Analysis of List Update Algorithms

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Abstract

These theories formalize the quantitative analysis of a number of classical algorithms for the list update problem: 2-competitiveness of move-to-front, the lower bound of 2 for the competitive- ness of deterministic list update algorithms and 1.6-competitiveness of the randomized COMB algorithm, the best randomized list update algorithm known to date.

An informal description is found in an accompanying report [HN16]. The material is based on the first two chapters of the book by Borodin and El-Yaniv [BEY98].

Contents

1 List Inversion

theory *Inversion*
imports *List−Index.List_Index*
begin

abbreviation *dist_perm xs ys* ≡ *distinct xs* ∧ *distinct ys* ∧ *set xs* = *set ys*

definition *before_in* :: *'a* ⇒ *'a* ⇒ *'a list* ⇒ *bool*
(*(_ </ _/ in _) [55,55,55] 55*) **where**
x < *y in xs* = (*index xs x* < *index xs y* ∧ *y* ∈ *set xs*)

definition *Inv* :: *'a list* ⇒ *'a list* ⇒ (*'a* ∗ *'a*) *set* **where**
Inv xs ys = {(*x*,*y*). *x* < *y in xs* ∧ *y* < *x in ys*}

lemma *before_in_setD1*: *x* < *y in xs* ⟹ *x* : *set xs*
⟨*proof*⟩

lemma *before_in_setD2*: *x* < *y in xs* ⟹ *y* : *set xs*
⟨*proof*⟩

lemma *not_before_in*:
x : *set xs* ⟹ *y* : *set xs* ⟹ ¬ *x* < *y in xs* ⟷ *y* < *x in xs* ∨ *x*=*y*
⟨*proof*⟩

lemma *before_in_ireft*: *x* < *x in xs* = *False*
⟨*proof*⟩

lemma *no_before_inI*[*simp*]: *x* < *y in xs* ⟹ (¬ *y* < *x in xs*) = *True*
⟨*proof*⟩

lemma *finite_Invs*[*simp*]: *finite*(*Inv xs ys*)
⟨*proof*⟩

lemma *Inv_id*[*simp*]: *Inv xs xs* = {}
⟨*proof*⟩

lemma *card_Inv_sym*: *card*(*Inv xs ys*) = *card*(*Inv ys xs*)
⟨*proof*⟩

lemma *Inv_tri_ineq*:
dist_perm xs ys ⟹ *dist_perm ys zs* ⟹
Inv xs zs ⊆ *Inv xs ys Un Inv ys zs*

⟨*proof*⟩

lemma *card_Inv_tri_ineq*:
dist_perm xs ys ⟹ *dist_perm ys zs* ⟹
card (*Inv xs zs*) ≤ *card*(*Inv xs ys*) + *card* (*Inv ys zs*)
⟨*proof*⟩

end

2 Swapping Adjacent Elements in a List

theory *Swaps*
imports *Inversion*
begin

Swap elements at index *n* and *Suc n*:

definition *swap n xs* =
(*if Suc n* < *size xs then xs*[*n* := *xs*!*Suc n*, *Suc n* := *xs*!*n*] *else xs*)

lemma *length_swap*[*simp*]: *length*(*swap i xs*) = *length xs*
⟨*proof*⟩

lemma *swap_id*[*simp*]: *Suc n* ≥ *size xs* ⟹ *swap n xs* = *xs*
⟨*proof*⟩

lemma *distinct_swap*[*simp*]:
distinct(*swap i xs*) = *distinct xs*
⟨*proof*⟩

lemma *swap_Suc*[*simp*]: *swap* (*Suc n*) (*a* # *xs*) = *a* # *swap n xs*
⟨*proof*⟩

lemma *index_swap_distinct*:
distinct xs ⟹ *Suc n* < *length xs* ⟹
index (*swap n xs*) *x* =
(*if x* = *xs*!*n then Suc n else if x* = *xs*!*Suc n then n else index xs x*)
⟨*proof*⟩

lemma *set_swap*[*simp*]: *set*(*swap n xs*) = *set xs*
⟨*proof*⟩

lemma *nth_swap_id*[*simp*]: *Suc i* < *length xs* ⟹ *swap i xs* ! *i* = *xs*!(*i*+*1*)
⟨*proof*⟩

lemma *before_in_swap*:
dist_perm xs ys $\Longrightarrow$ *Suc n* $<$ *size xs* $\Longrightarrow$
x $<$ *y in* (*swap n xs*) $\longleftrightarrow$
x $<$ *y in xs* $\wedge$ $\neg$ (*x* $=$ *xs*!*n* $\wedge$ *y* $=$ *xs*!*Suc n*) $\vee$ *x* $=$ *xs*!*Suc n* $\wedge$ *y* $=$ *xs*!*n*
⟨*proof*⟩

lemma *Inv_swap*: **assumes** *dist_perm xs ys*
shows *Inv xs* (*swap n ys*) $=$
(*if Suc n* $<$ *size xs*
then if ys!*n* $<$ *ys*!*Suc n in xs*
then Inv xs ys $\cup$ {(*ys*!*n*, *ys*!*Suc n*)}
else Inv xs ys $-$ {(*ys*!*Suc n*, *ys*!*n*)}
else Inv xs ys)
⟨*proof*⟩

Perform a list of swaps, from right to left:

abbreviation *swaps* **where** *swaps* $==$ *foldr swap*

lemma *swaps_inv*[*simp*]:
set (*swaps sws xs*) $=$ *set xs* $\wedge$
size(*swaps sws xs*) $=$ *size xs* $\wedge$
distinct(*swaps sws xs*) $=$ *distinct xs*
⟨*proof*⟩

lemma *swaps_eq_Nil_iff*[*simp*]: *swaps acts xs* $=$ [] $\longleftrightarrow$ *xs* $=$ []
⟨*proof*⟩

lemma *swaps_map_Suc*[*simp*]:
swaps (*map Suc sws*) (*a* # *xs*) $=$ *a* # *swaps sws xs*
⟨*proof*⟩

lemma *card_Inv_swaps_le*:
distinct xs $\Longrightarrow$ *card* (*Inv xs* (*swaps sws xs*)) $\leq$ *length sws*
⟨*proof*⟩

lemma *nth_swaps*: $\forall$ *i*$\in$*set is*. *j* $<$ *i* $\Longrightarrow$ *swaps is xs* ! *j* $=$ *xs* ! *j*
⟨*proof*⟩

lemma *not_before0*[*simp*]: $^{\sim}$ *x* $<$ *xs* ! *0 in xs*
⟨*proof*⟩

lemma *before_id*[*simp*]: ⟦ *distinct xs*; *i* $<$ *size xs*; *j* $<$ *size xs* ⟧ $\Longrightarrow$
xs ! *i* $<$ *xs* ! *j in xs* $\longleftrightarrow$ *i* $<$ *j*
⟨*proof*⟩

lemma *before_swaps*:
⟦ *distinct is*; ∀ *i*∈*set is*. *Suc i* < *size xs*; *distinct xs*; *i* ∉ *set is*; *i* < *j*; *j* < *size xs* ⟧ ⟹
swaps is xs ! *i* < *swaps is xs* ! *j in xs*
⟨*proof*⟩

lemma *card_Inv_swaps*:
⟦ *distinct is*; ∀ *i*∈*set is*. *Suc i* < *size xs*; *distinct xs* ⟧ ⟹
card(*Inv xs* (*swaps is xs*)) = *length is*
⟨*proof*⟩

lemma *swaps_eq_nth_take_drop*: *i* < *length xs* ⟹
swaps [*0*..<*i*] *xs* = *xs*!*i* # *take i xs* @ *drop* (*Suc i*) *xs*
⟨*proof*⟩

lemma *index_swaps_size*: *distinct s* ⟹
index s q ≤ *index* (*swaps sws s*) *q* + *length sws*
⟨*proof*⟩

lemma *index_swaps_last_size*: *distinct s* ⟹
size s ≤ *index* (*swaps sws s*) (*last s*) + *length sws* + *1*
⟨*proof*⟩

end

3 Deterministic Online and Offline Algorithms

theory *On_Off*
imports *Complex_Main*
begin

type_synonym ('*s*,'*r*,'*a*) *alg_off* = '*s* ⇒ '*r list* ⇒ '*a list*
type_synonym ('*s*,'*is*,'*r*,'*a*) *alg_on* = ('*s* ⇒ '*is*) ∗ ('*s* ∗ '*is* ⇒ '*r* ⇒ '*a* ∗ '*is*)

locale *On_Off* =
fixes *step* :: '*state* ⇒ '*request* ⇒ '*answer* ⇒ '*state*
fixes *t* :: '*state* ⇒ '*request* ⇒ '*answer* ⇒ *nat*
fixes *wf* :: '*state* ⇒ '*request list* ⇒ *bool*
begin

fun *T* :: '*state* ⇒ '*request list* ⇒ '*answer list* ⇒ *nat* **where**

$T\ s\ []\ []\ =\ 0\ |$
$T\ s\ (r\#rs)\ (a\#as)\ =\ t\ s\ r\ a\ +\ T\ (step\ s\ r\ a)\ rs\ as$

definition *Step* ::
$('state\ ,\ 'istate,\ 'request,\ 'answer)alg\_on$
$\Rightarrow\ 'state\ *\ 'istate \Rightarrow\ 'request \Rightarrow\ 'state\ *\ 'istate$
where
$Step\ A\ s\ r\ =\ (let\ (a,is')\ =\ snd\ A\ s\ r\ in\ (step\ (fst\ s)\ r\ a,\ is'))$

fun $config'$ :: $('state,'is,'request,'answer)\ alg\_on\ \Rightarrow\ ('state*'is) \Rightarrow\ 'request\ list$
$\Rightarrow\ ('state\ *\ 'is)$ **where**
$config'\ A\ s\ []\ =\ s\ |$
$config'\ A\ s\ (r\#rs)\ =\ config'\ A\ (Step\ A\ s\ r)\ rs$

lemma $config'\_snoc$: $config'\ A\ s\ (rs@[r])\ =\ Step\ A\ (config'\ A\ s\ rs)\ r$
⟨*proof*⟩

lemma $config'\_append2$: $config'\ A\ s\ (xs@ys)\ =\ config'\ A\ (config'\ A\ s\ xs)\ ys$
⟨*proof*⟩

lemma $config'\_induct$: $P\ (fst\ init) \Longrightarrow (\bigwedge s\ q\ a.\ P\ s \Longrightarrow P\ (step\ s\ q\ a))$
$\Longrightarrow P\ (fst\ (config'\ A\ init\ rs))$
⟨*proof*⟩

abbreviation *config* **where**
$config\ A\ s0\ rs\ ==\ config'\ A\ (s0,\ fst\ A\ s0)\ rs$

lemma $config\_snoc$: $config\ A\ s\ (rs@[r])\ =\ Step\ A\ (config\ A\ s\ rs)\ r$
⟨*proof*⟩

lemma $config\_append$: $config\ A\ s\ (xs@ys)\ =\ config'\ A\ (config\ A\ s\ xs)\ ys$
⟨*proof*⟩

lemma $config\_induct$: $P\ s0 \Longrightarrow (\bigwedge s\ q\ a.\ P\ s \Longrightarrow P\ (step\ s\ q\ a)) \Longrightarrow P\ (fst\ (config\ A\ s0\ qs))$
⟨*proof*⟩

fun T_on' :: $('state,'is,'request,'answer)\ alg\_on \Rightarrow ('state*'is) \Rightarrow\ 'request\ list \Rightarrow\ nat$ **where**
$T\_on'\ A\ s\ []\ =\ 0\ |$
$T\_on'\ A\ s\ (r\#rs)\ =\ (t\ (fst\ s)\ r\ (fst\ (snd\ A\ s\ r))) + T\_on'\ A\ (Step\ A\ s$

$r)\ rs$

lemma *T_on'_append*: $T\_on'\ A\ s\ (xs@ys) = T\_on'\ A\ s\ xs + T\_on'\ A\ (config'\ A\ s\ xs)\ ys$
⟨*proof*⟩

abbreviation T_on'' :: $('state,'is,'request,'answer)\ alg\_on \Rightarrow 'state \Rightarrow 'request\ list \Rightarrow nat$ **where**
$T\_on''\ A\ s\ rs == T\_on'\ A\ (s,fst\ A\ s)\ rs$

lemma *T_on_append*: $T\_on''\ A\ s\ (xs@ys) = T\_on''\ A\ s\ xs + T\_on'\ A\ (config\ A\ s\ xs)\ ys$
⟨*proof*⟩

abbreviation $T\_on\_n\ A\ s0\ xs\ n == T\_on'\ A\ (config\ A\ s0\ (take\ n\ xs))\ [xs!n]$

lemma *T_on___as_sum*: $T\_on''\ A\ s0\ rs = sum\ (T\_on\_n\ A\ s0\ rs)\ \{..<length\ rs\}$
⟨*proof*⟩

fun *off2* :: $('state,'is,'request,'answer)\ alg\_on \Rightarrow ('state * 'is,'request,'answer)\ alg\_off$ **where**
$off2\ A\ s\ [] = [] \mid$
$off2\ A\ s\ (r\#rs) = fst\ (snd\ A\ s\ r)\ \#\ off2\ A\ (Step\ A\ s\ r)\ rs$

abbreviation *off* :: $('state,'is,'request,'answer)\ alg\_on \Rightarrow ('state,'request,'answer)\ alg\_off$ **where**
$off\ A\ s0 \equiv off2\ A\ (s0,\ fst\ A\ s0)$

abbreviation T_off :: $('state,'request,'answer)\ alg\_off \Rightarrow 'state \Rightarrow 'request\ list \Rightarrow nat$ **where**
$T\_off\ A\ s0\ rs == T\ s0\ rs\ (A\ s0\ rs)$

abbreviation *T_on* :: $('state,'is,'request,'answer)\ alg\_on \Rightarrow 'state \Rightarrow 'request\ list \Rightarrow nat$ **where**
$T\_on\ A == T\_off\ (off\ A)$

lemma *T_on_on'*: *T_off* (λ*s0*. (*off2 A* (*s0*, *x*))) *s0 qs* = *T_on' A* (*s0*,*x*) *qs*
⟨*proof*⟩

lemma *T_on_on''*: *T_on A s0 qs* = *T_on'' A s0 qs*
⟨*proof*⟩

lemma *T_on_as_sum*: *T_on A s0 rs* = *sum* (*T_on_n A s0 rs*) {..<*length rs*}
⟨*proof*⟩

definition *T_opt* :: *'state* ⇒ *'request list* ⇒ *nat* **where**
T_opt s rs = *Inf* {*T s rs as* | *as*. *size as* = *size rs*}

definition *compet* :: (*'state*,*'is*,*'request*,*'answer*) *alg_on* ⇒ *real* ⇒ *'state set* ⇒ *bool* **where**
compet A c S = (∀ *s*∈*S*. ∃ *b* ≥ *0*. ∀ *rs*. *wf s rs* ⟶ *real*(*T_on A s rs*) ≤ *c* ∗ *T_opt s rs* + *b*)

lemma *length_off*[*simp*]: *length*(*off2 A s rs*) = *length rs*
⟨*proof*⟩

lemma *compet_mono*: **assumes** *compet A c S0* **and** *c* ≤ *c'*
shows *compet A c' S0*
⟨*proof*⟩

lemma *competE*: **fixes** *c* :: *real*
assumes *compet A c S0 c* ≥ *0* ∀ *s0 rs*. *size*(*aoff s0 rs*) = *length rs s0*∈*S0*
shows ∃ *b*≥*0*. ∀ *rs*. *wf s0 rs* ⟶ *T_on A s0 rs* ≤ *c* ∗ *T_off aoff s0 rs* + *b*
⟨*proof*⟩

end

end

4 Probability Theory

theory *Prob_Theory*
imports *HOL−Probability.Probability*

begin

lemma *integral_map_pmf*[*simp*]:
fixes *f*::*real* ⇒ *real*
shows (∫ *x*. *f x* ∂(*map_pmf g M*)) = (∫ *x*. *f* (*g x*) ∂*M*)
⟨*proof*⟩

4.1 function *E*

definition *E* :: *real pmf* ⇒ *real* **where**
E M = (∫ *x*. *x* ∂ *measure_pmf M*)

translations
∫ *x*. *f* ∂*M* <= *CONST lebesgue_integral M* (λ*x*. *f*)

notation (*latex* **output**) *E* (*E*[_] [*1*] *100*)

lemma *E_const*[*simp*]: *E* (*return_pmf a*) = *a*
⟨*proof*⟩

lemma *E_null*[*simp*]: *E* (*return_pmf 0*) = *0*
⟨*proof*⟩

lemma *E_finite_sum*: *finite* (*set_pmf X*) ⟹ *E X* = (∑ *x*∈(*set_pmf X*). *pmf X x* ∗ *x*)
⟨*proof*⟩

lemma *E_of_const*: *E*(*map_pmf* (λ*x*. *y*) (*X*::*real pmf*)) = *y* ⟨*proof*⟩

lemma *E_nonneg*:
shows (∀ *x*∈*set_pmf X*. *0*≤ *x*) ⟹ *0* ≤ *E X*
⟨*proof*⟩

lemma *E_nonneg_fun*: **fixes** *f*::′*a*⇒*real*
shows (∀ *x*∈*set_pmf X*. *0*≤*f x*) ⟹ *0* ≤ *E* (*map_pmf f X*)
⟨*proof*⟩

lemma *E_cong*:
fixes *f*::′*a* ⇒ *real*
shows *finite* (*set_pmf X*) ⟹ (∀ *x*∈ *set_pmf X*. (*f x*) = (*u x*)) ⟹ *E* (*map_pmf f X*) = *E* (*map_pmf u X*)
⟨*proof*⟩

lemma *E_mono3*:

fixes $f::'a \Rightarrow real$
shows *integrable* (*measure_pmf X*) $f \Longrightarrow$ *integrable* (*measure_pmf X*) u $\Longrightarrow (\forall x \in set\_pmf\ X.\ (f\ x) \leq (u\ x)) \Longrightarrow E\ (map\_pmf\ f\ X) \leq E\ (map\_pmf\ u\ X)$
⟨*proof*⟩

lemma *E_mono2*:
fixes $f::'a \Rightarrow real$
shows *finite* (*set_pmf X*) $\Longrightarrow (\forall x \in set\_pmf\ X.\ (f\ x) \leq (u\ x)) \Longrightarrow E\ (map\_pmf\ f\ X) \leq E\ (map\_pmf\ u\ X)$
⟨*proof*⟩

lemma *E_linear_diff2*: *finite* (*set_pmf A*) $\Longrightarrow E\ (map\_pmf\ f\ A) - E\ (map\_pmf\ g\ A) = E\ (map\_pmf\ (\lambda x.\ (f\ x) - (g\ x))\ A)$
⟨*proof*⟩

lemma *E_linear_plus2*: *finite* (*set_pmf A*) $\Longrightarrow E\ (map\_pmf\ f\ A) + E\ (map\_pmf\ g\ A) = E\ (map\_pmf\ (\lambda x.\ (f\ x) + (g\ x))\ A)$
⟨*proof*⟩

lemma *E_linear_sum2*: *finite* (*set_pmf D*) $\Longrightarrow E(map\_pmf\ (\lambda x.\ (\sum i{<}up.\ f\ i\ x))\ D)$
$= (\sum i{<}(up::nat).\ E(map\_pmf\ (f\ i)\ D))$
⟨*proof*⟩

lemma *E_linear_sum_allg*: *finite* (*set_pmf D*) $\Longrightarrow E(map\_pmf\ (\lambda x.\ (\sum i \in A.\ f\ i\ x))\ D)$
$= (\sum i \in (A::'a\ set).\ E(map\_pmf\ (f\ i)\ D))$
⟨*proof*⟩

lemma *E_finite_sum_fun*: *finite* (*set_pmf X*) $\Longrightarrow$
$E\ (map\_pmf\ f\ X) = (\sum x \in set\_pmf\ X.\ pmf\ X\ x * f\ x)$
⟨*proof*⟩

lemma *E_bernoulli*: $0 \leq p \Longrightarrow p \leq 1 \Longrightarrow$
$E\ (map\_pmf\ f\ (bernoulli\_pmf\ p)) = p*(f\ True) + (1-p)*(f\ False)$
⟨*proof*⟩

4.2 function *bv*

fun *bv*:: *nat* ⇒ *bool list pmf* **where**
bv 0 = *return_pmf* []
| *bv* (*Suc n*) = *do* {
(*xs*::*bool list*) ← *bv n*;

$(x::bool) \leftarrow (bernoulli\_pmf\ 0.5)$;
$return\_pmf\ (x\#xs)$
}

lemma *bv_finite*: *finite* $(bv\ n)$
⟨*proof*⟩

lemma *len_bv_n*: $\forall xs \in set\_pmf\ (bv\ n).\ length\ xs = n$
⟨*proof*⟩

lemma *bv_set*: $set\_pmf\ (bv\ n) = \{x::bool\ list.\ length\ x = n\}$
⟨*proof*⟩

lemma *len_not_in_bv*: $length\ xs \neq n \Longrightarrow xs \notin set\_pmf\ (bv\ n)$
⟨*proof*⟩

lemma *not_n_bv_0*: $length\ xs \neq n \Longrightarrow pmf\ (bv\ n)\ xs = 0$
⟨*proof*⟩

lemma *bv_comp_bernoulli*: $n < l$
$\Longrightarrow map\_pmf\ (\lambda y.\ y!n)\ (bv\ l) = bernoulli\_pmf\ (5\ /\ 10)$
⟨*proof*⟩

lemma *pmf_2elemlist*: $pmf\ (bv\ (Suc\ 0))\ ([x]) = pmf\ (bv\ 0)\ [] * pmf\ (bernoulli\_pmf\ (5\ /\ 10))\ x$
⟨*proof*⟩

lemma *pmf_moreelemlist*: $pmf\ (bv\ (Suc\ n))\ (x\#xs) = pmf\ (bv\ n)\ xs * pmf\ (bernoulli\_pmf\ (5\ /\ 10))\ x$
⟨*proof*⟩

lemma *list_pmf*: $length\ xs = n \Longrightarrow pmf\ (bv\ n)\ xs = (1\ /\ 2)\hat{}n$
⟨*proof*⟩

lemma *bv_0_notlen*: $pmf\ (bv\ n)\ xs = 0 \Longrightarrow length\ xs \neq n$
⟨*proof*⟩

lemma $length\ xs > n \Longrightarrow pmf\ (bv\ n)\ xs = 0$
⟨*proof*⟩

lemma *map_hd_list_pmf*: $map\_pmf\ hd\ (bv\ (Suc\ n)) = bernoulli\_pmf\ (5\ /\ 10)$
⟨*proof*⟩

lemma *map_tl_list_pmf*: *map_pmf tl* (*bv* (*Suc n*)) = *bv n*
⟨*proof*⟩

4.3 function *flip*

fun *flip* :: *nat* ⇒ *bool list* ⇒ *bool list* **where**
flip _ [] = []
| *flip 0* (*x*#*xs*) = (¬*x*)#*xs*
| *flip* (*Suc n*) (*x*#*xs*) = *x*#(*flip n xs*)

lemma *flip_length*[*simp*]: *length* (*flip i xs*) = *length xs*
⟨*proof*⟩

lemma *flip_out_of_bounds*: *y* ≥ *length X* ⟹ *flip y X* = *X*
⟨*proof*⟩

lemma *flip_other*: *y* < *length X* ⟹ *z* < *length X* ⟹ *z* ≠ *y* ⟹ *flip z X* ! *y* = *X* ! *y*
⟨*proof*⟩

lemma *flip_itself*: *y* < *length X* ⟹ *flip y X* ! *y* = (¬ *X* ! *y*)
⟨*proof*⟩

lemma *flip_twice*: *flip i* (*flip i b*) = *b*
⟨*proof*⟩

lemma *flipidiflip*: *y* < *length X* ⟹ *e* < *length X* ⟹ *flip e X* ! *y* = (*if e*=*y then* ~ (*X* ! *y*) *else X* ! *y*)
⟨*proof*⟩

lemma *bernoulli_Not*: *map_pmf Not* (*bernoulli_pmf* (*1* / *2*)) = (*bernoulli_pmf* (*1* / *2*))
⟨*proof*⟩

lemma *inv_flip_bv*: *map_pmf* (*flip i*) (*bv n*) = (*bv n*)
⟨*proof*⟩

4.4 Example for pmf

definition *twocoins* =
do {
x ← (*bernoulli_pmf 0.4*);
y ← (*bernoulli_pmf 0.5*);
return_pmf (*x* ∨ *y*)

}

lemma *experiment0_7*: *pmf twocoins True = 0.7*
⟨*proof*⟩

4.5 Sum Distribution

definition *Sum_pmf p Da Db* = (*bernoulli_pmf p*) ⤜ (%*b*. *if b then map_pmf Inl Da else map_pmf Inr Db*)

lemma *b0*: *bernoulli_pmf 0 = return_pmf False*
⟨*proof*⟩
lemma *b1*: *bernoulli_pmf 1 = return_pmf True*
⟨*proof*⟩

lemma *Sum_pmf_0*: *Sum_pmf 0 Da Db = map_pmf Inr Db*
⟨*proof*⟩

lemma *Sum_pmf_1*: *Sum_pmf 1 Da Db = map_pmf Inl Da*
⟨*proof*⟩

definition *Proj1_pmf D* = *map_pmf* (%*a*. *case a of Inl e* ⇒ *e*) (*cond_pmf D* {*f*. (∃ *e*. *Inl e* = *f*)})

lemma *A*: (*case_sum* (λ*e*. *e*) (λ*a*. *undefined*)) (*Inl e*) = *e*
⟨*proof*⟩

lemma *B*: *inj* (*case_sum* (λ*e*. *e*) (λ*a*. *undefined*))
⟨*proof*⟩

lemma *none*: *p* >*0* ⟹ *p* < *1* ⟹ (*set_pmf* (*bernoulli_pmf p* ⤜
(λ*b*. *if b then map_pmf Inl Da else map_pmf Inr Db*))
∩ {*f*. (∃ *e*. *Inl e* = *f*)}) ≠ {}
⟨*proof*⟩
lemma *none2*: *p* >*0* ⟹ *p* < *1* ⟹ (*set_pmf* (*bernoulli_pmf p* ⤜
(λ*b*. *if b then map_pmf Inl Da else map_pmf Inr Db*))
∩ {*f*. (∃ *e*. *Inr e* = *f*)}) ≠ {}
⟨*proof*⟩

lemma *C*: *set_pmf* (*Proj1_pmf* (*Sum_pmf 0.5 Da Db*)) = *set_pmf Da*
⟨*proof*⟩

thm *integral_measure_pmf*

thm *pmf_cond pmf_cond*[*OF none*]

lemma *proj1_pmf*: **assumes** *p>0 p<1* **shows** *Proj1_pmf* (*Sum_pmf p Da Db*) = *Da*
⟨*proof*⟩

definition *Proj2_pmf D* = *map_pmf* (%*a. case a of Inr e* ⇒ *e*) (*cond_pmf D* {*f*. (∃ *e. Inr e* = *f*)})

lemma *proj2_pmf*: **assumes** *p>0 p<1* **shows** *Proj2_pmf* (*Sum_pmf p Da Db*) = *Db*
⟨*proof*⟩

definition *invSum invA invB D x i* == *invA* (*Proj1_pmf D*) *x i* ∧ *invB* (*Proj2_pmf D*) *x i*

lemma *invSum_split*: *p>0* ⟹ *p<1* ⟹ *invA Da x i* ⟹ *invB Db x i* ⟹ *invSum invA invB* (*Sum_pmf p Da Db*) *x i*
⟨*proof*⟩

term (%*a. case a of Inl e* ⇒ *Inl* (*fa e*) | *Inr e* ⇒ *Inr* (*fb e*))
definition *f_on2 fa fb* = (%*a. case a of Inl e* ⇒ *map_pmf Inl* (*fa e*) | *Inr e* ⇒ *map_pmf Inr* (*fb e*))

term *bind_pmf*

lemma *Sum_bind_pmf*: **assumes** *a*: *bind_pmf Da fa* = *Da'* **and** *b*: *bind_pmf Db fb* = *Db'*
shows *bind_pmf* (*Sum_pmf p Da Db*) (*f_on2 fa fb*)
= *Sum_pmf p Da' Db'*
⟨*proof*⟩

definition *sum_map_pmf fa fb* = (%*a. case a of Inl e* ⇒ *Inl* (*fa e*) | *Inr e* ⇒ *Inr* (*fb e*))

lemma *Sum_map_pmf*: **assumes** *a*: *map_pmf fa Da* = *Da'* **and** *b*: *map_pmf fb Db* = *Db'*
shows *map_pmf* (*sum_map_pmf fa fb*) (*Sum_pmf p Da Db*)
= *Sum_pmf p Da' Db'*
⟨*proof*⟩

end

5 Randomized Online and Offline Algorithms

theory *Competitive_Analysis*
imports
Prob_Theory
On_Off
begin

5.1 Competitive Analysis Formalized

type_synonym ('s,'is,'r,'a)*alg_on_step* = ('s * 'is ⇒ 'r ⇒ ('a * 'is) *pmf*)
type_synonym ('s,'is)*alg_on_init* = ('s ⇒ 'is *pmf*)
type_synonym ('s,'is,'q,'a)*alg_on_rand* = ('s,'is)*alg_on_init* * ('s,'is,'q,'a)*alg_on_step*

5.1.1 classes of algorithms

definition *deterministic_init* :: ('s,'is)*alg_on_init* ⇒ *bool* **where**
deterministic_init I ⟷ (∀ *init*. *card*(*set_pmf* (*I init*)) = *1*)

definition *deterministic_step* :: ('s,'is,'q,'a)*alg_on_step* ⇒ *bool* **where**
deterministic_step S ⟷ (∀ *i is q*. *card*(*set_pmf* (*S* (*i*, *is*) *q*)) = *1*)

definition *random_step* :: ('s,'is,'q,'a)*alg_on_step* ⇒ *bool* **where**
random_step S ⟷ ~ *deterministic_step S*

5.1.2 Randomized Online and Offline Algorithms

context *On_Off*
begin

fun *steps* **where**

steps s [] [] = s
| *steps s (q#qs) (a#as) = steps (step s q a) qs as*

lemma *steps_append*: *length qs = length as ⟹ steps s (qs@qs′) (as@as′) = steps (steps s qs as) qs′ as′*
⟨*proof*⟩

lemma *T_append*: *length qs = length as ⟹ T s (qs@[q]) (as@[a]) = T s qs as + t (steps s qs as) q a*
⟨*proof*⟩

lemma *T_append2*: *length qs = length as ⟹ T s (qs@qs′) (as@as′) = T s qs as + T (steps s qs as) qs′ as′*
⟨*proof*⟩

abbreviation *Step_rand* :: (*′state*,*′is*,*′request*,*′answer*) *alg_on_rand* ⇒ *′request* ⇒ *′state* ∗ *′is* ⇒ (*′state* ∗ *′is*) *pmf* **where**
Step_rand A r s ≡ *bind_pmf* ((*snd A*) *s r*) (λ(*a*,*is′*). *return_pmf* (*step* (*fst s*) *r a*, *is′*))

fun *config′_rand* :: (*′state*,*′is*,*′request*,*′answer*) *alg_on_rand* ⇒ (*′state*∗*′is*) *pmf* ⇒ *′request list*
⇒ (*′state* ∗ *′is*) *pmf* **where**
config′_rand A s [] = s |
config′_rand A s (r#rs) = config′_rand A (*s* ⤜ *Step_rand A r*) *rs*

lemma *config′_rand_snoc*: *config′_rand A s* (*rs*@[*r*]) = *config′_rand A s rs* ⤜ *Step_rand A r*
⟨*proof*⟩

lemma *config′_rand_append*: *config′_rand A s* (*xs*@*ys*) = *config′_rand A* (*config′_rand A s xs*) *ys*
⟨*proof*⟩

abbreviation *config_rand* **where**
config_rand A s0 rs == *config′_rand A* ((*fst A s0*) ⤜ (λ*is*. *return_pmf* (*s0*, *is*))) *rs*

lemma *config′_rand_induct*: (∀ *x* ∈ *set_pmf init*. *P* (*fst x*)) ⟹ (⋀*s q a*. *P s* ⟹ *P* (*step s q a*))
⟹ ∀ *x*∈*set_pmf* (*config′_rand A init qs*). *P* (*fst x*)

⟨*proof*⟩

lemma *config_rand_induct*: $P\ s0 \Longrightarrow (\bigwedge s\ q\ a.\ P\ s \Longrightarrow P\ (step\ s\ q\ a)) \Longrightarrow \forall\, x{\in}set\_pmf\ (config\_rand\ A\ s0\ qs).\ P\ (fst\ x)$
⟨*proof*⟩

fun *T_on_rand'* :: (*'state*,*'is*,*'request*,*'answer*) *alg_on_rand* ⇒ (*'state*∗*'is*) *pmf* ⇒ *'request list* ⇒ *real* **where**
T_on_rand' A s [] = *0* |
T_on_rand' A s (*r*#*rs*) = *E* (*s* ⤜ (λ*s*. *bind_pmf* (*snd A s r*) (λ(*a*,*is'*). *return_pmf* (*real* (*t* (*fst s*) *r a*)))))
+ *T_on_rand' A* (*s* ⤜ *Step_rand A r*) *rs*

lemma *T_on_rand'_append*: *T_on_rand' A s* (*xs*@*ys*) = *T_on_rand' A s xs* + *T_on_rand' A* (*config'_rand A s xs*) *ys*
⟨*proof*⟩

abbreviation *T_on_rand* :: (*'state*,*'is*,*'request*,*'answer*) *alg_on_rand* ⇒ *'state* ⇒ *'request list* ⇒ *real* **where**
T_on_rand A s rs == *T_on_rand' A* (*fst A s* ⤜ (λ*is*. *return_pmf* (*s*,*is*))) *rs*

lemma *T_on_rand_append*: *T_on_rand A s* (*xs*@*ys*) = *T_on_rand A s xs* + *T_on_rand' A* (*config_rand A s xs*) *ys*
⟨*proof*⟩

abbreviation *T_on_rand'_n A s0 xs n* == *T_on_rand' A* (*config'_rand A s0* (*take n xs*)) [*xs*!*n*]

lemma *T_on_rand'_as_sum*: *T_on_rand' A s0 rs* = *sum* (*T_on_rand'_n A s0 rs*) {..<*length rs*}
⟨*proof*⟩

abbreviation *T_on_rand_n A s0 xs n* == *T_on_rand' A* (*config_rand A s0* (*take n xs*)) [*xs*!*n*]

lemma *T_on_rand_as_sum*: *T_on_rand A s0 rs* = *sum* (*T_on_rand_n A s0 rs*) {..<*length rs*}
⟨*proof*⟩

lemma *T_on_rand'_nn*: *T_on_rand' A s qs* $\geq$ *0*
⟨*proof*⟩

lemma *T_on_rand_nn*: *T_on_rand (I,S) s0 qs* $\geq$ *0*
⟨*proof*⟩

definition *compet_rand* :: (*'state*,*'is*,*'request*,*'answer*) *alg_on_rand* ⇒ *real* ⇒ *'state set* ⇒ *bool* **where**
compet_rand A c S0 = (∀ *s*∈*S0*. ∃ *b* $\geq$ *0*. ∀ *rs*. *wf s rs* ⟶ *T_on_rand A s rs* $\leq$ *c* ∗ *T_opt s rs* + *b*)

5.2 embeding of deterministic into randomized algorithms

fun *embed* :: (*'state*,*'is*,*'request*,*'answer*) *alg_on* ⇒ (*'state*,*'is*,*'request*,*'answer*) *alg_on_rand* **where**
embed A = ((λ*s*. *return_pmf* (*fst A s*)) ,
(λ*s r*. *return_pmf* (*snd A s r*)))

lemma *T_deter_rand*: *T_off* (λ*s0*. (*off2 A* (*s0*, *x*))) *s0 qs* = *T_on_rand'* (*embed A*) (*return_pmf* (*s0*,*x*)) *qs*
⟨*proof*⟩

lemma *config'_embed*: *config'_rand* (*embed A*) (*return_pmf s0*) *qs* = *return_pmf* (*config' A s0 qs*)
⟨*proof*⟩

lemma *config_embed*: *config_rand* (*embed A*) *s0 qs* = *return_pmf* (*config A s0 qs*)
⟨*proof*⟩

lemma *T_on_embed*: *T_on A s0 qs* = *T_on_rand* (*embed A*) *s0 qs*
⟨*proof*⟩

lemma *T_on'_embed*: *T_on' A* (*s0*,*x*) *qs* = *T_on_rand'* (*embed A*) (*return_pmf* (*s0*,*x*)) *qs*
⟨*proof*⟩

lemma *compet_embed*: *compet A c S0* = *compet_rand* (*embed A*) *c S0*
⟨*proof*⟩

end

end

6 Deterministic List Update

theory *Move_to_Front*
imports
Swaps
On_Off
Competitive_Analysis
begin

declare *Let_def*[*simp*]

6.1 Function *mtf*

definition *mtf* :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
mtf x xs =
(*if x* ∈ *set xs then x* # (*take* (*index xs x*) *xs*) @ *drop* (*index xs x* + *1*) *xs*
else xs)

lemma *mtf_id*[*simp*]: *x* ∉ *set xs* ⟹ *mtf x xs* = *xs*
⟨*proof*⟩

lemma *mtf0*[*simp*]: *x* ∈ *set xs* ⟹ *mtf x xs* ! *0* = *x*
⟨*proof*⟩

lemma *before_in_mtf*: **assumes** *z* ∈ *set xs*
shows *x* < *y in mtf z xs* ⟷
(*y* ≠ *z* ∧ (*if x*=*z then y* ∈ *set xs else x* < *y in xs*))
⟨*proof*⟩

lemma *Inv_mtf*: *set xs* = *set ys* ⟹ *z* : *set ys* ⟹ *Inv xs* (*mtf z ys*) =
Inv xs ys ∪ {(*x*,*z*)|*x*. *x* < *z in xs* ∧ *x* < *z in ys*}
− {(*z*,*x*)|*x*. *z* < *x in xs* ∧ *x* < *z in ys*}
⟨*proof*⟩

lemma *set_mtf*[*simp*]: *set*(*mtf x xs*) = *set xs*
⟨*proof*⟩

lemma *length_mtf*[*simp*]: *size* (*mtf x xs*) = *size xs*
⟨*proof*⟩

lemma *distinct_mtf*[*simp*]: *distinct* (*mtf x xs*) = *distinct xs*
⟨*proof*⟩

6.2 Function *mtf2*

definition *mtf2* :: *nat* ⇒ *'a* ⇒ *'a list* ⇒ *'a list* **where**
mtf2 n x xs =
(*if x* : *set xs then swaps* [*index xs x* − *n*..<*index xs x*] *xs else xs*)

lemma *mtf_eq_mtf2*: *mtf x xs* = *mtf2* (*length xs* − *1*) *x xs*
⟨*proof*⟩

lemma *mtf20*[*simp*]: *mtf2 0 x xs* = *xs*
⟨*proof*⟩

lemma *length_mtf2*[*simp*]: *length* (*mtf2 n x xs*) = *length xs*
⟨*proof*⟩

lemma *set_mtf2*[*simp*]: *set*(*mtf2 n x xs*) = *set xs*
⟨*proof*⟩

lemma *distinct_mtf2*[*simp*]: *distinct* (*mtf2 n x xs*) = *distinct xs*
⟨*proof*⟩

lemma *card_Inv_mtf2*: *xs*!*j* = *ys*!*0* ⟹ *j* < *length xs* ⟹ *dist_perm xs ys*
⟹
card (*Inv* (*swaps* [*i*..<*j*] *xs*) *ys*) = *card* (*Inv xs ys*) − *int*(*j*−*i*)
⟨*proof*⟩

6.3 Function Lxy

definition *Lxy* :: *'a list* ⇒ *'a set* ⇒ *'a list* **where**
Lxy xs S = *filter* (λ*z*. *z*∈*S*) *xs*
thm *inter_set_filter*

lemma *Lxy_length_cons*: *length* (*Lxy xs S*) ≤ *length* (*Lxy* (*x*#*xs*) *S*)
⟨*proof*⟩

lemma *Lxy_empty*[*simp*]: *Lxy* [] *S* = []
⟨*proof*⟩

lemma *Lxy_set_filter*: *set* (*Lxy xs S*) = *S* ∩ *set xs*
⟨*proof*⟩

lemma *Lxy_distinct*: *distinct xs* ⟹ *distinct* (*Lxy xs S*)
⟨*proof*⟩

lemma *Lxy_append*: *Lxy* (*xs*@*ys*) *S* = *Lxy xs S* @ *Lxy ys S*
⟨*proof*⟩

lemma *Lxy_snoc*: *Lxy* (*xs*@[*x*]) *S* = (*if x*∈*S then Lxy xs S* @ [*x*] *else Lxy xs S*)
⟨*proof*⟩

lemma *Lxy_not*: *S* ∩ *set xs* = {} ⟹ *Lxy xs S* = []
⟨*proof*⟩

lemma *Lxy_notin*: *set xs* ∩ *S* = {} ⟹ *Lxy xs S* = []
⟨*proof*⟩

lemma *Lxy_in*: *x*∈*S* ⟹ *Lxy* [*x*] *S* = [*x*]
⟨*proof*⟩

lemma *Lxy_project*:
assumes *x*≠*y x* ∈ *set xs* *y*∈*set xs distinct xs*
and *x* < *y in xs*
shows *Lxy xs* {*x*,*y*} = [*x*,*y*]
⟨*proof*⟩

lemma *Lxy_mono*: {*x*,*y*} ⊆ *set xs* ⟹ *distinct xs* ⟹ *x* < *y in xs* = *x* < *y in Lxy xs* {*x*,*y*}
⟨*proof*⟩

6.4 List Update as Online/Offline Algorithm

type_synonym *'a state* = *'a list*
type_synonym *answer* = *nat* ∗ *nat list*

definition *step* :: *'a state* ⇒ *'a* ⇒ *answer* ⇒ *'a state* **where**

step s r a =
(*let* (*k*,*sws*) = *a in mtf2 k r* (*swaps sws s*))

definition *t* :: *'a state* ⇒ *'a* ⇒ *answer* ⇒ *nat* **where**
t s r a = (*let* (*mf*,*sws*) = *a in index* (*swaps sws s*) *r* + *1* + *size sws*)

definition *static* **where** *static s rs* = (*set rs* ⊆ *set s*)

interpretation *On_Off step t static* ⟨*proof*⟩

type_synonym *'a alg_off* = *'a state* ⇒ *'a list* ⇒ *answer list*
type_synonym (*'a*,*'is*) *alg_on* = (*'a state*,*'is*,*'a*,*answer*) *alg_on*

lemma *T_ge_len*: *length as* = *length rs* ⟹ *T s rs as* ≥ *length rs*
⟨*proof*⟩

lemma *T_off_neq0*: (⋀*rs s0*. *size*(*alg s0 rs*) = *length rs*) ⟹
rs ≠ [] ⟹ *T_off alg s0 rs* ≠ *0*
⟨*proof*⟩

lemma *length_step*[*simp*]: *length* (*step s r as*) = *length s*
⟨*proof*⟩

lemma *step_Nil_iff*[*simp*]: *step xs r act* = [] ⟷ *xs* = []
⟨*proof*⟩

lemma *set_step2*: *set*(*step s r* (*mf*,*sws*)) = *set s*
⟨*proof*⟩

lemma *set_step*: *set*(*step s r act*) = *set s*
⟨*proof*⟩

lemma *distinct_step*: *distinct*(*step s r as*) = *distinct s*
⟨*proof*⟩

6.5 Online Algorithm Move-to-Front is 2-Competitive

definition *MTF* :: (*'a*,*unit*) *alg_on* **where**
MTF = (λ_. ()), λ*s r*. ((*size* (*fst s*) − *1*,[]), ()))

It was first proved by Sleator and Tarjan [**?**] that the Move-to-Front algorithm is 2-competitive.

lemma *potential*:
fixes *t* :: *nat* ⇒ *'a*::*linordered_ab_group_add* **and** *p* :: *nat* ⇒ *'a*

assumes *p0*: $p\ 0 = 0$ **and** *ppos*: $\bigwedge n.\ p\ n \geq 0$
and *ub*: $\bigwedge n.\ t\ n + p(n+1) - p\ n \leq u\ n$
shows $(\sum i{<}n.\ t\ i) \leq (\sum i{<}n.\ u\ i)$
⟨*proof*⟩

lemma *potential2*:
fixes $t :: nat \Rightarrow 'a{::}linordered\_ab\_group\_add$ **and** $p :: nat \Rightarrow 'a$
assumes *p0*: $p\ 0 = 0$ **and** *ppos*: $\bigwedge n.\ p\ n \geq 0$
and *ub*: $\bigwedge m.\ m{<}n \Longrightarrow t\ m + p(m+1) - p\ m \leq u\ m$
shows $(\sum i{<}n.\ t\ i) \leq (\sum i{<}n.\ u\ i)$
⟨*proof*⟩

abbreviation *before x xs* ≡ $\{y.\ y < x\ in\ xs\}$
abbreviation *after x xs* ≡ $\{y.\ x < y\ in\ xs\}$

lemma *finite_before*[*simp*]: *finite* (*before x xs*)
⟨*proof*⟩

lemma *finite_after*[*simp*]: *finite* (*after x xs*)
⟨*proof*⟩

lemma *before_conv_take*:
$x : set\ xs \Longrightarrow before\ x\ xs = set(take\ (index\ xs\ x)\ xs)$
⟨*proof*⟩

lemma *card_before*: $distinct\ xs \Longrightarrow x : set\ xs \Longrightarrow card\ (before\ x\ xs) = index\ xs\ x$
⟨*proof*⟩

lemma *before_Un*: $set\ xs = set\ ys \Longrightarrow x : set\ xs \Longrightarrow$
$before\ x\ ys = before\ x\ xs \cap before\ x\ ys\ Un\ after\ x\ xs \cap before\ x\ ys$
⟨*proof*⟩

lemma *phi_diff_aux*:
$card\ (Inv\ xs\ ys \cup$
$\{(y,\ x)\ |y.\ y < x\ in\ xs \wedge y < x\ in\ ys\} -$
$\{(x,\ y)\ |y.\ x < y\ in\ xs \wedge y < x\ in\ ys\}) =$
$card(Inv\ xs\ ys) + card(before\ x\ xs \cap before\ x\ ys)$
$- int(card(after\ x\ xs \cap before\ x\ ys))$
(**is** $card(?I \cup ?B - ?A) = card\ ?I + card\ ?b - int(card\ ?a)$)
⟨*proof*⟩

lemma *not_before_Cons*[*simp*]: $\neg\ x < y\ in\ y\ \#\ xs$

⟨*proof*⟩

lemma *before_Cons*[*simp*]:
  $y \in set\ xs \Longrightarrow y \neq x \Longrightarrow before\ y\ (x\#xs) = insert\ x\ (before\ y\ xs)$
⟨*proof*⟩

lemma *card_before_le_index*: $card\ (before\ x\ xs) \leq index\ xs\ x$
⟨*proof*⟩

lemma *config_config_length*: $length\ (fst\ (config\ A\ init\ qs)) = length\ init$
⟨*proof*⟩

lemma *config_config_distinct*:
  shows $distinct\ (fst\ (config\ A\ init\ qs)) = distinct\ init$
⟨*proof*⟩

lemma *config_config_set*:
  shows $set\ (fst\ (config\ A\ init\ qs)) = set\ init$
⟨*proof*⟩

lemma *config_config*:
  $set\ (fst\ (config\ A\ init\ qs)) = set\ init$
      $\wedge\ distinct\ (fst\ (config\ A\ init\ qs)) = distinct\ init$
      $\wedge\ length\ (fst\ (config\ A\ init\ qs)) = length\ init$
⟨*proof*⟩

lemma *config_dist_perm*:
  $distinct\ init \Longrightarrow dist\_perm\ (fst\ (config\ A\ init\ qs))\ init$
⟨*proof*⟩

lemma *config_rand_length*: $\forall x{\in}set\_pmf\ (config\_rand\ \ A\ init\ qs).\ length\ (fst\ x) = length\ init$
⟨*proof*⟩

lemma *config_rand_distinct*:
  shows $\forall x \in (config\_rand\ \ A\ init\ qs).\ distinct\ (fst\ x) = distinct\ init$
⟨*proof*⟩

lemma *config_rand_set*:
  shows $\forall x \in (config\_rand\ \ A\ init\ qs).\ set\ (fst\ x) = set\ init$
⟨*proof*⟩

lemma *config_rand*:
$\forall x \in (config\_rand \quad A \quad init\ qs).\ set\ (fst\ x) = set\ init$
$\quad \wedge\ distinct\ (fst\ x) = distinct\ init \wedge length\ (fst\ x) = length\ init$
⟨*proof*⟩

lemma *config_rand_dist_perm*:
$distinct\ init \Longrightarrow \forall x \in (config\_rand\ A\ init\ qs).\ dist\_perm\ (fst\ x)\ init$
⟨*proof*⟩

lemma *amor_mtf_ub*: **assumes** x : *set ys set xs* = *set ys*
shows $int(card(before\ x\ xs\ Int\ before\ x\ ys)) - card(after\ x\ xs\ Int\ before\ x\ ys)$
$\leq 2 * int(index\ xs\ x) - card\ (before\ x\ ys)$ (**is** $?m - ?n \leq 2 * ?j - ?k$)
⟨*proof*⟩

locale *MTF_Off* =
fixes *as* :: *answer list*
fixes *rs* :: *'a list*
fixes *s0* :: *'a list*
assumes *dist_s0*[*simp*]: *distinct s0*
assumes *len_as*: *length as* = *length rs*
begin

definition *mtf_A* :: *nat list* **where**
mtf_A = *map fst as*

definition *sw_A* :: *nat list list* **where**
sw_A = *map snd as*

fun *s_A* :: *nat* ⇒ *'a list* **where**
s_A 0 = *s0* |
$s\_A(Suc\ n) = step\ (s\_A\ n)\ (rs!n)\ (mtf\_A!n,\ sw\_A!n)$

lemma *length_s_A*[*simp*]: $length(s\_A\ n) = length\ s0$
⟨*proof*⟩

lemma *dist_s_A*[*simp*]: $distinct(s\_A\ n)$
⟨*proof*⟩

lemma *set_s_A*[*simp*]: *set*(*s_A n*) = *set s0*
⟨*proof*⟩

fun *s_mtf* :: *nat* ⇒ *'a list* **where**
s_mtf 0 = *s0* |
s_mtf (*Suc n*) = *mtf* (*rs*!*n*) (*s_mtf n*)

definition *t_mtf* :: *nat* ⇒ *int* **where**
t_mtf n = *index* (*s_mtf n*) (*rs*!*n*) + *1*

definition *T_mtf* :: *nat* ⇒ *int* **where**
T_mtf n = (∑ *i*<*n*. *t_mtf i*)

definition *c_A* :: *nat* ⇒ *int* **where**
c_A n = *index* (*swaps* (*sw_A*!*n*) (*s_A n*)) (*rs*!*n*) + *1*

definition *f_A* :: *nat* ⇒ *int* **where**
f_A n = *min* (*mtf_A*!*n*) (*index* (*swaps* (*sw_A*!*n*) (*s_A n*)) (*rs*!*n*))

definition *p_A* :: *nat* ⇒ *int* **where**
p_A n = *size*(*sw_A*!*n*)

definition *t_A* :: *nat* ⇒ *int* **where**
t_A n = *c_A n* + *p_A n*

definition *T_A* :: *nat* ⇒ *int* **where**
T_A n = (∑ *i*<*n*. *t_A i*)

lemma *length_s_mtf*[*simp*]: *length*(*s_mtf n*) = *length s0*
⟨*proof*⟩

lemma *dist_s_mtf*[*simp*]: *distinct*(*s_mtf n*)
⟨*proof*⟩

lemma *set_s_mtf*[*simp*]: *set* (*s_mtf n*) = *set s0*
⟨*proof*⟩

lemma *dperm_inv*: *dist_perm* (*s_A n*) (*s_mtf n*)
⟨*proof*⟩

definition *Phi* :: *nat* ⇒ *int* (Φ) **where**
Phi n = *card*(*Inv* (*s_A n*) (*s_mtf n*))

lemma *phi0*: *Phi 0 = 0*
⟨*proof*⟩

lemma *phi_pos*: *Phi n ≥ 0*
⟨*proof*⟩

lemma *mtf_ub*: *t_mtf n + Phi (n+1) − Phi n ≤ 2 ∗ c_A n − 1 + p_A n − f_A n*
⟨*proof*⟩

theorem *Sleator_Tarjan*: *T_mtf n ≤ (∑ i<n. 2∗c_A i + p_A i − f_A i) − n*
⟨*proof*⟩

corollary *Sleator_Tarjan′*: *T_mtf n ≤ 2∗T_A n − n*
⟨*proof*⟩

lemma *T_A_nneg*: *0 ≤ T_A n*
⟨*proof*⟩

lemma *T_mtf_ub*: *∀ i<n. rs!i ∈ set s0 ⟹ T_mtf n ≤ n ∗ size s0*
⟨*proof*⟩

corollary *T_mtf_competitive*: **assumes** *s0 ≠ []* **and** *∀ i<n. rs!i ∈ set s0*
shows *T_mtf n ≤ (2 − 1/(size s0)) ∗ T_A n*
⟨*proof*⟩

lemma *t_A_t*: *n < length rs ⟹ t_A n = int (t (s_A n) (rs ! n) (as ! n))*
⟨*proof*⟩

lemma *T_A_eq_lem*: *(∑ i=0..<length rs. t_A i) =*
T (s_A 0) (drop 0 rs) (drop 0 as)
⟨*proof*⟩

lemma *T_A_eq*: *T_A (length rs) = T s0 rs as*
⟨*proof*⟩

lemma *nth_off_MTF*: *n < length rs ⟹ off2 MTF s rs ! n = (size(fst s) − 1,[])*
⟨*proof*⟩

lemma *t_mtf_MTF*: *n < length rs ⟹*
t_mtf n = int (t (s_mtf n) (rs ! n) (off MTF s rs ! n))

⟨*proof*⟩

lemma *mtf_MTF*: $n < length\ rs \Longrightarrow length\ s = length\ s0 \Longrightarrow mtf\ (rs\ !\ n)\ s =$
  step s (*rs* ! *n*) (*off MTF s0 rs* ! *n*)
⟨*proof*⟩

lemma *T_mtf_eq_lem*: $(\sum i{=}0..{<}length\ rs.\ t\_mtf\ i) =$
  T (*s_mtf 0*) (*drop 0 rs*) (*drop 0* (*off MTF s0 rs*))
⟨*proof*⟩

lemma *T_mtf_eq*: *T_mtf* (*length rs*) = *T_on MTF s0 rs*
⟨*proof*⟩

corollary *MTF_competitive2*: $s0 \neq [] \Longrightarrow \forall i{<}length\ rs.\ rs!i \in set\ s0 \Longrightarrow$
  T_on MTF s0 rs ≤ (*2* − *1*/(*size s0*)) ∗ *T s0 rs as*
⟨*proof*⟩

corollary *MTF_competitive′*: *T_on MTF s0 rs* ≤ *2* ∗ *T s0 rs as*
⟨*proof*⟩

end

theorem *compet_MTF*: **assumes** *s0* ≠ [] *distinct s0 set rs* ⊆ *set s0*
shows *T_on MTF s0 rs* ≤ (*2* − *1*/(*size s0*)) ∗ *T_opt s0 rs*
⟨*proof*⟩

theorem *compet_MTF′*: **assumes** *distinct s0*
shows *T_on MTF s0 rs* ≤ (*2*::*real*) ∗ *T_opt s0 rs*
⟨*proof*⟩

theorem *MTF_is_2_competitive*: *compet MTF 2* {*s* . *distinct s*}
⟨*proof*⟩

6.6 Lower Bound for Competitiveness

This result is independent of MTF but is based on the list update problem defined in this theory.

lemma *rat_fun_lem*:
  fixes *l c* :: *real*
  assumes [*simp*]: *F* ≠ *bot*
  assumes *0* < *l*
  assumes *ev*:
    eventually (λ*n*. *l* ≤ *f n* / *g n*) *F*

eventually ($\lambda n.$ $(f\ n + c)$ / $(g\ n + d) \leq u$) F
and
g: *LIM* n F. $g\ n$:> *at_top*
shows $l \leq u$
⟨*proof*⟩

lemma *compet_lb0*:
fixes *a Aon Aoff cruel*
defines *f s0 rs* == *real*(*T_on Aon s0 rs*)
defines *g s0 rs* == *real*(*T_off Aoff s0 rs*)
assumes $\bigwedge$*rs s0*. *size*(*Aoff s0 rs*) = *length rs* **and** $\bigwedge n$. *cruel n* ≠ []
assumes *compet Aon c S0* **and** *c*≥*0* **and** *s0* ∈ *S0*
and *l*: *eventually* ($\lambda n.$ *f s0* (*cruel n*) / (*g s0* (*cruel n*) + *a*) ≥ *l*) *sequentially*
and *g*: *LIM n sequentially*. *g s0* (*cruel n*) :> *at_top*
and *l* > *0* **and** $\bigwedge n$. *static s0* (*cruel n*)
shows *l* ≤ *c*
⟨*proof*⟩

Sorting

fun *ins_sws* **where**
ins_sws k x [] = [] |
ins_sws k x (*y*#*ys*) = (*if k x* ≤ *k y then* [] *else map Suc* (*ins_sws k x ys*) @ [*0*])

fun *sort_sws* **where**
sort_sws k [] = [] |
sort_sws k (*x*#*xs*) =
ins_sws k x (*sort_key k xs*) @ *map Suc* (*sort_sws k xs*)

lemma *length_ins_sws*: *length*(*ins_sws k x xs*) ≤ *length xs*
⟨*proof*⟩

lemma *length_sort_sws_le*: *length*(*sort_sws k xs*) ≤ *length xs* ^ *2*
⟨*proof*⟩

lemma *swaps_ins_sws*:
swaps (*ins_sws k x xs*) (*x*#*xs*) = *insort_key k x xs*
⟨*proof*⟩

lemma *swaps_sort_sws*[*simp*]:
swaps (*sort_sws k xs*) *xs* = *sort_key k xs*
⟨*proof*⟩

The cruel adversary:

fun *cruel* :: ('*a*,'*is*) *alg_on* ⇒ '*a state* ∗ '*is* ⇒ *nat* ⇒ '*a list* **where**
cruel A s 0 = [] |
cruel A s (*Suc n*) = *last* (*fst s*) # *cruel A* (*Step A s* (*last* (*fst s*))) *n*

definition *adv* :: ('*a*,'*is*) *alg_on* ⇒ ('*a*::*linorder*) *alg_off* **where**
adv A s rs = (*if rs*=[] *then* [] *else*
let crs = *cruel A* (*Step A* (*s*, *fst A s*) (*last s*)) (*size rs* − *1*)
in (*0*,*sort_sws* (λ*x*. *size rs* − *1* − *count_list crs x*) *s*) # *replicate* (*size rs* − *1*) (*0*,[]))

lemma *set_cruel*: *s* ≠ [] ⟹ *set*(*cruel A* (*s*,*is*) *n*) ⊆ *set s*
⟨*proof*⟩

lemma *static_cruel*: *s* ≠ [] ⟹ *static s* (*cruel A* (*s*,*is*) *n*)
⟨*proof*⟩

lemma *T_cruel*:
s ≠ [] ⟹ *distinct s* ⟹
T s (*cruel A* (*s*,*is*) *n*) (*off2 A* (*s*,*is*) (*cruel A* (*s*,*is*) *n*)) ≥ *n*∗(*length s*)
⟨*proof*⟩

lemma *length_cruel*[*simp*]: *length* (*cruel A s n*) = *n*
⟨*proof*⟩

lemma *t_sort_sws*: *t s r* (*mf*, *sort_sws k s*) ≤ *size s* ^ *2* + *size s* + *1*
⟨*proof*⟩

lemma *T_noop*:
n = *length rs* ⟹ *T s rs* (*replicate n* (*0*, [])) = (∑ *r*←*rs*. *index s r* + *1*)
⟨*proof*⟩

lemma *sorted_asc*: *j*≤*i* ⟹ *i*<*size ss* ⟹ ∀ *x* ∈ *set ss*. ∀ *y* ∈ *set ss*. *k*(*x*) ≤ *k*(*y*) ⟶ *f y* ≤ *f x*
⟹ *sorted* (*map k ss*) ⟹ *f* (*ss* ! *i*) ≤ *f* (*ss* ! *j*)
⟨*proof*⟩

lemma *sorted_weighted_gauss_Ico_div2*:
fixes *f* :: *nat* ⇒ *nat*
assumes ⋀*i j*. *i* ≤ *j* ⟹ *j* < *n* ⟹ *f i* ≥ *f j*
shows (∑ *i*=*0*..<*n*. (*i* + *1*) ∗ *f i*) ≤ (*n* + *1*) ∗ *sum f* {*0*..<*n*} *div 2*
⟨*proof*⟩

lemma *T_adv*: **assumes** $l \neq 0$
shows *T_off* (*adv A*) [*0*..<*l*] (*cruel A* ([*0*..<*l*],*fst A* [*0*..<*l*]) (*Suc n*))
$\leq l^2 + l + 1 + (l + 1) * n$ *div* 2 (**is** *?l* $\leq$ *?r*)
⟨*proof*⟩

The main theorem:

theorem *compet_lb2*:
assumes *compet A c* {*xs*::*nat list*. *size xs* = *l*} **and** $l \neq 0$ **and** $c \geq 0$
shows $c \geq 2*l/(l+1)$
⟨*proof*⟩

end

theory *Bit_Strings*
imports *Complex_Main*
begin

7 Lemmas about BitStrings and sets theirof

7.1 the set of bitstring of length m is finite

lemma *bitstrings_finite*: *finite* {*xs*::*bool list*. *length xs* = *m*}
⟨*proof*⟩

7.2 how to calculate the cardinality of the set of bitstrings with certain bits already set

lemma *fbool*: *finite* {*xs*. (∀ *i*∈*X*. *xs* ! *i*) ∧ (∀ *i*∈*Y*. ¬ *xs* ! *i*) ∧ *length xs* = *m* ∧ *f* (*xs*!*e*)}
⟨*proof*⟩

fun *witness* :: *nat set* ⇒ *nat* ⇒ *bool list* **where**
witness X 0 = []
|*witness X* (*Suc n*) = (*witness X n*) @ [*n* ∈ *X*]

lemma *witness_length*: *length* (*witness X n*) = *n*
⟨*proof*⟩

lemma *iswitness*: *r*<*n* ⟹ ((*witness X n*)!*r*) = (*r*∈*X*)
⟨*proof*⟩

lemma *card1*: *finite* $S \Longrightarrow$ *finite* $X \Longrightarrow$ *finite* $Y \Longrightarrow X \cap Y = \{\} \Longrightarrow S \cap (X \cup Y) = \{\} \Longrightarrow S \cup X \cup Y = \{0..<m\} \Longrightarrow$
card $\{xs.\ (\forall i \in X.\ xs\ !\ i) \wedge (\forall i \in Y.\ \neg\ xs\ !\ i)\ \wedge\ length\ xs = m\} = 2\hat{}(m - card\ X - card\ Y)$
⟨*proof*⟩

lemma *card2*: **assumes** *finite* X **and** *finite* Y **and** $X \cap Y = \{\}$ **and** x: $X \cup Y \subseteq \{0..<m\}$
shows *card* $\{xs.\ (\forall i \in X.\ xs\ !\ i) \wedge (\forall i \in Y.\ \neg\ xs\ !\ i) \wedge length\ xs = m\} = 2\ \hat{}\ (m - card\ X - card\ Y)$
⟨*proof*⟩

7.3 Average out the second sum for free-absch

lemma *Expactation2or1*: *finite* $S \Longrightarrow$ *finite* $Tr \Longrightarrow$ *finite* $Fa \Longrightarrow card\ Tr + card\ Fa + card\ S \leq l \Longrightarrow$
$S \cap (Tr \cup Fa) = \{\} \Longrightarrow Tr \cap Fa = \{\} \Longrightarrow S \cup Tr \cup Fa \subseteq \{0..<l\} \Longrightarrow$
$(\sum x \in \{xs.\ (\forall i \in Tr.\ xs\ !\ i) \wedge (\forall i \in Fa.\ \neg\ xs\ !\ i) \wedge length\ xs = l\}.\ \sum j \in S.\ if\ x\ !\ j\ then\ 2\ else\ 1)$
$= 3\ /\ 2 * real\ (card\ S) * 2\ \hat{}\ (l - card\ Tr - card\ Fa)$
⟨*proof*⟩

end

8 Effect of mtf2

theory *MTF2_Effects*
imports *Move_to_Front*
begin

lemma *difind_difelem*:
$i < length\ xs \Longrightarrow distinct\ xs \Longrightarrow xs\ !\ j = a \Longrightarrow j < length\ xs \Longrightarrow i \neq j$
$\Longrightarrow\ ^{\sim}\ a = xs\ !\ i$
⟨*proof*⟩

lemma *fullchar*: **assumes** $index\ xs\ q < length\ xs$
shows
$(i < length\ xs) =$
$(index\ xs\ q < i \wedge i < length\ xs$
$\vee\ index\ xs\ q = i$

$\vee$ *index xs q* $-$ *n* $\leq$ *i* $\wedge$ *i* $<$ *index xs q*
$\vee$ *i* $<$ *index xs q* $-$ *n*)
⟨*proof*⟩

lemma *mtf2_effect*:
q $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ (*index xs q* $<$ *i* $\wedge$ *i* $<$ *length xs* $\longrightarrow$(*index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs* (*xs*!*i*) $\wedge$ *index xs q* $<$ *index* (*mtf2 n q xs*) (*xs*!*i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) $<$ *length xs*))
$\wedge$ (*index xs q* = *i* $\longrightarrow$ (*index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs q* $-$ *n* $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs q* $-$ *n*))
$\wedge$ (*index xs q* $-$ *n* $\leq$ *i* $\wedge$ *i* $<$ *index xs q* $\longrightarrow$ (*index* (*mtf2 n q xs*) (*xs*!*i*) = *Suc* (*index xs* (*xs*!*i*)) $\wedge$ *index xs q* $-$ *n* $<$ *index* (*mtf2 n q xs*) (*xs*!*i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) $\leq$ *index xs q*))
$\wedge$ (*i* $<$ *index xs q* $-$ *n* $\longrightarrow$ (*index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs* (*xs*!*i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) $<$ *index xs q* $-$ *n*))
⟨*proof*⟩

lemma *mtf2_forward_effect1*:
q $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ *index xs q* $<$ *i* $\wedge$ *i* $<$ *length xs*
$\Longrightarrow$ *index* (*mtf2 n q xs*) (*xs* ! *i*) = *index xs* (*xs* ! *i*) $\wedge$ *index xs q* $<$ *index* (*mtf2 n q xs*) (*xs* ! *i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs* ! *i*) $<$ *length xs* **and**

mtf2_forward_effect2: *q* $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ *index xs q* = *i*
$\Longrightarrow$ *index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs q* $-$ *n* $\wedge$ *index xs q* $-$ *n* = *index* (*mtf2 n q xs*) (*xs*!*i*) **and**
mtf2_forward_effect3: *q* $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ *index xs q* $-$ *n* $\leq$ *i* $\wedge$ *i* $<$ *index xs q*
$\Longrightarrow$ *index* (*mtf2 n q xs*) (*xs*!*i*) = *Suc* (*index xs* (*xs*!*i*)) $\wedge$ *index xs q* $-$ *n* $<$ *index* (*mtf2 n q xs*) (*xs*!*i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) $\leq$ *index xs q* **and**

mtf2_forward_effect4: *q* $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ *i* $<$ *index xs q* $-$ *n*
$\Longrightarrow$ *index* (*mtf2 n q xs*) (*xs*!*i*) = *index xs* (*xs*!*i*) $\wedge$ *index* (*mtf2 n q xs*) (*xs*!*i*) $<$ *index xs q* $-$ *n*
⟨*proof*⟩

lemma *yes*[*simp*]: *index xs x* $<$ *length xs*
$\Longrightarrow$ (*xs*!*index xs x*) = *x* ⟨*proof*⟩

lemma *mtf2_forward_effect1′*:
q $\in$ *set xs* $\Longrightarrow$ *distinct xs* $\Longrightarrow$ *index xs q* $<$ *index xs x* $\wedge$ *index xs x* $<$ *length xs*
$\Longrightarrow$ *index* (*mtf2 n q xs*) *x* = *index xs x* $\wedge$ *index xs q* $<$ *index* (*mtf2 n q xs*) *x* $\wedge$ *index* (*mtf2 n q xs*) *x* $<$ *length xs*
⟨*proof*⟩

lemma
mtf2_forward_effect2': $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow index\ xs\ q = index\ xs\ x$
$\Longrightarrow index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) = index\ xs\ q - n \wedge index\ xs\ q - n = index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x)$
⟨*proof*⟩

lemma
mtf2_forward_effect3': $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow index\ xs\ q - n \leq index\ xs\ x \Longrightarrow index\ xs\ x < index\ xs\ q$
$\Longrightarrow index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) = Suc\ (index\ xs\ (xs!index\ xs\ x)) \wedge index\ xs\ q - n < index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) \wedge index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) \leq index\ xs\ q$
⟨*proof*⟩

lemma
mtf2_forward_effect4': $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow index\ xs\ x < index\ xs\ q - n$
$\Longrightarrow index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) = index\ xs\ (xs!index\ xs\ x) \wedge index\ (mtf2\ n\ q\ xs)\ (xs!index\ xs\ x) < index\ xs\ q - n$
⟨*proof*⟩

lemma *splitit*: $(index\ xs\ q < i \wedge i < length\ xs \Longrightarrow P)$
$\Longrightarrow (index\ xs\ q = i \Longrightarrow P)$
$\Longrightarrow (index\ xs\ q - n \leq i \wedge i < index\ xs\ q \Longrightarrow P)$
$\Longrightarrow (i < index\ xs\ q - n \Longrightarrow P)$
$\Longrightarrow (i < length\ xs \Longrightarrow P)$
⟨*proof*⟩

lemma *mtf2_forward_beforeq*: $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow i < index\ xs\ q$
$\Longrightarrow index\ (mtf2\ n\ q\ xs)\ (xs!i) \leq index\ xs\ q$
⟨*proof*⟩

lemma *x_stays_before_y_if_y_not_moved_to_front*:
assumes $q \in set\ xs\ distinct\ xs\ x \in set\ xs\ y \in set\ xs\ y \neq q$
and $x < y\ in\ xs$
shows $x < y\ in\ (mtf2\ n\ q\ xs)$
⟨*proof*⟩

corollary *swapped_by_mtf2*: $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow$
$x < y\ in\ xs \Longrightarrow y < x\ in\ (mtf2\ n\ q\ xs) \Longrightarrow y = q$
⟨*proof*⟩

lemma *x_stays_before_y_if_y_not_moved_to_front_2dir*: $q \in set\ xs \Longrightarrow distinct\ xs \Longrightarrow x \in set\ xs \Longrightarrow\ y \in set\ xs \Longrightarrow y \neq q \Longrightarrow$
$x < y\ in\ xs = x < y\ in\ (mtf2\ n\ q\ xs)$
⟨*proof*⟩

lemma *mtf2_backwards_effect1*:
assumes $index\ xs\ q < length\ xs\ q \in set\ xs\ distinct\ xs$
$index\ xs\ q < index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) \wedge index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) < length\ xs$
$i < length\ xs$
shows $index\ xs\ q < i \wedge i < length\ xs$
⟨*proof*⟩

lemma *mtf2_backwards_effect2*:
assumes $index\ xs\ q < length\ xs\ q \in set\ xs\ distinct\ xs\ index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) = index\ xs\ q - n$
$i < length\ xs$
shows $index\ xs\ q = i$
⟨*proof*⟩

lemma *mtf2_backwards_effect3*:
assumes $index\ xs\ q < length\ xs\ q \in set\ xs\ distinct\ xs$
$index\ xs\ q - n < index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) \wedge index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) \leq index\ xs\ q$
$i < length\ xs$
shows $index\ xs\ q - n \leq i \wedge i < index\ xs\ q$
⟨*proof*⟩

lemma *mtf2_backwards_effect4*:
assumes $index\ xs\ q < length\ xs\ q \in set\ xs\ distinct\ xs$
$index\ (mtf2\ n\ q\ xs)\ (xs\ !\ i) < index\ xs\ q - n$
$i < length\ xs$
shows $i < index\ xs\ q - n$
⟨*proof*⟩

lemma *mtf2_backwards_effect4'*:
assumes $index\ xs\ q < length\ xs\ q \in set\ xs\ distinct\ xs$

index (mtf2 n q xs) x < index xs q − n
x ∈ set xs
shows *(index xs x) < index xs q − n*
⟨*proof*⟩

lemma
assumes *distA*: *distinct A* **and**
asm: *q ∈ set A*
shows
mtf2_mono: *q< x in A ⟹ q < x in (mtf2 n q A)* **and**
mtf2_q_after: *index (mtf2 n q A) q = index A q − n*
⟨*proof*⟩

8.1 effect of mtf2 on index

lemma *swapsthrough*: *distinct xs ⟹ q ∈ set xs ⟹ index (swaps [index xs q − entf..<index xs q] xs) q = index xs q − entf*
⟨*proof*⟩

term *mtf2*
lemma *mtf2_moves_to_front*: *distinct xs ⟹ q ∈ set xs ⟹ index (mtf2 (length xs) q xs) q = 0*
⟨*proof*⟩

lemma *xy_relativorder_mtf2*:
assumes
q≠x q≠y distinct xs x∈set xs y∈set xs q∈set xs
shows *x < y in mtf2 n q xs*
= x < y in xs
⟨*proof*⟩

lemma *mtf2_moves_to_frontm1*: *distinct xs ⟹ q ∈ set xs ⟹ index (mtf2 (length xs −1) q xs) q = 0*
⟨*proof*⟩

lemma *mtf2_moves_to_front'*: *distinct xs ⟹ y ∈ set xs ⟹ x ∈ set xs ⟹ x≠y ⟹ x < y in mtf2 (length xs−1) x xs = True*
⟨*proof*⟩

lemma *mtf2_moves_to_front''*: *distinct xs* $\Longrightarrow$ *y* $\in$ *set xs* $\Longrightarrow$ *x* $\in$ *set xs* $\Longrightarrow$ *x*$\neq$*y* $\Longrightarrow$ *x* $<$ *y in mtf2* (*length xs*) *x xs* = *True*
⟨*proof*⟩

end

9 BIT: an Online Algorithm for the List Update Problem

theory *BIT*
imports
Bit_Strings
MTF2_Effects
begin

abbreviation *config''* *A qs init n* == *config_rand A init* (*take n qs*)

lemma *sum_my*: **fixes** *f g*::*'b* $\Rightarrow$ *'a*::*ab_group_add*
assumes *finite A finite B*
shows ($\sum$ *x*$\in$*A*. *f x*) − ($\sum$ *x*$\in$*B*. *g x*)
= ($\sum$ *x*$\in$(*A* ∩ *B*). *f x* − *g x*) + ($\sum$ *x*$\in$*A*−*B*. *f x*) − ($\sum$ *x*$\in$*B*−*A*. *g x*)
⟨*proof*⟩

lemma *sum_my2*: ($\forall$ *x*$\in$*A*. *f x* = *g x*) $\Longrightarrow$ ($\sum$ *x*$\in$*A*. *f x*) = ($\sum$ *x*$\in$*A*. *g x*)
⟨*proof*⟩

9.1 Definition of BIT

definition *BIT_init* :: (*'a state*,*bool list* ∗ *'a list*)*alg_on_init* **where**
BIT_init init = *map_pmf* (λ*l*. (*l*,*init*)) (*bv* (*length init*))

lemma ~ *deterministic_init BIT_init*
⟨*proof*⟩

definition *BIT_step* :: (*'a state*, *bool list* ∗ *'a list*, *'a*, *answer*)*alg_on_step*
where

BIT_step s q = (let a=((if (fst (snd s))!(index (snd (snd s)) q) then 0 else (length (fst s))),[]) in
return_pmf (a , (flip (index (snd (snd s)) q) (fst (snd s)), snd (snd s))))

lemma *deterministic_step BIT_step*
⟨*proof*⟩

abbreviation *BIT* :: (*'a state*, *bool list*∗*'a list*, *'a*, *answer*)*alg_on_rand* **where**
BIT == (*BIT_init*, *BIT_step*)

9.2 Properties of BIT's state distribution

lemma *BIT_no_paid*: ∀((*free*,*paid*),_) ∈ (*BIT_step s q*). *paid*=[]
⟨*proof*⟩

9.2.1 About the Internal State

term (*config'_rand* (*BIT_init*, *BIT_step*) *s0 qs*)
lemma *config'_n_init*: **fixes** *qs init n*
shows *map_pmf* (*snd* ∘ *snd*) (*config'_rand* (*BIT_init*, *BIT_step*) *init qs*) = *map_pmf* (*snd* ∘ *snd*) *init*
⟨*proof*⟩

lemma *config_n_init*: *map_pmf* (*snd* ∘ *snd*) (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*) = *return_pmf s0*
⟨*proof*⟩

lemma *config_n_init2*: ∀(_,(_,*x*)) ∈ *set_pmf* (*config_rand* (*BIT_init*, *BIT_step*) *init qs*). *x* = *init*
⟨*proof*⟩
lemma *config_n_init3*: ∀*x* ∈ *set_pmf* (*config_rand* (*BIT_init*, *BIT_step*) *init qs*). *snd* (*snd x*) = *init*
⟨*proof*⟩

lemma *config'_n_bv*: **fixes** *qs init n*
shows *map_pmf* (*snd* ∘ *snd*) *init* = *return_pmf s0*
⟹ *map_pmf* (*fst* ∘ *snd*) *init* = *bv* (*length s0*)

$\Longrightarrow$ *map_pmf* (*snd* $\circ$ *snd*) (*config'_rand* (*BIT_init*, *BIT_step*) *init qs*) = *return_pmf s0*
$\wedge$ *map_pmf* (*fst* $\circ$ *snd*) (*config'_rand* (*BIT_init*, *BIT_step*) *init qs*) = *bv* (*length s0*)
⟨*proof*⟩

lemma *config_n_bv_2*: *map_pmf* (*snd* $\circ$ *snd*) (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*) = *return_pmf s0*
$\wedge$ *map_pmf* (*fst* $\circ$ *snd*) (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*) = *bv* (*length s0*)
⟨*proof*⟩

lemma *config_n_bv*: *map_pmf* (*fst* $\circ$ *snd*) (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*) = *bv* (*length s0*)
⟨*proof*⟩

lemma *config_n_fst_init_length*: $\forall$ (_,(*x*,_)) $\in$ *set_pmf* (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*). *length x* = *length s0*
⟨*proof*⟩

lemma *config_n_fst_init_length2*: $\forall$ *x* $\in$ *set_pmf* (*config_rand* (*BIT_init*, *BIT_step*) *s0 qs*). *length* (*fst* (*snd x*)) = *length s0*
⟨*proof*⟩

lemma *fperms*: *finite* {*x*::'*a list*. *length x* = *length init* $\wedge$ *distinct x* $\wedge$ *set x* = *set init*}
⟨*proof*⟩

lemma *finite_config_BIT*: **assumes** [*simp*]: *distinct init*
shows *finite* (*set_pmf* (*config_rand* (*BIT_init*, *BIT_step*) *init qs*)) (**is** *finite ?D*)
⟨*proof*⟩

9.3 BIT is 1.75-competitive (a combinatorial proof)

9.3.1 Definition of the Locale and Helper Functions

locale *BIT_Off* =

fixes *acts* :: *answer list*
fixes *qs* :: *'a list*
fixes *init* :: *'a list*
assumes *dist_init*[*simp*]: *distinct init*
assumes *len_acts*: *length acts* = *length qs*
begin

lemma *setinit*: (*index init*) ‘ *set init* = {*0*..<*length init*}
⟨*proof*⟩

definition *free_A* :: *nat list* **where**
free_A = *map fst acts*

definition *paid_A'* :: *nat list list* **where**
paid_A' = *map snd acts*

definition *paid_A* :: *nat list list* **where**
paid_A = *map* (*filter* (λ*x*. *Suc x* < *length init*)) *paid_A'*

lemma *len_paid_A*[*simp*]: *length paid_A* = *length qs*
⟨*proof*⟩
lemma *len_paid_A'*[*simp*]: *length paid_A'* = *length qs*
⟨*proof*⟩

lemma *paidAnm_inbound*: *n* < *length paid_A* ⟹ *m* < *length*(*paid_A*!*n*)
⟹ (*Suc* ((*paid_A*!*n*)!(*length* (*paid_A* ! *n*) − *Suc m*))) < *length init*
⟨*proof*⟩

fun *s_A'* :: *nat* ⇒ *'a list* **where**
s_A' 0 = *init* |
s_A'(*Suc n*) = *step* (*s_A' n*) (*qs*!*n*) (*free_A*!*n*, *paid_A'*!*n*)

lemma *length_s_A'*[*simp*]: *length*(*s_A' n*) = *length init*
⟨*proof*⟩

lemma *dist_s_A'*[*simp*]: *distinct*(*s_A' n*)
⟨*proof*⟩

lemma *set_s_A'*[*simp*]: *set*(*s_A' n*) = *set init*
⟨*proof*⟩

fun *s_A* :: *nat* ⇒ *'a list* **where**

$s\_A\ 0 = init\ |$
$s\_A(Suc\ n) = step\ (s\_A\ n)\ (qs!n)\ (free\_A!n,\ paid\_A!n)$

lemma *length_s_A*[*simp*]: $length(s\_A\ n) = length\ init$
⟨*proof*⟩

lemma *dist_s_A*[*simp*]: $distinct(s\_A\ n)$
⟨*proof*⟩

lemma *set_s_A*[*simp*]: $set(s\_A\ n) = set\ init$
⟨*proof*⟩

lemma *cost_paidAA'*: $n < length\ paid\_A' \Longrightarrow length\ (paid\_A!n) \leq length\ (paid\_A'!n)$
⟨*proof*⟩

lemma *swaps_filtered*: $swaps\ (filter\ (\lambda x.\ Suc\ x < length\ xs)\ ys)\ xs = swaps\ (ys)\ xs$
⟨*proof*⟩

lemma *sAsA'*: $n < length\ paid\_A' \Longrightarrow s\_A'\ n = s\_A\ n$
⟨*proof*⟩

lemma *sAsA''*: $n < length\ qs \Longrightarrow s\_A\ n = \ s\_A'\ n$
⟨*proof*⟩

definition $t\_BIT :: nat \Rightarrow real$ **where**
$t\_BIT\ n = T\_on\_rand\_n\ BIT\ init\ qs\ n$

definition $T\_BIT :: nat \Rightarrow real$ **where**
$T\_BIT\ n = (\sum i{<}n.\ t\_BIT\ i)$

definition $c\_A :: nat \Rightarrow int$ **where**
$c\_A\ n = index\ (swaps\ (paid\_A!n)\ (s\_A\ n))\ (qs!n) + 1$

definition $f\_A :: nat \Rightarrow int$ **where**
$f\_A\ n = min\ (free\_A!n)\ (index\ (swaps\ (paid\_A!n)\ (s\_A\ n))\ (qs!n))$

definition $p\_A :: nat \Rightarrow int$ **where**
$p\_A\ n = size(paid\_A!n)$

definition $t\_A :: nat \Rightarrow int$ **where**
$t\_A\ n = c\_A\ n + p\_A\ n$

definition $c\_A' :: nat \Rightarrow int$ **where**
$c\_A'\ n = index\ (swaps\ (paid\_A'!n)\ (s\_A'\ n))\ (qs!n) + 1$

definition $p\_A' :: nat \Rightarrow int$ **where**
$p\_A'\ n = size(paid\_A'!n)$
definition $t\_A' :: nat \Rightarrow int$ **where**
$t\_A'\ n = c\_A'\ n + p\_A'\ n$

lemma $t\_A\_A'\_leq$: $n < length\ paid\_A' \Longrightarrow t\_A\ n \leq t\_A'\ n$
⟨*proof*⟩

definition $T\_A' :: nat \Rightarrow int$ **where**
$T\_A'\ n = (\sum i{<}n.\ t\_A'\ i)$

definition $T\_A :: nat \Rightarrow int$ **where**
$T\_A\ n = (\sum i{<}n.\ t\_A\ i)$

lemma $T\_A\_A'\_leq$: $n \leq length\ paid\_A' \Longrightarrow T\_A\ n \leq T\_A'\ n$
⟨*proof*⟩

lemma $T\_A\_A'\_leq'$: $n \leq length\ qs \Longrightarrow T\_A\ n \leq T\_A'\ n$
⟨*proof*⟩

fun $s'\_A :: nat \Rightarrow nat \Rightarrow 'a\ list$ **where**
$s'\_A\ n\ 0 = s\_A\ n$
$|\ (s'\_A\ n\ (Suc\ m)) = swap\ ((paid\_A\ !\ n)!(length\ (paid\_A\ !\ n) - (Suc\ m))$
$)\ (s'\_A\ n\ m)$

lemma $set\_s'\_A[simp]$: $set\ (s'\_A\ n\ m) = set\ init$
⟨*proof*⟩

lemma $len\_s'\_A[simp]$: $length\ (s'\_A\ n\ m) = length\ init$
⟨*proof*⟩

lemma $distperm\_s'\_A[simp]$: $dist\_perm\ (s'\_A\ n\ m)\ init$
⟨*proof*⟩

lemma $s'A\_m\_le$: $m \leq (length\ (paid\_A\ !\ n)) \Longrightarrow swaps\ (drop\ (length$

$(paid\_A\ !\ n) - m)\ (paid\_A\ !\ n))\ (s\_A\ n) = s'\_A\ n\ m$
⟨*proof*⟩

lemma *s'A_m*: *swaps* (*paid_A* ! *n*) (*s_A n*) = *s'_A n* (*length* (*paid_A* ! *n*))
⟨*proof*⟩

definition *gebub* :: *nat* ⇒ *nat* ⇒ *nat* **where**
gebub n m = *index init* ((*s'_A n m*)!(*Suc* ((*paid_A*!*n*)!(*length* (*paid_A* ! *n*) − *Suc m*))))

lemma *gebub_inBound*: **assumes** *1*: *n* < *length paid_A* **and** *2*: *m* < *length* (*paid_A* ! *n*)
shows *gebub n m* < *length init*
⟨*proof*⟩

9.3.2 The Potential Function

fun *phi* :: *nat* ⇒'*a list*× (*bool list* × '*a list*) ⇒ *real* (*φ*) **where**
phi n (*c*,(*b*,_)) = (∑(*x*,*y*)∈(*Inv c* (*s_A n*)). (*if b*!(*index init y*) *then 2 else 1*))

lemma *phi'*: *phi n z* = (∑(*x*,*y*)∈(*Inv* (*fst z*) (*s_A n*)). (*if* (*fst* (*snd z*))!(*index init y*) *then 2 else 1*))
⟨*proof*⟩

lemma *Inv_empty2*: *length d* = *0* ⟹ *Inv c d* = {}
⟨*proof*⟩

corollary *Inv_empty3*: *length init* = *0* ⟹ *Inv c* (*s_A n*) = {}
⟨*proof*⟩

lemma *phi_empty2*: *length init* = *0* ⟹ *phi n* (*c*,(*b*,*i*)) = *0*
⟨*proof*⟩

lemma *phi_nonzero*: *phi n* (*c*,(*b*,*i*)) ≥ *0*
⟨*proof*⟩

definition *Phi* :: *nat* ⇒ *real* (Φ) **where**
Phi n = *E*(*map_pmf* (*φ n*) (*config'' BIT qs init n*))

definition *PhiPlus* :: *nat* ⇒ *real* (Φ^+) **where**

PhiPlus n = (let
nextconfig = bind_pmf (config'' BIT qs init n)
(λ(s,is). bind_pmf (BIT_step (s,is) (qs!n)) (λ(a,nis). return_pmf
(step s (qs!n) a,nis)))
in
E(map_pmf (phi (Suc n)) nextconfig))

lemma *PhiPlus_is_Phi_Suc*: *n<length qs ⟹ PhiPlus n = Phi (Suc n)*
⟨*proof*⟩

lemma *phi0*: *Phi 0 = 0* ⟨*proof*⟩

lemma *phi_pos*: *Phi n ≥ 0*
⟨*proof*⟩

9.3.3 Helper lemmas

lemma *swap_subs*: *dist_perm X Y ⟹ Inv X (swap z Y) ⊆ Inv X Y ∪*
{(Y ! z, Y ! Suc z)}
⟨*proof*⟩

9.3.4 InvOf

term *Inv*
abbreviation *InvOf y bits as ≡ {(x,y)|x. x < y in bits ∧ y < x in as}*

lemma *InvOf y xs ys = {(x,y)|x. (x,y)∈Inv xs ys}*
⟨*proof*⟩

lemma *InvOf y xs ys ⊆ Inv xs ys* ⟨*proof*⟩

lemma *numberofIsbeschr*: **assumes**
distxsys: *dist_perm xs ys* **and**
yinxs: *y ∈ set xs*
shows *index xs y ≤ index ys y + card (InvOf y xs ys)*
(**is** *?iBit ≤ ?iA + card ?I*)
⟨*proof*⟩

lemma *length init = 0 ⟹ length xs = length init ⟹ t xs q (mf, sws) =*
1 + length sws
⟨*proof*⟩

lemma *integr_index*: *integrable* (*measure_pmf* (*config''* (*BIT_init*, *BIT_step*) *qs init n*))
(λ(*s*, *is*). *real* (*Suc* (*index s* (*qs* ! *n*))))
⟨*proof*⟩

9.3.5 Upper Bound on the Cost of BIT

lemma *t_BIT_ub2*: (*qs*!*n*) ∉ *set init* ⟹ *t_BIT n* ≤ *Suc*(*size init*)
⟨*proof*⟩

lemma *t_BIT_ub*: (*qs*!*n*) ∈ *set init* ⟹ *t_BIT n* ≤ *size init*
⟨*proof*⟩

lemma *T_BIT_ub*: ∀ *i*<*n*. *qs*!*i* ∈ *set init* ⟹ *T_BIT n* ≤ *n* ∗ *size init*
⟨*proof*⟩

9.3.6 Main Lemma

lemma *myub*: *n* < *length qs* ⟹ *t_BIT n* + *Phi*(*n* + *1*) − *Phi n* ≤ (*7* / *4*) ∗ *t_A n* − *3*/*4*
⟨*proof*⟩

9.3.7 Lift the Result to the Whole Request List

lemma *T_BIT_absch_le*: **assumes** *nqs*: *n* ≤ *length qs*
shows *T_BIT n* ≤ (*7* / *4*) ∗ *T_A n* − *3*/*4*∗*n*
⟨*proof*⟩

lemma *T_BIT_absch*: **assumes** *nqs*: *n* ≤ *length qs*
shows *T_BIT n* ≤ (*7* / *4*) ∗ *T_A'* *n* − *3*/*4*∗*n*
⟨*proof*⟩

lemma *T_A_nneg*: *0* ≤ *T_A n*
⟨*proof*⟩

lemma *T_BIT_eq*: *T_BIT* (*length qs*) = *T_on_rand BIT init qs*
⟨*proof*⟩

corollary *T_BIT_competitive*: **assumes** *n* ≤ *length qs* **and** *init* ≠ [] **and** ∀ *i*<*n*. *qs*!*i* ∈ *set init*

shows *T_BIT n* ≤ ((*7* / *4*) − *3*/(*4* ∗ *size init*)) ∗ *T_A' n*
⟨*proof*⟩

lemma *t_A'_t*: *n* < *length qs* ⟹ *t_A' n* = *int* (*t* (*s_A' n*) (*qs*!*n*) (*acts* ! *n*))
⟨*proof*⟩

lemma *T_A'_eq_lem*: (∑ *i*=*0*..<*length qs*. *t_A' i*) =
T (*s_A' 0*) (*drop 0 qs*) (*drop 0 acts*)
⟨*proof*⟩

lemma *T_A'_eq*: *T_A'* (*length qs*) = *T init qs acts*
⟨*proof*⟩

corollary *BIT_competitive3*: *init* ≠ [] ⟹ ∀ *i*<*length qs*. *qs*!*i* ∈ *set init* ⟹
T_BIT (*length qs*) ≤ ((*7*/*4*) − *3* / (*4* ∗ *length init*)) ∗ *T init qs acts*
⟨*proof*⟩

corollary *BIT_competitive2*: *init* ≠ [] ⟹ ∀ *i*<*length qs*. *qs*!*i* ∈ *set init* ⟹
T_on_rand BIT init qs ≤ ((*7*/*4*) − *3* / (*4* ∗ *length init*)) ∗ *T init qs acts*
⟨*proof*⟩

corollary *BIT_absch_le*: *init* ≠ [] ⟹
T_on_rand BIT init qs ≤ (*7* / *4*) ∗ (*T init qs acts*) − *3*/*4* ∗ *length qs*
⟨*proof*⟩

end

9.3.8 Generalize Competitivness of BIT

lemma *setdi*: *set xs* = {*0*..<*length xs*} ⟹ *distinct xs*
⟨*proof*⟩

theorem *compet_BIT*: **assumes** *init* ≠ [] *distinct init set qs* ⊆ *set init*
shows *T_on_rand BIT init qs* ≤ ((*7*/*4*) − *3* / (*4* ∗ *length init*)) ∗ *T_opt init qs*
⟨*proof*⟩

theorem *compet_BIT4*: **assumes** *init* ≠ [] *distinct init*

shows *T_on_rand BIT init qs* $\leq$ *7/4* $*$ *T_opt init qs*
⟨*proof*⟩

theorem *compet_BIT_2*:
compet_rand BIT (7/4) {init. init $\neq$ [] $\wedge$ *distinct init}*
⟨*proof*⟩

end

10 Partial cost model

theory *Partial_Cost_Model*
imports *Move_to_Front*
begin

definition t_p :: *'a state* $\Rightarrow$ *'a* $\Rightarrow$ *answer* $\Rightarrow$ *nat* **where**
t_p *s q a* = (*let* (*mf*,*sws*) = *a in index* (*swaps sws s*) *q* + *size sws*)

notation (*latex*) t_p (t^*)

lemma $t_p t$: t_p *s q a* + *1* = *t s q a* ⟨*proof*⟩

interpretation *On_Off step* t_p *static* ⟨*proof*⟩

abbreviation T_p == *T*
abbreviation T_p*_opt* == *T_opt*
abbreviation T_p*_on* == *T_on*
abbreviation T_p*_on_rand'* == *T_on_rand'*
abbreviation T_p*_on_n* == *T_on_n*
abbreviation T_p*_on_rand* == *T_on_rand*
abbreviation T_p*_on_rand_n* == *T_on_rand_n*
abbreviation $config_p$ == *config*
abbreviation $compet_p$ == *compet*

end

11 Equivalence of Regular Expression with Variables

theory *RExp_Var*
imports *Regular−Sets.Equivalence_Checking*

begin

fun *castdown* :: *nat rexp* ⇒ *nat rexp* **where**
  castdown Zero = *Zero*
| *castdown One* = *One*
| *castdown* (*Plus a b*) = *Plus* (*castdown a*) (*castdown b*)
| *castdown* (*Times a b*) = *Times* (*castdown a*) (*castdown b*)
| *castdown* (*Star a*) = *Star* (*castdown a*)
| *castdown* (*Atom x*) = (*Atom* (*x div 2*))

fun *castup* :: *nat rexp* ⇒ *nat rexp* **where**
  castup Zero = *Zero*
| *castup One* = *One*
| *castup* (*Plus a b*) = *Plus* (*castup a*) (*castup b*)
| *castup* (*Times a b*) = *Times* (*castup a*) (*castup b*)
| *castup* (*Star a*) = *Star* (*castup a*)
| *castup* (*Atom x*) = *Atom* (*2*∗*x*)

lemma *castdown* (*castup r*) = *r*
⟨*proof*⟩

fun *substvar* :: *nat* ⇒ (*nat* ⇒ ((*nat rexp*) *option*)) ⇒ *nat rexp* **where**
  substvar i σ = (*case σ i of Some x* ⇒ *x*
                          | *None* ⇒ *Atom* (*2*∗*i*+*1*))

fun *w2rexp* :: *nat list* ⇒ *nat rexp* **where**
  w2rexp [] = *One*
| *w2rexp* (*a*#*as*) = *Times* (*Atom a*) (*w2rexp as*)

lemma *lang* (*w2rexp as*) = { *as* }
⟨*proof*⟩

fun *subst* :: *nat rexp* ⇒ (*nat* ⇒ *nat rexp option*) ⇒ *nat rexp* **where**
  subst Zero _ = *Zero*
| *subst One* _ = *One*
| *subst* (*Atom i*) *σ* = (*if i mod 2* = *0 then Atom i else substvar* (*i div 2*) *σ*)
| *subst* (*Plus a b*) *σ* = *Plus* (*subst a σ*) (*subst b σ*)
| *subst* (*Times a b*) *σ* = *Times* (*subst a σ*) (*subst b σ*)
| *subst* (*Star a*) *σ* = *Star* (*subst a σ*)

lemma *subst_w2rexp*: *lang* (*subst* (*w2rexp* (*xs* @ *ys*)) σ) = *lang* (*subst* (*w2rexp xs*) σ) @@ *lang* (*subst* (*w2rexp ys*) σ)
⟨*proof*⟩

fun *substW* :: *nat list* ⇒ (*nat* ⇒ *nat rexp option*) ⇒ *nat rexp* **where**
  substW as σ = *subst* (*w2rexp as*) σ

fun *substL* :: *nat lang* ⇒ (*nat* ⇒ *nat rexp option*) ⇒ *nat rexp set* **where**
  substL S σ = {*substW a* σ|*a*. *a* ∈ *S*}

fun *L* :: *nat rexp set* ⇒ *nat lang* **where**
  L S = (⋃ *r*∈*S*. *lang r*)

lemma *L_mono*: *S1* ⊆ *S2* ⟹ *L S1* ⊆ *L S2*
⟨*proof*⟩

definition *concS* :: *'b rexp set* ⇒ *'b rexp set* ⇒ *'b rexp set* **where**
  concS S1 S2 = {*Times a b*|*a b*. *a*∈*S1* ∧ *b*∈*S2*}

lemma *substL_conc*: *L* (*substL* (*L1* @@ *L2*) σ) = *L* (*concS* (*substL L1* σ) (*substL L2* σ))
⟨*proof*⟩

lemma *L_conc*: *L*(*concS M1 M2*) = (*L M1*) @@ (*L M2*)
⟨*proof*⟩

lemma *L*(*M1* ∪ *M2*) = (*L M1*) ∪ (*L M2*)
⟨*proof*⟩

fun *verund* :: *'b rexp list* ⇒ *'b rexp* **where**
  verund [] = *Zero*
| *verund* [*r*] = *r*
| *verund* (*r*#*rs*) = *Plus r* (*verund rs*)

lemma *lang_verund*: *r* ∈ *L* (*set rs*) = (*r* ∈ *lang* (*verund rs*))
⟨*proof*⟩

lemma *obtainit*:
  assumes *r* ∈ *lang* (*verund rs*)
  shows ∃ *x*∈(*set* (*rs*::*nat rexp list*)). *r* ∈ *lang x*
⟨*proof*⟩

lemma *lang_verund4*: $L\ (set\ rs) = lang\ (verund\ rs)$
⟨*proof*⟩

lemma *lang_verund1*: $r \in L\ (set\ rs) \Longrightarrow r \in lang\ (verund\ rs)$
⟨*proof*⟩
lemma *lang_verund2*: $r \in lang\ (verund\ rs) \Longrightarrow r \in L\ (set\ rs)$
⟨*proof*⟩

definition *starS* :: *'b rexp set* ⇒ *'b rexp set* **where**
starS S = {*Star* (*verund xs*)|*xs*. *set xs* ⊆ *S*}

lemma [] ∈ *L* (*starS S*)
⟨*proof*⟩

lemma *power_mono*: *L1* ⊆ *L2* ⟹ (*L1*::*'a lang*) ^^ *n* ⊆ *L2* ^^ *n*
⟨*proof*⟩

lemma *star_mono*: *L1* ⊆ *L2* ⟹ *star L1* ⊆ *star L2*
⟨*proof*⟩

lemma *Lstar*: *L*(*starS M*) = *star* (*L*(*M*))
⟨*proof*⟩

lemma *substL_star*: *L* (*substL* (*star L1*) *σ*) = *L* (*starS* (*substL L1 σ*))
⟨*proof*⟩

lemma *substituitionslemma*:
fixes *E* :: *nat rexp*
shows *L* (*substL* (*lang*(*E*)) *σ*) = *lang* (*subst E σ*)
⟨*proof*⟩

corollary *lift*: *lang e1* = *lang e2* ⟹ *lang* (*subst e1 σ*) = *lang* (*subst e2 σ*)
⟨*proof*⟩

11.1 Examples

lemma *lang* (*Plus* (*Atom* (*x*::*nat*)) (*Atom x*)) = *lang* (*Atom x*)
⟨*proof*⟩

fun *seq* :: *'a rexp list* ⇒ *'a rexp* **where**

seq [] = *One* |
seq [*r*] = *r* |
seq (*r*#*rs*) = *Times r* (*seq rs*)

abbreviation *question* **where** *question x* == *Plus x One*

definition *L_4cases* (*x*::*nat*) *y*=
verund [*seq*[*question* (*Atom x*),(*Atom y*), (*Atom y*)],
seq[*question* (*Atom x*),(*Atom y*),(*Atom x*),*Star*(*Times* (*Atom y*)(*Atom x*)),(*Atom y*),(*Atom y*)],
seq[*question* (*Atom x*),(*Atom y*),(*Atom x*),*Star*(*Times* (*Atom y*)(*Atom x*)),(*Atom x*)],
seq[(*Atom x*),(*Atom x*)]]

definition *L_A x y* = *seq*[*question* (*Atom x*),(*Atom y*), (*Atom y*)]
definition *L_B x y* = *seq*[*question* (*Atom x*),(*Atom y*),(*Atom x*),*Star*(*Times* (*Atom y*)(*Atom x*)),(*Atom y*),(*Atom y*)]
definition *L_C x y* = *seq*[*question* (*Atom x*),(*Atom y*),(*Atom x*),*Star*(*Times* (*Atom y*)(*Atom x*)),(*Atom x*)]
definition *L_D x y* = *seq*[(*Atom x*),(*Atom x*)]

lemma *L_4cases x y* = *verund* [*L_A x y*, *L_B x y*, *L_C x y*, *L_D x y*]
⟨*proof*⟩

definition *L_lasthasxx x y* = (*Plus* (*seq*[*question* (*Atom x*), *Star*(*Times* (*Atom y*)(*Atom x*)),(*Atom y*),(*Atom y*)])
(*seq*[*question* (*Atom y*), *Star*(*Times*(*Atom x*) (*Atom y*)),(*Atom x*),(*Atom x*)]))

lemma *lastxx_com*: *lang* (*L_lasthasxx* (*x*::*nat*) *y*) = *lang* (*L_lasthasxx y x*) (**is** *lang ?A* = *lang ?B*)
⟨*proof*⟩

lemma *lastxx_is_4cases*: *lang* (*L_4cases x y*) = *lang* (*L_lasthasxx x y*) (**is** *lang ?A* = *lang ?B*)
⟨*proof*⟩

definition *myUNIV x y* = *Star* (*Plus* (*Atom x*) (*Atom y*))

lemma *myUNIV_alle*: *lang* (*myUNIV x y*) = {*xs*. *set xs* ⊆ {*x*,*y*}}
⟨*proof*⟩

definition *nodouble x y* = (*Plus*
(*seq*[*question* (*Atom x*), *Star*(*Times*(*Atom y*)(*Atom x*)),(*Atom y*)])
(*seq*[*question* (*Atom y*), *Star*(*Times*(*Atom x*) (*Atom y*)),(*Atom x*)]))

lemma *myUNIV_char*: *lang* (*myUNIV* (*x*::*nat*) *y*) = *lang* (*Times* (*Star* (*L_lasthasxx x y*)) (*Plus One* (*nodouble x y*))) (**is** *lang ?A* = *lang ?B*)
⟨*proof*⟩

definition *mycasexxy x y* = *Plus* (*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom x*, *Atom x*, *Atom y*])
(*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom y*, *Atom y*, *Atom x*])
definition *mycasexyx x y* = *Plus* (*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom x*, *Atom y*, *Atom x*])
(*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom y*, *Atom x*, *Atom y*])
definition *mycasexx x y* = *Plus* (*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom x*, *Atom x*])
(*seq*[*Star* (*Plus* (*Atom x*) (*Atom y*)), *Atom y*, *Atom y*])
definition *mycasexy x y* = *Plus* (*seq*[*Atom x*, *Atom y*]) (*seq*[*Atom y*, *Atom x*])
definition *mycasex x y* = *Plus* (*Atom y*) (*Atom x*)

definition *mycases x y* = *Plus*
(*mycasexxy x y*)
(*Plus* (*mycasexyx x y*)
(*Plus* (*mycasexx x y*)
(*Plus* (*mycasexy x y*) (*Plus* (*mycasex x y*) (*One*)))))

lemma *mycases_char*: *lang* (*myUNIV* (*x*::*nat*) *y*) = *lang* (*mycases x y*) (**is** *lang ?A* = *lang ?B*)
⟨*proof*⟩

end

12 OPT2

theory *OPT2*
imports
Partial_Cost_Model
RExp_Var
begin

lemma (*N*::*nat set*) ≠ {} ⟹ *Inf N* : *N*
⟨*proof*⟩

lemma *nn_contains_Inf*:
fixes *S* :: *nat set*
assumes *nn*: *S* ≠ {}
shows *Inf S* ∈ *S*
⟨*proof*⟩

12.1 Definition

fun *OPT2* :: *'a list* ⇒ *'a list* ⇒ (*nat* ∗ *nat list*) *list* **where**
OPT2 [] [*x*,*y*] = []
| *OPT2* [*a*] [*x*,*y*] = [(*0*,[])]
| *OPT2* (*a*#*b*#*σ'*) [*x*,*y*] = (*if a*=*x then* (*0*,[]) # (*OPT2* (*b*#*σ'*) [*x*,*y*])
else (*if b*=*x then* (*0*,[])# (*OPT2* (*b*#*σ'*) [*x*,*y*])
else (*1*,[])# (*OPT2* (*b*#*σ'*) [*y*,*x*])))

lemma *OPT2_length*: *length* (*OPT2* *σ* [*x*, *y*]) = *length σ*
⟨*proof*⟩

lemma *OPT2x*: *OPT2* (*x*#*σ'*) [*x*,*y*] = (*0*,[])#(*OPT2* *σ'* [*x*,*y*])
⟨*proof*⟩

lemma *swapOpt*: $T_{p\_}opt$ [*x*,*y*] *σ* ≤ *1* + $T_{p\_}opt$ [*y*,*x*] *σ*
⟨*proof*⟩

lemma *tt*: *a* ∈ {*x*,*y*} ⟹ *OPT2* (*rest1*) (*step* [*x*,*y*] *a* (*hd* (*OPT2* (*a* # *rest1*) [*x*, *y*])))
= *tl* (*OPT2* (*a* # *rest1*) [*x*, *y*])
⟨*proof*⟩

lemma *splitqsallg*: *Strat* ≠ [] ⟹ *a* ∈ {*x*,*y*} ⟹

$t_p\ [x,\ y]\ a\ (hd\ (Strat)) +$
$(let\ L{=}step\ [x,y]\ a\ (hd\ (Strat))$
$in\ T_p\ L\ (rest1)\ (tl\ Strat)) =\ \ T_p\ [x,\ y]\ (a\ \#\ rest1)\ Strat$
⟨*proof*⟩

lemma *splitqs*: $a \in \{x,y\} \Longrightarrow T_p\ [x,\ y]\ (a\ \#\ rest1)\ (OPT2\ (a\ \#\ rest1)\ [x,\ y])$
$= \ t_p\ [x,\ y]\ a\ (hd\ (OPT2\ (a\ \#\ rest1)\ [x,\ y])) +$
$(let\ L{=}step\ [x,y]\ a\ (hd\ (OPT2\ (a\ \#\ rest1)\ [x,\ y]))$
$in\ T_p\ L\ (rest1)\ (OPT2\ (rest1)\ L))$
⟨*proof*⟩

lemma *tpx*: $t_p\ [x,\ y]\ x\ (hd\ (OPT2\ (x\ \#\ rest1)\ [x,\ y])) = 0$
⟨*proof*⟩

lemma *yup*: $T_p\ [x,\ y]\ (x\ \#\ rest1)\ (OPT2\ (x\ \#\ rest1)\ [x,\ y])$
$= (let\ L{=}step\ [x,y]\ x\ (hd\ (OPT2\ (x\ \#\ rest1)\ [x,\ y]))$
$in\ T_p\ L\ (rest1)\ (OPT2\ (rest1)\ L))$
⟨*proof*⟩

lemma *swapsxy*: $A \in \{\ [x,y],\ [y,x]\} \Longrightarrow swaps\ sws\ A \in \{\ [x,y],\ [y,x]\}$
⟨*proof*⟩

lemma *mtf2xy*: $A \in \{\ [x,y],\ [y,x]\} \Longrightarrow r{\in}\{x,y\} \Longrightarrow mtf2\ a\ r\ A \in \{\ [x,y],\ [y,x]\}$
⟨*proof*⟩

lemma *stepxy*: **assumes** $q \in \{x,y\}\ A \in \{\ [x,y],\ [y,x]\}$
shows $step\ A\ q\ a \in \{\ [x,y],\ [y,x]\}$
⟨*proof*⟩

12.2 Proof of Optimality

lemma *OPT2_is_lb*: $set\ \sigma \subseteq \{x,y\} \Longrightarrow x{\neq}y \Longrightarrow T_p\ [x,y]\ \sigma\ (OPT2\ \sigma\ [x,y]) \leq T_p\_opt\ [x,y]\ \sigma$
⟨*proof*⟩

lemma *OPT2_is_ub*: $set\ qs \subseteq \{x,y\} \Longrightarrow x{\neq}y \Longrightarrow T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) \geq T_p\_opt\ [x,y]\ qs$
⟨*proof*⟩

lemma *OPT2_is_opt*: *set qs* $\subseteq \{x,y\} \Longrightarrow x \neq y \Longrightarrow T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = T_p\_opt\ [x,y]\ qs$
⟨*proof*⟩

12.3 Performance on the four phase forms

lemma *OPT2_A*: **assumes** $x \neq y$ *qs* $\in$ *lang* (*seq* [*Plus* (*Atom x*) *One*, *Atom y*, *Atom y*])
shows $T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = 1$
⟨*proof*⟩

lemma *OPT2_A′*: **assumes** $x \neq y$ *qs* $\in$ *lang* (*seq* [*Plus* (*Atom x*) *One*, *Atom y*, *Atom y*])
shows *real* $(T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y])) = 1$
⟨*proof*⟩

lemma *OPT2_B*: **assumes** $x \neq y$ *qs*=*u*@*v* *u*=[] ∨ *u*=[*x*] *v* $\in$ *lang* (*seq*[*Times* (*Atom y*) (*Atom x*), *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom y*, *Atom y*])
shows $T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = (length\ v\ div\ 2)$
⟨*proof*⟩

lemma *OPT2_B1*: **assumes** $x \neq y$ *qs* $\in$ *lang* (*seq*[*Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom y*, *Atom y*])
shows *real* $(T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y])) = length\ qs\ /\ 2$
⟨*proof*⟩

lemma *OPT2_B2*: **assumes** $x \neq y$ *qs* $\in$ *lang* (*seq*[*Atom x*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom y*, *Atom y*])
shows $T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = ((length\ qs - 1)\ /\ 2)$
⟨*proof*⟩

lemma *OPT2_C*: **assumes** $x \neq y$ *qs*=*u*@*v* *u*=[] ∨ *u*=[*x*]
and *v* $\in$ *lang* (*seq*[*Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows $T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = (length\ v\ div\ 2)$
⟨*proof*⟩

lemma *OPT2_C1*: **assumes** $x \neq y$ *qs* $\in$ *lang* (*seq*[*Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows *real* $(T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y])) = (length\ qs - 1)\ /\ 2$
⟨*proof*⟩

lemma *OPT2_C2*: **assumes** $x \neq y$ $qs \in lang\ (seq[Atom\ x,\ Atom\ y,\ Atom\ x,\ Star(Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ x])$
shows $T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) = ((length\ qs - 2)\ /\ 2)$
⟨*proof*⟩

lemma *OPT2_ub*: $set\ qs \subseteq \{x,y\} \Longrightarrow T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y]) \leq length\ qs$
⟨*proof*⟩

lemma *OPT2_padded*: $R \in \{[x,y],[y,x]\} \Longrightarrow set\ qs \subseteq \{x,y\}$
$\Longrightarrow\ T_p\ R\ (qs@[x,x])\ (OPT2\ (qs@[x,x])\ R)$
$\leq T_p\ R\ (qs@[x])\ (OPT2\ (qs@[x])\ R) + 1$
⟨*proof*⟩

lemma *OPT2_split11*:
assumes *xy*: $x \neq y$
shows $R \in \{[x,y],[y,x]\} \Longrightarrow set\ xs \subseteq \{x,y\} \Longrightarrow set\ ys \subseteq \{x,y\} \Longrightarrow OPT2\ (xs@[x,x]@ys)\ R = OPT2\ (xs@[x,x])\ R\ @\ OPT2\ ys\ [x,y]$
⟨*proof*⟩

12.4 The function steps

lemma *steps_append*: $length\ qs = length\ as \Longrightarrow steps\ s\ (qs@[q])\ (as@[a]) = step\ (steps\ s\ qs\ as)\ q\ a$
⟨*proof*⟩

end

13 Phase Partitioning

theory *Phase_Partitioning*
imports *OPT2*
begin

13.1 Definition of Phases

definition $other\ a\ x\ y = (if\ a{=}x\ then\ y\ else\ x)$

definition *Lxx* **where**

$Lxx\ (x{::}nat)\ y = lang\ (L\_lasthasxx\ x\ y)$

lemma *Lxx_not_nullable*: $[] \notin Lxx\ x\ y$
⟨*proof*⟩

lemma *Lxx_ends_in_two_equal*: $xs \in Lxx\ x\ y \Longrightarrow \exists pref\ e.\ xs = pref\ @\ [e,e]$
⟨*proof*⟩

lemma $Lxx\ x\ y = Lxx\ y\ x$ ⟨*proof*⟩

definition $hideit\ x\ y = (Plus\ rexp.One\ (nodouble\ x\ y))$

lemma *Lxx_othercase*: $set\ qs \subseteq \{x,y\} \Longrightarrow \neg\ (\exists xs\ ys.\ qs = xs\ @\ ys \wedge xs \in Lxx\ x\ y) \Longrightarrow qs \in lang\ (hideit\ x\ y)$
⟨*proof*⟩

fun *pad* **where** $pad\ xs\ x\ y = (if\ xs{=}[]\ then\ [x,x]\ else\ (if\ last\ xs = x\ then\ xs\ @\ [x]\ else\ xs\ @\ [y]))$

lemma *pad_adds2*: $qs \neq [] \Longrightarrow set\ qs \subseteq \{x,y\} \Longrightarrow pad\ qs\ x\ y = qs\ @\ [last\ qs]$
⟨*proof*⟩

lemma *nodouble_padded*: $qs \neq [] \Longrightarrow qs \in lang\ (nodouble\ x\ y) \Longrightarrow pad\ qs\ x\ y \in Lxx\ x\ y$
⟨*proof*⟩

thm *UnE*
lemma $c \in A \cup B \Longrightarrow P$
⟨*proof*⟩

lemma *LxxE*: $qs \in Lxx\ x\ y$
$\Longrightarrow (qs \in lang\ (seq\ [Atom\ x,\ Atom\ x]) \Longrightarrow P\ x\ y\ qs)$
$\Longrightarrow (qs \in lang\ (seq\ [Plus\ (Atom\ x)\ rexp.One,\ Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y]) \Longrightarrow P\ x\ y\ qs)$
$\Longrightarrow (qs \in lang\ (seq\ [Plus\ (Atom\ x)\ rexp.One,\ Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ x]) \Longrightarrow P\ x\ y\ qs)$

$\Longrightarrow$ ($qs \in lang$ (seq [$Plus$ ($Atom\ x$) $rexp.One$, $Atom\ y$, $Atom\ y$]) $\Longrightarrow$ P $x\ y\ qs$)
$\Longrightarrow$ $P\ x\ y\ qs$
⟨*proof*⟩

thm *UnE LxxE*

lemma $qs \in Lxx\ x\ y \Longrightarrow P$
⟨*proof*⟩

lemma *LxxI*: ($qs \in lang$ (seq [$Atom\ x$, $Atom\ x$]) $\Longrightarrow$ $P\ x\ y\ qs$)
$\Longrightarrow$ ($qs \in lang$ (seq [$Plus$ ($Atom\ x$) $rexp.One$, $Atom\ y$, $Atom\ x$, $Star$ ($Times$ ($Atom\ y$) ($Atom\ x$)), $Atom\ y$, $Atom\ y$]) $\Longrightarrow$ $P\ x\ y\ qs$)
$\Longrightarrow$ ($qs \in lang$ (seq [$Plus$ ($Atom\ x$) $rexp.One$, $Atom\ y$, $Atom\ x$, $Star$ ($Times$ ($Atom\ y$) ($Atom\ x$)), $Atom\ x$]) $\Longrightarrow$ $P\ x\ y\ qs$)
$\Longrightarrow$ ($qs \in lang$ (seq [$Plus$ ($Atom\ x$) $rexp.One$, $Atom\ y$, $Atom\ y$]) $\Longrightarrow$ P $x\ y\ qs$)
$\Longrightarrow$ ($qs \in Lxx\ x\ y \Longrightarrow P\ x\ y\ qs$)
⟨*proof*⟩

lemma *Lxx1*: $xs \in Lxx\ x\ y \Longrightarrow length\ xs \geq 2$
⟨*proof*⟩

13.2 OPT2 Splitting

lemma *ayay*: $length\ qs = length\ as \Longrightarrow T_p\ s\ (qs@[q])\ (as@[a]) = T_p\ s\ qs$ $as + t_p\ (steps\ s\ qs\ as)\ q\ a$
⟨*proof*⟩

lemma *tlofOPT2*: $Q \in \{x,y\} \Longrightarrow set\ QS \subseteq \{x,y\} \Longrightarrow R \in \{[x, y], [y, x]\}$ $\Longrightarrow tl\ (OPT2\ ((Q\ \#\ QS)\ @\ [x, x])\ R) =$
$OPT2\ (QS\ @\ [x, x])\ (step\ R\ Q\ (hd\ (OPT2\ ((Q\ \#\ QS)\ @\ [x, x])\ R)))$
⟨*proof*⟩

lemma T_p*_split*: $length\ qs1 = length\ as1 \Longrightarrow T_p\ s\ (qs1@qs2)\ (as1@as2)$ $= T_p\ s\ qs1\ as1 + T_p\ (steps\ s\ qs1\ as1)\ qs2\ as2$
⟨*proof*⟩

lemma T_p*_spliting*: $x \neq y \Longrightarrow set\ xs \subseteq \{x,y\} \Longrightarrow set\ ys \subseteq \{x,y\} \Longrightarrow$
$R \in \{[x,y],[y,x]\} \Longrightarrow$
$T_p\ R\ (xs@[x,x])\ (OPT2\ (xs@[x,x])\ R) + T_p\ [x,y]\ ys\ (OPT2\ ys\ [x,y])$
$= T_p\ R\ (xs@[x,x]@ys)\ (OPT2\ (xs@[x,x]@ys)\ R)$

⟨*proof*⟩

lemma *OPTauseinander*: $x{\neq}y \Longrightarrow$ *set xs* $\subseteq \{x,y\} \Longrightarrow$ *set ys* $\subseteq \{x,y\} \Longrightarrow$
LTS $\in \{[x,y],[y,x]\} \Longrightarrow$ *hd LTS* = *last xs* $\Longrightarrow$
xs = (*pref* @ [*hd LTS*, *hd LTS*]) $\Longrightarrow$
T_p [*x*,*y*] *xs* (*OPT2 xs* [*x*,*y*]) + T_p *LTS ys* (*OPT2 ys LTS*)
= T_p [*x*,*y*] (*xs*@*ys*) (*OPT2* (*xs*@*ys*) [*x*,*y*])
⟨*proof*⟩

13.3 Phase Partitioning lemma

theorem *Phase_partitioning_general*:
fixes *P* :: (*nat state* ∗ *'is*) *pmf* ⇒ *nat* ⇒ *nat list* ⇒ *bool*
and *A* :: (*nat state*,*'is*,*nat*,*answer*) *alg_on_rand*
assumes *xny*: (*x0*::*nat*) ≠ *y0*
and *cpos*: (*c*::*real*)≥*0*
and *static*: *set* σ ⊆ {*x0*,*y0*}
and *initial*: *P* (*map_pmf* (%*is*. ([*x0*,*y0*],*is*)) (*fst A* [*x0*,*y0*])) *x0* [*x0*,*y0*]
and *D*: ⋀*a b* σ *s*. σ ∈ *Lxx a b* ⟹ *a*≠*b* ⟹ {*a*,*b*}={*x0*,*y0*} ⟹ *P s a* [*x0*,*y0*] ⟹ *set* σ ⊆ {*a*,*b*}
⟹ *T_on_rand'* *A s* σ ≤ *c* ∗ T_p [*a*,*b*] σ (*OPT2* σ [*a*,*b*]) ∧ *P* (*config'_rand A s* σ) (*last* σ) [*x0*,*y0*]
shows T_p*_on_rand A* [*x0*,*y0*] σ ≤ *c* ∗ T_p*_opt* [*x0*,*y0*] σ + *c*
⟨*proof*⟩

term *A*::(*nat*,*'is*) *alg_on*

theorem *Phase_partitioning_general_det*:
fixes *P* :: (*nat state* ∗ *'is*) ⇒ *nat* ⇒ *nat list* ⇒ *bool*
and *A* :: (*nat*,*'is*) *alg_on*
assumes *xny*: (*x0*::*nat*) ≠ *y0*
and *cpos*: (*c*::*real*)≥*0*
and *static*: *set* σ ⊆ {*x0*,*y0*}
and *initial*: *P* ([*x0*,*y0*],(*fst A* [*x0*,*y0*])) *x0* [*x0*,*y0*]
and *D*: ⋀*a b* σ *s*. σ ∈ *Lxx a b* ⟹ *a*≠*b* ⟹ {*a*,*b*}={*x0*,*y0*} ⟹ *P s a* [*x0*,*y0*] ⟹ *set* σ ⊆ {*a*,*b*}
⟹ *T_on'* *A s* σ ≤ *c* ∗ T_p [*a*,*b*] σ (*OPT2* σ [*a*,*b*]) ∧ *P* (*config'* *A s* σ) (*last* σ) [*x0*,*y0*]
shows T_p*_on A* [*x0*,*y0*] σ ≤ *c* ∗ T_p*_opt* [*x0*,*y0*] σ + *c*
⟨*proof*⟩

end

14 List factoring technique

theory *List_Factoring*
imports
Partial_Cost_Model
MTF2_Effects
begin

hide_const *config compet*

14.1 Helper functions

14.1.1 Helper lemmas

lemma *befaf*: **assumes** $q \in set\ s$ *distinct s*
shows $before\ q\ s \cup \{q\} \cup after\ q\ s = set\ s$
⟨*proof*⟩

lemma *index_sum*: **assumes** *distinct s* $q \in set\ s$
shows $index\ s\ q = (\sum e \in set\ s.\ if\ e < q\ in\ s\ then\ 1\ else\ 0)$
⟨*proof*⟩

14.1.2 ALG

fun $ALG :: 'a \Rightarrow 'a\ list \Rightarrow nat \Rightarrow ('a\ list * 'is) \Rightarrow nat$ **where**
$ALG\ x\ qs\ i\ s = (if\ x < (qs!i)\ in\ fst\ s\ then\ 1::nat\ else\ 0)$

lemma $t_p\_sumofALG$: $distinct\ (fst\ s) \Longrightarrow snd\ a = [] \Longrightarrow (qs!i) \in set\ (fst\ s)$

$\Longrightarrow t_p\ (fst\ s)\ (qs!i)\ a = (\sum e \in set\ (fst\ s).\ ALG\ e\ qs\ i\ s)$
⟨*proof*⟩

lemma $t_p\_sumofALGreal$: **assumes** $distinct\ (fst\ s)\ snd\ a = []\ qs!i \in set(fst\ s)$
shows $real(t_p\ (fst\ s)\ (qs!i)\ a) = (\sum e \in set\ (fst\ s).\ real(ALG\ e\ qs\ i\ s))$
⟨*proof*⟩

14.1.3 The function steps'

fun $steps'$ **where**
$steps'\ s\ \_\ \_\ 0 = s$
| $steps'\ s\ []\ []\ (Suc\ n) = s$

| *steps′ s (q#qs) (a#as) (Suc n) = steps′ (step s q a) qs as n*

lemma *steps′_steps*: *length as = length qs ⟹ steps′ s as qs (length as) = steps s as qs*
⟨*proof*⟩

lemma *steps′_length*: *length qs = length as ⟹ n ≤ length as*
⟹ length (steps′ s qs as n) = length s
⟨*proof*⟩

lemma *steps′_set*: *length qs = length as ⟹ n ≤ length as*
⟹ set (steps′ s qs as n) = set s
⟨*proof*⟩

lemma *steps′_distinct2*: *length qs = length as ⟹ n ≤ length as*
⟹ distinct s ⟹ distinct (steps′ s qs as n)
⟨*proof*⟩

lemma *steps′_distinct*: *length qs = length as ⟹ length as = n*
⟹ distinct (steps′ s qs as n) = distinct s
⟨*proof*⟩

lemma *steps′_dist_perm*: *length qs = length as ⟹ length as = n*
⟹ dist_perm s s ⟹ dist_perm (steps′ s qs as n) (steps′ s qs as n)
⟨*proof*⟩

lemma *steps′_rests*: *length qs = length as ⟹ n ≤ length as ⟹ steps′ s qs as n = steps′ s (qs@r1) (as@r2) n*
⟨*proof*⟩

lemma *steps′_append*: *length qs = length as ⟹ length qs = n ⟹ steps′ s (qs@[q]) (as@[a]) (Suc n) = step (steps′ s qs as n) q a*
⟨*proof*⟩

14.1.4 *ALG′_det*

definition *ALG′_det Strat qs init i x = ALG x qs i (swaps (snd (Strat!i)) (steps′ init qs Strat i),())*

lemma *ALG′_det_append*: *n < length Strat ⟹ n < length qs ⟹ ALG′_det Strat (qs@a) init n x*
= ALG′_det Strat qs init n x

⟨*proof*⟩

14.1.5 ALG'

abbreviation *config'' A qs init n == config_rand A init (take n qs)*

definition *ALG' A qs init i x = E(map_pmf (ALG x qs i) (config'' A qs init i))*

lemma *ALG'_refl*: *qs!i = x ⟹ ALG' A qs init i x = 0*
⟨*proof*⟩

14.1.6 *ALGxy_det*

definition *ALGxy_det* **where**
ALGxy_det A qs init x y = (∑ i∈{..<length qs}. (if (qs!i ∈ {y,x}) then ALG'_det A qs init i y + ALG'_det A qs init i x
else 0::nat))

lemma *ALGxy_det_alternativ*: *ALGxy_det A qs init x y*
= (∑ i∈{i. i<length qs ∧ (qs!i ∈ {y,x})}. ALG'_det A qs init i y + ALG'_det A qs init i x)
⟨*proof*⟩

14.1.7 ALGxy

definition *ALGxy* **where**
ALGxy A qs init x y = (∑ i∈{..<length qs} ∩ {i. (qs!i ∈ {y,x})}. ALG' A qs init i y + ALG' A qs init i x)

lemma *ALGxy_def2*:
ALGxy A qs init x y = (∑ i∈{i. i<length qs ∧ (qs!i ∈ {y,x})}. ALG' A qs init i y + ALG' A qs init i x)
⟨*proof*⟩
lemma *ALGxy_append*: *ALGxy A (rs@[r]) init x y =*
ALGxy A rs init x y + (if (r ∈ {y,x}) then ALG' A (rs@[r]) init (length rs) y + ALG' A (rs@[r]) init (length rs) x else 0)
⟨*proof*⟩

lemma *ALGxy_wholerange*: *ALGxy A qs init x y*
= (∑ i<(length qs). (if qs ! i ∈ {y, x}
then ALG' A qs init i y + ALG' A qs init i x
else 0))
⟨*proof*⟩

14.2 Transformation to Blocking Cost

lemma *umformung*:
fixes $A :: (('a::linorder)\ list, 'is, 'a, (nat * nat\ list))\ alg\_on\_rand$
assumes *no_paid*: $\bigwedge is\ s\ q.\ \forall ((free,paid),\_) \in (snd\ A\ (s,is)\ q).\ paid=[]$
assumes *inlist*: $set\ qs \subseteq set\ init$
assumes *dist*: $distinct\ init$
assumes $\bigwedge x.\ x < length\ qs \Longrightarrow finite\ (set\_pmf\ (config''\ A\ qs\ init\ x))$
shows $T_p\_on\_rand\ A\ init\ qs =$
$(\sum (x,y)\in\{(x,y).\ x \in set\ init \wedge y\in set\ init \wedge x<y\}.\ ALGxy\ A\ qs\ init\ x\ y)$
⟨*proof*⟩

lemma *before_in_index1*:
fixes l
assumes $set\ l = \{x,y\}$ **and** $length\ l = 2$ **and** $x\neq y$
shows $(if\ (x < y\ in\ l)\ then\ 0\ else\ 1) = index\ l\ x$
⟨*proof*⟩

lemma *before_in_index2*:
fixes l
assumes $set\ l = \{x,y\}$ **and** $length\ l = 2$ **and** $x\neq y$
shows $(if\ (x < y\ in\ l)\ then\ 1\ else\ 0) = index\ l\ y$
⟨*proof*⟩

lemma *before_in_index*:
fixes l
assumes $set\ l = \{x,y\}$ **and** $length\ l = 2$ **and** $x\neq y$
shows $(x < y\ in\ l) = (index\ l\ x = 0)$
⟨*proof*⟩

14.3 The pairwise property

definition *pairwise* **where**
$pairwise\ A = (\forall init.\ distinct\ init \longrightarrow (\forall qs\in\{xs.\ set\ xs \subseteq set\ init\}.\ \forall (x::('a::linorder),y)\in\{(x,y).\ x \in set\ init \wedge y\in set\ init \wedge x<y\}.\ T_p\_on\_rand\ A\ (Lxy\ init\ \{x,y\})\ (Lxy\ qs\ \{x,y\}) = ALGxy\ A\ qs\ init\ x\ y))$

definition $Pbefore\_in\ x\ y\ A\ qs\ init = map\_pmf\ (\lambda p.\ x < y\ in\ fst\ p)\ (config\_rand\ A\ init\ qs)$

lemma *T_on_n_no_paid*:
assumes
nopaid: $\bigwedge s\ n.\ map\_pmf\ (\lambda x.\ snd\ (fst\ x))\ (snd\ A\ s\ n) = return\_pmf\ []$
shows $T\_on\_rand\_n\ A\ init\ qs\ i = E\ (config''\ A\ qs\ init\ i \gg\!= (\lambda p.\ return\_pmf\ (real(index\ (fst\ p)\ (qs\ !\ i)))))$
⟨*proof*⟩

lemma *pairwise_property_lemma*:
assumes
relativeorder: $(\bigwedge init\ qs.\ distinct\ init \Longrightarrow qs \in \{xs.\ set\ xs \subseteq set\ init\}$
$\Longrightarrow (\bigwedge x\ y.\ (x,y) \in \{(x,y).\ x \in set\ init \wedge y \in set\ init \wedge x \neq y\}$
$\Longrightarrow x \neq y$
$\Longrightarrow Pbefore\_in\ x\ y\ A\ qs\ init = Pbefore\_in\ x\ y\ A\ (Lxy\ qs\ \{x,y\})\ (Lxy\ init\ \{x,y\})$
))
and *nopaid*: $\bigwedge xa\ r.\ \forall z \in set\_pmf(snd\ A\ xa\ r).\ snd(fst\ z) = []$
shows *pairwise A*
⟨*proof*⟩

lemma *umf_pair*: **assumes**
0: *pairwise A*
assumes *1*: $\bigwedge is\ s\ q.\ \forall((free,paid),\_) \in (snd\ A\ (s,\ is)\ q).\ paid=[]$
assumes *2*: $set\ qs \subseteq set\ init$
assumes *3*: *distinct init*
assumes *4*: $\bigwedge x.\ x<length\ qs \Longrightarrow finite\ (set\_pmf\ (config''\ A\ qs\ init\ x))$
shows $T_p\_on\_rand\ A\ init\ qs$
$= (\sum (x,y) \in \{(x,\ y).\ x \in set\ init \wedge y \in set\ init \wedge x < y\}.\ T_p\_on\_rand\ A\ (Lxy\ init\ \{x,y\})\ (Lxy\ qs\ \{x,y\}))$
⟨*proof*⟩

14.4 List Factoring for OPT

fun $ALG\_P :: nat\ list \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a\ list \Rightarrow nat$ **where**
$ALG\_P\ []\ x\ y\ xs = (0::nat)$
| $ALG\_P\ (s\#ss)\ x\ y\ xs = (if\ Suc\ s < length\ (swaps\ ss\ xs)$
$then\ (if\ ((swaps\ ss\ xs)!s=x \wedge (swaps\ ss\ xs)!(Suc\ s)=y) \vee ((swaps\ ss\ xs)!s=y \wedge (swaps\ ss\ xs)!(Suc\ s)=x)$
$then\ 1$
$else\ 0)$
$else\ 0) + ALG\_P\ ss\ x\ y\ xs$

lemma *ALG_P_erwischt_alle*:
assumes *dinit*: *distinct init*
shows
$\forall\, l{<}\ length\ sws.\ Suc\ (sws!l) < length\ init \Longrightarrow length\ sws$
$= (\sum (x,y){\in}\{(x,y).\ x \in set\ (init{::}('a{::}linorder)\ list) \wedge y{\in}set\ init \wedge x{<}y\}.\ ALG\_P\ sws\ x\ y\ init)$
⟨*proof*⟩

lemma t_p*_sumofALGALGP*:
assumes *distinct s* $(qs!i){\in}set\ s$
and $\forall\, l{<}\ length\ (snd\ a).\ Suc\ ((snd\ a)!l) < length\ s$
shows $t_p\ s\ (qs!i)\ a = (\sum e{\in}set\ s.\ ALG\ e\ qs\ i\ (swaps\ (snd\ a)\ s,()))$
$+ (\sum (x,y){\in}\{(x{::}('a{::}linorder),y).\ x \in set\ s \wedge y{\in}set\ s \wedge x{<}y\}.\ ALG\_P\ (snd\ a)\ x\ y\ s)$
⟨*proof*⟩

definition *ALG_P' Strat qs init i x y* = *ALG_P* (*snd* (*Strat*!*i*)) *x y* (*steps' init qs Strat i*)

lemma *ALG_P'_rest*: $n < length\ qs \Longrightarrow n < length\ Strat \Longrightarrow$
ALG_P' Strat (*take n qs* @ [*qs* ! *n*]) *init n x y* =
ALG_P' (*take n Strat* @ [*Strat* ! *n*]) (*take n qs* @ [*qs* ! *n*]) *init n x y*
⟨*proof*⟩

lemma *ALG_P'_rest2*: $n < length\ qs \Longrightarrow n < length\ Strat \Longrightarrow$
ALG_P' Strat qs init n x y =
ALG_P' (*Strat*@*r1*) (*qs*@*r2*) *init n x y*
⟨*proof*⟩

definition *ALG_Pxy* **where**
ALG_Pxy Strat qs init x y = $(\sum i{<}length\ qs.\ ALG\_P'\ Strat\ qs\ init\ i\ x\ y)$

lemma *wegdamit*: *length A* < *length Strat* ⟹ *b* ∉ {*x*,*y*} ⟹ *ALGxy_det Strat* (*A* @ [*b*]) *init x y*
= *ALGxy_det Strat A init x y*
⟨*proof*⟩

lemma *ALG_P_split*: *length qs* < *length Strat* ⟹ *ALG_Pxy Strat* (*qs*@[*q*]) *init x y* = *ALG_Pxy Strat qs init x y*
+ *ALG_P′ Strat* (*qs*@[*q*]) *init* (*length qs*) *x y*
⟨*proof*⟩

lemma *swap0in2*: **assumes** *set l* = {*x*,*y*} *x*≠*y length l* = *2 dist_perm l l*
shows
x < *y in* (*swap 0*) *l* = (~ *x* < *y in l*)
⟨*proof*⟩

lemma *before_in_swap2*:
dist_perm xs ys ⟹ *Suc n* < *size xs* ⟹ *x*≠*y* ⟹
x < *y in* (*swap n xs*) ⟷
(~ *x* < *y in xs* ∧ (*y* = *xs*!*n* ∧ *x* = *xs*!*Suc n*)
∨ *x* < *y in xs* ∧ ~(*y* = *xs*!*Suc n* ∧ *x* = *xs*!*n*))
⟨*proof*⟩

lemma *projected_paid_same_effect*:
assumes
d: *dist_perm s1 s1*
and *ee*: *x*≠*y*
and *f*: *set s2* = {*x*, *y*}
and *g*: *length s2* = *2*
and *h*: *dist_perm s2 s2*
shows *x* < *y in s1* = *x* < *y in s2* ⟹
x < *y in swaps acs s1* = *x* < *y in* (*swap 0* ^^ *ALG_P acs x y s1*) *s2*
⟨*proof*⟩

lemma *steps_steps′*:
length qs = *length as* ⟹ *steps s qs as* = *steps′ s qs as* (*length as*)
⟨*proof*⟩

lemma *T1_7′*: T_p *init qs Strat* = T_{p}*_opt init qs* ⟹ *length Strat* = *length qs*

$\Longrightarrow n \leq length\ qs \Longrightarrow$
$x \neq (y::('a::linorder)) \Longrightarrow$
$x \in set\ init \Longrightarrow y \in set\ init \Longrightarrow distinct\ init \Longrightarrow$
$set\ qs \subseteq set\ init \Longrightarrow$
$(\exists\ Strat2\ sws.$
$T_p\_opt\ (Lxy\ init\ \{x,y\})\ (Lxy\ (take\ n\ qs)\ \{x,y\}) \leq T_p\ (Lxy\ init\ \{x,y\})$
$(Lxy\ (take\ n\ qs)\ \{x,y\})\ Strat2 \wedge$ $length\ Strat2 = length\ (Lxy\ (take\ n\ qs)\ \{x,y\})$
$\wedge \quad (x < y\ in\ (steps'\ init\ (take\ n\ qs)\ (take\ n\ Strat)\ n))$
$= (x < y\ in\ (swaps\ sws\ (steps'\ (Lxy\ init\ \{x,y\})\ (Lxy\ (take\ n\ qs)\ \{x,y\})\ Strat2\ (length\ Strat2))))$
$\wedge\ T_p\ (Lxy\ init\ \{x,y\})\ (Lxy\ (take\ n\ qs)\ \{x,y\})\ Strat2 + length\ sws =$
$ALGxy\_det\ Strat\ (take\ n\ qs)\ init\ x\ y + ALG\_Pxy\ Strat\ (take\ n\ qs)\ init\ x\ y)$
⟨*proof*⟩

lemma *T1_7*:
assumes $T_p\ init\ qs\ Strat = T_p\_opt\ init\ qs\ length\ Strat = length\ qs$
$x \neq (y::('a::linorder))\ x \in set\ init\ y \in set\ init\ distinct\ init$
$set\ qs \subseteq set\ init$
shows $T_p\_opt\ (Lxy\ init\ \{x,y\})\ (Lxy\ qs\ \{x,y\})$
$\leq ALGxy\_det\ Strat\ qs\ init\ x\ y + ALG\_Pxy\ Strat\ qs\ init\ x\ y$
⟨*proof*⟩

lemma *T_snoc*: $length\ rs = length\ as$
$\Longrightarrow T\ init\ (rs@[r])\ (as@[a])$
$= T\ init\ rs\ as + t_p\ (steps'\ init\ rs\ as\ (length\ rs))\ r\ a$
⟨*proof*⟩

lemma *steps'_snoc*: $length\ rs = length\ as \Longrightarrow n = (length\ as)$
$\Longrightarrow steps'\ init\ (rs@[r])\ (as@[a])\ (Suc\ n) = step\ (steps'\ init\ rs\ as\ n)\ r\ a$
⟨*proof*⟩

lemma *steps'_take*:
assumes $n < length\ qs\ length\ qs = length\ Strat$
shows $steps'\ init\ (take\ n\ qs)\ (take\ n\ Strat)\ n$
$= steps'\ init\ qs\ Strat\ n$
⟨*proof*⟩

lemma *Tp_darstellung*: $length\ qs = length\ Strat$
$\Longrightarrow T_p\ init\ qs\ Strat =$

$(\sum i{\in}\{..{<}length\ qs\}.\ t_p\ (steps'\ init\ qs\ Strat\ i)\ (qs!i)\ (Strat!i))$
⟨*proof*⟩

lemma *umformung_OPT'*:
assumes *inlist*: *set qs* ⊆ *set init*
assumes *dist*: *distinct init*
assumes *qsStrat*: *length qs* = *length Strat*
assumes *noStupid*: ⋀*x l*. *x*<*length Strat* ⟹ *l*< *length* (*snd* (*Strat* ! *x*)) ⟹ *Suc* ((*snd* (*Strat* ! *x*))!*l*) < *length init*
shows T_p *init qs Strat* =
$(\sum (x,y){\in}\{(x,y{::}('a{::}linorder)).\ x \in set\ init \wedge y{\in}set\ init \wedge x{<}y\}.$
ALGxy_det Strat qs init x y + *ALG_Pxy Strat qs init x y*)
⟨*proof*⟩

lemma *nn_contains_Inf*:
fixes *S* :: *nat set*
assumes *nn*: *S* ≠ {}
shows *Inf S* ∈ *S*
⟨*proof*⟩

lemma *steps_length*: *length qs* = *length as* ⟹ *length* (*steps s qs as*) = *length s*
⟨*proof*⟩

lemma *OPT_noStupid*:
fixes *Strat*
assumes [*simp*]: *length Strat* = *length qs*
assumes *opt*: T_p *init qs Strat* = $T_{p\_}$*opt init qs*
assumes *init_nempty*: *init*≠[]
shows ⋀*x l*. *x* < *length Strat* ⟹
l < *length* (*snd* (*Strat* ! *x*)) ⟹
Suc ((*snd* (*Strat* ! *x*))!*l*) < *length init*
⟨*proof*⟩

lemma *umformung_OPT*:

assumes *inlist*: *set qs* $\subseteq$ *set init*
assumes *dist*: *distinct init*
assumes *a*: T_p_*opt init qs* = T_p *init qs Strat*
assumes *b*: *length qs* = *length Strat*
assumes *c*: *init*$\neq$[]
shows T_p_*opt init qs* =
($\sum$ (*x*,*y*)$\in${(*x*,*y*::('*a*::*linorder*)). *x* $\in$ *set init* $\wedge$ *y*$\in$*set init* $\wedge$ *x*<*y*}.
*ALGxy*_*det Strat qs init x y* + *ALG*_*Pxy Strat qs init x y*)
⟨*proof*⟩

corollary *OPT*_*zerlegen*:
assumes
dist: *distinct init*
and *c*: *init*$\neq$[]
and *setqsinit*: *set qs* $\subseteq$ *set init*
shows ($\sum$ (*x*,*y*)$\in${(*x*,*y*::('*a*::*linorder*)). *x* $\in$ *set init* $\wedge$ *y*$\in$*set init* $\wedge$ *x*<*y*}.
(T_p_*opt* (*Lxy init* {*x*,*y*}) (*Lxy qs* {*x*,*y*})))
$\leq$ T_p_*opt init qs*
⟨*proof*⟩

14.5 Factoring Lemma

lemma *cardofpairs*: *S* $\neq$ [] $\Longrightarrow$ *sorted S* $\Longrightarrow$ *distinct S* $\Longrightarrow$ *card* {(*x*,*y*). *x* $\in$ *set S* $\wedge$ *y*$\in$*set S* $\wedge$ *x*<*y*} = ((*length S*)$*$(*length S*−*1*)) / *2*
⟨*proof*⟩

lemma *factoringlemma*_*withconstant*:
fixes *A*
and *b*::*real*
and *c*::*real*
assumes *c*: *c* $\geq$ *1*
assumes *dist*: $\forall$ *e*$\in$*S0*. *distinct e*
assumes *notempty*: $\forall$ *e*$\in$*S0*. *length e* > *0*

assumes *pw*: *pairwise A*

assumes *on2*: $\forall$ *s0*$\in$*S0*. $\exists$ *b*$\geq$*0*. $\forall$ *qs*$\in${*x*. *set x* $\subseteq$ *set s0*}. $\forall$ (*x*,*y*)$\in${(*x*,*y*). *x* $\in$ *set s0* $\wedge$ *y*$\in$*set s0* $\wedge$ *x*<*y*}. T_p_*on*_*rand A* (*Lxy s0* {*x*,*y*}) (*Lxy qs* {*x*,*y*}) $\leq$ *c* $*$ (T_p_*opt* (*Lxy s0* {*x*,*y*}) (*Lxy qs* {*x*,*y*})) + *b*
assumes *nopaid*: $\bigwedge$*is s q*. $\forall$ ((*free*,*paid*),_) $\in$ (*snd A* (*s*, *is*) *q*). *paid*=[]
assumes *4*: $\bigwedge$*init qs*. *distinct init* $\Longrightarrow$ *set qs* $\subseteq$ *set init* $\Longrightarrow$ ($\bigwedge$*x*.

x<*length qs* $\Longrightarrow$ *finite* (*set_pmf* (*config''* *A qs init x*)))

shows $\forall$ *s0*$\in$*S0*. $\exists$ *b*$\geq$*0*. $\forall$ *qs*$\in\{x.$ *set x* $\subseteq$ *set s0*$\}$.
T_p*_on_rand A s0 qs* $\leq$ *c* $*$ *real* (T_p*_opt s0 qs*) + *b*
⟨*proof*⟩

lemma *factoringlemma_withconstant'*:
fixes *A*
and *b*::*real*
and *c*::*real*
assumes *c*: *c* $\geq$ *1*
assumes *dist*: $\forall$ *e*$\in$*S0*. *distinct e*
assumes *notempty*: $\forall$ *e*$\in$*S0*. *length e* > *0*

assumes *pw*: *pairwise A*

assumes *on2*: $\forall$ *s0*$\in$*S0*. $\exists$ *b*$\geq$*0*. $\forall$ *qs*$\in\{x.$ *set x* $\subseteq$ *set s0*$\}$. $\forall$ (*x*,*y*)$\in\{(x,y).$ *x* $\in$ *set s0* $\wedge$ *y*$\in$*set s0* $\wedge$ *x*<*y*$\}$. T_p*_on_rand A* (*Lxy s0* $\{x,y\}$) (*Lxy qs* $\{x,y\}$) $\leq$ *c* $*$ (T_p*_opt* (*Lxy s0* $\{x,y\}$) (*Lxy qs* $\{x,y\}$)) + *b*
assumes *nopaid*: $\bigwedge$*is s q*. $\forall$ ((*free*,*paid*),_) $\in$ (*snd A* (*s*, *is*) *q*). *paid*=[]
assumes *4*: $\bigwedge$*init qs*. *distinct init* $\Longrightarrow$ *set qs* $\subseteq$ *set init* $\Longrightarrow$ ($\bigwedge$*x*. *x*<*length qs* $\Longrightarrow$ *finite* (*set_pmf* (*config''* *A qs init x*)))

shows *compet_rand A c S0*
⟨*proof*⟩

end

15 TS: another 2-competitive Algorithm

theory *TS*
imports
OPT2
Phase_Partitioning
Move_to_Front
List_Factoring
RExp_Var
begin

15.1 Definition of TS

definition *TS_step_d* **where**
TS_step_d s q = ((

```
(
  let li = index (snd s) q in
  (if li = length  (snd s) then 0 — requested for first time
  else (let sincelast = take li  (snd s)
    in (let S={x. x < q in (fst s) ∧ count_list sincelast x ≤ 1}
      in
       (if S={} then 0
       else
        (index (fst s) q) − Min ( (index (fst s)) ‘ S)))
      )
  )
)
,[]), q#(snd s))
```

definition *rTS* :: *nat list* ⇒ (*nat*,*nat list*) *alg_on* **where** *rTS h* = ((λ*s*. *h*), *TS_step_d*)

```
fun TSstep where
  TSstep qs n (is,s)
    = ((qs!n)#is,
      step s (qs!n) ((
        let li = index is (qs!n) in
        (if li = length is then 0 — requested for first time
        else (let sincelast = take li is
          in (let S={x. x < (qs!n) in s ∧ count_list sincelast x ≤ 1}
            in
             (if S={} then 0
             else
              (index s (qs!n)) − Min ( (index s) ‘ S)))
            )
        )
        ),[]))
```

lemma *TSnopaid*: (*snd* (*fst* (*snd* (*rTS initH*) *is q*))) = []
⟨*proof*⟩

abbreviation *TSdet* **where**
TSdet init initH qs n == *config* (*rTS initH*) *init* (*take n qs*)

lemma *TSdet_Suc*: *Suc n* ≤ *length qs* ⟹ *TSdet init initH qs* (*Suc n*) = *Step* (*rTS initH*) (*TSdet init initH qs n*) (*qs*!*n*)

⟨*proof*⟩

definition *s_TS* **where** *s_TS init initH qs n* = *fst* (*TSdet init initH qs n*)

lemma *sndTSdet*: *n*≤*length xs* ⟹ *snd* (*TSdet init initH xs n*) = *rev* (*take n xs*) @ *initH*
⟨*proof*⟩

15.2 Behaviour of TS on lists of length 2

lemma
fixes *hs x y*
assumes *x*≠*y*
shows *oneTS_step* : *TS_step_d* ([*x*, *y*], *x*#*y*#*hs*) *y* = ((*1*, []), *y* # *x* # *y* # *hs*)
and *oneTS_stepyyy*: *TS_step_d* ([*x*, *y*], *y*#*x*#*hs*) *y* = ((*Suc 0*, []), *y*#*y*#*x*#*hs*)
and *oneTS_stepx*: *TS_step_d* ([*x*, *y*], *x*#*x*#*hs*) *y* = ((*0*, []), *y* # *x* # *x* # *hs*)
and *oneTS_stepy*: *TS_step_d* ([*x*, *y*], []) *y* = ((*0*, []), [*y*])
and *oneTS_stepxy*: *TS_step_d* ([*x*, *y*], [*x*]) *y* = ((*0*, []), [*y*, *x*])
and *oneTS_stepyy*: *TS_step_d* ([*x*, *y*], [*y*]) *y* = ((*Suc 0*, []), [*y*, *y*])
and *oneTS_stepyx*: *TS_step_d* ([*x*, *y*], *hs*) *x* = ((*0*, []), *x* # *hs*)
⟨*proof*⟩

lemmas *oneTS_steps* = *oneTS_stepx oneTS_stepxy oneTS_stepyx oneTS_stepy oneTS_stepyy oneTS_stepyyy oneTS_step*

15.3 Analysis of the Phases

definition *TS_inv c x i* ≡ (∃ *hs*. *c* = *return_pmf* ((*if x*=*hd i then i else rev i*),[*x*,*x*]@*hs*))
∨ *c* = *return_pmf* ((*if x*=*hd i then i else rev i*),[])

lemma *TS_inv_sym*: *a*≠*b* ⟹ {*a*,*b*}={*x*,*y*} ⟹ *z*∈{*x*,*y*} ⟹ *TS_inv c z* [*a*,*b*] = *TS_inv c z* [*x*,*y*]
⟨*proof*⟩

abbreviation *TS_inv′ s x i* == *TS_inv* (*return_pmf s*) *x i*

lemma *TS_inv'_det*: *TS_inv' s x i* = ((∃ *hs*. *s* = ((*if x=hd i then i else rev i*),[*x*,*x*]@*hs*))
∨ *s* = ((*if x=hd i then i else rev i*),[]))
⟨*proof*⟩

lemma *TS_inv'_det2*: *TS_inv'* (*s*,*h*) *x i* = (∃ *hs*. (*s*,*h*) = ((*if x=hd i then i else rev i*),[*x*,*x*]@*hs*))
∨ (*s*,*h*) = ((*if x=hd i then i else rev i*),[])
⟨*proof*⟩

15.3.1 (yx)*?

lemma *TS_yx'*: **assumes** *x* ≠ *y qs* ∈ *lang* (*Star*(*Times* (*Atom y*) (*Atom x*)))
∃ *hs*. *h*=[*x*,*y*]@*hs*
shows *T_on'* (*rTS h0*) ([*x*,*y*],*h*) (*qs*@*r*) = *length qs* + *T_on'* (*rTS h0*) ([*x*,*y*],((*rev qs*) @*h*)) *r*
∧ (∃ *hs*. ((*rev qs*) @*h*) = [*x*, *y*] @ *hs*)
∧ *config'* (*rTS h0*) ([*x*, *y*],*h*) *qs* = ([*x*,*y*],*rev qs* @ *h*)
⟨*proof*⟩

15.3.2 ?x

lemma *TS_x'*: *T_on'* (*rTS h0*) ([*x*,*y*],*h*) [*x*] = *0* ∧ *config'* (*rTS h0*) ([*x*, *y*],*h*) [*x*] = ([*x*,*y*], *rev* [*x*] @ *h*)
⟨*proof*⟩

15.3.3 ?yy

lemma *TS_yy'*: **assumes** *x* ≠ *y* ∃ *hs*. *h* = [*x*, *y*] @ *hs*
shows *T_on'* (*rTS h0*) ([*x*,*y*],*h*) [*y*, *y*] = *1 config'* (*rTS h0*) ([*x*, *y*],*h*) [*y*,*y*] = ([*y*,*x*],*rev* [*y*,*y*] @ *h*)
⟨*proof*⟩

15.3.4 yx(yx)*?

lemma *TS_yxyx'*: **assumes** [*simp*]: *x* ≠ *y* **and** *qs* ∈ *lang* (*seq*[*Times* (*Atom y*) (*Atom x*), *Star*(*Times* (*Atom y*) (*Atom x*))])
(∃ *hs*. *h*=[*x*,*x*]@*hs*) ∨ *index h y* = *length h*
shows *T_on'* (*rTS h0*) ([*x*,*y*],*h*) (*qs*@*r*) = *length qs* − *1* + *T_on'* (*rTS h0*) ([*x*,*y*],*rev qs* @ *h*) *r*
∧ (∃ *hs*. (*rev qs* @ *h*) = [*x*, *y*] @ *hs*)
∧ *config'* (*rTS h0*) ([*x*, *y*],*h*) *qs* = ([*x*,*y*], *rev qs* @ *h*)
⟨*proof*⟩

lemma *TS_xr'*: **assumes** $x \neq y$ $qs \in lang\ (Plus\ (Atom\ x)\ One)$
$h = [] \vee (\exists hs.\ h = [x,\ x] @ hs)$
shows $T\_on'\ (rTS\ h0)\ ([x,y],h)\ (qs@r) = T\_on'\ (rTS\ h0)\ ([x,y],rev\ qs@h)\ r$
$((\exists hs.\ (rev\ qs @ h) = [x,\ x] @ hs) \vee (rev\ qs @ h) = [x] \vee (rev\ qs @ h)=[])$
$config'\ (rTS\ h0)\ ([x,y],h)\ (qs@r) = config'\ (rTS\ h0)\ ([x,y],rev\ qs @ h)\ r$
⟨*proof*⟩

15.3.5 (x+1)yx(yx)*yy

lemma *ts_b'*: **assumes** $x \neq y$
$v \in lang\ (seq[Times\ (Atom\ y)\ (Atom\ x),\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
$(\exists hs.\ h = [x,\ x] @ hs) \vee h = [x] \vee h = []$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ v = (length\ v - 2)$
$\wedge\ (\exists hs.\ (rev\ v @ h) = [y,y]@hs) \wedge config'\ (rTS\ h0)\ ([x,y],\ h)\ v = ([y,x],\ rev\ v @ h)$
⟨*proof*⟩

lemma *TS_b'1*: **assumes** $x \neq y$ $h = [] \vee (\exists hs.\ h = [x,\ x] @ hs)$
$qs \in lang\ (seq\ [Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs = (length\ qs - 2)$
$\wedge\ TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ (last\ qs)\ [x,y]$
⟨*proof*⟩

lemma *TS_b1''*: **assumes**
$x \neq y$ $\{x,\ y\} = \{x0,\ y0\}$ $TS\_inv\ s\ x\ [x0,\ y0]$
$set\ qs \subseteq \{x,\ y\}$
$qs \in lang\ (seq\ [Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $TS\_inv\ (config'\_rand\ (embed\ (rTS\ h0))\ s\ qs)\ (last\ qs)\ [x0,\ y0]$
$\wedge\ T\_on\_rand'\ (embed\ (rTS\ h0))\ s\ qs = (length\ qs - 2)$
⟨*proof*⟩

lemma *ts_b2'*: **assumes** $x \neq y$

$qs \in lang\ (seq[Atom\ x,\ Times\ (Atom\ y)\ (Atom\ x),\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
$(\exists\, hs.\ h = [x,\ x]\ @\ hs) \vee h = []$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs = (length\ qs - 3)$
$\wedge\ \ config'\ (rTS\ h0)\ ([x,y],\ h)\ qs = ([y,x],rev\ qs@h) \wedge (\exists\, hs.\ (rev\ qs\ @\ h) = [y,y]@hs)$
⟨*proof*⟩

lemma *TS_b2''*: **assumes**
$x \neq y\ \{x,\ y\} = \{x0,\ y0\}\ TS\_inv\ s\ x\ [x0,\ y0]$
$set\ qs \subseteq \{x,\ y\}$
$qs \in lang\ (seq\ [Atom\ x,\ Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $TS\_inv\ (config'\_rand\ (embed\ (rTS\ h0))\ s\ qs)\ (last\ qs)\ [x0,\ y0]$
$\wedge\ T\_on\_rand'\ (embed\ (rTS\ h0))\ s\ qs = (length\ qs - 3)$
⟨*proof*⟩

lemma *TS_b'*: **assumes** $x \neq y\ h = [] \vee (\exists\, hs.\ h = [x,\ x]\ @\ hs)$
$qs \in lang\ (seq\ [Plus\ (Atom\ x)\ rexp.One,\ Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs$
$\leq 2 * T_p\ [x,\ y]\ qs\ (OPT2\ qs\ [x,\ y]) \wedge TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ (last\ qs)\ [x,y]$
⟨*proof*⟩

15.3.6 (x+1)yy

lemma *ts_a'*: **assumes** $x \neq y\ qs \in lang\ (seq\ [Plus\ (Atom\ x)\ One,\ Atom\ y,\ Atom\ y])$
$h = [] \vee (\exists\, hs.\ h = [x,\ x]\ @\ hs)$
shows $TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ (last\ qs)\ [x,y]$
$\wedge\ T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs = 2$
⟨*proof*⟩

lemma *TS_a'*: **assumes** $x \neq y$
$h = [] \vee (\exists\, hs.\ h = [x,\ x]\ @\ hs)$
and $qs \in lang\ (seq\ [Plus\ (Atom\ x)\ rexp.One,\ Atom\ y,\ Atom\ y])$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs \leq 2 * T_p\ [x,\ y]\ qs\ (OPT2\ qs\ [x,\ y])$
$\wedge\ TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ (last\ qs)\ [x,\ y]$
$\wedge\ T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs = 2$
⟨*proof*⟩

lemma *TS_a''*: **assumes**
$x \neq y$ $\{x, y\} = \{x0, y0\}$ *TS_inv s x* $[x0, y0]$
set qs $\subseteq \{x, y\}$ $qs \in lang$ (*seq* [*Plus* (*Atom x*) *One*, *Atom y*, *Atom y*])
shows
TS_inv (*config'_rand* (*embed* (*rTS h0*)) *s qs*) (*last qs*) $[x0, y0]$
$\wedge$ T_p*_on_rand'* (*embed* (*rTS h0*)) *s qs* = *2*
⟨*proof*⟩

15.3.7 x+yx(yx)*x

lemma *ts_c'*: **assumes** $x \neq y$
$v \in lang$ (*seq*[*Times* (*Atom y*) (*Atom x*), *Star* (*Times* (*Atom y*) (*Atom x*)), *Atom x*])
$(\exists\, hs.\ h = [x, x] @ hs) \vee h = [x] \vee h = []$
shows *T_on'* (*rTS h0*) $([x, y], h)$ *v* = (*length v* − *2*)
$\wedge$ *config'* (*rTS h0*) ([x,y], *h*) *v* = ([x,y],*rev v*@*h*) $\wedge$ $(\exists\, hs.$ (*rev v* @ *h*) = [x,x]@*hs*)
⟨*proof*⟩

lemma *TS_c1''*: **assumes**
$x \neq y$ $\{x, y\} = \{x0, y0\}$ *TS_inv s x* $[x0, y0]$
set qs $\subseteq \{x, y\}$
qs $\in$ *lang* (*seq* [*Atom y*, *Atom x*, *Star* (*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows *TS_inv* (*config'_rand* (*embed* (*rTS h0*)) *s qs*) (*last qs*) $[x0, y0]$
$\wedge$ *T_on_rand'* (*embed* (*rTS h0*)) *s qs* = (*length qs* − *2*)
⟨*proof*⟩

lemma *ts_c2'*: **assumes** $x \neq y$
qs $\in$ *lang* (*seq*[*Atom x*, *Times* (*Atom y*) (*Atom x*), *Star* (*Times* (*Atom y*) (*Atom x*)), *Atom x*])
$(\exists\, hs.\ h = [x, x] @ hs) \vee h = []$
shows *T_on'* (*rTS h0*) $([x, y], h)$ *qs* = (*length qs* − *3*)
$\wedge$ *config'* (*rTS h0*) ([x,y], *h*) *qs* = ([x,y],*rev qs*@*h*) $\wedge$ $(\exists\, hs.$ (*rev qs* @ *h*) = [x,x]@*hs*)
⟨*proof*⟩

lemma *TS_c2''*: **assumes**
$x \neq y$ $\{x, y\} = \{x0, y0\}$ *TS_inv s x* $[x0, y0]$
set qs $\subseteq \{x, y\}$
qs $\in$ *lang* (*seq* [*Atom x*, *Atom y*, *Atom x*, *Star* (*Times* (*Atom y*) (*Atom*

$x)),\ Atom\ x])$
shows $TS\_inv\ (config'\_rand\ (embed\ (rTS\ h0))\ s\ qs)\ (last\ qs)\ [x0,\ y0]$
$\wedge\ T\_on\_rand'\ (embed\ (rTS\ h0))\ s\ qs = (length\ qs - 3)$
⟨*proof*⟩

lemma *TS_c'*: **assumes** $x \neq y\ h = [] \vee (\exists hs.\ h = [x,\ x]\ @\ hs)$
$qs \in lang\ (seq\ [Plus\ (Atom\ x)\ rexp.One,\ Atom\ y,\ Atom\ x,\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ x])$
shows $T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs$
$\leq 2 * T_p\ [x,\ y]\ qs\ (OPT2\ qs\ [x,\ y]) \wedge\ TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ (last\ qs)\ [x,y]$
⟨*proof*⟩

15.3.8 xx

lemma *request_first*: $x{\neq}y \Longrightarrow Step\ (rTS\ h)\ ([x,\ y],\ is)\ x = ([x,y],x\#is)$
⟨*proof*⟩

lemma *ts_d'*: $qs \in Lxx\ x\ y \Longrightarrow$
$x \neq y \Longrightarrow$
$h = [] \vee (\exists hs.\ h = [x,\ x]\ @\ hs) \Longrightarrow$
$qs \in lang\ (seq\ [Atom\ x,\ Atom\ x]) \Longrightarrow$
$T\_on'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs = 0 \wedge$
$TS\_inv'\ (config'\ (rTS\ h0)\ ([x,\ y],\ h)\ qs)\ x\ [x,y]$
⟨*proof*⟩

lemma *TS_d'*: **assumes** *xny*: $x \neq y$ **and** $h = [] \vee (\exists hs.\ h = [x,\ x]\ @\ hs)$
and *qsis*: $qs \in lang\ (seq\ [Atom\ x,\ Atom\ x])$
shows $T\_on'\ (rTS\ h0)\ ([x,y],h)\ qs \leq 2 * T_p\ [x,\ y]\ qs\ (OPT2\ qs\ [x,\ y])$

and $TS\_inv'\ (config'\ (rTS\ h0)\ ([x,y],h)\ qs)\ \ (last\ qs)\ [x,\ y]$
and $T\_on'\ (rTS\ h0)\ ([x,y],h)\ qs = 0$
⟨*proof*⟩

lemma *TS_d''*: **assumes**
$x \neq y\ \{x,\ y\} = \{x0,\ y0\}\ TS\_inv\ s\ x\ [x0,\ y0]$
$set\ qs \subseteq \{x,\ y\}$
$qs \in lang\ (seq\ [Atom\ x,\ Atom\ x])$
shows $TS\_inv\ (config'\_rand\ (embed\ (rTS\ h0))\ s\ qs)\ (last\ qs)\ [x0,\ y0]$
$\wedge\ T\_on\_rand'\ (embed\ (rTS\ h0))\ s\ qs = 0$

⟨*proof*⟩

15.4 Phase Partitioning

lemma *D′*: **assumes** $\sigma' \in Lxx\ x\ y$ **and** $x \neq y$ **and** *TS_inv′* ([*x*, *y*], *h*) *x* [*x*, *y*]
shows *T_on′* (*rTS h0*) ([*x*, *y*], *h*) $\sigma' \leq 2 * T_p\ [x, y]\ \sigma'$ (*OPT2* σ' [*x*, *y*])
$\wedge$ *TS_inv* (*config′_rand* (*embed* (*rTS h0*)) (*return_pmf* ([*x*, *y*], *h*)) σ') (*last* σ') [*x*, *y*]
⟨*proof*⟩

theorem *TS_OPT2′*: (*x*::*nat*) $\neq y \Longrightarrow$ *set* $\sigma \subseteq \{x,y\}$
$\Longrightarrow T_p$*_on* (*rTS* []) [*x*,*y*] $\sigma \leq 2 *$ *real* (T_p*_opt* [*x*,*y*] σ) + *2*
⟨*proof*⟩

15.5 TS is pairwise

lemma *config′_distinct*[*simp*]:
shows *distinct* (*fst* (*config′ A S qs*)) = *distinct* (*fst S*)
⟨*proof*⟩

lemma *config′_set*[*simp*]:
shows *set* (*fst* (*config′ A S qs*)) = *set* (*fst S*)
⟨*proof*⟩

lemma *s_TS_append*: *i*≤*length as* ⟹*s_TS init h* (*as*@*bs*) *i* = *s_TS init h as i*
⟨*proof*⟩

lemma *s_TS_distinct*: *distinct init* ⟹ *i*<*length qs* ⟹ *distinct* (*fst* (*TSdet init h qs i*))
⟨*proof*⟩

lemma *othersdontinterfere*: *distinct init* ⟹ *i* < *length qs* ⟹ *a*∈*set init* ⟹ *b*∈*set init*
⟹ *set qs* ⊆ *set init* ⟹ *qs*!*i*∉{*a*,*b*} ⟹ *a* < *b in s_TS init h qs i* ⟹ *a* < *b in s_TS init h qs* (*Suc i*)
⟨*proof*⟩

lemma *TS_mono*:
fixes *l*::*nat*
assumes *1*: *x* < *y in s_TS init h xs* (*length xs*)
and *l_in_cs*: *l* ≤ *length cs*

and *firstocc*: $\forall j{<}l.\ cs\ !\ j \neq y$
and $x \notin set\ cs$
and *di*: *distinct init*
and *inin*: $set\ (xs\ @\ cs) \subseteq set\ init$
shows $x < y\ in\ s\_TS\ init\ h\ (xs@cs)\ (length\ (xs){+}l)$
⟨*proof*⟩

lemma *step_no_action*: $step\ s\ q\ (0{,}[]) = s$
⟨*proof*⟩

lemma *s_TS_set*: $i \leq length\ qs \Longrightarrow set\ (s\_TS\ init\ h\ qs\ i) = set\ init$
⟨*proof*⟩

lemma *count_notin2*: $count\_list\ xs\ x = 0 \Longrightarrow x \notin set\ xs$
⟨*proof*⟩

lemma *count_append*: $count\_list\ (xs@ys)\ x = count\_list\ xs\ x + count\_list\ ys\ x$
⟨*proof*⟩

lemma *count_rev*: $count\_list\ (rev\ xs)\ x = count\_list\ xs\ x$
⟨*proof*⟩

lemma *mtf2_q_passes*: **assumes** $q \in set\ xs\ distinct\ xs$
and $index\ xs\ q - n \leq index\ xs\ x\ index\ xs\ x < index\ xs\ q$
shows $q < x\ in\ (mtf2\ n\ q\ xs)$
⟨*proof*⟩

lemma *twotox*:
assumes $count\_list\ bs\ y \leq 1$
and *distinct init*
and $x \in set\ init$
and $y : set\ init$
and $x \notin set\ bs$
and $x{\neq}y$
shows $x < y\ in\ s\_TS\ init\ h\ (as@[x]@bs@[x])\ (length\ (as@[x]@bs@[x]))$
⟨*proof*⟩

lemma *count_drop*: $count\_list\ (drop\ n\ cs)\ x \leq count\_list\ cs\ x$
⟨*proof*⟩

lemma *count_take_less*: **assumes** $n{\leq}m$
shows $count\_list\ (take\ n\ cs)\ x \leq count\_list\ (take\ m\ cs)\ x$
⟨*proof*⟩

lemma *count_take*: *count_list* (*take n cs*) *x* ≤ *count_list cs x*
⟨*proof*⟩

lemma *casexxy*: **assumes** *σ*=*as*@[*x*]@*bs*@[*x*]@*cs*
and *x* ∉ *set cs*
and *set cs* ⊆ *set init*
and *x* ∈ *set init*
and *distinct init*
and *x* ∉ *set bs*
and *set as* ⊆ *set init*
and *set bs* ⊆ *set init*
shows (%*i*. *i*<*length cs* ⟶ (∀ *j*<*i*. *cs*!*j*≠*cs*!*i*) ⟶ *cs*!*i*≠*x*
⟶ (*cs*!*i*) ∉ *set bs*
⟶ *x* < (*cs*!*i*) *in* (*s_TS init h σ* (*length* (*as*@[*x*]@*bs*@[*x*]) + *i*+*1*))) *i*
⟨*proof*⟩

lemma *nopaid*: *snd* (*fst* (*TS_step_d s q*)) = [] ⟨*proof*⟩

lemma *staysuntouched*:
assumes *d*[*simp*]: *distinct* (*fst S*)
and *x*: *x* ∈ *set* (*fst S*)
and *y*: *y* ∈ *set* (*fst S*)
shows *set qs* ⊆ *set* (*fst S*) ⟹ *x* ∉ *set qs* ⟹ *y* ∉ *set qs*
⟹ *x* < *y in fst* (*config*′ (*rTS* []) *S qs*) = *x* < *y in fst S*
⟨*proof*⟩

lemma *staysuntouched*′:
assumes *d*[*simp*]: *distinct init*
and *x*: *x* ∈ *set init*
and *y*: *y* ∈ *set init*
and *set qs* ⊆ *set init*
and *x* ∉ *set qs* **and** *y* ∉ *set qs*
shows *x* < *y in fst* (*config* (*rTS* []) *init qs*) = *x* < *y in init*
⟨*proof*⟩

lemma *projEmpty*: *Lxy qs S* = [] ⟹ *x* ∈ *S* ⟹ *x* ∉ *set qs*
⟨*proof*⟩

lemma *Lxy_index_mono*:
assumes *x*∈*S y*∈*S*
and *index xs x* < *index xs y*
and *index xs y* < *length xs*
and *x*≠*y*

shows *index* (*Lxy xs S*) *x* < *index* (*Lxy xs S*) *y*
⟨*proof*⟩

lemma *proj_Cons*:
assumes *filterd_cons*: *Lxy qs S* = *a*#*as*
and *a_filter*: *a*∈*S*
obtains *pre suf* **where** *qs* = *pre* @ [*a*] @ *suf* **and** ⋀*x*. *x* ∈ *S* ⟹ *x* ∉ *set pre*
and *Lxy suf S* = *as*
⟨*proof*⟩

lemma *Lxy_rev*: *rev* (*Lxy qs S*) = *Lxy* (*rev qs*) *S*
⟨*proof*⟩

lemma *proj_Snoc*:
assumes *filterd_cons*: *Lxy qs S* = *as*@[*a*]
and *a_filter*: *a*∈*S*
obtains *pre suf* **where** *qs* = *pre* @ [*a*] @ *suf* **and** ⋀*x*. *x* ∈ *S* ⟹ *x* ∉ *set suf*
and *Lxy pre S* = *as*
⟨*proof*⟩

lemma *sndTSconfig′*: *snd* (*config′* (*rTS initH*) (*init*,[]) *qs*) = *rev qs* @ []
⟨*proof*⟩

lemma *projxx*:
fixes *e a bs*
assumes *axy*: *a*∈{*x*,*y*}
assumes *ane*: *a*≠*e*
assumes *exy*: *e*∈{*x*,*y*}
assumes *add*: *f*∈{[],[*e*]}
assumes *bsaxy*: *set* (*bs* @ [*a*] @ *f*) ⊆ {*x*,*y*}
assumes *Lxyinitxy*: *Lxy init* {*x*, *y*} ∈ {[*x*,*y*],[*y*,*x*]}
shows *a* < *e in fst* (*config*$_p$ (*rTS* []) (*Lxy init* {*x*, *y*}) ((*bs* @ [*a*] @ *f*) @ [*a*]))
⟨*proof*⟩

lemma *oneposs*:
assumes *set xs* = {*x*,*y*}
assumes *x*≠*y*
assumes *distinct xs*
assumes *True*: *x*<*y in xs*
shows *xs* = [*x*,*y*]
⟨*proof*⟩

lemma *twoposs*:
  assumes *set xs* = {*x*,*y*}
    assumes *x*≠*y*
    assumes *distinct xs*
    shows *xs* ∈ {[*x*,*y*], [*y*,*x*]}
⟨*proof*⟩

lemma *TS_pairwise'*: **assumes** *qs* ∈ {*xs*. *set xs* ⊆ *set init*}
      (*x*, *y*) ∈ {(*x*, *y*). *x* ∈ *set init* ∧ *y* ∈ *set init* ∧ *x* ≠ *y*}
      x ≠ *y distinct init*
  shows *Pbefore_in x y* (*embed* (*rTS* [])) *qs init* =
      Pbefore_in x y (*embed* (*rTS* [])) (*Lxy qs* {*x*, *y*}) (*Lxy init* {*x*, *y*})
⟨*proof*⟩

theorem *TS_pairwise*: *pairwise* (*embed* (*rTS* []))
⟨*proof*⟩

15.6 TS is 2-compet

lemma *TS_compet'*: *pairwise* (*embed* (*rTS* [])) ⟹
    ∀ *s0*∈{*init*::(*nat list*). *distinct init* ∧ *init*≠[]}. ∃ *b*≥*0*. ∀ *qs*∈{*x*. *set x* ⊆ *set s0*}. T_p_*on_rand* (*embed* (*rTS* [])) *s0 qs* ≤ (*2*::*real*) ∗ T_p_*opt s0 qs* + *b*
⟨*proof*⟩

lemma *TS_compet*: *compet_rand* (*embed* (*rTS* [])) *2* {*init*. *distinct init* ∧ *init* ≠ []}
⟨*proof*⟩

end

16 BIT is pairwise

theory *BIT_pairwise*
imports *List_Factoring BIT*
begin

lemma *L_nths*: *S* ⊆ {..<*length init*}
  ⟹ *map_pmf* (λ*l*. *nths l S*) (*Prob_Theory.bv* (*length init*))
    = (*Prob_Theory.bv* (*length* (*nths init S*)))
⟨*proof*⟩

lemma *L_nths_Lxy*:
assumes $x \in set\ init\ y \in set\ init\ x \neq y\ distinct\ init$
shows $map\_pmf\ (\lambda l.\ nths\ l\ \{index\ init\ x, index\ init\ y\})\ (Prob\_Theory.bv\ (length\ init)) = (Prob\_Theory.bv\ (length\ (Lxy\ init\ \{x,y\})))$
⟨*proof*⟩

lemma *nths_map*: $map\ f\ (nths\ xs\ S) = nths\ (map\ f\ xs)\ S$
⟨*proof*⟩

lemma *nths_empty*: $(\forall\, i \in S.\ i \geq length\ xs) \Longrightarrow nths\ xs\ S = []$
⟨*proof*⟩

lemma *nths_project'*: $i < length\ xs \Longrightarrow j < length\ xs \Longrightarrow i<j \Longrightarrow nths\ xs\ \{i,j\} = [xs!i,\ xs!j]$
⟨*proof*⟩

lemma *nths_project*:
assumes $i < length\ xs\ j < length\ xs\ i<j$
shows $nths\ xs\ \{i,j\}\ !\ 0 = xs\ !\ i \wedge nths\ xs\ \{i,j\}\ !\ 1 = xs\ !\ j$
⟨*proof*⟩

lemma *BIT_pairwise'*:
assumes $set\ qs \subseteq set\ init$
$(x,y) \in \{(x,y).\ x \in set\ init \wedge y \in set\ init \wedge x \neq y\}$
and $xny: x \neq y$ **and** $dinit: distinct\ init$
shows $Pbefore\_in\ x\ y\ BIT\ qs\ init = Pbefore\_in\ x\ y\ BIT\ (Lxy\ qs\ \{x,y\})\ (Lxy\ init\ \{x,y\})$
⟨*proof*⟩

theorem *BIT_pairwise*: *pairwise BIT*
⟨*proof*⟩

end

17 BIT is 1.75 competitive on lists of length 2

theory *BIT_2comp_on2*
imports *BIT Phase_Partitioning*
begin

17.1 auxliary lemmas

17.1.1 *E_bernoulli3*

lemma *E_bernoulli3*: **assumes** *0<p*
and *p<1*
and *finite* (*set_pmf* (*bind_pmf* (*bernoulli_pmf p*) *f*))
shows *E* (*bind_pmf* (*bernoulli_pmf p*) *f*) = *E*(*f True*)∗*p* + *E*(*f False*)∗(*1*−*p*)
(**is** *?L* = *?R*)
⟨*proof*⟩

17.1.2 types of configurations

definition *type0 init x y* = *do* {
(*a*::*bool*) ← (*bernoulli_pmf 0.5*);
(*b*::*bool*) ← (*bernoulli_pmf 0.5*);
return_pmf ([*x*,*y*], ([*a*,*b*],*init*))
}

definition *type1 init x y* = *do* {
(*a*::*bool*) ← (*bernoulli_pmf 0.5*);
(*b*::*bool*) ← (*bernoulli_pmf 0.5*);
return_pmf (*if* ~[*a*,*b*]!(*index init x*)∧[*a*,*b*]!(*index init y*) *then* ([*y*,*x*], ([*a*,*b*],*init*))
else ([*x*,*y*], ([*a*,*b*],*init*)))
}

definition *type3 init x y* = *do* {
(*a*::*bool*) ← (*bernoulli_pmf 0.5*);
(*b*::*bool*) ← (*bernoulli_pmf 0.5*);
return_pmf (*if* [*a*,*b*]!(*index init x*)∧~[*a*,*b*]!(*index init y*) *then* ([*x*,*y*], ([*a*,*b*],*init*))
else ([*y*,*x*], ([*a*,*b*],*init*)))
}

definition *type4 init x y* = *do* {
(*a*::*bool*) ← (*bernoulli_pmf 0.5*);
(*b*::*bool*) ← (*bernoulli_pmf 0.5*);
return_pmf (*if* ~[*a*,*b*]!(*index init y*) *then* ([*x*,*y*], ([*a*,*b*],*init*))
else ([*y*,*x*], ([*a*,*b*],*init*)))
}

definition *BIT_inv s x i* == (*s* = (*type0 i x* (*hd* (*filter* (λ*y*. *y*≠*x*) *i*))))

lemma *BIT_inv2*: *x*≠*y* ⟹ *z*∈{*x*,*y*} ⟹ *BIT_inv s z* [*x*,*y*] = (*s*= *type0*

$[x,y]$ z $(other\ z\ x\ y))$
⟨*proof*⟩

17.1.3 cost of BIT

lemma *costBIT_0x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
$E\ (type0\ [x0,\ y0]\ x\ y \gg\!=$
$(\lambda s.\ BIT\_step\ s\ x \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ x\ a)))))\ =\ 0$
⟨*proof*⟩

lemma *costBIT_0y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
$E\ (type0\ [x0,\ y0]\ x\ y \gg\!=$
$(\lambda s.\ BIT\_step\ s\ y \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ y\ a)))))\ =\ 1$
⟨*proof*⟩

lemma *costBIT_1x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
$E\ (type1\ [x0,\ y0]\ x\ y \gg\!=$
$(\lambda s.\ BIT\_step\ s\ x \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ x\ a)))))\ =\ 1/4$
⟨*proof*⟩

lemma *costBIT_1y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
$E\ (type1\ [x0,\ y0]\ x\ y \gg\!=$
$(\lambda s.\ BIT\_step\ s\ y \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ y\ a)))))\ =\ 3/4$
⟨*proof*⟩

lemma *costBIT_3x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
$E\ (type3\ [x0,\ y0]\ x\ y \gg\!=$
$(\lambda s.\ BIT\_step\ s\ x \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ x\ a)))))\ =\ 3/4$
⟨*proof*⟩

lemma *costBIT_3y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
E (*type3* $[x0,\ y0]$ x y $\gg\!=$
$(\lambda s.\ BIT\_step\ s\ y \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ y\ a)))))\ =\ 1/4$
⟨*proof*⟩

lemma *costBIT_4x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
E (*type4* $[x0,\ y0]$ x y $\gg\!=$
$(\lambda s.\ BIT\_step\ s\ x \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ x\ a)))))\ =\ 0.5$
⟨*proof*⟩

lemma *costBIT_4y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows
E (*type4* $[x0,\ y0]$ x y $\gg\!=$
$(\lambda s.\ BIT\_step\ s\ y \gg\!=$
$(\lambda(a,\ is').\ return\_pmf\ (real\ (t_p\ (fst\ s)\ y\ a)))))\ =\ 0.5$
⟨*proof*⟩

lemmas *costBIT = costBIT_0x costBIT_0y costBIT_1x costBIT_1y costBIT_3x costBIT_3y costBIT_4x costBIT_4y*

17.1.4 state transformation of BIT

abbreviation *BIT_Step s x == (s ≫= (λs. BIT_step s x ≫= (λ(a, is′). return_pmf (step (fst s) x a, is′))))*

lemma *oneBIT_step0x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type0* $[x0,\ y0]$ x y) x = *type0* $[x0,\ y0]$ x y
⟨*proof*⟩

lemma *oneBIT_step0y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type0* $[x0,\ y0]$ x y) y = *type4* $[x0,\ y0]$ x y
⟨*proof*⟩

lemma *oneBIT_step1x*:

assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type1* [*x0*, *y0*] *x y*) *x* = *type0* [*x0*, *y0*] *x y*
⟨*proof*⟩

lemma *oneBIT_step1y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type1* [*x0*, *y0*] *x y*) *y* = *type3* [*x0*, *y0*] *x y*
⟨*proof*⟩

lemma *oneBIT_step3x*:
assumes $x{\neq}y$ $x{:}\{x0,y0\}$ $y{:}\{x0,y0\}$
shows *BIT_Step* (*type3* [*x0*, *y0*] *x y*) *x* = *type1* [*x0*, *y0*] *x y*
⟨*proof*⟩

lemma *oneBIT_step3y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type3* [*x0*, *y0*] *x y*) *y* = *type0* [*x0*, *y0*] *y x*
⟨*proof*⟩

lemma *oneBIT_step4x*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type4* [*x0*, *y0*] *x y*) *x* = *type1* [*x0*, *y0*] *x y*
⟨*proof*⟩

lemma *oneBIT_step4y*:
assumes $x{\neq}y$ $x : \{x0,y0\}$ $y{\in}\{x0,y0\}$
shows *BIT_Step* (*type4* [*x0*, *y0*] *x y*) *y* = *type0* [*x0*, *y0*] *y x*
⟨*proof*⟩

lemmas *oneBIT_step* = *oneBIT_step0x oneBIT_step0y oneBIT_step1x oneBIT_step1y oneBIT_step3x oneBIT_step3y oneBIT_step4x oneBIT_step4y*

17.2 Analysis of the four phase forms

17.2.1 yx

lemma *bit_yx*: **assumes** $x \neq y$
and *kas*: *init* ∈ {[*x*,*y*],[*y*,*x*]}
and *qs* ∈ *lang* (*Star*(*Times* (*Atom y*) (*Atom x*)))
shows T_p*_on_rand′ BIT* (*type1 init x y*) (*qs*@*r*) = *0.75* ∗ *length qs* + T_p*_on_rand′ BIT* (*type1 init x y*) *r*
∧ *config′_rand BIT* (*type1 init x y*) *qs* = (*type1 init x y*)
⟨*proof*⟩

17.2.2 (yx)*yx

lemma *bit_yxyx*: **assumes** $x \neq y$ **and** *kas*: $init \in \{[x,y],[y,x]\}$ **and**
$qs \in lang\ (seq[Times\ (Atom\ y)\ (Atom\ x), Star(Times\ (Atom\ y)\ (Atom\ x))])$
shows $T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ (qs@r) = 0.75 * length\ qs + T_p\_on\_rand'\ BIT\ (type1\ init\ x\ y)\ r$
$\wedge\ config'\_rand\ BIT\ (type0\ init\ x\ y)\ qs\ = (type1\ init\ x\ y)$
⟨*proof*⟩

17.2.3 $x\hat{}+..$

lemma *BIT_x*: **assumes** $x \neq y$
$init \in \{[x,y],[y,x]\}\ qs \in lang\ (Plus\ (Atom\ x)\ One)$
shows $T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ (qs@r) = T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ r$
$\wedge\ config'\_rand\ BIT\ (type0\ init\ x\ y)\ qs = (type0\ init\ x\ y)$
⟨*proof*⟩

17.2.4 Phase Form A

lemma *BIT_a*: **assumes** $x \neq y$
$init \in \{[x,y],[y,x]\}$
$qs \in lang\ (seq\ [Plus\ (Atom\ x)\ One, Atom\ y, Atom\ y])$
shows $config'\_rand\ BIT\ (type0\ init\ x\ y)\ qs = (type0\ init\ y\ x)$ (**is** *?C*)
and *b*: $T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ qs = 1.5$ (**is** *?T*)
⟨*proof*⟩

lemma *bit_a*: **assumes**
$x \neq y\ \{x, y\} = \{x0, y0\}\ BIT\_inv\ s\ x\ [x0, y0]$
$set\ qs \subseteq \{x, y\}\ qs \in lang\ (seq\ [Plus\ (Atom\ x)\ One, Atom\ y, Atom\ y])$
shows
$T_p\_on\_rand'\ BIT\ s\ qs\ \leq 1.75 * T_p\ [x,y]\ qs\ (OPT2\ qs\ [x,y])$
$\wedge\ BIT\_inv\ (config'\_rand\ BIT\ s\ qs)\ (last\ qs)\ [x0, y0]$
$\wedge\ T_p\_on\_rand'\ BIT\ s\ qs = 1.5$
⟨*proof*⟩

lemma *bit_a''*: $a \neq b \Longrightarrow$
$\{a, b\} = \{x, y\} \Longrightarrow$
$BIT\_inv\ s\ a\ [x, y] \Longrightarrow$
$set\ qs \subseteq \{a, b\} \Longrightarrow$
$qs \in lang\ (seq\ [question\ (Atom\ a), Atom\ b, Atom\ b]) \Longrightarrow$
$BIT\_inv\ (Partial\_Cost\_Model.config'\_rand\ BIT\ s\ qs)\ (last\ qs)\ [x, y] \wedge T_p\_on\_rand'\ BIT\ s\ qs = 1.5$
⟨*proof*⟩

17.2.5 Phase Form B

lemma *BIT_b*: **assumes** *A*: $x \neq y$
$init \in \{[x,y],[y,x]\}$
$v \in lang\ (seq\ [Times\ (Atom\ y)\ (Atom\ x),\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ v = 0.75 * length\ v - 0.5$ (**is** *?T*)
and $config'\_rand\ BIT\ (type0\ init\ x\ y)\ v = (type0\ init\ y\ x)$ (**is** *?C*)
⟨*proof*⟩

lemma *bit_b''1*: **assumes**
$x \neq y\ \{x,\ y\} = \{x0,\ y0\}\ BIT\_inv\ s\ x\ [x0,\ y0]$
$set\ qs \subseteq \{x,\ y\}$
$qs \in lang\ (seq[Atom\ y,\ Atom\ x,\ Star(Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $BIT\_inv\ (config'\_rand\ BIT\ s\ qs)\ (last\ qs)\ [x0,\ y0] \wedge$
$T_p\_on\_rand'\ BIT\ s\ qs\ = 0.75 * length\ qs - 0.5$
⟨*proof*⟩

lemma *BIT_b2*: **assumes** *A*: $x \neq y$
$init \in \{[x,y],[y,x]\}$
$v \in lang\ (seq\ [Atom\ x,\ Times\ (Atom\ y)\ (Atom\ x),\ Star\ (Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $T_p\_on\_rand'\ BIT\ (type0\ init\ x\ y)\ v = 0.75 * (length\ v - 1) - 0.5$ (**is** *?T*)
and $config'\_rand\ BIT\ (type0\ init\ x\ y)\ v = (type0\ init\ y\ x)$ (**is** *?C*)
⟨*proof*⟩

lemma *bit_b''2*: **assumes**
$x \neq y\ \{x,\ y\} = \{x0,\ y0\}\ BIT\_inv\ s\ x\ [x0,\ y0]$
$set\ qs \subseteq \{x,\ y\}$
$qs \in lang\ (seq[Atom\ x,\ Atom\ y,\ Atom\ x,\ Star(Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$
shows $BIT\_inv\ (config'\_rand\ BIT\ s\ qs)\ (last\ qs)\ [x0,\ y0] \wedge$
$T_p\_on\_rand'\ BIT\ s\ qs\ = 0.75 * (length\ qs - 1) - 0.5$
⟨*proof*⟩

lemma *bit_b*: **assumes** $x \neq y$
$init \in \{[x,y],[y,x]\}$
$qs \in lang\ (seq[Plus\ (Atom\ x)\ One,\ Atom\ y,\ Atom\ x,\ Star(Times\ (Atom\ y)\ (Atom\ x)),\ Atom\ y,\ Atom\ y])$

shows $T_p\_on\_rand'$ *BIT* (*type0 init x y*) *qs* $\leq$ *1.75* $*$ T_p [*x*,*y*] *qs* (*OPT2 qs* [*x*,*y*])
and *config'_rand BIT* (*type0 init x y*) *qs* = *type0 init y x*
⟨*proof*⟩

lemma *bit_b''*: **assumes**
x $\neq$ *y* {*x*, *y*} = {*x0*, *y0*} *BIT_inv s x* [*x0*, *y0*]
set qs $\subseteq$ {*x*, *y*}
qs $\in$ *lang* (*seq*[*Plus* (*Atom x*) *One*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom y*, *Atom y*])
shows
$T_p\_on\_rand'$ *BIT s qs* $\leq$ *1.75* $*$ T_p [*x*,*y*] *qs* (*OPT2 qs* [*x*,*y*])
$\wedge$ *BIT_inv* (*config'_rand BIT s qs*) (*last qs*) [*x0*, *y0*]
⟨*proof*⟩

lemma *bit_b'''*: *a* $\neq$ *b* $\Longrightarrow$
{*a*, *b*} = {*x*, *y*} $\Longrightarrow$
BIT_inv s a [*x*, *y*] $\Longrightarrow$
set qs $\subseteq$ {*a*, *b*} $\Longrightarrow$
qs $\in$ *lang* (*seq*[*Plus* (*Atom x*) *One*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom y*, *Atom y*]) $\Longrightarrow$
BIT_inv (*Partial_Cost_Model.config'_rand BIT s qs*) (*last qs*) [*x*, *y*] $\wedge$ $T_p\_on\_rand'$ *BIT s qs* = *1.5*
⟨*proof*⟩

17.2.6 Phase Form C

lemma *BIT_c*: **assumes** *x* $\neq$ *y*
init $\in$ {[*x*,*y*],[*y*,*x*]}
v $\in$ *lang* (*seq* [*Times* (*Atom y*) (*Atom x*), *Star* (*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows $T_p\_on\_rand'$ *BIT* (*type0 init x y*) *v* = *0.75* $*$ *length v* $-$ *0.5*
and *config'_rand BIT* (*type0 init x y*) *v* = (*type0 init x y*) (**is** *?C*)
⟨*proof*⟩

lemma *bit_c''1*: **assumes**
x $\neq$ *y* {*x*, *y*} = {*x0*, *y0*} *BIT_inv s x* [*x0*, *y0*]
set qs $\subseteq$ {*x*, *y*}
qs $\in$ *lang* (*seq*[*Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows *BIT_inv* (*config'_rand BIT s qs*) (*last qs*) [*x0*, *y0*] $\wedge$
$T_p\_on\_rand'$ *BIT s qs* = *0.75* $*$ *length qs* $-$ *0.5*

⟨*proof*⟩

lemma *bit_c*: **assumes** $x \neq y$
init ∈ {[*x*,*y*],[*y*,*x*]}
qs ∈ *lang* (*seq*[*Plus* (*Atom x*) *One*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows T_p*_on_rand*′ *BIT* (*type0 init x y*) *qs* ≤ *1.75* ∗ T_p [*x*,*y*] *qs* (*OPT2 qs* [*x*,*y*])
and *config*′*_rand BIT* (*type0 init x y*) *qs* = *type0 init x y*
⟨*proof*⟩

lemma *bit_c*″: **assumes**
$x \neq y$ {*x*, *y*} = {*x0*, *y0*} *BIT_inv s x* [*x0*, *y0*]
set qs ⊆ {*x*, *y*}
qs ∈ *lang* (*seq*[*Plus* (*Atom x*) *One*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows
T_p*_on_rand*′ *BIT s qs* ≤ *1.75* ∗ T_p [*x*,*y*] *qs* (*OPT2 qs* [*x*,*y*])
∧ *BIT_inv* (*config*′*_rand BIT s qs*) (*last qs*) [*x0*, *y0*]
⟨*proof*⟩

lemma *BIT_c2*: **assumes** *A*: $x \neq y$
init ∈ {[*x*,*y*],[*y*,*x*]}
v ∈ *lang* (*seq* [*Atom x*, *Times* (*Atom y*) (*Atom x*), *Star* (*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows T_p*_on_rand*′ *BIT* (*type0 init x y*) *v* = *0.75* ∗ (*length v* − *1*) − *0.5* (**is** *?T*)
and *config*′*_rand BIT* (*type0 init x y*) *v* = (*type0 init x y*) (**is** *?C*)
⟨*proof*⟩

lemma *bit_c*″*2*: **assumes**
$x \neq y$ {*x*, *y*} = {*x0*, *y0*} *BIT_inv s x* [*x0*, *y0*]
set qs ⊆ {*x*, *y*}
qs ∈ *lang* (*seq*[*Atom x*, *Atom y*, *Atom x*, *Star*(*Times* (*Atom y*) (*Atom x*)), *Atom x*])
shows *BIT_inv* (*config*′*_rand BIT s qs*) (*last qs*) [*x0*, *y0*] ∧
T_p*_on_rand*′ *BIT s qs* = *0.75* ∗ (*length qs* − *1*) − *0.5*
⟨*proof*⟩

17.2.7 Phase Form D

lemma *bit_d*: **assumes**
$x \neq y$ $\{x, y\} = \{x0, y0\}$ *BIT_inv s x* $[x0, y0]$
set qs $\subseteq \{x, y\}$ *qs* $\in$ *lang* (*seq* [*Atom x*, *Atom x*])
shows T_p*_on_rand′ BIT s qs* $\leq$ *175 / 10*^{2} $*$ *real* (T_p $[x, y]$ *qs* (*OPT2 qs* $[x, y]$)) $\wedge$
BIT_inv (*config′_rand BIT s qs*) (*last qs*) $[x0, y0]$ $\wedge$
T_p*_on_rand′ BIT s qs* = *0*
⟨*proof*⟩

lemma *bit_d′*: **assumes**
$x \neq y$ $\{x, y\} = \{x0, y0\}$ *BIT_inv s x* $[x0, y0]$
set qs $\subseteq \{x, y\}$ *qs* $\in$ *lang* (*seq* [*Atom x*, *Atom x*])
shows *BIT_inv* (*config′_rand BIT s qs*) (*last qs*) $[x0, y0]$ $\wedge$
T_p*_on_rand′ BIT s qs* = *0*
⟨*proof*⟩

17.3 Phase Partitioning

lemma *BIT_inv_initial*: **assumes** (*x*::*nat*) $\neq$ *y*
shows *BIT_inv* (*map_pmf* (*Pair* $[x, y]$) (*fst BIT* $[x, y]$)) *x* $[x, y]$
⟨*proof*⟩

lemma *D″*: **assumes** *qs* $\in$ *Lxx a b*
$a \neq b$ $\{a, b\} = \{x, y\}$ *BIT_inv s a* $[x, y]$
set qs $\subseteq \{a, b\}$
shows T_p*_on_rand′ BIT s qs* $\leq$ *175 / 10*^{2} $*$ *real* (T_p $[a, b]$ *qs* (*OPT2 qs* $[a, b]$)) $\wedge$
BIT_inv (*Partial_Cost_Model.config′_rand BIT s qs*) (*last qs*) $[x, y]$
⟨*proof*⟩

theorem *BIT_175comp_on_2*:
assumes (*x*::*nat*) $\neq$ *y set* $\sigma \subseteq \{x,y\}$
shows T_p*_on_rand BIT* $[x,y]$ σ $\leq$ *1.75* $*$ *real* (T_p*_opt* $[x,y]$ σ) + *1.75*
⟨*proof*⟩

end

18 COMB

theory *Comb*
imports *TS BIT_2comp_on2 BIT_pairwise*

begin

18.1 Definition of COMB

type_synonym *CombState* = (*bool list* ∗ *nat list*) + (*nat list*)

definition *COMB_init* :: *nat list* ⇒ (*nat state*, *CombState*) *alg_on_init*
where
COMB_init h init =
Sum_pmf 0.8 (*fst BIT init*) (*fst* (*embed* (*rTS h*)) *init*)

lemma *COMB_init*[*simp*]: *COMB_init h init* =
do {
(*b*::*bool*) ← (*bernoulli_pmf 0.8*);
(*xs*::*bool list*) ← *Prob_Theory.bv* (*length init*);
return_pmf (*if b then Inl* (*xs*, *init*) *else Inr h*)
}
⟨*proof*⟩

definition *COMB_step* :: (*nat state*, *CombState*, *nat*, *answer*) *alg_on_step*
where
COMB_step s q = (*case snd s of Inl b* ⇒ *map_pmf* (λ((*a*,*b*),*c*). ((*a*,*b*),*Inl c*)) (*BIT_step* (*fst s*, *b*) *q*)
| *Inr b* ⇒ *map_pmf* (λ((*a*,*b*),*c*). ((*a*,*b*),*Inr c*)) (*return_pmf* (*TS_step_d* (*fst s*, *b*) *q*)))

definition *COMB h* = (*COMB_init h*, *COMB_step*)

18.2 Comb 1.6-competitive on 2 elements

abbreviation *noc* == (%*x*. *case x of Inl* (*s*,*is*) ⇒ (*s*,*Inl is*) | *Inr* (*s*,*is*) ⇒ (*s*,*Inr is*))
abbreviation *con* == (%(*s*,*is*). *case is of Inl is* ⇒ *Inl* (*s*,*is*) | *Inr is* ⇒ *Inr* (*s*,*is*))

definition *inv_COMB s x i* == (∃ *Da Db*. *finite* (*set_pmf Da*) ∧ *finite* (*set_pmf Db*) ∧
(*map_pmf con s*) = *Sum_pmf 0.8 Da Db* ∧ *BIT_inv Da x i* ∧ *TS_inv Db x i*)

lemma *noccon*: *noc o con* = *id*
⟨*proof*⟩

lemma *connoc*: *con o noc* = *id*

⟨*proof*⟩

lemma *obligation1′*: **assumes** *map_pmf con s* = *Sum_pmf* (*8* / *10*) *Da Db*
shows *config′_rand* (*COMB h*) *s qs* =
map_pmf noc (*Sum_pmf* (*8* / *10*) (*config′_rand BIT Da qs*)
(*config′_rand* (*embed* (*rTS h*)) *Db qs*))
⟨*proof*⟩

lemma *obligation1″*:
shows *config_rand* (*COMB h*) *init qs* =
map_pmf noc (*Sum_pmf* (*8* / *10*) (*config_rand BIT init qs*)
(*config_rand* (*embed* (*rTS h*)) *init qs*))
⟨*proof*⟩

lemma *obligation1*: **assumes** *map_pmf con s* = *Sum_pmf* (*8* / *10*) *Da Db*
shows *map_pmf con* (*config′_rand* (*COMB* []) *s qs*) =
Sum_pmf (*8* / *10*) (*config′_rand BIT Da qs*)
(*config′_rand* (*embed* (*rTS* [])) *Db qs*)
⟨*proof*⟩

lemma *BIT_config′_fin*: *finite* (*set_pmf s*) ⟹ *finite* (*set_pmf* (*config′_rand BIT s qs*))
⟨*proof*⟩

lemma *TS_config′_fin*: *finite* (*set_pmf s*) ⟹ *finite* (*set_pmf* (*config′_rand* (*embed* (*rTS h*)) *s qs*))
⟨*proof*⟩

lemma *obligation2*: **assumes** *map_pmf con s* = *Sum_pmf* (*8* / *10*) *Da Db*
and *finite* (*set_pmf Da*)
and *finite* (*set_pmf Db*)
shows T_{p}*_on_rand′* (*COMB* []) *s qs* =
2 / *10* ∗ T_{p}*_on_rand′* (*embed* (*rTS* [])) *Db qs* +
8 / *10* ∗ T_{p}*_on_rand′ BIT Da qs*
⟨*proof*⟩

lemma *Combination*:
fixes *bit*
assumes *qs* ∈ *pattern a* ≠ *b* {*a*, *b*} = {*x*, *y*} *set qs* ⊆ {*a*, *b*}
and *inv_COMB s a* [*x*,*y*]
and *TS*: ⋀*s h*. *a* ≠ *b* ⟹ {*a*, *b*} = {*x*, *y*} ⟹ *TS_inv s a* [*x*, *y*] ⟹

set qs $\subseteq \{a, b\}$
$\Longrightarrow$ *qs* $\in$ *pattern* $\Longrightarrow$
TS_inv (*config'_rand* (*embed* (*rTS h*)) *s qs*) (*last qs*) [*x*, *y*]
$\wedge$ T_p*_on_rand'* (*embed* (*rTS h*)) *s qs* = *ts*
and *BIT*: $\bigwedge$*s*. *a* $\neq$ *b* $\Longrightarrow$ $\{a, b\} = \{x, y\}$ $\Longrightarrow$ *BIT_inv s a* [*x*, *y*] $\Longrightarrow$ *set qs* $\subseteq \{a, b\}$
$\Longrightarrow$ *qs* $\in$ *pattern* $\Longrightarrow$
BIT_inv (*config'_rand BIT s qs*) (*last qs*) [*x*, *y*]
$\wedge$ T_p*_on_rand' BIT s qs* = *bit*
and *OPT_cost*: *a* $\neq$ *b* $\Longrightarrow$ *qs* $\in$ *pattern* $\Longrightarrow$ *real* (T_p [*a*, *b*] *qs* (*OPT2 qs* [*a*, *b*])) = *opt*
and *absch*: *qs* $\in$ *pattern* $\Longrightarrow$ *0.2* $*$ *ts* + *0.8* $*$ *bit* $\leq$ *1.6* $*$ *opt*
shows T_p*_on_rand'* (*COMB* []) *s qs* $\leq$ *16 / 10* $*$ *real* (T_p [*a*, *b*] *qs* (*OPT2 qs* [*a*, *b*])) $\wedge$
inv_COMB (*Partial_Cost_Model.config'_rand* (*COMB* []) *s qs*) (*last qs*) [*x*, *y*]
⟨*proof*⟩

theorem *COMB_OPT2'*: (*x*::*nat*) $\neq$ *y* $\Longrightarrow$ *set* $\sigma \subseteq \{x,y\}$
$\Longrightarrow$ T_p*_on_rand* (*COMB* []) [*x*,*y*] σ $\leq$ *1.6* $*$ *real* (T_p*_opt* [*x*,*y*] σ) + *1.6*
⟨*proof*⟩

18.3 COMB pairwise

lemma *config_rand_COMB*: *config_rand* (*COMB h*) *init qs* = *do* {
(*b*::*bool*) ← (*bernoulli_pmf 0.8*);
(*b1*,*b2*) ← (*config_rand BIT init qs*);
(*t1*,*t2*) ← (*config_rand* (*embed* (*rTS h*)) *init qs*);
return_pmf (*if b then* (*b1*, *Inl b2*) *else* (*t1*, *Inr t2*))
} (**is** *?LHS* = *?RHS*)
⟨*proof*⟩

lemma *COMB_no_paid*: $\forall$((*free*, *paid*), *t*)$\in$*set_pmf* (*snd* (*COMB* []) (*s*, *is*) *q*). *paid* = []
⟨*proof*⟩

lemma *COMB_pairwise*: *pairwise* (*COMB* [])
⟨*proof*⟩

18.4 COMB 1.6-competitive

lemma *finite_config_TS*: *finite* (*set_pmf* (*config''* (*embed* (*rTS h*)) *qs init n*)) (**is** *finite ?D*)
⟨*proof*⟩

lemma *COMB_has_finite_config_set*: **assumes** [*simp*]: *distinct init*
and *set qs* ⊆ *set init*
shows *finite* (*set_pmf* (*config_rand* (*COMB h*) *init qs*))
⟨*proof*⟩

theorem *COMB_competitive*: ∀ *s0*∈{*x*::*nat list*. *distinct x* ∧ *x*≠[]}.
∃ *b*≥*0*. ∀ *qs*∈{*x*. *set x* ⊆ *set s0*}.
T_p_*on_rand* (*COMB* []) *s0 qs* ≤ ((*8*::*nat*)/(*5*::*nat*)) ∗ T_p_*opt s0 qs* + *b*
⟨*proof*⟩

theorem *COMB_competitive_nice*: *compet_rand* (*COMB* []) ((*8*::*nat*)/(*5*::*nat*)) {*x*::*nat list*. *distinct x* ∧ *x*≠[]}
⟨*proof*⟩

end

References

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