
    using a2 by (metis dual-order.strict-implies-order min.absorb2 take-take)
  have f6: length (take j a) = j
    using a1 by (metis dual-order.strict-implies-order length-take min.absorb2)
  then have f7: ⋀n. min j n = length (take n (take j a))
    by (metis length-take)
  have f8: ⋀n ss. n = length (take n (ss::symbol list)) ∨ length ss < n
    by (metis leI length-take min.absorb2)
  have f9: ⋀ss. take j (ss::symbol list) = take j (take i ss)
    using f7 f6 f5 by (metis take-take)
  have f10: ⋀ss n. length (ss::symbol list) ≤ n ∨ length (take n ss) = n
    using f8 by (metis dual-order.strict-implies-order)
  then have f11: ⋀ss ssa. length (ss::symbol list) = length (take (length ss)
(ssa::symbol list)) ∨ length (take (length ssa) ss) = length ssa
    by (metis length-take min.absorb2)
  have f12: ⋀ss ssa n. take (length (ss::symbol list)) (ssa::symbol list) = take n
```

```
(take (length ss) ssa) ∨ length (take n ss) = n
    using f10 by (metis min.absorb2 take-take)
  { assume ¬ j < j
    { assume ¬ j < j ∧ i ≠ j
      moreover
      { assume length a ≠ j ∧ length (take i a) ≠ i
          then have ∃ss. length (take (length (take i (take (length a) (ss::symbol
list)))) (take j ss)) ≠ length (take i (take (length a) ss))
          using f12 f11 f6 f5 f4 by metis
        then have ∃ss ssa. take (length (ss::symbol list)) (take j (ssa::symbol list))
≠ take (length ss) (take i (take (length a) ssa))
          using f11 by metis
        then have length b ≠ j
          using f9 f4 f3 by metis }
      ultimately have length b ≠ j
        using f7 f6 f5 f3 a1 by (metis length-take) }
    then have length b = j ⟶ j < j
      using a2 by metis }
  then have j < length b
    using f9 f8 f7 f6 f4 f3 by metis
  then show j < length b ∧ is-word (take j b) ∧ is-nonterminal (b ! j)
    using f9 f3 a2 a1 by (metis nth-take)
qed

definition derivation-ge :: derivation ⇒ nat ⇒ bool
where
  derivation-ge D i = (∀ d ∈ set D. fst d ≥ i)

lemma derivation-ge-cons: derivation-ge (d#D) i = (fst d ≥ i ∧ derivation-ge D
i)
  by (auto simp add: derivation-ge-def)

lemma derivation-ge-append:
  derivation-ge (D@E) i = (derivation-ge D i ∧ derivation-ge E i)
  by (auto simp add: derivation-ge-def)

lemma leftmost-unaffected-Derivation:
  derivation-ge D (Suc i) ⟹ leftmost i a ⟹ Derivation a D b ⟹ leftmost i b
proof (induct D arbitrary: a)
  case Nil thus ?case by auto
next
  case (Cons d D)
  then have ∃ x. Derives1 a (fst d) (snd d) x ∧ Derivation x D b by simp
  then obtain x where x1: Derives1 a (fst d) (snd d) x and x2: Derivation x D
b by blast
  with Cons have leftmost-x: leftmost i x
    apply (rule-tac leftmost-unaffected-Derives1[
          where a=a and j=i and b=x and i=fst d and r=snd d])
    by (auto simp add: derivation-ge-def)
```

with *Cons x2* **show** *?case* **by** (*auto simp add*: *derivation-ge-def*)
qed

lemma *le-Derives1-take*:
assumes *le*: $i \leq j$
and *D*: *Derives1 a j r b*
shows *take i a* = *take i b*
proof −
note *Derives1-take*[**where** *a*=*a* **and** *i*=*j* **and** *r*=*r* **and** *b*=*b*]
with *le D* **show** *?thesis* **by** (*rule-tac le-take-same*[**where** *i*=*i* **and** *j*=*j*], *auto*)
qed

lemma *Derivation-take*: *derivation-ge D i* ⟹ *Derivation a D b* ⟹ *take i a* = *take i b*
proof(*induct D arbitrary*: *a b*)
case *Nil* **thus** *?case* **by** *auto*
next
case (*Cons d D*)
then have ∃ *x*. *Derives1 a* (*fst d*) (*snd d*) *x* ∧ *Derivation x D b*
by *simp*
then obtain *x* **where** *ax*: *Derives1 a* (*fst d*) (*snd d*) *x* **and** *xb*: *Derivation x D b* **by** *blast*
from *derivation-ge-cons Cons*(*2*) **have** *d*: $i \leq$ *fst d* **and** *D*: *derivation-ge D i* **by** *auto*
note *take-i-xb* = *Cons*(*1*)[*OF D xb*]
note *take-i-ax* = *le-Derives1-take*[*OF d ax*]
from *take-i-xb take-i-ax* **show** *?case* **by** *auto*
qed

lemma *leftmost-cons-less*: *i* < *length u* ⟹ *leftmost i* (*u*@*v*) = *leftmost i u*
by (*auto simp add*: *leftmost-def nth-append*)

lemma *leftmost-is-nonterminal*: *leftmost i u* ⟹ *is-nonterminal* (*u* ! *i*)
by (*metis leftmost-def*)

lemma *is-word-is-terminal*: *i* < *length u* ⟹ *is-word u* ⟹ *is-terminal* (*u* ! *i*)
by (*metis is-word-def list-all-length*)

lemma *leftmost-append*:
assumes *leftmost*: *leftmost i* (*u*@*v*)
and *is-word*: *is-word u*
shows *length u* ≤ *i*
proof (*cases i* < *length u*)
case *False* **thus** *?thesis* **by** *auto*
next
case *True*
with *leftmost* **have** *leftmost i u* **using** *leftmost-cons-less*[*OF True*] **by** *simp*
then have *is-nonterminal*: *is-nonterminal* (*u* ! *i*) **by** (*rule leftmost-is-nonterminal*)
note *is-terminal* = *is-word-is-terminal*[*OF True is-word*]

```
  note is-terminal-nonterminal[OF is-terminal is-nonterminal]
  then show ?thesis by auto
qed

lemma derivation-ge-empty[simp]: derivation-ge [] i
by (simp add: derivation-ge-def)

lemma leftmost-notword: leftmost i a ⟹ j > i ⟹ ¬ (is-word (take j a))
by (metis is-terminal-nonterminal is-word-def leftmost-def list-all-take)

lemma leftmost-unique: leftmost i a ⟹ leftmost j a ⟹ i = j
by (metis leftmost-def leftmost-notword linorder-neqE-nat)

lemma leftmost-Derives1: leftmost i a ⟹ Derives1 a j r b ⟹ i ≤ j
by (metis Derives1-leftmost leftmost-unique)

lemma leftmost-Derives1-propagate:
  assumes leftmost: leftmost i a
      and Derives1: Derives1 a j r b
    shows (is-word b ∧ i = j) ∨ (∃ k. leftmost k b ∧ i ≤ k)
proof −
  from leftmost-Derives1[OF leftmost Derives1] have ij: i ≤ j by auto
  show ?thesis
  proof (cases is-word b)
    case True show ?thesis
      by (metis Derives1 True ij le-neq-implies-less leftmost
          leftmost-unaffected-Derives1 order-refl)
  next
    case False show ?thesis
    by (metis (no-types, opaque-lifting) Derives1 Derives1-bound Derives1-sentence2

          Derives1-take append-take-drop-id ij le-neq-implies-less leftmost
          leftmost-append leftmost-cons-less leftmost-def length-take
          min.absorb2 nat-le-linear nonword-leftmost-exists not-le)
  qed
qed

lemma is-word-Derives1[elim]: is-word a ⟹ Derives1 a i r b ⟹ False
by (metis Derives1-leftmost is-terminal-nonterminal is-word-is-terminal leftmost-def)

lemma is-word-Derivation[elim]: is-word a ⟹ Derivation a D b ⟹ D = []
by (metis Derivation-leftmost is-terminal-nonterminal is-word-def
    leftmost-def list-all-length)

lemma leftmost-Derivation:
  leftmost i a ⟹ Derivation a D b ⟹ j ≤ i ⟹ derivation-ge D j
proof (induct D arbitrary: a b i j)
  case Nil thus ?case by auto
next
```

```
  case (Cons d D)
  then have ∃ x. Derives1 a (fst d) (snd d) x ∧ Derivation x D b by auto
  then obtain x where ax:Derives1 a (fst d) (snd d) x and xb:Derivation x D b
by blast
  note ji = Cons(4)
  note i-fstd = leftmost-Derives1[OF Cons(2) ax]
  note disj = leftmost-Derives1-propagate[OF Cons(2) ax]
  thus ?case
  proof(induct rule: disjCases2)
    case 1
    with xb have D = [] by auto
    with 1 ji show ?case by (simp add: derivation-ge-def)
  next
    case 2
    then obtain k where k: leftmost k x and ik: i ≤ k by blast
    note ge = Cons(1)[OF k xb, where j=j]
    from ji ik i-fstd ge show ?case
      by (simp add: derivation-ge-cons)
  qed
qed

lemma derivation-ge-list-all: derivation-ge D i = list-all (λ d. fst d ≥ i) D
by (simp add: Ball-set derivation-ge-def)

lemma split-derivation-leftmost:
  assumes derivation-ge D i
  and ¬ (derivation-ge D (Suc i))
  shows ∃ E F r. D = E@[(i, r)]@F ∧ derivation-ge E (Suc i)
proof −
  from assms have ∃ k. k < length D ∧ fst(D ! k) ≥ i ∧ ¬(fst(D ! k) ≥ Suc i)
    by (metis derivation-ge-def in-set-conv-nth)
  then have ∃ k. k < length D ∧ fst(D ! k) = i by auto
  then show ?thesis
  proof(induct rule: ex-minimal-witness)
    case (Minimal k)
      then have k-len: k < length D and k-i: fst (D ! k) = i by auto
      let ?E = take k D
      let ?r = snd (D ! k)
      let ?F = drop (Suc k) D
      note split = split-list-at[OF k-len]
      from k-i have D-k: D ! k = (i, ?r) by auto
      show ?case
        apply (rule exI[where x=?E])
        apply (rule exI[where x=?F])
        apply (rule exI[where x=?r])
        apply (subst D-k[symmetric])
        apply (rule conjI)
        apply (rule split)
        by (metis (mono-tags, lifting) Minimal.hyps(2) Suc-leI assms(1)
```

```
        derivation-ge-list-all le-neq-implies-less list-all-length list-all-take)
  qed
qed

lemma Derives1-Derives1-swap:
  assumes i < j
  and Derives1 a j p b
  and Derives1 b i q c
  shows ∃ b'. Derives1 a i q b' ∧ Derives1 b' (j − 1 + length (snd q)) p c
proof −
  from Derives1-split[OF assms(2)] obtain a1 a2 where
    a-src: a = a1 @ [fst p] @ a2 and a-dest: b = a1 @ snd p @ a2
    and a1-len: length a1 = j by blast
  note a = this
  from a have is-sentence-a1: is-sentence a1
    using Derives1-sentence2 assms(2) is-sentence-concat by blast
  from a have is-sentence-a2: is-sentence a2
    using Derives1-sentence2 assms(2) is-sentence-concat by blast
  from a have is-symbol-fst-p: is-symbol (fst p)
    by (metis Derives1-sentence1 assms(2) is-sentence-concat is-sentence-cons)
  from Derives1-split[OF assms(3)] obtain b1 b2 where
    b-src: b = b1 @ [fst q] @ b2 and b-dest: c = b1 @ snd q @ b2
    and b1-len: length b1 = i by blast
  have a-take-j: a1 = take j a by (metis a1-len a-src append-eq-conv-conj)
  have b-take-i: b1 @ [fst q] = take (Suc i) b
   by (metis append-assoc append-eq-conv-conj b1-len b-src length-append-singleton)

  from a-take-j b-take-i  take-eq-take-append[where j=j and i=Suc i and a=a]
  have ∃ u. a1 = (b1 @ [fst q]) @ u
   by (metis le-iff-add Suc-leI a1-len a-dest append-eq-conv-conj assms(1) take-add)

  then obtain u where u1: a1 = (b1 @ [fst q]) @ u by blast
  then have j-i-u: j = i + 1 + length u
    using Suc-eq-plus1 a1-len b1-len length-append length-append-singleton by auto
  from u1 is-sentence-a1 have is-sentence-b1-u: is-sentence b1 ∧ is-sentence u
    using is-sentence-concat by blast
 have u2: b2 = u @ snd p @ a2 by (metis a-dest append-assoc b-src same-append-eq
u1)
  let ?b = b1 @ (snd q) @ u @ [fst p] @ a2
  from assms have q-dom: q ∈ ℜ by auto
  have a-b': Derives1 a i q ?b
    apply (subst Derives1-def)
    apply (rule exI[where x=b1])
    apply (rule exI[where x=u@[fst p]@a2])
    apply (rule exI[where x=fst q])
    apply (rule exI[where x=snd q])
    apply (auto simp add: b1-len is-sentence-cons is-sentence-concat
          is-sentence-a2 is-symbol-fst-p is-sentence-b1-u a-src u1 q-dom)
    done
```

```
  from assms have p-dom: p ∈ ℜ by auto
  have is-sentence-snd-q: is-sentence (snd q)
    using Derives1-sentence2 a-b' is-sentence-concat by blast
  have b'-c: Derives1 ?b (j − 1 + length (snd q)) p c
    apply (subst Derives1-def)
    apply (rule exI[where x=b1 @ (snd q) @ u])
    apply (rule exI[where x=a2])
    apply (rule exI[where x=fst p])
    apply (rule exI[where x=snd p])
   apply (auto simp add: is-sentence-concat is-sentence-b1-u is-sentence-a2 p-dom
          is-sentence-snd-q b-dest u2 b1-len j-i-u)
    done
  show ?thesis
    apply (rule exI[where x=?b])
    apply (rule conjI)
    apply (rule a-b')
    apply (rule b'-c)
    done
qed

definition derivation-shift :: derivation ⇒ nat ⇒ nat ⇒ derivation
where
  derivation-shift D left right = map (λ d. (fst d − left + right, snd d)) D

lemma derivation-shift-empty[simp]: derivation-shift [] left right = []
  by (auto simp add: derivation-shift-def)

lemma derivation-shift-cons[simp]:
  derivation-shift (d#D) left right = ((fst d − left + right, snd d)#(derivation-shift
D left right))
by (simp add: derivation-shift-def)

lemma Derivation-append: Derivation a (D@E) c = (∃ b. Derivation a D b ∧
Derivation b E c)
proof(induct D arbitrary: a c E)
  case Nil thus ?case by auto
next
  case (Cons d D) thus ?case by auto
qed

lemma Derivation-implies-append:
  Derivation a D b ⟹ Derivation b E c ⟹ Derivation a (D@E) c
using Derivation-append by blast

lemma Derivation-swap-single-end-to-front:
  i < j ⟹ derivation-ge D j ⟹ Derivation a (D@[(i,r)]) b ⟹
   Derivation a ((i,r)#(derivation-shift D 1 (length (snd r)))) b
proof(induct D arbitrary: a)
  case Nil thus ?case by auto
```

next
case (*Cons d D*)
from *Cons* **have** ∃ *c*. *Derives1 a* (*fst d*) (*snd d*) *c* ∧ *Derivation c* (*D* @ [(*i*, *r*)]) *b*
by *simp*
then obtain *c* **where** *ac*: *Derives1 a* (*fst d*) (*snd d*) *c*
and *cb*: *Derivation c* (*D* @ [(*i*, *r*)]) *b* **by** *blast*
from *Cons*(*3*) **have** *D-j*: *derivation-ge D j* **by** (*simp add*: *derivation-ge-cons*)
from *Cons*(*1*)[*OF Cons*(*2*) *D-j cb*, *simplified*]
obtain *x* **where** *cx*: *Derives1 c i r x* **and**
xb: *Derivation x* (*derivation-shift D 1* (*length* (*snd r*))) *b* **by** *auto*
have *i-fst-d*: *i* < *fst d* **using** *Cons derivation-ge-cons* **by** *auto*
from *Derives1-Derives1-swap*[*OF i-fst-d ac cx*]
obtain *b′* **where** *ab′*: *Derives1 a i r b′* **and**
b′x: *Derives1 b′* (*fst d* − *1* + *length* (*snd r*)) (*snd d*) *x* **by** *blast*
show *?case* **using** *ab′ b′x xb* **by** *auto*
qed

lemma *Derivation-swap-single-mid-to-front*:
assumes *i* < *j*
and *derivation-ge D j*
and *Derivation a* (*D*@[(*i*,*r*)]@*E*) *b*
shows *Derivation a* ((*i*,*r*)#((*derivation-shift D 1* (*length* (*snd r*)))@*E*)) *b*
proof −
from *assms* **have** ∃ *x*. *Derivation a* (*D*@[(*i*, *r*)]) *x* ∧ *Derivation x E b*
using *Derivation-append* **by** *auto*
then obtain *x* **where** *ax*: *Derivation a* (*D*@[(*i*, *r*)]) *x* **and** *xb*: *Derivation x E b*
by *blast*
with *assms* **have** *Derivation a* ((*i*, *r*)#(*derivation-shift D 1* (*length* (*snd r*)))) *x*
using *Derivation-swap-single-end-to-front* **by** *blast*
then show *?thesis* **using** *Derivation-append xb* **by** *auto*
qed

lemma *length-derivation-shift*[*simp*]:
length(*derivation-shift D left right*) = *length D*
by (*simp add*: *derivation-shift-def*)

definition *LeftDerives1* :: *sentence* ⇒ *nat* ⇒ *rule* ⇒ *sentence* ⇒ *bool*
where
LeftDerives1 u i r v = (*leftmost i u* ∧ *Derives1 u i r v*)

lemma *LeftDerives1-implies-leftderives1*: *LeftDerives1 u i r v* ⟹ *leftderives1 u v*
by (*metis Derives1-def LeftDerives1-def append-eq-conv-conj leftderives1-def*
leftmost-def)

lemma *leftmost-Derives1-leftderives*:
leftmost i a ⟹ *Derives1 a i r b* ⟹ *leftderives b c* ⟹ *leftderives a c*
using *LeftDerives1-def LeftDerives1-implies-leftderives1*
leftderives1-implies-leftderives leftderives-trans **by** *blast*

```
theorem Derivation-implies-leftderives-gen:
  Derivation a D (u@v) ⟹ is-word u ⟹ (∃ w.
        leftderives a (u@w) ∧
        (v = [] ⟶ w = []) ∧
        (∀ X. is-first X v ⟶ is-first X w))
proof (induct length D arbitrary: D a u v)
  case 0
    then have a = u@v by auto
    thus ?case by (rule-tac x = v in exI, auto)
next
  case (Suc n)
    from Suc have D ≠ [] by auto
    with Suc Derivation-leftmost have ∃ i. leftmost i a by auto
    then obtain i where i: leftmost i a by blast
    show ?case
    proof (cases derivation-ge D (Suc i))
      case True
        with Suc have leftmost: leftmost i (u@v)
          by (rule-tac leftmost-unaffected-Derivation[OF True i], auto)
        have length-u: length u ≤ i
          using leftmost-append[OF leftmost Suc(4)] .
        have take-Suc: take (Suc i) a = take (Suc i) (u@v)
          using  Derivation-take[OF True Suc(3)] .
        with length-u have is-prefix-u: is-prefix u a
         by (metis append-assoc append-take-drop-id dual-order.strict-implies-order

              is-prefix-def le-imp-less-Suc take-all take-append)
        have a: u @ drop (length u) a = a
          using is-prefix-unsplit[OF is-prefix-u] .
        from take-Suc have length-take-Suc: length (take (Suc i) a) = Suc i
          by (metis Suc-leI i leftmost-def length-take min.absorb2)
        have v: v ≠ []
        proof(cases v = [])
          case False thus ?thesis by auto
        next
          case True
          with length-u have right: length(take (Suc i) (u@v)) = length u by simp
          note left = length-take-Suc
          from left right take-Suc have Suc i = length u by auto
          with length-u show ?thesis by auto
        qed
        then have ∃ X w. v = X#w by (cases v, auto)
        then obtain X w where v: v = X#w by blast
        have is-first-X: is-first X (drop (length u) a)
          apply (rule-tac is-first-drop-length[where v=v and w=w and k=Suc i])
          apply (simp-all add: take-Suc v)
          apply (metis Suc-leI i leftmost-def)
          apply (insert length-u)
```

```
        apply arith
        done
      show ?thesis
        apply (rule exI[where x=drop (length u) a])
        by (simp add: a v is-first-cons is-first-X)
    next
      case False
      have Di: derivation-ge D i
      using leftmost-Derivation[OF i Suc(3), where j=i, simplified] .
      from split-derivation-leftmost[OF Di False]
      obtain E F r where D-split: D = E @ [(i, r)] @ F
        and E-Suc: derivation-ge E (Suc i) by auto
      let ?D = (derivation-shift E 1 (length (snd r)))@F
      from D-split
      have Derivation a ((i,r) # ?D) (u @ v)
       using Derivation-swap-single-mid-to-front E-Suc Suc.prems(1) lessI by blast
      then have ∃ y. Derives1 a i r y ∧ Derivation y ?D (u @ v) by simp
      then obtain y where ay:Derives1 a i r y
        and yuv: Derivation y ?D (u @ v) by blast
      have length-D': length ?D = n using D-split Suc.hyps(2) by auto
      from Suc(1)[OF length-D'[symmetric] yuv Suc(4)]
      obtain w where leftderives y (u @ w) and (v = [] ⟶ w = [])
        and ∀ X. is-first X v ⟶ is-first X w by blast
      then show ?thesis using ay i leftmost-Derives1-leftderives by blast
    qed
qed

lemma derives-implies-leftderives-gen: derives a (u@v) ⟹ is-word u ⟹ (∃ w.
        leftderives a (u@w) ∧
        (v = [] ⟶ w = []) ∧
        (∀ X. is-first X v ⟶ is-first X w))
using Derivation-implies-leftderives-gen derives-implies-Derivation by blast

lemma derives-implies-leftderives: derives a b ⟹ is-word b ⟹ leftderives a b
using derives-implies-leftderives-gen by force

fun LeftDerivation :: sentence ⇒ derivation ⇒ sentence ⇒ bool
where
  LeftDerivation a [] b = (a = b)
| LeftDerivation a (d#D) b = (∃ x. LeftDerives1 a (fst d) (snd d) x ∧ LeftDeriva-
tion x D b)

lemma LeftDerives1-implies-Derives1: LeftDerives1 a i r b ⟹ Derives1 a i r b
by (metis LeftDerives1-def)

lemma LeftDerivation-implies-Derivation:
  LeftDerivation a D b ⟹ Derivation a D b
proof (induct D arbitrary: a)
  case (Nil) thus ?case by simp
```

next
case (*Cons d D*)
thus *?case*
using *CFG.LeftDerivation.simps*(*2*) *CFG-axioms Derivation.simps*(*2*)
LeftDerives1-implies-Derives1 **by** *blast*
qed

lemma *LeftDerivation-implies-leftderives*: *LeftDerivation a D b* ⟹ *leftderives a b*
proof (*induct D arbitrary*: *a b*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons d D*)
note *ihyps* = *this*
from *ihyps* **have** ∃ *x*. *LeftDerives1 a* (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x D b*
by *auto*
then obtain *x* **where** *LeftDerives1 a* (*fst d*) (*snd d*) *x* **and** *xb*: *LeftDerivation*
x D b **by** *blast*
with *LeftDerives1-implies-leftderives1* **have** *d1*: *leftderives a x* **by** *auto*
from *ihyps xb* **have** *d2*:*leftderives x b* **by** *simp*
show *leftderives a b* **by** (*rule leftderives-trans*[*OF d1 d2*])
qed

lemma *leftmost-witness*[*simp*]: *leftmost* (*length x*) (*x*@(*N*#*y*)) = (*is-word x* ∧
is-nonterminal N)
by (*auto simp add*: *leftmost-def*)

lemma *leftderives1-implies-LeftDerives1*:
assumes *leftderives1*: *leftderives1 u v*
shows ∃ *i r*. *LeftDerives1 u i r v*
proof −
from *leftderives1* **have**
∃*x y N α*. *u* = *x* @ [*N*] @ *y* ∧ *v* = *x* @ *α* @ *y* ∧ *is-word x* ∧ *is-sentence y* ∧
(*N*, *α*) ∈ ℜ
by (*simp add*: *leftderives1-def*)
then obtain *x y N α* **where**
u:*u* = *x* @ [*N*] @ *y* **and**
v:*v* = *x* @ *α* @ *y* **and**
x:*is-word x* **and**
y:*is-sentence y* **and**
r:(*N*, *α*) ∈ ℜ
by *blast*
show *?thesis*
apply (*rule-tac x*=*length x* **in** *exI*)
apply (*rule-tac x*=(*N*, *α*) **in** *exI*)
apply (*auto simp add*: *LeftDerives1-def*)
apply (*simp add*: *leftmost-def x u rule-nonterminal-type*[*OF r*])
apply (*simp add*: *Derives1-def u v*)
apply (*rule-tac x*=*x* **in** *exI*)
apply (*rule-tac x*=*y* **in** *exI*)

```
    apply (auto simp add: x y r)
    done
qed

lemma LeftDerivation-LeftDerives1:
  LeftDerivation a S y ⟹ LeftDerives1 y i r z ⟹ LeftDerivation a (S@[(i,r)]) z
proof (induct S arbitrary: a y z i r)
  case Nil thus ?case by simp
next
  case (Cons s S) thus ?case
    by (metis LeftDerivation.simps(2) append-Cons)
qed

lemma leftderives-implies-LeftDerivation: leftderives a b ⟹ ∃ D. LeftDerivation
a D b
proof (induct rule: leftderives-induct)
  case Base
  show ?case by (rule exI[where x=[]], simp)
next
  case (Step y z)
  note ihyps = this
  from ihyps obtain D where ay: LeftDerivation a D y by blast
  from ihyps leftderives1-implies-LeftDerives1 obtain i r where yz: LeftDerives1
y i r z by blast
  from LeftDerivation-LeftDerives1[OF ay yz] show ?case by auto
qed

lemma LeftDerivation-append:
  LeftDerivation a (D@E) c = (∃ b. LeftDerivation a D b ∧ LeftDerivation b E c)
proof(induct D arbitrary: a c E)
  case Nil thus ?case by auto
next
  case (Cons d D) thus ?case by auto
qed

lemma LeftDerivation-implies-append:
  LeftDerivation a D b ⟹ LeftDerivation b E c ⟹ LeftDerivation a (D@E) c
using LeftDerivation-append by blast

lemma Derivation-unique-dest: Derivation a D b ⟹ Derivation a D c ⟹ b = c
  apply (induct D arbitrary: a b c)
  apply auto
  using Derives1-unique-dest by blast

lemma Derivation-unique-src: Derivation a D c ⟹ Derivation b D c ⟹ a = b
  apply (induct D arbitrary: a b c)
  apply auto
  using Derives1-unique-src by blast
```

lemma *LeftDerives1-unique*: *LeftDerives1 a i r b* $\Longrightarrow$ *LeftDerives1 a j s b* $\Longrightarrow$ $i = j \wedge r = s$
using *Derives1-def LeftDerives1-def leftmost-unique* **by** *auto*

lemma *leftlang*: $\mathcal{L} = \{\ v \mid v.\ is\text{-}word\ v \wedge is\text{-}leftderivation\ v\ \}$
by (*metis* (*no-types*, *lifting*) *CFG.is-derivation-def CFG.is-leftderivation-def CFG.leftderivation-implies-derivation CFG-axioms Collect-cong* $\mathcal{L}$*-def derives-implies-leftderives*)

lemma *leftprefixlang*: $\mathcal{L}_P = \{\ u \mid u\ v.\ is\text{-}word\ u \wedge is\text{-}leftderivation\ (u@v)\ \}$
apply (*auto simp add*: $\mathcal{L}_P$*-def*)
using *derives-implies-leftderives-gen is-derivation-def is-leftderivation-def* **apply** *blast*
using *leftderivation-implies-derivation* **by** *blast*

lemma *derives-implies-leftderives-cons*:
is-word a $\Longrightarrow$ *derives u* $(a@X\#b) \Longrightarrow \exists\ c.$ *leftderives u* $(a@X\#c)$
by (*metis append-Cons append-Nil derives-implies-leftderives-gen is-first-def*)

lemma *is-word-append*[*simp*]: *is-word* $(a@b) =$ (*is-word a* $\wedge$ *is-word b*)
by (*auto simp add*: *is-word-def*)

lemma $\mathcal{L}_P$*-split*: $a@b \in \mathcal{L}_P \Longrightarrow a \in \mathcal{L}_P$
by (*auto simp add*: $\mathcal{L}_P$*-def*)

lemma $\mathcal{L}_P$*-is-word*: $a \in \mathcal{L}_P \Longrightarrow$ *is-word a*
by (*metis* (*no-types*, *lifting*) *leftprefixlang mem-Collect-eq*)

definition *Derive* :: *sentence* $\Rightarrow$ *derivation* $\Rightarrow$ *sentence*
where
Derive a D = (*THE b*. *Derivation a D b*)

lemma *Derivation-dest-ex-unique*: *Derivation a D b* $\Longrightarrow \exists!\ x.$ *Derivation a D x*
using *CFG.Derivation-unique-dest CFG-axioms* **by** *blast*

lemma *Derive*:
assumes *ab*: *Derivation a D b*
shows *Derive a D* = *b*
proof −
note *the1-equality*[*OF Derivation-dest-ex-unique*[*OF ab*] *ab*]
thus *?thesis* **by** (*simp add*: *Derive-def*)
qed

end

end
theory *Validity*
imports *LLEarleyParsing Derivations*
begin

context *LocalLexing* **begin**

definition *wellformed-token* :: *token* ⇒ *bool*
where
wellformed-token t = *is-terminal* (*terminal-of-token t*)

definition *wellformed-tokens* :: *tokens* ⇒ *bool*
where
wellformed-tokens ts = *list-all wellformed-token ts*

definition *doc-tokens* :: *tokens* ⇒ *bool*
where
doc-tokens p = (*wellformed-tokens p* ∧ *is-prefix* (*chars p*) *Doc*)

definition *wellformed-item* :: *item* ⇒ *bool*
where
wellformed-item x = (
item-rule x ∈ ℜ ∧
item-origin x ≤ *item-end x* ∧
item-end x ≤ *length Doc* ∧
item-dot x ≤ *length* (*item-rhs x*))

definition *wellformed-items* :: *items* ⇒ *bool*
where
wellformed-items X = (∀ *x* ∈ *X*. *wellformed-item x*)

lemma *is-word-terminals*: *wellformed-tokens p* ⟹ *is-word* (*terminals p*)
by (*simp add*: *is-word-def list-all-length terminals-def wellformed-token-def wellformed-tokens-def*)

lemma *is-word-subset*: *is-word x* ⟹ *set y* ⊆ *set x* ⟹ *is-word y*
by (*metis* (*mono-tags*, *opaque-lifting*) *in-set-conv-nth is-word-def list-all-length subsetCE*)

lemma *is-word-terminals-take*: *wellformed-tokens p* ⟹ *is-word*(*terminals* (*take n p*))
by (*metis append-take-drop-id is-word-terminals list-all-append wellformed-tokens-def*)

lemma *is-word-terminals-drop*: *wellformed-tokens p* ⟹ *is-word*(*terminals* (*drop n p*))
by (*metis append-take-drop-id is-word-terminals list-all-append wellformed-tokens-def*)

definition *pvalid* :: *tokens* ⇒ *item* ⇒ *bool*
where
pvalid p x = (∃ *u γ*.
wellformed-tokens p ∧
wellformed-item x ∧
u ≤ *length p* ∧

```
    charslength p = item-end x ∧
    charslength (take u p) = item-origin x ∧
    is-derivation (terminals (take u p) @ [item-nonterminal x] @ γ) ∧
    derives (item-α x) (terminals (drop u p)))

definition Gen :: tokens set ⇒ items
where
  Gen P = { x | x p. p ∈ P ∧ pvalid p x }

lemma wellformed-items (Gen P)
  by (auto simp add: Gen-def pvalid-def wellformed-items-def)

lemma wellformed-items (Init)
 by (auto simp add: wellformed-items-def Init-def init-item-def wellformed-item-def)

definition pvalid-left :: tokens ⇒ item ⇒ bool
where
  pvalid-left p x = (∃ u γ.
    wellformed-tokens p ∧
    wellformed-item x ∧
    u ≤ length p ∧
    charslength p = item-end x ∧
    charslength (take u p) = item-origin x ∧
    is-leftderivation (terminals (take u p) @ [item-nonterminal x] @ γ) ∧
    leftderives (item-α x) (terminals (drop u p)))

lemma pvalid-left: pvalid p x = pvalid-left p x
proof −
  have right-imp-left: pvalid-left p x ⟹ pvalid p x
   by (meson CFG.leftderives-implies-derives CFG-axioms LocalLexing.pvalid-def
     LocalLexing.pvalid-left-def LocalLexing-axioms leftderivation-implies-derivation)
  have left-imp-right: pvalid p x ⟹ pvalid-left p x
  proof −
    assume pvalid p x
    then obtain u γ where
      wellformed-tokens p ∧
       wellformed-item x ∧
       u ≤ length p ∧
       charslength p = item-end x ∧
       charslength (take u p) = item-origin x ∧
       is-derivation (terminals (take u p) @ [item-nonterminal x] @ γ) ∧
       derives (item-α x) (terminals (drop u p)) by (simp add: pvalid-def, blast)
    thus ?thesis
      apply (auto simp add: pvalid-left-def)
      apply (rule-tac x=u in exI)
      apply auto
      apply (simp add: is-leftderivation-def)
      apply (rule-tac derives-implies-leftderives-cons[where b=γ])
      apply (erule is-word-terminals-take)
```

apply (*simp add*: *is-derivation-def*)
by (*metis derives-implies-leftderives is-word-terminals-drop*)
qed
show *?thesis* **by** (*metis left-imp-right right-imp-left*)
qed

lemma $\mathcal{L}_P$*-wellformed-tokens*: *terminals* $p \in \mathcal{L}_P \Longrightarrow$ *wellformed-tokens* p
by (*metis* (*mono-tags*, *lifting*) $\mathcal{L}_P$*-is-word is-word-def length-map list-all-length nth-map terminals-def wellformed-token-def wellformed-tokens-def*)

end

end
theory *TheoremD2*
imports *LocalLexingLemmas Validity Derivations*
begin

context *LocalLexing* **begin**

definition *splits-at* :: *sentence* ⇒ *nat* ⇒ *sentence* ⇒ *symbol* ⇒ *sentence* ⇒ *bool*
where
splits-at δ i α N β = ($i <$ *length* $\delta \wedge \alpha =$ *take* i $\delta \wedge N = \delta$! $i \wedge \beta =$ *drop* (*Suc* i) δ)

lemma *splits-at-combine*: *splits-at* δ i α N $\beta \Longrightarrow \delta = \alpha$ @ [N] @ β
by (*simp add*: *id-take-nth-drop splits-at-def*)

lemma *splits-at-combine-dest*: *Derives1* a i r $b \Longrightarrow$ *splits-at* a i α N $\beta \Longrightarrow b = \alpha$ @ (*snd* r) @ β
by (*metis* (*no-types*, *lifting*) *Derives1-drop Derives1-split append-assoc append-eq-conv-conj length-append splits-at-def*)

lemma *Derives1-nonterminal*:
assumes *Derives1* a i r b
assumes *splits-at* a i α N β
shows *fst* $r = N \wedge$ *is-nonterminal* N
proof −
from *assms* **have** *fst*: *fst* $r = N$
by (*metis Derives1-split append-Cons nth-append-length splits-at-def*)
then have *is-nonterminal* N
by (*metis Derives1-def assms*(*1*) *prod.collapse rule-nonterminal-type*)
with *fst* **show** *?thesis* **by** *auto*
qed

lemma *splits-at-ex*: *Derives1* δ i r $s \Longrightarrow \exists$ α N β. *splits-at* δ i α N β
by (*simp add*: *Derives1-bound splits-at-def*)

lemma *splits-at-α*: *Derives1 δ i r s ⟹ splits-at δ i α N β ⟹*
α = take i δ ∧ α = take i s ∧ length α = i
by (*metis Derives1-split append-eq-conv-conj splits-at-def*)

lemma *LeftDerives1-splits-at-is-word*: *LeftDerives1 δ i r s ⟹ splits-at δ i α N β ⟹ is-word α*
by (*metis LeftDerives1-def leftmost-def splits-at-def*)

lemma *splits-at-β*: *Derives1 δ i r s ⟹ splits-at δ i α N β ⟹*
β = drop (Suc i) δ ∧ β = drop (i + length (snd r)) s ∧ length β = length δ − i − 1
by (*metis Derives1-drop Suc-eq-plus1 diff-diff-left length-drop splits-at-def*)

lemma *Derives1-prefix*:
assumes *ab*: *Derives1 δ i r (a@b)*
assumes *split*: *splits-at δ i α N β*
shows *is-prefix α a ∨ is-prefix a α*
proof −
have *take*: *α = take i (a@b)* **using** *ab split splits-at-α* **by** *blast*
show *?thesis*
proof (*cases i ≤ length a*)
case *True*
then have *α = take i a* **by** (*simp add*: *take*)
then have *is-prefix α a*
by (*metis append-take-drop-id is-prefix-def*)
with *True* **show** *?thesis* **by** *auto*
next
case *False*
then have *is-prefix a α*
by (*simp add*: *is-prefix-def nat-le-linear take*)
with *False* **show** *?thesis* **by** *auto*
qed
qed

lemma *Derives1-suffix*:
assumes *ab*: *Derives1 δ i r (a@b)*
assumes *split*: *splits-at δ i α N β*
shows *is-suffix β b ∨ is-suffix b β*
proof −
have *drop1*: *β = drop (i + length (snd r)) (a@b)* **using** *ab split splits-at-β* **by** *blast*
have *drop2*: *b = drop (length a) (a@b)* **by** *simp*
show *?thesis*
proof (*cases (i + length (snd r)) ≤ length a*)
case *True*
with *drop1 drop2* **have** *is-suffix b β* **by** (*simp add*: *is-suffix-def*)
then show *?thesis* **by** *auto*
next
case *False*

```
    then have length a ≤ (i + length (snd r)) by arith
    with drop1 drop2 have is-suffix β b
     by (metis append-Nil append-take-drop-id drop-append drop-eq-Nil is-suffix-def)
    then show ?thesis by auto
  qed
qed

lemma Derives1-skip-prefix:
  length a ≤ i ⟹ Derives1 (a@b) i r (a@c) ⟹ Derives1 b (i − length a) r c
apply (auto simp add: Derives1-def)
by (metis append-eq-append-conv-if is-sentence-concat is-sentence-cons is-symbol-def

    length-drop rule-nonterminal-type)

lemma cancel-suffix:
  assumes a @ c = b @ d
  assumes length c ≤ length d
  shows a = b @ (take (length d − length c) d)
proof −
  have a @ c = (b @ take (length d − length c) d) @ drop (length d − length c) d
    by (metis append-assoc append-take-drop-id assms(1))
  then show ?thesis
    by (metis append-eq-append-conv assms(2) diff-diff-cancel length-drop)
qed

lemma is-sentence-take:
  is-sentence y ⟹ is-sentence (take n y)
by (metis append-take-drop-id is-sentence-concat)

lemma Derives1-skip-suffix:
  assumes i: i < length a
  assumes D: Derives1 (a@c) i r (b@c)
  shows Derives1 a i r b
proof −
  note Derives1-def[where u=a@c and v=b@c and i=i and r=r]
  then have ∃ x y N α.
    a @ c = x @ [N] @ y ∧
    b @ c = x @ α @ y ∧ is-sentence x ∧ is-sentence y ∧ (N, α) ∈ ℜ ∧ r = (N,
α) ∧ i = length x
    using D by blast
  then obtain x y N α where split:
    a @ c = x @ [N] @ y ∧
     b @ c = x @ α @ y ∧ is-sentence x ∧ is-sentence y ∧ (N, α) ∈ ℜ ∧ r = (N,
α) ∧ i = length x
     by blast
  from split have length (a@c) = length (x @ [N] @ y) by auto
  then have length a + length c = length x + length y + 1 by simp
  with split have length a + length c = i + length y + 1 by simp
  with i have len-c-y: length c ≤ length y by arith
```

let *?y* = *take* (*length y* − *length c*) *y*
from *split* **have** *ac*: *a* @ *c* = (*x* @ [*N*]) @ *y* **by** *auto*
note *cancel-suffix*[**where** *a*=*a* **and** *c* = *c* **and** *b* = *x*@[*N*] **and** *d* = *y*, *OF ac len-c-y*]
then have *a*: *a* = *x* @ [*N*] @ *?y* **by** *auto*
from *split* **have** *bc*: *b* @ *c* = (*x* @ *α*) @ *y* **by** *auto*
note *cancel-suffix*[**where** *a*=*b* **and** *c* = *c* **and** *b* = *x*@*α* **and** *d* = *y*, *OF bc len-c-y*]
then have *b*: *b* = *x* @ *α* @ *?y* **by** *auto*
from *split len-c-y a b* **show** *?thesis*
apply (*simp only*: *Derives1-def*)
apply (*rule-tac x*=*x* **in** *exI*)
apply (*rule-tac x*=*?y* **in** *exI*)
apply (*rule-tac x*=*N* **in** *exI*)
apply (*rule-tac x*=*α* **in** *exI*)
apply *auto*
by (*rule is-sentence-take*)
qed

lemma *drop-cancel-suffix*: *a*@*c* = *drop n* (*b*@*c*) ⟹ *a* = *drop n b*
proof −
assume *a1*: *a* @ *c* = *drop n* (*b* @ *c*)
have *length* (*drop n b*) = *length b* + *length c* − *n* − *length c*
by (*metis add-diff-cancel-right′ diff-commute length-drop*)
then show *?thesis*
using *a1* **by** (*metis add-diff-cancel-right′ append-eq-append-conv drop-append length-append length-drop*)
qed

lemma *drop-keep-last*: *u* ≠ [] ⟹ *u* = *drop n* (*a*@[*X*]) ⟹ *u* = *drop n a* @ [*X*]
by (*metis append-take-drop-id drop-butlast last-appendR snoc-eq-iff-butlast*)

lemma *Derives1-X-is-part-of-rule*[*consumes 2*, *case-names Suffix Prefix*]:
assumes *aXb*: *Derives1 δ i r* (*a*@[*X*]@*b*)
assumes *split*: *splits-at δ i α N β*
assumes *prefix*: ⋀ *β*. *δ* = *a* @ [*X*] @ *β* ⟹ *length a* < *i* ⟹
Derives1 β (*i* − *length a* − *1*) *r b* ⟹ *False*
assumes *suffix*: ⋀ *α*. *δ* = *α* @ [*X*] @ *b* ⟹ *Derives1 α i r a* ⟹ *False*
shows ∃ *u v*. *a* = *α* @ *u* ∧ *b* = *v* @ *β* ∧ (*snd r*) = *u*@[*X*]@*v*
proof −
have *prefix-or*: *is-prefix α a* ∨ *is-proper-prefix a α*
by (*metis Derives1-prefix split aXb is-prefix-eq-proper-prefix*)
have *is-proper-prefix a α* ⟹ *False*
proof −
assume *proper*:*is-proper-prefix a α*
then have ∃ *u*. *u* ≠ [] ∧ *α* = *a*@*u* **by** (*metis is-proper-prefix-def*)
then obtain *u* **where** *u*: *u* ≠ [] ∧ *α* = *a*@*u* **by** *blast*
note *splits-at* = *splits-at-α*[*OF aXb split*] *splits-at-combine*[*OF split*]
from *splits-at* **have** *α1*: *α* = *take i δ* **by** *blast*

from *splits-at* **have** $\alpha 2$: $\alpha = take\ i\ (a@[X]@b)$ **by** *blast*
from *splits-at* **have** *lenα*: *length* $\alpha = i$ **by** *blast*
with *proper* **have** *lena*: *length* $a < i$
using *append-eq-conv-conj drop-eq-Nil leI u* **by** *auto*
from $u\ \alpha 2$ **have** $a@u = take\ i\ (a@[X]@b)$ **by** *auto*
with *lena* **have** $u = take\ (i - length\ a)\ ([X]@b)$ **by** (*simp add*: *less-or-eq-imp-le*)

with *lena* **have** *uX*: $u = [X]@(take\ (i - length\ a - 1)\ b)$ **by** (*simp add*: *not-less take-Cons′*)
let $?\beta = (take\ (i - length\ a - 1)\ b)\ @\ [N]\ @\ \beta$
from *splits-at* **have** *f1*: $\delta = \alpha\ @\ [N]\ @\ \beta$ **by** *blast*
with *u uX* **have** *f2*: $\delta = a\ @\ [X]\ @\ ?\beta$ **by** *simp*
note *skip* = *Derives1-skip-prefix*[**where** $a = a\ @\ [X]$ **and** $b = ?\beta$ **and** $r = r$ **and** $i = i$ **and** $c = b$]
then have *D*: *Derives1* $?\beta\ (i - length\ a - 1)\ r\ b$
using *One-nat-def Suc-leI aXb append-assoc diff-diff-left f2 lena length-Cons length-append length-append-singleton list.size*(*3*) **by** *fastforce*
note *prefix*[*OF f2 lena D*]
then show *False* **.**
qed
with *prefix-or* **have** *is-prefix*: *is-prefix* $\alpha\ a$ **by** *blast*

from *aXb* **have** *aXb′*: *Derives1* $\delta\ i\ r\ ((a@[X])@b)$ **by** *auto*
note *Derives1-suffix*[*OF aXb′ split*]
then have *suffix-or*: *is-suffix* $\beta\ b \vee$ *is-proper-suffix* $b\ \beta$
by (*metis is-suffix-eq-proper-suffix*)
have *is-proper-suffix* $b\ \beta \Longrightarrow$ *False*
proof −
assume *proper*: *is-proper-suffix* $b\ \beta$
then have $\exists\ u.\ u \neq [] \wedge \beta = u@b$ **by** (*metis is-proper-suffix-def*)
then obtain *u* **where** *u*: $u \neq [] \wedge \beta = u@b$ **by** *blast*
note *splits-at* = *splits-at-β*[*OF aXb split*] *splits-at-combine*[*OF split*]
from *splits-at* **have** $\beta 1$: $\beta = drop\ (Suc\ i)\ \delta$ **by** *blast*
from *splits-at* **have** $\beta 2$: $\beta = drop\ (i + length\ (snd\ r))\ (a\ @\ [X]\ @\ b)$ **by** *blast*
from *splits-at* **have** *lenβ*: *length* β = *length* $\delta - i - 1$ **by** *blast*
with *proper* **have** *lenb*: *length* $b <$ *length* β **by** (*metis is-proper-suffix-length-cmp*)

from $u\ \beta 2$ **have** $u@b = drop\ (i + length\ (snd\ r))\ ((a\ @\ [X])\ @\ b)$ **by** *auto*
hence $u = drop\ (i + length\ (snd\ r))\ (a\ @\ [X])$
by (*metis drop-cancel-suffix*)
hence *uX*: $u = drop\ (i + length\ (snd\ r))\ a\ @\ [X]$ **by** (*metis drop-keep-last u*)
let $?\alpha = \alpha\ @\ [N]\ @\ (drop\ (i + length\ (snd\ r))\ a)$
from *splits-at* **have** *f1*: $\delta = \alpha\ @\ [N]\ @\ \beta$ **by** *blast*
with *u uX* **have** *f2*: $\delta = ?\alpha\ @\ [X]\ @\ b$ **by** *simp*
note *skip* = *Derives1-skip-suffix*[**where** $a = ?\alpha$ **and** $c = [X]@b$ **and** $b{=}a$ **and** $r = r$ **and** $i = i$]
have *f3*: $i <$ *length* $(\alpha\ @\ [N]\ @\ drop\ (i + length\ (snd\ r))\ a)$
proof −
have *f1*: $1 + i + length\ b = length\ [X] + length\ b + i$

by (*metis Groups.add-ac(2) Suc-eq-plus1-left length-Cons list.size(3) list.size(4) semiring-normalization-rules(22)*)
have *f2*: *length* $\delta - i - 1 =$ *length* $((\alpha \mathbin{@} [N] \mathbin{@}$ *drop* $(i +$ *length* $(snd\ r))\ a) \mathbin{@} [X] \mathbin{@} b) -$ *Suc* i
by (*metis f2 length-drop splits-at(1)*)
have *length* ([]::*symbol list*) $\neq$ *length* $\delta - i - 1 -$ *length* b
by (*metis add-diff-cancel-right' append-Nil2 append-eq-append-conv lenβ length-append u*)
then have *length* ([]::*symbol list*) $\neq$ *length* $\alpha +$ *length* $([N] \mathbin{@}$ *drop* $(i +$ *length* $(snd\ r))\ a) - i$
using *f2 f1* **by** (*metis Suc-eq-plus1-left add-diff-cancel-right' diff-diff-left length-append*)
then show *?thesis*
by *auto*
qed
from *aXb f2* **have** *D*: *Derives1* $(?\alpha \mathbin{@} [X] \mathbin{@} b)\ i\ r\ (a@[X]@b)$ **by** *auto*
note *skip*[*OF f3 D*]
note *suffix*[*OF f2 skip*[*OF f3 D*]]
then show *False* **.**
qed
with *suffix-or* **have** *is-suffix*: *is-suffix* $\beta\ b$ **by** *blast*

from *is-prefix* **have** $\exists\ u.\ a = \alpha \mathbin{@} u$ **by** (*auto simp add*: *is-prefix-def*)
then obtain u **where** u: $a = \alpha \mathbin{@} u$ **by** *blast*
from *is-suffix* **have** $\exists\ v.\ b = v \mathbin{@} \beta$ **by** (*auto simp add*: *is-suffix-def*)
then obtain v **where** v: $b = v \mathbin{@} \beta$ **by** *blast*

from *u v splits-at-combine*[*OF split*] *aXb* **have** *D*:*Derives1* $(\alpha@[N]@\beta)\ i\ r\ (\alpha@(u@[X]@v)@\beta)$
by *simp*
from *splits-at-α*[*OF aXb split*] **have** *i*: *length* $\alpha = i$ **by** *blast*
from *i* **have** *i1*: *length* $\alpha \leq i$ **and** *i2*: $i \leq$ *length* α **by** *auto*
note *Derives1-skip-suffix*[*OF - Derives1-skip-prefix*[*OF i1 D*], *simplified*, *OF i2*]
then have *Derives1* $[N]\ 0\ r\ (u \mathbin{@} [X] \mathbin{@} v)$ **by** *auto*
then have *r*: *snd* $r = u \mathbin{@} [X] \mathbin{@} v$
by (*metis Derives1-split append-Cons append-Nil length-0-conv list.inject self-append-conv*)

show *?thesis* **using** $u\ v\ r$ **by** *auto*
qed

lemma $\mathcal{L}_P$*-derives*: $a \in \mathcal{L}_P \Longrightarrow \exists\ b.$ *derives* $[\mathfrak{S}]\ (a@b)$
by (*simp add*: $\mathcal{L}_P$*-def is-derivation-def*)

lemma $\mathcal{L}_P$*-leftderives*: $a \in \mathcal{L}_P \Longrightarrow \exists\ b.$ *leftderives* $[\mathfrak{S}]\ (a@b)$
by (*metis* $\mathcal{L}_P$*-derives* $\mathcal{L}_P$*-is-word derives-implies-leftderives-gen*)

lemma *Derives1-rule*: *Derives1 a i r b* $\Longrightarrow r \in \mathfrak{R}$
by (*auto simp add*: *Derives1-def*)

lemma *is-prefix-empty*[*simp*]: *is-prefix* [] *a*
by (*simp add*: *is-prefix-def*)

lemma *is-prefix-cons*: *is-prefix* (*x* # *a*) *b* = (∃ *c*. *b* = *x* # *c* ∧ *is-prefix a c*)
by (*metis append-Cons is-prefix-def*)

lemma *is-prefix-cancel*[*simp*]: *is-prefix* (*a*@*b*) (*a*@*c*) = *is-prefix b c*
by (*metis append-assoc is-prefix-def same-append-eq*)

lemma *is-prefix-chars*: *is-prefix a b* ⟹ *is-prefix* (*chars a*) (*chars b*)
proof (*induct a arbitrary*: *b*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x a*)
from *Cons*(*2*) **have** ∃ *c*. *b* = *x* # *c* ∧ *is-prefix a c*
by (*simp add*: *is-prefix-cons*)
then obtain *c* **where** *c*: *b* = *x* # *c* ∧ *is-prefix a c* **by** *blast*
from *c Cons*(*1*) **show** *?case* **by** *simp*
qed

lemma *is-prefix-length*: *is-prefix a b* ⟹ *length a* ≤ *length b*
by (*auto simp add*: *is-prefix-def*)

lemma *is-prefix-take*[*simp*]: *is-prefix* (*take n a*) *a*
apply (*auto simp add*: *is-prefix-def*)
apply (*rule-tac x*=*drop n a* **in** *exI*)
by *simp*

lemma *doc-tokens-length*: *doc-tokens p* ⟹ *length* (*chars p*) ≤ *length Doc*
by (*metis doc-tokens-def is-prefix-length*)

fun *count-terminals* :: *sentence* ⇒ *nat* **where**
count-terminals [] = *0*
| *count-terminals* (*x*#*xs*) = (*if* (*is-terminal x*) *then Suc* (*count-terminals xs*) *else* (*count-terminals xs*))

lemma *count-terminals-upper-bound*: *count-terminals p* ≤ *length p*
by (*induct p*, *auto*)

lemma *count-terminals-append*[*simp*]: *count-terminals* (*a*@*b*) = *count-terminals a* + *count-terminals b*
by (*induct a arbitrary*: *b*, *auto*)

lemma *Derives1-count-terminals*:
assumes *D*: *Derives1 a i r b*
shows *count-terminals b* = *count-terminals a* + *count-terminals* (*snd r*)
proof −
have ∃ *α N β*. *splits-at a i α N β* **using** *D splits-at-ex* **by** *simp*
then obtain *α N β* **where** *split*: *splits-at a i α N β* **by** *blast*

from *D split* **have** *N*: *is-nonterminal N* **by** (*simp add*: *Derives1-nonterminal*)
have *a*: $a = \alpha$ @ $[N]$ @ β **by** (*metis split splits-at-combine*)
from *D split* **have** *b*: $b = \alpha$ @ (*snd r*) @ β **using** *splits-at-combine-dest* **by** *simp*
show *?thesis*
apply (*simp add*: *a b*)
using *N* **by** (*metis is-terminal-nonterminal*)
qed

lemma *Derives1-count-terminals-leq*:
assumes *D*: *Derives1 a i r b*
shows *count-terminals a* $\leq$ *count-terminals b*
by (*metis Derives1-count-terminals assms le-less-linear not-add-less1*)

lemma *Derivation-count-terminals-leq*:
Derivation a E b $\Longrightarrow$ *count-terminals a* $\leq$ *count-terminals b*
proof (*induct E arbitrary*: *a*)
case *Nil* **thus** *?case* **by** *auto*
next
case (*Cons e E*)
then have $\exists$ *x i r*. *Derives1 a i r x* $\wedge$ *Derivation x E b* **using** *Derivation.simps*(*2*)
by *blast*
then obtain *x i r* **where** *axb*: *Derives1 a i r x* $\wedge$ *Derivation x E b* **by** *blast*
from *axb* **have** *ax*: *count-terminals a* $\leq$ *count-terminals x*
using *Derives1-count-terminals-leq* **by** *blast*
from *axb* **have** *xb*: *count-terminals x* $\leq$ *count-terminals b* **using** *Cons* **by** *simp*
show *?case* **using** *ax xb* **by** *arith*
qed

lemma *derives-count-terminals-leq*: *derives a b* $\Longrightarrow$ *count-terminals a* $\leq$ *count-terminals b*
using *Derivation-count-terminals-leq derives-implies-Derivation* **by** *force*

lemma *is-word-cons*[*simp*]: *is-word* (*x*#*xs*) = (*is-terminal x* $\wedge$ *is-word xs*)
by (*simp add*: *is-word-def*)

lemma *count-terminals-of-word*: *is-word w* $\Longrightarrow$ *count-terminals w* = *length w*
by (*induct w*, *auto*)

lemma *length-terminals*[*simp*]: *length* (*terminals p*) = *length p*
by (*auto simp add*: *terminals-def*)

lemma *path-length-is-upper-bound*:
assumes *p*: *wellformed-tokens p*
assumes α: *is-word* α
assumes *derives*: *derives* (α@*u*) (*terminals p*)
shows *length* α $\leq$ *length p*
proof −
have *counts*: *count-terminals* α $\leq$ *count-terminals* (*terminals p*)
using *derives derives-count-terminals-leq* **by** *fastforce*

```
 have len1: length α = count-terminals α by (simp add: α count-terminals-of-word)
 have len2: length (terminals p) = count-terminals (terminals p)
   by (simp add: count-terminals-of-word is-word-terminals p)
 show ?thesis using counts len1 len2 by auto
qed

lemma is-word-Derives1-index:
 assumes w: is-word w
 assumes derives1: Derives1 (w@a) i r b
 shows i ≥ length w
proof −
 from derives1 have n: is-nonterminal ((w@a) ! i)
   using Derives1-nonterminal splits-at-def splits-at-ex by auto
 from w have t: i < length w ⟹ is-terminal ((w@a) ! i)
   by (simp add: is-word-is-terminal nth-append)
 show ?thesis
   by (metis t n is-terminal-nonterminal less-le-not-le nat-le-linear)
qed

lemma is-word-Derivation-derivation-ge:
 assumes w: is-word w
 assumes D: Derivation (w@a) D b
 shows derivation-ge D (length w)
by (metis D Derivation-leftmost derivation-ge-empty leftmost-Derivation leftmost-append
w)

lemma derives-word-is-prefix:
 assumes w: is-word w
 assumes derives: derives (w@a) b
 shows is-prefix w b
by (metis Derivation-take append-eq-conv-conj derives derives-implies-Derivation
   is-prefix-take is-word-Derivation-derivation-ge w)

lemma terminals-take[simp]: terminals (take n p) = take n (terminals p)
by (simp add: take-map terminals-def)

lemma terminals-drop[simp]: terminals (drop n p) = drop n (terminals p)
by (simp add: drop-map terminals-def)

lemma take-prefix[simp]: is-prefix a b ⟹ take (length a) b = a
by (metis append-eq-conv-conj is-prefix-unsplit)

lemma Derives1-drop-prefixword:
 assumes w: is-word w
 assumes wa-b: Derives1 (w@a) i r b
 shows Derives1 a (i − length w) r (drop (length w) b)
proof −
 have i: length w ≤ i using wa-b is-word-Derives1-index w by blast
 have is-prefix w b by (metis append-eq-conv-conj i is-prefix-take le-Derives1-take
```

wa-b)
then have *b*: $b = w$ @ (*drop* (*length w*) *b*) **by** (*simp add*: *is-prefix-unsplit*)
show *?thesis*
apply (*rule-tac Derives1-skip-prefix*[*OF i*])
by (*simp add*: *b*[*symmetric*] *wa-b*)
qed

lemma *derives1-drop-prefixword*:
assumes *w*: *is-word w*
assumes *wa-b*: *derives1* (*w*@*a*) *b*
shows *derives1 a* (*drop* (*length w*) *b*)
by (*metis Derives1-drop-prefixword Derives1-implies-derives1 derives1-implies-Derives1 w wa-b*)

lemma *derives1-is-word-is-prefix-drop*:
assumes *w*: *is-word w*
assumes *w-a*: *is-prefix w a*
assumes *ab*: *derives1 a b*
shows *derives1* (*drop* (*length w*) *a*) (*drop* (*length w*) *b*)
by (*metis ab append-take-drop-id derives1-drop-prefixword take-prefix w w-a*)

lemma *derives-drop-prefixword-helper*:
derives a b $\Longrightarrow$ *is-word w* $\Longrightarrow$ *is-prefix w a* $\Longrightarrow$ *derives* (*drop* (*length w*) *a*) (*drop* (*length w*) *b*)
proof (*induct rule*: *derives-induct*)
case *Base* **thus** *?case* **by** *auto*
next
case (*Step y z*)
have *is-prefix-w-y*: *is-prefix w y*
by (*metis Step.hyps*(*1*) *Step.prems*(*1*) *Step.prems*(*2*) *derives-word-is-prefix is-prefix-def*)
thus *?case*
by (*metis Step.hyps*(*2*) *Step.hyps*(*3*) *Step.prems*(*1*) *Step.prems*(*2*) *derives1-implies-derives derives1-is-word-is-prefix-drop derives-trans*)
qed

lemma *derive-drop-prefixword*:
is-word w $\Longrightarrow$ *derives* (*w*@*a*) *b* $\Longrightarrow$ *derives a* (*drop* (*length w*) *b*)
by (*metis append-eq-conv-conj derives-drop-prefixword-helper is-prefix-take*)

lemma *thmD2'*:
assumes *X*: *is-terminal X*
assumes *p*: *doc-tokens p*
assumes *pX*: (*terminals p*)@[*X*] $\in \mathcal{L}_P$
shows $\exists$ *x*. *pvalid p x* $\wedge$ *next-symbol x* = *Some X*
proof −
from *p* **have** *wellformed-p*: *wellformed-tokens p* **by** (*simp add*: *doc-tokens-def*)
have $\exists$ ω. *leftderives* [$\mathfrak{S}$] (((*terminals p*)@[*X*]) @ ω) **using** $\mathcal{L}_P$*-leftderives pX* **by**

blast
then obtain ω **where** *leftderives* $[\mathfrak{S}]$ $(((terminals\ p)@[X])\ @\ \omega)$ **by** *blast*
then have $\exists\ D.\ LeftDerivation\ [\mathfrak{S}]\ D\ (((terminals\ p)@[X])\ @\ \omega)$
using *leftderives-implies-LeftDerivation* **by** *blast*
then obtain D **where** D: *LeftDerivation* $[\mathfrak{S}]\ D\ (((terminals\ p)@[X])\ @\ \omega)$ **by** *blast*
let $?P = \lambda\ k.\ (\exists\ a\ b.\ LeftDerivation\ [\mathfrak{S}]\ (take\ k\ D)\ (a@[X]@b) \wedge derives\ a\ (terminals\ p))$
have $?P\ (length\ D)$
apply (*rule-tac* x=*terminals* p **in** *exI*)
apply (*rule-tac* x=ω **in** *exI*)
using D **by** *simp*
then show *?thesis*
proof (*induct rule*: *minimal-witness*[**where** P=$?P$])
case (*Minimal* K)
from *Minimal*(*2*) **obtain** $a\ b$ **where**
aXb: *LeftDerivation* $[\mathfrak{S}]\ (take\ K\ D)\ (a\ @\ [X]\ @\ b)$ **and**
a: *derives* a (*terminals* p) **by** *blast*
have KD: $K > 0 \wedge length\ D > 0$
proof (*cases* $K = 0 \vee length\ D = 0$)
case *True*
hence $take\ K\ D = []$ **by** *auto*
with *True* aXb **have** $[\mathfrak{S}] = a\ @\ [X]\ @\ b$ **by** *simp*
hence $\mathfrak{S} = X$
by (*metis Nil-is-append-conv append-self-conv2 butlast.simps*(*2*)
butlast-append hd-append2 list.sel(*1*) *not-Cons-self2*)
then have *False*
using *X is-nonterminal-startsymbol is-terminal-nonterminal* **by** *auto*
then show *?thesis* **by** *blast*
next
case *False* **thus** *?thesis* **by** *arith*
qed
then have $take\ K\ D = take\ (K - 1)\ D\ @\ [D\ !\ (K - 1)]$
by (*metis Minimal.hyps*(*1*) *One-nat-def Suc-less-eq Suc-pred hd-drop-conv-nth*
le-imp-less-Suc take-hd-drop)
then have $\exists\ \delta.\ LeftDerivation\ [\mathfrak{S}]\ (take\ (K - 1)\ D)\ \delta \wedge LeftDerivation\ \delta\ [D\ !\ (K - 1)]\ (a@[X]@b)$
by (*metis LeftDerivation-append aXb*)
then obtain δ **where**
$\delta 1$: *LeftDerivation* $[\mathfrak{S}]\ (take\ (K - 1)\ D)\ \delta$
and $\delta 2$: *LeftDerivation* $\delta\ [D\ !\ (K - 1)]\ (a@[X]@b)$
by *blast*
from $\delta 2$ **have** $\exists\ i\ r.\ LeftDerives1\ \delta\ i\ r\ (a@[X]@b)$ **by** *fastforce*
then obtain $i\ r$ **where** *LeftDerives1-*δ: *LeftDerives1* $\delta\ i\ r\ (a@[X]@b)$ **by** *blast*
then have *Derives1-*δ: *Derives1* $\delta\ i\ r\ (a@[X]@b)$
by (*metis LeftDerives1-implies-Derives1*)
then have $\exists\ \alpha\ N\ \beta\ .\ splits\text{-}at\ \delta\ i\ \alpha\ N\ \beta$ **by** (*simp add*: *splits-at-ex*)
then obtain $\alpha\ N\ \beta$ **where** *split-*δ: *splits-at* $\delta\ i\ \alpha\ N\ \beta$ **by** *blast*

have *is-word-α*: *is-word α* **by** (*metis LeftDerives1-δ LeftDerives1-splits-at-is-word split-δ*)
have ¬ (*?P* (*K* − *1*)) **using** *KD Minimal*(*3*) **by** *auto*
with *δ1* **have** *min-δ*: ¬ (∃ *a b*. *δ* = *a*@[*X*]@*b* ∧ *derives a* (*terminals p*)) **by** *blast*
from *Derives1-δ split-δ* **have** ∃ *u v*. *a* = *α* @ *u* ∧ *b* = *v* @ *β* ∧ (*snd r*) = *u*@[*X*]@*v*
proof (*induction rule*: *Derives1-X-is-part-of-rule*)
case (*Suffix γ*)
from *min-δ Suffix*(*1*) *a* **show** *?case* **by** *auto*
next
case (*Prefix γ*)
have *derives γ* (*terminals p*)
by (*metis Derives1-implies-derives1 Prefix*(*2*) *a derives1-implies-derives derives-trans*)
with *min-δ Prefix*(*1*) **show** *?case* **by** *auto*
qed
then obtain *u v* **where** *uXv*: *a* = *α* @ *u* ∧ *b* = *v* @ *β* ∧ (*snd r*) = *u*@[*X*]@*v* **by** *blast*
let *?l* = *length α*
let *?q* = *take ?l p*
let *?x* = *Item r* (*length u*) (*charslength ?q*) (*charslength p*)
have *item-rhs ?x* = *snd r* **by** (*simp add*: *item-rhs-def*)
then have *item-rhs-x*: *item-rhs ?x* = *u*@[*X*]@*v* **using** *uXv* **by** *simp*
have *wellformed-x*: *wellformed-item ?x*
apply (*auto simp add*: *wellformed-item-def*)
apply (*metis Derives1-δ Derives1-rule*)
apply (*rule is-prefix-length*)
apply (*rule is-prefix-chars*)
apply *simp*
apply (*simp add*: *doc-tokens-length*[*OF p*])
using *item-rhs-x* **by** *simp*
from *item-rhs-x* **have** *next-symbol-x*: *next-symbol ?x* = *Some X*
by (*auto simp add*: *next-symbol-def is-complete-def*)
have *len-α-p*: *length α* ≤ *length p*
apply (*rule-tac path-length-is-upper-bound*[**where** *u*=*u*])
apply (*simp add*: *wellformed-p*)
apply (*simp add*: *is-word-α*)
using *a uXv* **by** *blast*
have *item-nonterminal-x*: *item-nonterminal ?x* = *N*
apply (*simp add*: *item-nonterminal-def*)
using *Derives1-δ Derives1-nonterminal split-δ* **by** *blast*
have *take-terminals*: *take* (*length α*) (*terminals p*) = *α*
apply (*rule-tac take-prefix*)
using *a derives-word-is-prefix is-word-α uXv* **by** *blast*
have *item-α-x*: *item-α ?x* = *u* **using** *item-α-def item-rhs-x* **by** *auto*
from *wellformed-x next-symbol-x len-α-p* **show** *?thesis*
apply (*rule-tac x*=*?x* **in** *exI*)
apply (*auto simp add*: *pvalid-def wellformed-p*)

```
    apply (rule-tac x=length α in exI)
    apply (auto)
    using item-nonterminal-x apply (simp)
    apply (simp add: take-terminals)
    apply (rule-tac x=β in exI)
   using LeftDerivation-implies-leftderives δ1 is-leftderivation-def split-δ splits-at-combine

    apply auto[1]
    using item-α-x apply simp
    by (metis a derive-drop-prefixword is-word-α uXv)
  qed
qed

lemma admissible-wellformed-tokens: admissible p ⟹ wellformed-tokens p
  by (auto simp add: admissible-def ℒ_P-wellformed-tokens)

lemma chars-append[simp]: chars (a@b) = (chars a)@(chars b)
  by (induct a arbitrary: b, auto)

lemma chars-of-token-simp[simp]: chars-of-token (a, b) = b
  by (simp add: chars-of-token-def)

lemma 𝒳-is-prefix: t ∈ 𝒳 k ⟹ is-prefix (snd t) (drop k Doc)
  by (auto simp add: 𝒳-def)

lemma is-prefix-append: is-prefix (a@b) D = (is-prefix a D ∧ is-prefix b (drop
(length a) D))
  by (metis append-assoc is-prefix-cancel is-prefix-def is-prefix-unsplit)

lemma 𝔓-are-doc-tokens: p ∈ 𝔓 ⟹ doc-tokens p
proof (induct rule: 𝔓-induct)
  case Base thus ?case
    by (simp add: doc-tokens-def wellformed-tokens-def)
next
  case (Induct p k u)
  from Induct(2)[simplified] show ?case
  proof (induct rule: limit-induct)
    case (Init p) from Induct(1)[OF Init] show ?case .
  next
    case (Iterate p Y)
    have Y-is-prefix: ⋀ p. p ∈ Y ⟹ is-prefix (chars p) Doc
      apply (drule Iterate(1))
      by (simp add: doc-tokens-def)
    have 𝒴 (𝒵 k u) (𝒫 k u) k ⊆ 𝒳 k by (metis 𝒵.simps(2) 𝒵-subset-𝒳)
    then have 1: Append (𝒴 (𝒵 k u) (𝒫 k u) k) k Y ⊆ Append (𝒳 k) k Y
      by (rule Append-mono, simp)
    have 2: p ∈ Append (𝒳 k) k Y ⟹ doc-tokens p
      apply (auto simp add: Append-def)
      apply (simp add: Iterate)
```

apply (*auto simp add*: *doc-tokens-def admissible-wellformed-tokens is-prefix-append Y-is-prefix*)
by (*metis* $\mathcal{X}$*-is-prefix snd-conv*)
show *?case*
apply (*rule 2*)
by (*metis* (*mono-tags*, *lifting*) *1 Iterate*(*2*) *subsetCE*)
qed
qed

theorem *thmD2*:
assumes *X*: *is-terminal X*
assumes *p*: $p \in \mathfrak{P}$
assumes *pX*: (*terminals p*)@[*X*] $\in \mathcal{L}_P$
shows $\exists$ *x*. *pvalid p x* $\wedge$ *next-symbol x* = *Some X*
by (*metis X* $\mathfrak{P}$*-are-doc-tokens p pX thmD2'*)

end

end
theory *TheoremD4*
imports *TheoremD2*
begin

context *LocalLexing* **begin**

lemma $\mathcal{X}$*-are-terminals*: $u \in \mathcal{X}$ *k* $\Longrightarrow$ *is-terminal* (*terminal-of-token u*)
by (*auto simp add*: $\mathcal{X}$*-def is-terminal-def terminal-of-token-def*)

lemma *terminals-append*[*simp*]: *terminals* (*a*@*b*) = ((*terminals a*) @ (*terminals b*))
by (*auto simp add*: *terminals-def*)

lemma *terminals-singleton*[*simp*]: *terminals* [*u*] = [*terminal-of-token u*]
by (*simp add*: *terminals-def*)

lemma *terminal-of-token-simp*[*simp*]: *terminal-of-token* (*a*, *b*) = *a*
by (*simp add*: *terminal-of-token-def*)

lemma *pvalid-item-end*: *pvalid p x* $\Longrightarrow$ *item-end x* = *charslength p*
by (*metis pvalid-def*)

lemma $\mathcal{W}$*-elem-in-TokensAt*:
assumes *P*: $P \subseteq \mathfrak{P}$
assumes *u-in-*$\mathcal{W}$: $u \in \mathcal{W}$ *P k*
shows *u* $\in$ *TokensAt k* (*Gen P*)
proof −
have *u*: $u \in \mathcal{X}$ *k* $\wedge$ ($\exists$ *p*$\in$*by-length k P*. *admissible* (*p* @ [*u*])) **using** *u-in-*$\mathcal{W}$
by (*auto simp add*: $\mathcal{W}$*-def*)
then obtain *p* **where** *p*: *p* $\in$ *by-length k P* $\wedge$ *admissible* (*p* @ [*u*]) **by** *blast*

```
  then have charslength-p: charslength p = k
   by (metis (mono-tags, lifting) by-length.simps charslength.simps mem-Collect-eq)

  from u have u: u ∈ 𝒳 k by blast
  from p have p-in-𝔓: p ∈ 𝔓
    by (metis (no-types, lifting) P by-length.simps mem-Collect-eq subsetCE)
  then have doc-tokens-p: doc-tokens p by (metis 𝔓-are-doc-tokens)
  let ?X = terminal-of-token u
  have X-is-terminal: is-terminal ?X by (metis 𝒳-are-terminals u)
  from p have terminals p @ [terminal-of-token u] ∈ ℒ P
    by (auto simp add: admissible-def)
  from thmD2[OF X-is-terminal p-in-𝔓 this] obtain x where
    x: pvalid p x ∧ next-symbol x = Some (terminal-of-token u) by blast
  have x-is-in-Gen-P: x ∈ Gen P
    by (metis (mono-tags, lifting) Gen-def by-length.simps mem-Collect-eq p x)
 have u-split[dest!]: ⋀ t s. u = (t, s) ⟹ t = terminal-of-token u ∧ s = chars-of-token
u
    by (metis chars-of-token-simp fst-conv terminal-of-token-def)
  show ?thesis
    apply (auto simp add: TokensAt-def bin-def)
    apply (rule-tac x=x in exI)
    apply (auto simp add: x-is-in-Gen-P x X-is-terminal)
    using x charslength-p pvalid-item-end apply (simp, blast)
    using u by (auto simp add: 𝒳-def)
qed

lemma is-derivation-is-sentence: is-derivation s ⟹ is-sentence s
by (metis (no-types, lifting) Derives1-sentence2 derives1-implies-Derives1
    derives-induct is-derivation-def is-nonterminal-startsymbol is-sentence-cons
    is-sentence-def is-symbol-def list.pred-inject(1))

lemma is-sentence-cons: is-sentence (N#s) = (is-symbol N ∧ is-sentence s)
  by (auto simp add: is-sentence-def)

lemma is-derivation-step:
  assumes uNv: is-derivation (u@[N]@v)
  assumes Nα: (N, α) ∈ ℜ
  shows is-derivation (u@α@v)
proof −
  from uNv have is-sentence (u@[N]@v) by (metis is-derivation-is-sentence)
  with is-sentence-concat is-sentence-cons
  have u-is-sentence: is-sentence u and v-is-sentence: is-sentence v
    by auto
  from Nα have derives1 (u@[N]@v) (u@α@v)
    apply (auto simp add: derives1-def)
    apply (rule-tac x=u in exI)
    apply (rule-tac x=v in exI)
    apply (rule-tac x=N in exI)
    by (auto simp add: u-is-sentence v-is-sentence)
```

then show *?thesis*
by (*metis derives1-implies-derives derives-trans is-derivation-def uNv*)
qed

lemma *is-derivation-derives*:
derives α $\beta \Longrightarrow$ *is-derivation* ($u@\alpha@v$) $\Longrightarrow$ *is-derivation* ($u@\beta@v$)
proof (*induct rule*: *derives-induct*)
case *Base* **thus** *?case* **by** *simp*
next
case (*Step y z*)
from *Step* **have** *1*: *is-derivation* (u @ y @ v) **by** *auto*
from *Step* **have** *2*: *derives1 y z* **by** *auto*
from *1 2* **show** *?case* **by** (*metis append-assoc derives1-def is-derivation-step*)
qed

lemma *item-rhs-split*: *item-rhs x* = (*item-*α *x*)@(*item-*β *x*)
by (*metis append-take-drop-id item-*α*-def item-*β*-def*)

lemma *pvalid-is-derivation-terminals-item-*β:
assumes *pvalid*: *pvalid p x*
shows $\exists$ δ. *is-derivation* ((*terminals p*)@(*item-*β *x*)@δ)
proof −
from *pvalid* **have** $\exists$ u γ. *is-derivation* (*terminals* (*take u p*) @ [*item-nonterminal x*] @ γ) ∧
derives (*item-*α *x*) (*terminals* (*drop u p*))
by (*auto simp add*: *pvalid-def*)
then obtain u γ **where** *1*: *is-derivation* (*terminals* (*take u p*) @ [*item-nonterminal x*] @ γ) ∧
derives (*item-*α *x*) (*terminals* (*drop u p*)) **by** *blast*
have *x-rule*: (*item-nonterminal x*, *item-rhs x*) ∈ $\mathfrak{R}$
by (*metis* (*no-types*, *lifting*) *LocalLexing.pvalid-def LocalLexing-axioms assms case-prodE item-nonterminal-def item-rhs-def prod.sel*(*1*) *snd-conv validRules well-formed-item-def*)
from *1 x-rule is-derivation-step* **have**
is-derivation ((*take u* (*terminals p*)) @ (*item-rhs x*) @ γ)
by *auto*
then have *is-derivation* ((*take u* (*terminals p*)) @ ((*item-*α *x*)@(*item-*β *x*)) @ γ)
by (*simp add*: *item-rhs-split*)
then have *is-derivation* ((*take u* (*terminals p*)) @ (*item-*α *x*) @ ((*item-*β *x*) @ γ))
by *simp*
then have *is-derivation* ((*take u* (*terminals p*)) @ (*drop u* (*terminals p*)) @ ((*item-*β *x*) @ γ))
by (*metis 1 is-derivation-derives terminals-drop*)
then have *is-derivation* ((*terminals p*) @ ((*item-*β *x*) @ γ))
by (*metis append-assoc append-take-drop-id*)
then show *?thesis* **by** *auto*
qed

lemma *next-symbol-not-complete*: *next-symbol x = Some t ⟹ ¬ (is-complete x)*
by (*metis next-symbol-def option.discI*)

lemma *next-symbol-starts-item-β*:
assumes *wf*: *wellformed-item x*
assumes *next-symbol*: *next-symbol x = Some t*
shows ∃ *δ*. *item-β x = t#δ*
proof −
from *next-symbol* **have** *nc*: ¬ (*is-complete x*) **using** *next-symbol-not-complete* **by** *auto*
from *next-symbol* **have** *atdot*: *item-rhs x* ! *item-dot x = t* **by** (*simp add*: *next-symbol-def nc*)
from *nc* **have** *inrange*: *item-dot x < length* (*item-rhs x*)
by (*simp add*: *is-complete-def*)
from *inrange atdot* **show** *?thesis*
apply (*simp add*: *item-β-def*)
by (*metis Cons-nth-drop-Suc*)
qed

lemma *pvalid-prefixlang*:
assumes *pvalid*: *pvalid p x*
assumes *is-terminal*: *is-terminal t*
assumes *next-symbol*: *next-symbol x = Some t*
shows (*terminals p*) @ [*t*] ∈ $\mathcal{L}_P$
proof −
have ∃ *δ*. *item-β x = t#δ*
by (*metis next-symbol next-symbol-starts-item-β pvalid pvalid-def*)
then obtain *δ* **where** *δ*:*item-β x = t#δ* **by** *blast*
have ∃ *ω*. *is-derivation* ((*terminals p*)@(*item-β x*)@*ω*)
by (*metis pvalid pvalid-is-derivation-terminals-item-β*)
then obtain *ω* **where** *is-derivation* ((*terminals p*)@(*item-β x*)@*ω*) **by** *blast*
then have *is-derivation* ((*terminals p*)@(*t#δ*)@*ω*) **by** (*metis δ*)
then have *is-derivation* (((*terminals p*)@[*t*])@(*δ*@*ω*)) **by** *simp*
then show *?thesis*
by (*metis* (*no-types*, *lifting*) *CFG.*$\mathcal{L}_P$*-def CFG-axioms*
append-Nil2 is-terminal is-word-append is-word-cons
is-word-terminals mem-Collect-eq pvalid pvalid-def)
qed

lemma *TokensAt-elem-in-*$\mathcal{W}$:
assumes *P*: *P* ⊆ $\mathfrak{P}$
assumes *u-in-Tokens-at*: *u* ∈ *TokensAt k* (*Gen P*)
shows *u* ∈ $\mathcal{W}$ *P k*
proof −
have ∃ *t s x l*.
u = (*t*, *s*) ∧
x ∈ *bin* (*Gen P*) *k* ∧
next-symbol x = Some t ∧ *is-terminal t* ∧ *l* ∈ *Lex t Doc k* ∧ *s = take l*

```
(drop k Doc)
  using u-in-Tokens-at by (auto simp add: TokensAt-def)
  then obtain t s x l where
    u: u = (t, s) ∧
       x ∈ bin (Gen P) k ∧
        next-symbol x = Some t ∧ is-terminal t ∧ l ∈ Lex t Doc k ∧ s = take l
(drop k Doc)
      by blast
  from u have t: t = terminal-of-token u by (metis terminal-of-token-simp)
  from u have s: s = chars-of-token u by (metis chars-of-token-simp)
  from u have item-end-x: item-end x = k by (metis (mono-tags, lifting) bin-def
mem-Collect-eq)
  from u have ∃ p ∈ P. pvalid p x by (auto simp add: bin-def Gen-def)
  then obtain p where p: p ∈ P and pvalid: pvalid p x by blast
  have p-len: length (chars p) = k
    by (metis charslength.simps item-end-x pvalid pvalid-item-end)
  have u-in-𝒳: u ∈ 𝒳 k
    apply (simp add: 𝒳-def)
    apply (rule-tac x=t in exI)
    apply (rule-tac x=l in exI)
    using u by (simp add: is-terminal-def)
  show ?thesis
    apply (auto simp add: 𝒲-def)
    apply (simp add: u-in-𝒳)
    apply (rule-tac x=p in exI)
    apply (simp add: p p-len)
    apply (simp add: admissible-def t[symmetric])
    apply (rule pvalid-prefixlang[where x=x])
    apply (simp add: pvalid)
    apply (simp add: u)
    apply (simp add: u)
    done
qed

theorem thmD4:
  assumes P: P ⊆ 𝔓
  shows 𝒲 P k = TokensAt k (Gen P)
using 𝒲-elem-in-TokensAt TokensAt-elem-in-𝒲
by (metis Collect-cong Collect-mem-eq assms)

end

end
theory TheoremD5
imports TheoremD4
begin

context LocalLexing begin
```

lemma *Scan-empty*: *Scan* {} *k I* = *I*
by (*simp add*: *Scan-def*)

lemma *π-no-tokens*: *π k* {} *I* = *limit* (λ *I*. *Complete k* (*Predict k I*)) *I*
by (*simp add*: *π-def Scan-empty*)

lemma *bin-elem*: *x* ∈ *bin I k* ⟹ *x* ∈ *I*
by (*auto simp add*: *bin-def*)

lemma *Gen-implies-pvalid*: *x* ∈ *Gen P* ⟹ ∃ *p* ∈ *P*. *pvalid p x*
by (*auto simp add*: *Gen-def*)

lemma *wellformed-init-item*[*simp*]: *r* ∈ ℜ ⟹ *k* ≤ *length Doc* ⟹ *wellformed-item* (*init-item r k*)
by (*simp add*: *init-item-def wellformed-item-def*)

lemma *init-item-origin*[*simp*]: *item-origin* (*init-item r k*) = *k*
by (*auto simp add*: *item-origin-def init-item-def*)

lemma *init-item-end*[*simp*]: *item-end* (*init-item r k*) = *k*
by (*auto simp add*: *item-end-def init-item-def*)

lemma *init-item-nonterminal*[*simp*]: *item-nonterminal* (*init-item r k*) = *fst r*
by (*auto simp add*: *init-item-def item-nonterminal-def*)

lemma *init-item-α*[*simp*]: *item-α* (*init-item r k*) = []
by (*auto simp add*: *init-item-def item-α-def*)

lemma *Predict-elem-in-Gen*:
assumes *I-in-Gen-P*: *I* ⊆ *Gen P*
assumes *k*: *k* ≤ *length Doc*
assumes *x-in-Predict*: *x* ∈ *Predict k I*
shows *x* ∈ *Gen P*
proof −
have *x* ∈ *I* ∨ (∃ *r y*. *r* ∈ ℜ ∧ *x* = *init-item r k* ∧ *y* ∈ *bin I k* ∧ *next-symbol y* = *Some*(*fst r*))
using *x-in-Predict* **by** (*auto simp add*: *Predict-def*)
then show *?thesis*
proof (*induct rule*: *disjCases2*)
case *1* **thus** *?case* **using** *I-in-Gen-P* **by** *blast*
next
case *2*
then obtain *r y* **where** *ry*: *r* ∈ ℜ ∧ *x* = *init-item r k* ∧ *y* ∈ *bin I k* ∧
next-symbol y = *Some* (*fst r*) **by** *blast*
then have ∃ *p* ∈ *P*. *pvalid p y*
using *Gen-implies-pvalid I-in-Gen-P bin-elem subsetCE* **by** *blast*
then obtain *p* **where** *p*: *p* ∈ *P* ∧ *pvalid p y* **by** *blast*
have *wellformed-p*: *wellformed-tokens p* **using** *p* **by** (*auto simp add*: *pvalid-def*)
have *wellformed-x*: *wellformed-item x*

by (*simp add*: *ry k*)
from *ry* **have** *item-end y* = *k* **by** (*auto simp add*: *bin-def*)
with *p* **have** *charslength-p*[*simplified*]: *charslength p* = *k* **by** (*auto simp add*: *pvalid-def*)
have *item-end-x*: *item-end x* = *k* **by** (*simp add*: *ry*)
have *pvalid-x*: *pvalid p x*
apply (*auto simp add*: *pvalid-def*)
apply (*simp add*: *wellformed-p*)
apply (*simp add*: *wellformed-x*)
apply (*rule-tac x*=*length p* **in** *exI*)
apply (*auto simp add*: *charslength-p ry*)
by (*metis append-Cons next-symbol-starts-item-β p pvalid-def*
pvalid-is-derivation-terminals-item-β ry)
then show *?case* **using** *Gen-def mem-Collect-eq p* **by** *blast*
qed
qed

lemma *Predict-subset-Gen*:
assumes *I* ⊆ *Gen P*
assumes *k* ≤ *length Doc*
shows *Predict k I* ⊆ *Gen P*
using *Predict-elem-in-Gen assms* **by** *blast*

lemma *nth-superfluous-append*[*simp*]: *i* < *length a* ⟹ (*a*@*b*)!*i* = *a*!*i*
by (*simp add*: *nth-append*)

lemma *tokens-nth-in-*$\mathcal{Z}$:
p ∈ $\mathfrak{P}$ ⟹ ∀ *i*. *i* < *length p* ⟶ (∃ *u*. *p* ! *i* ∈ $\mathcal{Z}$ (*charslength* (*take i p*)) *u*)
proof (*induct rule*: $\mathfrak{P}$-*induct*)
case *Base* **thus** *?case* **by** *simp*
next
case (*Induct p k u*)
then have *p* ∈ *limit* (*Append* ($\mathcal{Z}$ *k* (*Suc u*)) *k*) ($\mathcal{P}$ *k u*) **by** *simp*
then show *?case*
proof (*induct rule*: *limit-induct*)
case (*Init p*) **thus** *?case* **using** *Induct* **by** *auto*
next
case (*Iterate p Y*)
from *Iterate*(*2*) **have** *p* ∈ *Y* ∨ (∃ *q t*. *p* = *q*@[*t*] ∧ *q* ∈ *by-length k Y* ∧ *t* ∈ $\mathcal{Z}$ *k* (*Suc u*) ∧
admissible (*q* @ [*t*]))
by (*auto simp add*: *Append-def*)
then show *?case*
proof (*induct rule*: *disjCases2*)
case *1* **thus** *?case* **using** *Iterate*(*1*) **by** *auto*
next
case *2*
then obtain *q t* **where**
qt: *p* = *q* @ [*t*] ∧ *q* ∈ *by-length k Y* ∧ *t* ∈ $\mathcal{Z}$ *k* (*Suc u*) ∧ *admissible* (*q* @

```
[t]) by blast
      then have q-in-Y: q ∈ Y by auto
      with qt have k: k = charslength q by auto
      with qt have t: t ∈ 𝒵 k (Suc u) by auto
      show ?case
      proof(auto simp add: qt)
        fix i
        assume i: i < Suc (length q)
        then have i < length q ∨ i = length q by arith
        then show ∃ u. (q @ [t]) ! i ∈ 𝒵 (length (chars (take i q))) u
        proof (induct rule: disjCases2)
          case 1
            from Iterate(1)[OF q-in-Y]
            show ?case by (simp add: 1)
        next
          case 2
            show ?case
              apply (auto simp add: 2)
              apply (rule-tac x=Suc u in exI)
              using k t by auto
        qed
      qed
    qed
  qed
qed

lemma path-append-token:
  assumes p: p ∈ 𝒫 k u
  assumes t: t ∈ 𝒵 k (Suc u)
  assumes pt: admissible (p@[t])
  assumes k: charslength p = k
  shows p@[t] ∈ 𝒫 k (Suc u)
apply (simp only: 𝒫.simps)
apply (rule-tac n=Suc 0 in limit-elem)
using p t pt k apply (auto simp only: Append-def funpower.simps)
by fastforce

definition indexlt-rel :: ((nat × nat) × (nat × nat)) set where
  indexlt-rel = less-than <*lex*> less-than

definition indexlt :: nat ⇒ nat ⇒ nat ⇒ nat ⇒ bool where
  indexlt k' u' k u = (((k', u'), (k, u)) ∈ indexlt-rel)

lemma indexlt-simp: indexlt k' u' k u = (k' < k ∨ (k' = k ∧ u' < u))
  by (auto simp add: indexlt-def indexlt-rel-def)

lemma wf-indexlt-rel: wf indexlt-rel
  using indexlt-rel-def pair-less-def by auto
```

lemma $\mathcal{P}$*-induct*[*consumes 1*, *case-names Induct*]:
assumes $p \in \mathcal{P}\ k\ u$
assumes *induct*: $\bigwedge p\ k\ u\ .\ (\bigwedge p'\ k'\ u'.\ p' \in \mathcal{P}\ k'\ u' \Longrightarrow \mathit{indexlt}\ k'\ u'\ k\ u \Longrightarrow P\ p'\ k'\ u') \Longrightarrow p \in \mathcal{P}\ k\ u \Longrightarrow P\ p\ k\ u$
shows $P\ p\ k\ u$
proof −
let *?R* = *indexlt-rel* <∗*lex*∗> {}
have *wf-R*: *wf ?R* **by** (*auto simp add*: *wf-indexlt-rel*)
let *?P* = $\lambda\ a.\ \mathit{snd}\ a \in \mathcal{P}\ (\mathit{fst}\ (\mathit{fst}\ a))\ (\mathit{snd}\ (\mathit{fst}\ a))\ \longrightarrow\ P\ (\mathit{snd}\ a)\ (\mathit{fst}\ (\mathit{fst}\ a))\ (\mathit{snd}\ (\mathit{fst}\ a))$
have $p \in \mathcal{P}\ k\ u \longrightarrow P\ p\ k\ u$
apply (*rule wf-induct*[*OF wf-R*, **where** P = *?P* **and** $a = ((k, u), p)$, *simplified*])
apply (*auto simp add*: *indexlt-def*[*symmetric*])
apply (*rule-tac p*=*ba* **and** *k*=*a* **and** *u*=*b* **in** *induct*)
by *auto*
thus *?thesis* **using** *assms* **by** *auto*
qed

lemma *nonempty-path-indices*:
assumes *p*: $p \in \mathcal{P}\ k\ u$
assumes *nonempty*: $p \neq []$
shows $k > 0 \vee u > 0$
proof (*cases* $u = 0$)
case *True*
note u = *True*
have $k > 0$
proof (*cases* $k = 0$)
case *True*
with $p\ u$ **have** $p = []$ **by** *simp*
with *nonempty* **have** *False* **by** *auto*
then show *?thesis* **by** *auto*
next
case *False*
then show *?thesis* **by** *arith*
qed
then show *?thesis* **by** *blast*
next
case *False*
then show *?thesis* **by** *arith*
qed

lemma *base-paths*:
assumes *p*: $p \in \mathcal{P}\ k\ 0$
assumes *k*: $k > 0$
shows $\exists\ u.\ p \in \mathcal{P}\ (k - 1)\ u$
proof −
from k **have** $\exists\ i.\ k = \mathit{Suc}\ i$ **by** *arith*
then obtain i **where** *i*: $k = \mathit{Suc}\ i$ **by** *blast*

```
  from p show ?thesis
    by (auto simp add: i natUnion-def)
qed

lemma indexlt-trans: indexlt k'' u'' k' u' ⟹ indexlt k' u' k u ⟹ indexlt k'' u''
k u
using dual-order.strict-trans indexlt-simp by auto

definition is-continuation :: nat ⇒ nat ⇒ tokens ⇒ tokens ⇒ bool where
  is-continuation k u q ts = (q ∈ 𝒫 k u ∧ charslength q = k ∧ admissible (q@ts)
∧
    (∀ t ∈ set ts. t ∈ 𝒵 k (Suc u)) ∧ (∀ t ∈ set (butlast ts). chars-of-token t = []))

lemma limit-Append-path-nonelem-split: p ∈ limit (Append T k) (𝒫 k u) ⟹ p ∉
𝒫 k u ⟹
  ∃ q ts. p = q@ts ∧ q ∈ 𝒫 k u ∧ charslength q = k ∧ admissible (q@ts) ∧ (∀ t
∈ set ts. t ∈ T) ∧
    (∀ t ∈ set (butlast ts). chars-of-token t = [])
proof (induct rule: limit-induct)
  case (Init p) thus ?case by auto
next
  case (Iterate p Y)
  show ?case
  proof (cases p ∈ Y)
    case True
    from Iterate(1)[OF True Iterate(3)] show ?thesis by blast
  next
    case False
    with Append-def Iterate(2)
    have ∃ q t. p = q@[t] ∧ q ∈ by-length k Y ∧ t ∈ T ∧ admissible (q @ [t]) by
auto
   then obtain q t where qt: p = q@[t] ∧ q ∈ by-length k Y ∧ t ∈ T ∧ admissible
(q @ [t])
      by blast
    from qt have qlen: charslength q = k by auto
    have q ∈ 𝒫 k u ∨ q ∉ 𝒫 k u by blast
    then show ?thesis
    proof(induct rule: disjCases2)
      case 1
      show ?case
        apply (rule-tac x=q in exI)
        apply (rule-tac x=[t] in exI)
        using qlen by (simp add: qt 1)
    next
      case 2
      have q-in-Y: q ∈ Y using qt by auto
      from Iterate(1)[OF q-in-Y 2]
      obtain q' ts where
        q'ts: q = q' @ ts ∧ q' ∈ 𝒫 k u ∧ charslength q' = k ∧ (∀ t∈set ts. t ∈ T) ∧
```

```
            (∀ t∈set(butlast ts). chars-of-token t = [])
        by blast
      with qlen have charslength ts = 0 by auto
      hence empty: ∀ t ∈ set(ts). chars-of-token t = []
        apply (induct ts)
        by auto
      show ?case
        apply (rule-tac x=q' in exI)
        apply (rule-tac x=ts@[t] in exI)
        using qt q'ts empty by auto
    qed
  qed
qed

lemma limit-Append-path-nonelem-split':
  p ∈ limit (Append (𝒵 k (Suc u)) k) (𝒫 k u) ⟹ p ∉ 𝒫 k u ⟹
   ∃ q ts. p = q@ts ∧ is-continuation k u q ts
apply (simp only: is-continuation-def)
apply (rule-tac limit-Append-path-nonelem-split)
by auto

lemma final-step-of-path: p ∈ 𝒫 k u ⟹ p ≠ [] ⟹ (∃ q ts k' u'. p = q@ts ∧
indexlt k' u' k u
  ∧ is-continuation k' u' q ts)
proof (induct rule: 𝒫-induct)
  case (Induct p k u)
  from Induct(2) Induct(3) have ku-0: k > 0 ∨ u > 0
    using nonempty-path-indices by blast
  show ?case
  proof (cases u = 0)
    case True
    with ku-0 have k-0: k > 0 by arith
    with True Induct(2) base-paths have ∃ u'. p ∈ 𝒫 (k − 1) u' by auto
    then obtain u' where u': p ∈ 𝒫 (k − 1) u' by blast
    have indexlt: indexlt (k − 1) u' k u by (simp add: indexlt-simp k-0)
    from Induct(1)[OF u' indexlt Induct(3)] show ?thesis
      using indexlt indexlt-trans by blast
  next
    case False
    then have ∃ u'. u = Suc u' by arith
    then obtain u' where u': u = Suc u' by blast
    with Induct(2) have p-limit: p ∈ limit (Append (𝒵 k (Suc u')) k) (𝒫 k u')
      using 𝒫.simps(2) by blast
    from u' have indexlt: indexlt k u' k u by (simp add: indexlt-simp)
    have p ∈ 𝒫 k u' ∨ p ∉ 𝒫 k u' by blast
    then show ?thesis
    proof (induct rule: disjCases2)
      case 1
      from Induct(1)[OF 1 indexlt Induct(3)] show ?case
```

```
      using indexlt indexlt-trans by blast
    next
      case 2
      from limit-Append-path-nonelem-split'[OF p-limit 2]
      show ?case using indexlt u' by auto
    qed
  qed
qed

lemma terminals-empty[simp]: terminals [] = []
  by (auto simp add: terminals-def)

lemma empty-in-\mathcal{L}_P[simp]: [] ∈ \mathcal{L}_P
  apply (simp add: \mathcal{L}_P-def is-derivation-def)
  apply (rule-tac x=[\mathfrak{S}] in exI)
  by simp

lemma admissible-empty[simp]: admissible []
  by (auto simp add: admissible-def)

lemma \mathfrak{P}-are-admissible: p ∈ \mathfrak{P} ⟹ admissible p
proof (induct rule: \mathfrak{P}-induct)
  case Base thus ?case by simp
next
  case (Induct p k u)
  from Induct(2)[simplified] show ?case
  proof (induct rule: limit-induct)
    case (Init p) from Induct(1)[OF Init] show ?case .
  next
    case (Iterate p Y)
    have \mathcal{Y} (\mathcal{Z} k u) (\mathcal{P} k u) k ⊆ \mathcal{X} k by (metis \mathcal{Z}.simps(2) \mathcal{Z}-subset-\mathcal{X})
    then have 1: Append (\mathcal{Y} (\mathcal{Z} k u) (\mathcal{P} k u) k) k Y ⊆ Append (\mathcal{X} k) k Y
      by (rule Append-mono, simp)
    have 2: p ∈ Append (\mathcal{X} k) k Y ⟹ admissible p
      apply (auto simp add: Append-def)
      by (simp add: Iterate)
    show ?case
      apply (rule 2)
      using 1 Iterate(2) by blast
  qed
qed

lemma prefix-of-empty-is-empty: is-prefix q [] ⟹ q = []
by (metis  is-prefix-cons neq-Nil-conv)

lemma subset-\mathcal{P} :
  assumes leq: k' < k ∨ (k' = k ∧ u' ≤ u)
  shows \mathcal{P} k' u' ⊆ \mathcal{P} k u
proof −
```

```
  from leq show ?thesis
  proof (induct rule: disjCases2)
    case 1
    have s1: 𝒫 k' u' ⊆ 𝒬 k' by (rule-tac subset-𝒫𝒬k)
    have s2: 𝒬 k' ⊆ 𝒬 (k − 1)
      apply (rule-tac subset-𝒬)
      using 1 by arith
    from subset-𝒬𝒫Suc[where k=k − 1] 1 have s3: 𝒬 (k − 1) ⊆ 𝒫 k 0
      by simp
    have s4: 𝒫 k 0 ⊆ 𝒫 k u by (rule-tac subset-𝒫k, simp)
    from s1 s2 s3 s4 subset-trans show ?case by blast
  next
    case 2 thus ?case by (simp add : subset-𝒫k)
  qed
qed

lemma empty-path-is-elem[simp]: [] ∈ 𝒫 k u
proof −
  have [] ∈ 𝒫 0 0 by simp
  then show [] ∈ 𝒫 k u by (metis le0 not-gr0 subsetCE subset-𝒫)
qed

lemma is-prefix-of-append:
  assumes is-prefix p (a@b)
  shows is-prefix p a ∨ (∃ b'. b' ≠ [] ∧ is-prefix b' b ∧ p = a@b')
apply (auto simp add: is-prefix-def)
by (metis append-Nil2 append-eq-append-conv2 assms is-prefix-cancel is-prefix-def)

lemma prefix-is-continuation: is-continuation k u p ts ⟹ is-prefix ts' ts ⟹
  is-continuation k u p ts'
apply (auto simp add: is-continuation-def is-prefix-def)
apply (metis ℒ_P-split admissible-def append-assoc terminals-append)
using in-set-butlast-appendI by fastforce

lemma charslength-0: (∀ t ∈ set ts. chars-of-token t = []) = (charslength ts = 0)
by (induct ts, auto)

lemma is-continuation-in-𝒫: is-continuation k u p ts ⟹ p@ts ∈ 𝒫 k (Suc u)
proof(induct ts rule: rev-induct)
  case Nil thus ?case
    apply (auto simp add: is-continuation-def)
    using subset-𝒫Suc by fastforce
next
  case (snoc t ts)
  from snoc(2) have is-continuation k u p ts
    by (metis append-Nil2 is-prefix-cancel is-prefix-empty prefix-is-continuation)
  note induct = snoc(1)[OF this]
  then have pts: p@ts ∈ limit (Append (𝒵 k (Suc u)) k) (𝒫 k u) by simp
  note is-cont = snoc(2)
```

```
 then have admissible: admissible (p@ts@[t]) by (simp add: is-continuation-def)
 from is-cont have t: t ∈ 𝒵 k (Suc u) by (simp add: is-continuation-def)
from is-cont have ∀ t ∈ set ts. chars-of-token t = [] by (simp add: is-continuation-def)
 then have charslength-ts: charslength ts = 0 by (simp only: charslength-0)
 from is-cont have plen: charslength p = k by (simp add: is-continuation-def)
 show ?case
   apply (simp only: 𝒫.simps)
   apply (rule-tac limit-step-pointwise[OF pts])
   apply (simp add: pointwise-Append)
   apply (auto simp add: Append-def)
   apply (rule-tac x=fst t in exI)
   apply (rule-tac x=snd t in exI)
   apply (auto simp add: admissible)
   using charslength-ts apply simp
   using plen apply simp
   using t by simp
qed

lemma indexlt-subset-𝒫: indexlt k' u' k u ⟹ 𝒫 k' (Suc u') ⊆ 𝒫 k u
apply (rule-tac subset-𝒫)
apply (simp add: indexlt-simp)
apply arith
done

lemma prefixes-are-paths: p ∈ 𝒫 k u ⟹ is-prefix x p ⟹ x ∈ 𝒫 k u
proof (induct arbitrary: x rule: 𝒫-induct)
  case (Induct p k u)
  show ?case
  proof (cases p = [])
    case True
    then have x = []
      using Induct.prems prefix-of-empty-is-empty by blast
    then show x ∈ 𝒫 k u by simp
  next
    case False
    from final-step-of-path[OF Induct(2) False]
    obtain q ts k' u' where step: p = q@ts ∧ indexlt k' u' k u ∧ is-continuation
k' u' q ts
      by blast
    have subset: 𝒫 k' u' ⊆ 𝒫 k u
      by (metis indexlt-simp less-or-eq-imp-le step subset-𝒫)
    have is-prefix x q ∨ (∃ ts'. ts' ≠ [] ∧ is-prefix ts' ts ∧ x = q@ts')
      apply (rule-tac is-prefix-of-append)
      using Induct(3) step by auto
    then show ?thesis
    proof (induct rule: disjCases2)
      case 1
        have x: x ∈ 𝒫 k' u'
        using 1 Induct step by (auto simp add: is-continuation-def)
```

```
      then show x ∈ 𝒫 k u using subset subsetCE by blast
    next
      case 2
      then obtain ts' where ts': is-prefix ts' ts ∧ x = q@ts' by blast
      have is-continuation k' u' q ts' using step prefix-is-continuation ts' by blast
      with ts' have x ∈ 𝒫 k' (Suc u')
        apply (simp only: ts')
        apply (rule-tac is-continuation-in-𝒫)
        by simp
      with subset show x ∈ 𝒫 k u using indexlt-subset-𝒫 step by blast
    qed
  qed
qed

lemma empty-or-last-of-suffix:
  assumes q = q' @ [t]
  assumes q = p @ ts
  shows ts = [] ∨ (∃ ts'. q' = p @ ts' ∧ ts'@[t] = ts)
by (metis assms(1) assms(2) butlast-append last-appendR snoc-eq-iff-butlast)

lemma is-prefix-butlast: is-prefix q (butlast p) ⟹ is-prefix q p
by (metis butlast-conv-take is-prefix-append is-prefix-def is-prefix-take)

lemma last-step-of-path:
  q ∈ 𝒫 k u ⟹ q = q'@[t] ⟹
  ∃ k' u'. indexlt k' u' k u ∧ q ∈ 𝒫 k' (Suc u') ∧ charslength q' = k' ∧ t ∈ 𝒵 k'
(Suc u')
proof (induct arbitrary: q' t rule: 𝒫-induct)
  case (Induct q k u)
    have ∃ p ts k' u'. q = p@ts ∧ indexlt k' u' k u ∧ is-continuation k' u' p ts
      apply (rule-tac final-step-of-path)
      apply (simp add: Induct(2))
      apply (simp add: Induct(3))
      done
   then obtain p ts k' u' where pts: q = p@ts ∧ indexlt k' u' k u ∧ is-continuation
k' u' p ts
      by blast
    then have indexlt: indexlt k' u' k u by auto
    from pts have ts = [] ∨ (∃ ts'. q' = p @ ts' ∧ ts'@[t] = ts)
      by (metis empty-or-last-of-suffix Induct(3))
    then show ?case
    proof (induct rule: disjCases2)
      case 1
        with pts have q: q ∈ 𝒫 k' u' by (auto simp add: is-continuation-def)
        from Induct(1)[OF this indexlt Induct(3)] show ?case
          using indexlt indexlt-trans by blast
    next
      case 2
        then obtain ts' where ts': q' = p @ ts' ∧ ts'@[t] = ts by blast
```

```
      then have is-prefix ts' ts using is-prefix-def by blast
      then have is-continuation k' u' p ts' by (metis prefix-is-continuation pts)
      have charslength ts' = 0 using charslength-0 is-continuation-def pts ts' by
auto
       then have q'len: charslength q' = k' using is-continuation-def pts ts' by
auto
      have t ∈ set ts using ts' by auto
     with pts have t-in-𝒵: t ∈ 𝒵 k' (Suc u') using is-continuation-def by blast
      have q-dom: q ∈ 𝒫 k' (Suc u') using pts is-continuation-in-𝒫 by blast
      show ?case
        apply (rule-tac x=k' in exI)
        apply (rule-tac x=u' in exI)
        by (simp only: indexlt q'len t-in-𝒵 q-dom)
   qed
qed

lemma charslength-of-butlast-0: p ∈ 𝒫 k 0 ⟹ p = q@[t] ⟹ charslength q < k
using last-step-of-path LocalLexing-axioms indexlt-simp by blast

lemma charslength-of-butlast: p ∈ 𝒫 k u ⟹ p = q@[t] ⟹ charslength q ≤ k
by (metis indexlt-simp last-step-of-path eq-imp-le less-imp-le-nat)

lemma last-token-of-path:
  assumes q ∈ 𝒫 k u
  assumes q = q'@[t]
  assumes charslength q' = k
  shows t ∈ 𝒵 k u
proof −
  from assms have ∃ k' u'. indexlt k' u' k u ∧ q ∈ 𝒫 k' (Suc u') ∧ charslength q'
= k' ∧
    t ∈ 𝒵 k' (Suc u') using last-step-of-path by blast
  then obtain k' u' where th: indexlt k' u' k u ∧ q ∈ 𝒫 k' (Suc u') ∧ charslength
q' = k' ∧
    t ∈ 𝒵 k' (Suc u') by blast
  with assms(3) have k': k' = k by blast
  with th have t ∈ 𝒵 k' (Suc u') ∧ u' < u using indexlt-simp by auto
  then show ?thesis
   by (metis (no-types, opaque-lifting) 𝒵-subset-Suc k' linorder-neqE-nat not-less-eq

      subsetCE subset-fSuc-strict)
qed

lemma final-step-of-path': p ∈ 𝒫 k u ⟹ p ∉ 𝒫 k (u − 1) ⟹
  ∃ q ts. u > 0 ∧ p = q@ts ∧ is-continuation k (u − 1) q ts
by (metis Suc-diff-1 𝒫.simps(2) diff-0-eq-0 limit-Append-path-nonelem-split' not-gr0)

lemma is-continuation-continue:
  assumes is-continuation k u q ts
  assumes charslength ts = 0
```

```
  assumes t ∈ 𝒵 k (Suc u)
  assumes admissible (q @ ts @ [t])
  shows is-continuation k u q (ts@[t])
proof −
  from assms show ?thesis
    by (simp add: is-continuation-def charslength-0)
qed



theorem compatibility-def:
  assumes p-in-dom: p ∈ 𝒫 k u
  assumes q-in-dom: q ∈ 𝒫 k u
  assumes p-charslength: charslength p = k
  assumes q-split: q = q'@[t]
  assumes q'len: charslength q' = k
  assumes admissible: admissible (p @ [t])
  shows p @ [t] ∈ 𝒫 k u
proof −
  have u: u > 0
  proof (cases u = 0)
    case True
      then have charslength q' < k
        using charslength-of-butlast-0 q-in-dom q-split by blast
      with q'len have False by arith
      then show ?thesis by blast
  next
    case False
      then show ?thesis by arith
  qed
  have t-dom: t ∈ 𝒵 k u using last-token-of-path q'len q-in-dom q-split by blast
  have p ∈ 𝒫 k (u − 1) ∨ p ∉ 𝒫 k (u − 1) by blast
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      with t-dom p-charslength admissible u have is-continuation k (u − 1) p [t]
        by (auto simp add: is-continuation-def)
      with u show p@[t] ∈ 𝒫 k u
        by (metis One-nat-def Suc-pred is-continuation-in-𝒫)
  next
    case 2
      from final-step-of-path'[OF p-in-dom 2]
      obtain p' ts where p': p = p' @ ts ∧ is-continuation k (u − 1) p' ts
        by blast
      from p' p-charslength is-continuation-def have charslength-ts: charslength ts
= 0
        by auto
      from u have u': Suc (u − 1) = u by arith
      have is-continuation k (u − 1) p' (ts@[t])
```

```
      apply (rule-tac is-continuation-continue)
      using p' apply blast
      using charslength-ts apply blast
      apply (simp only: u' t-dom)
      using admissible p' apply auto
      done
    from is-continuation-in-𝒫[OF this] show ?case by (simp only: p' u', simp)
  qed
qed

lemma is-prefix-admissible:
  assumes is-prefix a b
  assumes admissible b
  shows admissible a
proof −
  from assms show ?thesis
    by (auto simp add: is-prefix-def admissible-def ℒ_P-def)
qed

lemma butlast-split: n < length q ⟹ butlast q = (take n q)@(drop n (butlast q))
by (metis append-take-drop-id take-butlast)

lemma in-𝒫-charslength:
  assumes p-dom: p ∈ 𝒫 k u
  shows ∃ v. p ∈ 𝒫 (charslength p) v
proof (cases charslength p ≥ k)
  case True
    show ?thesis
      apply (rule-tac x=u in exI)
      by (metis True le-neq-implies-less p-dom subsetCE subset-𝒫)
next
  case False
    then have charslength: charslength p < k by arith
    have p = [] ∨ p ≠ [] by blast
    thus ?thesis
    proof (induct rule: disjCases2)
      case 1 thus ?case by simp
    next
      case 2
        from final-step-of-path[OF p-dom 2] obtain q ts k' u' where
          step: p = q @ ts ∧ indexlt k' u' k u ∧ is-continuation k' u' q ts by blast
        from step have k': charslength q = k' using is-continuation-def by blast
        from step have charslength q ≤ charslength p  by simp
        with k' have k': k' ≤ charslength p by simp
        from step have p ∈ 𝒫 k' (Suc u') using is-continuation-in-𝒫 by blast
        with k' have p ∈ 𝒫 (charslength p) (Suc u')
          by (metis le-neq-implies-less subsetCE subset-𝒫)
        then show ?case by blast
    qed
```

qed

theorem *general-compatibility*:
  $p \in \mathcal{P}\ k\ u \Longrightarrow q \in \mathcal{P}\ k\ u \Longrightarrow charslength\ p = charslength\ (take\ n\ q)$
    $\Longrightarrow charslength\ p \leq k \Longrightarrow admissible\ (p\ @\ (drop\ n\ q)) \Longrightarrow p\ @\ (drop\ n\ q) \in \mathcal{P}\ k\ u$
proof (*induct length q − n arbitrary*: *p q n k u*)
  case *0*
    from *0* **have** *0* = *length q − n* **by** *auto*
    then have *n*: $n \geq length\ q$ **by** *arith*
    then have *drop n q* = [] **by** *auto*
    then show *?case* **by** (*simp add*: *0.prems*(*1*))
next
  case (*Suc l*)
    have $n \geq length\ q \vee n < length\ q$ **by** *arith*
    then show *?case*
    proof (*induct rule*: *disjCases2*)
      case *1*
        then have *drop n q* = [] **by** *auto*
        then show *?case* **by** (*simp add*: *Suc.prems*(*1*))
    next
      case *2*
        then have *length q* > *0* **by** *auto*
        then have *q-nonempty*: $q \neq []$ **by** *auto*
        let $?q' = butlast\ q$
        from *q-nonempty Suc*(*2*) **have** *h1*: $l = length\ ?q' - n$ **by** *auto*
        have *h2*: $?q' \in \mathcal{P}\ k\ u$
          by (*metis Suc.prems*(*2*) *butlast-conv-take is-prefix-take prefixes-are-paths*)
        have *h3*: *charslength p* = *charslength* (*take n ?q*′)
          using *2.hyps Suc.prems*(*3*) *take-butlast* **by** *force*
        have *is-prefix* (*p* @ *drop n ?q*′) (*p* @ *drop n q*)
          by (*simp add*: *butlast-conv-take drop-take*)
        note *h4* = *is-prefix-admissible*[*OF this Suc.prems*(*5*)]
        note *induct* = *Suc*(*1*)[*OF h1 Suc*(*3*) *h2 h3 Suc.prems*(*4*) *h4*]
        let $?p' = p\ @\ (drop\ n\ (butlast\ q))$
        from *induct* **have** $?p' \in \mathcal{P}\ k\ u$ **.**
        let $?i = charslength\ ?p'$
        have *charslength-i*[*symmetric*]: *charslength ?q*′ = *?i*
          using *Suc.prems*(*3*) **apply** *simp*
          apply (*subst butlast-split*[*OF 2*])
          by *simp*
        have *q-split*: $q = ?q'@[last\ q]$ **by** (*simp add*: *q-nonempty*)
         with *Suc.prems*(*2*) *charslength-of-butlast* **have** *charslength-q*′: *charslength* $?q' \leq k$
          by *blast*
        from *q-nonempty* **have** *p′last*: $?p'@[last\ q] = p@(drop\ n\ q)$
          by (*metis 2.hyps append-assoc drop-eq-Nil drop-keep-last not-le q-split*)
        have $?i \leq k$ **by** (*simp only*: *charslength-i charslength-q*′)
        then have $?i = k \vee ?i < k$ **by** *auto*

```
      then show ?case
      proof (induct rule: disjCases2)
        case 1
          have charslength-q': charslength ?q' = k using charslength-i[symmetric]
1 by blast
          from compatibility-def[OF induct Suc.prems(2) 1 q-split charslength-q']
          show ?case by (simp only: p'last Suc.prems(5))
      next
        case 2
          from in-P-charslength[OF induct]
          obtain v1 where v1: ?p' ∈ P ?i v1 by blast
          from last-step-of-path[OF Suc.prems(2) q-split]
          have ∃ u. q ∈ P ?i u by (metis charslength-i)
          then obtain v2 where v2: q ∈ P ?i v2 by blast
          let ?v = max v1 v2
          have v1 ≤ ?v by auto
          with v1 have dom1: ?p' ∈ P ?i ?v by (metis (no-types, opaque-lifting)
subsetCE subset-Pk)
          have v2 ≤ ?v by auto
           with v2 have dom2: q ∈ P ?i ?v by (metis (no-types, opaque-lifting)
subsetCE subset-Pk)
          from compatibility-def[OF dom1 dom2 - q-split]
          have p @ drop n q ∈ P ?i ?v
            by (simp only: p'last charslength-i[symmetric] Suc.prems(5))
         then show p @ drop n q ∈ P k u by (meson 2.hyps subsetCE subset-P)
      qed
   qed
qed

lemma wellformed-item-derives:
  assumes wellformed: wellformed-item x
  shows derives [item-nonterminal x] (item-rhs x)
proof −
  from wellformed have (item-nonterminal x, item-rhs x) ∈ ℜ
    by (simp add: item-nonterminal-def item-rhs-def wellformed-item-def)
  then show ?thesis
    by (metis append-Nil2 derives1-def derives1-implies-derives is-sentence-concat
        rule-α-type self-append-conv2)
qed

lemma wellformed-complete-item-β:
  assumes wellformed: wellformed-item x
  assumes complete: is-complete x
  shows item-β x = []
using complete is-complete-def item-β-def by auto

lemma wellformed-complete-item-derives:
  assumes wellformed: wellformed-item x
  assumes complete: is-complete x
```

shows *derives* [*item-nonterminal x*] (*item-α x*)
using *complete is-complete-def item-α-def wellformed wellformed-item-derives* **by** *auto*

lemma *is-derivation-implies-admissible*:
is-derivation (*terminals p* @ *δ*) ⟹ *is-word* (*terminals p*) ⟹ *admissible p*
using $\mathcal{L}_P$*-def admissible-def* **by** *blast*

lemma *item-rhs-of-inc-item*[*simp*]: *item-rhs* (*inc-item x k*) = *item-rhs x*
by (*auto simp add*: *inc-item-def item-rhs-def*)

lemma *item-rule-of-inc-item*[*simp*]: *item-rule* (*inc-item x k*) = *item-rule x*
by (*simp add*: *inc-item-def*)

lemma *item-origin-of-inc-item*[*simp*]: *item-origin* (*inc-item x k*) = *item-origin x*
by (*simp add*: *inc-item-def*)

lemma *item-end-of-inc-item*[*simp*]: *item-end* (*inc-item x k*) = *k*
by (*simp add*: *inc-item-def*)

lemma *item-dot-of-inc-item*[*simp*]: *item-dot* (*inc-item x k*) = (*item-dot x*) + *1*
by (*simp add*: *inc-item-def*)

lemma *item-nonterminal-of-inc-item*[*simp*]: *item-nonterminal* (*inc-item x k*) = *item-nonterminal x*
by (*simp add*: *inc-item-def item-nonterminal-def*)

lemma *wellformed-inc-item*:
assumes *wellformed*: *wellformed-item x*
assumes *next-symbol*: *next-symbol x* = *Some s*
assumes *k-upper-bound*: *k* ≤ *length Doc*
assumes *k-lower-bound*: *k* ≥ *item-end x*
shows *wellformed-item* (*inc-item x k*)
proof −
have *k-lower-bound′*: *k* ≥ *item-origin x*
using *k-lower-bound wellformed wellformed-item-def* **by** *auto*
show *?thesis*
apply (*auto simp add*: *wellformed-item-def k-upper-bound k-lower-bound′*)
using *wellformed wellformed-item-def* **apply** *blast*
using *is-complete-def next-symbol next-symbol-not-complete not-less-eq-eq* **by** *blast*
qed

lemma *item-α-of-inc-item*:
assumes *wellformed*: *wellformed-item x*
assumes *next-symbol*: *next-symbol x* = *Some s*
shows *item-α* (*inc-item x k*) = *item-α x* @ [*s*]
by (*metis* (*mono-tags*, *lifting*) *item-dot-of-inc-item item-rhs-of-inc-item*
One-nat-def add.right-neutral add-Suc-right is-complete-def item-α-def item-β-def

le-neq-implies-less list.sel(1) next-symbol next-symbol-not-complete next-symbol-starts-item-β

take-hd-drop wellformed wellformed-item-def)

lemma *derives1-pad*:
assumes *derives1*: *derives1* α β
assumes *u*: *is-sentence u*
assumes *v*: *is-sentence v*
shows *derives1* ($u@\alpha@v$) ($u@\beta@v$)
proof −
from *derives1* **have**
$\exists x\ y\ N\ \delta.\ \alpha = x\ @\ [N]\ @\ y \wedge \beta = x\ @\ \delta\ @\ y \wedge$ *is-sentence x* $\wedge$ *is-sentence y* $\wedge$ $(N, \delta) \in \mathfrak{R}$
by (*auto simp add*: *derives1-def*)
then obtain *x y N* δ **where**
1: $\alpha = x\ @\ [N]\ @\ y \wedge \beta = x\ @\ \delta\ @\ y \wedge$ *is-sentence x* $\wedge$ *is-sentence y* $\wedge$ $(N, \delta) \in \mathfrak{R}$ **by** *blast*
show *?thesis*
apply (*simp only*: *derives1-def*)
apply (*rule-tac x=u@x* **in** *exI*)
apply (*rule-tac x=y@v* **in** *exI*)
apply (*rule-tac x=N* **in** *exI*)
apply (*rule-tac x=*δ **in** *exI*)
using *1 u v is-sentence-concat* **by** *auto*
qed

lemma *derives-pad*:
derives α $\beta \Longrightarrow$ *is-sentence u* $\Longrightarrow$ *is-sentence v* $\Longrightarrow$ *derives* ($u@\alpha@v$) ($u@\beta@v$)
proof (*induct rule*: *derives-induct*)
case *Base* **thus** *?case* **by** *simp*
next
case (*Step y z*)
from *Step* **have** *1*: *derives* ($u@\alpha@v$) ($u@y@v$) **by** *auto*
from *Step* **have** *2*: *derives1 y z* **by** *auto*
then have *derives1* ($u@y@v$) ($u@z@v$) **by** (*simp add*: *Step.prems derives1-pad*)

then show *?case*
using *1 derives1-implies-derives derives-trans* **by** *blast*
qed

lemma *derives1-is-sentence*: *derives1* α $\beta \Longrightarrow$ *is-sentence* α $\wedge$ *is-sentence* β
using *Derives1-sentence1 Derives1-sentence2 derives1-implies-Derives1* **by** *blast*

lemma *derives-is-sentence*: *derives* α $\beta \Longrightarrow$ $(\alpha = \beta) \vee$ (*is-sentence* α $\wedge$ *is-sentence* β)
proof (*induct rule*: *derives-induct*)
case *Base* **thus** *?case* **by** *simp*
next

```
  case (Step y z)
    show ?case using Step.hyps(2) Step.hyps(3) derives1-is-sentence by blast
qed

lemma derives-append:
  assumes au: derives a u
  assumes bv: derives b v
  assumes is-sentence-a: is-sentence a
  assumes is-sentence-b: is-sentence b
  shows derives (a@b) (u@v)
proof −
  from au have a = u ∨ (is-sentence a ∧ is-sentence u)
    using derives-is-sentence by blast
  then have au-sentences: is-sentence a ∧ is-sentence u using is-sentence-a by
blast
  from bv have b = v ∨ (is-sentence b ∧ is-sentence v)
    using derives-is-sentence by blast
  then have bv-sentences: is-sentence b ∧ is-sentence v using is-sentence-b by
blast
  have 1: derives (a@b) (u@b)
    apply (rule-tac derives-pad[OF au, where u=[], simplified])
    using is-sentence-b by auto
  have 2: derives (u@b) (u@v)
    apply (rule-tac derives-pad[OF bv, where v=[], simplified])
    apply (simp add: au-sentences)
    done
  from 1 2 derives-trans show ?thesis by blast
qed

lemma is-sentence-item-α: wellformed-item x ⟹ is-sentence (item-α x)
  by (metis is-sentence-take item-α-def item-rhs-def prod.collapse rule-α-type well-
formed-item-def)

lemma is-nonterminal-item-nonterminal: wellformed-item x ⟹ is-nonterminal
(item-nonterminal x)
  by (metis item-nonterminal-def prod.collapse rule-nonterminal-type wellformed-item-def)

lemma Complete-elem-in-Gen:
  assumes I-in-Gen: I ⊆ Gen (𝒫 k u)
  assumes k: k ≤ length Doc
  assumes x-in-Complete: x ∈ Complete k I
  shows x ∈ Gen (𝒫 k u)
proof −
  let ?P = 𝒫 k u
  from x-in-Complete have x ∈ I ∨ (∃ x1 x2. x = inc-item x1 k ∧
    x1 ∈ bin I (item-origin x2) ∧ x2 ∈ bin I k ∧ is-complete x2 ∧
    next-symbol x1 = Some (item-nonterminal x2))
    by (auto simp add: Complete-def)
  then show ?thesis
```

proof (*induct rule*: *disjCases2*)
case *1* **thus** *?case* **using** *I-in-Gen subsetCE* **by** *blast*
next
case *2*
then obtain *x1 x2* **where** *x12*: $x = inc\text{-}item\ x1\ k \wedge$
$x1 \in bin\ I\ (item\text{-}origin\ x2) \wedge x2 \in bin\ I\ k \wedge is\text{-}complete\ x2 \wedge$
$next\text{-}symbol\ x1 = Some\ (item\text{-}nonterminal\ x2)$ **by** *blast*
from *x12* **have** $\exists\ p1\ p2.\ p1 \in ?P \wedge pvalid\ p1\ x1 \wedge p2 \in ?P \wedge pvalid\ p2\ x2$
by (*meson Gen-implies-pvalid I-in-Gen bin-elem subsetCE*)
then obtain *p1 p2* **where** *p1*: $p1 \in ?P \wedge pvalid\ p1\ x1$ **and** *p2*: $p2 \in ?P \wedge pvalid\ p2\ x2$
by *blast*
from *p1* **obtain** $w\ \delta$ **where** *p1valid*:
$wellformed\text{-}tokens\ p1 \wedge$
$wellformed\text{-}item\ x1 \wedge$
$w \leq length\ p1 \wedge$
$charslength\ p1 = item\text{-}end\ x1 \wedge$
$charslength\ (take\ w\ p1) = item\text{-}origin\ x1 \wedge$
$is\text{-}derivation\ (terminals\ (take\ w\ p1)\ @\ [item\text{-}nonterminal\ x1]\ @\ \delta) \wedge$
$derives\ (item\text{-}\alpha\ x1)\ (terminals\ (drop\ w\ p1))$
using *pvalid-def* **by** *blast*
from *p2* **obtain** $y\ \gamma$ **where** *p2valid*:
$wellformed\text{-}tokens\ p2 \wedge$
$wellformed\text{-}item\ x2 \wedge$
$y \leq length\ p2 \wedge$
$charslength\ p2 = item\text{-}end\ x2 \wedge$
$charslength\ (take\ y\ p2) = item\text{-}origin\ x2 \wedge$
$is\text{-}derivation\ (terminals\ (take\ y\ p2)\ @\ [item\text{-}nonterminal\ x2]\ @\ \gamma) \wedge$
$derives\ (item\text{-}\alpha\ x2)\ (terminals\ (drop\ y\ p2))$
using *pvalid-def* **by** *blast*
let $?r = p1\ @\ (drop\ y\ p2)$
have *charslength-p1-eq*: $charslength\ p1 = item\text{-}end\ x1$ **by** (*simp only*: *p1valid*)
from *x12* **have** *item-end-x1*: $item\text{-}end\ x1 = item\text{-}origin\ x2$
using *bin-def mem-Collect-eq* **by** *blast*
have *item-end-x2*: $item\text{-}end\ x2 = k$ **using** *bin-def x12* **by** *blast*
then have *charslength-p1-leq*: $charslength\ p1 \leq k$
using *charslength-p1-eq item-end-x1 p2valid wellformed-item-def* **by** *auto*
have $\exists \delta'.\ item\text{-}\beta\ x1 = [item\text{-}nonterminal\ x2]\ @\ \delta'$
by (*simp add*: *next-symbol-starts-item-β p1valid x12*)
then obtain δ' **where** δ': $item\text{-}\beta\ x1 = [item\text{-}nonterminal\ x2]\ @\ \delta'$ **by** *blast*
have $is\text{-}derivation\ ((terminals\ (take\ w\ p1))@(item\text{-}rhs\ x1)@\delta)$
using *is-derivation-derives p1valid wellformed-item-derives* **by** *blast*
then have $is\text{-}derivation\ ((terminals\ (take\ w\ p1))@(item\text{-}\alpha\ x1\ @\ item\text{-}\beta\ x1)@\delta)$
by (*simp add*: *item-rhs-split*)
then have $is\text{-}derivation\ ((terminals\ (take\ w\ p1))@((terminals\ (drop\ w\ p1))\ @\ item\text{-}\beta\ x1)@\delta)$
using *is-derivation-derives p1valid* **by** *auto*
then have $is\text{-}derivation\ ((terminals\ p1)@(item\text{-}\beta\ x1)@\delta)$
by (*metis append-assoc append-take-drop-id terminals-append*)

then have *is-derivation* ((*terminals p1*)@([*item-nonterminal x2*] @ δ')@δ)
using *is-derivation-derives* δ' **by** *auto*
then have *is-derivation* ((*terminals p1*)@(*terminals* (*drop y p2*)) @ δ' @δ)
using *is-complete-def is-derivation-derives is-derivation-step item-α-def item-nonterminal-def item-rhs-def p2valid wellformed-item-def x12* **by** *auto*
then have *is-derivation* (*terminals* (*p1* @ (*drop y p2*)) @ (δ' @ δ)) **by** *simp*
then have *admissible-r*: *admissible* (*p1* @ (*drop y p2*))
apply (*rule-tac is-derivation-implies-admissible*)
apply *auto*
apply (*rule is-word-terminals*)
apply (*simp add*: *p1valid*)
using *p2valid* **using** *is-word-terminals-drop terminals-drop* **by** *auto*
have *r-in-dom*: *?r* $\in \mathcal{P}$ *k u*
apply (*rule-tac general-compatibility*)
apply (*simp add*: *p1*)
apply (*simp add*: *p2*)
apply (*simp only*: *p2valid charslength-p1-eq item-end-x1*)
apply (*simp only*: *charslength-p1-leq*)
by (*simp add*: *admissible-r*)
have *wellformed-r*: *wellformed-tokens ?r*
using *admissible-r admissible-wellformed-tokens* **by** *blast*
have *wellformed-x*: *wellformed-item x*
apply (*simp add*: *x12*)
apply (*rule-tac wellformed-inc-item*)
apply (*simp add*: *p1valid*)
apply (*simp add*: *x12*)
apply (*simp add*: *k*)
using *charslength-p1-eq charslength-p1-leq* **by** *auto*
have *charslength-p1-as-p2*: *charslength p1* = *charslength* (*take y p2*)
using *charslength-p1-eq item-end-x1 p2valid* **by** *linarith*
then have *charslength-r*: *charslength ?r* = *item-end x*
apply (*simp add*: *x12*)
apply (*subst length-append*[*symmetric*])
apply (*subst chars-append*[*symmetric*])
apply (*subst append-take-drop-id*)
using *item-end-x2 p2valid* **by** *auto*
have *item-α-x*: *item-α x* = *item-α x1* @ [*item-nonterminal x2*]
using *x12 p1valid* **by** (*simp add*: *item-α-of-inc-item*)
from *p2valid* **have** *derives-item-nonterminal-x2*:
derives [*item-nonterminal x2*] (*terminals* (*drop y p2*))
using *derives-trans wellformed-complete-item-derives x12* **by** *blast*
have *pvalid ?r x*
apply (*auto simp only*: *pvalid-def*)
apply (*rule-tac x*=*w* **in** *exI*)
apply (*rule-tac x*=δ **in** *exI*)
apply (*auto simp only*:)
apply (*simp add*: *wellformed-r*)
apply (*simp add*: *wellformed-x*)
using *p1valid* **apply** *simp*

```
    apply (simp only: charslength-r)
    using x12 p1valid apply simp
    using x12 p1valid apply simp
    apply (simp add:  item-α-x)
    apply (rule-tac derives-append)
    using p1valid apply simp
    using derives-item-nonterminal-x2 p1valid apply auto[1]
    using is-sentence-item-α p1valid apply blast
    using is-derivation-is-sentence is-sentence-concat p2valid by blast
  with r-in-dom show ?case using Gen-def mem-Collect-eq by blast
 qed
qed

lemma Complete-subset-Gen:
 assumes I-in-Gen-P: I ⊆ Gen (𝒫 k u)
 assumes k: k ≤ length Doc
 shows Complete k I ⊆ Gen (𝒫 k u)
using Complete-elem-in-Gen I-in-Gen-P k by blast

lemma 𝒫-are-admissible: p ∈ 𝒫 k u ⟹ admissible p
apply (rule-tac 𝔓-are-admissible)
using 𝔓-covers-𝒫 subsetCE by blast

lemma is-continuation-base:
 assumes p-dom: p ∈ 𝒫 k u
 assumes charslength-p: charslength p = k
 shows is-continuation k u p []
apply (auto simp add: is-continuation-def)
apply (simp add: p-dom)
using charslength-p apply simp
using 𝒫-are-admissible p-dom by blast

lemma is-continuation-empty-chars:
 is-continuation k u q ts ⟹ charslength (q@ts) = k ⟹ chars ts = []
by (simp add: is-continuation-def)

lemma 𝒵-subset: u ≤ v ⟹ 𝒵 k u ⊆ 𝒵 k v
using 𝒵-subset-Suc subset-fSuc by blast

lemma is-continuation-increase-u:
 assumes cont: is-continuation k u q ts
 assumes uv: u ≤ v
 shows is-continuation k v q ts
proof −
 have q ∈ 𝒫 k u using cont is-continuation-def by blast
 with uv have q-dom: q ∈ 𝒫 k v by (meson subsetCE subset-𝒫k)
 from uv have 𝒵: ⋀ t. t ∈ 𝒵 k (Suc u) ⟹ t ∈ 𝒵 k (Suc v)
  using 𝒵-subset le-neq-implies-less less-imp-le-nat not-less-eq subsetCE by blast
```

```
  show ?thesis
    apply (auto simp only: is-continuation-def)
    apply (simp add: q-dom)
    using cont is-continuation-def apply simp
    using cont is-continuation-def apply simp
    using cont is-continuation-def 𝒵 apply simp
    using cont is-continuation-def apply (simp only:)
    done
qed

lemma pvalid-next-symbol-derivable:
  assumes pvalid: pvalid p x
  assumes next-symbol: next-symbol x = Some s
  shows ∃ δ. is-derivation((terminals p)@[s]@δ)
proof −
  from pvalid pvalid-def have wellformed-x: wellformed-item x by auto
  from next-symbol-starts-item-β[OF wellformed-x next-symbol]
  obtain ω where ω: item-β x = [s] @ ω by auto
  from pvalid have ∃ γ. is-derivation((terminals p)@(item-β x)@γ)
    using pvalid-is-derivation-terminals-item-β by blast
  then obtain γ where is-derivation((terminals p)@(item-β x)@γ) by blast
  with ω have is-derivation((terminals p)@[s]@ω@γ) by auto
  then show ?thesis by blast
qed

lemma pvalid-admissible:
  assumes pvalid: pvalid p x
  shows admissible p
proof −
  have ∃ δ. is-derivation((terminals p)@(item-β x)@δ)
    by (simp add: pvalid pvalid-is-derivation-terminals-item-β)
  then obtain δ where δ: is-derivation((terminals p)@(item-β x)@δ) by blast
  have is-word: is-word (terminals p)
    using pvalid-def is-word-terminals pvalid by blast
  show ?thesis using δ is-derivation-implies-admissible is-word by blast
qed

lemma pvalid-next-terminal-admissible:
  assumes pvalid: pvalid p x
  assumes next-symbol: next-symbol x = Some t
  assumes terminal: is-terminal t
  shows admissible (p@[(t, c)])
proof −
  have is-word (terminals p)
    using is-word-terminals pvalid pvalid-def by blast
  then show ?thesis
   using is-derivation-implies-admissible next-symbol pvalid pvalid-next-symbol-derivable

     terminal by fastforce
```

qed

lemma $\mathcal{X}$*-wellformed*: $t \in \mathcal{X}\ k \Longrightarrow$ *wellformed-token t*
by (*simp add*: $\mathcal{X}$*-are-terminals wellformed-token-def*)

lemma $\mathcal{Z}$*-wellformed*: $t \in \mathcal{Z}\ k\ u \Longrightarrow$ *wellformed-token t*
using $\mathcal{X}$*-wellformed* $\mathcal{Z}$*-subset-*$\mathcal{X}$ **by** *blast*

lemma *Scan-elem-in-Gen*:
assumes *I-in-Gen*: $I \subseteq$ *Gen* ($\mathcal{P}\ k\ u$)
assumes k: $k \leq$ *length Doc*
assumes T: $T \subseteq \mathcal{Z}\ k\ u$
assumes *x-in-Scan*: $x \in$ *Scan* $T\ k\ I$
shows $x \in$ *Gen* ($\mathcal{P}\ k\ u$)
proof −
have $u = 0 \Longrightarrow x \in I$
proof −
assume $u = 0$
then have $\mathcal{Z}\ k\ u = \{\}$ **by** *simp*
then have $T = \{\}$ **using** T **by** *blast*
then have *Scan* $T\ k\ I = I$ **by** (*simp add*: *Scan-empty*)
then show $x \in I$ **using** *x-in-Scan* **by** *simp*
qed
then have $x \in I \vee (u > 0 \wedge (\exists\ y\ t\ c.\ x =$ *inc-item* $y\ (k +$ *length* $c) \wedge y \in$ *bin* $I\ k \wedge$
$(t,\ c) \in T \wedge$ *next-symbol* $y =$ *Some* $t))$ **using** *x-in-Scan Scan-def* **by** *auto*
then show *?thesis*
proof (*induct rule*: *disjCases2*)
case *1* **thus** *?case* **using** *I-in-Gen* **by** *blast*
next
case *2*
then obtain $y\ t\ c$ **where** *x-is-scan*: $x =$ *inc-item* $y\ (k +$ *length* $c) \wedge y \in$ *bin* I $k \wedge$
$(t,\ c) \in T \wedge$ *next-symbol* $y =$ *Some* t **by** *blast*
have *u-gt-0*: $0 < u$ **using** *2* **by** *blast*
have $\exists\ p \in \mathcal{P}\ k\ u.$ *pvalid* $p\ y$ **using** *Gen-implies-pvalid I-in-Gen bin-elem x-is-scan* **by** *blast*
then obtain p **where** p: $p \in \mathcal{P}\ k\ u \wedge$ *pvalid* $p\ y$ **by** *blast*
have *p-dom*: $p \in \mathcal{P}\ k\ u$ **using** p **by** *blast*
from p *pvalid-def x-is-scan* **have** *charslength-p*: *charslength* $p = k$
using *bin-def mem-Collect-eq* **by** *auto*
obtain *tok* **where** *tok*: *tok* $= (t,\ c)$ **using** *x-is-scan* **by** *blast*
have *tok-dom*: *tok* $\in \mathcal{Z}\ k\ u$ **using** *tok x-is-scan* T **by** *blast*
have $p = [] \vee p \neq []$ **by** *blast*
then have $\exists\ q\ ts\ u'.\ p = q@ts \wedge u' < u \wedge$ *charslength ts* $= 0 \wedge$ *is-continuation* $k\ u'\ q\ ts$
proof (*induct rule*: *disjCases2*)
case *1* **thus** *?case*
apply (*rule-tac x*=p **in** *exI*)

```
      apply (rule-tac x=[] in exI)
      apply (rule-tac x=0 in exI)
      apply (simp add: 2 is-continuation-def)
      using charslength-p by simp
    next
     case 2
     from final-step-of-path[OF p-dom 2] obtain q ts k' u'
      where final-step: p = q @ ts ∧ indexlt k' u' k u ∧ is-continuation k' u' q ts
by blast
     then have k' ≤ k using indexlt-simp by auto
     then have k' < k ∨ k' = k by arith
     then show ?case
     proof (induct rule: disjCases2)
      case 1
      have p ∈ 𝒫 k' (Suc u') using final-step is-continuation-in-𝒫 by blast
      then have p-dom: p ∈ 𝒫 k 0 by (meson 1 subsetCE subset-𝒫)
      with charslength-p have is-continuation k 0 p [] using is-continuation-base
by blast
      then show ?case
       apply (rule-tac x=p in exI)
       apply (rule-tac x=[] in exI)
       apply (rule-tac x=0 in exI)
       apply (simp add: u-gt-0)
       done
     next
      case 2
      with final-step indexlt-simp have u' < u by auto
      then show ?case
       apply (rule-tac x=q in exI)
       apply (rule-tac x=ts in exI)
       apply (rule-tac x=u' in exI)
       using final-step 2 apply auto
       using charslength-p is-continuation-empty-chars by blast
     qed
   qed
   then obtain q ts u' where
     p-split: p = q@ts ∧ u' < u ∧ charslength ts = 0 ∧ is-continuation k u' q ts
by blast
   then have ∃ u''. u' ≤ u'' ∧ Suc u'' = u by (auto, arith)
   then obtain u'' where u'':  u' ≤ u'' ∧ Suc u'' = u by blast
   with p-split have cont-u'': is-continuation k u'' q ts
    using is-continuation-increase-u by blast
   have admissible: admissible (p@[tok])
    apply (simp add: tok)
    apply (rule-tac pvalid-next-terminal-admissible[where x=y])
    apply (simp add: p)
    apply (simp add: x-is-scan)
    using 𝒵-wellformed tok tok-dom wellformed-token-def by auto
   have is-continuation k u'' q (ts@[tok])
```

apply (*rule is-continuation-continue*)
apply (*simp add*: *cont-u''*)
using *p-split* **apply** *simp*
using *u''* *tok-dom* **apply** *simp*
using *admissible p-split* **by** *auto*
with *p-split u''* **have** *ptok-dom*: $p@[tok] \in \mathcal{P}\ k\ u$
using *append-assoc is-continuation-in-*$\mathcal{P}$ **by** *auto*
from *p* **obtain** *i* γ **where** *valid*:
wellformed-tokens p $\wedge$
wellformed-item y $\wedge$
$i \leq$ *length p* $\wedge$
charslength p = *item-end y* $\wedge$
charslength (*take i p*) = *item-origin y* $\wedge$
is-derivation (*terminals* (*take i p*) @ [*item-nonterminal y*] @ γ) $\wedge$
derives (*item-*α *y*) (*terminals* (*drop i p*)) **using** *pvalid-def* **by** *blast*
have *clen-ptok*: *k* + *length c* = *charslength* ($p@[tok]$)
using *charslength-p tok* **by** *simp*
from *ptok-dom* **have** *ptok-doc-tokens*: *doc-tokens* ($p@[tok]$)
using $\mathfrak{P}$*-are-doc-tokens* $\mathfrak{P}$*-covers-*$\mathcal{P}$ *rev-subsetD* **by** *blast*
have *wellformed-x*: *wellformed-item x*
apply (*simp add*: *x-is-scan*)
apply (*rule-tac wellformed-inc-item*)
apply (*simp add*: *valid*)
apply (*simp add*: *x-is-scan*)
apply (*simp only*: *clen-ptok*)
using *ptok-doc-tokens charslength.simps doc-tokens-length* **apply** *presburger*
apply (*simp only*: *clen-ptok*)
using *valid* **by** *auto*
have *pvalid* ($p@[tok]$) *x*
apply (*auto simp only*: *pvalid-def*)
apply (*rule-tac x=i* **in** *exI*)
apply (*rule-tac x=*γ **in** *exI*)
apply (*auto simp only*:)
using *ptok-dom admissible admissible-wellformed-tokens* **apply** *blast*
apply (*simp add*: *wellformed-x*)
using *valid* **apply** *simp*
apply (*simp add*: *x-is-scan clen-ptok*)
using *valid* **apply** (*simp add*: *x-is-scan*)
using *valid* **apply** (*simp add*: *x-is-scan*)
using *valid* **apply** (*simp add*: *x-is-scan*)
apply (*subst item-*α*-of-inc-item*)
using *valid* **apply** *simp*
using *x-is-scan* **apply** *simp*
apply (*rule-tac derives-append*)
apply *simp*
apply (*simp add*: *tok*)
using *is-sentence-item-*α **apply** *blast*
by (*meson pvalid-next-symbol-derivable LocalLexing-axioms is-derivation-is-sentence*

```
        is-sentence-concat p x-is-scan)
    with ptok-dom show ?thesis
      using Gen-def mem-Collect-eq by blast
  qed
qed

lemma Scan-subset-Gen:
  assumes I-in-Gen: I ⊆ Gen (𝒫 k u)
  assumes k: k ≤ length Doc
  assumes T: T ⊆ 𝒵 k u
  shows Scan T k I ⊆ Gen (𝒫 k u)
using I-in-Gen Scan-elem-in-Gen T k by blast

theorem thmD5:
  assumes I: I ⊆ Gen (𝒫 k u)
  assumes k: k ≤ length Doc
  assumes T: T ⊆ 𝒵 k u
  shows π k T I ⊆ Gen (𝒫 k u)
apply (simp add: π-def)
apply (rule-tac limit-upperbound)
using I k T Predict-subset-Gen Complete-subset-Gen Scan-subset-Gen apply metis
by (simp add: I)

end

end
theory TheoremD6
imports TheoremD5
begin

context LocalLexing begin

definition inc-dot :: nat ⇒ item ⇒ item
where
  inc-dot d x = Item (item-rule x) (item-dot x + d) (item-origin x) (item-end x)

lemma inc-dot-0[simp]: inc-dot 0 x = x
  by (simp add: inc-dot-def)

lemma Predict-mk-regular1:
  ∃ (P :: rule ⇒ item ⇒ bool) F. Predict k = mk-regular1 P F
proof −
  let ?P = λ r x::item. r ∈ ℜ ∧ item-end x = k ∧ next-symbol x = Some(fst r)
  let ?F = λ r (x::item). init-item r k
  show ?thesis
    apply (rule-tac x=?P in exI)
    apply (rule-tac x=?F in exI)
    apply (rule-tac ext)
    by (auto simp add: mk-regular1-def bin-def Predict-def)
```

qed

lemma *Complete-mk-regular2*:
∃ (*P* :: *dummy* ⇒ *item* ⇒ *item* ⇒ *bool*) *F*. *Complete k* = *mk-regular2 P F*
proof −
let *?P* = λ (*r*::*dummy*) *x y*. *item-end x* = *item-origin y* ∧ *item-end y* = *k* ∧ *is-complete y* ∧
next-symbol x = *Some* (*item-nonterminal y*)
let *?F* = λ (*r*::*dummy*) *x y*. *inc-item x k*
show *?thesis*
apply (*rule-tac x*=*?P* **in** *exI*)
apply (*rule-tac x*=*?F* **in** *exI*)
apply (*rule-tac ext*)
by (*auto simp add*: *mk-regular2-def bin-def Complete-def*)
qed

lemma *Scan-mk-regular1*:
∃ (*P* :: *token* ⇒ *item* ⇒ *bool*) *F*. *Scan T k* = *mk-regular1 P F*
proof −
let *?P* = λ (*tok*::*token*) (*x*::*item*). *item-end x* = *k* ∧ *tok* ∈ *T* ∧ *next-symbol x* = *Some* (*fst tok*)
let *?F* = λ (*tok*::*token*) (*x*::*item*). *inc-item x* (*k* + *length* (*snd tok*))
show *?thesis*
apply (*rule-tac x*=*?P* **in** *exI*)
apply (*rule-tac x*=*?F* **in** *exI*)
apply (*rule-tac ext*)
by (*auto simp add*: *mk-regular1-def bin-def Scan-def*)
qed

lemma *Predict-regular*: *regular* (*Predict k*)
by (*metis Predict-mk-regular1 regular1*)

lemma *Complete-regular*: *regular* (*Complete k*)
by (*metis Complete-mk-regular2 regular2*)

lemma *Scan-regular*: *regular* (*Scan T k*)
by (*metis Scan-mk-regular1 regular1*)

lemma *π-functional*: *π k T* = *limit* ((*Scan T k*) *o* (*Complete k*) *o* (*Predict k*))
proof −
have *π k T* = *limit* (λ *I*. *Scan T k* (*Complete k* (*Predict k I*)))
using *π-def* **by** *blast*
moreover have (λ *I*. *Scan T k* (*Complete k* (*Predict k I*))) =
(*Scan T k*) *o* (*Complete k*) *o* (*Predict k*)
apply (*rule ext*)
by *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *π-step-regular*: *regular* ((*Scan T k*) *o* (*Complete k*) *o* (*Predict k*))
by (*simp add*: *Complete-regular Predict-regular Scan-regular regular-comp*)

lemma *π-regular*: *regular* (*π k T*)
by (*simp add*: *π-functional π-step-regular regular-limit*)

lemma *π-fix*: *Scan T k* (*Complete k* (*Predict k* (*π k T I*))) = *π k T I*
using *π-functional π-step-regular regular-fixpoint* **by** *fastforce*

lemma *π-fix'*: ((*Scan T k*) *o* (*Complete k*) *o* (*Predict k*)) (*π k T I*) = *π k T I*
using *π-functional π-step-regular regular-fixpoint* **by** *fastforce*

lemma *setmonotone-cases*:
assumes *setmonotone f*
shows *f X* = *X* ∨ *X* ⊂ *f X*
using *assms elem-setmonotone* **by** *fastforce*

lemma *distribute-fixpoint-over-setmonotone-comp*:
assumes *f*: *setmonotone f*
assumes *g*: *setmonotone g*
assumes *fixpoint*: (*f o g*) *I* = *I*
shows *f I* = *I* ∧ *g I* = *I*
proof −
from *setmonotone-cases*[*OF g*, **where** *X*=*I*] **show** *?thesis*
proof(*induct rule*: *disjCases2*)
case *1*
thus *?case* **using** *fixpoint* **by** *simp*
next
case *2*
with *f* **have** *I* ⊂ (*f o g*) *I*
by (*metis comp-apply fixpoint less-asym' setmonotone-cases*)
with *fixpoint* **have** *False* **by** *simp*
then show *?case* **by** *blast*
qed
qed

lemma *distribute-fixpoint-over-setmonotone-comp-3*:
assumes *f*: *setmonotone f*
assumes *g*: *setmonotone g*
assumes *h*: *setmonotone h*
assumes *fixpoint*: (*f o g o h*) *I* = *I*
shows *f I* = *I* ∧ *g I* = *I* ∧ *h I* = *I*
by (*meson distribute-fixpoint-over-setmonotone-comp f fixpoint g h setmonotone-comp*)

lemma *Predict-π-fix*: *Predict k* (*π k T I*) = *π k T I*
by (*meson Complete-regular Predict-regular Scan-regular π-fix'*
distribute-fixpoint-over-setmonotone-comp-3 regular-implies-setmonotone)

lemma *Scan-π-fix*: *Scan T k* (*π k T I*) = *π k T I*

by (*meson Complete-regular Predict-regular Scan-regular π-fix′ distribute-fixpoint-over-setmonotone-comp-3 regular-implies-setmonotone*)

lemma *Complete-π-fix*: *Complete k* (*π k T I*) = *π k T I*
by (*meson Complete-regular Predict-regular Scan-regular π-fix′ distribute-fixpoint-over-setmonotone-comp-3 regular-implies-setmonotone*)

lemma *π-idempotent*: *π k T* (*π k T I*) = *π k T I*
by (*simp add*: *π-functional π-step-regular limit-is-idempotent*)

lemma *derivation-shift-identity*[*simp*]: *derivation-shift D 0 0* = *D*
by (*simp add*: *derivation-shift-def*)

lemma *Derivation-skip-prefix*: *Derivation* (*u*@*v*) *D w* ⟹ *derivation-ge D* (*length u*) ⟹
Derivation v (*derivation-shift D* (*length u*) *0*) (*drop* (*length u*) *w*)
proof (*induct D arbitrary*: *u v w*)
case *Nil*
thus *?case* **by** (*simp add*: *append-eq-conv-conj*)
next
case (*Cons d D*)
from *Cons* **have** ∃ *x*. *Derives1* (*u*@*v*) (*fst d*) (*snd d*) *x* ∧ *Derivation x D w* **by** *auto*
then obtain *x* **where** *x*: *Derives1* (*u*@*v*) (*fst d*) (*snd d*) *x* ∧ *Derivation x D w* **by** *blast*
from *Cons* **have** *d*: *fst d* ≥ *length u* **and** *D*: *derivation-ge D* (*length u*)
using *derivation-ge-cons* **apply** *blast*
using *Cons.prems*(*2*) *derivation-ge-cons* **by** *blast*
have ∃ *x′*. *x* = *u*@*x′* **by** (*metis append-eq-conv-conj d le-Derives1-take x*)
then obtain *x′* **where** *x′*: *x* = *u*@*x′* **by** *blast*
show *?case*
apply *simp*
apply (*rule-tac x*=*x′* **in** *exI*)
using *Cons.hyps D Derives1-skip-prefix d x x′* **by** *blast*
qed

lemma *leftmost-skip-prefix*: *leftmost i* (*u*@*v*) ⟹ *i* ≥ *length u* ⟹ *leftmost* (*i* − *length u*) *v*
by (*simp add*: *leftmost-def less-diff-conv2 nth-append*)

lemma *LeftDerivation-skip-prefix*: *LeftDerivation* (*u*@*v*) *D w* ⟹ *derivation-ge D* (*length u*) ⟹
LeftDerivation v (*derivation-shift D* (*length u*) *0*) (*drop* (*length u*) *w*)
proof (*induct D arbitrary*: *u v w*)
case *Nil*
thus *?case* **by** (*simp add*: *append-eq-conv-conj*)
next
case (*Cons d D*)
from *Cons* **have** ∃ *x*. *LeftDerives1* (*u*@*v*) (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x*

D w **by** *auto*
then obtain *x* **where** *x*: *LeftDerives1* (*u@v*) (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x D w* **by** *blast*
from *Cons* **have** *d*: *fst d* ≥ *length u* **and** *D*: *derivation-ge D* (*length u*)
using *derivation-ge-cons* **apply** *blast*
using *Cons.prems*(*2*) *derivation-ge-cons* **by** *blast*
have ∃ *x'*. *x* = *u@x'*
by (*metis LeftDerives1-implies-Derives1 append-eq-conv-conj d le-Derives1-take x*)
then obtain *x'* **where** *x'*: *x* = *u@x'* **by** *blast*
have *leftmost*: *leftmost* (*fst d*) (*u@v*) **using** *LeftDerives1-def x* **by** *blast*
have *1*: *LeftDerives1 v* (*fst d* − *length u*) (*snd d*) *x'*
apply (*auto simp add*: *LeftDerives1-def*)
apply (*simp add*: *leftmost d leftmost-skip-prefix*)
using *Derives1-skip-prefix LeftDerives1-implies-Derives1 d x x'* **by** *blast*
have *2*: *LeftDerivation x'* (*derivation-shift D* (*length u*) *0*) (*drop* (*length u*) *w*)
using *Cons.hyps D x x'* **by** *blast*
show *?case*
apply *simp*
apply (*rule-tac x=x'* **in** *exI*)
using *1 2* **by** *blast*
qed

lemma *splits-at-append*: *splits-at u i u1 N u2* ⟹ *splits-at* (*u@v*) *i u1 N* (*u2@v*)
by (*auto simp add*: *splits-at-def*)

lemma *LeftDerives1-append-leftmost-unique*: *LeftDerives1* (*a@b*) *i r c* ⟹ *leftmost j a* ⟹ *i* = *j*
by (*meson LeftDerives1-def leftmost-cons-less leftmost-def leftmost-unique*)

lemma *drop-derivation-shift*:
drop n (*derivation-shift D left right*) = *derivation-shift* (*drop n D*) *left right*
by (*auto simp add*: *derivation-shift-def drop-map*)

lemma *take-derivation-shift*:
take n (*derivation-shift D left right*) = *derivation-shift* (*take n D*) *left right*
by (*auto simp add*: *derivation-shift-def take-map*)

lemma *derivation-shift-0-shift*: *derivation-shift* (*derivation-shift D left1 0*) *left2 right2* =
derivation-shift D (*left1* + *left2*) *right2*
by (*auto simp add*: *derivation-shift-def*)

lemma *splits-at-append-prefix*:
splits-at v i α *N* β ⟹ *splits-at* (*u@v*) (*i* + *length u*) (*u@*α) *N* β
apply (*auto simp add*: *splits-at-def*)
by (*simp add*: *nth-append*)

lemma *splits-at-implies-Derives1*: *splits-at* δ *i* α *N* β ⟹ *is-sentence* δ ⟹ *r*∈ ℜ

$\Longrightarrow$ *fst r = N*
$\Longrightarrow$ *Derives1 δ i r (α@(snd r)@β)*
by (*metis* (*no-types*, *lifting*) *Derives1-def is-sentence-concat length-take less-or-eq-imp-le min.absorb2 prod.collapse splits-at-combine splits-at-def*)

lemma *Derives1-append-prefix*:
assumes *Derives1*: *Derives1 v i r w*
assumes *u*: *is-sentence u*
shows *Derives1 (u@v) (i + length u) r (u@w)*
proof −
have ∃ *α N β*. *splits-at v i α N β* **using** *assms splits-at-ex* **by** *auto*
then obtain *α N β* **where** *split-v*: *splits-at v i α N β* **by** *blast*
have *split-w*: *w = α@(snd r)@β* **using** *assms split-v splits-at-combine-dest* **by** *blast*
have *split-uv*: *splits-at (u@v) (i + length u) (u@α) N β*
by (*simp add*: *split-v splits-at-append-prefix*)
have *is-sentence-uv*: *is-sentence (u@v)*
using *Derives1 Derives1-sentence1 is-sentence-concat u* **by** *blast*
show *?thesis*
by (*metis Derives1 Derives1-nonterminal Derives1-rule append-assoc is-sentence-uv split-uv split-v split-w splits-at-implies-Derives1*)
qed

lemma *leftmost-prepend-word*: *leftmost i v* $\Longrightarrow$ *is-word u* $\Longrightarrow$ *leftmost (i + length u) (u@v)*
by (*simp add*: *leftmost-def nth-append*)

lemma *LeftDerives1-append-prefix*:
assumes *Derives1*: *LeftDerives1 v i r w*
assumes *u*: *is-word u*
shows *LeftDerives1 (u@v) (i + length u) r (u@w)*
proof −
have *1*: *Derives1 v i r w*
by (*simp add*: *Derives1 LeftDerives1-implies-Derives1*)
have *2*: *leftmost i v*
using *Derives1 LeftDerives1-def* **by** *blast*
have *3*: *is-sentence u* **using** *u* **by** *fastforce*
have *4*: *Derives1 (u@v) (i + length u) r (u@w)*
by (*simp add*: *1 3 Derives1-append-prefix*)
have *5*: *leftmost (i + length u) (u@v)*
by (*simp add*: *2 leftmost-prepend-word u*)
show *?thesis*
by (*simp add*: *4 5 LeftDerives1-def*)
qed

lemma *Derivation-append-prefix*: *Derivation v D w* $\Longrightarrow$ *is-sentence u* $\Longrightarrow$ *Derivation (u@v) (derivation-shift D 0 (length u)) (u@w)*
proof (*induct D arbitrary*: *u v w*)

```
  case Nil thus ?case by auto
next
  case (Cons d D)
    then have ∃ x. Derives1 v (fst d) (snd d) x ∧ Derivation x D w by auto
    then obtain x where x: Derives1 v (fst d) (snd d) x ∧ Derivation x D w by
blast
     with Cons have induct: Derivation (u@x) (derivation-shift D 0 (length u))
(u@w) by auto
    have Derives1: Derives1 (u@v) ((fst d) + length u) (snd d) (u@x)
      by (simp add: Cons.prems(2) Derives1-append-prefix x)
    show ?case
      apply simp
      apply (rule-tac x=u@x in exI)
      by (simp add: Cons.hyps Cons.prems(2) Derives1 x)
qed

lemma LeftDerivation-append-prefix: LeftDerivation v D w ⟹ is-word u ⟹
  LeftDerivation (u@v) (derivation-shift D 0 (length u)) (u@w)
proof (induct D arbitrary: u v w)
  case Nil thus ?case by auto
next
  case (Cons d D)
    then have ∃ x. LeftDerives1 v (fst d) (snd d) x ∧ LeftDerivation x D w by
auto
    then obtain x where x: LeftDerives1 v (fst d) (snd d) x ∧ LeftDerivation x
D w by blast
   with Cons have induct: LeftDerivation (u@x) (derivation-shift D 0 (length u))
(u@w) by auto
    have Derives1: LeftDerives1 (u@v) ((fst d) + length u) (snd d) (u@x)
      by (simp add: Cons.prems(2) LeftDerives1-append-prefix x)
    show ?case
      apply simp
      apply (rule-tac x=u@x in exI)
      by (simp add: Cons.hyps Cons.prems(2) Derives1 x)
qed

lemma derivation-ge-shift-simp: derivation-ge D i ⟹ i ≥ l ⟹ r ≥ l ⟹
   derivation-shift D l r = derivation-shift D 0 (r − l)
proof (induct D)
  case Nil thus ?case by auto
next
  case (Cons d D)
  have fst-d: fst d ≥ l
    using Cons.prems(1) Cons.prems(2) derivation-ge-cons le-trans by blast
  show ?case
    apply auto
    using Cons fst-d apply arith
    using Cons derivation-ge-cons apply auto
    done
```

```
qed

lemma append-dropped-prefix: is-prefix u v ⟹ drop (length u) v = w ⟹ u@w
= v
using is-prefix-unsplit by blast

lemma derivation-ge-shift-plus:
  assumes derivation-ge D u
  assumes derivation-ge (derivation-shift D u 0) v
  shows derivation-ge D (u + v)
proof −
  from assms show ?thesis
    apply (auto simp add: derivation-ge-def derivation-shift-def)
    by fastforce
qed

lemma LeftDerivation-breakdown:
  LeftDerivation (u@v) D w ⟹ ∃ n w1 w2. w = w1 @ w2 ∧
     LeftDerivation u (take n D) w1 ∧
     derivation-ge (drop n D) (length w1) ∧
     LeftDerivation v (derivation-shift (drop n D) (length w1) 0) w2
proof (induct length D arbitrary: u v D w)
  case 0
    then have D: D = [] by auto
    with 0 have u@v = w by auto
    with D show ?case
      apply (rule-tac x=0 in exI)
      apply (rule-tac x=u in exI)
      apply (rule-tac x=v in exI)
      by auto
next
  case (Suc l)
    then have ∃ d D'. D = d#D'
      by (metis LeftDerivation.elims(2) length-0-conv nat.simps(3))
    then obtain d D' where D-split: D = d#D' by blast
    from Suc have is-sentence-uv: is-sentence (u@v)
    by (metis D-split Derives1-sentence1 LeftDerivation.simps(2) LeftDerives1-implies-Derives1)
    then have is-sentence-u: is-sentence u and is-sentence-v: is-sentence v
      by (simp add: is-sentence-concat)+
    have is-word u ∨ (¬ is-word u) by blast
    then show ?case
      proof(induct rule: disjCases2)
        case 1
          then have derivation-ge-u: derivation-ge D (length u)
         using LeftDerivation-implies-Derivation Suc.prems is-word-Derivation-derivation-ge
by blast
          have is-prefix: is-prefix u w
            using 1.hyps LeftDerivation-implies-leftderives Suc.prems
              derives-word-is-prefix leftderives-implies-derives by blast
```

```
        have u-w: w = u @ (drop (length u) w)
          by (metis 1.hyps LeftDerivation-implies-leftderives Suc.prems
            derives-word-is-prefix is-prefix-unsplit leftderives-implies-derives)
        show ?case
          apply (rule-tac x=0 in exI)
          apply (rule-tac x=u in exI)
          apply (rule-tac x=drop (length u) w in exI)
          apply (auto)
          apply (rule u-w)
          apply (rule derivation-ge-u)
          by (simp add: LeftDerivation-skip-prefix Suc.prems derivation-ge-u)
      next
        case 2
        with is-sentence-u have ∃ i u1 N u2. splits-at u i u1 N u2 ∧ leftmost i u
          using leftmost-def nonword-leftmost-exists splits-at-def by auto
        then obtain i u1 N u2 where split-u: splits-at u i u1 N u2 ∧ leftmost i
u by blast
        have is-word-u1: is-word u1 by (metis leftmost-def split-u splits-at-def)
        have LeftDerivation (u@v) (d#D') w  using D-split Suc.prems by blast
        then have ∃ x. LeftDerives1 (u@v) (fst d) (snd d) x ∧ LeftDerivation x
D' w
          by simp
        then obtain x where x: LeftDerives1 (u@v) (fst d) (snd d) x ∧ Left-
Derivation x D' w
          by blast
        then have fst-d-eq-i: fst d = i using
          splits-at-combine LeftDerives1-append-leftmost-unique split-u
          by metis
          have split-uv: splits-at (u@v) i u1 N (u2@v) by (simp add: split-u
splits-at-append)
        have split-x: x = u1 @ ((snd (snd d)) @ u2 @ v)
          using LeftDerives1-implies-Derives1 fst-d-eq-i split-uv
            splits-at-combine-dest x by blast
        have derivation-ge-D': derivation-ge D' (length u1)
        using LeftDerivation-implies-Derivation is-word-Derivation-derivation-ge

            leftmost-def split-u split-x splits-at-def x by fastforce
        have D1: LeftDerivation ((snd (snd d)) @ u2 @ v) (derivation-shift D'
(length u1) 0)
          (drop (length u1) w)
          using LeftDerivation-skip-prefix derivation-ge-D' split-x x by blast
      then have D2: LeftDerivation (((snd (snd d)) @ u2) @ v) (derivation-shift
D' (length u1) 0)
          (drop (length u1) w) by auto
        have l = length (derivation-shift D' (length u1) 0)
          using D-split Suc.hyps(2) by auto
        from Suc(1)[OF this D2] obtain n w1 w2 where induct:
          drop (length u1) w = w1 @ w2 ∧
           LeftDerivation (snd (snd d) @ u2)
```

```
        (take n (derivation-shift D' (length u1) 0)) w1 ∧
       derivation-ge (drop n (derivation-shift D' (length u1) 0)) (length w1) ∧
        LeftDerivation v (derivation-shift (drop n (derivation-shift D' (length
u1) 0))
        (length w1) 0) w2 by blast
    have derivation-ge-D'-u1-w1: derivation-ge (drop n D') (length u1 + length
w1)
     proof −
      from induct have 1: derivation-ge (derivation-shift (drop n D') (length
u1) 0) (length w1)
       apply (subst drop-derivation-shift[symmetric])
       by blast
      have 2: derivation-ge (drop n D') (length u1)
       by (metis append-take-drop-id derivation-ge-D' derivation-ge-append)
      show ?thesis using 1 2 derivation-ge-shift-plus by blast
     qed
     have LeftDerivation (u1@(snd (snd d) @ u2)) (derivation-shift
         (take n (derivation-shift D' (length u1) 0)) 0 (length u1)) (u1@w1)
     using induct LeftDerivation-append-prefix is-word-u1 by blast
     then have der1: LeftDerivation (u1@(snd (snd d) @ u2))
        (derivation-shift (take n D') (length u1) (length u1)) (u1@w1)
      using take-derivation-shift derivation-shift-0-shift by auto
    have eq1: derivation-shift (take n D') (length u1) (length u1) = take n D'
      apply (subst derivation-ge-shift-simp[where i = length u1])
      apply auto
      by (metis append-take-drop-id derivation-ge-D' derivation-ge-append)
     from der1 eq1 have der2: LeftDerivation (u1@(snd (snd d) @ u2)) (take
n D') (u1@w1)
      by auto
     have eq2: take (Suc n) D = d#(take n D')
      by (simp add: D-split)
     have der3: LeftDerivation u (take (Suc n) D) (u1@w1)
      apply (simp add: eq2)
      apply (rule-tac x=u1@(snd (snd d) @ u2) in exI)
     by (metis Derives1-skip-suffix LeftDerives1-def append-assoc der2 fst-d-eq-i

        split-u split-x splits-at-def x)
     have is-prefix u1 w
      using LeftDerivation-implies-leftderives derives-word-is-prefix is-word-u1

        leftderives-implies-derives split-x x by blast
     then have eq3: u1 @ (w1@w2) = w
      apply (rule-tac append-dropped-prefix)
      apply (auto simp add: induct)
      done
     show ?case
      apply (rule-tac x=Suc n in exI)
      apply (rule-tac x=u1@w1 in exI)
      apply (rule-tac x=w2 in exI)
```

```
        apply auto
        apply (simp add: eq3)
        apply (simp add: der3)
        apply (simp add: D-split)
        apply (rule derivation-ge-D'-u1-w1)
        apply (simp add: D-split)
        using induct derivation-shift-0-shift drop-derivation-shift apply auto
        done
    qed
qed

lemma Derives1-terminals-stay:
  assumes Derives1: Derives1 u i r v
  assumes t-dom: t ∈ set u
  assumes terminal: is-terminal t
  shows t ∈ set v
proof −
  have ∃ α β N. splits-at u i α N β using Derives1 splits-at-ex by blast
  then obtain α β N where split-u: splits-at u i α N β by blast
  then have t ∈ set (α @ [N] @ β) using splits-at-combine t-dom by auto
  then have t-possible-locations: t ∈ set α ∨ t = N ∨ t ∈ set β by auto
   have is-nonterminal: is-nonterminal N using Derives1 Derives1-nonterminal
split-u by auto
  with t-possible-locations terminal have t-locations: t ∈ set α ∨ t ∈ set β
    using is-terminal-nonterminal by blast
 from Derives1 split-u have v = α @ (snd r) @ β by (simp add: splits-at-combine-dest)

  with t-locations show ?thesis by auto
qed

lemma Derivation-terminals-stay: Derivation u D v ⟹ t ∈ set u ⟹ is-terminal
t ⟹ t ∈ set v
proof (induct D arbitrary: u v)
  case Nil thus ?case by auto
next
  case (Cons d D)
  then have ∃ x. Derives1 u (fst d) (snd d) x ∧ Derivation x D v by auto
  then obtain x where x: Derives1 u (fst d) (snd d) x ∧ Derivation x D v by
auto
  show ?case using Cons Derives1-terminals-stay x by blast
qed

lemma Derivation-empty-no-terminals: Derivation u D [] ⟹ t ∈ set u ⟹ is-nonterminal
t
  by (metis Ball-set Derivation-implies-derives Derivation-terminals-stay
    derives-is-sentence is-sentence-def is-symbol-distinct list.pred-inject(1))

lemma mono-subset-elem: mono f ⟹ A ⊆ B ⟹ x ∈ f A ⟹ x ∈ f B using
mono-def by blast
```

lemma *wellformed-inc-dot*: *wellformed-item x* $\Longrightarrow$ *item-dot x* + *d* $\leq$ *length* (*item-rhs x*) $\Longrightarrow$
wellformed-item(*inc-dot d x*)
by (*simp add*: *inc-dot-def item-rhs-def wellformed-item-def*)

lemma *init-item-dot*[*simp*]: *item-dot* (*init-item r k*) = *0*
by (*simp add*: *init-item-def*)

lemma *init-item-rhs*[*simp*]: *item-rhs* (*init-item r k*) = *snd r*
by (*simp add*: *init-item-def item-rhs-def*)

lemma *init-item-β*[*simp*]: *item-β* (*init-item r k*) = *snd r*
by (*simp add*: *item-β-def*)

lemma *mono-π*: *mono* (*π k T*)
by (*simp add*: *π-regular regular-implies-mono*)

lemma *π-subset-elem-trans*:
assumes *Y*: *Y* $\subseteq$ *π k T X*
assumes *z*: *z* $\in$ *π k T Y*
shows *z* $\in$ *π k T X*
proof −
from *Y* **have** *π k T Y* $\subseteq$ *π k T* (*π k T X*) **by** (*simp add*: *monoD mono-π*)
then have *π k T Y* $\subseteq$ *π k T X* **using** *π-idempotent* **by** *blast*
with *z* **show** *?thesis* **using** *contra-subsetD* **by** *blast*
qed

lemma *inc-dot-origin*[*simp*]: *item-origin* (*inc-dot d x*) = *item-origin x*
by (*simp add*: *inc-dot-def*)

lemma *inc-dot-end*[*simp*]: *item-end* (*inc-dot d x*) = *item-end x*
by (*simp add*: *inc-dot-def*)

lemma *inc-dot-rhs*[*simp*]: *item-rhs* (*inc-dot d x*) = *item-rhs x*
by (*simp add*: *inc-dot-def item-rhs-def*)

lemma *inc-dot-dot*[*simp*]: *item-dot* (*inc-dot d x*) = *item-dot x* + *d*
by (*simp add*: *inc-dot-def*)

lemma *inc-dot-nonterminal*[*simp*]: *item-nonterminal* (*inc-dot d x*) = *item-nonterminal x*
by (*simp add*: *inc-dot-def item-nonterminal-def*)

lemma *Predict-subset-π*: *Predict k X* $\subseteq$ *π k T X*
proof −
have *setmonotone* (*π k T*)
by (*simp add*: *π-regular regular-implies-setmonotone*)
then have *s*: *X* $\subseteq$ *π k T X* **by** (*simp add*: *subset-setmonotone*)

```
  have mono (Predict k) by (simp add: Predict-regular regular-implies-mono)
  with s have Predict k X ⊆ Predict k (π k T X) by (simp add: monoD)
  then show Predict k X ⊆ π k T X by (simp add: Predict-π-fix)
qed

lemma Complete-subset-π: Complete k X ⊆ π k T X
proof −
  have setmonotone (π k T)
    by (simp add: π-regular regular-implies-setmonotone)
  then have s: X ⊆ π k T X by (simp add: subset-setmonotone)
  have mono (Complete k) by (simp add: Complete-regular regular-implies-mono)

  with s have Complete k X ⊆ Complete k (π k T X) by (simp add: monoD)
  then show Complete k X ⊆ π k T X by (simp add: Complete-π-fix)
qed

lemma inc-inc-dot[simp]: inc-dot a (inc-dot b x) = inc-dot (a + b) x
  by (simp add: inc-dot-def)

lemma thmD6-Left: wellformed-item x ⟹ item-β x = δ @ ω ⟹ item-end x =
k ⟹
  LeftDerivation δ D [] ⟹ inc-dot (length δ) x ∈ π k {} {x}
proof (induct length D arbitrary: x δ ω D rule: less-induct)
  case less
    have length δ = 0 ∨ length δ = 1 ∨ length δ ≥ 2 by arith
    then show ?case
    proof (induct rule: disjCases3)
      case 1
        then have δ = [] by auto
      then show ?case by (simp add: π-regular elem-setmonotone regular-implies-setmonotone)
    next
      case 2
        then have ∃ N. δ = [N]
          by (metis One-nat-def append-self-conv2 drop-all id-take-nth-drop
            le-numeral-extra(4) lessI take-0)
        then obtain N where N: δ = [N] by blast
        then have N ∈ set δ by auto
      then have is-nonterminal-N: is-nonterminal N using Derivation-empty-no-terminals

          LeftDerivation-implies-Derivation less.prems(4) by blast
        have D ≠ [] using LeftDerivation.elims(2) N less.prems(4) by blast
        then have ∃ e E. D = e#E using LeftDerivation.elims(2) less.prems(4)
by blast
        then obtain e E where eE: D = e#E by blast
        then have ∃ γ. LeftDerives1 δ (fst e) (snd e) γ ∧
         LeftDerivation γ E [] using LeftDerivation.simps(2) less.prems(4) by blast

        then obtain γ where γ: LeftDerives1 δ (fst e) (snd e) γ ∧ LeftDerivation
γ E [] by blast
```

with *N* **have** *γ-def*: *γ* = *snd* (*snd e*)
by (*metis 2.hyps Derives1-split LeftDerives1-def One-nat-def append-Cons*

append-Nil append-Nil2 leftmost-def length-0-conv less-nat-zero-code linorder-neqE-nat
list.inject not-less-eq)
have *next-symbol-x*: *next-symbol x* = *Some N*
using *N less.prems*(*1*) *less.prems*(*2*) *next-symbol-def next-symbol-starts-item-β*

wellformed-complete-item-β **by** *fastforce*
have *x-subset*: {*x*} ⊆ *π k* {} {*x*}
using *π-regular regular-implies-setmonotone subset-setmonotone* **by** *blast*
let *?y* = *init-item* (*snd e*) *k*
have *?y* ∈ *Predict k* {*x*}
apply (*simp add*: *Predict-def*)
apply (*rule disjI2*)
apply (*rule-tac x*=*fst* (*snd e*) **in** *exI*)
apply (*rule-tac x*=*snd* (*snd e*) **in** *exI*)
apply *auto*
using *Derives1-rule LeftDerives1-implies-Derives1 γ* **apply** *blast*
apply (*rule-tac x*=*x* **in** *exI*)
by (*metis* (*mono-tags*, *lifting*) *Derives1-split LeftDerives1-def N γ*
append.simps(*1*) *append.simps*(*2*) *bin-def is-nonterminal-N leftmost-cons-nonterminal*
leftmost-unique length-greater-0-conv less.prems(*3*) *less-nat-zero-code*
list.inject mem-Collect-eq next-symbol-x singletonI)
then have *y-dom*: *?y* ∈ *π k* {} {*x*} **using** *Predict-subset-π* **by** *blast*
let *?z* = *inc-dot* (*length γ*) *?y*
have *item-dot ?y* = *0* **and** *item-rhs ?y* = *γ* **by** (*auto simp add*: *γ-def*)
note *y-props* = *this*
then have *wellformed-y*: *wellformed-item ?y*
using *Derives1-rule LeftDerives1-implies-Derives1 γ less.prems*(*1*) *less.prems*(*3*)

wellformed-init-item wellformed-item-def **by** *blast*
with *y-props* **have** *wellformed-z*: *wellformed-item ?z* **by** (*simp add*: *wellformed-inc-dot*)
have *item-β-y*: *item-β ?y* = *γ* @ [] **using** *item-rhs-split y-props*(*2*) **by** *auto*
have *is-complete-z*: *is-complete ?z* **by** (*simp add*: *is-complete-def γ-def*)
have *?z* ∈ *π k* {} {*?y*}
apply (*rule less*(*1*)[**where** *D*=*E*])
apply (*auto simp add*: *eE wellformed-y γ*)
apply (*simp add*: *γ-def*)
done
with *y-dom* **have** *z-dom*: *?z* ∈ *π k* {} {*x*}
using *π-subset-elem-trans empty-subsetI insert-subset* **by** *blast*
let *?w* = *inc-dot* (*length δ*) *x*
have *?w* ∈ *Complete k* {*x*, *?z*}
apply (*simp add*: *Complete-def*)
apply (*rule-tac disjI2*)+

```
      apply (rule-tac x=x in exI)
      apply (auto simp add: 2)
      apply (simp add: inc-dot-def inc-item-def less)
      apply (rule-tac x=?z in exI)
      apply (auto simp add: bin-def less is-complete-z next-symbol-x)
     by (metis Derives1-split LeftDerives1-def N γ append-Cons append-self-conv2

        is-nonterminal-N leftmost-cons-nonterminal leftmost-unique length-0-conv
list.inject)
     then have ?w ∈ π k {} {x, ?z} using Complete-subset-π by blast
    then show ?case by (meson π-subset-elem-trans insert-subset x-subset z-dom)

  next
    case 3
     then have ∃ N α. δ = [N] @ α
     by (metis append-Cons append-Nil count-terminals.cases le0 le-0-eq list.size(3)

         numeral-le-iff semiring-norm(69))
     then obtain N α where δ-split: δ = [N] @ α by blast
     with 3 have α-nonempty: α ≠ []
      by (metis (full-types) One-nat-def Suc-eq-plus1 append-Nil2 impossible-Cons
length-Cons
        list.size(3) nat-1-add-1)
     have LeftDerivation ([N] @ α) D [] using δ-split less.prems(4) by blast
     from LeftDerivation-breakdown[OF this, simplified]
    obtain n where n: LeftDerivation [N] (take n D) [] ∧ LeftDerivation α (drop
n D) [] by blast
     let ?E = take n D
     from n have E: LeftDerivation [N] ?E [] by auto
     let ?F = drop n D
     from n have F: LeftDerivation α ?F [] by auto
     have length-add: length ?E + length ?F = length D by simp
     have ?E ≠ [] using E by force
     then have length-E-0: length ?E > 0 by auto
     have ?F ≠ [] using F α-nonempty by force
     then have length-F-0: length ?F > 0 by auto
     from length-add length-E-0 length-F-0
     have length ?E < length D ∧ length ?F < length D
        using add.commute nat-add-left-cancel-less nat-neq-iff not-add-less2 by
linarith
    then have length-E: length ?E < length D and length-F: length ?F < length
D by auto
     let ?y = inc-dot (length [N]) x
     have y-dom: ?y ∈ π k {} {x}
      apply (rule-tac less(1)[where D=?E and ω=α@ω])
      apply (rule length-E)
      by (auto simp add: less δ-split E)
    let ?z = inc-dot (length α) ?y
    have wellformed-y: wellformed-item ?y
```

```
    using δ-split is-complete-def less.prems(1) less.prems(2) wellformed-complete-item-β

      wellformed-inc-dot by fastforce
    have ?z ∈ π k {} {?y}
      apply (rule-tac less(1)[where D=?F and ω=ω])
      apply (rule length-F)
      apply (rule wellformed-y)
      apply (auto simp add: F less)
      by (metis δ-split add.commute append-assoc append-eq-conv-conj drop-drop
inc-dot-dot
        inc-dot-rhs item-β-def length-Cons less.prems(2) list.size(3))
    then have z-dom: ?z ∈ π k {} {x} using π-subset-elem-trans y-dom by blast

    have ?z = inc-dot (length δ) x by (simp add: δ-split)
    with z-dom show ?case by auto
  qed
qed

lemma derives-empty-implies-LeftDerivation: derives δ [] ⟹ ∃ D. LeftDerivation
δ D []
  using derives-implies-leftderives is-word-def leftderives-implies-LeftDerivation
    list.pred-inject(1) by blast

lemma thmD6: wellformed-item x ⟹ item-β x = δ @ ω ⟹ item-end x = k ⟹

  derives δ [] ⟹ inc-dot (length δ) x ∈ π k {} {x}
using derives-empty-implies-LeftDerivation thmD6-Left by blast

end

end
theory TheoremD7
imports TheoremD6
begin

context LocalLexing begin

lemma Derives1-keep-first-terminal: Derives1 (x#u) i r (y#v) ⟹ is-terminal x
⟹ x = y
  by (metis Derives1-leftmost Derives1-take leftmost-cons-terminal list.sel(1) not-le
take-Cons')

lemma Derives1-nonterminal-head:
  assumes Derives1 u i r (N#v)
  assumes is-nonterminal N
  shows ∃ u' M. u = M#u' ∧ is-nonterminal M
proof −
  from assms have nonempty-u: u ≠ []
    by (metis Derives1-bound less-nat-zero-code list.size(3))
```

```
  have ∃ u' M. u = M#u'
    using count-terminals.cases nonempty-u by blast
  then obtain u' M where split-u: u = M#u' by blast
  have is-sentence-u: is-sentence u using assms
    using Derives1-sentence1 by blast
  then have is-terminal M ∨ is-nonterminal M
    using is-sentence-cons is-symbol-distinct split-u by blast
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      have is-terminal N
        using 1.hyps Derives1-keep-first-terminal
        assms(1) split-u by blast
      with assms have False using is-terminal-nonterminal by blast
      then show ?case by blast
  next
    case 2 with split-u show ?case by blast
  qed
qed

lemma sentence-starts-with-nonterminal:
  assumes is-nonterminal N
  assumes derives u []
  shows ∃ X r. u@[N] = X#r ∧ is-nonterminal X
proof (cases u = [])
  case True thus ?thesis using assms(1) by blast
next
  case False
  then have ∃ X r. u = X#r using count-terminals.cases by blast
  then obtain X r where Xr: u = X#r by blast
  then have is-nonterminal X
    by (metis False assms(2) count-terminals.simps derives-count-terminals-leq
      derives-is-sentence is-sentence-cons is-symbol-distinct not-le zero-less-Suc)
  with Xr show ?thesis by auto
qed

lemma Derives1-nonterminal-head':
  assumes Derives1 u i r (v1@[N]@v2)
  assumes is-nonterminal N
  assumes derives v1 []
  shows ∃ u' M. u = M#u' ∧ is-nonterminal M
proof −
  from sentence-starts-with-nonterminal[OF assms(2,3)]
  obtain X r where v1 @ [N] = X # r ∧ is-nonterminal X by blast
  then show ?thesis
    by (metis Derives1-nonterminal-head append-Cons append-assoc assms(1))
qed

lemma thmD7-helper:
```

```
  assumes LeftDerivation [𝔖] D (N#v)
  assumes is-nonterminal N
  assumes 𝔖 ≠ N
  shows ∃ n M a a1 a2 w. n < length D ∧ (M, a) ∈ ℜ ∧ LeftDerivation [𝔖] (take
n D) (M#w) ∧
    a = a1 @ [N] @ a2 ∧ derives a1 []
proof −
  have ∃ n u v. LeftDerivation [𝔖] (take n D) (u@[N]@v) ∧ derives u []
    apply (rule-tac x=length D in exI)
    apply (rule-tac x=[] in exI)
    apply (rule-tac x=v in exI)
    using assms by simp
  then show ?thesis
  proof (induct rule: ex-minimal-witness)
    case (Minimal K)
      have nonzero-K: K ≠ 0
      proof (cases K = 0)
        case True
          with Minimal have ∃ u v. [𝔖] = u@[N]@v
            using LeftDerivation.simps(1) take-0 by auto
          with assms have False by (simp add: Cons-eq-append-conv)
          then show ?thesis by simp
      next
        case False
          then show ?thesis by arith
      qed
      from Minimal(1)
     obtain u v where uv: LeftDerivation [𝔖] (take K D) (u @ [N] @ v) ∧ derives
u [] by blast
      from nonzero-K have take K D ≠ []
       using Minimal.hyps(2) less-nat-zero-code nat-neq-iff take-eq-Nil uv by force

      then have ∃ E e. (take K D) = E@[e] using rev-exhaust by blast
      then obtain E e where Ee: take K D = E@[e] by blast
       with uv have ∃ x. LeftDerivation [𝔖] E x ∧ LeftDerives1 x (fst e) (snd e)
(u @ [N] @ v)
        by (simp add: LeftDerivation-append)
     then obtain x where x: LeftDerivation [𝔖] E x ∧ LeftDerives1 x (fst e) (snd
e) (u @ [N] @ v)
        by blast
      then have ∃ w M. x = M#w ∧ is-nonterminal M
        using Derives1-nonterminal-head' LeftDerives1-implies-Derives1 assms(2)
uv by blast
    then obtain w M where split-x: x = M#w and is-nonterminal-M: is-nonterminal
M by blast
      from Ee nonzero-K have E: E = take (K − 1) D
      by (metis Minimal.hyps(2) butlast-snoc butlast-take dual-order.strict-implies-order

          le-less-linear take-all uv)
```

```
    have leftmost (fst e) (M#w) using x LeftDerives1-def split-x by blast
    with is-nonterminal-M have fst-e: fst e = 0
      by (simp add: leftmost-cons-nonterminal leftmost-unique)
    have Derives1: Derives1 x 0 (snd e) (u @ [N] @ v)
      using LeftDerives1-implies-Derives1 fst-e x by auto
    have x-splits-at: splits-at x 0 [] M w
      by (simp add: split-x splits-at-def)
    from Derives1 x-splits-at
    have pq: ∃ p q. u = [] @ p ∧ v = q @ w ∧ snd (snd e) = p @ [N] @ q
    proof (induct rule: Derives1-X-is-part-of-rule)
      case (Suffix α) thus ?case by blast
    next
      case (Prefix β)
        then have derives-β: derives β []
         using Derives1-implies-derives1 derives1-implies-derives derives-trans uv
by blast
        with Prefix(1) x Minimal E nonzero-K show False
          by (meson diff-less less-nat-zero-code less-one nat-neq-iff)
    qed
    from this[simplified] obtain q where q: v = q @ w ∧ snd (snd e) = u @ N
# q by blast
    have M-def: fst (snd e) = M
      using Derives1 Derives1-nonterminal x-splits-at by blast
    show ?case
      apply (rule-tac x=K−1 in exI)
      apply (rule-tac x=M in exI)
      apply (rule-tac x=snd (snd e) in exI)
      apply (rule-tac x=u in exI)
      apply (rule-tac x=q in exI)
      apply (rule-tac x=w in exI)
        by (metis Derives1 Derives1-rule E Ee M-def One-nat-def Suc-pred pq
append-Nil
          append-same-eq dual-order.strict-implies-order le-less-linear nonzero-K
not-Cons-self2
        not-gr0 not-less-eq prod.collapse q self-append-conv split-x take-all uv x)
  qed
qed

lemma head-of-item-β-is-next-symbol:
  wellformed-item x ⟹ item-β x = t#δ ⟹ next-symbol x = Some t
  using next-symbol-def next-symbol-starts-item-β wellformed-complete-item-β by
fastforce

lemma next-symbol-predicts: next-symbol x = Some N ⟹ (N, a) ∈ ℜ ⟹ k =
item-end x ⟹
  init-item (N, a) k ∈ Predict k {x}
using Predict-def bin-def by auto

lemma thmD7-LeftDerivation: LeftDerivation [𝔖] D (N#γ) ⟹ is-nonterminal N
```

```
⟹ (N, α) ∈ ℜ ⟹
  init-item (N, α) 0 ∈ π 0 {} Init
proof (induct length D arbitrary: D N γ α rule: less-induct)
  case less
    let ?trivial = 𝔖 = N
    have ?trivial ∨ ¬ ?trivial by blast
    then show ?case
    proof (induct rule: disjCases2)
      case 1
        then have init-item (N, α) 0 ∈ Init
          apply (subst Init-def)
          by (auto simp add: less)
        then show ?case
            by (meson π-regular contra-subsetD regular-implies-setmonotone sub-
set-setmonotone)
    next
      case 2
        from thmD7-helper[OF less(2) less(3) 2]
        obtain n M a a1 a2 w where n < length D and  (M, a) ∈ ℜ and
           LeftDerivation [𝔖] (take n D) (M # w) and a = a1 @ [N] @ a2 and
derives a1 []
          by blast
        note M = this
        let ?x = init-item (M, a) 0
        have x-dom: ?x ∈ π 0 {} Init
          apply (rule less(1)[OF - M(3) - M(2)])
          using M(1) apply simp
          using M(2) by simp
        have wellformed-x: wellformed-item ?x by (simp add: M(2))
        let ?y = inc-dot (length a1) ?x
        have ?y ∈ π 0 {} {?x}
          apply (rule thmD6[where ω=[N] @ a2])
          using wellformed-x by (auto simp add: M)
        with x-dom have y-dom: ?y ∈ π 0 {} Init
          using π-subset-elem-trans empty-subsetI insert-subset by blast
        have wellformed-y: wellformed-item ?y
          using M(4) wellformed-inc-dot wellformed-x by auto
        have item-β ?y = N#a2 by (simp add: M(4) item-β-def)
        then have next-symbol-y: next-symbol ?y = Some N
          by (simp add: head-of-item-β-is-next-symbol wellformed-y)
        let ?z = init-item (N, α) 0
        have ?z ∈ Predict 0 {?y}
          by (simp add: less.prems(3) next-symbol-predicts next-symbol-y)
        then have ?z ∈ π 0 {} {?y} using Predict-subset-π by auto
        with y-dom show ?z ∈ π 0 {} Init
          using π-subset-elem-trans empty-subsetI insert-subset by blast
    qed
qed
```

theorem *thmD7*: *is-derivation* $(N\#\gamma) \Longrightarrow$ *is-nonterminal* $N \Longrightarrow (N, \alpha) \in \mathfrak{R} \Longrightarrow$

init-item (N, α) *0* $\in \pi$ *0* $\{\}$ *Init*
by (*metis* $\mathcal{L}_P$*-is-word derives-implies-leftderives-cons empty-in-*$\mathcal{L}_P$ *is-derivation-def*

leftderives-implies-LeftDerivation self-append-conv2 thmD7-LeftDerivation)

end

end
theory *TheoremD8*
imports *TheoremD7*
begin

context *LocalLexing* **begin**

lemma *wellformed-tokens-empty-path*[*simp*]: *wellformed-tokens* []
by (*simp add*: *wellformed-tokens-def*)

lemma $\mathcal{P}$*-0-0-Gen*: *Gen* $(\mathcal{P}$ *0 0*$) = \{$ *x . wellformed-item x* $\wedge$ *item-origin x = 0* $\wedge$ *item-end x = 0* $\wedge$
derives (*item-*α *x*) [] $\wedge$ ($\exists$ γ. *is-derivation* ([*item-nonterminal x*] @ γ)) $\}$
by (*auto simp add*: *Gen-def pvalid-def*)

lemma *Init-subset-Gen*: *Init* $\subseteq$ *Gen* $(\mathcal{P}$ *0 0*$)$
apply (*subst* $\mathcal{P}$*-0-0-Gen*)
apply (*auto simp add*: *Init-def*)
apply (*rule-tac x*=[] **in** *exI*)
apply (*simp add*: *is-derivation-def*)
done

lemma $\mathcal{J}$*-0-0-subset-Gen*: $\mathcal{J}$ *0 0* $\subseteq$ *Gen* $(\mathcal{P}$ *0 0*$)$
apply (*simp only*: $\mathcal{J}$*.simps*)
apply (*rule-tac thmD5*)
apply (*rule Init-subset-Gen*)
by *auto*

lemma *inc-dot-rule*[*simp*]: *item-rule* (*inc-dot d x*) = *item-rule x*
by (*simp add*: *inc-dot-def*)

lemma *init-item-rule*[*simp*]: *item-rule* (*init-item r k*) = *r*
by (*simp add*: *init-item-def*)

lemma *item-dot-is-*α*-length*: *wellformed-item x* $\Longrightarrow$ *item-dot x* = *length* (*item-*α *x*)
apply (*simp add*: *item-*α*-def*)
by (*simp add*: *min-absorb2 wellformed-item-def*)

lemma *Gen-subset-*$\mathcal{J}$*-0-0-helper*:

```
  assumes wellformed-item x
  assumes item-origin x = 0
  assumes item-end x = 0
  assumes derives (item-α x) []
  assumes is-derivation (item-nonterminal x # γ)
  shows x ∈ π 0 {} Init
proof −
  let ?y = init-item (item-nonterminal x, item-rhs x) 0
  have y-dom: ?y ∈ π 0 {} Init
    apply (rule-tac thmD7)
    using assms apply auto
    using is-nonterminal-item-nonterminal apply blast
    by (simp add: item-nonterminal-def item-rhs-def wellformed-item-def)
  let ?x = inc-dot (length (item-α x)) ?y
  have x1: item-rule x = item-rule ?x
    apply (simp)
    by (simp add: item-nonterminal-def item-rhs-def)
  have x2: item-dot x = item-dot ?x
    apply simp
    by (simp add: assms(1) item-dot-is-α-length)
  have x3: item-origin x = item-origin ?x
    using assms by auto
  have x4: item-end x = item-end ?x
    using assms by auto
  from x1 x2 x3 x4 have x-is-inc: x = ?x using item.expand by blast
  have wellformed-item-y: wellformed-item ?y
   using assms(1) item-nonterminal-def item-rhs-def wellformed-item-def by auto
  have x ∈ π 0 {} {?y}
    apply (subst x-is-inc)
    apply (rule-tac thmD6)
    apply (simp add: wellformed-item-y)
    apply (simp add: item-rhs-split)
    apply simp
    using assms apply simp
    done
  with y-dom show ?thesis
    using π-subset-elem-trans empty-subsetI insert-subset by blast
qed

lemma Gen-subset-J-0-0: Gen (P 0 0) ⊆ J 0 0
  apply (subst P-0-0-Gen)
  apply auto
  using Gen-subset-J-0-0-helper by blast

theorem thmD8: J 0 0 = Gen (P 0 0)
  using Gen-subset-J-0-0 J-0-0-subset-Gen by blast

end
```

end
theory *TheoremD9*
imports *TheoremD8*
begin

context *LocalLexing* **begin**

definition *items-le* :: *nat* ⇒ *items* ⇒ *items*
where
items-le k I = { *x* . *x* ∈ *I* ∧ *item-end x* ≤ *k* }

definition *items-eq* :: *nat* ⇒ *items* ⇒ *items*
where
items-eq k I = { *x* . *x* ∈ *I* ∧ *item-end x* = *k* }

definition *paths-le* :: *nat* ⇒ *tokens set* ⇒ *tokens set*
where
paths-le k P = { *p* . *p* ∈ *P* ∧ *charslength p* ≤ *k* }

definition *paths-eq* :: *nat* ⇒ *tokens set* ⇒ *tokens set*
where
paths-eq k P = { *p* . *p* ∈ *P* ∧ *charslength p* = *k* }

lemma *items-le-pointwise*: *pointwise* (*items-le k*)
by (*auto simp add*: *pointwise-def items-le-def*)

lemma *items-le-is-filter*: *items-le k I* ⊆ *I*
by (*auto simp add*: *items-le-def*)

lemma *items-eq-pointwise*: *pointwise* (*items-eq k*)
by (*auto simp add*: *pointwise-def items-eq-def*)

lemma *items-eq-is-filter*: *items-eq k I* ⊆ *I*
by (*auto simp add*: *items-eq-def*)

lemma *paths-le-pointwise*: *pointwise* (*paths-le k*)
by (*auto simp add*: *pointwise-def paths-le-def*)

lemma *paths-le-continuous*: *continuous* (*paths-le k*)
by (*simp add*: *paths-le-pointwise pointbased-implies-continuous pointwise-implies-pointbased*)

lemma *paths-le-mono*: *mono* (*paths-le k*)
by (*simp add*: *continuous-imp-mono paths-le-continuous*)

lemma *paths-le-is-filter*: *paths-le k P* ⊆ *P*
by (*auto simp add*: *paths-le-def*)

lemma *paths-eq-pointwise*: *pointwise* (*paths-eq k*)
by (*auto simp add*: *pointwise-def paths-eq-def*)

lemma *paths-eq-is-filter*: *paths-eq k P* $\subseteq$ *P*
by (*auto simp add*: *paths-eq-def*)

lemma *Predict-item-end*: *x* $\in$ *Predict k Y* $\Longrightarrow$ *item-end x* = *k* $\vee$ *x* $\in$ *Y*
using *Predict-def* **by** *auto*

lemma *Complete-item-end*: *x* $\in$ *Complete k Y* $\Longrightarrow$ *item-end x* = *k* $\vee$ *x* $\in$ *Y*
using *Complete-def* **by** *auto*

lemma $\mathcal{J}$*-0-0-item-end*: *x* $\in$ $\mathcal{J}$ *0 0* $\Longrightarrow$ *item-end x* = *0*
apply (*simp add*: π*-def*)
proof (*induct rule*: *limit-induct*)
case (*Init x*) **thus** *?case* **by** (*auto simp add*: *Init-def*)
next
case (*Iterate x Y*)
then have *x* $\in$ *Complete 0* (*Predict 0 Y*) **by** (*simp add*: *Scan-empty*)
then have *item-end x* = *0* $\vee$ *x* $\in$ *Predict 0 Y* **using** *Complete-item-end* **by** *blast*
then have *item-end x* = *0* $\vee$ *x* $\in$ *Y* **using** *Predict-item-end* **by** *blast*
then show *?case* **using** *Iterate* **by** *blast*
qed

lemma *items-le-*$\mathcal{J}$*-0-0*: *items-le 0* ($\mathcal{J}$ *0 0*) = $\mathcal{J}$ *0 0*
using *LocalLexing.*$\mathcal{J}$*-0-0-item-end LocalLexing.items-le-def LocalLexing-axioms* **by** *blast*

lemma *paths-le-*$\mathcal{P}$*-0-0*: *paths-le 0* ($\mathcal{P}$ *0 0*) = $\mathcal{P}$ *0 0*
by (*auto simp add*: *paths-le-def*)

definition *empty-tokens* :: *token set* $\Rightarrow$ *token set*
where
empty-tokens T = { *t* . *t* $\in$ *T* $\wedge$ *chars-of-token t* = [] }

lemma *items-le-Predict*: *items-le k* (*Predict k I*) = *Predict k* (*items-le k I*)
by (*auto simp add*: *items-le-def Predict-def bin-def*)

lemma *items-le-Complete*:
wellformed-items I $\Longrightarrow$ *items-le k* (*Complete k I*) = *Complete k* (*items-le k I*)
by (*auto simp add*: *items-le-def Complete-def bin-def is-complete-def wellformed-items-def*

wellformed-item-def)

lemma *items-le-Scan*:
items-le k (*Scan T k I*) = *Scan* (*empty-tokens T*) *k* (*items-le k I*)
by (*auto simp add*: *items-le-def Scan-def bin-def empty-tokens-def*)

lemma *wellformed-items-Gen*: *wellformed-items* (*Gen P*)
using *Gen-implies-pvalid pvalid-def wellformed-items-def* **by** *blast*

lemma *wellformed-$\mathcal{J}$-0-0*: *wellformed-items* ($\mathcal{J}$ *0 0*)
using *thmD8 wellformed-items-Gen* **by** *auto*

lemma *wellformed-items-Predict*:
wellformed-items I $\Longrightarrow$ *wellformed-items* (*Predict k I*)
by (*auto simp add*: *wellformed-items-def wellformed-item-def Predict-def bin-def*)

lemma *wellformed-items-Complete*:
wellformed-items I $\Longrightarrow$ *wellformed-items* (*Complete k I*)
apply (*auto simp add*: *wellformed-items-def wellformed-item-def Complete-def bin-def*)
apply (*metis dual-order.trans*)
using *is-complete-def next-symbol-not-complete not-less-eq-eq* **by** *blast*

lemma *$\mathcal{X}$-length-bound*: (*t*, *c*) $\in$ $\mathcal{X}$ *k* $\Longrightarrow$ *k* + *length c* $\leq$ *length Doc*
using *$\mathcal{X}$-is-prefix is-prefix-length not-le* **by** *fastforce*

lemma *wellformed-items-Scan*:
wellformed-items I $\Longrightarrow$ *T* $\subseteq$ $\mathcal{X}$ *k* $\Longrightarrow$ *wellformed-items* (*Scan T k I*)
apply (*auto simp add*: *wellformed-items-def wellformed-item-def Scan-def bin-def $\mathcal{X}$-length-bound*)
using *is-complete-def next-symbol-not-complete not-less-eq-eq* **by** *blast*

lemma *wellformed-items-π*:
assumes *wellformed-items I*
assumes *T* $\subseteq$ $\mathcal{X}$ *k*
shows *wellformed-items* (π *k T I*)
proof −
{
fix *x* :: *item*
have *x* $\in$ π *k T I* $\Longrightarrow$ *wellformed-item x*
proof (*simp add*: *π-def*, *induct rule*: *limit-induct*)
case (*Init x*) **thus** *?case* **using** *assms*(*1*) **by** (*simp add*: *wellformed-items-def*)

next
case (*Iterate x Y*)
have *wellformed-items Y* **by** (*simp add*: *Iterate.hyps*(*1*) *wellformed-items-def*)
then have *wellformed-items* (*Scan T k* (*Complete k* (*Predict k Y*)))
by (*simp add*: *assms*(*2*) *wellformed-items-Complete wellformed-items-Predict*

wellformed-items-Scan)
then show *?case* **by** (*simp add*: *Iterate.hyps*(*2*) *wellformed-items-def*)
qed
}
then show *?thesis* **using** *wellformed-items-def* **by** *auto*
qed

lemma *$\mathcal{J}$-subset-Suc-u*: $\mathcal{J}$ *k u* $\subseteq$ $\mathcal{J}$ *k* (*Suc u*)

by (*metis Complete-π-fix Complete-subset-π* $\mathcal{J}$*.simps*(*1*) $\mathcal{J}$*.simps*(*2*) $\mathcal{J}$*.simps*(*3*) *not0-implies-Suc*)

lemma *mono-TokensAt*: *mono* (*TokensAt k*)
by (*auto simp add*: *mono-def TokensAt-def bin-def*)

lemma $\mathcal{T}$*-subset-TokensAt*: $\mathcal{T}$ *k u* ⊆ *TokensAt k* ($\mathcal{J}$ *k u*)
proof (*induct u*)
case *0* **thus** *?case* **by** *simp*
next
case (*Suc u*)
have *1*: *Tokens k* ($\mathcal{T}$ *k u*) ($\mathcal{J}$ *k u*) = *Sel* ($\mathcal{T}$ *k u*) (*TokensAt k* ($\mathcal{J}$ *k u*))
by (*simp add*: *Tokens-def*)
have *2*: *Sel* ($\mathcal{T}$ *k u*) (*TokensAt k* ($\mathcal{J}$ *k u*)) ⊆ *TokensAt k* ($\mathcal{J}$ *k u*)
by (*simp add*: *Sel-upper-bound Suc.hyps*)
have $\mathcal{T}$ *k* (*Suc u*) ⊆ *TokensAt k* ($\mathcal{J}$ *k u*)
by (*simp add*: *1 2*)
then show *?case*
by (*meson* $\mathcal{J}$*-subset-Suc-u mono-TokensAt mono-subset-elem subset-iff*)
qed

lemma *TokensAt-subset-*$\mathcal{X}$: *TokensAt k I* ⊆ $\mathcal{X}$ *k*
using *TokensAt-def* $\mathcal{X}$*-def is-terminal-def* **by** *auto*

lemma *wellformed-items-*$\mathcal{J}$*-induct-u*:
assumes *wellformed-items* ($\mathcal{J}$ *k u*)
shows *wellformed-items* ($\mathcal{J}$ *k* (*Suc u*))
proof −
{
fix *x* :: *item*
have *x* ∈ $\mathcal{J}$ *k* (*Suc u*) ⟹ *wellformed-item x*
proof (*simp add*: *π-def*, *induct rule*: *limit-induct*)
case (*Init x*)
with *assms* **show** *?case* **by** (*auto simp add*: *wellformed-items-def*)
next
case (*Iterate p Y*)
from *Iterate*(*1*) **have** *wellformed-Y*: *wellformed-items Y*
by (*auto simp add*: *wellformed-items-def*)
then have *wellformed-items* (*Complete k* (*Predict k Y*))
by (*simp add*: *wellformed-items-Complete wellformed-items-Predict*)
then have *wellformed-items* (*Scan* (*Tokens k* ($\mathcal{T}$ *k u*) ($\mathcal{J}$ *k u*)) *k* (*Complete k* (*Predict k Y*)))
apply (*rule-tac wellformed-items-Scan*)
apply *simp*
apply (*simp add*: *Tokens-def*)
by (*meson Sel-upper-bound TokensAt-subset-*$\mathcal{X}$ $\mathcal{T}$*-subset-TokensAt subset-trans*)
then show *?case*
using *Iterate.hyps*(*2*) *wellformed-items-def* **by** *blast*

```
    qed
  }
  then show ?thesis
    using wellformed-items-def by blast
qed

lemma wellformed-items-𝒥-k-u-if-0: wellformed-items (𝒥 k 0) ⟹ wellformed-items
(𝒥 k u)
  apply (induct u)
  apply (simp)
  using wellformed-items-𝒥-induct-u by blast

lemma wellformed-items-natUnion: (⋀ k. wellformed-items (I k)) ⟹ wellformed-items
(natUnion I)
  by (auto simp add: natUnion-def wellformed-items-def)

lemma wellformed-items-ℐ-k-if-0: wellformed-items (𝒥 k 0) ⟹ wellformed-items
(ℐ k)
  apply (simp)
  apply (rule wellformed-items-natUnion)
  using wellformed-items-𝒥-k-u-if-0 by blast

lemma wellformed-items-𝒥-ℐ: wellformed-items (𝒥 k u) ∧ wellformed-items (ℐ k)
proof (induct k arbitrary: u)
  case 0
    show ?case
     using wellformed-𝒥-0-0 wellformed-items-ℐ-k-if-0 wellformed-items-𝒥-k-u-if-0
by blast
next
  case (Suc k)
    have 0: wellformed-items (𝒥 (Suc k) 0)
      apply simp
      using Suc.hyps wellformed-items-π by auto
    then show ?case
      using wellformed-items-ℐ-k-if-0 wellformed-items-𝒥-k-u-if-0 by blast
qed

lemma wellformed-items-𝒥: wellformed-items (𝒥 k u)
by (simp add: wellformed-items-𝒥-ℐ)

lemma wellformed-items-ℐ: wellformed-items (ℐ k)
using wellformed-items-𝒥-ℐ by blast

lemma funpower-consume-function:
  assumes law: ⋀ X. P X ⟹ f (g X) = h (f X) ∧ P (g X)
  shows P I ⟹ P (funpower g n I) ∧ f (funpower g n I) = funpower h n (f I)
proof (induct n)
  case 0
  then show ?case by simp
```

```
next
  case (Suc n)
  then have S1: P (funpower g n I) and S2: f (funpower g n I) = funpower h n
(f I)
    by auto
  have law1: ⋀ X. P X ⟹ f (g X) = h (f X) using law by auto
  have law2: ⋀ X. P X ⟹ P (g X) using law by auto
  show ?case
    apply simp
    apply (subst law1[where X=funpower g n I])
    apply (simp add: S1)
    apply (subst S2)
    apply auto
    apply (rule law2)
    apply (simp add: S1)
    done
qed

lemma limit-consume-function:
  assumes continuous: continuous f
  assumes law: ⋀ X. P X ⟹ f (g X) = h (f X) ∧ P (g X)
  assumes setmonotone: setmonotone g
  shows P I ⟹ f (limit g I) = limit h (f I)
proof −
  have 1: f (limit g I) = f (natUnion (λn. funpower g n I))
    by (simp add: limit-def)
 have chain (λn. funpower g n I) by (simp add: setmonotone setmonotone-implies-chain-funpower)
  from continuous-apply[OF continuous this]
  have swap: f (natUnion (λn. funpower g n I)) = natUnion (f ∘ (λn. funpower
g n I)) by blast
  have f o (λn. funpower g n I) = (λ n. f (funpower g n I)) by auto
  also have P I ⟹ (λ n. f (funpower g n I)) = (λ n. funpower h n (f I))
    by (metis funpower-consume-function[where P=P and f=f and g=g and
h=h, OF law, simplified])
  ultimately have P I ⟹ f o (λn. funpower g n I) = (λ n. funpower h n (f I))
by auto
  with swap have 2: P I ⟹ f (natUnion (λn. funpower g n I)) = natUnion (λ
n. funpower h n (f I))
    by auto
  have 3: natUnion (λ n. funpower h n (f I)) = limit h (f I)
    by (simp add: limit-def)
  assume P I
  with 1 2 3 show ?thesis by auto
qed

lemma items-le-π-swap:
  assumes wellformed-I: wellformed-items I
  assumes T: T ⊆ 𝒳 k
  shows items-le k (π k T I) = π k (empty-tokens T) (items-le k I)
```

```
proof −
  let ?g = (Scan T k) o (Complete k) o (Predict k)
  let ?h = (Scan (empty-tokens T) k) o (Complete k) o (Predict k)
  have law1: ⋀ I. wellformed-items I ⟹ items-le k (?g I) = ?h (items-le k I)
   using LocalLexing.wellformed-items-Predict LocalLexing-axioms items-le-Complete

      items-le-Predict items-le-Scan by auto
  have law2: ⋀ I. wellformed-items I ⟹ wellformed-items (?g I)
     by (simp add: T wellformed-items-Complete wellformed-items-Predict well-
formed-items-Scan)
  show ?thesis
    apply (subst π-functional)
    apply (subst limit-consume-function[where P=wellformed-items and h=?h])
   apply (simp add: items-le-pointwise pointbased-implies-continuous pointwise-implies-pointbased)
    using law1 law2 apply blast
    apply (simp add: π-step-regular regular-implies-setmonotone)
    apply (rule wellformed-I)
    apply (subst π-functional)
    apply blast
    done
qed

lemma items-le-idempotent: items-le k (items-le k I) = items-le k I
  using items-le-def by auto

lemma paths-le-idempotent: paths-le k (paths-le k P) = paths-le k P
  using paths-le-def by auto

lemma items-le-fix-D:
  assumes items-le-fix: items-le k I = I
  assumes x-dom: x ∈ I
  shows item-end x ≤ k
using items-le-def items-le-fix x-dom by blast

lemma remove-paths-le-in-subset-Gen:
  assumes items-le k I = I
  assumes I ⊆ Gen P
  shows I ⊆ Gen (paths-le k P)
proof −
  {
    fix x :: item
    assume x-dom: x ∈ I
    then have x-item-end:  item-end x ≤ k using assms items-le-fix-D by auto
    have x ∈ Gen P using assms x-dom by auto
   then obtain p where p: p ∈ P ∧ pvalid p x using Gen-implies-pvalid by blast

    have charslength-p: charslength p ≤ k using p pvalid-item-end x-item-end by
auto
    then have p ∈ paths-le k P by (simp add: p paths-le-def)
```

```
    then have x ∈ Gen (paths-le k P) using Gen-def p by blast
  }
  then show ?thesis by blast
qed

lemma mono-Gen: mono Gen
  by (auto simp add: mono-def Gen-def)

lemma empty-tokens-idempotent: empty-tokens (empty-tokens T) = empty-tokens
T
  by (auto simp add: empty-tokens-def)

lemma empty-tokens-is-filter: empty-tokens T ⊆ T
  by (auto simp add: empty-tokens-def)

lemma items-le-paths-le: items-le k (Gen P) = Gen (paths-le k P)
 using LocalLexing.Gen-def LocalLexing.items-le-def LocalLexing-axioms paths-le-def

  pvalid-item-end by auto

lemma bin-items-le[symmetric]: bin I k = bin (items-le k I) k
  by (auto simp add: bin-def items-le-def)

lemma TokensAt-items-le[symmetric]: TokensAt k I = TokensAt k (items-le k I)
  using TokensAt-def bin-items-le by blast

lemma by-length-paths-le[symmetric]: by-length k P = by-length k (paths-le k P)
  using by-length.simps paths-le-def by auto

lemma 𝒲-paths-le[symmetric]: 𝒲 P k = 𝒲 (paths-le k P) k
  using 𝒲-def by-length-paths-le by blast

theorem 𝒯-equals-𝒵-induct-step:
  assumes induct: items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k u))
  assumes induct-tokens: 𝒯 k u = 𝒵 k u
  shows 𝒯 k (Suc u) = 𝒵 k (Suc u)
proof −
  have TokensAt k (𝒥 k u) = TokensAt k (items-le k (𝒥 k u))
    using TokensAt-items-le by blast
  also have TokensAt k (items-le k (𝒥 k u)) = TokensAt k (Gen (paths-le k (𝒫 k
u)))
    using induct by auto
  ultimately have TokensAt-le: TokensAt k (𝒥 k u) = TokensAt k (Gen (paths-le
k (𝒫 k u)))
    by auto
  have TokensAt k (𝒥 k u) = 𝒲 (𝒫 k u) k
    apply (subst TokensAt-le)
    apply (subst 𝒲-paths-le[symmetric])
    apply (rule-tac thmD4[symmetric])
```

```
    using 𝔓-covers-𝒫 paths-le-is-filter by blast
  then show ?thesis
    by (simp add: induct-tokens Tokens-def 𝒴-def)
qed

theorem thmD9:
  assumes induct: items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k u))
  assumes induct-tokens: 𝒯 k u = 𝒵 k u
  assumes k: k ≤ length Doc
  shows items-le k (𝒥 k (Suc u)) ⊆ Gen (paths-le k (𝒫 k (Suc u)))
proof −
  have t1: items-le k (𝒥 k (Suc u)) = items-le k (π k (𝒯 k (Suc u)) (𝒥 k u))
    by auto
  have t2: items-le k (π k (𝒯 k (Suc u)) (𝒥 k u)) =
    π k (empty-tokens (𝒯 k (Suc u))) (items-le k (𝒥 k u))
    apply (subst items-le-π-swap)
    apply (simp add: wellformed-items-𝒥)
    using TokensAt-subset-𝒳 𝒯-subset-TokensAt apply blast
    by blast
  have t3: π k (empty-tokens (𝒯 k (Suc u))) (items-le k (𝒥 k u)) =
    π k (empty-tokens (𝒯 k (Suc u))) (Gen (paths-le k (𝒫 k u)))
    using induct by auto
  have 𝒫-subset: 𝒫 k u ⊆ 𝒫 k (Suc u) using subset-𝒫Suc by blast
  then have paths-le k (𝒫 k u) ⊆ paths-le k (𝒫 k (Suc u))
    by (simp add: mono-subset-elem paths-le-mono subsetI)
  with mono-Gen have Gen (paths-le k (𝒫 k u)) ⊆ Gen (paths-le k (𝒫 k (Suc u)))
    by (simp add: mono-subset-elem subsetI)
  then have t4: π k (empty-tokens (𝒯 k (Suc u))) (Gen (paths-le k (𝒫 k u))) ⊆
    π k (empty-tokens (𝒯 k (Suc u))) (Gen (paths-le k (𝒫 k (Suc u))))
    by (rule monoD[OF mono-π])
  have 𝒯-eq-𝒵: 𝒯 k (Suc u) = 𝒵 k (Suc u)
    using 𝒯-equals-𝒵-induct-step assms(1) induct-tokens by blast
  have t5: π k (empty-tokens (𝒯 k (Suc u))) (Gen (paths-le k (𝒫 k (Suc u)))) ⊆
    Gen (paths-le k (𝒫 k (Suc u)))
    apply (rule-tac remove-paths-le-in-subset-Gen)
    apply (subst items-le-π-swap)
    using wellformed-items-Gen apply blast
    using 𝒯-eq-𝒵 𝒵-subset-𝒳 empty-tokens-is-filter apply blast
   apply (simp only: empty-tokens-idempotent paths-le-idempotent items-le-paths-le)
    apply (rule-tac thmD5)
    using items-le-is-filter items-le-paths-le apply blast
    apply (rule k)
    using 𝒯-eq-𝒵 empty-tokens-is-filter by blast
  from t1 t2 t3 t4 t5 show ?thesis using subset-trans by blast
qed

end

end
```

theory *Ladder*
imports *TheoremD9*
begin

context *LocalLexing* **begin**

definition *LeftDerivationFix* :: *sentence* ⇒ *nat* ⇒ *derivation* ⇒ *nat* ⇒ *sentence* ⇒ *bool*
where
LeftDerivationFix α *i D j* β = (*is-sentence* α ∧ *is-sentence* β
∧ *LeftDerivation* α *D* β ∧ *i* < *length* α ∧ *j* < *length* β
∧ α ! *i* = β ! *j* ∧ (∃ *E F*. *D* = *E*@(*derivation-shift F 0* (*Suc j*)) ∧
LeftDerivation (*take i* α) *E* (*take j* β) ∧
LeftDerivation (*drop* (*Suc i*) α) *F* (*drop* (*Suc j*) β)))

definition *LeftDerivationIntro* ::
sentence ⇒ *nat* ⇒ *rule* ⇒ *nat* ⇒ *derivation* ⇒ *nat* ⇒ *sentence* ⇒ *bool*
where
LeftDerivationIntro α *i r ix D j* γ = (∃ β. *LeftDerives1* α *i r* β ∧
ix < *length* (*snd r*) ∧ (*snd r*) ! *ix* = γ ! *j* ∧
LeftDerivationFix β (*i* + *ix*) *D j* γ)

lemma *LeftDerivationFix-empty*[*simp*]: *is-sentence* α ⟹ *i* < *length* α ⟹ *LeftDerivationFix* α *i* [] *i* α
apply (*auto simp add*: *LeftDerivationFix-def*)
apply (*rule-tac x*=[] **in** *exI*)
apply *auto*
done

lemma *Derive-empty*[*simp*]: *Derive a* [] = *a*
by (*auto simp add*: *Derive-def*)

lemma *LeftDerivation-append1*: *LeftDerivation a* (*D*@[(*i*, *r*)]) *c* ⟹ ∃ *b*. *LeftDerivation a D b*
∧ *LeftDerives1 b i r c*
by (*simp add*: *LeftDerivation-append*)

lemma *Derivation-append1*: *Derivation a* (*D*@[(*i*, *r*)]) *c* ⟹ ∃ *b*. *Derivation a D b*
∧ *Derives1 b i r c*
by (*simp add*: *Derivation-append*)

lemma *Derivation-take-derive*:
assumes *Derivation a D b*
shows *Derivation a* (*take n D*) (*Derive a* (*take n D*))
by (*metis Derivation-append Derive append-take-drop-id assms*)

lemma *LeftDerivation-take-derive*:
assumes *LeftDerivation a D b*

shows *LeftDerivation a* (*take n D*) (*Derive a* (*take n D*))
by (*metis Derive LeftDerivation-append LeftDerivation-implies-Derivation append-take-drop-id assms*)

lemma *Derivation-Derive-take-Derives1*:
assumes $N \neq 0$
assumes $N \leq$ *length D*
assumes *Derivation a D b*
assumes α: α = *Derive a* (*take* $(N - 1)$ *D*)
assumes β = *Derive a* (*take N D*)
shows *Derives1* α (*fst* (*D* ! $(N - 1)$)) (*snd* (*D* ! $(N - 1)$)) β
proof −
let *?D1* = *take* $(N - 1)$ *D*
let *?D2* = *take N D*
from *assms* **have** *app*: *?D2* = *?D1* @ [*D* ! $(N - 1)$]
apply *auto*
by (*metis Suc-less-eq Suc-pred le-imp-less-Suc take-Suc-conv-app-nth*)
from *assms* **have** *Derivation a ?D2* β
using *Derivation-take-derive* **by** *blast*
with *app* **show** *?thesis*
using *Derivation.simps Derivation-append Derive* α **by** *auto*
qed

lemma *LeftDerivation-Derive-take-LeftDerives1*:
assumes $N \neq 0$
assumes $N \leq$ *length D*
assumes *LeftDerivation a D b*
assumes α: α = *Derive a* (*take* $(N - 1)$ *D*)
assumes β = *Derive a* (*take N D*)
shows *LeftDerives1* α (*fst* (*D* ! $(N - 1)$)) (*snd* (*D* ! $(N - 1)$)) β
proof −
let *?D1* = *take* $(N - 1)$ *D*
let *?D2* = *take N D*
from *assms* **have** *app*: *?D2* = *?D1* @ [*D* ! $(N - 1)$]
apply *auto*
by (*metis Suc-less-eq Suc-pred le-imp-less-Suc take-Suc-conv-app-nth*)
from *assms* **have** *LeftDerivation a ?D2* β
using *LeftDerivation-take-derive* **by** *blast*
with *app* **show** *?thesis*
by (*metis Derive LeftDerivation-append1 LeftDerivation-implies-Derivation* α *prod.collapse*)
qed

lemma *LeftDerives1-skip-prefix*:
length a $\leq i \Longrightarrow$ *LeftDerives1* (*a*@*b*) *i r* (*a*@*c*) $\Longrightarrow$ *LeftDerives1 b* (*i* − *length a*) *r c*
apply (*auto simp add*: *LeftDerives1-def*)
using *leftmost-skip-prefix* **apply** *blast*
by (*simp add*: *Derives1-skip-prefix*)

lemma *LeftDerives1-skip-suffix*:
assumes *i*: $i < length\ a$
assumes *D*: *LeftDerives1* $(a@c)\ i\ r\ (b@c)$
shows *LeftDerives1* $a\ i\ r\ b$
proof −
note *Derives1-def*[**where** $u{=}a@c$ **and** $v{=}b@c$ **and** $i{=}i$ **and** $r{=}r$]
then have $\exists x\ y\ N\ \alpha.$
$a\ @\ c = x\ @\ [N]\ @\ y\ \wedge$
$b\ @\ c = x\ @\ \alpha\ @\ y \wedge$ *is-sentence* $x \wedge$ *is-sentence* $y \wedge (N, \alpha) \in \mathfrak{R} \wedge r = (N, \alpha) \wedge i = length\ x$
using *D LeftDerives1-implies-Derives1* **by** *auto*
then obtain $x\ y\ N\ \alpha$ **where** *split*:
$a\ @\ c = x\ @\ [N]\ @\ y\ \wedge$
$b\ @\ c = x\ @\ \alpha\ @\ y \wedge$ *is-sentence* $x \wedge$ *is-sentence* $y \wedge (N, \alpha) \in \mathfrak{R} \wedge r = (N, \alpha) \wedge i = length\ x$
by *blast*
from *split* **have** *length* $(a@c) = length\ (x\ @\ [N]\ @\ y)$ **by** *auto*
then have *length* $a + length\ c = length\ x + length\ y + 1$ **by** *simp*
with *split* **have** *length* $a + length\ c = i + length\ y + 1$ **by** *simp*
with *i* **have** *len-c-y*: *length* $c \leq length\ y$ **by** *arith*
let *?y* $= take\ (length\ y - length\ c)\ y$
from *split* **have** *ac*: $a\ @\ c = (x\ @\ [N])\ @\ y$ **by** *auto*
note *cancel-suffix*[**where** $a{=}a$ **and** $c = c$ **and** $b = x@[N]$ **and** $d = y$, *OF ac len-c-y*]
then have *a*: $a = x\ @\ [N]\ @$ *?y* **by** *auto*
from *split* **have** *bc*: $b\ @\ c = (x\ @\ \alpha)\ @\ y$ **by** *auto*
note *cancel-suffix*[**where** $a{=}b$ **and** $c = c$ **and** $b = x@\alpha$ **and** $d = y$, *OF bc len-c-y*]
then have *b*: $b = x\ @\ \alpha\ @$ *?y* **by** *auto*
from *split len-c-y a b* **show** *?thesis*
apply (*simp only*: *LeftDerives1-def Derives1-def*)
apply (*rule-tac conjI*)
using *D LeftDerives1-def i leftmost-cons-less* **apply** *blast*
apply (*rule-tac* $x{=}x$ **in** *exI*)
apply (*rule-tac* $x{=}$*?y* **in** *exI*)
apply (*rule-tac* $x{=}N$ **in** *exI*)
apply (*rule-tac* $x{=}\alpha$ **in** *exI*)
apply *auto*
by (*rule is-sentence-take*)
qed

lemma *LeftDerives1-X-is-part-of-rule*[*consumes 2*, *case-names Suffix Prefix*]:
assumes *aXb*: *LeftDerives1* $\delta\ i\ r\ (a@[X]@b)$
assumes *split*: *splits-at* $\delta\ i\ \alpha\ N\ \beta$
assumes *prefix*: $\bigwedge \beta.\ \delta = a\ @\ [X]\ @\ \beta \Longrightarrow length\ a < i \Longrightarrow$ *is-word* $(a\ @\ [X])$ $\Longrightarrow$
LeftDerives1 $\beta\ (i - length\ a - 1)\ r\ b \Longrightarrow False$
assumes *suffix*: $\bigwedge \alpha.\ \delta = \alpha\ @\ [X]\ @\ b \Longrightarrow$ *LeftDerives1* $\alpha\ i\ r\ a \Longrightarrow False$

shows $\exists\ u\ v.\ a = \alpha\ @\ u \wedge b = v\ @\ \beta \wedge (snd\ r) = u@[X]@v$
proof −
have *aXb-old*: *Derives1* $\delta\ i\ r\ (a@[X]@b)$
using *LeftDerives1-implies-Derives1 aXb* **by** *blast*
have *prefix-or*: *is-prefix* $\alpha\ a \vee$ *is-proper-prefix* $a\ \alpha$
by (*metis Derives1-prefix split aXb-old is-prefix-eq-proper-prefix*)
have *is-word-*α: *is-word* α
using *LeftDerives1-splits-at-is-word aXb assms*(*2*) **by** *blast*
have *is-proper-prefix a* $\alpha \Longrightarrow$ *False*
proof −
assume *proper*:*is-proper-prefix a* α
then have $\exists\ u.\ u \neq [] \wedge \alpha = a@u$ **by** (*metis is-proper-prefix-def*)
then obtain u **where** u: $u \neq [] \wedge \alpha = a@u$ **by** *blast*
note *splits-at* = *splits-at-*α[*OF aXb-old split*] *splits-at-combine*[*OF split*]
from *splits-at* **have** $\alpha 1$: $\alpha =$ *take i* δ **by** *blast*
from *splits-at* **have** $\alpha 2$: $\alpha =$ *take i* $(a@[X]@b)$ **by** *blast*
from *splits-at* **have** *len*α: *length* $\alpha = i$ **by** *blast*
with *proper* **have** *lena*: *length* $a < i$
using *append-eq-conv-conj drop-eq-Nil leI u* **by** *auto*
with *is-word-*α $\alpha 2$ **have** *is-word-aX*: *is-word* $(a@[X])$
by (*simp add*: *is-word-terminals not-less take-Cons*′ *u*)
from *u* $\alpha 2$ **have** $a@u =$ *take i* $(a@[X]@b)$ **by** *auto*
with *lena* **have** $u =$ *take* $(i -$ *length a*$)$ $([X]@b)$ **by** (*simp add*: *less-or-eq-imp-le*)

with *lena* **have** *uX*: $u = [X]@($*take* $(i -$ *length a* $- 1)$ $b)$ **by** (*simp add*: *not-less take-Cons*′)
let $?\beta = ($*take* $(i -$ *length a* $- 1)$ $b) @ [N] @ \beta$
from *splits-at* **have** *f1*: $\delta = \alpha @ [N] @ \beta$ **by** *blast*
with *u uX* **have** *f2*: $\delta = a @ [X] @ ?\beta$ **by** *simp*
note *skip* = *LeftDerives1-skip-prefix*[**where** $a = a @ [X]$ **and** $b = ?\beta$ **and** $r = r$ **and** $i = i$ **and** $c = b$]
then have *D*: *LeftDerives1* $?\beta$ $(i -$ *length a* $- 1)$ *r b*
using *One-nat-def Suc-leI aXb append-assoc diff-diff-left f2 lena length-Cons length-append length-append-singleton list.size*(*3*) **by** *fastforce*
note *prefix*[*OF f2 lena is-word-aX D*]
then show *False* **.**
qed
with *prefix-or* **have** *is-prefix*: *is-prefix* $\alpha\ a$ **by** *blast*

from *aXb* **have** *aXb*′: *LeftDerives1* $\delta\ i\ r\ ((a@[X])@b)$ **by** *auto*
then have *aXb*′*-old*: *Derives1* $\delta\ i\ r\ ((a@[X])@b)$ **by** (*simp add*: *LeftDerives1-implies-Derives1*)

note *Derives1-suffix*[*OF aXb*′*-old split*]
then have *suffix-or*: *is-suffix* $\beta\ b \vee$ *is-proper-suffix* $b\ \beta$
by (*metis is-suffix-eq-proper-suffix*)
have *is-proper-suffix* $b\ \beta \Longrightarrow$ *False*
proof −
assume *proper*: *is-proper-suffix* $b\ \beta$
then have $\exists\ u.\ u \neq [] \wedge \beta = u@b$ **by** (*metis is-proper-suffix-def*)

```
    then obtain u where u: u ≠ [] ∧ β = u@b by blast
    note splits-at = splits-at-β[OF aXb-old split] splits-at-combine[OF split]
    from splits-at have β1: β = drop (Suc i) δ by blast
    from splits-at have β2: β = drop (i + length (snd r)) (a @ [X] @ b) by blast
    from splits-at have lenβ: length β = length δ − i − 1 by blast
   with proper have lenb: length b < length β by (metis is-proper-suffix-length-cmp)

    from u β2 have u@b = drop (i + length (snd r)) ((a @ [X]) @ b) by auto
    hence u = drop (i + length (snd r)) (a @ [X])
      by (metis drop-cancel-suffix)
    hence uX: u = drop (i + length (snd r)) a @ [X] by (metis drop-keep-last u)
    let ?α = α @ [N] @ (drop (i + length (snd r)) a)
    from splits-at have f1: δ = α @ [N] @ β by blast
    with u uX have f2: δ = ?α @ [X] @ b by simp
    note skip = LeftDerives1-skip-suffix[where a = ?α and c = [X]@b and b=a
and
      r = r and i = i]
    have f3: i < length (α @ [N] @ drop (i + length (snd r)) a)
    proof −
      have f1: 1 + i + length b = length [X] + length b + i
      by (metis Groups.add-ac(2) Suc-eq-plus1-left length-Cons list.size(3) list.size(4)
semiring-normalization-rules(22))
      have f2: length δ − i − 1 = length ((α @ [N] @ drop (i + length (snd r)) a)
@ [X] @ b) − Suc i
        by (metis f2 length-drop splits-at(1))
      have length ([]::symbol list) ≠ length δ − i − 1 − length b
         by (metis add-diff-cancel-right' append-Nil2 append-eq-append-conv lenβ
length-append u)
     then have length ([]::symbol list) ≠ length α + length ([N] @ drop (i + length
(snd r)) a) − i
        using f2 f1 by (metis Suc-eq-plus1-left add-diff-cancel-right' diff-diff-left
length-append)
      then show ?thesis
        by auto
    qed
    from aXb f2 have D: LeftDerives1 (?α @ [X] @ b) i r (a@[X]@b) by auto
    note skip[OF f3 D]
    note suffix[OF f2  skip[OF f3 D]]
    then show False .
  qed
  with suffix-or have is-suffix: is-suffix β b by blast

  from is-prefix have ∃ u. a = α @ u by (auto simp add: is-prefix-def)
  then obtain u where u: a = α @ u by blast
  from is-suffix have ∃ v. b = v @ β by (auto simp add: is-suffix-def)
  then obtain v where v: b = v @ β by blast

  from u v splits-at-combine[OF split] aXb have D:LeftDerives1 (α@[N]@β) i r
(α@(u@[X]@v)@β)
```

by *simp*
from *splits-at-α*[*OF aXb-old split*] **have** *i*: *length α* = *i* **by** *blast*
from *i* **have** *i1*: *length α* ≤ *i* **and** *i2*: *i* ≤ *length α* **by** *auto*
note *LeftDerives1-skip-suffix*[*OF - LeftDerives1-skip-prefix*[*OF i1 D*], *simplified*, *OF i2*]
then have *LeftDerives1* [*N*] *0 r* (*u* @ [*X*] @ *v*) **by** *auto*
then have *Derives1* [*N*] *0 r* (*u* @ [*X*] @ *v*)
using *LeftDerives1-implies-Derives1* **by** *auto*
then have *r*: *snd r* = *u* @ [*X*] @ *v*
by (*metis Derives1-split append-Cons append-Nil length-0-conv list.inject self-append-conv*)

show *?thesis* **using** *u v r* **by** *auto*
qed

lemma *LeftDerivationFix-grow-suffix*:
assumes *LDF*: *LeftDerivationFix* (*b1*@[*X*]@*b2*) (*length b1*) *D j c*
assumes *suffix-b2*: *LeftDerives1 suffix e r b2*
assumes *is-word-b1X*: *is-word* (*b1*@[*X*])
shows *LeftDerivationFix* (*b1*@[*X*]@*suffix*) (*length b1*) ((*e* + *length* (*b1*@[*X*]), *r*)#*D*) *j c*
proof −
from *LDF* **have** *LDF'*: *is-sentence* (*b1*@[*X*]@*b2*) ∧ *is-sentence c* ∧
LeftDerivation (*b1* @ [*X*] @ *b2*) *D c* ∧ *length b1* < *length* (*b1* @ [*X*] @ *b2*) ∧
j < *length c* ∧
(*b1* @ [*X*] @ *b2*) ! *length b1* = *c* ! *j* ∧
(∃ *E F*. *D* = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take* (*length b1*) (*b1* @ [*X*] @ *b2*)) *E* (*take j c*) ∧
LeftDerivation (*drop* (*Suc* (*length b1*)) (*b1* @ [*X*] @ *b2*)) *F* (*drop* (*Suc j*) *c*))
using *LeftDerivationFix-def* **by** *blast*
then obtain *E F* **where** *EF*: *D* = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take* (*length b1*) (*b1* @ [*X*] @ *b2*)) *E* (*take j c*) ∧
LeftDerivation (*drop* (*Suc* (*length b1*)) (*b1* @ [*X*] @ *b2*)) *F* (*drop* (*Suc j*) *c*)
by *blast*
then have *LD-b1-c*: *LeftDerivation b1 E* (*take j c*) **by** *simp*
with *is-word-b1X* **have** *E*: *E* = []
using *LeftDerivation-implies-Derivation is-word-Derivation is-word-append* **by** *blast*
then have *b1-def*: *b1* = *take j c* **using** *LD-b1-c* **by** *auto*
then have *b1-len*: *j* = *length b1*
by (*simp add*: *LDF' dual-order.strict-implies-order min.absorb2*)
have *D*: *D* = *derivation-shift F 0* (*Suc j*) **using** *EF E* **by** *simp*
have *step*: *LeftDerives1* (*b1* @ [*X*] @ *suffix*) (*Suc* (*e* + *length b1*)) *r* (*b1* @ [*X*] @ *b2*) ∧
LeftDerivation (*b1* @ [*X*] @ *b2*) *D c*
by (*metis LDF' LeftDerives1-append-prefix add-Suc-right append-assoc assms*(*2*) *is-word-b1X*
length-append-singleton)
then have *is-sentence-b1Xsuffix*: *is-sentence* (*b1* @ [*X*] @ *suffix*)
using *Derives1-sentence1 LeftDerives1-implies-Derives1* **by** *blast*

```
  have X-eq-cj: X = c ! j using LDF' by auto
  show ?thesis
    apply (simp add: LeftDerivationFix-def)
    apply (rule conjI)
    using is-sentence-b1Xsuffix apply simp
    apply (rule conjI)
    using LDF' apply simp
    apply (rule conjI)
    using step apply force
    apply (rule conjI)
    using LDF' apply simp
    apply (rule conjI)
    apply (rule X-eq-cj)
    apply (rule-tac x=[] in exI)
    apply (rule-tac x=(e, r)#F in exI)
    apply auto
    apply (rule b1-len[symmetric])
    apply (rule D)
    apply (rule b1-def)
    apply (rule-tac x=b2 in exI)
    apply (simp add: suffix-b2)
    using EF by auto
qed

lemma Derives1-append-suffix:
  assumes Derives1: Derives1 v i r w
  assumes u: is-sentence u
  shows Derives1 (v@u) i r (w@u)
proof −
  have ∃ α N β. splits-at v i α N β using assms splits-at-ex by auto
  then obtain α N β where split-v: splits-at v i α N β by blast
  have split-w: w = α@(snd r)@β using assms split-v splits-at-combine-dest by
blast
  have split-uv: splits-at (v@u) i α N (β@u)
    by (simp add: split-v splits-at-append)
  have is-sentence-uv: is-sentence (v@u)
    using Derives1 Derives1-sentence1 is-sentence-concat u by blast
  show ?thesis
   by (metis Derives1 Derives1-nonterminal Derives1-rule append-assoc is-sentence-uv

        split-uv split-v split-w splits-at-implies-Derives1 )
qed

lemma leftmost-append-suffix: leftmost i v ⟹ leftmost i (v@u)
by (simp add: leftmost-def nth-append)

lemma LeftDerives1-append-suffix:
  assumes Derives1: LeftDerives1 v i r w
  assumes u: is-sentence u
```

shows *LeftDerives1* (*v*@*u*) *i r* (*w*@*u*)
proof −
have *1*: *Derives1 v i r w*
by (*simp add*: *Derives1 LeftDerives1-implies-Derives1*)
have *2*: *leftmost i v*
using *Derives1 LeftDerives1-def* **by** *blast*
have *3*: *is-sentence u* **using** *u* **by** *fastforce*
have *4*: *Derives1* (*v*@*u*) *i r* (*w*@*u*)
by (*simp add*: *1 3 Derives1-append-suffix*)
have *5*: *leftmost i* (*v*@*u*)
by (*simp add*: *2 leftmost-append-suffix u*)
show *?thesis*
by (*simp add*: *4 5 LeftDerives1-def*)
qed

lemma *LeftDerivationFix-is-sentence*:
LeftDerivationFix a i D j b ⟹ *is-sentence a* ∧ *is-sentence b*
using *LeftDerivationFix-def* **by** *blast*

lemma *LeftDerivationIntro-is-sentence*:
LeftDerivationIntro α *i r ix D j* γ ⟹ *is-sentence* α ∧ *is-sentence* γ
by (*meson Derives1-sentence1 LeftDerivationFix-is-sentence LeftDerivationIntro-def*
LeftDerives1-implies-Derives1)

lemma *LeftDerivationFix-grow-prefix*:
assumes *LDF*: *LeftDerivationFix* (*b1*@[*X*]@*b2*) (*length b1*) *D j c*
assumes *prefix-b1*: *LeftDerives1 prefix e r b1*
shows *LeftDerivationFix* (*prefix*@[*X*]@*b2*) (*length prefix*) ((*e*, *r*)#*D*) *j c*
proof −
from *LDF* **have** *LDF'*: *LeftDerivation* (*b1* @ [*X*] @ *b2*) *D c* ∧
length b1 < *length* (*b1* @ [*X*] @ *b2*) ∧
j < *length c* ∧
(*b1* @ [*X*] @ *b2*) ! *length b1* = *c* ! *j* ∧
(∃ *E F*. *D* = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take* (*length b1*) (*b1* @ [*X*] @ *b2*)) *E* (*take j c*) ∧
LeftDerivation (*drop* (*Suc* (*length b1*)) (*b1* @ [*X*] @ *b2*)) *F* (*drop* (*Suc j*) *c*))
using *LeftDerivationFix-def* **by** *blast*
then obtain *E F* **where** *EF*: *D* = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take* (*length b1*) (*b1* @ [*X*] @ *b2*)) *E* (*take j c*) ∧
LeftDerivation (*drop* (*Suc* (*length b1*)) (*b1* @ [*X*] @ *b2*)) *F* (*drop* (*Suc j*) *c*)
by *blast*
then have *E-b1-c*: *LeftDerivation b1 E* (*take j c*) **by** *simp*
with *EF* **have** *F-b2-c*: *LeftDerivation b2 F* (*drop* (*Suc j*) *c*) **by** *simp*
have *step*: *LeftDerives1* (*prefix* @ [*X*] @ *b2*) *e r* (*b1* @ [*X*] @ *b2*)
using *LDF LeftDerivationFix-is-sentence LeftDerives1-append-suffix*
is-sentence-concat prefix-b1 **by** *blast*
show *?thesis*
apply (*simp add*: *LeftDerivationFix-def*)

```
    apply (rule conjI)
   apply (metis Derives1-sentence1 LDF LeftDerivationFix-def LeftDerives1-implies-Derives1

     is-sentence-concat is-sentence-cons prefix-b1)
    apply (rule conjI)
    using LDF LeftDerivationFix-is-sentence apply blast
    apply (rule conjI)
    apply (rule-tac x=b1@[X]@b2 in exI)
    using step apply simp
    using LDF' apply auto[1]
    apply (rule conjI)
    using LDF' apply simp
    apply (rule conjI)
    using LDF' apply auto[1]
    apply (rule-tac x=(e,r)#E in exI)
    apply (rule-tac x=F in exI)
    apply (auto simp add: EF F-b2-c)
    apply (rule-tac x=b1 in exI)
    apply (simp add: prefix-b1 E-b1-c)
    done
qed

lemma LeftDerivationFixOrIntro:
  LeftDerivation a D γ ⟹ is-sentence γ ⟹ j < length γ ⟹
  (∃ i. LeftDerivationFix a i D j γ) ∨
  (∃ d α ix. d < length D ∧ LeftDerivation a (take d D) α ∧
    LeftDerivationIntro α (fst (D ! d)) (snd (D ! d)) ix (drop (Suc d) D) j γ)
proof (induct length D arbitrary: a D γ j rule: less-induct)

  case less
  have length D = 0 ∨ length D ≠ 0 by blast
  then show ?case
  proof (induct rule: disjCases2)
    case 1
    then have D: D = [] by auto
    with less have ∃ i. LeftDerivationFix a i D j γ
      apply (rule-tac x=j in exI)
      by auto
    then show ?case by blast
  next
    case 2
    note less2 = 2
    have ∃ n β i. n ≤ length D ∧ β = Derive a (take n D) ∧ LeftDerivationFix β
i (drop n D) j γ
      apply (rule-tac x=length D in exI)
      apply auto
      using Derive LeftDerivationFix-empty LeftDerivation-implies-Derivation less
by blast
    then show ?case
```

```
  proof (induct rule: ex-minimal-witness)
    case (Minimal N)
    then obtain β i where Minimal-N:
      N ≤ length D ∧ β = Derive a (take N D) ∧ LeftDerivationFix β i (drop N
D) j γ by blast
    have N = 0 ∨ N ≠ 0 by blast
    then show ?case
    proof (induct rule: disjCases2)
      case 1
      with Minimal-N have β = a by auto
      with 1 Minimal-N show ?case
        apply (rule-tac disjI1)
        by auto
    next
      case 2
      let ?δ = Derive a (take (N − 1) D)
       have LeftDerives1-δ: LeftDerives1 ?δ (fst (D ! (N − 1))) (snd (D ! (N −
1))) β
       using 2.hyps LeftDerivation-Derive-take-LeftDerives1 Minimal-N less.prems(1)
by blast
      then have Derives1-δ: Derives1 ?δ (fst (D ! (N − 1))) (snd (D ! (N − 1)))
β
        using LeftDerives1-implies-Derives1 by blast
      have i-len: i < length β using Minimal-N
        by (auto simp add: LeftDerivationFix-def)
      then have ∃ X β-1 β-2. splits-at β i β-1 X β-2
        using splits-at-def by blast
      then obtain X β-1 β-2 where β-split: splits-at β i β-1 X β-2 by blast
       then have β-combine: β = β-1 @ [X] @ β-2 using splits-at-combine by
blast
      then have LeftDerives1-δ-hyp:
        LeftDerives1 ?δ (fst (D ! (N − 1))) (snd (D ! (N − 1))) (β-1 @ [X] @
β-2)
        using LeftDerives1-δ by blast
      from β-split have i-def: i = length β-1
        by (simp add: dual-order.strict-implies-order min.absorb2 splits-at-def)
      have ∃ Y δ-1 δ-2. splits-at ?δ (fst (D ! (N − 1))) δ-1 Y δ-2
        using Derives1-δ splits-at-ex by blast
      then obtain Y δ-1 δ-2 where δ-split: splits-at ?δ (fst (D ! (N − 1))) δ-1
Y δ-2 by blast
      have NFix: LeftDerivationFix (β-1 @ [X] @ β-2) (length β-1) (drop N D)
j γ
        using Minimal-N β-combine i-def by auto
      from LeftDerives1-δ-hyp δ-split
      have ∃ u v. β-1 = δ-1 @ u ∧ β-2 = v @ δ-2 ∧ snd (snd (D ! (N − 1))) =
u @ [X] @ v
      proof (induct rule: LeftDerives1-X-is-part-of-rule)
        case (Suffix suffix)
          let ?k = N − 1
```

```
          let ?β = Derive a (take ?k D)
          let ?i = length β-1
          have k-less: ?k < length D using 2.hyps Minimal-N by linarith
          then have k-leq: ?k ≤ length D by auto
          have drop-k-d: drop ?k D = (D ! (N − 1))#(drop N D)
            using 2.hyps Cons-nth-drop-Suc k-less by fastforce
        from LeftDerivationFix-grow-suffix[OF NFix Suffix(4) Suffix(3)] Suffix(1)
Suffix(2) 2
          have LeftDerivationFix ?β ?i (drop ?k D) j γ
            apply auto
            by (metis One-nat-def drop-k-d)
          with Minimal(2)[where k=?k] show False
            using 2.hyps k-leq by auto
      next
        case (Prefix prefix)
          have collapse: (fst (D ! (N − 1)), snd (D ! (N − 1))) # drop N D =
drop (N − 1) D
            by (metis 2.hyps Cons-nth-drop-Suc Minimal-N Suc-diff-1 neq0-conv
not-less
              not-less-eq prod.collapse)
        from LeftDerivationFix-grow-prefix[OF NFix Prefix(2)] Prefix(1) collapse
          have LeftDerivationFix ?δ (length prefix) (drop (N − 1) D) j γ by auto
          with Minimal(2)[where k = N − 1] show False
        by (metis Minimal-N collapse diff-le-self le-neq-implies-less less-imp-diff-less

            less-or-eq-imp-le not-Cons-self2)
      qed
      then obtain u v where uv:
        β-1 = δ-1 @ u ∧ β-2 = v @ δ-2 ∧ snd (snd (D ! (N − 1))) = u @ [X]
@ v by blast
      have X-1: snd (snd (D ! (N − Suc 0))) ! length u = X using uv by auto
      have X-2: γ ! j = X using LeftDerivationFix-def NFix by auto
      show ?case
        apply (rule disjI2)
        apply (rule-tac x=N − 1 in exI)
        apply (rule-tac x=?δ in exI)
        apply (rule-tac x=length u in exI)
        apply (rule conjI)
        using Minimal-N less2 apply linarith
        apply (rule conjI)
        using LeftDerivation-take-derive less.prems(1) apply blast
        apply (subst LeftDerivationIntro-def)
        apply (rule-tac x=β in exI)
        apply auto
        using LeftDerives1-δ One-nat-def apply presburger
        using uv apply auto[1]
        using X-1 X-2 apply auto[1]
       by (metis (no-types, lifting) 2.hyps Derives1-δ Derives1-split Minimal-N
One-nat-def
```

Suc-diff-1 δ-split append-eq-conv-conj i-def length-append neq0-conv splits-at-def uv)
qed
qed
qed
qed

type-synonym *deriv* = *nat* × *nat* × *nat*
type-synonym *ladder* = *deriv list*

definition *deriv-n* :: *deriv* ⇒ *nat* **where**
deriv-n d = *fst d*

definition *deriv-j* :: *deriv* ⇒ *nat* **where**
deriv-j d = *fst* (*snd d*)

definition *deriv-ix* :: *deriv* ⇒ *nat* **where**
deriv-ix d = *snd* (*snd d*)

definition *deriv-i* :: *deriv* ⇒ *nat* **where**
deriv-i d = *snd* (*snd d*)

definition *ladder-j* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-j L index = *deriv-j* (*L* ! *index*)

definition *ladder-i* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-i L index = (*if index* = *0 then deriv-i* (*hd L*) *else ladder-j L* (*index* − *1*))

definition *ladder-n* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-n L index = *deriv-n* (*L* ! *index*)

definition *ladder-prev-n* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-prev-n L index = (*if index* = *0 then 0 else* (*ladder-n L* (*index* − *1*)))

definition *ladder-ix* :: *ladder* ⇒ *nat* ⇒ *nat* **where**
ladder-ix L index = (*if index* = *0 then undefined else deriv-ix* (*L* ! *index*))

definition *ladder-last-j* :: *ladder* ⇒ *nat* **where**
ladder-last-j L = *ladder-j L* (*length L* − *1*)

definition *ladder-last-n* :: *ladder* ⇒ *nat* **where**
ladder-last-n L = *ladder-n L* (*length L* − *1*)

definition *is-ladder* :: *derivation* ⇒ *ladder* ⇒ *bool* **where**
is-ladder D L = (*L* ≠ [] ∧
(∀ *u*. *u* < *length L* ⟶ *ladder-n L u* ≤ *length D*) ∧
(∀ *u v*. *u* < *v* ∧ *v* < *length L* ⟶ *ladder-n L u* < *ladder-n L v*) ∧
ladder-last-n L = *length D*)

definition *ladder-γ* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒ *sentence* **where**
ladder-γ a D L index = *Derive a* (*take* (*ladder-n L index*) *D*)

definition *ladder-α* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒ *sentence* **where**
ladder-α a D L index = (*if index* = *0 then a else ladder-γ a D L* (*index* − *1*))

definition *LeftDerivationIntrosAt* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *nat* ⇒ *bool* **where**
LeftDerivationIntrosAt a D L index = (
let α = *ladder-α a D L index in*
let i = *ladder-i L index in*
let j = *ladder-j L index in*
let ix = *ladder-ix L index in*
let γ = *ladder-γ a D L index in*
let n = *ladder-n L* (*index* − *1*) *in*
let m = *ladder-n L index in*
let e = *D* ! *n in*
let E = *drop* (*Suc n*) (*take m D*) *in*
i = *fst e* ∧
LeftDerivationIntro α i (*snd e*) *ix E j γ*)

definition *LeftDerivationIntros* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *bool* **where**
LeftDerivationIntros a D L = (
∀ *index*. *1* ≤ *index* ∧ *index* < *length L* ⟶ *LeftDerivationIntrosAt a D L index*)

definition *LeftDerivationLadder* :: *sentence* ⇒ *derivation* ⇒ *ladder* ⇒ *sentence* ⇒ *bool* **where**
LeftDerivationLadder a D L b = (
LeftDerivation a D b ∧
is-ladder D L ∧
LeftDerivationFix a (*ladder-i L 0*) (*take* (*ladder-n L 0*) *D*) (*ladder-j L 0*) (*ladder-γ a D L 0*) ∧
LeftDerivationIntros a D L)

definition *mk-deriv-fix* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *deriv* **where**
mk-deriv-fix i n j = (*n*, *j*, *i*)

definition *mk-deriv-intro* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *deriv* **where**
mk-deriv-intro ix n j = (*n*, *j*, *ix*)

lemma *mk-deriv-fix-i*[*simp*]: *deriv-i* (*mk-deriv-fix i n j*) = *i*
by (*simp add*: *deriv-i-def mk-deriv-fix-def*)

lemma *mk-deriv-fix-j*[*simp*]: *deriv-j* (*mk-deriv-fix i n j*) = *j*
by (*simp add*: *deriv-j-def mk-deriv-fix-def*)

lemma *mk-deriv-fix-n*[*simp*]: *deriv-n* (*mk-deriv-fix i n j*) = *n*
by (*simp add*: *deriv-n-def mk-deriv-fix-def*)

lemma *mk-deriv-intro-i*[*simp*]: *deriv-i* (*mk-deriv-intro i n j*) = *i*
by (*simp add*: *deriv-i-def mk-deriv-intro-def*)

lemma *mk-deriv-intro-ix*[*simp*]: *deriv-ix* (*mk-deriv-intro ix n j*) = *ix*
by (*simp add*: *deriv-ix-def mk-deriv-intro-def*)

lemma *mk-deriv-intro-j*[*simp*]: *deriv-j* (*mk-deriv-intro i n j*) = *j*
by (*simp add*: *deriv-j-def mk-deriv-intro-def*)

lemma *mk-deriv-intro-n*[*simp*]: *deriv-n* (*mk-deriv-intro i n j*) = *n*
by (*simp add*: *deriv-n-def mk-deriv-intro-def*)

lemma *LeftDerivationFix-implies-ex-ladder*:
LeftDerivationFix a i D j $\gamma \Longrightarrow \exists$ *L*. *LeftDerivationLadder a D L* $\gamma \wedge$
ladder-last-j L = *j* $\wedge$ *ladder-last-n L* = *length D*
apply (*rule-tac x*=[*mk-deriv-fix i* (*length D*) *j*] **in** *exI*)
apply (*auto simp add*: *LeftDerivationLadder-def*)
apply (*simp add*: *LeftDerivationFix-def*)
apply (*simp add*: *is-ladder-def*)
apply (*auto simp add*: *ladder-i-def ladder-j-def ladder-n-def ladder-*γ*-def*)
apply (*simp add*: *ladder-last-n-def ladder-n-def*)
using *Derive LeftDerivationFix-def LeftDerivation-implies-Derivation* **apply** *blast*
apply (*simp add*: *LeftDerivationIntros-def*)
apply (*simp add*: *ladder-last-j-def ladder-j-def*)
apply (*simp add*: *ladder-last-n-def ladder-n-def*)
done

lemma *trivP*[*case-names prems*]: *P* $\Longrightarrow$ *P* **by** *blast*

lemma *LeftDerivationLadder-ladder-n-bound*:
assumes *LeftDerivationLadder a D L b*
assumes *index* < *length L*
shows *ladder-n L index* $\leq$ *length D*
using *LeftDerivationLadder-def assms*(*1*) *assms*(*2*) *is-ladder-def* **by** *blast*

lemma *LeftDerivationLadder-deriv-n-bound*:
assumes *LeftDerivationLadder a D L b*
assumes *index* < *length L*
shows *deriv-n* (*L* ! *index*) $\leq$ *length D*
using *LeftDerivationLadder-def assms*(*1*) *assms*(*2*) *is-ladder-def ladder-n-def* **by** *auto*

lemma *ladder-n-simp1*[*simp*]: *u* < *length L* $\Longrightarrow$ *ladder-n* (*L*@*L'*) *u* = *ladder-n L u*
by (*simp add*: *ladder-n-def*)

lemma *ladder-n-simp2*[*simp*]: *ladder-n* (*L*@[*d*]) (*length L*) = *deriv-n d*
by (*simp add*: *ladder-n-def*)

lemma *ladder-j-simp1*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{ladder-j } (L@L') \ u = \textit{ladder-j } L \ u$
by (*simp add*: *ladder-j-def*)

lemma *ladder-j-simp2*[*simp*]: *ladder-j* $(L@[d])$ (*length* L) = *deriv-j* d
by (*simp add*: *ladder-j-def*)

lemma *ladder-i-simp1*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{ladder-i } (L@L') \ u = \textit{ladder-i } L \ u$
by (*auto simp add*: *ladder-i-def*)

lemma *ladder-ix-simp1*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{ladder-ix } (L@L') \ u = \textit{ladder-ix } L$
u
by (*auto simp add*: *ladder-ix-def*)

lemma *ladder-ix-simp2*[*simp*]: $L \neq [] \Longrightarrow \textit{ladder-ix } (L@[d]) \ (\textit{length } L) = \textit{deriv-ix}$
d
by (*auto simp add*: *ladder-ix-def*)

lemma *ladder-γ-simp1*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{ladder-}\gamma \ a \ D \ (L@L') \ u = \textit{ladder-}\gamma$
$a \ D \ L \ u$
by (*simp add*: *ladder-γ-def*)

lemma *ladder-γ-simp2*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{is-ladder } D \ L \Longrightarrow$
$\textit{ladder-}\gamma \ a \ (D@D') \ L \ u = \textit{ladder-}\gamma \ a \ D \ L \ u$
by (*simp add*: *is-ladder-def ladder-γ-def*)

lemma *ladder-α-simp1*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{ladder-}\alpha \ a \ D \ (L@L') \ u = \textit{ladder-}\alpha$
$a \ D \ L \ u$
by (*simp add*: *ladder-α-def*)

lemma *ladder-α-simp2*[*simp*]: $u < \textit{length } L \Longrightarrow \textit{is-ladder } D \ L \Longrightarrow$
$\textit{ladder-}\alpha \ a \ (D@D') \ L \ u = \textit{ladder-}\alpha \ a \ D \ L \ u$
by (*simp add*: *is-ladder-def ladder-α-def*)

lemma *ladder-n-minus-1-bound*: *is-ladder* $D \ L \Longrightarrow \textit{index} \geq 1 \Longrightarrow \textit{index} < \textit{length}$
$L \Longrightarrow$
$\textit{ladder-n } L \ (\textit{index} - \textit{Suc } 0) < \textit{length } D$
by (*metis* (*no-types*, *lifting*) *One-nat-def Suc-diff-1 Suc-le-lessD dual-order.strict-implies-order*

is-ladder-def le-neq-implies-less not-less)

lemma *LeftDerivationIntrosAt-ignore-appendix*:
assumes *is-ladder*: *is-ladder* $D \ L$
assumes *hyp*: *LeftDerivationIntrosAt* $a \ D \ L \ \textit{index}$
assumes *index-ge*: $\textit{index} \geq 1$
assumes *index-less*: $\textit{index} < \textit{length } L$
shows *LeftDerivationIntrosAt* $a \ (D \ @ \ D') \ (L \ @ \ L') \ \textit{index}$
proof −
have *index-minus-1*: $\textit{index} - \textit{Suc } 0 < \textit{length } L$

```
    using index-less by arith
  have is-0: ladder-n L index − length D = 0
    using index-less is-ladder is-ladder-def by auto
  from index-ge index-less show ?thesis
    apply (simp add: LeftDerivationIntrosAt-def Let-def)
    apply (simp add: index-minus-1 is-ladder ladder-n-minus-1-bound is-0)
    using hyp apply (auto simp add: LeftDerivationIntrosAt-def Let-def)
    done
qed

lemma ladder-i-eq-last-j: L ≠ [] ⟹ ladder-i (L @ L') (length L) = ladder-last-j
L
by (simp add: ladder-i-def ladder-last-j-def)

lemma ladder-last-n-intro: L ≠ [] ⟹ ladder-n L (length L − Suc 0) = lad-
der-last-n L
by (simp add: ladder-last-n-def)

lemma is-ladder-not-empty: is-ladder D L ⟹ L ≠ []
using is-ladder-def by blast

lemma last-ladder-γ:
  assumes is-ladder: is-ladder D L
  assumes ladder-last-n: ladder-last-n L = length D
  shows ladder-γ a D L (length L − Suc 0) = Derive a D
proof −
  from is-ladder is-ladder-not-empty have L ≠ [] by blast
  then show ?thesis
    by (simp add: ladder-γ-def ladder-last-n-intro ladder-last-n)
qed

lemma ladder-α-full:
  assumes is-ladder: is-ladder D L
  assumes ladder-last-n: ladder-last-n L = length D
  shows ladder-α a (D @ D') (L @ L') (length L) = Derive a D
proof −
  from is-ladder have L-not-empty: L ≠ [] by (simp add: is-ladder-def)
  with is-ladder ladder-last-n show ?thesis
    apply (simp add: ladder-α-def)
    apply (simp add: last-ladder-γ)
    done
qed

lemma LeftDerivationIntro-implies-LeftDerivation:
  LeftDerivationIntro α i r ix D j γ ⟹ LeftDerivation α ((i,r)#D) γ
using LeftDerivationFix-def LeftDerivationIntro-def by auto

lemma LeftDerivationLadder-grow:
  LeftDerivationLadder a D L α ⟹ ladder-last-j L = i ⟹
```

```
   LeftDerivationIntro α i r ix E j γ ⟹
   LeftDerivationLadder a (D@[(i, r)]@E) (L@[mk-deriv-intro ix (Suc(length D +
length E)) j]) γ
proof (induct arbitrary: a D L α i r ix E j γ rule: trivP)
  case prems
  {
    fix u :: nat
    assume u < Suc (length L)
    then have u < length L ∨ u = length L by arith
    then have ladder-n (L @ [mk-deriv-intro ix (Suc (length D + length E)) j]) u
≤
      Suc (length D + length E)
    proof (induct rule: disjCases2)
      case 1
      then show ?case
        apply simp
         by (meson LeftDerivationLadder-ladder-n-bound le-Suc-eq le-add1 le-trans
prems(1))
    next
      case 2
      then show ?case
        by (simp add: ladder-n-def)
    qed
  }
  note ladder-n-ineqs = this
  {
    fix u :: nat
    fix v :: nat
    assume u-less-v: u < v
    assume v < Suc (length L)
    then have v < length L ∨ v = length L by arith
    then have ladder-n (L @ [mk-deriv-intro ix (Suc (length D + length E)) j]) u
          < ladder-n (L @ [mk-deriv-intro ix (Suc (length D + length E)) j]) v
    proof (induct rule: disjCases2)
      case 1
      with u-less-v have u-bound: u < length L by arith
      show ?case using 1 u-bound apply simp
      using prems u-less-v LeftDerivationLadder-def is-ladder-def by auto
    next
      case 2
      with u-less-v have u-bound: u < length L by arith
      have deriv-n (L ! u) ≤ length D
        using LeftDerivationLadder-deriv-n-bound prems(1) u-bound by blast
      then show ?case
        apply (simp add: u-bound)
        apply (simp add: ladder-n-def 2)
        done
    qed
  }
```

```
  note ladder-n-ineqs = ladder-n-ineqs this
  have is-ladder:
    is-ladder (D @ (i, r) # E) (L @ [mk-deriv-intro ix (Suc (length D + length E))
j])
    apply (auto simp add: is-ladder-def)
    using ladder-n-ineqs apply auto
    apply (simp add: ladder-last-n-def)
    done
  have is-ladder-L: is-ladder D L
    using LeftDerivationLadder-def prems.prems(1) by blast
  have ladder-last-n-eq-length: ladder-last-n L = length D
    using is-ladder-L is-ladder-def by blast
  have L-not-empty: L ≠ []
    using LeftDerivationLadder-def is-ladder-def prems(1) by blast
  {
    fix index :: nat
    assume index-ge: Suc 0 ≤ index
    assume index < Suc (length L)
    then have index < length L ∨ index = length L by arith
    then have LeftDerivationIntrosAt a (D @ (i, r) # E)
      (L @ [mk-deriv-intro ix (Suc (length D + length E)) j]) index
    proof (induct rule: disjCases2)
      case 1
      then show ?case
        using LeftDerivationIntrosAt-ignore-appendix
          LeftDerivationIntros-def LeftDerivationLadder-def One-nat-def
          index-ge prems.prems(1) by presburger
    next
      case 2
      have min-simp: ⋀ n E. min n (Suc (n + length E)) = n
        by auto
      with 2 prems is-ladder-L ladder-last-n-eq-length show ?case
        apply (simp add: LeftDerivationIntrosAt-def)
        apply (simp add: L-not-empty ladder-i-eq-last-j ladder-last-n-intro)
        apply (simp add: ladder-α-full min-simp)
        apply (simp add: ladder-γ-def)
       by (metis Derive LeftDerivationIntro-implies-LeftDerivation LeftDerivation-
Ladder-def
          LeftDerivation-implies-Derivation LeftDerivation-implies-append)
    qed
  }
  then show ?case
    apply (auto simp add: LeftDerivationLadder-def)
    using prems apply (auto simp add: LeftDerivationLadder-def)[1]
   using LeftDerivationFix-def LeftDerivationIntro-def LeftDerivation-append ap-
ply auto[1]
    using is-ladder apply simp
    using L-not-empty apply simp
   using LeftDerivationLadder-def LeftDerivationLadder-ladder-n-bound ladder-γ-def
```

```
      prems.prems(1) apply auto[1]
    apply (subst LeftDerivationIntros-def)
    apply auto
    done
qed

lemma LeftDerivationIntro-bounds-ij:
  LeftDerivationIntro α i r ix D j β ⟹ i < length α ∧ j < length β
  by (meson Derives1-bound LeftDerivationFix-def LeftDerivationIntro-def
    LeftDerives1-implies-Derives1)

theorem LeftDerivationLadder-exists: LeftDerivation a D γ ⟹ is-sentence γ ⟹
j < length γ ⟹
  ∃ L. LeftDerivationLadder a D L γ ∧ ladder-last-j L = j
proof (induct length D arbitrary: a D γ j rule: less-induct)
  case less
  from LeftDerivationFixOrIntro[OF less(2,3,4)] show ?case
  proof (induct rule: disjCases2)
    case 1
    then obtain i where LeftDerivationFix a i D j γ by blast
    show ?case
    using 1.hyps LeftDerivationFix-implies-ex-ladder by blast
  next
    case 2
    then obtain d α ix where inductrule: d < length D ∧
      LeftDerivation a (take d D) α ∧
      LeftDerivationIntro α (fst (D ! d)) (snd (D ! d)) ix (drop (Suc d) D) j γ by
blast
    then have less-length-D: length (take d D) < length D
      and LeftDerivation-α: LeftDerivation a (take d D) α by auto
    have is-sentence-α: is-sentence α using LeftDerivationIntro-is-sentence induc-
trule by blast
    have fst (D ! d) < length α using LeftDerivationIntro-bounds-ij inductrule by
blast
   from less(1)[OF less-length-D LeftDerivation-α is-sentence-α, where j= fst (D
! d), OF this]
    obtain L where induct-Ladder:
      LeftDerivationLadder a (take d D) L α and induct-last: ladder-last-j L = fst
(D ! d)
      by blast
     have induct-intro: LeftDerivationIntro α (fst (D ! d)) (snd (D ! d)) ix (drop
(Suc d) D) j γ
      using inductrule by blast
    have d < length D using inductrule by blast
    then have simp-to-D: take d D @ D ! d # drop (Suc d) D = D
      using id-take-nth-drop by force
     from LeftDerivationLadder-grow[OF induct-Ladder induct-last induct-intro]
simp-to-D
```

```
    show ?case
      apply auto
      apply (rule-tac x=
       L @ [mk-deriv-intro ix (Suc (min (length D) d + (length D − Suc d))) j] in
exI)
      apply (simp add: ladder-last-j-def)
      done
  qed
qed

lemma LeftDerivationLadder-L-0:
  assumes LeftDerivationLadder α D L β
  assumes length L = 1
  shows ∃ i. LeftDerivationFix α i D (ladder-last-j L) β
proof −
  have is-ladder D L using assms by (auto simp add: LeftDerivationLadder-def)
  then have ladder-n: ladder-n L 0 = length D
    by (simp add: assms(2) is-ladder-def ladder-last-n-def)
  show ?thesis
    apply (rule-tac x = ladder-i L 0 in exI)
    using assms(1) apply (auto simp add: LeftDerivationLadder-def)
   by (metis Derive LeftDerivationFix-def LeftDerivation-implies-Derivation One-nat-def
assms(2)
      diff-Suc-1 ladder-last-j-def ladder-n order-refl take-all)
qed

lemma LeftDerivationFix-splits-at-derives:
  assumes LeftDerivationFix a i D j b
  shows ∃ U a1 a2 b1 b2. splits-at a i a1 U a2 ∧ splits-at b j b1 U b2 ∧
    derives a1 b1 ∧ derives a2 b2
proof −
   note hyp = LeftDerivationFix-def[where α=a and β=b and D=D and i=i
and j=j]
  from hyp obtain E F where EF:
    D = E @ derivation-shift F 0 (Suc j) ∧
       LeftDerivation (take i a) E (take j b) ∧ LeftDerivation (drop (Suc i) a) F
(drop (Suc j) b)
    using assms by blast
  show ?thesis
    apply (rule-tac x=a ! i in exI)
    apply (rule-tac x=take i a in exI)
    apply (rule-tac x=drop (Suc i) a in exI)
    apply (rule-tac x=take j b in exI)
    apply (rule-tac x=drop (Suc j) b in exI)
    using Derivation-implies-derives LeftDerivation-implies-Derivation assms hyp
      splits-at-def by blast
qed

lemma  LeftDerivation-append-suffix:
```

```
  LeftDerivation a D b ⟹ is-sentence c ⟹ LeftDerivation (a@c) D (b@c)
proof (induct D arbitrary: a b c)
  case Nil
  then show ?case by auto
next
  case (Cons d D)
  then show ?case
    apply auto
    apply (rule-tac x=x@c in exI)
    apply auto
    using LeftDerives1-append-suffix by simp
qed

lemma LeftDerivation-impossible: LeftDerivation a D b ⟹ i < length a ⟹
  is-nonterminal (a ! i) ⟹ derivation-ge D (Suc i) ⟹ D = []
proof (induct D)
  case Nil then show ?case by auto
next
  case (Cons d D)
  then have lm: ⋀ j. leftmost j a ⟹ j ≤ i
   by (metis Derives1-sentence1 LeftDerivation.simps(2) LeftDerives1-implies-Derives1

      leftmost-exists leftmost-unique)
  from Cons show ?case
    apply auto
    apply (auto simp add: derivation-ge-def LeftDerives1-def)
    using lm[where j=fst d] by arith
qed

lemma derivation-ge-shift: derivation-ge (derivation-shift F 0 j) j
  apply (induct F)
  apply (auto simp add: derivation-ge-def)
  done

lemma LeftDerivationFix-splits-at-nonterminal:
  assumes LeftDerivationFix a i D j b
  assumes is-nonterminal (a ! i)
 shows ∃ U a1 a2 b1. splits-at a i a1 U a2 ∧ splits-at b j b1 U a2 ∧ LeftDerivation
a1 D b1
proof −
  note hyp = LeftDerivationFix-def[where α=a and β=b and D=D and i=i
and j=j]
  from hyp obtain E F where EF:
    D = E @ derivation-shift F 0 (Suc j) ∧ LeftDerivation (take i a) E (take j b)
∧
     LeftDerivation (drop (Suc i) a) F (drop (Suc j) b)
    using assms by blast
  have ∃ β. LeftDerivation a E β ∧ LeftDerivation β (derivation-shift F 0 (Suc
j)) b
```

```
    using EF LeftDerivation-append assms(1) hyp by blast
  then obtain β where β-intro:
     LeftDerivation a E β ∧ LeftDerivation β (derivation-shift F 0 (Suc j)) b by
blast
  have LeftDerivation ((take i a)@(drop i a)) E ((take j b)@(drop i a))
     by (metis EF LeftDerivation-append-suffix append-take-drop-id assms(1) hyp
is-sentence-concat)
  then have LeftDerivation a E ((take j b)@(drop i a)) by simp
  then have β-decomposed: β = (take j b)@(drop i a)
   using Derivation-unique-dest LeftDerivation-implies-Derivation β-intro by blast

  then have β ! j = a ! i
   by (metis Cons-nth-drop-Suc assms(1) hyp length-take min.absorb2 nth-append-length

       order.strict-implies-order)
  then have is-nt: is-nonterminal (β ! j) by (simp add: assms(2))
  have index-j: j < length β using β-decomposed assms(1) hyp by auto
  have derivation: LeftDerivation β (derivation-shift F 0 (Suc j)) b
    by (simp add: β-intro)
  from LeftDerivation-impossible[OF derivation index-j is-nt derivation-ge-shift]
  have F: F = [] by (metis length-0-conv length-derivation-shift)
  then have β-is-b: β = b using β-intro by auto
  show ?thesis
    apply (rule-tac x=a ! i in exI)
    apply (rule-tac x=take i a in exI)
    apply (rule-tac x=drop (Suc i) a in exI)
    apply (rule-tac x=take j b in exI)
    using EF F assms(1) hyp splits-at-def by auto
qed

lemma LeftDerivationIntro-implies-nonterminal:
  LeftDerivationIntro α i (snd e) ix E j γ ⟹ is-nonterminal (α ! i)
by (simp add: LeftDerivationIntro-def LeftDerives1-def leftmost-is-nonterminal)

lemma LeftDerivationIntrosAt-implies-nonterminal:
  LeftDerivationIntrosAt a D L index ⟹ is-nonterminal((ladder-α a D L index) !
(ladder-i L index))
by (meson LeftDerivationIntro-implies-nonterminal LeftDerivationIntrosAt-def)

lemma LeftDerivationIntro-examine-rule:
  LeftDerivationIntro α i r ix D j γ ⟹ splits-at α i α1 M α2 ⟹
    ∃ η. M = fst r ∧ η = snd r ∧ (M, η) ∈ ℜ
by (metis Derives1-nonterminal Derives1-rule LeftDerivationIntro-def LeftDerives1-implies-Derives1

  prod.collapse)

lemma LeftDerivation-skip-prefixword-ex:
  assumes LeftDerivation (u@v) D w
  assumes is-word u
```

shows $\exists\ w'.\ w = u@w' \wedge$ *LeftDerivation v* (*derivation-shift D* (*length u*) *0*) *w'*
by (*metis LeftDerivation.simps*(*1*) *LeftDerivation-breakdown LeftDerivation-implies-Derivation*

LeftDerivation-skip-prefix append-eq-conv-conj assms(*1*) *assms*(*2*) *is-word-Derivation*

is-word-Derivation-derivation-ge)

definition *ladder-cut* :: *ladder* ⇒ *nat* ⇒ *ladder*
where *ladder-cut L n* = (*let i* = *length L* − *1 in L*[*i* := (*n*, *snd* (*L* ! *i*))])

fun *deriv-shift* :: *nat* ⇒ *nat* ⇒ *deriv* ⇒ *deriv*
where *deriv-shift dn dj* (*n*, *j*, *i*) = (*n* − *dn*, *j* − *dj*, *i*)

definition *ladder-shift* :: *ladder* ⇒ *nat* ⇒ *nat* ⇒ *ladder*
where *ladder-shift L dn dj* = *map* (*deriv-shift dn dj*) *L*

lemma *splits-at-append-suffix-prevails*:
assumes *splits-at* (*a*@*b*) *i u N v*
assumes *i* < *length a*
shows $\exists\ v'.\ v = v'@b \wedge a{=}u@[N]@v'$
proof −
have *min* (*length a*) (*Suc i*) = *Suc i*
using *Suc-leI assms*(*2*) *min.absorb2* **by** *blast*
then show *?thesis*
by (*metis* (*no-types*) *append-assoc append-eq-conv-conj append-take-drop-id assms*(*1*)
hd-drop-conv-nth length-take splits-at-def take-hd-drop)
qed

lemma *derivation-shift-right-left-cancel*:
derivation-shift (*derivation-shift D 0 r*) *r 0* = *D*
by (*induct D*, *auto*)

lemma *derivation-shift-left-right-cancel*:
assumes *derivation-ge D r*
shows *derivation-shift* (*derivation-shift D r 0*) *0 r* = *D*
using *assms derivation-ge-shift-simp derivation-shift-0-shift* **by** *auto*

lemma *LeftDerivation-ge-take*:
assumes *derivation-ge D k*
assumes *LeftDerivation a D b*
assumes *D* ≠ []
shows *take k a* = *take k b* ∧ *is-word* (*take k a*)
proof −
obtain *d D'* **where** *d*: *d*#*D'* = *D* **using** *assms*(*3*) *list.exhaust* **by** *blast*
then have $\exists\ x.$ *LeftDerives1 a* (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x D' b*
using *LeftDerivation.simps*(*2*) *assms*(*2*) **by** *blast*
then obtain *x* **where** *x*: *LeftDerives1 a* (*fst d*) (*snd d*) *x* ∧ *LeftDerivation x D'*
b **by** *blast*

have *fst-d-k*: *fst d* ≥ *k* **using** *d assms*(*1*) *derivation-ge-cons* **by** *blast*
from *x fst-d-k* **have** *is-word*: *is-word* (*take k a*)
by (*metis LeftDerives1-def append-take-drop-id is-word-append leftmost-def min.absorb2 take-append take-take*)
have *is-eq*: *take k a* = *take k b*
using *Derivation-take LeftDerivation-implies-Derivation assms*(*1*) *assms*(*2*) **by** *blast*
show *?thesis* **using** *is-word is-eq* **by** *blast*
qed

lemma *LeftDerivationFix-splits-at-symbol*:
assumes *LeftDerivationFix a i D j b*
shows ∃ *U a1 a2 b1 b2 n*. *splits-at a i a1 U a2* ∧ *splits-at b j b1 U b2* ∧
n ≤ *length D* ∧ *LeftDerivation a1* (*take n D*) *b1* ∧ *derivation-ge* (*drop n D*) (*Suc*(*length b1*)) ∧
LeftDerivation a2 (*derivation-shift* (*drop n D*) (*Suc*(*length b1*)) *0*) *b2* ∧
(*n* = *length D* ∨ (*n* < *length D* ∧ *is-word* (*b1*@[*U*])))
proof −
note *hyp* = *LeftDerivationFix-def*[**where** α=*a* **and** β=*b* **and** *D*=*D* **and** *i*=*i* **and** *j*=*j*]
from *hyp* **obtain** *E F* **where** *EF*:
D = *E* @ *derivation-shift F 0* (*Suc j*) ∧ *LeftDerivation* (*take i a*) *E* (*take j b*) ∧
LeftDerivation (*drop* (*Suc i*) *a*) *F* (*drop* (*Suc j*) *b*)
using *assms* **by** *blast*
have ∃ β. *LeftDerivation a E* β ∧ *LeftDerivation* β (*derivation-shift F 0* (*Suc j*)) *b*
using *EF LeftDerivation-append assms*(*1*) *hyp* **by** *blast*
then obtain β **where** β-*intro*:
LeftDerivation a E β ∧ *LeftDerivation* β (*derivation-shift F 0* (*Suc j*)) *b* **by** *blast*
have *LeftDerivation* ((*take i a*)@(*drop i a*)) *E* ((*take j b*)@(*drop i a*))
by (*metis EF LeftDerivation-append-suffix append-take-drop-id assms*(*1*) *hyp is-sentence-concat*)
then have *LeftDerivation a E* ((*take j b*)@(*drop i a*)) **by** *simp*
then have β-*decomposed*: β = (*take j b*)@(*drop i a*)
using *Derivation-unique-dest LeftDerivation-implies-Derivation* β-*intro* **by** *blast*

have *derivation*: *LeftDerivation* β (*derivation-shift F 0* (*Suc j*)) *b*
by (*simp add*: β-*intro*)
have ∃ *n*. *n* ≤ *length D* ∧ *E* = *take n D*
by (*metis EF append-eq-conv-conj is-prefix-length is-prefix-take*)
then obtain *n* **where** *n*: *n* ≤ *length D* ∧ *E* = *take n D* **by** *blast*
have *F-def*: *drop n D* = *derivation-shift F 0* (*Suc j*)
by (*metis EF append-eq-conv-conj length-take min.absorb2 n*)
have *min-j*: *min* (*length b*) *j* = *j* **using** *assms hyp* **by** *linarith*
have *derivation-ge-Suc-j*: *derivation-ge* (*drop n D*) (*Suc j*)
using *F-def derivation-ge-shift* **by** *simp*
have *LeftDerivation-*β*-b*: *LeftDerivation* β (*drop n D*) *b* **by** (*simp add*: *F-def*

```
β-intro)
  have is-word-Suc-j-b: n ≠ length D ⟹ is-word (take (Suc j) b)
   by (metis EF F-def LeftDerivation-ge-take β-intro append-Nil2 derivation-ge-Suc-j

      length-take min.absorb2 n)
  have take-Suc-j-b-decompose: take (Suc j) b = take j b @ [a ! i]
    using assms hyp take-Suc-conv-app-nth by auto
  show ?thesis
    apply (rule-tac x=a ! i in exI)
    apply (rule-tac x=take i a in exI)
    apply (rule-tac x=drop (Suc i) a in exI)
    apply (rule-tac x=take j b in exI)
    apply (rule-tac x=drop (Suc j) b in exI)
    apply (rule-tac x=n in exI)
    apply (auto simp add: min-j)
    using assms hyp splits-at-def apply blast
    using assms hyp splits-at-def apply blast
    using n apply blast
    using EF n apply simp
    using F-def apply simp
    using derivation-ge-shift apply blast
    using F-def  derivation-shift-right-left-cancel apply simp
    using EF apply blast
    using n apply arith
    using is-word-Suc-j-b  take-Suc-j-b-decompose is-word-append apply simp+
    done
qed

lemma LeftDerivation-breakdown′: LeftDerivation (u @ v) D w ⟹
  ∃ n w1 w2.
    n ≤ length D ∧
    w = w1 @ w2 ∧
    LeftDerivation u (take n D) w1 ∧
    derivation-ge (drop n D) (length w1) ∧
    LeftDerivation v (derivation-shift (drop n D) (length w1) 0) w2
proof −
  assume hyp: LeftDerivation (u @ v) D w
  from LeftDerivation-breakdown[OF hyp] obtain n w1 w2 where breakdown:
   w = w1 @ w2 ∧
    LeftDerivation u (take n D) w1 ∧
    derivation-ge (drop n D) (length w1) ∧
    LeftDerivation v (derivation-shift (drop n D) (length w1) 0) w2 by blast
  obtain m where m: m = min (length D) n by blast
  have take-m: take m D = take n D using m is-prefix-take take-prefix by fastforce
  have drop-m: drop m D = drop n D
    by (metis append-eq-conv-conj append-take-drop-id length-take m)
  have m-bound: m ≤ length D by (simp add: m)
  show ?thesis
    apply (rule-tac x=m in exI)
```

```
    apply (rule-tac x=w1 in exI)
    apply (rule-tac x=w2 in exI)
    using breakdown m-bound take-m drop-m by auto
qed

lemma LeftDerives1-append-replace-in-left:
  assumes ld1: LeftDerives1 (α@δ) i r β
  assumes i-bound: i < length α
  shows ∃ α'. β = α'@δ ∧ LeftDerives1 α i r α' ∧ i + length (snd r) ≤ length α'
proof −
  obtain α' where α': α' = (take i α)@(snd r)@(drop (i+1) α) by blast
  have fst-r: fst r = α ! i
  proof −
    have ∀ ss n p ssa. ¬ LeftDerives1 ss n p ssa ∨ Derives1 ss n p ssa
      using LeftDerives1-implies-Derives1 by blast
    then have Derives1 (α @ δ) i r β
      using ld1 by blast
    then show ?thesis
      using Derives1-nonterminal i-bound splits-at-def by auto
  qed
  have Derives1 α i r α'
    using i-bound ld1
    apply (auto simp add: α' Derives1-def)
    apply (rule-tac x=take i α in exI)
    apply (rule-tac x=drop (i+1) α in exI)
    apply (rule-tac x=fst r in exI)
    apply auto
    apply (simp add: fst-r)
    using id-take-nth-drop apply blast
    using Derives1-sentence1 LeftDerives1-implies-Derives1 is-sentence-concat
      is-sentence-take apply blast
   apply (metis Derives1-sentence1 LeftDerives1-implies-Derives1 append-take-drop-id

      is-sentence-concat)
    using Derives1-rule LeftDerives1-implies-Derives1 by blast
  then have leftderives1-α-α': LeftDerives1 α i r α'
    using LeftDerives1-def i-bound ld1 leftmost-cons-less by auto
  have i-bound-α': i + length (snd r) ≤ length α'
    using α' i-bound
    by (simp add: add-mono-thms-linordered-semiring(2) le-add1 less-or-eq-imp-le
min.absorb2)
  have is-sentence-δ: is-sentence δ
    using Derives1-sentence1 LeftDerives1-implies-Derives1 is-sentence-concat ld1
by blast
  then have β: β = α'@δ
    using ld1 leftderives1-α-α' Derives1-append-suffix Derives1-unique-dest
      LeftDerives1-implies-Derives1 by blast
  show ?thesis
    apply (rule-tac x=α' in exI)
```

```
    using β i-bound-α′ leftderives1-α-α′ by blast
qed

lemma LeftDerivationIntro-propagate:
  assumes intro: LeftDerivationIntro (α@δ) i r ix D j γ
  assumes i-α: i < length α
  assumes non: is-nonterminal (γ ! j)
  shows ∃ ω. LeftDerivation α ((i,r)#D) ω ∧ γ = ω@δ ∧ j < length ω
proof −
  from intro LeftDerivationIntro-def[where α=α@δ and i=i and r=r and ix=ix
and D=D and
    j=j and γ=γ]
  obtain β where ld-β: LeftDerives1 (α @ δ) i r β and
     ix: ix < length (snd r) ∧ snd r ! ix = γ ! j and
     β-fix: LeftDerivationFix β (i + ix) D j γ
     by blast
  from LeftDerives1-append-replace-in-left[OF ld-β i-α]
  obtain α′ where α′: β = α′ @ δ ∧ LeftDerives1 α i r α′ ∧ i + length (snd r)
≤ length α′
    by blast
  have i-plus-ix-bound: i + ix < length α′ using α′ ix by linarith
  have ld-γ: LeftDerivationFix (α′ @ δ) (i + ix) D j γ
    using β-fix α′ by simp
  then have non-i-ix: is-nonterminal ((α′ @ δ) ! (i + ix))
    by (simp add: LeftDerivationFix-def non)
  from LeftDerivationFix-splits-at-nonterminal[OF ld-γ non-i-ix]
  obtain U a1 a2 b1 where U:
    splits-at (α′ @ δ) (i + ix) a1 U a2 ∧ splits-at γ j b1 U a2 ∧ LeftDerivation a1
D b1
    by blast
  have ∃ q. a2 = q@δ ∧ α′ = a1 @ [U] @ q
    using splits-at-append-suffix-prevails[OF - i-plus-ix-bound, where b=δ] U by
blast
  then obtain q where q: a2 = q@δ ∧ α′ = a1 @ [U] @ q by blast
  show ?thesis
    apply (rule-tac x=b1@[U]@q in exI)
    apply auto
    apply (rule-tac x=α′ in exI)
    apply (metis LeftDerivationFix-def LeftDerivation-append-suffix U α′
      q append-Cons append-Nil is-sentence-concat ld-γ)
    using U q splits-at-combine apply auto[1]
    using U splits-at-def by auto
qed

lemma LeftDerivationIntro-finish:
  assumes intro: LeftDerivationIntro (α@δ) i r ix D j γ
  assumes i-α: i < length α
  shows ∃ k ω δ′.
    k ≤ length D ∧
```

```
    LeftDerivation α ((i, r)#(take k D)) ω ∧
    LeftDerivation (α @ δ) ((i, r)#(take k D)) (ω @ δ) ∧
    derivation-ge (drop k D) (length ω) ∧
    LeftDerivation δ (derivation-shift (drop k D) (length ω) 0) δ' ∧
    γ = ω @ δ' ∧ j < length ω
proof −
  from intro LeftDerivationIntro-def[where α=α@δ and i=i and r=r and ix=ix
and D=D and
    j=j and γ=γ]
  obtain β where ld-β: LeftDerives1 (α @ δ) i r β and
    ix: ix < length (snd r) ∧ snd r ! ix = γ ! j and
    β-fix: LeftDerivationFix β (i + ix) D j γ
    by blast
  from LeftDerives1-append-replace-in-left[OF ld-β i-α]
  obtain α' where α': β = α' @ δ ∧ LeftDerives1 α i r α' ∧ i + length (snd r)
≤ length α'
    by blast
  have i-plus-ix-bound: i + ix < length α' using α' ix by linarith
  have ld-γ: LeftDerivationFix (α' @ δ) (i + ix) D j γ
    using β-fix α' by simp
  from LeftDerivationFix-splits-at-symbol[OF ld-γ]
  obtain U a1 a2 b1 b2 n where U:
    splits-at (α' @ δ) (i + ix) a1 U a2 ∧
    splits-at γ j b1 U b2 ∧
    n ≤ length D ∧
    LeftDerivation a1 (take n D) b1 ∧
    derivation-ge (drop n D) (Suc (length b1)) ∧
    LeftDerivation a2 (derivation-shift (drop n D) (Suc (length b1)) 0) b2 ∧
    (n = length D ∨ n < length D ∧ is-word (b1 @ [U]))
    by blast
  have n-bound: n ≤ length D using U by blast
  have ∃ q. a2 = q@δ ∧ α' = a1 @ [U] @ q
    using splits-at-append-suffix-prevails[OF - i-plus-ix-bound, where b=δ] U by
blast
  then obtain q where q: a2 = q@δ ∧ α' = a1 @ [U] @ q by blast
  have j: j = length b1
   using U by (simp add: dual-order.strict-implies-order min.absorb2 splits-at-def)
  have n = length D ∨ n < length D ∧ is-word (b1 @ [U]) using U by blast
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      from 1 have drop-n-D: drop n D = [] by (simp add: U)
      then have LeftDerivation a2 [] b2 using U by simp
      then have a2-eq-b2: a2 = b2 by simp
        show ?case
          apply (rule-tac x=n in exI)
          apply (rule-tac x=b1@[U]@q in exI)
          apply (rule-tac x=δ in exI)
          apply auto
```

```
        apply (simp add: 1)
        apply (rule-tac x=α' in exI)
       apply (metis LeftDerivationFix-is-sentence LeftDerivation-append-suffix U
α'
          append-Cons append-Nil is-sentence-concat ld-γ q)
        apply (rule-tac x=α' @ δ in exI)
        apply (metis 1.hyps LeftDerivationFix-def U α' a2-eq-b2 id-take-nth-drop
ld-β
          ld-γ q splits-at-def take-all)
        apply (simp add: drop-n-D)+
        apply (metis U a2-eq-b2 id-take-nth-drop q splits-at-def)
        using j apply arith
        done
  next
    case 2
     obtain E where E: E = (derivation-shift (drop n D) (Suc (length b1)) 0)
by blast
     then have LeftDerivation (q@δ) E b2 using U q  by simp
     from LeftDerivation-breakdown'[OF this] obtain n' w1 w2 where w1w2:
       n' ≤ length E ∧
        b2 = w1 @ w2 ∧
        LeftDerivation q (take n' E) w1 ∧
        derivation-ge (drop n' E) (length w1) ∧
        LeftDerivation δ (derivation-shift (drop n' E) (length w1) 0) w2 by blast
     have length-E-D: length E = length D − n using E n-bound by simp
      have n-plus-n'-bound: n + n' ≤ length D using length-E-D w1w2 n-bound
by arith
     have take-breakdown: take (n + n') D = (take n D) @ (take n' (drop n D))
       using take-add by blast
     have q-w1: LeftDerivation q (take n' E) w1 using w1w2 by blast
     have isw: is-word (b1 @ [U]) using 2 by blast
    have take-n': take n' (drop n D) = derivation-shift (take n' E) 0 (Suc (length
b1))
       by (metis E U derivation-shift-left-right-cancel take-derivation-shift)
      have α'-derives-b1-U-w1: LeftDerivation α' (take (n + n') D) (b1 @ U #
w1)
       apply (subst take-breakdown)
       apply (rule-tac LeftDerivation-implies-append[where b=b1@[U]@q])
       apply (metis LeftDerivationFix-is-sentence LeftDerivation-append-suffix U
         is-sentence-concat ld-γ q)
       apply (simp add: take-n')
        by (rule LeftDerivation-append-prefix[OF q-w1, where u = b1@[U], OF
isw, simplified])
     have dge: derivation-ge (drop (n + n') D) (Suc (length b1 + length w1))
     proof −
       have derivation-ge (drop n' (drop n D)) (length b1 + 1 + length w1)
            by (metis (no-types) E Suc-eq-plus1 U append-take-drop-id deriva-
tion-ge-append derivation-ge-shift-plus drop-derivation-shift w1w2)
       then show derivation-ge (drop (n + n') D) (Suc (length b1 + length w1))
```

```
      by (metis (no-types) Suc-eq-plus1 add.commute drop-drop semiring-normalization-rules(23))
    qed
    show ?case
      apply (rule-tac x=n+n' in exI)
      apply (rule-tac x=b1 @ [U] @ w1 in exI)
      apply (rule-tac x=w2 in exI)
      apply auto
      using n-plus-n'-bound apply simp
      apply (rule-tac x=α' in exI)
      using α' α'-derives-b1-U-w1 apply blast
      apply (rule-tac x=α' @ δ in exI)
        apply (metis Cons-eq-appendI LeftDerivationFix-is-sentence LeftDeriva-
tion-append-suffix
        α' α'-derives-b1-U-w1 append-assoc is-sentence-concat ld-β ld-γ)
      apply (rule dge)
    apply (metis E Suc-eq-plus1 add.commute derivation-shift-0-shift drop-derivation-shift

      drop-drop w1w2)
      using U splits-at-combine w1w2 apply auto[1]
      by (simp add: j)
  qed
qed

lemma LeftDerivationLadder-propagate:
  LeftDerivationLadder (α@δ) D L γ ⟹ ladder-i L 0 < length α ⟹ n = ladder-n
L index
  ⟹ index < length L ⟹
    if (index + 1 < length L) then
      (∃ β. LeftDerivation α (take n D) β ∧ ladder-γ (α@δ) D L index = β@δ ∧
        ladder-j L index < length β)
    else
    (∃ n' β δ'. (index = 0 ∨ ladder-prev-n L index < n') ∧ n' ≤ n ∧ LeftDerivation
α (take n' D) β ∧
        LeftDerivation (α@δ) (take n' D) (β@δ) ∧
        derivation-ge (drop n' D) (length β) ∧
        LeftDerivation δ (derivation-shift (drop n' D) (length β) 0) δ' ∧
        ladder-γ (α@δ) D L index = β@δ' ∧ ladder-j L index < length β)
proof (induct index arbitrary: n)
  case 0
  have ldfix:
    LeftDerivationFix (α@δ) (ladder-i L 0) (take n D) (ladder-j L 0) (ladder-γ
(α@δ) D L 0)
    using 0.prems(1) 0.prems(3) LeftDerivationLadder-def by blast
  from 0 have 1 < length L ∨ 1 = length L by arith
  then show ?case
  proof (induct rule: disjCases2)
    case 1
    have LeftDerivationIntrosAt (α@δ) D L 1
      using 0.prems(1) 1.hyps LeftDerivationIntros-def LeftDerivationLadder-def
```

by *blast*
from *LeftDerivationIntrosAt-implies-nonterminal*[*OF this*]
have *is-nonterminal* (*ladder-γ* (*α* @ *δ*) *D L 0* ! *ladder-j L 0*)
by (*simp add*: *ladder-α-def ladder-i-def*)
with *ldfix* **have** *is-nonterminal* ((*α*@*δ*) ! (*ladder-i L 0*)) **by** (*simp add*: *LeftDerivationFix-def*)
from *LeftDerivationFix-splits-at-nonterminal*[*OF ldfix this*] **obtain** *U a1 a2 b* **where** *thesplit*:
splits-at (*α* @ *δ*) (*ladder-i L 0*) *a1 U a2* ∧
splits-at (*ladder-γ* (*α* @ *δ*) *D L 0*) (*ladder-j L 0*) *b U a2* ∧
LeftDerivation a1 (*take n D*) *b* **by** *blast*
have ∃ *δ′*. *a2* = *δ′* @ *δ* ∧ *α* = *a1* @ [*U*] @ *δ′*
using *thesplit splits-at-append-suffix-prevails* **using** *0.prems*(*2*) **by** *blast*
then obtain *δ′* **where** *δ′*: *a2* = *δ′* @ *δ* ∧ *α* = *a1* @ ([*U*] @ *δ′*) **by** *blast*
obtain *β* **where** *β*: *β* = *b* @ ([*U*] @ *δ′*) **by** *blast*
have *is-sentence α* **using** *LeftDerivationFix-is-sentence is-sentence-concat ldfix* **by** *blast*
then have *is-sentence* ([*U*] @ *δ′*) **using** *δ′ is-sentence-concat* **by** *blast*
with *δ′ thesplit* **have** *LeftDerivation* (*a1* @ ([*U*] @ *δ′*)) (*take n D*) (*b* @ ([*U*] @ *δ′*))
using *LeftDerivation-append-suffix* **by** *blast*
then have *α-derives-β*: *LeftDerivation α* (*take n D*) *β* **using** *β δ′* **by** *blast*
have *β-append-δ*: *ladder-γ* (*α* @ *δ*) *D L 0* = *β*@*δ*
by (*metis β δ′ append-assoc splits-at-combine thesplit*)
have *ladder-j-bound*: *ladder-j L 0* < *length β*
by (*metis One-nat-def β diff-add-inverse dual-order.strict-implies-order leD le-add1*
length-Cons length-append length-take list.size(*3*) *min.absorb2 neq0-conv splits-at-def*
thesplit zero-less-diff zero-less-one)
show *?case*
using *1* **apply** *simp*
apply (*rule-tac x*=*β* **in** *exI*)
by (*auto simp add*: *α-derives-β β-append-δ ladder-j-bound*)
next
case *2*
note *case-2* = *2*
have *n-def*: *n* = *length D*
by (*metis 0.prems*(*1*) *0.prems*(*3*) *2.hyps LeftDerivationLadder-def One-nat-def*
diff-Suc-1 is-ladder-def ladder-last-n-intro)
then have *take-n-D*: *take n D* = *D* **by** (*simp add*: *eq-imp-le*)
from *LeftDerivationFix-splits-at-symbol*[*OF ldfix*] **obtain** *U a1 a2 b1 b2 m* **where** *U*:
splits-at (*α* @ *δ*) (*ladder-i L 0*) *a1 U a2* ∧
splits-at (*ladder-γ* (*α* @ *δ*) *D L 0*) (*ladder-j L 0*) *b1 U b2* ∧
m ≤ *length* (*take n D*) ∧
LeftDerivation a1 (*take m* (*take n D*)) *b1* ∧
derivation-ge (*drop m* (*take n D*)) (*Suc* (*length b1*)) ∧

```
      LeftDerivation a2 (derivation-shift (drop m (take n D)) (Suc (length b1)) 0)
b2 ∧
       (m = length (take n D) ∨ (m < length (take n D) ∧ is-word (b1 @ [U])))
by blast
   obtain D' where D': D' = derivation-shift (drop m D) (Suc (length b1)) 0 by
blast
    then have a2-derives-b2: LeftDerivation a2 D' b2 using U take-n-D by auto
    from U have m-leq-n: m ≤ n
      by (simp add: 0.prems(1) 0.prems(3) 0.prems(4) LeftDerivationLadder-def
is-ladder-def
        min.absorb2)
    from U have splits-at (α @ δ) (ladder-i L 0) a1 U a2 by blast
    from splits-at-append-suffix-prevails[OF this 0(2)] obtain v' where
      v': a2 = v' @ δ ∧ α = a1 @ [U] @ v' by blast
    have a1-derives-b1: LeftDerivation a1 (take m D) b1 using m-leq-n U
     by (metis 0.prems(1) 0.prems(3) 2.hyps LeftDerivationLadder-def One-nat-def

         cancel-comm-monoid-add-class.diff-cancel is-ladder-def ladder-last-n-intro
order-refl
        take-all)
    have LeftDerivation (v' @ δ) D' b2 using a2-derives-b2 v' by simp
    from LeftDerivation-breakdown'[OF this] obtain m' w1 w2 where w12:
      b2 = w1 @ w2 ∧
       m' ≤ length D' ∧
       LeftDerivation v' (take m' D') w1 ∧
       derivation-ge (drop m' D') (length w1) ∧
       LeftDerivation δ (derivation-shift (drop m' D') (length w1) 0) w2 by blast
    have length D' ≤ length D − m by (simp add: D')
    then have m' ≤ length D − m using w12 dual-order.trans by blast
  then have m-m'-leq-n: m + m' ≤ n using n-def m-leq-n le-diff-conv2 add.commute

      by linarith
    obtain β where β: β = b1 @ ([U] @ w1) by blast
    have is-sentence ([U] @ v')
      using LeftDerivationFix-is-sentence is-sentence-concat ldfix v' by blast
    then have LeftDerivation (a1 @ ([U] @ v')) (take m D) (b1 @ ([U] @ v'))
      using LeftDerivation-append-suffix a1-derives-b1 by blast
    then have α-derives-pre-β: LeftDerivation α (take m D) (b1 @ ([U] @ v'))
      using v' by blast
    have m = n ∨ (m < n ∧ is-word (b1 @ [U]))
      using U n-def[symmetric] take-n-D by simp
   then have pre-β-derives-β: LeftDerivation (b1 @ ([U] @ v')) (take m' (drop m
D)) β
    proof (induct rule: disjCases2)
      case 1
       then have m' = 0 using m-m'-leq-n by arith
       then show ?case
        apply (simp add: β)
        using w12 by simp
```

```
    next
      case 2
        then have is-word-prefix: is-word (b1 @ [U]) by blast
        have take-drop-eq: take m' (drop m D) = derivation-shift (take m' D')
            0 (length (b1 @ [U]))
            apply (simp add: D' take-derivation-shift)
         by (metis U derivation-shift-left-right-cancel take-derivation-shift take-n-D)
        have v'-derives-w1: LeftDerivation v' (take m' D') w1
          by (simp add: w12)
        with is-word-prefix have
          LeftDerivation ((b1 @ [U]) @ v') (derivation-shift (take m' D')
            0 (length (b1 @ [U]))) ((b1 @ [U]) @ w1)
          using  LeftDerivation-append-prefix by blast
        with take-drop-eq show ?case by (simp add: β)
    qed
    have (take m D) @ (take m' (drop m D)) = (take (m + m') D)
      by (simp add: take-add)
    then have α-derives-β: LeftDerivation α (take (m + m') D) β
     using LeftDerivation-implies-append α-derives-pre-β pre-β-derives-β by fast-
force
    have derivation-ge-drop-m-m': derivation-ge (drop (m + m') D) (length β)
      proof −
        have f1: drop m' (drop m D) = drop (m + m') D
          by (simp add: add.commute)
        have derivation-ge (drop m' (drop m D)) (Suc (length b1))
         by (metis (no-types) U append-take-drop-id derivation-ge-append take-n-D)
        then show derivation-ge (drop (m + m') D) (length β)
          using f1 by (metis (no-types) D' β append-assoc derivation-ge-shift-plus
            drop-derivation-shift length-append length-append-singleton w12)
      qed
   have δ-derives-w2: LeftDerivation δ (derivation-shift (drop (m + m') D) (length
β) 0) w2
      proof −
        have derivation-shift (drop m' D') (length w1) 0 = derivation-shift (drop
(m + m') D) (length β) 0
       by (simp add: D' β add.commute derivation-shift-0-shift drop-derivation-shift)
       then show LeftDerivation δ (derivation-shift (drop (m + m') D) (length β)
0) w2
          using w12 by presburger
      qed
    have ladder-γ-def: ladder-γ (α @ δ) D L 0 = β @ w2
      using U β splits-at-combine w12 by auto
    have ladder-j-bound: ladder-j L 0 < length β using U β splits-at-def by auto
    show ?case
      using 2 apply simp
      apply (rule-tac x=m + m' in exI)
      apply (auto simp add: m-m'-leq-n)
      apply (rule-tac x=β in exI)
      apply (auto simp add: α-derives-β)
```

```
      using LeftDerivationFix-is-sentence LeftDerivation-append-suffix α-derives-β

        is-sentence-concat ldfix apply blast
      using derivation-ge-drop-m-m' apply blast
      apply (rule-tac x=w2 in exI)
      apply auto
      using δ-derives-w2 apply blast
      using ladder-γ-def apply blast
      using ladder-j-bound apply blast
      done
  qed
next
  case (Suc index)
  have step: LeftDerivationIntrosAt (α@δ) D L (Suc index)
   by (metis LeftDerivationIntros-def LeftDerivationLadder-def Suc.prems(1) Suc.prems(4)

      Suc-eq-plus1-left le-add1)
  have index-plus-1-bound: index + 1 < length L
    using Suc.prems(4) by linarith
  then have index-bound: index < length L by arith
  obtain n' where n': n' = ladder-n L index by blast
  from Suc.hyps[OF Suc.prems(1) Suc.prems(2) n' index-bound] index-plus-1-bound

  have ∃ α'. LeftDerivation α (take n' D) α' ∧
    ladder-γ (α@δ) D L index = α'@δ ∧ ladder-j L index < length α'
    by auto
  then obtain α' where α': LeftDerivation α (take n' D) α' ∧
    ladder-γ (α@δ) D L index = α'@δ ∧ ladder-j L index < length α'
    by blast
  have Suc-index-bound: Suc index < length L using Suc.prems by auto
  have is-ladder: is-ladder D L using Suc.prems LeftDerivationLadder-def by auto

  have n-def: n = ladder-n L (Suc index)
    using Suc-index-bound n' by (simp add: Suc.prems(3))
   with n' have n'-less-n: n' < n using is-ladder Suc-index-bound is-ladder-def
lessI by blast
  have ladder-α-is-γ: ladder-α (α@δ) D L (Suc index) = ladder-γ (α@δ) D L index
    by (simp add: ladder-α-def)
  obtain i where i: i = ladder-i L (Suc index) by blast
  obtain e where e: e = (D ! n') by blast
  obtain E where E: E = drop (Suc n') (take n D) by blast
  obtain ix where ix: ix = ladder-ix L (Suc index) by blast
  obtain j where j: j = ladder-j L (Suc index) by blast
  obtain γ where γ: γ = ladder-γ (α@δ) D L (Suc index) by blast
  have intro: LeftDerivationIntro (α'@δ) i (snd e) ix E j γ
    by (metis E LeftDerivationIntrosAt-def α' γ ladder-α-is-γ
      diff-Suc-1 e i ix j local.step n' n-def)
  have is-eq-fst-e: i = fst e
    by (metis LeftDerivationIntrosAt-def diff-Suc-1 e i local.step n')
```

```
  have i-less-α': i < length α' using i α' ladder-i-def by simp
  have (Suc index) + 1 < length L ∨ (Suc index) + 1 = length L
    using Suc-index-bound by arith
  then show ?case
  proof (induct rule: disjCases2)
    case 1
      from 1 have LeftDerivationIntrosAt (α@δ) D L (Suc (Suc index))
        by (metis LeftDerivationIntros-def LeftDerivationLadder-def Suc.prems(1)
          Suc-eq-plus1 Suc-eq-plus1-left le-add1)
      from LeftDerivationIntrosAt-implies-nonterminal[OF this] have
        is-nonterminal (ladder-α (α @ δ) D L (Suc (Suc index)) ! ladder-i L (Suc
(Suc index)))
        by blast
       then have non-γ-j: is-nonterminal (γ ! j) by (simp add: γ j ladder-α-def
ladder-i-def)
      from LeftDerivationIntro-propagate[OF intro i-less-α' non-γ-j]
      obtain ω where ω: LeftDerivation α' ((i, snd e) # E) ω ∧ γ = ω @ δ ∧ j
< length ω
        by blast
      have α-ω: LeftDerivation α ((take n' D)@((i, snd e) # E)) ω
        using α' ω LeftDerivation-implies-append by blast
      have i-e: (i, snd e) = e by (simp add: is-eq-fst-e)
      have take-n-D-e: ((take n' D)@(e # E)) = take n D
      proof −
       obtain nn :: (nat × symbol × symbol list) list ⇒ (nat × nat × nat) list ⇒
nat and
          nna :: (nat × symbol × symbol list) list ⇒ (nat × nat × nat) list ⇒ nat
and
          nnb :: (nat × symbol × symbol list) list ⇒ (nat × nat × nat) list ⇒ nat
where
          f1: (∀ ps psa. ¬ is-ladder ps psa ∨ psa ≠ [] ∧ (∀ n. ¬ n < length psa ∨
            ladder-n psa n ≤ length ps) ∧ (∀ n na. (¬ n < na ∨ ¬ na < length psa)
∨
            ladder-n psa n < ladder-n psa na) ∧ ladder-last-n psa = length ps) ∧
            (∀ ps psa. (psa = [] ∨ nn ps psa < length psa ∧ ¬ ladder-n psa (nn ps
psa) ≤
              length ps ∨ (nna ps psa < nnb ps psa ∧ nnb ps psa < length psa) ∧
              ¬ ladder-n psa (nna ps psa) < ladder-n psa (nnb ps psa) ∨
              ladder-last-n psa ≠ length ps) ∨ is-ladder ps psa)
          using is-ladder-def by moura
        then have f2: ladder-last-n L = length D
          using is-ladder by blast
        have f3: min (ladder-last-n L) n = n
          using f1 by (metis (no-types) Suc-eq-plus1 index-plus-1-bound is-ladder
            min.absorb2 n-def)
        then have take n' (take n D) @ take n D ! n' # E = take n D
          using f2 by (metis E id-take-nth-drop length-take n'-less-n)
        then show ?thesis
          using f3 f2 by (metis (no-types) append-assoc append-eq-conv-conj
```

```
                dual-order.strict-implies-order e length-take min.absorb2 n'-less-n
nth-append)
      qed
      from 1 show ?case
        apply auto
        apply (rule-tac x=ω in exI)
        apply auto
        using α-ω i-e take-n-D-e apply auto[1]
        using γ ω apply blast
        using ω j by blast
  next
    case 2
    from LeftDerivationIntro-finish[OF intro i-less-α'] obtain k ω δ' where kwδ':
      k ≤ length E ∧
       LeftDerivation α' ((i, snd e) # take k E) ω ∧
       LeftDerivation (α' @ δ) ((i, snd e) # take k E) (ω @ δ) ∧
       derivation-ge (drop k E) (length ω) ∧
       LeftDerivation δ (derivation-shift (drop k E) (length ω) 0) δ' ∧
       γ = ω @ δ' ∧ j < length ω by blast
    have ladder-last-n-1: ladder-last-n L = n
      by (metis 2.hyps Suc-eq-plus1 diff-Suc-1 ladder-last-n-def n-def)
    from is-ladder have ladder-last-n-2: ladder-last-n L = length D
      using is-ladder-def by blast
   from ladder-last-n-1 ladder-last-n-2 have n-eq-length-D: n = length D by blast

    have take-split: take (Suc (n' + k)) D = (take n' D) @ ((i, snd e) # take k E)
      apply (simp add: E n-eq-length-D)
      by (metis (no-types, lifting) Cons-eq-appendI add-Suc append-eq-appendI e
        is-eq-fst-e n'-less-n n-eq-length-D prod.collapse
        self-append-conv2 take-Suc-conv-app-nth take-add)
    have α-ω: LeftDerivation α (take (Suc (n' + k)) D) ω
      apply (subst take-split)
      apply (rule LeftDerivation-implies-append[where b=α'])
      apply (simp add: α')
      using kwδ' by blast
    have Suc-n'-k-bound: Suc (n' + k) ≤ n using E kwδ' n'-less-n by auto[1]
    from 2 show ?case
      apply auto
      apply (rule-tac x=Suc (n' + k) in exI)
      apply auto
      apply (simp add: ladder-prev-n-def n')
      using Suc-n'-k-bound apply blast
      apply (rule-tac x=ω in exI)
      apply auto
      using α-ω apply blast
   using α-ω LeftDerivationFix-def LeftDerivationLadder-def LeftDerivation-append-suffix

      Suc.prems(1) is-sentence-concat apply auto[1]
      apply (metis E add.commute add-Suc-right drop-drop kwδ' n-eq-length-D
```

```
nat-le-linear
        take-all)
      apply (rule-tac x=δ' in exI)
      apply auto
    apply (metis E LeftDerivationLadder-ladder-n-bound Suc.prems(1) Suc-index-bound

        add.commute add-Suc-right drop-drop kwδ' n-def n-eq-length-D take-all)
      using γ kwδ' apply blast
      using j kwδ' by blast
  qed
qed

lemma ladder-i-of-cut-at-0:
  assumes L-non-empty: L ≠ []
  shows ladder-i (ladder-cut L n) 0 = ladder-i L 0
proof −
  have length L ≠ 0 using L-non-empty by auto
  then have length L = 1 ∨ length L > 1 by arith
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      then show ?case
        apply (simp add: ladder-cut-def ladder-i-def deriv-i-def)
        by (simp add: assms hd-conv-nth)
  next
    case 2
      then show ?case
        apply (simp add: ladder-cut-def ladder-i-def deriv-i-def)
      by (metis diff-is-0-eq hd-conv-nth leD list-update-nonempty nth-list-update-neq)
  qed
qed

lemma ladder-last-j-of-cut:
  assumes L-non-empty: L ≠ []
  shows ladder-last-j (ladder-cut L n) = ladder-last-j L
proof −
  have length-L-nonzero: length L ≠ 0 using L-non-empty by auto
  then have length-ladder-cut: length (ladder-cut L n) = length L
    by (metis ladder-cut-def length-list-update)
  show ?thesis
    apply (simp add: ladder-last-j-def length-ladder-cut)
    apply (simp add: ladder-cut-def ladder-j-def deriv-j-def)
   by (metis length-L-nonzero diff-less neq0-conv nth-list-update-eq snd-conv zero-less-Suc)
qed

lemma length-ladder-cut:
  assumes L-non-empty: L ≠ []
  shows length (ladder-cut L n) = length L
by (metis ladder-cut-def length-list-update)
```

lemma *ladder-last-n-of-cut*:
assumes *L-non-empty*: $L \neq []$
shows *ladder-last-n* (*ladder-cut L n*) = *n*
proof −
show *?thesis*
apply (*simp add*: *ladder-last-n-def length-ladder-cut*[*OF L-non-empty*])
apply (*simp add*: *ladder-n-def ladder-cut-def deriv-n-def*)
by (*metis assms diff-Suc-less fst-conv length-greater-0-conv nth-list-update-eq*)
qed

lemma *ladder-n-of-cut*:
assumes *L-non-empty*: $L \neq []$
assumes *index* < *length L* − *1*
shows *ladder-n* (*ladder-cut L n*) *index* = *ladder-n L index*
by (*metis assms*(*2*) *ladder-cut-def ladder-n-def nat-neq-iff nth-list-update-neq*)

lemma *ladder-n-prev-bound*:
assumes *ladder*: *is-ladder D L*
assumes *u-bound*: *u* < *length L* − *1*
shows *ladder-n L u* ≤ *ladder-prev-n L* (*length L* − *1*)
proof −
have *ladder-n L u* ≤ *ladder-n L* (*length L* − *2*)
proof −
have *u* < *Suc* (*length L* − *2*)
using *u-bound* **by** *linarith*
then show *?thesis*
by (*metis* (*no-types*) *diff-Suc-less is-ladder-def ladder leI length-greater-0-conv*

not-less-eq numeral-2-eq-2 order.order-iff-strict)
qed
then show *?thesis*
by (*metis One-nat-def Suc-diff-Suc diff-Suc-1 ladder-prev-n-def neq0-conv not-less0*

numeral-2-eq-2 u-bound zero-less-diff)
qed

lemma *ladder-n-last-is-length*:
assumes *is-ladder D L*
shows *ladder-n L* (*length L* − *1*) = *length D*
using *assms is-ladder-def ladder-last-n-intro* **by** *auto*

lemma *derivation-ge-shift-implies-derivation-ge*:
assumes *dge*: *derivation-ge* (*derivation-shift F 0 j*) *k*
shows *derivation-ge F* (*k* − *j*)
proof −
have ⋀ *i r*. (*i*, *r*) ∈ *set* (*derivation-shift F 0 j*) ⟹ *i* ≥ *k*
using *dge derivation-ge-def* **by** *auto*
{

```
    fix i :: nat
    fix r :: symbol × (symbol list)
    assume ir: (i, r) ∈ set F
    then have (i + j, r) ∈ set (derivation-shift F 0 j)
    proof −
      have (i + j, r) ∈ (λp. (fst p − 0 + j, snd p)) ‘ set F
        by (metis (lifting) ir diff-zero image-eqI prod.collapse prod.inject)
      then show ?thesis
        by (simp add: derivation-shift-def)
    qed
    then have i + j ≥ k using dge derivation-ge-def by auto
    then have i ≥ k − j by auto
  }
  then show ?thesis using derivation-ge-def by auto
qed

lemma Derives1-bound′: Derives1 a i r b ⟹ i ≤ length b
  by (metis Derives1-bound Derives1-take append-Nil2 append-take-drop-id drop-eq-Nil

    dual-order.strict-implies-order length-take min.absorb2 nat-le-linear)

lemma LeftDerivation-Derives1-last:
  assumes LeftDerivation a D b
  assumes D ≠ []
  shows Derives1 (Derive a (take (length D − 1) D)) (fst (last D)) (snd (last D))
b
by (metis Derive LeftDerivation-Derive-take-LeftDerives1 LeftDerivation-implies-Derivation

  LeftDerives1-implies-Derives1 assms(1) assms(2) last-conv-nth le-refl length-0-conv
take-all)

lemma last-of-prefix-in-set:
  assumes n < length E
  assumes D = E@F
  shows last E ∈ set (drop n D)
proof −
  have f1: last (drop n E) = last E
    by (simp add: assms(1))
  have drop n E ≠ []
    by (metis (no-types) Cons-nth-drop-Suc assms(1) list.simps(3))
  then show ?thesis
   using f1 by (metis (no-types) append.simps(2) append-butlast-last-id append-eq-conv-conj
assms(2) drop-append in-set-dropD insertCI list.set(2))
qed

lemma LeftDerivationFix-cut-appendix:
  assumes ldfix: LeftDerivationFix (α@δ) i D j (β@δ′)
  assumes α-β: LeftDerivation α (take n D) β
  assumes n-bound: n ≤ length D
```

assumes *dge*: *derivation-ge* (*drop n D*) (*length* β)
assumes *i-in*: $i <$ *length* α
assumes *j-in*: $j <$ *length* β
shows *LeftDerivationFix* α *i* (*take n D*) *j* β
proof −
from *LeftDerivationFix-def*[**where** α=α@δ **and** *i*=*i* **and** *D*=*D* **and** *j*=*j* **and** β=β@δ']
obtain *E F* **where** *EF*:
is-sentence (α @ δ) ∧
is-sentence (β @ δ') ∧
LeftDerivation (α @ δ) *D* (β @ δ') ∧
$i <$ *length* (α @ δ) ∧
$j <$ *length* (β @ δ') ∧
(α @ δ) ! i = (β @ δ') ! j ∧
D = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take i* (α @ δ)) *E* (*take j* (β @ δ')) ∧
LeftDerivation (*drop* (*Suc i*) (α @ δ)) *F* (*drop* (*Suc j*) (β @ δ')) **using** *ldfix*
by *auto*
with *i-in j-in* **have** *take-i-E-take-j*: *LeftDerivation* (*take i* α) *E* (*take j* β)
by (*simp add*: *less-or-eq-imp-le*)
obtain *m* **where** *m*: *m* = *length E* **by** *blast*
{
assume *n-less-m*: $n < m$
then have *E-nonempty*: *E* ≠ [] **using** *gr-implies-not0 list.size*(*3*) *m* **by** *auto*
have *last-E-in-set*: *last E* ∈ *set* (*drop n D*)
using *last-of-prefix-in-set n-less-m m EF* **by** *blast*
obtain *k r* **where** *kr*: (*k*, *r*) = *last E* **by** (*metis old.prod.exhaust*)
have *k-lower-bound*: $k \geq$ *length* β **using** *dge last-E-in-set kr*
by (*metis derivation-ge-def fst-conv*)
have ∃ α'. *Derives1* α' *k r* (*take j* β) **using** *LeftDerivation-Derives1-last take-i-E-take-j*
by (*metis E-nonempty kr fst-conv snd-conv*)
then have $k \leq j$ **by** (*metis Derives1-bound' j-in length-take less-imp-le-nat min.absorb2*)
then have *k-upper-bound*: $k <$ *length* β **using** *j-in* **by** *arith*
from *k-lower-bound k-upper-bound* **have** *False* **by** *arith*
}
then have *m-le-n*: $m \leq n$ **by** *arith*
have *take-i-E-take-j*: *LeftDerivation* (*take i* α) *E* (*take j* β)
by (*simp add*: *take-i-E-take-j*)
have *take n D* = *E* @ (*take* ($n - m$) (*derivation-shift F 0* (*Suc j*)))
using *EF m m-le-n* **by** *auto*
then have *take-n-D*: *take n D* = *E* @ (*derivation-shift* (*take* ($n - m$) *F*) *0* (*Suc j*))
using *take-derivation-shift* **by** *auto*
obtain *F'* **where** *F'*: *F'* = *derivation-shift* (*take* ($n - m$) *F*) *0* (*Suc j*) **by** *blast*

have *LeftDerivation* ((*take i* α)@(*drop i* α)) *E* ((*take j* β)@(*drop i* α))
using *take-i-E-take-j*

```
  by (metis EF LeftDerivation-append-suffix append-take-drop-id is-sentence-concat)

  then have LeftDerivation α E ((take j β)@(drop i α)) by simp
  with take-n-D have take-j-drop-i: LeftDerivation ((take j β)@(drop i α)) F' β
using F'
  by (metis Derivation-unique-dest LeftDerivation-append LeftDerivation-implies-Derivation
α-β)
  have F'-ge: derivation-ge F' (Suc j) using F' derivation-ge-shift by blast
 have drop-append: drop i α = [α!i] @ (drop (Suc i) α) by (simp add: Cons-nth-drop-Suc
i-in)
  have take-append: take j β @ [α!i] = take (Suc j) β
  by (metis LeftDerivationFix-def i-in j-in ldfix nth-superfluous-append take-Suc-conv-app-nth)
  have take-drop-Suc: (take j β)@(drop i α) = (take (Suc j) β)@(drop (Suc i) α)
    by (simp add: drop-append take-append)
  with take-drop-Suc take-j-drop-i have LeftDerivation ((take (Suc j) β)@(drop
(Suc i) α)) F' β
    by auto
  note helper = LeftDerivation-skip-prefix[OF this]
  have len-take: length (take (Suc j) β) = Suc j by (simp add: Suc-leI j-in
min.absorb2)
  have F'-shift: derivation-shift F' (Suc j) 0 = take (n − m) F
    using F' derivation-shift-right-left-cancel by blast
  have LeftDerivation-drop: LeftDerivation (drop (Suc i) α) (take (n − m) F)
(drop (Suc j) β)
    using helper len-take F'-shift F'-ge by auto
  show ?thesis
    apply (auto simp add: LeftDerivationFix-def)
    using LeftDerivationFix-is-sentence is-sentence-concat ldfix apply blast
    using LeftDerivationFix-is-sentence is-sentence-concat ldfix apply blast
    using α-β apply blast
    using i-in apply blast
    using j-in apply blast
    using LeftDerivationFix-def i-in j-in ldfix apply auto[1]
    apply (rule-tac x=E in exI)
    apply (rule-tac x=take (n − m) F in exI)
    apply auto
    using take-n-D apply blast
    using take-i-E-take-j apply blast
    using LeftDerivation-drop by blast
qed

lemma LeftDerivationFix-cut-appendix':
  assumes ldfix: LeftDerivationFix (α@δ) i D j (β@δ')
  assumes α-β: LeftDerivation α D β
  assumes i-in: i < length α
  assumes j-in: j < length β
  shows LeftDerivationFix α i D j β
proof −
  obtain n where n: n = length D by blast
```

```
  have LeftDerivationFix α i (take n D) j β
    apply (rule-tac LeftDerivationFix-cut-appendix)
    apply (rule ldfix)
    using α-β n apply auto[1]
    using n apply auto[1]
    using n apply auto[1]
    using i-in apply blast
    using j-in apply blast
    done
  then show ?thesis using n by auto
qed

lemma LeftDerivationIntro-cut-appendix:
  assumes ldfix: LeftDerivationIntro (α@δ) i r ix D j (β@δ')
  assumes α-β: LeftDerivation α ((i,r)#(take n D)) β
  assumes n-bound: n ≤ length D
  assumes dge: derivation-ge (drop n D) (length β)
  assumes i-in: i < length α
  assumes j-in: j < length β
  shows LeftDerivationIntro α i r ix (take n D) j β
proof −
  from LeftDerivationIntro-def[where α=α@δ and i=i and r=r and ix=ix and
D=D and j=j and γ=β@δ']
  obtain ω where ω: LeftDerives1 (α @ δ) i r ω ∧
      ix < length (snd r) ∧ snd r ! ix = (β @ δ') ! j ∧ LeftDerivationFix ω (i +
ix) D j (β @ δ')
      using ldfix by blast
  then have ∃ α'. ω = α' @ δ ∧ i + length (snd r) ≤ length α'
    using i-in using LeftDerives1-append-replace-in-left by blast
  then obtain α' where α': ω = α' @ δ ∧ i + length (snd r) ≤ length α' by blast
  have α-α': LeftDerives1 α i r α' using α' ω using LeftDerives1-skip-suffix i-in
by blast
  from α-β obtain u where u: LeftDerives1 α i r u ∧ LeftDerivation u (take n
D) β by auto
 with α-α' have u = α' using Derives1-unique-dest LeftDerives1-implies-Derives1
by blast
  with u have α'-β: LeftDerivation α' (take n D) β by auto
  have ldfix-append: LeftDerivationFix (α' @ δ) (i + ix) D j (β @ δ') using α' ω
by blast
  have i-plus-ix-bound: i + ix < length α' using ω α'
   using add-lessD1 le-add-diff-inverse less-asym' linorder-neqE-nat nat-add-left-cancel-less

    by linarith
 from LeftDerivationFix-cut-appendix[OF ldfix-append α'-β n-bound dge i-plus-ix-bound
j-in]
  have ldfix: LeftDerivationFix α' (i + ix) (take n D) j β .
  show ?thesis
    apply (simp add: LeftDerivationIntro-def)
    apply (rule-tac x=α' in exI)
```

```
    apply auto
    using α-α′ apply blast
    using ω apply blast
    apply (simp add: ω j-in)
    using ldfix by blast
qed

lemma LeftDerivationIntro-cut-appendix′:
  assumes ldfix: LeftDerivationIntro (α@δ) i r ix D j (β@δ′)
  assumes α-β: LeftDerivation α ((i,r)#D) β
  assumes i-in: i < length α
  assumes j-in: j < length β
  shows LeftDerivationIntro α i r ix D j β
proof −
  obtain n where n: n = length D by blast
  have LeftDerivationIntro α i r ix (take n D) j β
    apply (rule-tac LeftDerivationIntro-cut-appendix)
    apply (rule ldfix)
    using α-β n apply auto[1]
    using n apply auto[1]
    using n apply auto[1]
    using i-in apply blast
    using j-in apply blast
    done
  then show ?thesis using n by auto
qed

lemma ladder-n-monotone: is-ladder D L ⟹ u ≤ v ⟹ v < length L ⟹ ladder-n
L u ≤ ladder-n L v
by (metis is-ladder-def le-neq-implies-less linear not-less)

lemma ladder-i-cut:
  assumes index-bound: index < length L
  shows ladder-i (ladder-cut L n) index = ladder-i L index
proof −
  have index = 0 ∨ index > 0 by arith
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      with index-bound have L ≠ [] by (simp add: less-numeral-extra(3))
      then show ?case using 1 by (simp add: ladder-i-of-cut-at-0)
  next
    case 2
      then show ?case
        apply (auto simp add: ladder-i-def ladder-cut-def ladder-j-def deriv-j-def
Let-def)
       using index-bound by auto
  qed
qed
```

lemma *ladder-j-cut*:
assumes *index-bound*: *index* < *length L*
shows *ladder-j* (*ladder-cut L n*) *index* = *ladder-j L index*
by (*metis gr-implies-not0 index-bound ladder-cut-def ladder-j-def ladder-last-j-def ladder-last-j-of-cut length-ladder-cut list.size*(*3*) *nth-list-update-neq*)

lemma *ladder-ix-cut*:
assumes *index-lower-bound*: *index* > *0*
assumes *index-upper-bound*: *index* < *length L*
shows *ladder-ix* (*ladder-cut L n*) *index* = *ladder-ix L index*
proof −
show *?thesis*
using *index-lower-bound* **apply** (*simp add*: *ladder-ix-def deriv-ix-def*)
by (*metis index-upper-bound ladder-cut-def nth-list-update-eq nth-list-update-neq snd-conv*)
qed

lemma *LeftDerivation-from-in-between*:
assumes *α-β*: *LeftDerivation α* (*take u D*) *β*
assumes *α-γ*: *LeftDerivation α* (*take v D*) *γ*
assumes *u-le-v*: *u* ≤ *v*
shows *LeftDerivation β* (*drop u* (*take v D*)) *γ*
proof −
have *take-split*: *take v D* = *take u D* @ (*drop u* (*take v D*))
by (*metis u-le-v add-diff-cancel-left′ drop-take le-Suc-ex take-add*)
then show *?thesis* **using** *α-γ α-β*
by (*metis* (*no-types*, *lifting*) *Derivation-unique-dest LeftDerivation-append LeftDerivation-implies-Derivation*)
qed

lemma *LeftDerivationLadder-cut-appendix-helper*:
assumes *LDLadder*: *LeftDerivationLadder* (*α*@*δ*) *D L γ*
assumes *ladder-i-in-α*: *ladder-i L 0* < *length α*
shows ∃ *E F γ1 γ2 L′*. *D* = *E*@*F* ∧
γ = *γ1* @ *γ2* ∧
LeftDerivationLadder α E L′ γ1 ∧
derivation-ge F (*length γ1*) ∧
LeftDerivation δ (*derivation-shift F* (*length γ1*) *0*) *γ2* ∧
L′ = *ladder-cut L* (*length E*)
proof −
have *is-ladder-D-L*: *is-ladder D L* **using** *LDLadder LeftDerivationLadder-def* **by** *blast*
obtain *n* **where** *n*: *n* = *ladder-last-n L* **by** *blast*
then have *n-eq-ladder-n*: *n* = *ladder-n L* (*length L* − *1*) **using** *ladder-last-n-def* **by** *blast*
have *length-L-nonzero*: *length L* > *0*
using *LeftDerivationLadder-def assms*(*1*) *is-ladder-def* **by** *blast*
from *LeftDerivationLadder-propagate*[*OF LDLadder ladder-i-in-α n-eq-ladder-n*]

```
  obtain n' β δ' where finish:
    (length L − 1 = 0 ∨ ladder-prev-n L (length L − 1) < n') ∧
     n' ≤ n ∧
     LeftDerivation α (take n' D) β ∧
     LeftDerivation (α @ δ) (take n' D) (β @ δ) ∧
     derivation-ge (drop n' D) (length β) ∧
     LeftDerivation δ (derivation-shift (drop n' D) (length β) 0) δ' ∧
     ladder-γ (α @ δ) D L (length L − 1) = β @ δ' ∧ ladder-j L (length L − 1) <
length β
     using length-L-nonzero by auto
  obtain E where E: E = take n' D by blast
  obtain F where F: F = drop n' D by blast
  obtain L' where L': L' = ladder-cut L (length E) by blast
  have γ-ladder: γ = ladder-γ (α @ δ) D L (length L − 1)
   by (metis Derive LDLadder LeftDerivationLadder-def LeftDerivation-implies-Derivation

       append-Nil2 append-take-drop-id drop-eq-Nil is-ladder-def ladder-γ-def le-refl
n
      n-eq-ladder-n)
  then have γ: γ = β @ δ' using finish by auto
  have is-sentence (α@δ)
     using LDLadder LeftDerivationFix-is-sentence LeftDerivationLadder-def by
blast
  then have is-sentence-α: is-sentence α using is-sentence-concat by blast
  have is-sentence γ
    using Derivation-implies-derives LDLadder LeftDerivationFix-is-sentence
     LeftDerivationLadder-def LeftDerivation-implies-Derivation derives-is-sentence
by blast
  then have is-sentence-β: is-sentence β using γ is-sentence-concat by blast
  have length-L': length L' = length L
    by (metis L' ladder-cut-def length-list-update)
  have ladder-last-n-L': ladder-last-n L' = length E
    using L' ladder-last-n-of-cut length-L-nonzero by blast
  have length-D-eq-n: length D = n
    using LDLadder LeftDerivationLadder-def is-ladder-def n by auto
  then have length-E-eq-n': length E = n' by (simp add: E finish min.absorb2)
  {
    fix u :: nat
    assume u < length L'
    then have u < length L' − 1 ∨ u = length L' − 1 by arith
    then have ladder-n L' u ≤ length E
    proof (induct rule: disjCases2)
      case 1
        have u-bound: u < length L − 1 using 1 by (simp add: length-L')
        then have L'-eq-L: ladder-n L' u = ladder-n L u using L' ladder-n-of-cut
          length-L-nonzero length-greater-0-conv by blast
        from u-bound have ladder-n L u ≤ ladder-prev-n L (length L − 1)
          using ladder-n-prev-bound LeftDerivationLadder-def assms(1) by blast
        then show ?case
```

```
        using L'-eq-L finish length-E-eq-n' u-bound by linarith
    next
      case 2
       then have ladder-n L' u = length E using ladder-last-n-L' ladder-last-n-def
by auto
        then show ?case by auto
    qed
  }
  note ladder-n-bound = this
  {
    fix u :: nat
    fix v :: nat
    assume u-less-v: u < v
    assume v-bound: v < length L'
    have v < length L' - 1 ∨ v = length L' - 1 using v-bound by arith
    then have ladder-n L' u < ladder-n L' v
    proof (induct rule: disjCases2)
      case 1
        show ?case
            using 1.hyps L' LeftDerivationLadder-def assms(1) is-ladder-def lad-
der-n-of-cut
          length-L' u-less-v by auto
    next
      case 2
        note v-def = 2
        have v = 0 ∨ v ≠ 0 by arith
        then show ?case
        proof (induct rule: disjCases2)
          case 1
          then show ?case using u-less-v by auto
        next
          case 2
         then have ladder-prev-n L (length L - 1) < n' using finish v-def length-L'

            by linarith
          then show ?case
            by (metis (no-types, lifting) L' LeftDerivationLadder-def assms(1)
              ladder-last-n-L' ladder-last-n-def ladder-n-of-cut ladder-n-prev-bound
              le-neq-implies-less length-E-eq-n' length-L' length-greater-0-conv
              less-trans u-less-v v-def)
        qed
    qed
  }
  note ladder-n-pairwise-bound = this

  have is-ladder-E-L': is-ladder E L'
    apply (auto simp add: is-ladder-def ladder-last-n-L')
    using length-L-nonzero length-L' apply simp
    using ladder-n-bound apply blast
```

using *ladder-n-pairwise-bound* **by** *blast*

{
fix *index* :: *nat*
assume *index-bound*: *index* + *1* < *length L*
then have *index-le*: *index* < *length L* **by** *arith*
from *index-bound* **have** *len-L-minus-1*: *length L* − *1* ≠ *0* **by** *arith*
obtain *m* **where** *m*: *m* = *ladder-n L index* **by** *blast*
from *LeftDerivationLadder-propagate*[*OF LDLadder ladder-i-in-α m index-le*]
obtain *ω* **where**
ω: *LeftDerivation α* (*take m D*) *ω* ∧ *ladder-γ* (*α* @ *δ*) *D L index* = *ω* @ *δ* ∧
ladder-j L index < *length ω* **using** *index-bound* **by** *auto*
have *L′-Derive*: *ladder-γ α E L′ index* = *Derive α* (*take* (*ladder-n L′ index*) *E*)
by (*simp add*: *ladder-γ-def*)
have *ladder-n-L′-eq-L*: *ladder-n L′ index* = *ladder-n L index*
using *L′ index-bound ladder-n-of-cut length-L-nonzero* **by** *auto*
have *ladder-prev-n L* (*length L* − *1*) < *n′* **using** *finish len-L-minus-1* **by** *blast*
then have *n′-is-upper-bound*: *ladder-n L* (*length L* − *2*) < *n′* **using** *index-bound*
by (*metis diff-diff-left ladder-prev-n-def len-L-minus-1 one-add-one*)
have *index-upper-bound*: *index* ≤ *length L* − *2* **using** *index-bound* **by** *linarith*

have *ladder-n-upper-bound*: *ladder-n L index* ≤ *ladder-n L* (*length L* − *2*)
apply (*rule-tac ladder-n-monotone*)
apply (*rule-tac is-ladder-D-L*)
apply (*rule index-upper-bound*)
using *length-L-nonzero* **by** *linarith*
with *n′-is-upper-bound* **have** *m-le-n′*: *m* ≤ *n′*
using *dual-order.strict-implies-order le-less-trans m* **by** *linarith*
then have *take m E* = *take m D*
by (*metis E le-take-same length-E-eq-n′ order-refl take-all*)
then have *take-helper*: (*take* (*ladder-n L′ index*) *E*) = *take m D*
by (*simp add*: *ladder-n-L′-eq-L m*)
then have *Derive-eq-ω*: *Derive α* (*take* (*ladder-n L′ index*) *E*) = *ω*
by (*simp add*: *Derive LeftDerivation-implies-Derivation ω*)
then have *t1*: *ladder-γ* (*α*@*δ*) *D L index* = (*ladder-γ α E L′ index*) @ *δ*
by (*simp add*: *L′-Derive ω*)
have *ω-eq*: *ω* = *ladder-γ α E L′ index* **by** (*simp add*: *Derive-eq-ω L′-Derive*)
then have *t2*: *LeftDerivation α* (*take* (*ladder-n L index*) *D*) (*ladder-γ α E L′ index*)
using *ω m* **by** *blast*
have *t3*: (*take* (*ladder-n L′ index*) *E*) = *take* (*ladder-n L index*) *D*
by (*simp add*: *m take-helper*)
have *t4*: *ladder-j L index* < *length* (*ladder-γ α E L′ index*)
using *ω ω-eq* **by** *blast*
have *t5*: *E* ! (*ladder-n L′ index*) = *D* ! (*ladder-n L index*)
using *E ladder-n-L′-eq-L ladder-n-upper-bound n′-is-upper-bound* **by** *auto*
note *t* = *t1 t2 t3 t4 t5*
}

note *ladder-early-stage* = *this*

{
fix *index* :: *nat*
assume *index-bound*: *index* < *length L*
have *i*: *ladder-i L′ index* = *ladder-i L index*
using *L′ ladder-i-cut* **by** (*simp add*: *index-bound*)
have *j*: *ladder-j L′ index* = *ladder-j L index*
using *L′ ladder-j-cut* **by** (*simp add*: *index-bound*)
have *ix*: *index* > *0* ⟹ *ladder-ix L′ index* = *ladder-ix L index*
using *L′ ladder-ix-cut* **by** (*simp add*: *index-bound*)
have *α*: *ladder-α* (*α*@*δ*) *D L index* = (*ladder-α α E L′ index*) @ *δ*
by (*simp add*: *index-bound ladder-α-def ladder-early-stage*(*1*))
have *i-bound*: *index* > *0* ⟹ *ladder-i L′ index* < *length* (*ladder-α α E L′ index*)
using *i index-bound ladder-α-def ladder-early-stage*(*4*) *ladder-i-def* **by** *auto*

note *ij* = *i j ix α i-bound*
}
note *ladder-every-stage* = *this*

{
fix *u* :: *nat*
fix *v* :: *nat*
assume *u-le-v*: *u* ≤ *v*
assume *v-bound*: *v* < *length L*
have *ladder-n L u* ≤ *ladder-n L v*
using *is-ladder-D-L ladder-n-monotone u-le-v v-bound* **by** *blast*
}
note *ladder-L-n-pairwise-le* = *this*

{
fix *index* :: *nat*
assume *index-lower-bound*: *index* > *0*
assume *index-upper-bound*: *index* + *1* < *length L*
note *derivation* = *ladder-early-stage*(*2*)
have *derivation1*:
LeftDerivation α (*take* (*ladder-n L* (*index* − *Suc 0*)) *D*) (*ladder-γ α E L′* (*index* − *Suc 0*))
using *derivation*[*of index* − *Suc 0*] *index-lower-bound index-upper-bound*
using *One-nat-def Suc-diff-1 Suc-eq-plus1 le-less-trans lessI less-or-eq-imp-le* **by** *linarith*
have *derivation2*:
LeftDerivation α (*take* (*ladder-n L index*) *D*) (*ladder-γ α E L′ index*)
using *derivation*[*of index*] *index-upper-bound* **by** *blast*
have *ladder-α-is-γ*[*symmetric*]: *ladder-γ α E L′* (*index* − *Suc 0*) = *ladder-α α E L′ index*
using *index-lower-bound ladder-α-def* **by** *auto*
have *ladder-n-le*: *ladder-n L* (*index* − *Suc 0*) ≤ *ladder-n L index*
apply (*rule-tac ladder-L-n-pairwise-le*)

```
    apply arith
    using index-upper-bound by arith
   from LeftDerivation-from-in-between[OF derivation1 derivation2 ladder-n-le]
ladder-α-is-γ
  have LeftDerivation (ladder-α α E L' index) (drop (ladder-n L' (index − Suc
0))
    (take (ladder-n L' index) E)) (ladder-γ α E L' index)
     by (metis L' Suc-leI add-lessD1 index-lower-bound index-upper-bound lad-
der-early-stage(3)
       ladder-n-of-cut le-add-diff-inverse2 length-L-nonzero length-greater-0-conv
less-diff-conv)
 }
 note LeftDerivation-delta-early = this

 have LeftDerivationFix-α-0: LeftDerivationFix α (ladder-i L' 0) (take (ladder-n
L' 0) E)
  (ladder-j L' 0) (ladder-γ α E L' 0)
 proof −
   have ldfix: LeftDerivationFix (α@δ) (ladder-i L 0) (take (ladder-n L 0) D)
(ladder-j L 0)
    (ladder-γ (α@δ) D L 0)
    using LDLadder LeftDerivationLadder-def by blast
  have ladder-i-L': ladder-i L' 0 = ladder-i L 0
    using L' ladder-i-of-cut-at-0 length-L-nonzero by blast
  have ladder-j-L': ladder-j L' 0 = ladder-j L 0
    by (metis L' ladder-cut-def ladder-j-def ladder-last-j-def ladder-last-j-of-cut
      length-L' length-greater-0-conv nth-list-update-neq)
  have length L = 1 ∨ length L > 1 using length-L-nonzero by linarith
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
     from 1 have ladder-n-L'-0: ladder-n L' 0 = n'
      using diff-is-0-eq' ladder-last-n-L' ladder-last-n-def length-E-eq-n'
        length-L' less-or-eq-imp-le by auto
     have take-n'-E: take n' E = E by (simp add: length-E-eq-n')
     from ladder-n-L'-0 take-n'-E have take-ladder-n-L': take (ladder-n L' 0) E
= E by auto
     have ladder-n L 0 = length D
      by (simp add: 1.hyps length-D-eq-n n-eq-ladder-n)
     then have take-ladder-n-L-0: take (ladder-n L 0) D = D by simp
     have ladder-γ-α: ladder-γ α E L' 0 = β
      apply (simp add: ladder-γ-def take-ladder-n-L')
      by (simp add: Derive E LeftDerivation-implies-Derivation finish)
     have ladder-j-in-β: ladder-j L 0 < length β
      using finish 1.hyps by auto
    have ldfix-1: LeftDerivationFix (α@δ) (ladder-i L 0) D (ladder-j L 0) (β@δ')
      using 1.hyps γ γ-ladder ldfix take-ladder-n-L-0 by auto
     then have LeftDerivationFix α (ladder-i L 0) E (ladder-j L 0) β
       by (simp add: E LeftDerivationFix-cut-appendix finish ladder-i-in-α lad-
```

der-j-in-β
length-D-eq-n)
then show *?case*
by (*simp add*: *ladder-i-L′ ladder-j-L′ take-ladder-n-L′ ladder-γ-α*)
next
case *2*
have *h*: *0 + 1 < length L* **using** *2.hyps* **by** *auto*
note *stage = ladder-early-stage*[*of 0*, *OF h*]
have *ldfix0*: *LeftDerivationFix* (*α@δ*) (*ladder-i L 0*) (*take* (*ladder-n L 0*) *D*) (*ladder-j L 0*)
((*ladder-γ α E L′ 0*) @ *δ*)
using *ladder-i-L′ ladder-j-L′ ldfix stage*(*1*) *stage*(*3*) **by** *auto*
from *LeftDerivationFix-cut-appendix′*[*OF ldfix0 stage*(*2*) *ladder-i-in-α stage*(*4*)]
show *?case*
by (*simp add*: *ladder-i-L′ ladder-j-L′ stage*(*3*))
qed
qed

{
fix *index* :: *nat*
assume *index-bounds*: *1 ≤ index ∧ index + 1 < length L*
have *introsAt-appendix*: *LeftDerivationIntrosAt* (*α@δ*) *D L index*
using *LDLadder LeftDerivationIntros-def LeftDerivationLadder-def add-lessD1 index-bounds*
by *blast*
have *index-plus-1-upper-bound*: *index + 1 < length L* **using** *index-bounds* **by** *arith*
note *early-stage = ladder-early-stage*[*of index*, *OF index-plus-1-upper-bound*]
have *ladder-i-L-index-eq-fst*: *ladder-i L index = fst* (*D ! ladder-n L* (*index − Suc 0*))
using *introsAt-appendix LeftDerivationIntrosAt-def index-bounds* **by** (*metis One-nat-def*)
have *E-at-D-at*: (*E ! ladder-n L′* (*index − Suc 0*)) = (*D ! ladder-n L* (*index − Suc 0*))
using *ladder-early-stage*[*of index − Suc 0*]
by (*metis One-nat-def add-lessD1 index-bounds le-add-diff-inverse2*)
then have *ladder-i-L′-index-eq-fst*: *ladder-i L′ index = fst* (*E ! ladder-n L′* (*index − Suc 0*))
using *index-bounds ladder-i-L-index-eq-fst ladder-every-stage*(*1*) *le-add1 le-less-trans* **by** *auto*
have *same-derivation*: (*drop* (*Suc* (*ladder-n L′* (*index − Suc 0*))) (*take* (*ladder-n L′ index*) *E*)) =
(*drop* (*Suc* (*ladder-n L* (*index − Suc 0*))) (*take* (*ladder-n L index*) *D*))
using *L′ early-stage*(*3*) *index-bounds ladder-n-of-cut length-L-nonzero* **by** *auto*
have *deriv-split*: (*drop* (*ladder-n L′* (*index − Suc 0*)) (*take* (*ladder-n L′ index*) *E*)) =
((*ladder-i L′ index*, *snd* (*E ! ladder-n L′* (*index − Suc 0*))) #
drop (*Suc* (*ladder-n L′* (*index − Suc 0*))) (*take* (*ladder-n L′ index*) *E*))

by (*metis Cons-nth-drop-Suc One-nat-def Suc-le-lessD add-lessD1 diff-Suc-less index-bounds*
ladder-i-L'-index-eq-fst ladder-n-bound ladder-n-pairwise-bound length-L'
length-take min.absorb2 nth-take prod.collapse)
have *LeftDerivationIntrosAt* α *E L' index*
apply (*auto simp add*: *LeftDerivationIntrosAt-def Let-def*)
using *ladder-i-L'-index-eq-fst* **apply** *blast*
apply (*rule-tac LeftDerivationIntro-cut-appendix'*[**where** δ=δ **and** $\delta' = \delta$])
apply (*metis E-at-D-at LeftDerivationIntrosAt-def One-nat-def Suc-le-lessD add-lessD1*
early-stage(*1*) *index-bounds introsAt-appendix ladder-every-stage*(*2*)
ladder-every-stage(*3*) *ladder-every-stage*(*4*) *ladder-i-L'-index-eq-fst same-derivation*)
defer *1*
using *index-bounds ladder-every-stage*(*5*) **apply** *auto*[*1*]
using *early-stage*(*4*) *index-bounds ladder-every-stage*(*2*) **apply** *auto*[*1*]
using *LeftDerivation-delta-early deriv-split*
by (*metis One-nat-def Suc-le-eq index-bounds*)
}
note *LeftDerivationIntrosAt-early* = *this*

{
fix *index* :: *nat*
assume *index-bounds*: 1 $\leq$ *index* $\wedge$ *index* + 1 = *length L*
have *introsAt-appendix*: *LeftDerivationIntrosAt* (*α*@*δ*) *D L index*
using *LDLadder LeftDerivationIntros-def LeftDerivationLadder-def add-lessD1 index-bounds*
by (*metis Suc-eq-plus1 not-less-eq*)
have *ladder-i-L-index-eq-fst*: *ladder-i L index* = *fst* (*D* ! *ladder-n L* (*index* − *Suc 0*))
using *introsAt-appendix LeftDerivationIntrosAt-def index-bounds* **by** (*metis One-nat-def*)
have *E-at-D-at*: (*E* ! *ladder-n L'* (*index* − *Suc 0*)) = (*D* ! *ladder-n L* (*index* − *Suc 0*))
using *ladder-early-stage*[*of index* − *Suc 0*]
by (*metis One-nat-def Suc-eq-plus1 index-bounds le-add-diff-inverse2 lessI*)
then have *ladder-i-L'-index-eq-fst*: *ladder-i L' index* = *fst* (*E* ! *ladder-n L'* (*index* − *Suc 0*))
using *index-bounds ladder-i-L-index-eq-fst ladder-every-stage*(*1*) *le-add1 le-less-trans* **by** *auto*
obtain *D'* **where** *D'*: *D'* = (*drop* (*Suc* (*ladder-n L* (*index* − *Suc 0*))) (*take* (*ladder-n L index*) *D*))
by *blast*
obtain *k* **where** *k*: *k* = (*ladder-n L' index*) − (*Suc* (*ladder-n L'* (*index* − *Suc 0*)))
by *blast*
have *ladder-n-L'-index*: *ladder-n L' index* = *length E*
by (*metis diff-add-inverse2 index-bounds ladder-last-n-L' ladder-last-n-def length-L'*)
have *take-is-E*: *take* (*ladder-n L' index*) *E* = *E* **by** (*simp add*: *ladder-n-L'-index*)

have *ladder-n-L-index*: *ladder-n L index = length D*
by (*metis diff-add-inverse2 index-bounds length-D-eq-n n-eq-ladder-n*)
have *take-is-D*: *take* (*ladder-n L index*) *D = D*
by (*simp add*: *ladder-n-L-index*)
have *write-as-take-k-D'*: (*drop* (*Suc* (*ladder-n L'* (*index − Suc 0*))) *E*) = *take k D'*
using *take-is-D*
by (*metis D' E L' One-nat-def Suc-le-lessD add-diff-cancel-right' diff-Suc-less drop-take index-bounds k ladder-n-L'-index ladder-n-of-cut length-E-eq-n' length-L-nonzero length-greater-0-conv*)
have *k-bound*: *k* ≤ *length D'*
by (*metis le-iff-add append-take-drop-id k ladder-n-L'-index length-append length-drop write-as-take-k-D'*)
have *D'-alt*: *D' = drop* (*Suc* (*ladder-n L* (*index − Suc 0*))) *D*
by (*simp add*: *D' take-is-D*)
have *LeftDerivationIntrosAt α E L' index*
apply (*auto simp add*: *LeftDerivationIntrosAt-def Let-def*)
using *ladder-i-L'-index-eq-fst* **apply** *blast*
apply (*subst take-is-E*)
apply (*subst write-as-take-k-D'*)
apply (*rule-tac LeftDerivationIntro-cut-appendix*[**where** *δ*=*δ* **and** *δ' = δ'*])
apply (*metis D' Derive E E-at-D-at LeftDerivationIntrosAt-def LeftDerivation-implies-Derivation One-nat-def Suc-le-lessD add-diff-cancel-right' diff-Suc-less finish index-bounds introsAt-appendix ladder-γ-def ladder-every-stage*(*2*) *ladder-every-stage*(*3*) *ladder-every-stage*(*4*) *ladder-i-L'-index-eq-fst length-L-nonzero take-is-E*)
apply (*metis Cons-nth-drop-Suc E L' LeftDerivation-from-in-between LeftDerivation-take-derive One-nat-def Suc-le-lessD add-diff-cancel-right' diff-Suc-less finish index-bounds ladder-α-def ladder-γ-def ladder-i-L'-index-eq-fst ladder-n-L'-index ladder-n-of-cut ladder-prev-n-def length-E-eq-n' length-L-nonzero less-imp-le-nat less-numeral-extra*(*3*) *list.size*(*3*) *prod.collapse take-is-E write-as-take-k-D'*)
using *k-bound* **apply** *blast*
using *D'-alt* **apply** (*metis* (*no-types*, *lifting*) *Derive E L' LeftDerivation-implies-Derivation One-nat-def Suc-leI Suc-le-lessD add-diff-cancel-right' diff-Suc-less drop-drop finish index-bounds k ladder-γ-def ladder-n-L'-index ladder-n-of-cut ladder-prev-n-def le-add-diff-inverse2 length-E-eq-n' length-L-nonzero length-greater-0-conv less-not-refl2 take-is-E*)
using *index-bounds ladder-every-stage*(*5*) **apply** *auto*[*1*]

```
    by (metis Derive E LeftDerivation-implies-Derivation One-nat-def add-diff-cancel-right'
      diff-Suc-less finish index-bounds ladder-γ-def ladder-every-stage(2) length-L-nonzero
        take-is-E)
  }
  note LeftDerivationIntrosAt-last = this

  have ladder-E-L': LeftDerivationLadder α E L' β
    apply (auto simp add: LeftDerivationLadder-def)
    using finish E apply blast
    using is-ladder-E-L' apply blast
    using LeftDerivationFix-α-0 apply blast
     using LeftDerivationIntros-def LeftDerivationIntrosAt-early LeftDerivationIn-
trosAt-last
    by (metis Suc-eq-plus1 Suc-leI le-neq-implies-less length-L')

  show ?thesis
    apply (rule-tac x=E in exI)
    apply (rule-tac x=F in exI)
    apply (rule-tac x=β in exI)
    apply (rule-tac x=δ' in exI)
    apply (rule-tac x=L' in exI)
    apply auto
    using E F apply simp
    apply (rule γ)
    using ladder-E-L' apply blast
    using F finish apply blast
    using F finish apply blast
    by (rule L')
qed

theorem LeftDerivationLadder-cut-appendix:
  assumes LDLadder: LeftDerivationLadder (α@δ) D L γ
  assumes ladder-i-in-α: ladder-i L 0 < length α
  shows ∃ E F γ1 γ2 L'. D = E@F ∧
    γ = γ1 @ γ2 ∧
    LeftDerivationLadder α E L' γ1 ∧
    derivation-ge F (length γ1) ∧
    LeftDerivation δ (derivation-shift F (length γ1) 0) γ2 ∧
    length L' = length L ∧ ladder-i L' 0 = ladder-i L 0 ∧
    ladder-last-j L' = ladder-last-j L
proof −
  from LeftDerivationLadder-cut-appendix-helper[OF LDLadder ladder-i-in-α]
  obtain E F γ1 γ2 L' where helper:
    D = E @ F ∧
    γ = γ1 @ γ2 ∧
    LeftDerivationLadder α E L' γ1 ∧
    derivation-ge F (length γ1) ∧
```

```
    LeftDerivation δ (derivation-shift F (length γ1) 0) γ2 ∧ L' = ladder-cut L
(length E)
    by blast
  show ?thesis
    apply (rule-tac x=E in exI)
    apply (rule-tac x=F in exI)
    apply (rule-tac x=γ1 in exI)
    apply (rule-tac x=γ2 in exI)
    apply (rule-tac x=L' in exI)
   using helper LDLadder LeftDerivationLadder-def is-ladder-def ladder-i-of-cut-at-0

     ladder-last-j-of-cut length-ladder-cut by force
qed

definition ladder-stepdown-diff :: ladder ⇒ nat where
  ladder-stepdown-diff L = Suc (ladder-n L 0)

definition ladder-stepdown-α-0 :: sentence ⇒ derivation ⇒ ladder ⇒ sentence
where
  ladder-stepdown-α-0 a D L = Derive a (take (ladder-stepdown-diff L) D)

lemma LeftDerivationIntro-LeftDerives1:
  assumes LeftDerivationIntro α i r ix D j γ
  assumes splits-at α i a1 A a2
  shows LeftDerives1 α i r (a1@(snd r)@a2)
by (metis LeftDerivationIntro-def LeftDerivationIntro-examine-rule LeftDerivation-
Intro-is-sentence
  LeftDerives1-def assms(1) assms(2) prod.collapse splits-at-implies-Derives1)

lemma LeftDerives1-Derive:
  assumes LeftDerives1 α i r γ
  shows Derive α [(i, r)] = γ
by (metis Derive LeftDerivation.simps(1) LeftDerivation-LeftDerives1
  LeftDerivation-implies-Derivation append-Nil assms)

lemma ladder-stepdown-α-0-altdef:
  assumes ladder: LeftDerivationLadder α D L γ
  assumes length-L: length L > 1
  assumes split: splits-at (ladder-α α D L 1) (ladder-i L 1) a1 A a2
  shows ladder-stepdown-α-0 α D L = a1 @ (snd (snd (D ! (ladder-n L 0)))) @
a2
proof −
  have 1: ladder-α α D L 1 = Derive α (take (ladder-n L 0) D)
    by (simp add: ladder-α-def ladder-γ-def)
  obtain rule  where rule: rule = snd (D ! (ladder-n L 0)) by blast
 have ∃ E ω. LeftDerivationIntro (ladder-α α D L 1) (ladder-i L 1) rule (ladder-ix
L 1)
    E (ladder-j L 1) ω
    by (metis LeftDerivationIntrosAt-def LeftDerivationIntros-def LeftDerivation-
```

Ladder-def
One-nat-def diff-Suc-1 ladder length-L order-refl rule)
then obtain *E ω* **where** *intro*:
LeftDerivationIntro (*ladder-α α D L 1*) (*ladder-i L 1*) *rule* (*ladder-ix L 1*) *E* (*ladder-j L 1*) *ω*
by *blast*
then have *2*: *LeftDerives1* (*ladder-α α D L 1*) (*ladder-i L 1*) *rule* (*a1*@(*snd rule*)@*a2*)
using *LeftDerivationIntro-LeftDerives1 split* **by** *blast*
have *fst-D*: *fst* (*D* ! (*ladder-n L 0*)) = *ladder-i L 1*
by (*metis LeftDerivationIntrosAt-def LeftDerivationIntros-def LeftDerivation-Ladder-def*
One-nat-def diff-Suc-1 ladder le-numeral-extra(*4*) *length-L*)
have *derive-derive*: *Derive α* (*take* (*Suc* (*ladder-n L 0*)) *D*) =
Derive (*Derive α* (*take* (*ladder-n L 0*) *D*)) [*D* ! (*ladder-n L 0*)]
proof −
have *f1*: *Derivation α* (*take* (*Suc* (*ladder-n L 0*)) *D*) (*Derive α* (*take* (*Suc* (*ladder-n L 0*)) *D*))
using *Derivation-take-derive LeftDerivationLadder-def LeftDerivation-implies-Derivation ladder* **by** *blast*
have *f2*: *length L* − *1* < *length L*
using *length-L* **by** *linarith*
have *0* < *length L* − *1*
using *length-L* **by** *linarith*
then have *f3*: *take* (*Suc* (*ladder-n L 0*)) *D* = *take* (*ladder-n L 0*) *D* @ [*D* ! *ladder-n L 0*]
using *f2* **by** (*metis* (*full-types*) *LeftDerivationLadder-def is-ladder-def ladder ladder-last-n-def take-Suc-conv-app-nth*)
obtain *sss* :: *symbol list* ⇒ (*nat* × *symbol* × *symbol list*) *list* ⇒ (*nat* × *symbol* × *symbol list*) *list* ⇒ *symbol list* ⇒ *symbol list* **where**
∀ *x0 x1 x2 x3*. (∃ *v4*. *Derivation x3 x2 v4* ∧ *Derivation v4 x1 x0*) = (*Derivation x3 x2* (*sss x0 x1 x2 x3*) ∧ *Derivation* (*sss x0 x1 x2 x3*) *x1 x0*)
by *moura*
then show *?thesis*
using *f3 f1 Derivation-append Derive* **by** *auto*
qed
then have *3*: *ladder-stepdown-α-0 α D L* = *Derive* (*ladder-α α D L 1*) [*D* ! (*ladder-n L 0*)]
using *1* **by** (*simp add*: *ladder-stepdown-α-0-def ladder-stepdown-diff-def*)
have *4*: *D* ! (*ladder-n L 0*) = (*ladder-i L 1*, *rule*)
using *rule fst-D* **by** (*metis prod.collapse*)
then show *?thesis* **using** *2 3 4 LeftDerives1-Derive snd-conv* **by** *auto*
qed

lemma *ladder-i-0-bound*:
assumes *ld*: *LeftDerivationLadder α D L γ*
shows *ladder-i L 0* < *length α*
proof −
have *LeftDerivationFix α* (*ladder-i L 0*) (*take* (*ladder-n L 0*) *D*)

```
    (ladder-j L 0) (ladder-γ α D L 0)
    using ld LeftDerivationLadder-def by simp
  then show ?thesis using LeftDerivationFix-def by simp
qed

lemma ladder-j-bound:
  assumes ld: LeftDerivationLadder α D L γ
  assumes index-bound: index < length L
  shows ladder-j L index < length (ladder-γ α D L index)
proof −
  have ld': LeftDerivationLadder (α@[]) D L γ using ld by simp
  have ladder-i-0: ladder-i L 0 < length α using ladder-i-0-bound ld by auto
  obtain n where n: n = ladder-n L index by blast
 note propagate = LeftDerivationLadder-propagate[OF ld' ladder-i-0 n index-bound]
  from index-bound have index + 1 < length L ∨ index + 1 = length L by arith
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      then have ∃ β. LeftDerivation α (take n D) β ∧
        ladder-γ (α @ []) D L index = β @ [] ∧ ladder-j L index < length β
        using propagate by auto
      then show ?case by auto
  next
    case 2
      then have
       ∃ n' β δ'.
         (index = 0 ∨ ladder-prev-n L index < n') ∧
         n' ≤ n ∧
         LeftDerivation α (take n' D) β ∧
         LeftDerivation (α @ []) (take n' D) (β @ []) ∧
         derivation-ge (drop n' D) (length β) ∧
         LeftDerivation [] (derivation-shift (drop n' D) (length β) 0) δ' ∧
         ladder-γ (α @ []) D L index = β @ δ' ∧ ladder-j L index < length β
         using propagate by auto
      then show ?case by auto
  qed
qed

lemma ladder-last-j-bound:
  assumes ld: LeftDerivationLadder α D L γ
  shows ladder-last-j L < length γ
proof −
  have length L − 1 < length L
    using LeftDerivationLadder-def assms is-ladder-def by auto
  from ladder-j-bound[OF ld this]
  show ?thesis
    by (metis Derive LeftDerivationLadder-def LeftDerivation-implies-Derivation
One-nat-def
      is-ladder-def ladder-last-j-def last-ladder-γ ld)
```

qed

fun *ladder-shift-n* :: *nat* ⇒ *ladder* ⇒ *ladder* **where**
  ladder-shift-n N [] = []
| *ladder-shift-n N* ((*n*, *j*, *i*)#*L*) = ((*n* − *N*, *j*, *i*)#(*ladder-shift-n N L*))

fun *ladder-stepdown* :: *ladder* ⇒ *ladder*
where
  ladder-stepdown [] = *undefined*
| *ladder-stepdown* [*v*] = *undefined*
| *ladder-stepdown* ((*n0*, *j0*, *i0*)#(*n1*, *j1*, *ix1*)#*L*) =
  (*n1* − *Suc n0*, *j1*, *j0* + *ix1*) # (*ladder-shift-n* (*Suc n0*) *L*)

lemma *ladder-shift-n-length*:
  length (*ladder-shift-n N L*) = *length L*
  by (*induct L*, *auto*)

lemma *ladder-stepdown-prepare*:
  assumes *length L* > *1*
  shows *L* = (*ladder-n L 0*, *ladder-j L 0*, *ladder-i L 0*)#
    (*ladder-n L 1*, *ladder-j L 1*, *ladder-ix L 1*)#(*drop 2 L*)
proof −
  have ∃ *n0 j0 i0 n1 j1 ix1 L′*. *L* = ((*n0*, *j0*, *i0*)#(*n1*, *j1*, *ix1*)#*L′*)
    by (*metis One-nat-def Suc-eq-plus1 assms ladder-stepdown.cases less-not-refl*
list.size(*3*)
      list.size(*4*) *not-less0*)
  then obtain *n0 j0 i0 n1 j1 ix1 L′* **where** *L′*: *L* = ((*n0*, *j0*, *i0*)#(*n1*, *j1*, *ix1*)#*L′*)
by *blast*
  have *n0*: *n0* = *ladder-n L 0* **using** *L′*
    by (*auto simp add*: *ladder-n-def deriv-n-def*)
  show *?thesis* **using** *L′*
    by (*auto simp add*: *ladder-n-def deriv-n-def ladder-j-def deriv-j-def*
      ladder-i-def deriv-i-def ladder-ix-def deriv-ix-def)
qed

lemma *ladder-stepdown-length*:
  assumes *length L* > *1*
  shows *length* (*ladder-stepdown L*) = *length L* − *1*
apply (*subst ladder-stepdown-prepare*[*OF assms*(*1*)])
apply (*simp add*: *ladder-shift-n-length*)
using *assms* **apply** *arith*
done

lemma *ladder-stepdown-i-0*:
  assumes *length L* > *1*
  shows *ladder-i* (*ladder-stepdown L*) *0* = *ladder-i L 1* + *ladder-ix L 1*
  using *ladder-stepdown-prepare*[*OF assms*] *ladder-i-def ladder-j-def deriv-j-def*
  by (*metis One-nat-def deriv-i-def diff-Suc-1 ladder-stepdown.simps*(*3*) *list.sel*(*1*)

snd-conv zero-neq-one)

lemma *ladder-shift-n-cons*: *ladder-shift-n N* (*x*#*L*) = (*fst x* − *N*, *snd x*)#(*ladder-shift-n N L*)
apply (*induct L*)
by (*cases x*, *simp*)+

lemma *ladder-shift-n-drop*: *ladder-shift-n N* (*drop n L*) = *drop n* (*ladder-shift-n N L*)
proof (*induct L arbitrary*: *n*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x L*)
show *?case*
proof (*cases n*)
case *0* **then show** *?thesis*
by *simp*
next
case (*Suc n*) **then show** *?thesis*
by (*simp add*: *ladder-shift-n-cons Cons*)
qed
qed

lemma *drop-2-shift*:
assumes *index* > *0*
assumes *length L* > *1*
shows *drop 2 L* ! (*index* − *Suc 0*) = *L* ! *Suc index*
proof −
define *l1 l2* **and** *L*′ **where** *l1* = *L* ! *0 l2* = *L* ! *1*
and *L*′ = *drop 2 L*
with ‹*length L* > *1*› **have** *L* = *l1* # *l2* # *L*′
by (*cases L*) (*auto simp add*: *neq-Nil-conv*)
with ‹*index* > *0*› **show** *?thesis*
by *simp*
qed

lemma *ladder-shift-n-at*:
index < *length L* ⟹ (*ladder-shift-n N L*) ! *index* = (*fst* (*L* ! *index*) − *N*, *snd* (*L* ! *index*))
proof (*induct L arbitrary*: *index*)
case *Nil* **then show** *?case* **by** *auto*
next
case (*Cons x L*)
show *?case*
apply (*simp add*: *ladder-shift-n-cons*)
apply (*cases index*)
apply (*auto*)
apply (*rule-tac Cons*(*1*))
using *Cons*(*2*) **by** *auto*

qed

lemma *ladder-stepdown-j*:
assumes *length-L-greater-1*: *length L > 1*
assumes *L'*: *L' = ladder-stepdown L*
assumes *index-bound*: *index < length L'*
shows *ladder-j L' index = ladder-j L (Suc index)*
proof −
note *L-prepare = ladder-stepdown-prepare*[*OF length-L-greater-1*]
have *ladder-stepdown-L-def*: *ladder-stepdown L = ((ladder-n L (Suc 0)) − Suc (ladder-n L 0), ladder-j L (Suc 0), ladder-j L 0 + ladder-ix L (Suc 0)) # ladder-shift-n (Suc (ladder-n L 0)) (drop 2 L))*
by (*subst L-prepare*, *simp*)
have *index = 0 ∨ index > 0* **by** *arith*
then show *ladder-j L' index = ladder-j L (Suc index)*
proof (*induct rule*: *disjCases2*)
case *1*
show *?case*
by (*simp add*: *L' ladder-stepdown-L-def 1 ladder-j-def deriv-j-def*)
next
case *2*
have *index-bound'*: *Suc index < length L*
using *index-bound L' ladder-stepdown-length length-L-greater-1* **by** *auto*
show *?case*
apply (*simp add*: *L' ladder-stepdown-L-def 2 ladder-j-def ladder-shift-n-drop drop-2-shift*)
apply (*subst drop-2-shift*)
apply (*simp add*: *2*)
using *length-L-greater-1* **apply** (*simp add*: *ladder-shift-n-length*)
apply (*simp add*: *deriv-j-def*)
apply (*simp add*: *ladder-shift-n-at*[*OF index-bound'*])
done
qed
qed

lemma *ladder-stepdown-last-j*:
assumes *length-L-greater-1*: *length L > 1*
shows *ladder-last-j* (*ladder-stepdown L*) = *ladder-last-j L*
using *ladder-stepdown-j Suc-diff-Suc diff-Suc-1 ladder-last-j-def ladder-stepdown-length*

length-L-greater-1 lessI **by** *auto*

lemma *ladder-stepdown-n*:
assumes *length-L-greater-1*: *length L > 1*
assumes *L'*: *L' = ladder-stepdown L*
assumes *index-bound*: *index < length L'*
shows *ladder-n L' index = ladder-n L (Suc index) − ladder-stepdown-diff L*
proof −
note *L-prepare = ladder-stepdown-prepare*[*OF length-L-greater-1*]

```
  have ladder-stepdown-L-def: ladder-stepdown L = ((ladder-n L (Suc 0) - Suc
(ladder-n L 0), ladder-j L (Suc 0), ladder-j L 0 + ladder-ix L (Suc 0)) #
    ladder-shift-n (Suc (ladder-n L 0)) (drop 2 L))
    by (subst L-prepare, simp)
  have index = 0 ∨ index > 0 by arith
  then show ladder-n L' index = ladder-n L (Suc index) - ladder-stepdown-diff
L
  proof (induct rule: disjCases2)
    case 1
      show ?case
          by (simp add: L' ladder-stepdown-L-def 1 ladder-n-def deriv-n-def lad-
der-stepdown-diff-def)
  next
    case 2
      have index-bound': Suc index < length L
        using index-bound L' ladder-stepdown-length length-L-greater-1 by auto
      show ?case
       apply (simp add: L' ladder-stepdown-L-def 2 ladder-n-def ladder-shift-n-drop
drop-2-shift
          ladder-stepdown-diff-def)
        apply (subst drop-2-shift)
        apply (simp add: 2)
        using length-L-greater-1 apply (simp add: ladder-shift-n-length)
        apply (simp add: deriv-n-def)
        apply (simp add: ladder-shift-n-at[OF index-bound'])
        done
  qed
qed

lemma ladder-stepdown-ix:
  assumes length-L-greater-1: length L > 1
  assumes L': L' = ladder-stepdown L
  assumes index-lower-bound: 0 < index
  assumes index-upper-bound: index < length L'
  shows ladder-ix L' index = ladder-ix L (Suc index)
proof -
  note L-prepare = ladder-stepdown-prepare[OF length-L-greater-1]
  have ladder-stepdown-L-def: ladder-stepdown L = ((ladder-n L (Suc 0) - Suc
(ladder-n L 0), ladder-j L (Suc 0), ladder-j L 0 + ladder-ix L (Suc 0)) #
    ladder-shift-n (Suc (ladder-n L 0)) (drop 2 L))
    by (subst L-prepare, simp)

  have index-bound': Suc index < length L
    using index-upper-bound L' ladder-stepdown-length length-L-greater-1 by auto
  show ?thesis
       apply (simp add: L' ladder-stepdown-L-def index-lower-bound ladder-ix-def
ladder-shift-n-drop)
       apply (subst drop-2-shift)
       apply (simp add: index-lower-bound)
```

using *length-L-greater-1* **apply** (*simp add*: *ladder-shift-n-length*)
apply (*simp add*: *deriv-ix-def*)
apply (*simp add*: *ladder-shift-n-at*[*OF index-bound'*])
using *index-lower-bound* **by** *arith*
qed

lemma *Derive-Derive*:
assumes *Derivation* α (*D*@*E*) γ
shows *Derive* (*Derive* α *D*) *E* = *Derive* α (*D*@*E*)
using *Derivation-append Derive assms* **by** *fastforce*

lemma *drop-at-shift*:
assumes $n \leq$ *index*
assumes *index* < *length D*
shows *drop n D* ! (*index* − *n*) = *D* ! *index*
using *assms*(*1*) *assms*(*2*) **by** *auto*

theorem *LeftDerivationLadder-stepdown*:
assumes *ldl*: *LeftDerivationLadder* α *D L* γ
assumes *length-L*: *length L* > *1*
shows ∃ *L'*. *LeftDerivationLadder* (*ladder-stepdown-*α*-0* α *D L*) (*drop* (*ladder-stepdown-diff L*) *D*)
L' γ ∧ *length L'* = *length L* − *1* ∧ *ladder-i L' 0* = *ladder-i L 1* + *ladder-ix L 1* ∧
ladder-last-j L' = *ladder-last-j L*
proof −
obtain *L'* **where** *L'*: *L'* = *ladder-stepdown L* **by** *blast*
have *ldl1*: *LeftDerivation* (*ladder-stepdown-*α*-0* α *D L*) (*drop* (*ladder-stepdown-diff L*) *D*) γ
proof −
have *D-split*: *D* = (*take* (*ladder-stepdown-diff L*) *D*) @ (*drop* (*ladder-stepdown-diff L*) *D*)
by *simp*
show *?thesis* **using** *D-split ldl*
proof −
obtain *sss* :: *symbol list* ⇒ (*nat* × *symbol* × *symbol list*) *list* ⇒ (*nat* × *symbol* × *symbol list*) *list* ⇒ *symbol list* ⇒ *symbol list* **where**
∀ *x0 x1 x2 x3*. (∃ *v4*. *LeftDerivation x3 x2 v4* ∧ *LeftDerivation v4 x1 x0*) = (*LeftDerivation x3 x2* (*sss x0 x1 x2 x3*) ∧ *LeftDerivation* (*sss x0 x1 x2 x3*) *x1 x0*)
by *moura*
then have (¬ *LeftDerivation* α (*take* (*ladder-stepdown-diff L*) *D* @ *drop* (*ladder-stepdown-diff L*) *D*) γ ∨ *LeftDerivation* α (*take* (*ladder-stepdown-diff L*) *D*) (*sss* γ (*drop* (*ladder-stepdown-diff L*) *D*) (*take* (*ladder-stepdown-diff L*) *D*) α) ∧ *LeftDerivation* (*sss* γ (*drop* (*ladder-stepdown-diff L*) *D*) (*take* (*ladder-stepdown-diff L*) *D*) α) (*drop* (*ladder-stepdown-diff L*) *D*) γ) ∧ (*LeftDerivation* α (*take* (*ladder-stepdown-diff L*) *D* @ *drop* (*ladder-stepdown-diff L*) *D*) γ ∨ (∀ *ss*. ¬ *LeftDerivation* α (*take* (*ladder-stepdown-diff L*) *D*) *ss* ∨ ¬ *LeftDerivation ss* (*drop* (*ladder-stepdown-diff L*) *D*) γ))
using *LeftDerivation-append* **by** *blast*

```
      then show ?thesis
       by (metis (no-types) D-split Derivation-take-derive Derivation-unique-dest
LeftDerivationLadder-def LeftDerivation-implies-Derivation ladder-stepdown-α-0-def
ldl)
    qed
  qed
  have L'-nonempty: L' ≠ [] using L' ladder-stepdown-length length-L by fastforce
  {
    fix u :: nat
    assume u': u < length L'
    then have Suc-u: Suc u < length L using L' ladder-stepdown-length length-L
by auto
    have ladder-n L (Suc u) ≤ length D
      using ldl Suc-u by (simp add: LeftDerivationLadder-ladder-n-bound)
    then have ladder-n L' u ≤ length D − ladder-stepdown-diff L
      apply (subst ladder-stepdown-n[OF length-L L' u'])
      by auto
  }
  note is-ladder-prop1 = this
  {
    fix u :: nat
    fix v :: nat
    assume u-less-v: u < v
    assume v-L': v < length L'
    from u-less-v v-L' have u-L': u < length L' by arith
    have ladder-n L (Suc u) < ladder-n L (Suc v)
     using ldl by (metis (no-types, lifting) L' LeftDerivationLadder-def One-nat-def
Suc-diff-1
        Suc-lessD Suc-mono is-ladder-def ladder-stepdown-length length-L u-less-v
v-L')
    then have ladder-n L' u < ladder-n L' v
      apply (simp add: ladder-stepdown-n[OF length-L L'] u-L' v-L')
     by (metis (no-types, lifting) L' LeftDerivationLadder-def Suc-eq-plus1 Suc-leI
        diff-less-mono is-ladder-def ladder-stepdown-diff-def ladder-stepdown-length
ldl
       length-L less-diff-conv u-L' zero-less-Suc)
  }
  note is-ladder-prop2 = this
  have is-ladder-L': is-ladder (drop (ladder-stepdown-diff L) D) L'
    apply (auto simp add: is-ladder-def)
    using L'-nonempty apply blast
    using is-ladder-prop1 apply blast
    using is-ladder-prop2 apply blast
   using ladder-last-n-def ladder-stepdown-n L' LeftDerivationLadder-def Suc-diff-Suc
diff-Suc-1
     ladder-n-last-is-length ladder-stepdown-length ldl length-L lessI by auto
  have ldfix: LeftDerivationFix (ladder-stepdown-α-0 α D L) (ladder-i L' 0)
    (take (ladder-n L' 0) (drop (ladder-stepdown-diff L) D)) (ladder-j L' 0)
     (ladder-γ (ladder-stepdown-α-0 α D L) (drop (ladder-stepdown-diff L) D) L'
```

```
0)
  proof −
    have introsAt-L-1: LeftDerivationIntrosAt α D L 1
      using LeftDerivationIntros-def LeftDerivationLadder-def ldl length-L by blast
    thm LeftDerivationIntrosAt-def
    obtain n where n: n = ladder-n L 0 by blast
    obtain m where m: m = ladder-n L 1 by blast
    have LeftDerivationIntro (ladder-α α D L 1) (ladder-i L 1) (snd (D ! n))
      (ladder-ix L 1) (drop (Suc n) (take m D)) (ladder-j L 1) (ladder-γ α D L 1)
       using n m introsAt-L-1 by (metis LeftDerivationIntrosAt-def One-nat-def
diff-Suc-1)
    from iffD1[OF  LeftDerivationIntro-def this] obtain β where β:
      LeftDerives1 (ladder-α α D L 1) (ladder-i L 1) (snd (D ! n)) β ∧
       ladder-ix L 1 < length (snd (snd (D ! n))) ∧
       snd (snd (D ! n)) ! ladder-ix L 1 = ladder-γ α D L 1 ! ladder-j L 1 ∧
        LeftDerivationFix β (ladder-i L 1 + ladder-ix L 1) (drop (Suc n) (take m
D)) (ladder-j L 1)
       (ladder-γ α D L 1)
       by blast
    have β = Derive (ladder-α α D L 1) [D ! n]
     by (metis (no-types, opaque-lifting) LeftDerivationIntrosAt-def LeftDerives1-Derive
β
        cancel-comm-monoid-add-class.diff-cancel introsAt-L-1 n prod.collapse)
    then have β-def: β = ladder-stepdown-α-0 α D L
      proof −
        obtain sss :: nat ⇒ symbol list ⇒ symbol list and ss :: nat ⇒ symbol list
⇒ symbol and sssa :: nat ⇒ symbol list ⇒ symbol list where
          ∀ x2 x3. (∃ v4 v5 v6. splits-at x3 x2 v4 v5 v6) = splits-at x3 x2 (sss x2 x3)
(ss x2 x3) (sssa x2 x3)
          by moura
        then have f1: ∀ ssa n p ssb. ¬ Derives1 ssa n p ssb ∨ splits-at ssa n (sss n
ssa) (ss n ssa) (sssa n ssa)
          using splits-at-ex by presburger
        then have β = sss (ladder-i L 1) (ladder-α α D L 1) @ snd (snd (D ! n))
@ sssa (ladder-i L 1) (ladder-α α D L 1)
          by (meson LeftDerives1-implies-Derives1 β splits-at-combine-dest)
        then show ?thesis
            using f1 by (metis (no-types) LeftDerives1-implies-Derives1 β lad-
der-stepdown-α-0-altdef ldl length-L n)
      qed
    have ladder-i-L'-0: ladder-i L' 0 = ladder-i L 1 + ladder-ix L 1
      using L' ladder-stepdown-i-0 length-L by blast
    have derivation-eq: (take (ladder-n L' 0) (drop (ladder-stepdown-diff L) D)) =

      (drop (Suc n) (take m D)) using n m
       by (metis L' L'-nonempty One-nat-def drop-take ladder-stepdown-diff-def
ladder-stepdown-n
        length-L length-greater-0-conv)
    have ladder-j-L'-0: ladder-j L' 0 = ladder-j L 1
```

```
    using L' L'-nonempty ladder-stepdown-j length-L by auto
  have ladder-γ: (ladder-γ (ladder-stepdown-α-0 α D L) (drop (ladder-stepdown-diff
L) D) L' 0) =
    ladder-γ α D L 1
   by (metis Derivation-take-derive Derivation-unique-dest LeftDerivationFix-def

     LeftDerivation-implies-Derivation β β-def derivation-eq ladder-γ-def ldl1)
  from β-def β ladder-i-L'-0 derivation-eq ladder-j-L'-0 ladder-γ
  show ?thesis by auto
 qed
 {
  fix index :: nat
  assume index-lower-bound: Suc 0 ≤ index
  assume index-upper-bound: index < length L'
  then have Suc-index-upper-bound: Suc index < length L
    using L' Suc-diff-Suc Suc-less-eq diff-Suc-1 ladder-stepdown-length length-L
less-Suc-eq
    by auto
  then have intros-at-Suc-index: LeftDerivationIntrosAt α D L (Suc index)
    by (metis LeftDerivationIntros-def LeftDerivationLadder-def Suc-eq-plus1-left
ldl le-add1)
  from iffD1[OF LeftDerivationIntrosAt-def this] have ldintro:
    let α' = ladder-α α D L (Suc index); i = ladder-i L (Suc index); j = ladder-j
L (Suc index);
     ix = ladder-ix L (Suc index); γ = ladder-γ α D L (Suc index); n = ladder-n
L (Suc index − 1);
     m = ladder-n L (Suc index); e = D ! n; E = drop (Suc n) (take m D)
     in i = fst e ∧ LeftDerivationIntro α' i (snd e) ix E j γ by blast
  have index-minus-Suc-0-bound: index − Suc 0 < length L'
    by (simp add: index-upper-bound less-imp-diff-less)
  note helpers = length-L L' index-minus-Suc-0-bound
  have ladder-i-L'-index:
    ladder-i L' index = fst (drop (ladder-stepdown-diff L) D ! ladder-n L' (index
− Suc 0))
    apply (auto simp add: ladder-i-def)
    using index-lower-bound apply arith
  apply (simp add: ladder-stepdown-n[OF helpers] ladder-stepdown-j[OF helpers])
    apply (subst drop-at-shift)
     using LeftDerivationLadder-def Suc-index-upper-bound Suc-leI Suc-lessD
is-ladder-def
      ladder-stepdown-diff-def ldl apply presburger
  apply (metis LeftDerivationLadder-def One-nat-def Suc-eq-plus1 Suc-index-upper-bound

     add.commute add-diff-cancel-right' ladder-n-minus-1-bound ldl le-add1)
  by (metis LeftDerivationIntrosAt-def intros-at-Suc-index diff-Suc-1 ladder-i-def
nat.simps(3))
  have intro-at-index:
  LeftDerivationIntro (ladder-α (ladder-stepdown-α-0 α D L) (drop (ladder-stepdown-diff
L) D) L' index)
```

```
    (ladder-i L' index) (snd (drop (ladder-stepdown-diff L) D ! ladder-n L' (index
− Suc 0)))
    (ladder-ix L' index)
    (drop (Suc (ladder-n L' (index − Suc 0)))
      (take (ladder-n L' index) (drop (ladder-stepdown-diff L) D)))
    (ladder-j L' index) (ladder-γ (ladder-stepdown-α-0 α D L)
      (drop (ladder-stepdown-diff L) D) L' index)
  proof −
    have arg1: (ladder-α (ladder-stepdown-α-0 α D L)
      (drop (ladder-stepdown-diff L) D) L' index) = ladder-α α D L (Suc index)
      apply (auto simp add: ladder-α-def ladder-γ-def)
      using index-lower-bound apply arith
      apply (simp add: ladder-stepdown-n[OF helpers] ladder-stepdown-α-0-def)
      apply (subst Derive-Derive[where γ=ladder-γ α D L index])
       apply (metis (no-types, lifting) Derivation-take-derive LeftDerivationLad-
der-def
        LeftDerivation-implies-Derivation Suc-index-upper-bound Suc-leI Suc-lessD

         add.commute is-ladder-def ladder-γ-def ladder-stepdown-diff-def ldl
         le-add-diff-inverse2 take-add)
     by (metis LeftDerivationLadder-def Suc-index-upper-bound Suc-leI Suc-lessD
add.commute
        is-ladder-def ladder-stepdown-diff-def ldl le-add-diff-inverse2 take-add)
    have arg2: ladder-i L' index = ladder-i L (Suc index)
       using L' index-lower-bound index-minus-Suc-0-bound ladder-i-def lad-
der-stepdown-j
        length-L by auto
    obtain n where n: n = ladder-n L (Suc index − 1) by blast
    have arg3: (snd (drop (ladder-stepdown-diff L) D ! ladder-n L' (index − Suc
0))) =
      snd (D ! n)
      apply (simp add: ladder-stepdown-n[OF helpers] index-lower-bound)
      apply (subst drop-at-shift)
      using index-lower-bound
    apply (metis (no-types, opaque-lifting) L' LeftDerivationLadder-def One-nat-def
Suc-eq-plus1
     add.commute diff-Suc-1 index-upper-bound is-ladder-def ladder-stepdown-diff-def

       ladder-stepdown-length ldl le-add-diff-inverse2 length-L less-or-eq-imp-le n
       nat.simps(3) neq0-conv not-less not-less-eq-eq)
      using index-lower-bound
     apply (metis LeftDerivationLadder-def One-nat-def Suc-index-upper-bound
Suc-le-lessD
      Suc-pred diff-Suc-1 ladder-n-minus-1-bound ldl le-imp-less-Suc less-imp-le)

      using index-lower-bound n by (simp add: Suc-diff-le)
    have arg4: ladder-ix L' index = ladder-ix L (Suc index)
    using ladder-stepdown-ix L' Suc-le-lessD index-lower-bound index-upper-bound
length-L
```

```
      by auto
    obtain m where m: m = ladder-n L (Suc index) by blast
    have Suc-index-Suc: Suc (index − Suc 0) = index
      using index-lower-bound by arith
     have arg5: (drop (Suc (ladder-n L′ (index − Suc 0))) (take (ladder-n L′
index)
      (drop (ladder-stepdown-diff L) D))) = drop (Suc n) (take m D)
      apply (simp add: ladder-stepdown-n[OF helpers]
        ladder-stepdown-n[OF length-L L′ index-upper-bound] n m Suc-index-Suc)
      by (metis (no-types, lifting) LeftDerivationLadder-def Suc-eq-plus1-left
        Suc-index-upper-bound Suc-leI Suc-le-lessD Suc-lessD drop-drop drop-take
       index-lower-bound is-ladder-def ladder-stepdown-diff-def ldl le-add-diff-inverse2)
    have arg6: ladder-j L′ index = ladder-j L (Suc index)
      using L′ index-upper-bound ladder-stepdown-j length-L by blast
    have arg7: (ladder-γ (ladder-stepdown-α-0 α D L)
      (drop (ladder-stepdown-diff L) D) L′ index) = ladder-γ α D L (Suc index)
      apply (simp add: ladder-γ-def)
       apply (simp add: ladder-stepdown-n[OF length-L L′ index-upper-bound]
ladder-stepdown-α-0-def)
      apply (subst Derive-Derive[where γ=ladder-γ α D L (Suc index)])
      apply (metis (no-types, lifting) L′ LeftDerivationLadder-def
        LeftDerivation-implies-Derivation LeftDerivation-take-derive Suc-le-lessD
            add-diff-inverse-nat diff-is-0-eq index-lower-bound index-upper-bound
is-ladder-L′
       is-ladder-def ladder-γ-def ladder-stepdown-n ldl le-0-eq length-L less-numeral-extra(3)

        less-or-eq-imp-le take-add)
      by (metis (no-types, lifting) L′ One-nat-def add-diff-inverse-nat diff-is-0-eq
       index-lower-bound index-upper-bound is-ladder-L′ is-ladder-def ladder-stepdown-n
le-0-eq
        le-neq-implies-less length-L less-numeral-extra(3) less-or-eq-imp-le take-add
zero-less-one)
    from ldintro arg1 arg2 arg3 arg4 arg5 arg6 arg7 show ?thesis
      by (metis m n)
   qed
  have LeftDerivationIntrosAt (ladder-stepdown-α-0 α D L) (drop (ladder-stepdown-diff
L) D)
    L′ index
    apply (auto simp add: LeftDerivationIntrosAt-def Let-def)
    using ladder-i-L′-index apply blast
    using intro-at-index by blast
 }
 note introsAt = this
 show ?thesis
   apply (rule-tac x=L′ in exI)
   apply auto
   defer 1
   using L′ ladder-stepdown-length length-L apply auto[1]
   using ladder-stepdown-i-0 length-L L′ apply auto[1]
```

```
    using ladder-stepdown-last-j L' length-L apply auto[1]
    apply (auto simp add: LeftDerivationLadder-def)
    using ldl1 apply blast
    using is-ladder-L' apply blast
    using ldfix apply blast
    apply (auto simp add: LeftDerivationIntros-def)
    apply (simp add: introsAt)
    done
qed

fun ladder-shift-j :: nat ⇒ ladder ⇒ ladder where
  ladder-shift-j d [] = []
| ladder-shift-j d ((n, j, i)#L) = ((n, j − d, i)#(ladder-shift-j d L))

definition ladder-cut-prefix :: nat ⇒ ladder ⇒ ladder
where
  ladder-cut-prefix d L =
    (ladder-shift-j d L)[0 := (ladder-n L 0, ladder-j L 0 − d, ladder-i L 0 − d)]

lemma ladder-shift-j-length:
  length (ladder-shift-j d L) = length L
  by (induct L, auto)

lemma ladder-cut-prefix-length:
  shows length (ladder-cut-prefix d L) = length L
apply (simp add: ladder-cut-prefix-def)
apply (simp add: ladder-shift-j-length)
done

lemma ladder-shift-j-cons: ladder-shift-j d (x#L) = (fst x, fst (snd x) − d, snd(snd
x))#
  (ladder-shift-j d L)
  apply (induct L)
  by (cases x, simp)+

lemma deriv-j-ladder-shift-j:
  index < length L ⟹ deriv-j (ladder-shift-j d L ! index) = deriv-j (L ! index) −
d
proof (induct L arbitrary: index)
  case Nil
    then show ?case by auto
next
  case (Cons x L)
    show ?case
      apply (subst ladder-shift-j-cons)
      apply (cases index)
      using Cons by (auto simp add: deriv-j-def)
qed
```

lemma *deriv-n-ladder-shift-j*:
index < *length L* ⟹ *deriv-n* (*ladder-shift-j d L* ! *index*) = *deriv-n* (*L* ! *index*)
proof (*induct L arbitrary*: *index*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons x L*)
show *?case*
apply (*subst ladder-shift-j-cons*)
apply (*cases index*)
using *Cons* **by** (*auto simp add*: *deriv-n-def*)
qed

lemma *deriv-ix-ladder-shift-j*:
index < *length L* ⟹ *deriv-ix* (*ladder-shift-j d L* ! *index*) = *deriv-ix* (*L* ! *index*)
proof (*induct L arbitrary*: *index*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons x L*)
show *?case*
apply (*subst ladder-shift-j-cons*)
apply (*cases index*)
using *Cons* **by** (*auto simp add*: *deriv-ix-def*)
qed

lemma *ladder-cut-prefix-j*:
assumes *index-bound*: *index* < *length L*
assumes *length-L*: *length L* > *0*
shows *ladder-j* (*ladder-cut-prefix d L*) *index* = *ladder-j L index* − *d*
apply (*simp add*: *ladder-j-def ladder-cut-prefix-def*)
apply (*cases index*)
apply (*auto simp add*: *length-L*)
apply (*subst nth-list-update-eq*)
apply (*simp only*: *ladder-shift-j-length length-L*)
apply (*simp add*: *deriv-j-def*)
apply (*subst deriv-j-ladder-shift-j*)
using *index-bound* **apply** *arith*
by *blast*

lemma *hd-0-subst*: *length L* > *0* ⟹ *hd* (*L* [*0* := *x*]) = *x*
using *hd-conv-nth* **by** (*simp add*: *upd-conv-take-nth-drop*)

lemma *ladder-cut-prefix-i*:
assumes *index-bound*: *index* < *length L*
assumes *length-L*: *length L* > *0*
shows *ladder-i* (*ladder-cut-prefix d L*) *index* = *ladder-i L index* − *d*
apply (*simp add*: *ladder-i-def ladder-cut-prefix-def*)
apply (*cases index*)

```
  apply auto[1]
  apply (subst hd-0-subst)
  using length-L ladder-shift-j-length apply metis
  apply (auto simp add: deriv-i-def)
  apply (case-tac nat)
  apply (simp add: ladder-j-def deriv-j-def)
  apply auto
  apply (subst nth-list-update-eq)
  using length-L ladder-shift-j-length apply auto[1]
  apply simp
  apply (simp add: ladder-j-def)
  apply (subst deriv-j-ladder-shift-j)
  using index-bound apply arith
  apply simp
  done

lemma ladder-cut-prefix-n:
  assumes index-bound: index < length L
  assumes length-L: length L > 0
  shows ladder-n (ladder-cut-prefix d L) index = ladder-n L index
  apply (simp add: ladder-cut-prefix-def)
  apply (cases index)
  apply (auto simp add: ladder-n-def)
  apply (subst nth-list-update-eq)
  apply (simp add: ladder-shift-j-length)
  using length-L apply blast
  apply (simp add: deriv-n-def )
  apply (rule-tac deriv-n-ladder-shift-j)
  using index-bound by arith

lemma ladder-cut-prefix-ix:
  assumes index-bound: index < length L
  assumes length-L: length L > 0
  shows ladder-ix (ladder-cut-prefix d L) index = ladder-ix L index
  apply (simp add: ladder-cut-prefix-def)
  apply (cases index)
  apply (auto simp add: ladder-ix-def)
  apply (rule-tac deriv-ix-ladder-shift-j)
  using index-bound by arith

lemma LeftDerivationFix-derivation-ge-is-nonterminal:
  assumes ldfix: LeftDerivationFix α i D j γ
  assumes derivation-ge-d: derivation-ge D d
  assumes is-nonterminal: is-nonterminal (γ ! j)
  shows (D = [] ∧ α = γ ∧ i = j) ∨ (i > d ∧ j ≥ d)
proof −
  have is-nonterminal (α ! i) using ldfix is-nonterminal
    by (simp add: LeftDerivationFix-def)
  from LeftDerivationFix-splits-at-nonterminal[OF ldfix this] obtain U a1 a2 b1
```

```
where U:
    splits-at α i a1 U a2 ∧ splits-at γ j b1 U a2 ∧ LeftDerivation a1 D b1 by blast
  have D = [] ∨ D ≠ [] by auto
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
      then have a1 = b1 using U by auto
      then have i-eq-j: i = j using U
      by (metis dual-order.strict-implies-order length-take min.absorb2 splits-at-def)

      from 1 have α = γ using ldfix LeftDerivationFix-def by auto
      with 1 i-eq-j show ?case by blast
  next
    case 2
      have ∃ a1'. LeftDerives1 a1 (fst (hd D)) (snd (hd D)) a1' using U 2
        by (metis LeftDerivation.elims(2) list.sel(1))
      then obtain a1' where a1': LeftDerives1 a1 (fst (hd D)) (snd (hd D)) a1'
by blast
    then have (fst (hd D)) < length a1 using Derives1-bound LeftDerives1-implies-Derives1
by blast
      then have fst-less-i: (fst (hd D)) < i using U
        by (simp add: leD min.absorb2 nat-le-linear splits-at-def)
      have d-le-fst: d ≤ fst (hd D) using derivation-ge-d 2 by (simp add: deriva-
tion-ge-def)
      with fst-less-i have d-less-i: d < i using le-less-trans by blast
      have ∃ b1'. LeftDerives1 b1' (fst (last D)) (snd (last D)) b1 using U 2
      by (metis Derive LeftDerivation-Derive-take-LeftDerives1 LeftDerivation-implies-Derivation

          last-conv-nth length-0-conv order-refl take-all)
     then obtain b1' where b1': LeftDerives1 b1' (fst (last D)) (snd (last D)) b1
by blast
      then have fst (last D) ≤ length b1
        using Derives1-bound' LeftDerives1-implies-Derives1 by blast
      then have fst-le-j: fst (last D) ≤ j using U splits-at-def by auto
        have d ≤ fst (last D) using derivation-ge-d 2 using derivation-ge-def
last-in-set by blast
      with fst-le-j have d ≤ j using order.trans by blast
      with d-less-i show ?thesis by auto
  qed
qed

lemma LeftDerivationFix-derivation-ge:
  assumes ldfix: LeftDerivationFix α i D j γ
  assumes derivation-ge-d: derivation-ge D d
  shows i = j ∨ (i > d ∧ j ≥ d)
proof −
  from LeftDerivationFix-splits-at-symbol[OF ldfix] obtain U a1 a2 b1 b2 n where
U:
    splits-at α i a1 U a2 ∧
```

```
    splits-at γ j b1 U b2 ∧
    n ≤ length D ∧
    LeftDerivation a1 (take n D) b1 ∧
    derivation-ge (drop n D) (Suc (length b1)) ∧
    LeftDerivation a2 (derivation-shift (drop n D) (Suc (length b1)) 0) b2 ∧
    (n = length D ∨ n < length D ∧ is-word (b1 @ [U])) by blast
  have n = 0 ∨ n > 0 by auto
  then show ?thesis
  proof (induct rule: disjCases2)
    case 1
     then have a1 = b1 using U by auto
     then have i-eq-j: i = j using U
     by (metis dual-order.strict-implies-order length-take min.absorb2 splits-at-def)

     then show ?case by blast
  next
    case 2
     obtain E where E: E = take n D by blast
     have E-nonempty: E ≠ [] using E 2
       using U less-nat-zero-code list.size(3) take-eq-Nil by auto
       have ∃ a1'. LeftDerives1 a1 (fst (hd E)) (snd (hd E)) a1' using U E
E-nonempty
       by (metis LeftDerivation.simps(2) list.exhaust list.sel(1))
     then obtain a1' where a1': LeftDerives1 a1 (fst (hd E)) (snd (hd E)) a1'
by blast
    then have (fst (hd E)) < length a1 using Derives1-bound LeftDerives1-implies-Derives1
by blast
     then have fst-less-i: (fst (hd E)) < i using U
       by (simp add: leD min.absorb2 nat-le-linear splits-at-def)
     have d-le-fst: d ≤ fst (hd E) using derivation-ge-d E-nonempty E
       by (simp add: LeftDerivation.elims(2) U derivation-ge-def hd-conv-nth)
     with fst-less-i have d-less-i: d < i using le-less-trans by blast
    have ∃ b1'. LeftDerives1 b1' (fst (last E)) (snd (last E)) b1 using E-nonempty
U E
          by (metis LeftDerivation-append1 append-butlast-last-id prod.collapse)

     then obtain b1' where b1': LeftDerives1 b1' (fst (last E)) (snd (last E)) b1
by blast
     then have fst (last E) ≤ length b1
       using Derives1-bound' LeftDerives1-implies-Derives1 by blast
     then have fst-le-j: fst (last E) ≤ j using U splits-at-def by auto
     have d ≤ fst (last E) using derivation-ge-d E-nonempty E
       using derivation-ge-d last-in-set by (metis derivation-ge-def set-take-subset
subsetCE)
     with fst-le-j have d ≤ j using order.trans by blast
     with d-less-i show ?thesis by auto
  qed
qed
```

lemma *LeftDerivationIntro-derivation-ge*:
assumes *ldintro*: *LeftDerivationIntro* α *i r ix D j* γ
assumes *i-ge-d*: $i \geq d$
assumes *derivation-ge-d*: *derivation-ge D d*
shows $j \geq d$
proof −
from *iffD1*[*OF LeftDerivationIntro-def ldintro*] **obtain** β **where** β:
LeftDerives1 α *i r* $\beta \wedge$ *ix* < *length* (*snd r*) $\wedge$ *snd r* ! *ix* = γ ! *j* $\wedge$
LeftDerivationFix β (*i* + *ix*) *D j* γ **by** *blast*
then have (*i* + *ix* = *j*) $\vee$ (*i* + *ix* > *d* $\wedge$ *j* $\geq$ *d*)
using *LeftDerivationFix-derivation-ge derivation-ge-d* **by** *blast*
then show *?thesis*
proof (*induct rule*: *disjCases2*)
case *1* **then show** *?case* **using** *i-ge-d trans-le-add1* **by** *blast*
next
case *2* **then show** *?case* **by** *simp*
qed
qed

lemma *derivation-ge-LeftDerivationLadder*:
assumes *derivation-ge-d*: *derivation-ge D d*
assumes *ladder*: *LeftDerivationLadder* α *D L* γ
assumes *ladder-i-0*: *ladder-i L 0* $\geq$ *d*
shows *index* < *length L* $\Longrightarrow$ *ladder-i L index* $\geq$ *d* $\wedge$ *ladder-j L index* $\geq$ *d*
proof (*induct index*)
case *0*
from *iffD1*[*OF LeftDerivationLadder-def ladder*]
have *ldfix*: *LeftDerivationFix* α (*ladder-i L 0*)
(*take* (*ladder-n L 0*) *D*) (*ladder-j L 0*) (*ladder-*γ α *D L 0*) **by** *blast*
have *derivation-ge* (*take* (*ladder-n L 0*) *D*) *d*
using *derivation-ge-d* **by** (*metis append-take-drop-id derivation-ge-append*)
from *ladder-i-0 derivation-ge-d LeftDerivationFix-derivation-ge*[*OF ldfix this*]
show *?case* **by** *linarith*
next
case (*Suc n*)
have *ladder-i-Suc*: *ladder-i L* (*Suc n*) $\geq$ *d*
apply (*auto simp add*: *ladder-i-def*)
using *Suc* **by** *auto*
from *iffD1*[*OF LeftDerivationLadder-def ladder*] **have** *LeftDerivationIntros* α *D L*
by *blast*
then have *LeftDerivationIntrosAt* α *D L* (*Suc n*)
using *Suc.prems*
by (*metis LeftDerivationIntros-def Suc-eq-plus1-left le-add1*)
from *iffD1*[*OF LeftDerivationIntrosAt-def this*]
show *?case* **using** *ladder-i-Suc LeftDerivationIntro-derivation-ge*
by (*metis append-take-drop-id derivation-ge-append derivation-ge-d*)
qed

lemma *derivation-shift-append*:
derivation-shift (*A*@*B*) *left right* =
(*derivation-shift A left right*) @ (*derivation-shift B left right*)
by (*induct A*, *simp*+)

lemma *derivation-shift-right-left-subtract*:
right ≥ *left* ⟹ *derivation-shift* (*derivation-shift L 0 right*) *left 0* =
derivation-shift L 0 (*right* − *left*)
by (*induct L*, *simp*+)

lemma *LeftDerivationFix-cut-prefix*:
assumes *LeftDerivationFix* (*δ*@*α*) *i D j γ*
assumes *derivation-ge D* (*length δ*)
assumes *i* ≥ *length δ*
assumes *is-word-δ*: *is-word δ*
shows ∃ *γ′*. *γ* = *δ* @ *γ′* ∧
LeftDerivationFix α (*i* − *length δ*) (*derivation-shift D* (*length δ*) *0*) (*j* − *length δ*) *γ′*
proof −
have *j-ge-d*: *j* ≥ *length δ*
using *assms*(*3*) *LeftDerivationFix-derivation-ge*[*OF assms*(*1*) *assms*(*2*)] **by** *arith*
obtain *γ′* **where** *γ′*: *γ′* = *drop* (*length δ*) *γ* **by** *blast*
from *iffD1*[*OF LeftDerivationFix-def assms*(*1*)] **obtain** *E F* **where** *EF*:
is-sentence (*δ* @ *α*) ∧
is-sentence γ ∧
LeftDerivation (*δ* @ *α*) *D γ* ∧
i < *length* (*δ* @ *α*) ∧
j < *length γ* ∧
(*δ* @ *α*) ! *i* = *γ* ! *j* ∧
D = *E* @ *derivation-shift F 0* (*Suc j*) ∧
LeftDerivation (*take i* (*δ* @ *α*)) *E* (*take j γ*) ∧
LeftDerivation (*drop* (*Suc i*) (*δ* @ *α*)) *F* (*drop* (*Suc j*) *γ*) **by** *blast*
then have *LeftDerivation* (*δ* @ *α*) *D γ* **by** *blast*
from *LeftDerivation-skip-prefixword-ex*[*OF this is-word-δ*]
obtain *γ′* **where** *γ′*: *γ* = *δ* @ *γ′* ∧ *LeftDerivation α* (*derivation-shift D* (*length δ*) *0*) *γ′* **by** *blast*
have *ldf1*: *is-sentence α* **using** *EF is-sentence-concat* **by** *blast*
have *ldf2*: *is-sentence γ′* **using** *EF γ′ is-sentence-concat* **by** *blast*
have *ldf3*: *i* − *length δ* < *length α*
by (*metis EF append-Nil assms*(*3*) *drop-append drop-eq-Nil not-le*)
have *ldf4*: *j* − *length δ* < *length γ′*
by (*metis EF append-Nil j-ge-d γ′ drop-append drop-eq-Nil not-le*)
have *ldf5*: *α* ! (*i* − *length δ*) = *γ′* ! (*j* − *length δ*)
by (*metis γ′ EF assms*(*3*) *j-ge-d leD nth-append*)
have *D-split*: *D* = *E* @ *derivation-shift F 0* (*Suc j*) **using** *EF* **by** *blast*
show *?thesis*
apply (*rule-tac x*=*γ′* **in** *exI*)
apply (*auto simp add*: *γ′*)

apply (*auto simp add*: *LeftDerivationFix-def*)
using *ldf1* **apply** *blast*
using *ldf2* **apply** *blast*
using γ' **apply** *blast*
using *ldf3* **apply** *blast*
using *ldf4* **apply** *blast*
using *ldf5* **apply** *blast*
apply (*rule-tac x=derivation-shift E* (*length* δ) *0* **in** *exI*)
apply (*rule-tac x=F* **in** *exI*)
apply *auto*
apply (*subst D-split*)
apply (*simp add*: *derivation-shift-append*)
apply (*subst derivation-shift-right-left-subtract*)
apply (*simp add*: *j-ge-d le-Suc-eq*)
using *j-ge-d* **apply** (*simp add*: *Suc-diff-le*)
apply (*metis EF LeftDerivation-implies-Derivation LeftDerivation-skip-prefix* γ'
append-eq-conv-conj assms(*3*) *drop-take is-word-Derivation-derivation-ge is-word-*δ
take-all take-append)
using *EF Suc-diff-le* γ' *assms*(*3*) *j-ge-d* **by** *auto*
qed

lemma *LeftDerives1-propagate-prefix*:
LeftDerives1 (δ @ α) *i r* $\beta \Longrightarrow$ *i* $\geq$ *length* $\delta \Longrightarrow$ *is-prefix* δ β
proof −
assume *a1*: *LeftDerives1* (δ @ α) *i r* β
assume *a2*: *length* $\delta \leq$ *i*
have *f3*: *take i* (δ @ α) = *take i* β
using *a1 Derives1-take LeftDerives1-implies-Derives1* **by** *blast*
then have *f4*: *length* (*take i* β) = *i*
using *a1* **by** (*metis* (*no-types*) *Derives1-bound LeftDerives1-implies-Derives1 dual-order.strict-implies-order length-take min.absorb2*)
have *take* (*length* δ) (*take i* β) = δ
using *f3 a2* **by** (*simp add*: *append-eq-conv-conj*)
then show *?thesis*
using *f4 a2* **by** (*metis* (*no-types*) *append-Nil2 append-eq-conv-conj diff-is-0-eq*′ *is-prefix-take take-0 take-append*)
qed

lemma *LeftDerivationIntro-cut-prefix*:
assumes *LeftDerivationIntro* (δ@α) *i r ix D j* γ
assumes *derivation-ge D* (*length* δ)
assumes *i* $\geq$ *length* δ
assumes *is-word-*δ: *is-word* δ
shows $\exists$ γ'. γ = δ @ γ' $\wedge$
LeftDerivationIntro α (*i* − *length* δ) *r ix* (*derivation-shift D* (*length* δ) *0*) (*j* − *length* δ) γ'
proof −

```
  from iffD1[OF LeftDerivationIntro-def assms(1)] obtain β where β:
    LeftDerives1 (δ @ α) i r β ∧
     ix < length (snd r) ∧ snd r ! ix = γ ! j ∧ LeftDerivationFix β (i + ix) D j γ
by blast
  have ∃ β'. β = δ @ β'
   using LeftDerives1-propagate-prefix β assms(3) by (metis append-dropped-prefix)

  then obtain β' where β': β = δ @ β' by blast
  with β have LeftDerives1 (δ @ α) i r (δ @ β') by simp
  from LeftDerives1-skip-prefix[OF assms(3) this]
  have α-β': LeftDerives1 α (i − length δ) r β' by blast
  have ldfix: LeftDerivationFix (δ @ β') (i + ix) D j γ using β β' by auto
  have δ-le-i-plus-ix: length δ ≤ i + ix using assms(3) by arith
  from LeftDerivationFix-cut-prefix[OF ldfix assms(2) δ-le-i-plus-ix assms(4)]
  obtain γ' where γ': γ = δ @ γ' ∧
     LeftDerivationFix β' (i + ix − length δ) (derivation-shift D (length δ) 0) (j −
length δ) γ'
      by blast
  have same-symbol: γ ! j = γ' ! (j − length δ)
    by (metis LeftDerivationFix-def β β' δ-le-i-plus-ix γ' leD nth-append)
  have β'-γ': LeftDerivationFix β' (i − length δ + ix)
    (derivation-shift D (length δ) 0) (j − length δ) γ' by (simp add: γ' assms(3))

  show ?thesis
    apply (simp add: LeftDerivationIntro-def)
    apply (rule-tac x=γ' in exI)
    apply (auto simp add: γ')
    apply (rule-tac x=β' in exI)
    by (auto simp add: β α-β' same-symbol β'-γ')
qed

lemma LeftDerivationLadder-implies-LeftDerivation-at-index:
  assumes LeftDerivationLadder α D L γ
  assumes index < length L
  shows LeftDerivation α (take (ladder-n L index) D) (ladder-γ α D L index)
using LeftDerivationLadder-def LeftDerivation-take-derive assms(1) ladder-γ-def
by auto

lemma LeftDerivationLadder-cut-prefix-propagate:
  assumes ladder: LeftDerivationLadder (δ@α) D L γ
  assumes is-word-δ: is-word δ
  assumes derivation-ge-δ: derivation-ge D (length δ)
  assumes ladder-i-0: ladder-i L 0 ≥ length δ
  assumes L': L' = ladder-cut-prefix (length δ) L
  assumes D': D' = derivation-shift D (length δ) 0
  shows index < length L ⟹
    LeftDerivation α (take (ladder-n L' index) D') (ladder-γ α D' L' index) ∧
    ladder-α (δ@α) D L index = δ@(ladder-α α D' L' index) ∧
    ladder-γ (δ@α) D L index = δ@(ladder-γ α D' L' index)
```

```
proof (induct index)
  case 0
    have ladder-α: ladder-α (δ@α) D L 0 = δ@(ladder-α α D' L' 0)
      by (simp add: ladder-α-def)
    have ldfix: LeftDerivationFix (δ@α) (ladder-i L 0) (take (ladder-n L 0) D)
     (ladder-j L 0) (ladder-γ (δ@α) D L 0) using ladder LeftDerivationLadder-def
by blast
    have dge-take: derivation-ge (take (ladder-n L 0) D) (length δ)
      using derivation-ge-δ by (metis append-take-drop-id derivation-ge-append)
    from LeftDerivationFix-cut-prefix[OF ldfix dge-take ladder-i-0 is-word-δ]
    obtain γ' where γ': ladder-γ (δ @ α) D L 0 = δ @ γ' ∧
     LeftDerivationFix α (ladder-i L 0 − length δ) (derivation-shift (take (ladder-n
L 0) D) (length δ) 0)
      (ladder-j L 0 − length δ) γ' by blast
    have ladder-γ: ladder-γ (δ@α) D L 0 = δ@(ladder-γ α D' L' 0)
      using γ' by (metis 0.prems D' Derive L' LeftDerivationFix-def
     LeftDerivation-implies-Derivation ladder-γ-def ladder-cut-prefix-n take-derivation-shift)
    have LeftDerivation α (take (ladder-n L' 0) D') (ladder-γ α D' L' 0)
    proof −
      have LeftDerivation (δ@α) (take (ladder-n L 0) D) (ladder-γ (δ@α) D L 0)
        using LeftDerivationLadder-implies-LeftDerivation-at-index ladder 0.prems
by blast
      then show ?thesis
        by (metis D' LeftDerivationLadder-def LeftDerivation-skip-prefix
          LeftDerivation-take-derive derivation-ge-δ ladder ladder-γ-def)
    qed
    then show ?case using ladder-α ladder-γ by auto
next
  case (Suc index)
    have index-less-L: index < length L using Suc(2) by arith
    then have induct: ladder-γ (δ@α) D L index = δ@(ladder-γ α D' L' index)
      using Suc by blast
    then have ladder-α: ladder-α (δ@α) D L (Suc index) = δ@(ladder-α α D' L'
(Suc index))
      by (simp add: ladder-α-def)
    have introsAt: LeftDerivationIntrosAt (δ@α) D L (Suc index)
      using Suc(2) ladder
      by (metis LeftDerivationIntros-def LeftDerivationLadder-def Suc-eq-plus1-left
le-add1)
    obtain n m e E where n: n = ladder-n L (Suc index − 1) and
      m: m = ladder-n L (Suc index) and e: e = D ! n and E: E = drop (Suc n)
(take m D)
      by blast
    from iffD1[OF LeftDerivationIntrosAt-def introsAt] have
       LeftDerivationIntro (ladder-α (δ @ α) D L (Suc index)) (ladder-i L (Suc
index)) (snd e)
      (ladder-ix L (Suc index)) E (ladder-j L (Suc index)) (ladder-γ (δ @ α) D L
(Suc index))
      using n m e E Let-def by meson
```

then have *ldintro*:
LeftDerivationIntro (*δ*@(*ladder-α α D′ L′* (*Suc index*))) (*ladder-i L* (*Suc index*)) (*snd e*)
(*ladder-ix L* (*Suc index*)) *E* (*ladder-j L* (*Suc index*)) (*ladder-γ* (*δ* @ *α*) *D L* (*Suc index*))
by (*simp add*: *ladder-α*)
have *dge-E-δ*: *derivation-ge E* (*length δ*)
apply (*simp add*: *E*)
using *derivation-ge-δ*
by (*metis append-take-drop-id derivation-ge-append*)
have *ladder-i-Suc*: *length δ* ≤ *ladder-i L* (*Suc index*)
using *Suc.prems derivation-ge-LeftDerivationLadder derivation-ge-δ ladder ladder-i-0*
by *blast*
from *LeftDerivationIntro-cut-prefix*[*OF ldintro dge-E-δ ladder-i-Suc is-word-δ*]
obtain *γ′* **where** *γ′*: *ladder-γ* (*δ* @ *α*) *D L* (*Suc index*) = *δ* @ *γ′* ∧
LeftDerivationIntro (*ladder-α α D′ L′* (*Suc index*)) (*ladder-i L* (*Suc index*) − *length δ*) (*snd e*)
(*ladder-ix L* (*Suc index*)) (*derivation-shift E* (*length δ*) *0*) (*ladder-j L* (*Suc index*) − *length δ*) *γ′*
by *blast*
then have *LeftDerivation* (*ladder-α α D′ L′* (*Suc index*))
((*ladder-i L* (*Suc index*) − *length δ*, *snd e*) # (*derivation-shift E* (*length δ*) *0*)) *γ′*
using *LeftDerivationIntro-implies-LeftDerivation* **by** *blast*
then have *LeftDerivation* (*ladder-γ α D′ L′ index*)
((*ladder-i L* (*Suc index*) − *length δ*, *snd e*) # (*derivation-shift E* (*length δ*) *0*)) *γ′*
by (*auto simp add*: *ladder-α-def*)
have *ld*: *LeftDerivation α* (*take* (*ladder-n L′* (*Suc index*)) *D′*) (*ladder-γ α D′ L′* (*Suc index*))
proof −
have *LeftDerivation* (*δ*@*α*) (*take* (*ladder-n L* (*Suc index*)) *D*) (*ladder-γ* (*δ*@*α*) *D L* (*Suc index*))
using *LeftDerivationLadder-implies-LeftDerivation-at-index ladder Suc.prems*
by *blast*
then show *?thesis*
by (*metis D′ LeftDerivationLadder-def LeftDerivation-skip-prefix*
LeftDerivation-take-derive derivation-ge-δ ladder ladder-γ-def)
qed
then show *?case*
using *γ′ D′ Derive L′ LeftDerivationIntro-def n m e E ld*
LeftDerivation-implies-Derivation ladder-γ-def ladder-cut-prefix-n take-derivation-shift
by (*metis* (*no-types*, *lifting*) *LeftDerivationLadder-implies-LeftDerivation-at-index*

LeftDerivation-skip-prefixword-ex Suc.prems Suc-leI index-less-L is-word-δ ladder
ladder-α le-0-eq neq0-conv)
qed

theorem *LeftDerivationLadder-cut-prefix*:
assumes *ladder*: *LeftDerivationLadder* (δ@α) *D L* γ
assumes *is-word-δ*: *is-word* δ
assumes *ladder-i-0*: *ladder-i L 0* $\geq$ *length* δ
shows $\exists$ *D' L'* γ'. $\gamma = \delta$ @ $\gamma' \wedge$
LeftDerivationLadder α *D' L'* $\gamma' \wedge$
D' = *derivation-shift D* (*length* δ) *0* $\wedge$
length L' = *length L* $\wedge$ *ladder-i L' 0* + *length* δ = *ladder-i L 0* $\wedge$
ladder-last-j L' + *length* δ = *ladder-last-j L*
proof −
obtain *D'* **where** *D'*: *D'* = *derivation-shift D* (*length* δ) *0* **by** *blast*
obtain *L'* **where** *L'*: *L'* = *ladder-cut-prefix* (*length* δ) *L* **by** *blast*
obtain γ' **where** γ': γ' = *drop* (*length* δ) γ **by** *blast*
have *ladder-last-j-upper-bound*: *ladder-last-j L* < *length* γ **using** *ladder*
using *ladder-last-j-bound* **by** *blast*
have *derivation-ge-δ*: *derivation-ge D* (*length* δ) **using** *is-word-δ LeftDerivation-Ladder-def*
LeftDerivation-implies-Derivation is-word-Derivation-derivation-ge ladder **by** *blast*
note *derivation-ge-ladder* =
derivation-ge-LeftDerivationLadder[*OF derivation-ge-δ ladder ladder-i-0*]
have *ladder-last-j-lower-bound*: *ladder-last-j L* $\geq$ *length* δ
using *LeftDerivationLadder-def derivation-ge-ladder is-ladder-def ladder*
ladder-last-j-def **by** *auto*
from *ladder-last-j-upper-bound ladder-last-j-lower-bound*
have *δ-less-γ*: *length* δ < *length* γ **by** *arith*
then have *γ-def*: $\gamma = \delta$ @ γ'
by (*metis LeftDerivation.simps*(*1*) *LeftDerivationLadder-def LeftDerivation-ge-take* γ'
append-eq-conv-conj derivation-ge-δ ladder)
have *length-L-nonzero*: *length L* $\neq$ *0*
using *LeftDerivationLadder-def is-ladder-def ladder* **by** *auto*
have *ladder-i-L'-thm*: $\bigwedge$ *index*. *index* < *length L* $\Longrightarrow$ *ladder-i L' index* + *length* δ = *ladder-i L index*
apply (*simp add*: *L'*)
apply (*subst ladder-cut-prefix-i*)
apply *simp*
using *length-L-nonzero* **apply** *blast*
using *derivation-ge-ladder* **by** *auto*
have *ladder-j-L'-thm*: $\bigwedge$ *index*. *index* < *length L* $\Longrightarrow$ *ladder-j L' index* + *length* δ = *ladder-j L index*
apply (*simp add*: *L'*)
apply (*subst ladder-cut-prefix-j*)
using *LeftDerivationLadder-def is-ladder-def ladder* **apply** *blast*
using *LeftDerivationLadder-def is-ladder-def ladder* **apply** *blast*
using *derivation-ge-ladder* **by** *auto*
have *length-L'*: *length L'* = *length L* **using** *L' ladder-cut-prefix-length* **by** *simp*
have *α-γ'*: *LeftDerivation* α *D'* γ'

```
  using D' LeftDerivationLadder-def LeftDerivation-skip-prefix γ' derivation-ge-δ
ladder
    by blast
  have length-D': length D' = length D by (simp add: D')
   have is-ladder-D-L: is-ladder D L using LeftDerivationLadder-def ladder by
blast
  {
    fix u :: nat
    assume u-bound-L': u < length L'
    have u-bound-L: u < length L using length-L' using u-bound-L' by simp
    have ladder-n L' u ≤ length D'
      apply (simp add: length-D' L')
      apply (subst ladder-cut-prefix-n)
      apply (simp add: u-bound-L)
      using length-L-nonzero apply arith
      using u-bound-L is-ladder-D-L
      by (simp add: is-ladder-def)
  }
  note is-ladder-1 = this
  {
    fix u :: nat
    fix v :: nat
    assume u-less-v: u < v
    assume v-bound-L': v < length L'
    then have v-bound-L: v < length L by (simp add: length-L')
    with u-less-v have u-bound-L: u < length L by arith
    have ladder-n L' u < ladder-n L' v
      apply (simp add: L')
      apply (subst ladder-cut-prefix-n)
      using u-bound-L apply blast
      using length-L-nonzero apply blast
      apply (subst ladder-cut-prefix-n)
      using v-bound-L apply blast
      using length-L-nonzero apply blast
      using u-less-v v-bound-L is-ladder-D-L by (simp add: is-ladder-def)
  }
  note is-ladder-2 = this
  have is-ladder-3: ladder-last-n L' = length D'
    apply (simp add: length-D' ladder-last-n-def L')
    apply (subst ladder-cut-prefix-n)
    apply (simp add: ladder-cut-prefix-length)
    using length-L-nonzero apply auto[1]
    using length-L-nonzero apply blast
    apply (simp add: ladder-cut-prefix-length)
    using is-ladder-D-L by (simp add: is-ladder-def ladder-last-n-def)
  have is-ladder-4: LeftDerivationFix α (ladder-i L' 0) (take (ladder-n L' 0) D')
    (ladder-j L' 0) (ladder-γ α D' L' 0)
  proof −
    have ldfix: LeftDerivationFix (δ@α) (ladder-i L 0) (take (ladder-n L 0) D)
```

(*ladder-j L 0*) (*ladder-γ* (*δ@α*) *D L 0*)
using *ladder LeftDerivationLadder-def* **by** *blast*
have *dge*: *derivation-ge* (*take* (*ladder-n L 0*) *D*) (*length δ*)
using *derivation-ge-δ* **by** (*metis append-take-drop-id derivation-ge-append*)
from *LeftDerivationFix-cut-prefix*[*OF ldfix dge ladder-i-0 is-word-δ*]
obtain *γ′* **where** *γ′*: *ladder-γ* (*δ* @ *α*) *D L 0* = *δ* @ *γ′* ∧
LeftDerivationFix α (*ladder-i L 0* − *length δ*) (*derivation-shift* (*take* (*ladder-n L 0*) *D*) (*length δ*) *0*)
(*ladder-j L 0* − *length δ*) *γ′* **by** *blast*
then show *?thesis*
using *LeftDerivationLadder-cut-prefix-propagate D′ L′ append-eq-conv-conj derivation-ge-δ*
is-word-δ ladder ladder-cut-prefix-i ladder-cut-prefix-j ladder-cut-prefix-n ladder-i-0
length-0-conv length-L-nonzero length-greater-0-conv take-derivation-shift **by** *auto*
qed
{
fix *index* :: *nat*
assume *index-lower-bound*: *Suc 0* ≤ *index*
assume *index-upper-bound*: *index* < *length L′*
have *introsAt*: *LeftDerivationIntrosAt* (*δ@α*) *D L index*
by (*metis LeftDerivationIntros-def LeftDerivationLadder-def One-nat-def index-lower-bound*
index-upper-bound ladder length-L′)
then have *ladder-i-L*: *ladder-i L index* = *fst* (*D* ! *ladder-n L* (*index* − *Suc 0*))
by (*metis LeftDerivationIntrosAt-def One-nat-def* ‹*LeftDerivationIntrosAt* (*δ* @ *α*) *D L index*›)
have *ladder-i-L′-as-L*: *ladder-i L′ index* = *ladder-i L index* − (*length δ*)
using *ladder-cut-prefix-i L′ index-upper-bound is-ladder-D-L is-ladder-not-empty length-L′*
length-greater-0-conv **by** *auto*
have *length-L-gr-0*: *length L* > *0* **using** *length-L′ length-L-nonzero* **by** *arith*
have *index-Suc-upper-bound-L*: *index* − *Suc 0* < *length L* **using** *index-upper-bound length-L′* **by** *arith*
have *fst* (*D′* ! *ladder-n L′* (*index* − *Suc 0*)) = *fst* (*D* ! *ladder-n L* (*index* − *Suc 0*)) − (*length δ*)
apply (*subst D′*, *subst L′*)
apply (*subst ladder-cut-prefix-n*[*OF index-Suc-upper-bound-L length-L-gr-0*])
apply (*simp add*: *derivation-shift-def*)
using *index-lower-bound index-upper-bound is-ladder-D-L ladder-n-minus-1-bound length-L′* **by** *auto*
then have *ladder-i-L′*: *ladder-i L′ index* = *fst* (*D′* ! *ladder-n L′* (*index* − *Suc 0*))
using *ladder-i-L ladder-i-L′-as-L* **by** *auto*
have *LeftDerivationIntro* (*ladder-α α D′ L′ index*) (*ladder-i L′ index*)
(*snd* (*D′* ! *ladder-n L′* (*index* − *Suc 0*))) (*ladder-ix L′ index*)
(*drop* (*Suc* (*ladder-n L′* (*index* − *Suc 0*))) (*take* (*ladder-n L′ index*) *D′*)) (*ladder-j L′ index*)

```
    (ladder-γ α D' L' index)
  proof −
    have LeftDerivationIntro (ladder-α (δ@α) D L index) (ladder-i L index)
      (snd (D ! ladder-n L (index − Suc 0))) (ladder-ix L index)
       (drop (Suc (ladder-n L (index − Suc 0))) (take (ladder-n L index) D))
(ladder-j L index)
      (ladder-γ (δ@α) D L index) using introsAt
      by (metis LeftDerivationIntrosAt-def One-nat-def)
   then have ldintro: LeftDerivationIntro (δ@(ladder-α α D' L' index)) (ladder-i
L index)
      (snd (D ! ladder-n L (index − Suc 0))) (ladder-ix L index)
       (drop (Suc (ladder-n L (index − Suc 0))) (take (ladder-n L index) D))
(ladder-j L index)
      (ladder-γ (δ@α) D L index)
       using D' L' LeftDerivationLadder-cut-prefix-propagate derivation-ge-δ in-
dex-upper-bound
         is-word-δ ladder ladder-i-0 length-L' by auto
    have dge: derivation-ge (drop (Suc (ladder-n L (index − Suc 0))))
      (take (ladder-n L index) D)) (length δ) using derivation-ge-δ
      by (metis append-take-drop-id derivation-ge-append)
    have δ-le-i-L: length δ ≤ ladder-i L index
      using derivation-ge-ladder index-upper-bound length-L' by auto
   from LeftDerivationIntro-cut-prefix[OF ldintro dge δ-le-i-L is-word-δ] obtain
γ' where γ':
      ladder-γ (δ @ α) D L index = δ @ γ' ∧
        LeftDerivationIntro (ladder-α α D' L' index) (ladder-i L index − length
δ)
         (snd (D ! ladder-n L (index − Suc 0))) (ladder-ix L index)
        (derivation-shift (drop (Suc (ladder-n L (index − Suc 0))) (take (ladder-n
L index) D))
         (length δ) 0) (ladder-j L index − length δ) γ' by blast
    have h1: ladder-i L' index = ladder-i L index − length δ
      using L' ladder-cut-prefix-i ladder-i-L'-as-L by blast
     have h2: (snd (D' ! ladder-n L' (index − Suc 0))) = (snd (D ! ladder-n L
(index − Suc 0)))
      apply (subst L', subst ladder-cut-prefix-n)
      apply (simp add: index-Suc-upper-bound-L)
      using length-L-gr-0 apply auto[1]
      apply (subst D')
      apply (simp add: derivation-shift-def)
    using index-lower-bound index-upper-bound is-ladder-D-L ladder-n-minus-1-bound

       length-L' by auto
    have h3: ladder-ix L' index = ladder-ix L index
      using ladder-cut-prefix-ix L' index-upper-bound length-L' length-L-gr-0 by
auto
    have h4: (drop (Suc (ladder-n L' (index − Suc 0))) (take (ladder-n L' index)
D')) =
      (derivation-shift (drop (Suc (ladder-n L (index − Suc 0))) (take (ladder-n
```

```
L index) D))
          (length δ) 0)
        apply (subst D′)
        apply (subst L′)+
        apply (subst ladder-cut-prefix-n, simp add: index-Suc-upper-bound-L)
        using length-L-gr-0 apply blast
        apply (subst ladder-cut-prefix-n)
        using index-upper-bound length-L′ apply arith
        using length-L-gr-0 apply blast
        apply (simp add: derivation-shift-def)
        by (simp add: drop-map take-map)
      have h5: ladder-j L′ index = ladder-j L index − length δ
        using ladder-cut-prefix-j L′ index-upper-bound length-L′ length-L-gr-0 by
auto
      have h6: ladder-γ α D′ L′ index = γ′
        using D′ L′ LeftDerivationLadder-cut-prefix-propagate γ′ derivation-ge-δ
index-upper-bound
          is-word-δ ladder ladder-i-0 length-L′ by auto
      show ?thesis using h1 h2 h3 h4 h5 h6 γ′ by simp
    qed
    then have LeftDerivationIntrosAt α D′ L′ index
      apply (auto simp add: LeftDerivationIntrosAt-def Let-def)
      using ladder-i-L′ by blast
  }
  note is-ladder-5 = this
  show ?thesis
    apply (rule-tac x=D′ in exI)
    apply (rule-tac x=L′ in exI)
    apply (rule-tac x=γ′ in exI)
    apply auto
    using γ-def apply blast
    defer 1
    using D′ apply blast
    using L′ ladder-cut-prefix-length apply auto[1]
    apply (subst ladder-i-L′-thm)
    using LeftDerivationLadder-def is-ladder-def ladder apply blast
    apply simp
    apply (simp add: ladder-last-j-def)
    apply (subst ladder-j-L′-thm)
    apply (simp add: length-L′)
    using length-L-nonzero apply arith
    apply (simp add: length-L′)
    apply (auto simp add: LeftDerivationLadder-def)
    using α-γ′ apply blast
    apply (auto simp add: is-ladder-def)
    using length-L-nonzero length-L′ apply auto[1]
    using is-ladder-1 apply blast
    using is-ladder-2 apply blast
    using is-ladder-3 apply blast
```

using *is-ladder-4* **apply** *blast*
by (*auto simp add*: *LeftDerivationIntros-def is-ladder-5*)
qed

end

end
theory *TheoremD10*
imports *TheoremD9 Ladder*
begin

context *LocalLexing* **begin**

lemma $\mathcal{P}$*-wellformed*: $p \in \mathcal{P}\ k\ u \Longrightarrow$ *wellformed-tokens p*
using $\mathcal{P}$*-are-admissible admissible-wellformed-tokens* **by** *blast*

lemma $\mathcal{X}$*-token-length*: $t \in \mathcal{X}\ k \Longrightarrow k +$ *length* (*chars-of-token t*) $\leq$ *length Doc*
by (*metis le-diff-conv2* $\mathcal{X}$*-is-prefix add.commute chars-of-token-def empty-*$\mathcal{X}$
empty-iff is-prefix-length le-neq-implies-less length-drop linear)

lemma *mono-Scan*: *mono* (*Scan T k*)
by (*simp add*: *Scan-regular regular-implies-mono*)

lemma π*-apply-setmonotone*: $x \in I \Longrightarrow x \in \pi\ k\ T\ I$
using *Complete-subset-*π *LocalLexing.Complete-def LocalLexing-axioms* **by** *blast*

lemma *Scan-apply-setmonotone*: $x \in I \Longrightarrow x \in$ *Scan T k I*
by (*simp add*: *Scan-def*)

lemma *leftderives-padfront*:
assumes *leftderives* α β
assumes *is-word u*
shows *leftderives* ($u@\alpha$) ($u@\beta$)
using *LeftDerivation-append-prefix LeftDerivation-implies-leftderives assms*(*1*) *assms*(*2*)

leftderives-implies-LeftDerivation **by** *blast*

lemma *leftderives-padback*:
assumes *leftderives* α β
assumes *is-sentence u*
shows *leftderives* ($\alpha@u$) ($\beta@u$)
using *LeftDerivation-append-suffix LeftDerivation-implies-leftderives assms*(*1*) *assms*(*2*)

leftderives-implies-LeftDerivation **by** *blast*

lemma *leftderives-pad*:
assumes α-β: *leftderives* α β
assumes *is-word*: *is-word u*

assumes *is-sentence*: *is-sentence v*
shows *leftderives* (*u*@*α*@*v*) (*u*@*β*@*v*)
by (*simp add*: *α-β is-sentence is-word leftderives-padback leftderives-padfront*)

lemma *leftderives-rule*:
assumes (*N*, *w*) ∈ ℜ
shows *leftderives* [*N*] *w*
by (*metis append-Nil append-Nil2 assms is-sentence-def is-word-terminals leftderives1-def*

leftderives1-implies-leftderives list.pred-inject(*1*) *terminals-empty wellformed-tokens-empty-path*)

lemma *leftderives-rule-step*:
assumes *ld*: *leftderives a* (*u*@[*N*]@*v*)
assumes *rule*: (*N*, *w*) ∈ ℜ
assumes *is-word*: *is-word u*
assumes *is-sentence*: *is-sentence v*
shows *leftderives a* (*u*@*w*@*v*)
proof −
have *N-w*: *leftderives* [*N*] *w* **using** *rule leftderives-rule* **by** *blast*
then have *leftderives* (*u*@[*N*]@*v*) (*u*@*w*@*v*) **using** *leftderives-pad is-word is-sentence*
by *blast*
then show *leftderives a* (*u*@*w*@*v*) **using** *leftderives-trans ld* **by** *blast*
qed

lemma *leftderives-trans-step*:
assumes *ld*: *leftderives a* (*u*@*b*@*v*)
assumes *rule*: *leftderives b c*
assumes *is-word*: *is-word u*
assumes *is-sentence*: *is-sentence v*
shows *leftderives a* (*u*@*c*@*v*)
proof −
have *leftderives* (*u*@*b*@*v*) (*u*@*c*@*v*) **using** *leftderives-pad ld rule is-word is-sentence*
by *blast*
then show *?thesis* **using** *leftderives-trans ld* **by** *blast*
qed

lemma *charslength-of-prefix*:
assumes *is-prefix a b*
shows *charslength a* ≤ *charslength b*
by (*simp add*: *assms is-prefix-chars is-prefix-length*)

lemma *item-rhs-simp*[*simp*]: *item-rhs* (*Item* (*N*, *α*) *d i j*) = *α*
by (*simp add*: *item-rhs-def*)

definition *Prefixes* :: *'a list* ⇒ *'a list set*
where
Prefixes p = { *q* . *is-prefix q p* }

lemma ℘-*wellformed*: *p* ∈ ℘ ⟹ *wellformed-tokens p*

by (*simp add*: $\mathfrak{P}$*-are-admissible admissible-wellformed-tokens*)

lemma *Prefixes-reflexive*[*simp*]: $p \in$ *Prefixes* p
by (*simp add*: *Prefixes-def is-prefix-def*)

lemma *Prefixes-is-prefix*: $q \in$ *Prefixes* $p =$ *is-prefix* $q\ p$
by (*simp add*: *Prefixes-def*)

lemma *prefixes-are-paths'*: $p \in \mathfrak{P} \Longrightarrow$ *is-prefix* $q\ p \Longrightarrow q \in \mathfrak{P}$
using $\mathcal{P}$*.simps*(*3*) $\mathfrak{P}$*-def prefixes-are-paths* **by** *blast*

lemma *thmD10-ladder*:
$p \in \mathfrak{P} \Longrightarrow$
charslength $p = k \Longrightarrow$
$X \in T \Longrightarrow$
$T \subseteq \mathcal{X}\ k \Longrightarrow$
$(N, \alpha @ \beta) \in \mathfrak{R} \Longrightarrow$
$r \leq$ *length* $p \Longrightarrow$
leftderives $[\mathfrak{S}]$ $((terminals\ (take\ r\ p)) @ [N] @ \gamma) \Longrightarrow$
LeftDerivationLadder $\alpha\ D\ L$ $(terminals\ ((drop\ r\ p) @ [X])) \Longrightarrow$
ladder-last-j $L =$ *length* $(drop\ r\ p) \Longrightarrow$
$k' = k +$ *length* (*chars-of-token* $X) \Longrightarrow$
$x =$ *Item* $(N, \alpha @ \beta)$ (*length* α) (*charslength* $(take\ r\ p))\ k' \Longrightarrow$
$I =$ *items-le* k' $(\pi\ k'\ \{\}\ (Scan\ T\ k\ (Gen\ (Prefixes\ p))))$
$\Longrightarrow x \in I$
proof (*induct length L arbitrary*: $N\ \alpha\ \beta\ r\ \gamma\ D\ L\ x$ *rule*: *less-induct*)
case *less*
have *item-origin-x-def*: *item-origin* $x =$ (*charslength* $(take\ r\ p)$)
by (*simp add*: *less.prems*(*11*))
then have *x-k*: *item-origin* $x \leq k$
using *charslength.simps is-prefix-chars is-prefix-length is-prefix-take less.prems*(*2*)
by *force*
have *item-end-x-def*: *item-end* $x = k'$ **by** (*simp add*: *less.prems*(*11*))
have *item-dot-x-def*: *item-dot* $x =$ *length* α **by** (*simp add*: *less.prems*(*11*))
have k'*-upperbound*: $k' \leq$ *length Doc*
using $\mathcal{X}$*-token-length less.prems*(*10*) *less.prems*(*3*) *less.prems*(*4*) **by** *blast*
note *item-def* = *less.prems*(*11*)
note k' = *less.prems*(*10*)
note *rule-dom* = *less.prems*(*5*)
note *p-charslength* = *less.prems*(*2*)
note *p-dom* = *less.prems*(*1*)
note r = *less.prems*(*6*)
note *leftderives-start* = *less.prems*(*7*)
note *X-dom* = *less.prems*(*3*)
have *wellformed-x*: *wellformed-item* x
apply (*auto simp add*: *wellformed-item-def item-def rule-dom p-charslength*)
apply (*subst* k')
apply (*subst charslength.simps*[*symmetric*])
apply (*subst p-charslength*[*symmetric*])

```
      using item-origin-x-def p-charslength x-k apply linarith
      apply (rule k'-upperbound)
      done
    have  leftderives-α: leftderives α (terminals ((drop r p)@[X]))
    using LeftDerivationLadder-def LeftDerivation-implies-leftderives less.prems(8)
by blast
  have is-sentence-drop-pX: is-sentence (drop r (terminals p) @ [terminal-of-token
X])
    by (metis derives-is-sentence is-sentence-concat leftderives-α leftderives-implies-derives

        rule-α-type rule-dom terminals-append terminals-drop terminals-singleton)
    have snd-item-rule-x: snd (item-rule x) = α@β by (simp add: item-def)
    from less have is-ladder D L using LeftDerivationLadder-def by blast
    then have length L ≠ 0 by (simp add: is-ladder-not-empty)
    then have length L = 1 ∨ length L > 1 by arith
    then show ?case
    proof (induct rule: disjCases2)
      case 1
        have ∃ i. LeftDerivationFix α i D (length (drop r p)) (terminals ((drop r
p)@[X]))
          using 1.hyps LeftDerivationLadder-L-0 less.prems(8) less.prems(9) by
fastforce
       then obtain i where LDF:
         LeftDerivationFix α i D (length (drop r p)) (terminals ((drop r p)@[X]))
by blast
         from LeftDerivationFix-splits-at-derives[OF this] obtain U a1 a2 b1 b2
where decompose:
         splits-at α i a1 U a2 ∧ splits-at (terminals (drop r p @ [X]))
           (length (drop r p)) b1 U b2 ∧ derives a1 b1 ∧ derives a2 b2 by blast
       then have b1: b1 = terminals (drop r p)
         by (simp add: append-eq-conv-conj splits-at-def)
       with decompose have U: U = fst X
      by (metis length-terminals nth-append-length splits-at-def terminal-of-token-def

          terminals-append terminals-singleton)
       from decompose b1 U have b2: b2 = []
         by (simp add: splits-at-combine splits-at-def)
       have D: LeftDerivation α D (terminals ((drop r p)@[X]))
         using LDF LeftDerivationLadder-def less.prems(8) by blast
       let ?y = Item (item-rule x) (length a1) (item-origin x) k
       have wellformed-y: wellformed-item ?y
         using wellformed-x
         apply (auto simp add: wellformed-item-def x-k)
         using k' k'-upperbound apply arith
         apply (simp add: item-rhs-def snd-item-rule-x)
         using decompose splits-at-def
         by (simp add: is-prefix-length trans-le-add1)
       have y-nonterminal: item-nonterminal ?y = N
         by (simp add: item-def item-nonterminal-def)
```

have *item-α-y*: *item-α ?y = a1*
by (*metis append-assoc append-eq-conv-conj append-take-drop-id decompose item.sel(1) item.sel(2) item-α-def item-rhs-def snd-item-rule-x splits-at-def*)
have *pvalid-y*: *pvalid p ?y*
apply (*auto simp add*: *pvalid-def*)
using *p-dom* 𝔓*-wellformed* **apply** *blast*
using *wellformed-y* **apply** *auto*[*1*]
apply (*rule-tac x=r* **in** *exI*)
apply (*auto simp add*: *r*)
using *p-charslength* **apply** *simp*
using *item-def* **apply** *simp*
apply (*rule-tac x=γ* **in** *exI*)
using *y-nonterminal* **apply** *simp*
using *is-derivation-def leftderives-start* **apply** *auto*[*1*]
apply (*simp add*: *item-α-y*)
using *b1 decompose* **by** *auto*
let *?z = inc-item ?y k′*
have *item-rhs-y*: *item-rhs ?y = α@β*
by (*simp add*: *item-def item-rhs-def*)
have *split-α*: *α = a1@*[*U*]*@a2* **using** *decompose splits-at-combine* **by** *blast*
have *next-symbol-y*: *next-symbol ?y = Some*(*fst X*)
by (*auto simp add*: *next-symbol-def is-complete-def item-rhs-y split-α U*)
have *z-in-Scan-y*: *?z ∈ Scan T k* {*?y*}
apply (*simp add*: *Scan-def*)
apply (*rule disjI2*)
apply (*rule-tac x=?y* **in** *exI*)
apply (*rule-tac x=fst X* **in** *exI*)
apply (*rule-tac x=snd X* **in** *exI*)
apply (*auto simp add*: *bin-def X-dom*)
using *k′ chars-of-token-def* **apply** *simp*
using *next-symbol-y* **apply** *simp*
done
from *pvalid-y* **have** *?y ∈ Gen*(*Prefixes p*)
apply (*simp add*: *Gen-def*)
apply (*rule-tac x=p* **in** *exI*)
by (*auto simp add*: *paths-le-def p-dom*)
then have *Scan T k* {*?y*} ⊆ *Scan T k* (*Gen*(*Prefixes p*))
apply (*rule-tac monoD*[*OF mono-Scan*])
apply *blast*
done
with *z-in-Scan-y* **have** *z-in-Scan-Gen*: *?z ∈ Scan T k* (*Gen*(*Prefixes p*))
using *rev-subsetD* **by** *blast*
have *wellformed-z*: *wellformed-item ?z*
using *k′ k′-upperbound next-symbol-y wellformed-inc-item wellformed-y* **by** *auto*
have *item-β-z*: *item-β ?z = a2@β*
apply (*simp add*: *item-β-def*)
apply (*simp add*: *item-rhs-y split-α*)

```
      done
    have item-end-z: item-end ?z = k' by simp
    have x-via-z: x = inc-dot (length a2) ?z
      by (simp add: inc-dot-def less.prems(11) split-α)
    have x-in-z: x ∈ π k' {} {?z}
      apply (subst x-via-z)
      apply (rule-tac thmD6[OF wellformed-z item-β-z item-end-z])
      using decompose b2 by blast
    have π k' {} {?z} ⊆ π k' {} (Scan T k (Gen(Prefixes p)))
      apply (rule-tac monoD[OF mono-π])
      using z-in-Scan-Gen using empty-subsetI insert-subset by blast
    then have x-in-Scan-Gen: x ∈ π k' {} (Scan T k (Gen(Prefixes p)))
      using x-in-z by blast
    have item-end x = k' by (simp add: item-end-x-def)
    with x-in-Scan-Gen show ?case
      using items-le-def less.prems(12) mem-Collect-eq nat-le-linear by blast
  next
   case 2
    obtain i where i: i = ladder-i L 0 by blast
    obtain i' where i': i' = ladder-j L 0 by blast
    obtain α' where α': α' = ladder-γ α D L 0 by blast
    obtain n where n: n = ladder-n L 0 by blast
    have ldfix: LeftDerivationFix α i (take n D) i' α'
      using LeftDerivationLadder-def α' i i' n less.prems(8) by blast
   have α'-alt: α' = ladder-α α D L 1 using 2 by (simp add: α' ladder-α-def)

    have i'-alt: i' = ladder-i L 1 using 2 by (simp add: i' ladder-i-def)
    obtain e where e: e = D ! n by blast
    obtain ix where ix: ix = ladder-ix L 1 by blast
    obtain m where m: m = ladder-n L 1 by blast
    obtain E where E: E = drop (Suc n) (take m D) by blast
   have ldintro: LeftDerivationIntro α' i' (snd e) ix E (ladder-j L 1) (ladder-γ
α D L 1)
      by (metis 2.hyps LeftDerivationIntrosAt-def LeftDerivationIntros-def
        LeftDerivationLadder-def One-nat-def α'-alt E diff-Suc-1 e i'-alt ix leI
        less.prems(8) m n not-less-eq zero-less-one)
    have is-nonterminal-α'-at-i': is-nonterminal (α' ! i')
      using LeftDerivationIntro-implies-nonterminal ldintro by blast
    then have is-nonterminal-α-at-i: is-nonterminal (α ! i)
      using LeftDerivationFix-def ldfix by auto
   then have ∃ A a1 a2 a1'. splits-at α i a1 A a2 ∧ splits-at α' i' a1' A a2 ∧
      LeftDerivation a1 (take n D) a1'
      using LeftDerivationFix-splits-at-nonterminal ldfix by auto
    then obtain A a1 a2 a1' where A: splits-at α i a1 A a2 ∧ splits-at α' i'
a1' A a2 ∧
      LeftDerivation a1 (take n D) a1' by blast
    have A-def: A = α' ! i' using A splits-at-def by blast
  have is-nonterminal-A: is-nonterminal A using A-def is-nonterminal-α'-at-i'
by blast
```

have $\exists$ *rule*. $e = (i', rule)$
by (*metis e 2.hyps LeftDerivationIntrosAt-def LeftDerivationIntros-def LeftDerivationLadder-def One-nat-def Suc-leI diff-Suc-1 i'-alt less.prems*(8) *n prod.collapse zero-less-one*)
then obtain *rule* **where** *rule*: $e = (i', rule)$ **by** *blast*
obtain w **where** w: $w = snd\ rule$ **by** *blast*
obtain α'' **where** α'': $\alpha'' = a1' @ w @ a2$ **by** *blast*
have α'-α'': *LeftDerives1* $\alpha'\ i'\ rule\ \alpha''$
by (*metis A w LeftDerivationFix-is-sentence LeftDerivationIntro-def LeftDerivationIntro-examine-rule LeftDerives1-def* α'' *ldfix ldintro prod.collapse rule snd-conv splits-at-implies-Derives1*)
then have *is-word-a1'*: *is-word a1'* **using** *LeftDerives1-splits-at-is-word A* **by** *blast*
have *A-eq-fst-rule*: $A = fst\ rule$
using *A LeftDerivationIntro-examine-rule ldintro rule* **by** *fastforce*
have *ix-bound*: $ix < length\ w$ **using** *ix w rule ldintro LeftDerivationIntro-def snd-conv*
by *auto*
then have $\exists$ *w1 W w2*. *splits-at w ix w1 W w2* **using** *splits-at-def* **by** *blast*
then obtain *w1 W w2* **where** *W*: *splits-at w ix w1 W w2* **by** *blast*
have *a1'-w-a2*: *a1'@w@a2 = ladder-stepdown-α-0 α D L*
using *ladder-stepdown-α-0-altdef 2.hyps A α'-alt e i'-alt less.prems*(8) *n rule snd-conv w* **by** *force*
from *LeftDerivationLadder-stepdown*[*OF less.prems*(8) *2*] **obtain** L' **where** L':
LeftDerivationLadder (*a1'@*(*w@a2*)) (*drop* (*ladder-stepdown-diff L*) *D*) L'
(*terminals* (*drop r p* @ [*X*])) ∧
length $L' = length\ L - 1$ ∧
ladder-i L' *0* = *ladder-i L 1* + *ladder-ix L 1* ∧ *ladder-last-j* L' = *ladder-last-j L*
using *a1'-w-a2* **by** *auto*
have *ladder-i-L'-0*: *ladder-i* L' *0* = i' + *ix* **using** L' *i'-alt ix* **by** *auto*
have *ladder-last-j-L'*: *ladder-last-j* L' = *length* (*drop r p*) **using** L' *less.prems* **by** *auto*
from L' **have** *this1*: *LeftDerivationLadder* (*a1'@*(*w@a2*)) (*drop* (*ladder-stepdown-diff L*) *D*) L'
(*terminals* (*drop r p* @ [*X*])) **by** *blast*
have *this2*: *length a1'* ≤ *ladder-i* L' *0* **using** *A ladder-i-L'-0 splits-at-def* **by** *auto*
from *LeftDerivationLadder-cut-prefix*[*OF this1 is-word-a1' this2*]
obtain $D'\ L''\ \gamma'$ **where** L'':
terminals (*drop r p* @ [*X*]) = *a1'* @ γ' ∧
LeftDerivationLadder (*w* @ *a2*) $D'\ L''\ \gamma'$ ∧
D' = *derivation-shift* (*drop* (*ladder-stepdown-diff L*) *D*) (*length a1'*) *0* ∧
length L'' = *length* L' ∧

ladder-i L'' 0 + length a1' = ladder-i L' 0 ∧
ladder-last-j L'' + length a1' = ladder-last-j L' **by** *blast*
have *length-a1'-bound*: *length a1' ≤ length (drop r p)* **using** *L''* *ladder-last-j-L'*
by *linarith*
then have *is-prefix-a1'-drop-r-p*: *is-prefix a1' (terminals (drop r p))*
proof −
have *f1*: *∀ ss ssa ssb. ¬ is-prefix (ss::symbol list) (ssa @ ssb) ∨ is-prefix ss ssa ∨ (∃ ssc. ssc ≠ [] ∧ is-prefix ssc ssb ∧ ss = ssa @ ssc)*
by (*simp add*: *is-prefix-of-append*)
have *f2*: *⋀ss ssa. is-prefix ((ss::symbol list) @ ssa) ss ∨ ¬ is-prefix ssa []*
by (*metis* (*no-types*) *append-Nil2 is-prefix-cancel*)
have *f3*: *⋀ss. is-prefix ss [] ∨ ¬ is-prefix (terminals (drop r p) @ ss) a1'*
by (*metis* (*no-types*) *drop-eq-Nil is-prefix-append length-a1'-bound length-terminals*)
have *is-prefix a1' (a1' @ γ') ∧ is-prefix a1' a1'*
by (*metis* (*no-types*) *append-Nil2 is-prefix-cancel is-prefix-empty*)
then show *?thesis*
using *f3 f2 f1* **by** (*metis L'' terminals-append*)
qed
obtain *r'* **where** *r'*: *r' = r + i'* **by** *blast*
have *length-a1'-eq-i'*: *length a1' = i'*
using *A less-or-eq-imp-le min.absorb2 splits-at-def* **by** *auto*
have *a1'-r'*: *r ≤ r' ∧ r' ≤ length p ∧*
terminals (drop r p) = a1' @ (terminals (drop r' p))
using *is-prefix-a1'-drop-r-p r'*
proof −
have *∃ q. terminals (drop r p) = a1' @ q*
using *is-prefix-a1'-drop-r-p* **by** (*metis is-prefix-unsplit*)
then obtain *q* **where** *q*: *terminals (drop r p) = a1' @ q* **by** *blast*
have *q = drop i' (terminals (drop r p))*
using *length-a1'-eq-i' q* **by** (*simp add*: *append-eq-conv-conj*)
then have *q = terminals (drop i' (drop r p))* **by** *simp*
then have *q = terminals (drop r' p)* **by** (*simp add*: *r' add.commute*)
with *q* **show** *?thesis*
using *add.commute diff-add-inverse le-Suc-ex le-add1 le-diff-conv length-a1'-bound*
length-a1'-eq-i' length-drop r r' **by** *auto*
qed
have *ladder-i-L''*: *ladder-i L'' 0 = ix* **using** *L'' ladder-i-L'-0 length-a1'-eq-i'*
add.commute add-left-cancel **by** *linarith*
have *L''-ladder*: *LeftDerivationLadder (w @ a2) D' L'' γ'* **using** *L''* **by** *blast*
have *ladder-i L'' 0 < length w* **using** *ladder-i-L'' ix-bound* **by** *blast*
from *LeftDerivationLadder-cut-appendix*[*OF L''-ladder this*]
obtain *E' F' γ1' γ2' L'''* **where** *L'''*:
D' = E' @ F' ∧
γ' = γ1' @ γ2' ∧

LeftDerivationLadder w E′ L‴ γ1′ ∧
derivation-ge F′ (length γ1′) ∧
LeftDerivation a2 (derivation-shift F′ (length γ1′) 0) γ2′ ∧
length L‴ = length L″ ∧ ladder-i L‴ 0 = ladder-i L″ 0 ∧
ladder-last-j L‴ = ladder-last-j L″
by *blast*
obtain *z* **where** *z*: *z = Item (A, w) (length w) (charslength (take r′ p)) k′*
by *blast*
have *z1*: *length L‴ < length L* **using** *2.hyps L′ L″ L‴* **by** *linarith*
have *z2*: *p ∈ 𝔓* **by** (*simp add*: *p-dom*)
have *z3*: *charslength p = k* **using** *p-charslength* **by** *auto*
have *z4*: *X ∈ T* **by** (*simp add*: *X-dom*)
have *z5*: *T ⊆ 𝒳 k* **by** (*simp add*: *less.prems*(*4*))
have *rule ∈ ℜ*
using *Derives1-rule LeftDerives1-implies-Derives1 α′-α″* **by** *blast*
then have *z6*: *(A, w @ []) ∈ ℜ* **using** *w A-eq-fst-rule* **by** *auto*
have *z7*: *r′ ≤ length p* **using** *a1′-r′* **by** *linarith*
have *leftderives* [𝔖] *((terminals (take r p))@[N]@γ)*
using *leftderives-start* **by** *blast*
then have *leftderives* [𝔖] *((terminals (take r p))@(α@β)@γ)*
by (*metis 𝔓-wellformed is-derivation-def is-derivation-is-sentence is-sentence-concat*
is-word-terminals-take leftderives-implies-derives leftderives-rule-step
p-dom rule-dom)
then have *leftderives* [𝔖] *((terminals (take r p))@a1@([A]@a2@β)@γ)*
using *A splits-at-combine append-assoc* **by** *fastforce*
then have *z8-helper*: *leftderives* [𝔖] *((terminals (take r p))@a1′@([A]@a2@β)@γ)*
by (*meson A LeftDerivation-implies-leftderives 𝔓-wellformed is-derivation-def*
is-derivation-is-sentence is-sentence-concat is-word-terminals-take
leftderives-implies-derives leftderives-trans-step p-dom)
have *join-terminals*: *(terminals (take r p))@a1′ = terminals (take r′ p)*
by (*metis is-prefix-a1′-drop-r-p length-a1′-eq-i′ r′ take-add take-prefix*
terminals-drop terminals-take)
from *z8-helper join-terminals* **have** *z8*:
leftderives [𝔖] *(terminals (take r′ p) @ [A] @ a2 @ β @ γ)*
by (*metis append-assoc*)
have *γ′-altdef*: *γ′ = terminals (drop r′ p @ [X])*
using *L″ a1′-r′* **by** *auto*
have *ladder-last-j L‴ + length a1′ = length (drop r p)*
using *L″ L‴ ladder-last-j-L′* **by** *linarith*
then have *ladder-last-j-L‴-γ′*: *ladder-last-j L‴ = length γ′ − 1*
by (*simp add*: *γ′-altdef length-a1′-eq-i′ r′*)
then have *length γ′ − 1 < length γ1′*
using *L‴ ladder-last-j-bound* **by** *fastforce*
then have *length γ1′ + length γ2′ − 1 < length γ1′*
using *L‴* **by** *simp*
then have *length γ2′ = 0* **by** *arith*
then have *γ2′*: *γ2′ = []* **by** *simp*

then have $\gamma 1'$: $\gamma 1' = terminals\ (drop\ r'\ p\ @\ [X])$ **using** γ'*-altdef* L''' **by** *auto*
then have *z9*: *LeftDerivationLadder w E' L''' (terminals (drop r' p @ [X]))*
using L''' **by** *blast*
have *z10*: *ladder-last-j L''' = length (drop r' p)*
using γ'*-altdef ladder-last-j-L'''-*γ' **by** *auto*
note *z11* = k'
have *z12*: *z = Item (A, w @ []) (length w) (charslength (take r' p)) k'*
using *z* **by** *simp*
note *z13* = *less.prems(12)*
note *induct* = *less.hyps(1)[of L''' A w [] r' a2@β@γ E' z]*
note *z-in-I* = *induct[OF z1 z2 z3 z4 z5 z6 z7 z8 z9 z10 z11 z12 z13]*
have *a2-derives-empty*: *derives a2 []* **using** L''' $\gamma 2'$
using *LeftDerivation-implies-leftderives leftderives-implies-derives* **by** *blast*

obtain *x1* **where** *x1*: *x1 = Item (N, α@β) (length a1)*
(charslength (take r p)) (charslength (take r' p)) **by** *blast*
obtain *q* **where** *q*: *q = take r' p* **by** *blast*
then have *is-prefix-q-p*: *is-prefix q p* **by** *simp*
then have *q-in-Prefixes*: *q ∈ Prefixes p* **using** *Prefixes-is-prefix* **by** *blast*
then have *wellformed-q*: *wellformed-tokens q*
using *p-dom is-prefix-q-p prefixes-are-paths' 𝔓-wellformed* **by** *blast*
have *item-rule-x1*: *item-rule x1 = (N, α@β)*
using *x1* **by** *simp*
have *is-prefix-r-r'*: *is-prefix (take r p) (take r' p)*
by (*metis append-eq-conv-conj is-prefix-take r' take-add*)
then have *charslength-le-r-r'*: *charslength (take r p) ≤ charslength (take r' p)*
using *charslength-of-prefix* **by** *blast*
have *is-prefix (take r' p) p* **by** *auto*
then have *charslength-r'-p*: *charslength (take r' p) ≤ charslength p*
using *charslength-of-prefix* **by** *blast*
have *charslength p ≤ length Doc*
using *less.prems(1) add-leE k' k'-upperbound z3* **by** *blast*
with *charslength-r'-p* **have** *charslength-r'-Doc*:
charslength (take r' p) ≤ length Doc **by** *arith*
have *α-decompose*: *α = a1 @ [A] @ a2* **using** *A splits-at-combine* **by** *blast*
have *wellformed-x1*: *wellformed-item x1*
apply (*auto simp add*: *wellformed-item-def*)
using *item-rule-x1 less.prems* **apply** *auto[1]*
using *charslength-le-r-r' x1* **apply** *auto[1]*
using *charslength-r'-Doc x1* **apply** *auto[1]*
using *x1 α-decompose* **by** *simp*
have *item-nonterminal-x1*: *item-nonterminal x1 = N*
by (*simp add*: *x1 item-nonterminal-def*)
have *r-q-p*: *take r (terminals q) = terminals (take r p)*
by (*metis q is-prefix-r-r' length-take min.absorb2 r take-prefix terminals-take*)

have *item-α-x1*: *item-α x1 = a1* **by** (*simp add*: *α-decompose item-α-def x1*)

```
      have a1': a1' = drop r (terminals q)
        by (metis append-eq-conv-conj join-terminals length-take length-terminals
min.absorb2 q r)
      have pvalid-q-x1: pvalid q x1
     apply (simp add: pvalid-def wellformed-q wellformed-x1 item-nonterminal-x1)
        apply (rule-tac x=r in exI)
        apply auto
        apply (simp add: a1'-r' min.absorb2 q)
        apply (simp add: q x1)
        apply (simp add: q x1 r')
        using r-q-p less.prems(7) append-Cons is-leftderivation-def
          leftderivation-implies-derivation apply fastforce
        apply (simp add: item-α-x1)
         using a1' A LeftDerivation-implies-leftderives leftderives-implies-derives
by blast
      have item-end-x1-le-k': item-end x1 ≤ k'
        using charslength-r'-p
        apply (simp add: x1)
        using less.prems by auto
      have x1-in-I: x1 ∈ I
        apply (subst less.prems(12))
        apply (auto simp add: items-le-def item-end-x1-le-k')
        apply (rule π-apply-setmonotone)
        apply (rule Scan-apply-setmonotone)
        apply (simp add: Gen-def)
        apply (rule-tac x=q in exI)
        by (simp add: pvalid-q-x1 paths-le-def q-in-Prefixes)
      obtain x2 where x2: x2 = inc-item x1 k' by blast
      have x1-in-bin: x1 ∈ bin I (item-origin z)
        using bin-def x1 x1-in-I z12 by auto
      have x2-in-Complete: x2 ∈ Complete k' I
        apply (simp add: Complete-def)
        apply (rule disjI2)
        apply (rule-tac x=x1 in exI)
        apply (simp add: x2)
        apply (rule-tac x=z in exI)
        apply auto
        using x1-in-bin apply blast
        using bin-def z12 z-in-I apply auto[1]
        apply (simp add: is-complete-def z12)
     by (simp add: α-decompose is-complete-def item-nonterminal-def next-symbol-def
x1 z12)
      have wf-I': wellformed-items (π k' {} (Scan T k (Gen (Prefixes p))))
     by (simp add: wellformed-items-Gen wellformed-items-Scan wellformed-items-π
z5)
      from items-le-Complete[OF this] x2-in-Complete
      have x2-in-I: x2 ∈ I by (metis Complete-π-fix z13)
      have wellformed-items I
        using wf-I' items-le-is-filter wellformed-items-def z13 by auto
```

```
        with x2-in-I have wellformed-x2: wellformed-item x2
          by (simp add: wellformed-items-def)
        have item-dot-x2: item-dot x2 = Suc (length a1)
          by (simp add: x2 x1)
        have item-β-x2: item-β x2 = a2 @ β
          apply (simp add: item-β-def item-dot-x2)
          apply (simp add: item-rhs-def x2 x1 α-decompose)
          done
        have item-end-x2: item-end x2 = k' by (simp add: x2)
         note inc-dot-x2-by-a2 = thmD6[OF wellformed-x2 item-β-x2 item-end-x2
a2-derives-empty]
        have x = inc-dot (length a2) x2
          by (simp add: α-decompose inc-dot-def less.prems(11) x1 x2)
        with inc-dot-x2-by-a2 have x ∈ π k' {} {x2} by auto
        then have x ∈ π k' {} I using x2-in-I
          by (meson contra-subsetD empty-subsetI insert-subset monoD mono-π)
        then show x ∈ I
       by (metis (no-types, lifting) π-subset-elem-trans dual-order.refl item-end-x-def

           items-le-def items-le-is-filter mem-Collect-eq z13)
    qed
qed

theorem thmD10:
  assumes p-dom: p ∈ 𝔓
  assumes p-charslength: charslength p = k
  assumes X-dom: X ∈ T
  assumes T-dom: T ⊆ 𝒳 k
  assumes rule-dom: (N, α@β) ∈ ℜ
  assumes r: r ≤ length p
  assumes leftderives-start: leftderives [𝔖] ((terminals (take r p))@[N]@γ)
  assumes leftderives-α: leftderives α (terminals ((drop r p)@[X]))
  assumes k': k' = k + length (chars-of-token X)
  assumes item-def: x = Item (N, α@β) (length α) (charslength (take r p)) k'
  assumes I: I = items-le k' (π k' {} (Scan T k (Gen (Prefixes p))))
  shows x ∈ I
proof −
  have ∃ D. LeftDerivation α D (terminals ((drop r p)@[X]))
    using leftderives-α leftderives-implies-LeftDerivation by blast
  then obtain D where D: LeftDerivation α D (terminals ((drop r p)@[X])) by
blast
  have is-sentence: is-sentence (terminals (drop r p @ [X]))
   using derives-is-sentence is-sentence-concat leftderives-α leftderives-implies-derives

      rule-α-type rule-dom by blast
  have ∃ L. LeftDerivationLadder α D L  (terminals ((drop r p)@[X])) ∧
        ladder-last-j L = length (drop r p)
    apply (rule LeftDerivationLadder-exists)
    apply (rule D)
```

```
    apply (rule is-sentence)
    by auto
  then obtain L where L: LeftDerivationLadder α D L  (terminals ((drop r
p)@[X])) and
    L-ladder-last-j: ladder-last-j L = length (drop r p) by blast
 from thmD10-ladder[OF assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
assms(7)
    L L-ladder-last-j k' item-def I]
  show ?thesis .
qed

end

end
theory TheoremD11
imports TheoremD10
begin

context LocalLexing begin

lemma LeftDerivationLadder-length-1:
  assumes ladder: LeftDerivationLadder α D L γ
  assumes singleton-L: length L = 1
  shows LeftDerivationFix α (ladder-i L 0) D (ladder-last-j L) γ
proof −
  have ldfix: LeftDerivationFix α (ladder-i L 0) (take (ladder-n L 0) D) (ladder-j
L 0)
    (ladder-γ α D L 0)
    using ladder LeftDerivationLadder-def by blast
  have ladder-n-0: ladder-n L 0 = length D
    using ladder singleton-L
     by (metis LeftDerivationLadder-def One-nat-def diff-Suc-1 is-ladder-def lad-
der-last-n-intro)
  from ldfix ladder-n-0 ladder singleton-L show ?thesis
    by (metis Derivation-unique-dest LeftDerivationLadder-def
    LeftDerivationLadder-implies-LeftDerivation-at-index LeftDerivationLadder-ladder-n-bound

    LeftDerivation-implies-Derivation One-nat-def diff-Suc-1 ladder-last-j-def take-all

      zero-less-one)
qed

lemma LeftDerivationFix-from-singleton-helper:
  assumes LeftDerivationFix [A] 0 D (length u) (u @ [B] @ v)
  shows D = []
proof −
  from iffD1[OF LeftDerivationFix-def assms] obtain E F where EF:
    is-sentence [A] ∧
     is-sentence (u @ [B] @ v) ∧
```

```
    LeftDerivation [A] D (u @ [B] @ v) ∧
    0 < length [A] ∧
    length u < length (u @ [B] @ v) ∧
    [A] ! 0 = (u @ [B] @ v) ! length u ∧
    D = E @ derivation-shift F 0 (Suc (length u)) ∧
    LeftDerivation (take 0 [A]) E (take (length u) (u @ [B] @ v)) ∧
    LeftDerivation (drop (Suc 0) [A]) F (drop (Suc (length u)) (u @ [B] @ v))
   by blast
  from EF have E: E = []
   by (metis Derivation.elims(2) Derives1-split LeftDerivation-implies-Derivation

     Nil-is-append-conv list.distinct(1) take-0)
  from EF have F: F = []
   by (metis E LeftDerivation.simps(1) LeftDerivation-ge-take LeftDerivation-implies-Derivation

    append-eq-conv-conj derivation-ge-shift is-word-Derivation length-Cons length-derivation-shift

     list.size(3) nth-Cons-0 nth-append self-append-conv2 take-0)
  from EF E F show D = []  by auto
qed

lemma LeftDerivationFix-from-singleton:
  assumes LeftDerivationFix [A] i D j γ
  shows D = []
proof −
  have ∃ u B v. splits-at γ j u B v  using assms
    using LeftDerivationFix-splits-at-derives by blast
  then obtain u B v where s: splits-at γ j u B v by blast
  from s have s1: γ = u @ [B] @ v using splits-at-combine by blast
  from s have s2: j = length u using splits-at-def by auto
  from assms s1 s2 LeftDerivationFix-from-singleton-helper
  show ?thesis by (metis LeftDerivationFix-def length-Cons less-Suc0 list.size(3))
qed

lemma LeftDerivationLadder-ladder-γ-last:
  assumes LeftDerivationLadder α D L γ
  shows γ = ladder-γ α D L (length L − 1)
by (metis Derive LeftDerivationLadder-def LeftDerivation-implies-Derivation One-nat-def
assms
  is-ladder-def last-ladder-γ)

theorem thmD11-helper:
  p ∈ 𝔓 ⟹
   charslength p = k ⟹
   X ∈ T ⟹
   T ⊆ 𝒳 k ⟹
   q = p @ [X] ⟹
   (N, α@β) ∈ ℜ ⟹
   r ≤ length q ⟹
```

```
  LeftDerivation [𝔖] D ((terminals (take r q))@[N]@γ) ⟹
  leftderives α (terminals (drop r q)) ⟹
  k' = k + length (chars-of-token X) ⟹
  x = Item (N, α@β) (length α) (charslength (take r q)) k' ⟹
  I = items-le k' (π k' {} (Scan T k (Gen (Prefixes p)))) ⟹
  x ∈ I
proof (induct length D arbitrary: D r N γ α β x rule: less-induct)
 case less
  have D = [] ∨ D ≠ [] by blast
  then show ?case
  proof (induct rule: disjCases2)
    case 1
     then have r: r = 0
          by (metis LeftDerivation.simps(1) diff-add-0 diff-add-inverse2 le-0-eq
length-0-conv
           length-append length-terminals less.prems(7) less.prems(8) list.size(4)
take-eq-Nil)
      with 1 have γ: γ = []
      using LeftDerivation.simps(1) append-Cons append-self-conv2 less.prems(8)
list.inject
          take-eq-Nil terminals-empty by auto
      from r γ 1 have start-is-N: 𝔖 = N
      using LeftDerivation.simps(1) append-eq-Cons-conv less.prems(8) list.inject
take-eq-Nil
          terminals-empty by auto
      have h1: r ≤ length p using r by auto
      have h2: leftderives [𝔖] (terminals (take r p) @ [N] @ γ)
       by (simp add: r γ start-is-N)
      have h3: leftderives α (terminals (drop r p @ [X]))
       using less.prems by (simp add: r less.prems)
      have h4: x = Item (N, α @ β) (length α) (charslength (take r p)) k'
       using less.prems by (simp add: r less.prems)
       from thmD10[OF less.prems(1, 2, 3, 4, 6) h1 h2 h3 less.prems(10) h4
less.prems(12)]
      show ?case .
   next
    case 2
      note D-non-empty = 2
      have r < length q ∨ r = length q using less by arith
      then show ?case
      proof (induct rule: disjCases2)
        case 1
         have h1: r ≤ length p using less.prems 1 by auto
         have take-q-p: take r q = take r p
          using 1 less.prems
          by (simp add: drop-keep-last le-neq-implies-less nat-le-linear not-less-eq
not-less-eq-eq)
          have h2: leftderives [𝔖] (terminals (take r p) @ [N] @ γ)
           apply (simp only: take-q-p[symmetric])
```

```
          using less.prems LeftDerivation-implies-leftderives by blast
        have h3: leftderives α (terminals (drop r p @ [X]))
          using less.prems(5, 9) h1 by simp
        have h4: k' = k + length (chars-of-token X) using less.prems by blast
        have h5: x = Item (N, α @ β) (length α) (charslength (take r p)) k'
        using less.prems take-q-p by simp
      from thmD10[OF less.prems(1, 2, 3, 4, 6) h1 h2 h3 h4 h5 less.prems(12)]
show ?case .
    next
      case 2
        from 2 have ld: LeftDerivation [𝔖] D (terminals q @ [N] @ γ)
          using less.prems(8) by auto
        from 2 have α-derives-empty: derives α []
          using less.prems(9) by auto
        have is-sentence-p: is-sentence (terminals p)
              using less.prems(1) ℒ_P-derives 𝔓-are-admissible admissible-def
is-derivation-def
            is-derivation-is-sentence is-sentence-concat by blast
        have is-symbol-X: is-symbol (terminal-of-token X)
          using less.prems(3, 4) 𝒳-are-terminals is-symbol-def rev-subsetD by
blast
            have is-sentence-q: is-sentence (terminals q) using is-sentence-p
is-symbol-X
          less.prems LeftDerivation-implies-leftderives is-derivation-def
         is-derivation-is-sentence is-sentence-concat ld leftderives-implies-derives
by blast
        have is-symbol-N: is-symbol N
          using less.prems(6) is-symbol-def rule-nonterminal-type by blast
        have is-sentence-γ: is-sentence γ
      by (meson LeftDerivation-implies-leftderives is-derivation-def is-derivation-is-sentence

            is-sentence-concat ld leftderives-implies-derives)
        have ld-exists-h1: is-sentence (terminals q @ [N] @ γ)
          using is-sentence-q is-sentence-γ is-symbol-N is-sentence-concat
        LeftDerivation-implies-leftderives is-derivation-def is-derivation-is-sentence
ld
            leftderives-implies-derives by blast
        have ld-exists-h2: length q < length (terminals q @ [N] @ γ) by simp
       from LeftDerivationLadder-exists[OF ld ld-exists-h1 ld-exists-h2] obtain
L where
          L: LeftDerivationLadder [𝔖] D L (terminals q @ [N] @ γ) and
          L-last-j: ladder-last-j L = length q by blast
        note r-eq-length-q = 2
      have ladder-i-0-eq-0: ladder-i L 0 = 0 using L append-Nil ladder-i-0-bound

          length-append-singleton less-Suc0 list.size(3) by fastforce
        have length L = 1 ∨ length L > 1 using L
            by (metis LeftDerivationLadder-def Suc-eq-plus1 Suc-eq-plus1-left
butlast-conv-take
```

```
        butlast-snoc diff-add-inverse2 is-ladder-def le-add1 le-neq-implies-less
        length-append-singleton old.nat.exhaust take-0)
    then show ?case
    proof (induct rule: disjCases2)
     case 1
      from LeftDerivationLadder-length-1[OF L 1] ladder-i-0-eq-0
       have ldfix: LeftDerivationFix [𝔖] 0 D (ladder-last-j L) (terminals q
@ [N] @ γ)
        by auto
      with LeftDerivationFix-from-singleton have D = [] by blast
      with D-non-empty have False by auto
      then show ?case by blast
    next
     case 2
      obtain a where a: a = ladder-α [𝔖] D L (length L − 1) by blast
       then have a-as-γ: a = ladder-γ [𝔖] D L (length L − 2) using 2
ladder-α-def
        by (metis diff-diff-left diff-is-0-eq not-le one-add-one)
      have introsAt: LeftDerivationIntrosAt [𝔖] D L (length L − 1) using
L
       by (metis 2.hyps LeftDerivationIntros-def LeftDerivationLadder-def
One-nat-def
          Suc-leI Suc-lessD diff-less less-not-refl not-less-eq zero-less-diff)
      obtain i where i: i = ladder-i L (length L − 1) by blast
      obtain j where j: j = ladder-j L (length L − 1) by blast
      obtain ix where ix: ix = ladder-ix L (length L − 1) by blast
      obtain c where c: c = ladder-γ [𝔖] D L (length L − 1) by blast
      obtain n where n: n = ladder-n L (length L − 1 − 1) by blast
      obtain m where m: m = ladder-n L (length L − 1) by blast
      obtain e where e: e = D ! n by blast
      obtain E where E: E = drop (Suc n) (take m D) by blast
      from iffD1[OF LeftDerivationIntrosAt-def introsAt]
      have i = fst e ∧ LeftDerivationIntro a i (snd e) ix E j c
       by (metis E a c e i ix j m n)
      then have i-eq-fst-e: i = fst e and
       ldintro: LeftDerivationIntro a i (snd e) ix E j c by auto
      have c-def: c = terminals q @ [N] @ γ
       using c L LeftDerivationLadder-ladder-γ-last by simp
      from iffD1[OF LeftDerivationIntro-def ldintro] obtain b where b:
       LeftDerives1 a i (snd e) b ∧ ix < length (snd (snd e)) ∧
      snd (snd e) ! ix = c ! j ∧ LeftDerivationFix b (i + ix) E j c by blast
       obtain M ω where Mω: (M, ω) = snd e using prod.collapse by
blast
      have j-q: j = length q using L-last-j j ladder-last-j-def by auto
      with c-def have c-at-j: c ! j = N
       by (metis append-Cons length-terminals nth-append-length)
      with b Mω have ω-at-ix: ω ! ix = N by (metis snd-conv)
      obtain μ1 μ2 where split-ω: splits-at ω ix μ1 N μ2
       by (metis Mω ω-at-ix b snd-conv splits-at-def)
```

obtain *a1 a2* **where** *split-a*: *splits-at a i a1 M a2* **using** *b*
by (*metis LeftDerivationIntro-bounds-ij LeftDerivationIntro-examine-rule Mω*
fst-conv ldintro splits-at-def)
then have *is-word-a1*: *is-word a1*
using *LeftDerives1-splits-at-is-word b* **by** *blast*
have *b = a1 @ ω @ a2* **using** *split-a b Mω*
by (*metis LeftDerives1-implies-Derives1 snd-conv splits-at-combine-dest*)

then have *b-def*: *b = a1 @ μ1 @ [N] @ μ2 @ a2* **using** *split-ω splits-at-combine*
by *simp*
have *is-nonterminal-N*: *is-nonterminal N*
using *less.prems*(*6*) *rule-nonterminal-type* **by** *blast*
with *LeftDerivationFix-splits-at-nonterminal split-a b*
have ∃ *U b1 b2 c1*. *splits-at b* (*i + ix*) *b1 U b2* ∧ *splits-at c j c1 U b2* ∧
LeftDerivation b1 E c1 **by** (*simp add*: *LeftDerivationFix-def c-at-j*)
then obtain *b1 b2 c1* **where** *b1b2c1*:
splits-at b (*i + ix*) *b1 N b2* ∧ *splits-at c j c1 N b2* ∧
LeftDerivation b1 E c1 **using** *c-at-j splits-at-def* **by** *auto*
then have *c1-q*: *c1 = terminals q* **using** *c-def j-q*
by (*simp add*: *append-eq-conv-conj splits-at-def*)
have *length-a1-eq-i*: *length a1 = i* **using** *split-a splits-at-def* **by** *auto*
have *length-μ1-eq-ix*: *length μ1 = ix* **using** *split-ω splits-at-def* **by** *auto*
have *b1 = take* (*i + ix*) *b* **using** *b1b2c1 splits-at-def* **by** *blast*
then have *b1-eq-a1-μ1*: *b1 = a1 @ μ1* **using** *b-def length-a1-eq-i length-μ1-eq-ix*
by (*simp add*: *append-eq-conv-conj take-add*)
have *LeftDerivation* (*a1 @ μ1*) *E c1* **using** *b1b2c1 b1-eq-a1-μ1* **by** *blast*
from *LeftDerivation-skip-prefixword-ex*[*OF this is-word-a1*]
obtain *w* **where** *w*: *c1 = a1 @ w* ∧
LeftDerivation μ1 (*derivation-shift E* (*length a1*) *0*) *w* **by** *blast*
have *a1-eq-take-i*: *a1 = take i* (*terminals q*)
using *w c1-q split-a append-eq-conv-conj length-a1-eq-i* **by** *blast*
have *μ1-leftderives*: *leftderives μ1* (*terminals* (*drop i q*)) **using** *w c1-q split-a*
LeftDerivation-implies-leftderives append-eq-conv-conj length-a1-eq-i **by** *auto*
have *LeftDerivation* [𝔖] (*take n D*) *a*
by (*metis 2.hyps L LeftDerivationLadder-implies-LeftDerivation-at-index*
One-nat-def Suc-lessD a-as-γ diff-Suc-eq-diff-pred diff-Suc-less n numeral-2-eq-2)
then have *LD-to-M*: *LeftDerivation* [𝔖] (*take n D*) ((*terminals* (*take i q*))@[*M*]@*a2*)
using *split-a splits-at-combine a1-eq-take-i terminals-take* **by** *auto*

have *is-ladder*: *is-ladder D L* **using** *L* **by** (*simp add*: *LeftDerivation-Ladder-def*)
then have *n-less-m*: $n < m$ **using** *n m is-ladder-def*
by (*metis* (*no-types*, *lifting*) *2.hyps One-nat-def diff-Suc-less length-greater-0-conv zero-less-diff*)
have *m-le-D*: $m \leq$ *length D* **using** *m is-ladder-def is-ladder dual-order.refl ladder-n-last-is-length* **by** *auto*
have *length* (*take n D*) = *n* **using** *n-less-m m-le-D*
using *length-take less-irrefl less-le-trans linear min.absorb2* **by** *auto*
then have *length-take-n-D*: *length* (*take n D*) < *length D*
using *n-less-m m-le-D less-le-trans* **by** *linarith*
have *ω-decompose*: $\omega = \mu 1 @ (N \# \mu 2)$ **using** *split-ω splits-at-combine* **by** *simp*
have $(M, \omega) \in \mathfrak{R}$ **by** (*metis Derives1-rule LeftDerives1-implies-Derives1 Mω b*)
with *ω-decompose* **have** *M-rule*: $(M, \mu 1 @ (N \# \mu 2)) \in \mathfrak{R}$ **by** *simp*
have *i-le-q*: $i \leq$ *length q* **using** *a1-eq-take-i length-a1-eq-i* **by** *auto*
obtain *y* **where**
y: $y = Item\ (M, \mu 1\ @\ N\ \#\ \mu 2)$ (*length* $\mu 1$) (*charslength* (*take i q*)) k' **by** *blast*
from *less.hyps*[*OF length-take-n-D less.prems*(*1*, *2*, *3*, *4*, *5*) *M-rule i-le-q LD-to-M*
$\mu 1$*-leftderives less.prems*(*10*) *y less.prems*(*12*)]
have *y-in-I*: $y \in I$ **by** *blast*
obtain *z* **where** *z*: $z = Item\ (N, \alpha @ \beta)\ 0\ k'\ k'$ **by** *blast*
then have *z-is-init-item*: $z =$ *init-item* $(N, \alpha @ \beta)\ k'$ **by** (*simp add*: *init-item-def*)
have $z \in$ *Predict* k' $\{y\}$
apply (*simp add*: *z-is-init-item*)
apply (*rule next-symbol-predicts*)
apply (*simp add*: *is-complete-def next-symbol-def y*)
apply (*simp add*: *less.prems*(*6*))
apply (*simp add*: *y item-end-def*)
done
then have $z \in$ *Predict* k' *I* **using** *Predict-def bin-def y-in-I* **by** *auto*
then have *z-in-I*: $z \in I$ **by** (*metis Predict-π-fix items-le-Predict less.prems*(*12*))
have *length-chars-q*: *length* (*chars q*) = k' **using** *less.prems* **by** *simp*
have *x-is-inc-dot*: $x =$ *inc-dot* (*length* α) *z*
by (*simp add*: *less.prems*(*11*) *r-eq-length-q length-chars-q z inc-dot-def*)
have *wellformed-items-I*: *wellformed-items I*
apply (*subst less.prems*(*12*))
by (*meson LocalLexing.items-le-is-filter LocalLexing.wellformed-items-Gen LocalLexing-axioms empty-subsetI less.prems*(*4*) *subsetCE wellformed-items-Scan wellformed-items-π wellformed-items-def*)
with *z-in-I* **have** *wellformed-z*: *wellformed-item z*

```
        using wellformed-items-def by blast
      have item-β-z: item-β z = α@β by (simp add: z-is-init-item)
        have item-end-z: item-end z = k' by (simp add: z-is-init-item)

      have x ∈ π k' {} {z}
        apply (simp add: x-is-inc-dot)
        apply (rule thmD6)
        apply (rule wellformed-z)
        apply (rule item-β-z)
        apply (rule item-end-z)
        by (simp add: α-derives-empty)
      then have x ∈ π k' {} I using z-in-I
    by (meson contra-subsetD empty-subsetI insert-subset monoD mono-π)

      then show ?case
          by (metis (no-types, lifting) LocalLexing.wellformed-item-def
LocalLexing-axioms
    π-subset-elem-trans item.sel(3) item.sel(4) items-le-def items-le-is-filter

        less.prems(11) less.prems(12) mem-Collect-eq wellformed-z z)
  qed
 qed
qed

theorem thmD11:
  assumes p-dom: p ∈ 𝔓
  assumes p-charslength: charslength p = k
  assumes X-dom: X ∈ T
  assumes T-dom: T ⊆ 𝒳 k
  assumes q-def: q = p @ [X]
  assumes rule-dom: (N, α@β) ∈ ℜ
  assumes r: r ≤ length q
  assumes leftderives-start: leftderives [𝔖] ((terminals (take r q))@[N]@γ)
  assumes leftderives-α: leftderives α (terminals (drop r q))
  assumes k': k' = k + length (chars-of-token X)
  assumes item-def: x = Item (N, α@β) (length α) (charslength (take r q)) k'
  assumes I: I = items-le k' (π k' {} (Scan T k (Gen (Prefixes p))))
  shows x ∈ I
proof −
  have ∃ D. LeftDerivation [𝔖] D ((terminals (take r q))@[N]@γ)
    using leftderives-start leftderives-implies-LeftDerivation by blast
  then obtain D where D: LeftDerivation [𝔖] D ((terminals (take r q))@[N]@γ)
by blast
  from thmD11-helper[OF assms(1, 2, 3, 4, 5, 6, 7) D assms(9, 10, 11, 12)]
  show ?thesis .
qed

end
```

```
end
theory TheoremD12
imports TheoremD11
begin

context LocalLexing begin

lemma charslength-appendix-is-empty:
  charslength (p@ts) = charslength p ⟹ (⋀ t. t ∈ set ts ⟹ chars-of-token t =
[])
proof (induct ts)
  case Nil then show ?case by auto
next
  case (Cons s ts)
    have charslength (p @ s # ts) = charslength (p @ ts) + length (chars-of-token
s)
      by simp
    then have charslength (p @ s # ts) = charslength p + charslength ts + length
(chars-of-token s)
      by simp
    with Cons.prems(1) have charslength ts + length (chars-of-token s) = 0 by
arith
    then show ?case using Cons.prems(2) charslength-0 by auto
qed

lemma empty-tokens-have-charslength-0:
  (⋀ t. t ∈ set ts ⟹ chars-of-token t = []) ⟹ charslength ts = 0
proof (induct ts)
  case Nil then show ?case by simp
next
  case (Cons t ts)
    then show ?case by auto
qed

lemma π-idempotent': π k {} (π k T I) = π k T I
  apply (simp add: π-no-tokens)
  by (simp add: Complete-π-fix Predict-π-fix fix-is-fix-of-limit)

theorem thmD12:
  assumes induct: items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k u))
  assumes induct-tokens: 𝒯 k u = 𝒵 k u
  shows items-le k (𝒥 k (Suc u)) ⊇ Gen (paths-le k (𝒫 k (Suc u)))
proof −
  {
    fix x :: item
    assume x-dom: x ∈ Gen (paths-le k (𝒫 k (Suc u)))
    have ∃ q. pvalid q x ∧ q ∈ 𝒫 k (Suc u) ∧ charslength q ≤ k
    proof −
```

```
    have ⋀i I n. i ∈ I ∨ i ∉ items-le n I
      by (meson items-le-is-filter subsetCE)
    then show ?thesis
        by (metis Gen-implies-pvalid x-dom items-le-fix-D items-le-idempotent
items-le-paths-le
        pvalid-item-end)
  qed
  then obtain q where q: pvalid q x ∧ q ∈ 𝒫 k (Suc u) ∧ charslength q ≤ k by
blast
  have q ∈ 𝒫 k u ∨ q ∉ 𝒫 k u by blast
  then have x ∈ items-le k (𝒥 k (Suc u))
  proof (induct rule: disjCases2)
    case 1
    with q have x ∈ Gen (paths-le k (𝒫 k u))
      apply (auto simp add: Gen-def)
      apply (rule-tac x=q in exI)
      by (simp add: paths-le-def)
    with induct have x ∈ items-le k (𝒥 k u) by simp
    then show ?case
        using LocalLexing.items-le-def LocalLexing-axioms 𝒥-subset-Suc-u by
fastforce
  next
    case 2
    have q-is-limit: q ∈ limit (Append (𝒴 (𝒵 k u) (𝒫 k u) k) k) (𝒫 k u) using
q by auto
     from limit-Append-path-nonelem-split[OF q-is-limit 2] obtain p ts where
p-ts:
       q = p @ ts ∧
       p ∈ 𝒫 k u ∧
       charslength p = k ∧
       admissible (p @ ts) ∧
       (∀ t∈set ts. t ∈ 𝒴 (𝒵 k u) (𝒫 k u) k) ∧ (∀ t∈set (butlast ts). chars-of-token
t = [])
        by blast
    then have ts-nonempty: ts ≠ [] using 2 using self-append-conv by auto
    obtain T where T: T = 𝒴 (𝒵 k u) (𝒫 k u) k by blast
    obtain J where J: J = π k T (Gen (paths-le k (𝒫 k u))) by blast
    from q p-ts have chars-of-token-is-empty: ⋀ t. t∈set ts ⟹ chars-of-token
t = []
     using charslength-appendix-is-empty chars-append charslength.simps le-add1
le-imp-less-Suc
        le-neq-implies-less length-append not-less-eq by auto
    {
      fix sss :: token list
      have is-prefix sss ts ⟹ pvalid (p @ sss) x ⟹ x ∈ J
      proof (induct length sss arbitrary: sss x rule: less-induct)
        case less
          have sss = [] ∨ sss ≠ [] by blast
          then show ?case
```

proof (*induct rule*: *disjCases2*)
case *1*
with *less* **have** *pvalid-p-x*: *pvalid p x* **by** *auto*
have *charslength-p*: *charslength p* $\leq$ *k* **using** *p-ts* **by** *blast*
with *p-ts* **have** *p* $\in$ *paths-le k* ($\mathcal{P}$ *k u*)
by (*simp add*: *paths-le-def*)
with *pvalid-p-x* **have** *x* $\in$ *Gen* (*paths-le k* ($\mathcal{P}$ *k u*))
using *Gen-def mem-Collect-eq* **by** *blast*
then have *x* $\in$ π *k T* (*Gen* (*paths-le k* ($\mathcal{P}$ *k u*))) **using** π*-apply-setmonotone*
by *blast*
then show *x* $\in$ *J* **using** *pvalid-item-end q J LocalLexing.items-le-def*
LocalLexing-axioms charslength-p mem-Collect-eq pvalid-p-x **by** *auto*
next
case *2*
then have $\exists$ *a ss. sss* = *ss*@[*a*] **using** *rev-exhaust* **by** *blast*
then obtain *a ss* **where** *snoc*: *sss* = *ss*@[*a*] **by** *blast*
obtain *p*′ **where** *p*′: *p*′ = *p* @ *ss* **by** *auto*
then have *pvalid-left* (*p*′@[*a*]) *x* **using** *snoc less pvalid-left* **by** *simp*
from *iffD1*[*OF pvalid-left-def this*] **obtain** *r* ω **where** *pvalid*:
wellformed-tokens (*p*′ @ [*a*]) $\wedge$
wellformed-item x $\wedge$
r $\leq$ *length* (*p*′ @ [*a*]) $\wedge$
charslength (*p*′ @ [*a*]) = *item-end x* $\wedge$
charslength (*take r* (*p*′ @ [*a*])) = *item-origin x* $\wedge$
is-leftderivation (*terminals* (*take r* (*p*′ @ [*a*])) @ [*item-nonterminal x*] @ ω) $\wedge$
leftderives (*item-*α *x*) (*terminals* (*drop r* (*p*′ @ [*a*]))) **by** *blast*
obtain *q*′ **where** *q*′: *q*′ = *p*′@[*a*] **by** *blast*
have *is-prefix-ss-ts*: *is-prefix ss ts* **using** *snoc less*
by (*simp add*: *is-prefix-append*)
then have *is-prefix* (*p*@*ss*) *q* **using** *p*′ *snoc p-ts* **by** *simp*
then have *is-prefix p*′ *q* **using** *p*′ **by** *simp*
then have *h1*: *p*′ $\in$ $\mathfrak{P}$ **using** *q* $\mathfrak{P}$*-covers-*$\mathcal{P}$ *prefixes-are-paths*′ *subsetCE* **by** *blast*
have *charslength-ss*: *charslength ss* = *0*
apply (*rule empty-tokens-have-charslength-0*)
by (*metis is-prefix-ss-ts append-is-Nil-conv chars-append chars-of-token-is-empty*
charslength.simps charslength-0 is-prefix-def length-greater-0-conv list.size(*3*))
then have *h2*: *charslength p*′ = *k* **using** *p*′ *p-ts* **by** *auto*
have *a-in-ts*: *a* $\in$ *set ts*
by (*metis in-set-dropD is-prefix-append is-prefix-cons list.set-intros*(*1*)
snoc less(*2*))
then have *h3*: *a* $\in$ *T* **using** *T p-ts* **by** *blast*

```
                have h4: T ⊆ 𝒳 k
                  using LocalLexing.𝒵.simps(2) LocalLexing-axioms T 𝒵-subset-𝒳
by blast
                note h5 = q'
                obtain N where N: N = item-nonterminal x by blast
                obtain α where α: α = item-α x by blast
                obtain β where β: β = item-β x by blast
                have item-rule-x: item-rule x = (N, α @ β)
                  using N α β item-nonterminal-def item-rhs-def item-rhs-split by
auto
                have wellformed-item x using pvalid by blast
                then have h6: (N, α@β) ∈ ℜ using item-rule-x
                  by (simp add: wellformed-item-def)
                have h7: r ≤ length q' using pvalid q' by blast
                have h8: leftderives [𝔖] (terminals (take r q') @ [N] @ ω)
                  using N is-leftderivation-def pvalid q' by blast
                have h9: leftderives α (terminals (drop r q')) using α pvalid q' by
blast
                have h10: k = k + length (chars-of-token a)
                  by (simp add: a-in-ts chars-of-token-is-empty)
                have h11: x = Item (N, α @ β) (length α) (charslength (take r q'))
k
                    by (metis α charslength-ss a-in-ts append-Nil2 chars.simps(2)
chars-append
                        chars-of-token-is-empty charslength.simps h2 item.collapse
item-dot-is-α-length
                      item-rule-x length-greater-0-conv list.size(3) pvalid q')
              have x-dom: x ∈ items-le k (π k {} (Scan T k (Gen (Prefixes p'))))
                  using thmD11[OF h1 h2 h3 h4 h5 h6 h7 h8 h9 h10 h11] by auto
                {
                  fix y :: item
                  fix toks :: token list
                  assume pvalid-toks-y: pvalid toks y
                  assume is-prefix-toks-p': is-prefix toks p'
                  then have charslength-toks: charslength toks ≤ k
                    using charslength-of-prefix h2 by blast
                     then have item-end-y: item-end y ≤ k using pvalid-item-end
pvalid-toks-y
                    by auto
                  have is-prefix toks p ∨ (∃ ss'. is-prefix ss' ss ∧ toks = p@ss')
                    using is-prefix-of-append is-prefix-toks-p' p' by auto
                  then have y ∈ J
                  proof (induct rule: disjCases2)
                    case 1
                     then have toks ∈ 𝒫 k u using p-ts prefixes-are-paths by blast
                      with charslength-toks have toks ∈ paths-le k (𝒫 k u)
                        by (simp add: paths-le-def)
                      then have y ∈ Gen (paths-le k (𝒫 k u)) using pvalid-toks-y
                        Gen-def mem-Collect-eq by blast
```

```
                    then have y ∈ π k T (Gen (paths-le k (𝒫 k u))) using
π-apply-setmonotone
                      by blast
                    then show y ∈ J by (simp add: J items-le-def item-end-y)
                  next
                    case 2
                    then obtain ss' where ss': is-prefix ss' ss ∧ toks = p@ss' by
blast
                    then have l1: length ss' < length sss
                      using append-eq-conv-conj append-self-conv is-prefix-length
length-append
                        less-le-trans nat-neq-iff not-Cons-self2 not-add-less1 snoc by
fastforce
                    have l2: is-prefix ss' ts using ss' is-prefix-ss-ts
                      by (metis append-dropped-prefix is-prefix-append)
                    have l3: pvalid (p @ ss') y using ss' pvalid-toks-y by simp
                    show ?case using less.hyps[OF l1 l2 l3] by blast
                  qed
                }
                then have Gen (Prefixes p') ⊆ J
                  by (meson Gen-implies-pvalid Prefixes-is-prefix subsetI)
                with x-dom have r0: x ∈ items-le k (π k {} (Scan T k J))
                    by (metis (no-types, lifting) LocalLexing.items-le-def LocalLex-
ing-axioms
                      mem-Collect-eq mono-Scan mono-π mono-subset-elem subsetI)
                then have x-in-π: x ∈ π k {} (Scan T k J)
                  using LocalLexing.items-le-is-filter LocalLexing-axioms subsetCE
by blast
                have r1: Scan T k J = J
                  by (simp add: J Scan-π-fix)
                have r2: π k {} J = J using π-idempotent' using J by blast
                from x-in-π r1 r2 show x ∈ J by auto
              qed
            qed
          }
          note th = this
          have x-in-J: x ∈ J
            apply (rule th[of ts])
            apply (simp add: is-prefix-eq-proper-prefix)
            using p-ts q by blast
          have 𝒯-eq-𝒵: 𝒯 k (Suc u) = 𝒵 k (Suc u)
            using induct induct-tokens 𝒯-equals-𝒵-induct-step by blast
          have T-alt: T = 𝒯 k (Suc u) using 𝒯-eq-𝒵 T by simp
          have J = π k T (items-le k (𝒥 k u)) using induct J by simp
            then have J ⊆ π k T (𝒥 k u) by (simp add: items-le-is-filter monoD
mono-π)
          with T-alt have J ⊆ 𝒥 k (Suc u) using 𝒥.simps(2) by blast
          with x-in-J have x ∈ 𝒥 k (Suc u) by blast
          then show ?case
```

```
        using LocalLexing.items-le-def LocalLexing-axioms pvalid-item-end q by
auto
    qed
  }
  then show ?thesis by auto
qed

end

end
theory TheoremD13
imports TheoremD12
begin

context LocalLexing begin

lemma pointwise-natUnion-swap:
  assumes pointwise-f: pointwise f
  shows f (natUnion G) = natUnion (λ u. f (G u))
proof −
  note f-simp = pointwise-simp[OF pointwise-f]
  have h1: f (natUnion G) = ⋃ {f {x} |x. x ∈ (natUnion G)} using f-simp by
blast
  have h2: ⋀ x. f (G x) = ⋃ {f {y} |y. y ∈ G x} using f-simp by metis
  show ?thesis
    apply (subst h1)
    apply (subst h2)
    apply (simp add: natUnion-def)
    by blast
qed

lemma pointwise-Gen: pointwise Gen
  by (simp add: pointwise-def Gen-def, blast)

lemma thmD13-part1:
  assumes start: items-le k (𝒥 k 0) = Gen (paths-le k (𝒫 k 0))
  assumes valid-k: k ≤ length Doc
  shows items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k u)) ∧ 𝒯 k u = 𝒵 k u
proof (induct u)
  case 0
    then show ?case using start by auto
next
  case (Suc u)
    from Suc have sub: items-le k (𝒥 k (Suc u)) ⊆ Gen (paths-le k (𝒫 k (Suc u)))
      using thmD9 valid-k by blast
    from Suc have sup: items-le k (𝒥 k (Suc u)) ⊇ Gen (paths-le k (𝒫 k (Suc u)))
      using thmD12 by blast
    from Suc have tokens: 𝒯 k (Suc u) = 𝒵 k (Suc u)
      using 𝒯-equals-𝒵-induct-step by blast
```

```
  from sub sup tokens show ?case by blast
qed

lemma thmD13-part2:
  assumes start: items-le k (𝒥 k 0) = Gen (paths-le k (𝒫 k 0))
  assumes valid-k: k ≤ length Doc
  shows items-le k (ℐ k) = Gen (paths-le k (𝒬 k))
proof −
  note part1 = thmD13-part1[OF start valid-k]
  have e1: items-le k (ℐ k) = natUnion (λ u. items-le k (𝒥 k u))
    using items-le-pointwise pointwise-natUnion-swap by auto
  have e2: natUnion (λ u. items-le k (𝒥 k u)) = natUnion (λ u. Gen (paths-le k
(𝒫 k u)))
    using part1 by auto
  have e3: natUnion (λ u. Gen (paths-le k (𝒫 k u))) = Gen (natUnion (λ u.
paths-le k (𝒫 k u)))
    using pointwise-Gen pointwise-natUnion-swap by fastforce
  have e4: natUnion (λ u. paths-le k (𝒫 k u)) = paths-le k (natUnion (λ u. 𝒫 k
u))
    using paths-le-pointwise pointwise-natUnion-swap by fastforce
  from e1 e2 e3 e4 show ?thesis by simp
qed

theorem thmD13:
  assumes start: items-le k (𝒥 k 0) = Gen (paths-le k (𝒫 k 0))
  assumes valid-k: k ≤ length Doc
  shows items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k u)) ∧ 𝒯 k u = 𝒵 k u
    ∧ items-le k (ℐ k) = Gen (paths-le k (𝒬 k))
using thmD13-part1[OF start valid-k] thmD13-part2[OF start valid-k] by blast

end

end
theory TheoremD14
imports TheoremD13
begin

context LocalLexing begin

lemma empty-tokens-of-empty[simp]: empty-tokens {} = {}
  using empty-tokens-is-filter by blast

lemma items-le-split-via-eq: items-le (Suc k) J = items-le k J ∪ items-eq (Suc k)
J
  by (auto simp add: items-le-def items-eq-def)

lemma paths-le-split-via-eq: paths-le (Suc k) P = paths-le k P ∪ paths-eq (Suc k)
P
  by (auto simp add: paths-le-def paths-eq-def)
```

lemma *natUnion-superset*:
shows $g\ i \subseteq natUnion\ g$
by (*meson natUnion-elem subset-eq*)

definition *indexle* :: $nat \Rightarrow nat \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
$indexle\ k'\ u'\ k\ u = ((indexlt\ k'\ u'\ k\ u) \vee (k' = k \wedge u' = u))$

definition *produced-by-scan-step* :: $item \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
$produced\text{-}by\text{-}scan\text{-}step\ x\ k\ u = (\exists\ k'\ u'\ y\ X.\ indexle\ k'\ u'\ k\ u \wedge y \in \mathcal{J}\ k'\ u' \wedge$
$item\text{-}end\ y = k' \wedge X \in (\mathcal{T}\ k'\ u') \wedge x = inc\text{-}item\ y\ (k' + length\ (chars\text{-}of\text{-}token\ X)) \wedge$
$next\text{-}symbol\ y = Some\ (terminal\text{-}of\text{-}token\ X))$

lemma *indexle-trans*: $indexle\ k''\ u''\ k'\ u' \Longrightarrow indexle\ k'\ u'\ k\ u \Longrightarrow indexle\ k''\ u''\ k\ u$
using *indexle-def indexlt-trans*
proof −
assume *a1*: $indexle\ k''\ u''\ k'\ u'$
assume *a2*: $indexle\ k'\ u'\ k\ u$
then have *f3*: $\bigwedge n\ na.\ u' = u \vee indexlt\ n\ na\ k\ u \vee \neg\ indexlt\ n\ na\ k'\ u'$
by (*meson indexle-def indexlt-trans*)
have $\bigwedge n\ na.\ k' = k \vee indexlt\ n\ na\ k\ u \vee \neg\ indexlt\ n\ na\ k'\ u'$
using *a2* **by** (*meson indexle-def indexlt-trans*)
then show *?thesis*
using *f3 a2 a1 indexle-def* **by** *auto*
qed

lemma *produced-by-scan-step-trans*:
assumes $indexle\ k'\ u'\ k\ u$
assumes $produced\text{-}by\text{-}scan\text{-}step\ x\ k'\ u'$
shows $produced\text{-}by\text{-}scan\text{-}step\ x\ k\ u$
proof −
from *iffD1*[*OF produced-by-scan-step-def assms*(*2*)] **obtain** $k'a\ u'a\ y\ X$ **where** *produced-k'-u'*:
$indexle\ k'a\ u'a\ k'\ u' \wedge$
$y \in \mathcal{J}\ k'a\ u'a \wedge$
$item\text{-}end\ y = k'a \wedge$
$X \in \mathcal{T}\ k'a\ u'a \wedge$
$x = inc\text{-}item\ y\ (k'a + length\ (chars\text{-}of\text{-}token\ X)) \wedge next\text{-}symbol\ y = Some\ (terminal\text{-}of\text{-}token\ X)$
by *blast*
then show *?thesis* **using** *indexle-trans assms*(*1*) *produced-by-scan-step-def* **by** *blast*
qed

lemma $\mathcal{J}$*-induct*[*consumes 1*, *case-names Induct*]:
assumes $x \in \mathcal{J}\ k\ u$
assumes *induct*: $\bigwedge\ x\ k\ u\ .\ (\bigwedge\ x'\ k'\ u'.\ x' \in \mathcal{J}\ k'\ u' \Longrightarrow indexlt\ k'\ u'\ k\ u \Longrightarrow P$

```
x' k' u')
                    ⟹ x ∈ 𝒥 k u ⟹ P x k u
  shows P x k u
proof −
  let ?R = indexlt-rel <*lex*> {}
  have wf-R: wf ?R by (auto simp add: wf-indexlt-rel)
  let ?P = λ a. snd a ∈ 𝒥 (fst (fst a)) (snd (fst a)) ⟶ P (snd a) (fst (fst a))
(snd (fst a))
  have x ∈ 𝒥 k u ⟶ P x k u
   apply (rule wf-induct[OF wf-R, where P = ?P and a = ((k, u), x), simplified])
    apply (auto simp add: indexlt-def[symmetric])
    apply (rule-tac x=ba and k=a and u=b in induct)
    by auto
  thus ?thesis using assms by auto
qed

lemma π-no-tokens-item-end:
  assumes x-in-π: x ∈ π k {} I
  shows item-end x = k ∨ x ∈ I
proof −
  have x-in-limit: x ∈ limit (λI. Complete k (Predict k I)) I
    using x-in-π π-no-tokens by auto
  then show ?thesis
  proof (induct rule: limit-induct)
    case (Init x) then show ?case by auto
  next
    case (Iterate x J)
      from Iterate(2) have item-end x = k ∨ x ∈ Predict k J
        using Complete-item-end by auto
      then show ?case
      proof (induct rule: disjCases2)
        case 1 then show ?case by blast
      next
        case 2
          then have item-end x = k ∨ x ∈ J
            using Predict-item-end by auto
          then show ?case
          proof (induct rule: disjCases2)
            case 1 then show ?case by blast
          next
            case 2 then show ?case using Iterate(1)[OF 2] by blast
          qed
      qed
  qed
qed

lemma natUnion-ex: x ∈ natUnion f ⟹ ∃ i. x ∈ f i
  by (metis (no-types, opaque-lifting) mk-disjoint-insert natUnion-superset natU-
nion-upperbound
```

```
    subsetCE subset-insert)

lemma locate-in-limit:
  assumes x-in-limit: x ∈ limit f X
  assumes x-notin-X: x ∉ X
  shows ∃ n. x ∈ funpower f (Suc n) X ∧ x ∉ funpower f n X
proof −
  have ∃ N. x ∈ funpower f N X using x-in-limit limit-def natUnion-ex by fastforce
  then obtain N where N: x ∈ funpower f N X by blast
  {
    fix n :: nat
    have x ∈ funpower f n X ⟹ ∃ m < n. x ∈ funpower f (Suc m) X ∧ x ∉
funpower f m X
    proof (induct n)
      case 0
        with x-notin-X show ?case by auto
    next
      case (Suc n)
        have x ∉ funpower f n X ∨ x ∈ funpower f n X by blast
        then show ?case
        proof (induct rule: disjCases2)
          case 1
            then show ?case using Suc by fastforce
        next
          case 2
            from Suc(1)[OF 2] show ?case using less-SucI by blast
        qed
    qed
  }
  with N show ?thesis by auto
qed

lemma produced-by-scan-step:
  x ∈ 𝒥 k u ⟹ item-end x > k ⟹ produced-by-scan-step x k u
proof (induct rule: 𝒥-induct)
  case (Induct x k u)
    have (k = 0 ∧ u = 0) ∨ (k > 0 ∧ u = 0) ∨ (u > 0) by arith
    then show ?case
    proof (induct rule: disjCases3)
      case 1
        with Induct have item-end x = 0 using 𝒥-0-0-item-end by blast
        with Induct have False by arith
        then show ?case by  blast
    next
      case 2
        then obtain k′ where k′: k = Suc k′ using Suc-pred′ by blast
        with Induct 2 have x ∈ 𝒥 (Suc k′) 0 by auto
        then have x ∈ π k {} (ℐ k′) by (simp add: k′)
       then have item-end x = k ∨ x ∈ ℐ k′ using π-no-tokens-item-end by blast
```

```
      then show ?case
      proof (induct rule: disjCases2)
       case 1
        with Induct have False by auto
        then show ?case by blast
      next
       case 2
        then have ∃ u'. x ∈ 𝒥 k' u' using 𝓘.simps natUnion-ex by fastforce
        then obtain u' where u': x ∈ 𝒥 k' u' by blast
        have k'-bound: k' < item-end x using k' Induct by arith
        have indexlt: indexlt k' u' k u by (simp add: indexlt-simp k')
        from Induct(1)[OF u' this k'-bound]
        have pred-produced: produced-by-scan-step x k' u' .
         then show ?case using indexlt produced-by-scan-step-trans indexle-def
by blast
      qed
   next
    case 3
      then have ex-u': ∃ u'. u = Suc u' by arith
      then obtain u' where u': u = Suc u' by blast
      with Induct have x ∈ 𝒥 k (Suc u') by metis
      then have x-in-π: x ∈ π k (𝒯 k u) (𝒥 k u') using u' 𝒥.simps by metis
      have x ∈ 𝒥 k u' ∨ x ∉ 𝒥 k u' by blast
      then show ?case
      proof (induct rule: disjCases2)
       case 1
        have indexlt: indexlt k u' k u by (simp add: indexlt-simp u')
        with Induct(1)[OF 1 indexlt Induct(3)] show ?case
          using indexle-def produced-by-scan-step-trans by blast
      next
       case 2
        have item-end-x: k < item-end x using Induct by auto
       obtain f where f: f = Scan (𝒯 k u) k ∘ Complete k ∘ Predict k by blast
        have x ∈ limit f (𝒥 k u')
          using x-in-π π-functional f by simp
        from locate-in-limit[OF this 2] obtain n where n:
          x ∈ funpower f (Suc n) (𝒥 k u') ∧
           x ∉ funpower f n (𝒥 k u') by blast
        obtain Y where Y: Y = funpower f n (𝒥 k u')
          by blast
        have x-f-Y: x ∈ f Y ∧ x ∉ Y using Y n by auto
          then have x ∈ Scan (𝒯 k u) k (Complete k (Predict k Y)) using
comp-apply f by simp
        then have x ∈ (Complete k (Predict k Y)) ∨
          x ∈ { inc-item x' (k + length c) | x' t c. x' ∈ bin (Complete k (Predict
k Y)) k ∧
                  (t, c) ∈ (𝒯 k u) ∧ next-symbol x' = Some t } using Scan-def by
simp
        then show ?case
```

```
      proof (induct rule: disjCases2)
        case 1
          then have False using item-end-x x-f-Y Complete-item-end Predict-item-end
            using less-not-refl3 by blast
          then show ?case by auto
      next
        case 2
          have Y ⊆ limit f (𝒥 k u') using Y limit-def natUnion-superset by fastforce
            then have Y ⊆ π k (𝒯 k u) (𝒥 k u') using f by (simp add: π-functional)
          then have Y-in-𝒥: Y ⊆ 𝒥 k u using u' by simp
          then have in-𝒥: Complete k (Predict k Y) ⊆ 𝒥 k u
          proof −
            have f1: ∀ f I Ia i. (¬ mono f ∨ ¬ (I::item set) ⊆ Ia ∨ (i::item) ∉ f I) ∨ i ∈ f Ia
              by (meson mono-subset-elem)
            obtain ii :: item set ⇒ item set ⇒ item where
              ∀ x0 x1. (∃ v2. v2 ∈ x1 ∧ v2 ∉ x0) = (ii x0 x1 ∈ x1 ∧ ii x0 x1 ∉ x0)
              by moura
            then have f2: ∀ I Ia. ii Ia I ∈ I ∧ ii Ia I ∉ Ia ∨ I ⊆ Ia
              by blast
            obtain nn :: nat where
              f3: u = Suc nn
              using ex-u' by presburger
            moreover
            { assume ii (𝒥 k u) (Complete k (Predict k Y)) ∈ Complete k (π k (𝒯 k (Suc nn)) (𝒥 k nn))
              then have ?thesis
                using f3 f2 Complete-π-fix by auto }
            ultimately show ?thesis
              using f2 f1 by (metis (full-types) Complete-regular Predict-π-fix Predict-regular
                𝒥.simps(2) Y-in-𝒥 regular-implies-mono)
          qed
          from 2 obtain x' t c where x'-t-c:
            x = inc-item x' (k + length c) ∧ x' ∈ bin (Complete k (Predict k Y)) k ∧
            (t, c) ∈ 𝒯 k u ∧ next-symbol x' = Some t by blast
          show ?case
            apply (simp add: produced-by-scan-step-def)
            apply (rule-tac x=k in exI)
            apply (rule-tac x=u in exI)
            apply (simp add: indexle-def)
            apply (rule-tac x=x' in exI)
            apply auto
            using x'-t-c bin-def in-𝒥 apply auto[1]
```

```
            using x'-t-c bin-def apply blast
            apply (rule-tac x=t in exI)
            apply (rule-tac x=c in exI)
            using x'-t-c by auto
        qed
      qed
    qed
qed

lemma limit-single-step:
  assumes x ∈ f X
  shows x ∈ limit f X
by (metis assms elem-limit-simp funpower.simps(1) funpower.simps(2))

lemma Gen-union: Gen (A ∪ B) = Gen A ∪ Gen B
  by (simp add: Gen-def, blast)

lemma is-prefix-Prefixes-subset:
  assumes is-prefix q p
  shows Prefixes q ⊆ Prefixes p
proof −
  show ?thesis
    apply (auto simp add: Prefixes-def)
    using assms by (metis is-prefix-append is-prefix-def)
qed

lemma Prefixes-subset-𝒫:
  assumes p ∈ 𝒫 k u
  shows Prefixes p ⊆ 𝒫 k u
using Prefixes-is-prefix assms prefixes-are-paths by blast

lemma Prefixes-subset-paths-le:
  assumes Prefixes p ⊆ P
  shows Prefixes p ⊆ paths-le (charslength p) P
using Prefixes-is-prefix assms charslength-of-prefix paths-le-def by auto

lemma Scan-𝒥-subset-𝒥:
  Scan (𝒯 k (Suc u)) k (𝒥 k u) ⊆ 𝒥 k (Suc u)
by (metis (no-types, lifting) Scan-π-fix 𝒥.simps(2) 𝒥-subset-Suc-u monoD mono-Scan)

lemma subset-𝒥k: u ≤ v ⟹ 𝒥 k u ⊆ 𝒥 k v
  thm 𝒥-subset-Suc-u
  by (rule subset-fSuc, rule 𝒥-subset-Suc-u)

lemma subset-𝒥ℐk: 𝒥 k u ⊆ ℐ k by (auto simp add: natUnion-def)

lemma subset-ℐ𝒥Suc: ℐ k ⊆ 𝒥 (Suc k) u
proof −
  have a: ℐ k ⊆ 𝒥 (Suc k) 0
```

```
    apply (simp only: 𝒥.simps)
    using π-apply-setmonotone by blast
  show ?thesis
    apply (case-tac u = 0)
    apply (simp only: a)
    apply (rule subset-trans[OF a subset-𝒥k])
    by auto
qed

lemma subset-ℐSuc: ℐ k ⊆ ℐ (Suc k)
  by (rule subset-trans[OF subset-ℐ𝒥Suc subset-𝒥ℐk])

lemma subset-ℐ: i ≤ j ⟹ ℐ i ⊆ ℐ j
  by (rule subset-fSuc[where u=i and v=j and f = ℐ, OF subset-ℐSuc])

lemma subset-𝒥 :
  assumes leq: k' < k ∨ (k' = k ∧ u' ≤ u)
  shows 𝒥 k' u' ⊆ 𝒥 k u
proof −
  from leq show ?thesis
  proof (induct rule: disjCases2)
    case 1
    have s1: 𝒥 k' u' ⊆ ℐ k' by (rule-tac subset-𝒥ℐk)
    have s2: ℐ k' ⊆ ℐ (k − 1)
      apply (rule-tac subset-ℐ)
      using 1 by arith
    from subset-ℐ𝒥Suc[where k=k − 1] 1 have s3: ℐ (k − 1) ⊆ 𝒥 k 0
      by simp
    have s4: 𝒥 k 0 ⊆ 𝒥 k u by (rule-tac subset-𝒥k, simp)
    from s1 s2 s3 s4 subset-trans show ?case by blast
  next
    case 2 thus ?case by (simp add : subset-𝒥k)
  qed
qed

lemma 𝒥-subset:
  assumes indexle k' u' k u
  shows 𝒥 k' u' ⊆ 𝒥 k u
using subset-𝒥 indexle-def indexlt-simp
by (metis assms less-imp-le-nat order-refl)

lemma Scan-items-le:
  assumes bounded-T: ⋀ t . t ∈ T ⟹ length (chars-of-token t) ≤ l
  shows Scan T k (items-le k P) ⊆ items-le (k + l) (Scan T k P)
proof −
  {
    fix x :: item
    assume x-dom: x ∈ Scan T k (items-le k P)
    then have x-dom': x ∈ Scan T k P
```

```
    by (meson items-le-is-filter mono-Scan mono-subset-elem)
  from x-dom have x ∈ items-le k P ∨
    (∃ y t c. x = inc-item y (k + length c) ∧ y ∈ bin (items-le k P) k ∧ (t, c) ∈
T
     ∧ next-symbol y = Some t)
    using Scan-def using UnE mem-Collect-eq by auto
  then have item-end x ≤ k + l
  proof (induct rule: disjCases2)
    case 1 then show ?case
      by (meson items-le-fix-D items-le-idempotent trans-le-add1)
  next
    case 2
      then obtain y t c where y: x = inc-item y (k + length c) ∧ y ∈ bin
(items-le k P) k ∧
        (t, c) ∈ T ∧ next-symbol y = Some t by blast
      then have item-end-x: item-end x = (k + length c) by simp
      from bounded-T y have length c ≤ l
        using chars-of-token-simp by auto
      with item-end-x show ?case by arith
  qed
  with x-dom' have x ∈ items-le (k + l) (Scan T k P)
    using items-le-def mem-Collect-eq by blast
 }
 then show ?thesis by blast
qed

lemma Scan-mono-tokens:
 P ⊆ Q ⟹ Scan P k I ⊆ Scan Q k I
by (auto simp add: Scan-def)

theorem thmD14: k ≤ length Doc ⟹ items-le k (𝒥 k u) = Gen (paths-le k (𝒫 k
u)) ∧ 𝒯 k u = 𝒵 k u
   ∧ items-le k (ℐ k) = Gen (paths-le k (𝒬 k))
proof (induct k arbitrary: u rule: less-induct)
 case (less k)
  have k = 0 ∨ k ≠ 0 by arith
  then show ?case
  proof (induct rule: disjCases2)
    case 1
      have 𝒥-eq-𝒫: items-le k (𝒥 k 0) = Gen (paths-le k (𝒫 k 0))
        by (simp only: 1 thmD8 items-le-paths-le)
      show ?case using thmD13[OF 𝒥-eq-𝒫 less.prems] by blast
  next
    case 2
      have ∃ k'. k = Suc k' using 2 by arith
      then obtain k' where k': k = Suc k' by blast
      have k'-less-k: k' < k using k' by arith
      have items-le k (𝒥 k 0) = Gen (paths-le k (𝒫 k 0))
      proof −
```

```
        have simp-left: items-le k (𝒥 k 0) = π k {} (items-le k (ℐ k'))
          using items-le-π-swap k' wellformed-items-ℐ by auto
        have simp-right: Gen (paths-le k (𝒫 k 0)) = natUnion (λ v. Gen (paths-le
k (𝒫 k' v)))
       by (simp add: k' paths-le-pointwise pointwise-Gen pointwise-natUnion-swap)

        {
          fix v :: nat
         have split-𝒥: items-le k (𝒥 k' v) = items-le k' (𝒥 k' v) ∪ items-eq k (𝒥
k' v)
           using k'  items-le-split-via-eq by blast
         have sub1: items-le k' (𝒥 k' v) ⊆ natUnion (λ v. Gen (paths-le k (𝒫 k'
v)))
         proof −
           have h: items-le k' (𝒥 k' v) ⊆ Gen (paths-le k (𝒫 k' v))
           proof −
             have f1: items-le k' (Gen (𝒫 k' v)) ∪ items-eq (Suc k') (Gen (𝒫 k'
v)) =
               Gen (paths-le k (𝒫 k' v))
            using LocalLexing.items-le-split-via-eq LocalLexing-axioms items-le-paths-le
k'
               by blast
             have k' ≤ length Doc
            by (metis (no-types) dual-order.trans k' less.prems lessI less-imp-le-nat)
             then have items-le k' (𝒥 k' v) = items-le k' (Gen (𝒫 k' v))
               by (simp add: items-le-paths-le k' less.hyps)
             then show ?thesis
               using f1 by blast
           qed
            have Gen (paths-le k (𝒫 k' v)) ⊆ natUnion (λ v. Gen (paths-le k (𝒫
k' v)))
             using natUnion-superset by fastforce
           then show ?thesis using h by blast
         qed
         {
           fix x :: item
           assume x-dom: x ∈ items-eq k (𝒥 k' v)
           have x-in-𝒥: x ∈ 𝒥 k' v using x-dom items-eq-def by auto
           have item-end-x: item-end x = k using x-dom items-eq-def by auto
           then have k'-bound: k' < item-end x using k' by arith
           from produced-by-scan-step[OF x-in-𝒥 k'-bound]
           have produced-by-scan-step x k' v .
         from iffD1[OF produced-by-scan-step-def this] obtain k'' v'' y X where
scan-step:
             indexle k'' v'' k' v ∧ y ∈ 𝒥 k'' v'' ∧ item-end y = k'' ∧ X ∈ 𝒯 k''
v'' ∧
             x = inc-item y (k'' + length (chars-of-token X)) ∧
             next-symbol y = Some (terminal-of-token X) by blast
           then have y-in-items-le: y ∈ items-le k'' (𝒥 k'' v'')
```

```
          using items-le-def LocalLexing-axioms le-refl mem-Collect-eq by blast
        have y-in-Gen: y ∈ Gen(paths-le k'' (𝒫 k'' v''))
        proof −
          have f1: ⋀n. k' < n ∨ ¬ k < n
            using Suc-lessD k' by blast
          have f2: k'' = k' ∨ k'' < k'
            using indexle-def indexlt-simp scan-step by force
          have f3: k' < k
            using k' by blast
          have f4: k' ≤ length Doc
            using f1 by (meson less.prems less-Suc-eq-le)
          have k'' ≤ length Doc ∨ k' = k''
                  using f2 f1 by (meson Suc-lessD less.prems less-Suc-eq-le
less-trans-Suc)
          then show ?thesis
             using f4 f3 f2 Suc-lessD y-in-items-le less.hyps less-trans-Suc by
blast
        qed
        then have ∃ p. p ∈ 𝒫 k'' v'' ∧ pvalid p y
          by (meson Gen-implies-pvalid paths-le-is-filter rev-subsetD)
        then obtain p where p: p ∈ 𝒫 k'' v'' ∧ pvalid p y by blast
         then have charslength-p: charslength p = k'' using pvalid-item-end
scan-step by auto
        have pvalid-p-y: pvalid p y using p by blast
        have admissible (p@[(fst X, snd X)])
          apply (rule pvalid-next-terminal-admissible)
          apply (rule pvalid-p-y)
          using scan-step apply (simp add: terminal-of-token-def)
            using scan-step by (metis TokensAt-subset-𝒳 𝒯-subset-TokensAt
𝒳-are-terminals
              rev-subsetD terminal-of-token-def)
        then have admissible-p-X: admissible (p@[X]) by simp
      have X-in-𝒵: X ∈ 𝒵 k'' (Suc v'') by (metis (no-types, lifting) Suc-lessD
𝒵-subset-Suc
          k'-bound dual-order.trans indexle-def indexlt-simp item-end-of-inc-item
item-end-x
            le-add1 le-neq-implies-less less.hyps less.prems not-less-eq scan-step
subsetCE)
        have pX-in-𝒫-k''-v'': p@[X] ∈ 𝒫 k'' (Suc v'')
          apply (simp only: 𝒫.simps)
          apply (rule limit-single-step)
          apply (auto simp only: Append-def)
          apply (rule-tac x=p in exI)
          apply (rule-tac x=X in exI)
          apply (simp only: admissible-p-X X-in-𝒵)
          using charslength-p p by auto
        have indexle k'' v'' k' v using scan-step by simp
        then have indexle k'' (Suc v'') k' (Suc v)
          by (simp add: indexle-def indexlt-simp)
```

then have $\mathcal{P}\ k''\ (Suc\ v'') \subseteq \mathcal{P}\ k'\ (Suc\ v)$
by (*metis indexle-def indexlt-simp less-or-eq-imp-le subset-*$\mathcal{P}$)
with *pX-in-*$\mathcal{P}$*-k''-v''* **have** *pX-in-*$\mathcal{P}$*-k'*: $p@[X] \in \mathcal{P}\ k'\ (Suc\ v)$ **by** *blast*
have *charslength* $(p@[X]) = k'' +$ *length* (*chars-of-token X*)
using *charslength-p* **by** *auto*
then have *charslength* $(p@[X])$ = *item-end x* **using** *scan-step* **by** *simp*
then have *charslength-p-X*: *charslength* $(p@[X]) = k$ **using** *item-end-x* **by** *simp*
then have *pX-dom*: $p@[X] \in$ *paths-le k* $(\mathcal{P}\ k'\ (Suc\ v))$
using *lessI less-Suc-eq-le mem-Collect-eq pX-in-*$\mathcal{P}$*-k' paths-le-def* **by** *auto*
have *wellformed-x*: *wellformed-item x*
using *item-end-x less.prems scan-step wellformed-inc-item wellformed-items-*$\mathcal{J}$
wellformed-items-def **by** *auto*
have *wellformed-p-X*: *wellformed-tokens* $(p@[X])$
using $\mathcal{P}$*-wellformed pX-in-*$\mathcal{P}$*-k''-v''* **by** *blast*
from *iffD1*[*OF pvalid-def pvalid-p-y*] **obtain** $r\ \gamma$ **where** *r-*γ:
wellformed-tokens p $\wedge$
wellformed-item y $\wedge$
$r \leq$ *length p* $\wedge$
charslength p = *item-end y* $\wedge$
charslength (*take r p*) = *item-origin y* $\wedge$
is-derivation (*terminals* (*take r p*) @ [*item-nonterminal y*] @ γ) $\wedge$
derives (*item-*α *y*) (*terminals* (*drop r p*)) **by** *blast*
have *r-le-p*: $r \leq$ *length p* **by** (*simp add*: *r-*γ)
have *item-nonterminal-x*: *item-nonterminal x* = *item-nonterminal y*
by (*simp add*: *scan-step*)
have *item-*α*-x*: *item-*α *x* = (*item-*α *y*) @ [*terminal-of-token X*]
by (*simp add*: *item-*α*-of-inc-item r-*γ *scan-step*)
have *pvalid-x*: *pvalid* $(p@[X])$ *x*
apply (*auto simp add*: *pvalid-def wellformed-x wellformed-p-X*)
apply (*rule-tac x=r* **in** *exI*)
apply *auto*
apply (*simp add*: *le-SucI r-*γ)
using *r-*γ *scan-step* **apply** *auto*[*1*]
using *r-*γ *scan-step* **apply** *auto*[*1*]
apply (*rule-tac x=*γ **in** *exI*)
apply (*simp add*: *r-le-p item-nonterminal-x*)
using *r-*γ **apply** *simp*
apply (*simp add*: *r-le-p item-*α*-x*)
by (*metis terminals-singleton append-Nil2*
derives-implies-leftderives derives-is-sentence is-sentence-concat
is-sentence-cons is-symbol-def is-word-append is-word-cons is-word-terminals
is-word-terminals-drop leftderives-implies-derives leftderives-padback
leftderives-refl r-γ *terminals-append terminals-drop wellformed-p-X*)
then have *x* $\in$ *Gen* (*paths-le k* $(\mathcal{P}\ k'\ (Suc\ v))$) **using** *pX-dom Gen-def*
LocalLexing-axioms mem-Collect-eq **by** *auto*

```
      }
      then have sub2: items-eq k (𝒥 k' v) ⊆ natUnion (λ v. Gen (paths-le k
(𝒫 k' v)))
    by (meson dual-order.trans natUnion-superset subsetI)
     have suffices3: items-le k (𝒥 k' v) ⊆ natUnion (λ v. Gen (paths-le k (𝒫
k' v)))
       using split-𝒥 sub1 sub2 by blast
      have items-le k (𝒥 k' v) ⊆ Gen (paths-le k (𝒫 k 0))
       using suffices3 simp-right by blast
    }
    note suffices2 = this
    have items-le-natUnion-swap: items-le k (ℐ k') = natUnion(λ v. items-le
k (𝒥 k' v))
      by (simp add: items-le-pointwise pointwise-natUnion-swap)
    then have suffices1: items-le k (ℐ k') ⊆ Gen (paths-le k (𝒫 k 0))
      using suffices2 natUnion-upperbound by metis
    have sub-lemma: items-le k (𝒥 k 0) ⊆ Gen (paths-le k (𝒫 k 0))
    proof −
      have items-le k (𝒥 k 0) ⊆ Gen (𝒫 k 0)
        apply (subst simp-left)
        apply (rule thmD5)
        apply (auto simp only: less)
        using suffices1 items-le-is-filter items-le-paths-le subsetCE by blast
      then show ?thesis
        by (simp add: items-le-idempotent remove-paths-le-in-subset-Gen)
    qed
     have eq1: π k {} (items-le k (ℐ k')) = π k {} (items-le k (natUnion (𝒥
k'))) by simp
    then have eq2: π k {} (items-le k (natUnion (𝒥 k'))) =
      π k {} (natUnion (λ v. items-le k (𝒥 k' v)))
      using items-le-natUnion-swap by auto
    from simp-left eq1 eq2
     have simp-left': items-le k (𝒥 k 0) = π k {} (natUnion (λ v. items-le k
(𝒥 k' v)))
      by metis
    {
      fix v :: nat
      fix q :: token list
      fix x :: item
      assume q-dom: q ∈ paths-eq k (𝒫 k' v)
      assume pvalid-q-x: pvalid q x
      have q-in-𝒫: q ∈ 𝒫 k' v using q-dom paths-eq-def by auto
     have charslength-q: charslength q = k using q-dom paths-eq-def by auto
      with k'-less-k have q-nonempty: q ≠ []
        using 2.hyps chars.simps(1) charslength.simps list.size(3) by auto
      then have ∃ p X. q = p @ [X] by (metis append-butlast-last-id)
      then obtain p X where pX: q = p @ [X] by blast
      from last-step-of-path[OF q-in-𝒫 pX] obtain k'' v'' where k'':
        indexlt k'' v'' k' v ∧ q ∈ 𝒫 k'' (Suc v'') ∧ charslength p = k'' ∧
```

$X \in \mathcal{Z}\ k''\ (Suc\ v'')$ **by** *blast*
have *h1*: $p \in \mathfrak{P}$
by (*metis* (*no-types*, *lifting*) *LocalLexing.*$\mathfrak{P}$*-covers-*$\mathcal{P}$ *LocalLexing-axioms*

append-Nil2 is-prefix-cancel is-prefix-empty pX prefixes-are-paths q-in-$\mathcal{P}$ *subsetCE*)
have *h2*: *charslength* $p = k''$ **using** k'' **by** *blast*
obtain T **where** *T*: $T = \{X\}$ **by** *blast*
have *h3*: $X \in T$ **using** T **by** *blast*
have *h4*: $T \subseteq \mathcal{X}\ k''$ **using** $\mathcal{Z}$*-subset-*$\mathcal{X}$ $T\ k''$ **by** *blast*
obtain N **where** *N*: $N =$ *item-nonterminal* x **by** *blast*
obtain α **where** α: $\alpha =$ *item-*α x **by** *blast*
obtain β **where** β: $\beta =$ *item-*β x **by** *blast*
have *wellformed-x*: *wellformed-item* x **using** *pvalid-def pvalid-q-x* **by** *blast*
then have *h5*: $(N, \alpha\ @\ \beta) \in \mathfrak{R}$
using $N\ \alpha\ \beta$ *item-nonterminal-def item-rhs-def item-rhs-split prod.collapse*

wellformed-item-def **by** *auto*
have *pvalid-left-q-x*: *pvalid-left* $q\ x$ **using** *pvalid-q-x* **by** (*simp add*: *pvalid-left*)
from *iffD1*[*OF pvalid-left-def pvalid-left-q-x*] **obtain** $r\ \gamma$ **where** *r-*γ:
wellformed-tokens $q \wedge$
wellformed-item $x \wedge$
$r \leq$ *length* $q \wedge$
charslength $q =$ *item-end* $x \wedge$
charslength (*take* $r\ q$) = *item-origin* $x \wedge$
is-leftderivation (*terminals* (*take* $r\ q$) @ [*item-nonterminal* x] @ γ) $\wedge$
leftderives (*item-*α x) (*terminals* (*drop* $r\ q$)) **by** *blast*
have *h6*: $r \leq$ *length* q **using** *r-*γ **by** *blast*
have *h7*: *leftderives* [$\mathfrak{S}$] (*terminals* (*take* $r\ q$) @ [N] @ γ)
using *r-*γ N *is-leftderivation-def* **by** *blast*
have *h8*: *leftderives* α (*terminals* (*drop* $r\ q$)) **using** *r-*γ α **by** *metis*
have *h9*: $k = k'' +$ *length* (*chars-of-token* X) **using** *r-*γ
using *charslength-q h2 pX* **by** *auto*
have *h10*: $x =$ *Item* $(N, \alpha\ @\ \beta)$ (*length* α) (*charslength* (*take* $r\ q$)) k
by (*metis* $N\ \alpha\ \beta$ *charslength-q item.collapse item-dot-is-*α*-length item-nonterminal-def*
item-rhs-def item-rhs-split prod.collapse r-γ)
from *thmD11*[*OF h1 h2 h3 h4 pX h5 h6 h7 h8 h9 h10*]
have $x \in$ *items-le* k ($\pi\ k\ \{\}$ (*Scan* $T\ k''$ (*Gen* (*Prefixes* p))))
by *blast*
then have *x-in*: $x \in \pi\ k\ \{\}$ (*Scan* $T\ k''$ (*Gen* (*Prefixes* p)))
using *items-le-is-filter* **by** *blast*
have *subset1*: *Prefixes* $p \subseteq$ *Prefixes* q
apply (*rule is-prefix-Prefixes-subset*)
by (*simp add*: *pX is-prefix-def*)
have *subset2*: *Prefixes* $q \subseteq \mathcal{P}\ k''$ (*Suc* v'')
apply (*rule Prefixes-subset-*$\mathcal{P}$)

using k'' **by** *blast*
from *subset1 subset2* **have** *Prefixes* $p \subseteq \mathcal{P}\ k''\ (Suc\ v'')$ **by** *blast*
then have *Prefixes* $p \subseteq$ *paths-le* $k''\ (\mathcal{P}\ k''\ (Suc\ v''))$
using k'' *Prefixes-subset-paths-le* **by** *blast*
then have *subset3*: *Gen* (*Prefixes* p) $\subseteq$ *Gen* (*paths-le* $k''\ (\mathcal{P}\ k''\ (Suc\ v''))$)
using *Gen-def LocalLexing-axioms* **by** *auto*
have k''*-less-k*: $k'' < k$ **using** $k''\ k'$ **using** *indexlt-simp less-Suc-eq* **by** *auto*
then have k''*-Doc-bound*: $k'' \leq$ *length Doc* **using** *less* **by** *auto*
from *less*(*1*)[*OF* k''*-less-k* k''*-Doc-bound*, *of Suc* v'']
have *induct1*: *items-le* $k''\ (\mathcal{J}\ k''\ (Suc\ v'')) = Gen\ (paths\text{-}le\ k''\ (\mathcal{P}\ k''\ (Suc\ v'')))$
by *blast*
from *less*(*1*)[*OF* k''*-less-k* k''*-Doc-bound*, *of Suc*(*Suc* v'')]
have *induct2*: $\mathcal{T}\ k''\ (Suc\ (Suc\ v'')) = \mathcal{Z}\ k''\ (Suc\ (Suc\ v''))$ **by** *blast*
have *subset4*: *Gen* (*Prefixes* p) $\subseteq$ *items-le* $k''\ (\mathcal{J}\ k''\ (Suc\ v''))$
using *subset3 induct1* **by** *auto*
from *induct1 subset4*
have *subset6*: *Scan* $T\ k''$ (*Gen* (*Prefixes* p)) $\subseteq$
Scan $T\ k''$ (*items-le* $k''\ (\mathcal{J}\ k''\ (Suc\ v''))$)
apply (*rule-tac monoD*[*OF mono-Scan*])
by *blast*
have $k'' +$ *length* (*chars-of-token* X) $= k$
by (*simp add*: *h9*)
have $\bigwedge\ t.\ t \in T \Longrightarrow$ *length* (*chars-of-token* t) $\leq$ *length* (*chars-of-token* X)
using T **by** *auto*
from *Scan-items-le*[*of* T, *OF this*, *simplified*, *of* $k''\ \mathcal{J}\ k''\ (Suc\ v'')$] *h9*
have *subset7*: *Scan* $T\ k''$ (*items-le* $k''\ (\mathcal{J}\ k''\ (Suc\ v''))$)
$\subseteq$ *items-le* k (*Scan* $T\ k''\ (\mathcal{J}\ k''\ (Suc\ v''))$) **by** *simp*
have $T \subseteq \mathcal{Z}\ k''\ (Suc\ (Suc\ v''))$ **using** $T\ k''$
using $\mathcal{Z}$*-subset-Suc rev-subsetD singletonD subsetI* **by** *blast*
then have T*-subset-*$\mathcal{T}$: $T \subseteq \mathcal{T}\ k''\ (Suc\ (Suc\ v''))$ **using** *induct2* **by** *auto*
have *subset8*: *Scan* $T\ k''\ (\mathcal{J}\ k''\ (Suc\ v'')) \subseteq$
Scan $(\mathcal{T}\ k''\ (Suc\ (Suc\ v'')))\ k''\ (\mathcal{J}\ k''\ (Suc\ v''))$
using T*-subset-*$\mathcal{T}$ *Scan-mono-tokens* **by** *blast*
have *subset9*: *Scan* $(\mathcal{T}\ k''\ (Suc\ (Suc\ v')))\ k''\ (\mathcal{J}\ k''\ (Suc\ v'')) \subseteq \mathcal{J}\ k''\ (Suc\ (Suc\ v''))$
by (*rule Scan-*$\mathcal{J}$*-subset-*$\mathcal{J}$)
have *subset10*: $(Scan\ T\ k''\ (\mathcal{J}\ k''\ (Suc\ v''))) \subseteq \mathcal{J}\ k''\ (Suc\ (Suc\ v''))$
using *subset8 subset9* **by** *blast*
have $k'' \leq k'$ **using** k'' *indexlt-simp* **by** *auto*
then have *indexle* $k''\ (Suc\ (Suc\ v''))\ k'\ (Suc\ (Suc\ v''))$ **using** *indexlt-simp*
using *indexle-def le-neq-implies-less* **by** *auto*
then have *subset11*: $\mathcal{J}\ k''\ (Suc\ (Suc\ v'')) \subseteq \mathcal{J}\ k'\ (Suc\ (Suc\ v''))$
using $\mathcal{J}$*-subset* **by** *blast*
have *subset12*: *Scan* $T\ k''\ (\mathcal{J}\ k''\ (Suc\ v'')) \subseteq \mathcal{J}\ k'\ (Suc\ (Suc\ v''))$
using *subset8 subset9 subset10 subset11* **by** *blast*

```
        then have subset13: items-le k (Scan T k'' (𝒥 k'' (Suc v''))) ⊆
          items-le k (𝒥 k' (Suc (Suc v'')))
          using items-le-def mem-Collect-eq rev-subsetD subsetI by auto
         have subset14: Scan T k'' (Gen (Prefixes p)) ⊆ items-le k (𝒥 k' (Suc
(Suc v'')))
          using subset6 subset7 subset13 by blast
        then have x-in': x ∈ π k {} (items-le k (𝒥 k' (Suc (Suc v''))))
          using x-in
         by (meson π-apply-setmonotone π-subset-elem-trans subsetCE subsetI)
        from x-in' have x ∈ π k {} (natUnion (λ v. items-le k (𝒥 k' v)))
          by (meson k' mono-π mono-subset-elem natUnion-superset)
      }
      note suffices6 = this
      {
       fix v :: nat
       have Gen (paths-eq k (𝒫 k' v)) ⊆ π k {} (natUnion (λ v. items-le k (𝒥
k' v)))
         using suffices6 by (meson Gen-implies-pvalid subsetI)
      }
      note suffices5 = this
      {
       fix v :: nat
       have paths-le k (𝒫 k' v) = paths-le k' (𝒫 k' v) ∪ paths-eq k (𝒫 k' v)
         using  paths-le-split-via-eq k' by metis
       then have Gen-split: Gen (paths-le k (𝒫 k' v)) =
        Gen (paths-le k' (𝒫 k' v)) ∪ Gen(paths-eq k (𝒫 k' v)) using Gen-union
by metis
         have case-le: Gen (paths-le k' (𝒫 k' v)) ⊆  π k {} (natUnion (λ v.
items-le k (𝒥 k' v)))
       proof −
         from less k'-less-k have k' ≤ length Doc by arith
         from less(1)[OF k'-less-k this]
         have items-le k' (𝒥 k' v) = Gen (paths-le k' (𝒫 k' v)) by blast
         then have Gen (paths-le k' (𝒫 k' v)) ⊆ natUnion (λ v. items-le k (𝒥
k' v))
          using items-le-def LocalLexing-axioms k'-less-k natUnion-superset by
fastforce
         then show ?thesis using π-apply-setmonotone by blast
       qed
       have Gen (paths-le k (𝒫 k' v)) ⊆ π k {} (natUnion (λ v. items-le k (𝒥
k' v)))
         using Gen-split case-le suffices5 UnE rev-subsetD subsetI by blast
      }
      note suffices4 = this
      have super-lemma: Gen (paths-le k (𝒫 k 0)) ⊆ items-le k (𝒥 k 0)
       apply (subst simp-right)
       apply (subst simp-left')
       using suffices4 by (meson natUnion-ex rev-subsetD subsetI)
      from super-lemma sub-lemma show ?thesis by blast
```

```
      qed
      then show ?case using thmD13 less.prems by blast
    qed
qed

end

end
theory PathLemmas
imports TheoremD14
begin

context LocalLexing begin

lemma characterize-𝒫:
  (∀ i < length p. ∃ u. p ! i ∈ 𝒵 (charslength (take i p)) u) ⟹ admissible p ⟹
  ∃ u. p ∈ 𝒫 (charslength p) u
proof (induct p rule: rev-induct)
  case Nil
    show ?case by simp
next
  case (snoc a p)
    from snoc.prems have admissible-p: admissible p
      by (metis append-Nil2 is-prefix-admissible is-prefix-cancel is-prefix-empty)
    {
      fix i :: nat
      assume ilen: i < length p
      then have i < length (p@[a])
        by (simp add: Suc-leI Suc-le-lessD trans-le-add1)
      with snoc have ∃ u. (p @ [a]) ! i ∈ 𝒵 (charslength (take i (p @ [a]))) u
        by blast
      then obtain u where u: (p @ [a]) ! i ∈ 𝒵 (charslength (take i (p @ [a]))) u
by blast
      from ilen have p-at: (p @ [a]) ! i = p ! i by (simp add: nth-append)
    from ilen have p-take: take i (p @ [a]) = take i p by (simp add: less-or-eq-imp-le)

      from u p-at p-take have p-i: p ! i ∈ 𝒵 (charslength (take i p)) u by simp
      then have ∃ u. p ! i ∈ 𝒵 (charslength (take i p)) u by blast
    }
    then have ∀ i < length p. ∃ u. p ! i ∈ 𝒵 (charslength (take i p)) u by auto
    with admissible-p snoc.hyps obtain u where u: p ∈ 𝒫 (charslength p) u by
blast
    have ∃ u. (p @ [a]) ! (length p) ∈ 𝒵 (charslength (take (length p) (p @ [a]))) u
      using snoc
      by (metis (no-types, opaque-lifting) add-Suc-right append-Nil2 length-Cons
length-append
        less-Suc-eq-le less-or-eq-imp-le)
    then obtain v where (p @ [a]) ! (length p) ∈ 𝒵 (charslength (take (length p)
(p @ [a]))) v
```

```
    by blast
  then have v: a ∈ 𝒵 (charslength p) v by simp
  {
    assume v-leq-u: v ≤ u
    then have a ∈ 𝒵 (charslength p) (Suc u) using v
      by (meson LocalLexing.subset-fSuc LocalLexing-axioms 𝒵-subset-Suc subsetCE)
    from path-append-token[OF u this snoc.prems(2)]
    have p @ [a] ∈ 𝒫 (charslength p) (Suc u) by blast
    then have ?case using in-𝒫-charslength by blast
  }
  note case-v-leq-u = this
  {
    assume u-less-v: u < v
    then obtain w where w: v = Suc w using less-imp-Suc-add by blast
    with u-less-v have u ≤ w by arith
    with u have p ∈ 𝒫 (charslength p) w by (meson subsetCE subset-𝒫k)
    from v w path-append-token[OF this - snoc.prems(2)]
    have p @ [a] ∈ 𝒫 (charslength p) (Suc w) by blast
    then have ?case using in-𝒫-charslength by blast
  }
  note case-u-less-v = this

  show ?case using case-v-leq-u case-u-less-v not-le by blast
qed

lemma drop-empty-tokens:
  assumes p: p ∈ 𝔓
  assumes r: r ≤ length p
  assumes empty: charslength (take r p) = 0
  assumes admissible: admissible (drop r p)
  shows drop r p ∈ 𝔓
proof −
  have p-𝒵: ∀ i<length p. ∃ u. p ! i ∈ 𝒵 (charslength (take i p)) u using p
    using tokens-nth-in-𝒵 by blast
  obtain q where q: q = drop r p by blast
  {
    fix j :: nat
    assume j : j < length q
    have length-p-q-r: length p = length q + r
      using r q add.commute diff-add-inverse le-Suc-ex length-drop by simp
    have j-plus-r-bound: j + r < length p by (simp add: j length-p-q-r)
    with p-𝒵 have ∃ u. p ! (j + r) ∈ 𝒵 (charslength (take (j + r) p)) u by blast
    then obtain u where u: p ! (j + r) ∈ 𝒵 (charslength (take (j + r) p)) u by blast
    have p-at-is-q-at: p ! (j + r) = q ! j by (simp add: add.commute q r)
   have take (j + r) p = (take r p) @ (take j q) by (metis add.commute q take-add)
    with empty have charslength (take (j + r) p) = charslength (take j q) by auto
    with u p-at-is-q-at have q ! j ∈ 𝒵 (charslength (take j q)) u by simp
```

```
    then have ∃ u. q ! j ∈ 𝒵 (charslength (take j q)) u by auto
  }
  then have ∀ i<length q. ∃ u. q ! i ∈ 𝒵 (charslength (take i q)) u by blast
  from characterize-𝒫[OF this] have ∃ u. q ∈ 𝒫 (charslength q) u using admissible
q by auto
  then show ?thesis using 𝔓-covers-𝒫 q by blast
qed

end

end
theory MainTheorems
imports PathLemmas
begin

context LocalLexing begin

theorem 𝔍-is-generated-by-𝔓: 𝔍 = Gen 𝔓
proof −
  have wellformed-items (ℐ (length Doc))
    using wellformed-items-ℐ by auto
  then have ⋀ x. x ∈ ℐ (length Doc) ⟹ item-end x ≤ length Doc
    using wellformed-item-def wellformed-items-def by blast
  then have ℐ (length Doc) ⊆ items-le (length Doc) (ℐ (length Doc))
    by (auto simp only: items-le-def)
  then have ℐ (length Doc) = items-le (length Doc) (ℐ (length Doc))
    using items-le-is-filter by blast
  then have 𝔍-altdef: 𝔍 = items-le (length Doc) (ℐ (length Doc)) using 𝔍-def by
auto
  have ⋀ p. p ∈ (𝒬 (length Doc)) ⟹ charslength p ≤ length Doc
    using 𝔓-are-doc-tokens 𝔓-def doc-tokens-length by auto
  then have 𝒬 (length Doc) ⊆ paths-le (length Doc) (𝒬 (length Doc))
    by (auto simp only: paths-le-def)
  then have 𝒬 (length Doc) = paths-le (length Doc) (𝒬 (length Doc))
    using paths-le-is-filter by blast
  then have 𝔓-altdef: 𝔓 = paths-le (length Doc) (𝒬 (length Doc)) using 𝔓-def
by auto
  show ?thesis using 𝔍-altdef 𝔓-altdef thmD14 by auto
qed

definition finished-item :: symbol list ⇒ item
where
  finished-item α = Item (𝔖, α) (length α) 0 (length Doc)

lemma item-rule-finished-item[simp]: item-rule (finished-item α) = (𝔖, α)
  by (simp add: finished-item-def)

lemma item-origin-finished-item[simp]: item-origin (finished-item α) = 0
  by (simp add: finished-item-def)
```

lemma *item-end-finished-item*[*simp*]: *item-end* (*finished-item* α) = *length Doc*
by (*simp add*: *finished-item-def*)

lemma *item-dot-finished-item*[*simp*]: *item-dot* (*finished-item* α) = *length* α
by (*simp add*: *finished-item-def*)

lemma *item-rhs-finished-item*[*simp*]: *item-rhs* (*finished-item* α) = α
by (*simp add*: *finished-item-def*)

lemma *item-α-finished-item*[*simp*]: *item-α* (*finished-item* α) = α
by (*simp add*: *finished-item-def item-α-def*)

lemma *item-nonterminal-finished-item*[*simp*]: *item-nonterminal* (*finished-item* α)
= $\mathfrak{S}$
by (*simp add*: *finished-item-def item-nonterminal-def*)

lemma *Derives1-of-singleton*:
assumes *Derives1* [*N*] *i r* α
shows *i* = *0* $\wedge$ *r* = (*N*, α)
proof −
from *assms* **have** *i* = *0* **using** *Derives1-bound*
using *length-Cons less-Suc0 list.size*(*3*) **by** *fastforce*
then show *?thesis* **using** *assms*
using *Derives1-def append-Cons append-self-conv append-self-conv2 length-0-conv*

list.inject **by** *auto*
qed

definition *pvalid-with* :: *tokens* ⇒ *item* ⇒ *nat* ⇒ *symbol list* ⇒ *bool*
where
pvalid-with p x u γ =
(*wellformed-tokens p* $\wedge$
wellformed-item x $\wedge$
u ≤ *length p* $\wedge$
charslength p = *item-end x* $\wedge$
charslength (*take u p*) = *item-origin x* $\wedge$
is-derivation (*terminals* (*take u p*) @ [*item-nonterminal x*] @ γ) $\wedge$
derives (*item-α x*) (*terminals* (*drop u p*)))

lemma *pvalid-with*: *pvalid p x* = (∃ *u* γ. *pvalid-with p x u* γ)
using *pvalid-def pvalid-with-def* **by** *blast*

theorem *Completeness*:
assumes *p-in-ll*: *p* ∈ *ll*
shows ∃ α. *pvalid-with p* (*finished-item* α) *0* [] $\wedge$ *finished-item* α ∈ $\mathfrak{I}$
proof −
have *p*: *p* ∈ $\mathfrak{P}$ $\wedge$ *charslength p* = *length Doc* $\wedge$ *terminals p* ∈ $\mathcal{L}$
using *p-in-ll ll-def* **by** *auto*

```
  then have derives-p: derives [𝔖] (terminals p)
    using ℒ-def is-derivation-def mem-Collect-eq by blast
  then have ∃ D. Derivation [𝔖] D (terminals p)
    by (simp add: derives-implies-Derivation)
  then obtain D where D: Derivation [𝔖] D (terminals p) by blast
  have is-word-p: is-word (terminals p)  using leftlang p by blast
 have not-is-word-𝔖: ¬ (is-word [𝔖]) using is-nonterminal-startsymbol is-terminal-nonterminal

    is-word-cons by blast
  have D ≠ [] using D is-word-p not-is-word-𝔖 using Derivation.simps(1) by
force
  then have ∃ d D'. D = d#D' using D Derivation.elims(2) by blast
  then obtain d D' where d: D = d#D' by blast
  have ∃ α. Derives1 [𝔖] (fst d) (snd d) α ∧ derives α (terminals p)
    using d D Derivation.simps(2) Derivation-implies-derives by blast
  then obtain α where α: Derives1 [𝔖] (fst d) (snd d) α ∧ derives α (terminals
p) by blast
  then have snd-d-in-𝔑: snd d ∈ 𝔑 using Derives1-rule by blast
  with α have snd-d: snd d = (𝔖, α) using Derives1-of-singleton by blast
  have wellformed-p: wellformed-tokens p by (simp add: 𝔓-wellformed p)
  have wellformed-finished-item: wellformed-item (finished-item α)
    apply (auto simp add: wellformed-item-def)
    using snd-d snd-d-in-𝔑 by metis
  have pvalid-with: pvalid-with p (finished-item α) 0 []
    apply (auto simp add: pvalid-with-def)
    using wellformed-p apply blast
    using wellformed-finished-item apply blast
    using p apply (simp add: finished-item-def)
    apply (simp add: is-derivation-def)
    by (simp add: α)
  then have pvalid p (finished-item α) using pvalid-def pvalid-with-def by blast
  then have finished-item α ∈ Gen 𝔓 using Gen-def mem-Collect-eq p by blast
  then have finished-item α ∈ 𝔍 using 𝔍-is-generated-by-𝔓 by blast
  with pvalid-with show ?thesis by blast
qed

theorem Soundness:
  assumes finished-item-α: finished-item α ∈ 𝔍
  shows ∃ p. pvalid-with p (finished-item α) 0 [] ∧ p ∈ ll
proof −
  have finished-item α ∈ Gen 𝔓
    using 𝔍-is-generated-by-𝔓 finished-item-α by auto
  then obtain p where p: p ∈ 𝔓 ∧ pvalid p (finished-item α)
    using Gen-implies-pvalid by blast
  have pvalid-p-finished-item: pvalid  p (finished-item α) using p by blast
  from iffD1[OF pvalid-def this, simplified] obtain r γ where pvalid:
    wellformed-tokens p ∧
     wellformed-item (finished-item α) ∧
     r ≤ length p ∧
```

```
    length (chars p) = length Doc ∧
    chars (take r p) = [] ∧
    is-derivation (take r (terminals p) @ 𝔖 # γ) ∧ derives α (drop r (terminals
p))
    by  blast
  have item-rule (finished-item α) ∈ ℜ using pvalid
    using wellformed-item-def by blast
  then have (𝔖, α) ∈ ℜ by simp
  then have is-derivation-α: is-derivation α by (simp add: is-derivation-def left-
derives-rule)
  have drop-r-p-in-𝔓: drop r p ∈ 𝔓
    apply (rule drop-empty-tokens)
    using p apply blast
    using pvalid apply blast
    using pvalid apply simp
    by (metis append-Nil2 derives-trans is-derivation-α is-derivation-def
      is-derivation-implies-admissible is-word-terminals-drop pvalid terminals-drop)
  then have in-ll: drop r p ∈ ll
    apply (auto simp add: ll-def)
    apply (metis append-Nil append-take-drop-id chars-append pvalid)
    using is-derivation-α pvalid
    by (metis (no-types, lifting) ℒ-def derives-trans is-derivation-def
      is-word-terminals-drop mem-Collect-eq terminals-drop)
  have pvalid-with (drop r p) (finished-item α) 0 []
    apply (auto simp add: pvalid-with-def)
    using 𝔓-wellformed drop-r-p-in-𝔓 apply blast
    using pvalid apply blast
    apply (metis append-Nil append-take-drop-id chars-append pvalid)
    apply (simp add: is-derivation-def)
    using pvalid by blast
  with in-ll show ?thesis by auto
qed

lemma is-finished-and-finished-item:
  assumes wellformed-x: wellformed-item x
  shows is-finished x = (∃ α. x = finished-item α)
proof −
  {
    assume is-finished-x: is-finished x
    obtain α where α: α = item-rhs x by blast
    have x = finished-item α
      apply (rule item.expand)
      apply auto
      using α is-finished-def is-finished-x item-nonterminal-def item-rhs-def apply
auto[1]
     using α assms is-complete-def is-finished-def is-finished-x wellformed-item-def
apply auto[1]
      using is-finished-def is-finished-x apply blast
      using is-finished-def is-finished-x by auto
```

```
    then have ∃ α. x = finished-item α by blast
  }
  note left-implies-right = this
  {
    assume ∃ α. x = finished-item α
    then obtain α where α: x = finished-item α by blast
    have is-finished x by (simp add: α is-finished-def is-complete-def)
  }
  note right-implies-left = this
  show ?thesis using left-implies-right right-implies-left by blast
qed

theorem Correctness:
  shows (ll ≠ {}) = earley-recognised
proof −
  have 1: (ll ≠ {}) = (∃ α. finished-item α ∈ 𝔍)
    using Soundness Completeness ex-in-conv by fastforce
  have 2: (∃ α. finished-item α ∈ 𝔍) = (∃ x ∈ 𝔍. is-finished x)
   using 𝔍-def is-finished-and-finished-item wellformed-items-ℐ wellformed-items-def
by auto
  show ?thesis using earley-recognised-def 1 2 by blast
qed

end

end
```